SOME APPROXIMATION PROPERTIES OF SINGULAR INTEGRAL OPERATORS

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ÖZLEM DUMAN

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Prof.Dr. Nihat ÇELEBİ

Director

I certify that this thesis satisfies all the requirements as a thesis for the degree of Master of Science.

Assoc. Prof. Dr. Harun KARSLI

Head of Department

This is to certify that we have read this thesis and that in our opinion it is fully adequate, in scope and quality as a thesis for the degree of Master of Science.

Assoc. Prof. Dr. Harun KARSLI

Supervisor

Examining Committee Members

Assoc. Prof. Dr. Ayşegül ERENÇİN

Assoc. Prof. Dr. Harun KARSLI

Assist.Prof. Dr. M. Seyyit SEYYİDOĞLU

ABSTRACT

SOME APPROXIMATION PROPERTIES OF SINGULAR INTEGRAL OPERATORS

Duman , Özlem

Master, Department of Mathematics

Supervisor: Assoc. Prof. Dr. Harun Karslı

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This thesis is a survey on some approximation properties of singular integral operators in the approximation theory. This thesis consists of four chapters. Firstly, a short history of the studies on the approximation properties of singular integral and some basic definitions and theorems were given. In the second chapter, the solution of the heat equation is the starting point of our study of Fourier series. Next a closer look at the partial sums of Fourier series were given. Using the formula for the Fourier coefficients, the key observation was made such that these sums could be written as integrals called singular integral of f with Dirichlet kernel or the convolution of f with Dirichlet kernel, which led us to investigate the limiting properties of the convolutions as n tends to infinity. In this investigation, there found the kernels satisfying Approximate Identity whose convolutions with f tends to f. Unfortunately, Dirichlet kernel was not Approximate Identity. At this stage of the problem of convergence, various other methods of summing the Fourier series of a function f could be considered that led to convolutions of the function f with kernels satisfying Approximate Identity and in the third chapter some important theorems about the convolutions of f with some kernels were given. In the last chapter, the study of Assoc. Dr. Harun Karslı, which deals with the theory of approximation of singular integral operators was further investigated.

Keywords: Heat equation, Fourier series, singular integrals, Approximate Identity, Dirichlet kernel, Césaro summation, Poisson kernel, Abel summation, Féjer theorem

ÖZET

SİNGULAR İNTEGRAL OPERATÖRLERİNİN YAKLAŞIM ÖZELLİKLERİ

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Bu tez, yaklaşım teorisindeki singular integral operatörlerinin yaklaşım özellikleri üzerine yapılan bir çalışmadır. Bu tez dört bölümden oluşmaktadır. İlk olarak, singular integrallerin yaklaşım özellikleri üzerine yapılan çalışmaların kısa bir tarihçesi ve bazı temel tanım ve teoremler verilmiştir. İkinci bölümde, ısı denkleminin çözümü, Fourier serileri üzerindeki çalışmamızın başlangıç noktası olmuştur. Sonrasında Fourier serilerinin kısmi toplamı daha yakından incelenmiştir. Fourier katsayılarının formülü kullanılarak bu toplamların Dirichlet çekirdekli f fonksiyonunun singular integrali veya Dirichlet çekirdekli f fonksiyonunun konvolüsyonu olarak adlandırılan integraller şeklinde yazılabileceği gözlemi yapılmıştır. Bu da bizi n sonsuza giderken konvolüsyonun limit özelliklerini incelemeye sevk etmiştir. Bu araştırmada f fonksiyonu ile konvolüsyonu f fonksiyonuna giden Approximate Identity özelliğini sağlayan çekirdekler bulunmuştur. Ama Dirichlet çekirdeği Approximate Identity özelliklerini sağlamamaktadır. Yaklaşım probleminin bu aşamasında Approximate Identitiy özelliğini sağlayan çekirdeklerle konvolüsyona götüren birçok farklı Fourier serilerinin toplama metodu araştırıldı ve üçüncü bölümde f fonksiyonu ile oluşturulan konvolüsyonunun yaklaşım özellikleri ile ilgili önemli teoremler verilmiştir. Son bölümde Doç. Dr. Harun Karslı hocamın çalışması olan singular integral operatorlerin yaklaşım teorisi üzerinde çalışılmıştır.

Anahtar Kelimeler: Isı denklemi, Fourier serileri, singular integralleri, Approximate Identity, Dirichlet çekirdeği, Césaro toplamı, Poisson çekirdeği, Abel toplamı, Féjer teoremi.

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LIST OF SYMBOLS

SYMBOLS

$\mathbb{C}$	Complex plane or the field of complex numbers
C [a,b]	Space of continuous functions
C(f)	The set of points of continuity
D(T)	Domain of an operator T
D(f)	The set of d-points
$D_N(x)$	Dirichlet kernel
$F_N(x)$	Fejér kernel
L(f)	The set of Lebesgue points
$L^p[a,b]$	A function space
$\mathbb{R}$	Real line or the field of real numbers
$\mathbb{R}^n$	Euclidean n-space
R(T)	Range of an operator T
$R_r(\theta)$	Poisson kernel
$S_N(f)$	Nth partial sum
f*g	Convolution of f and g
$\|x\|$	Norm of x
a.e	Almost everywhere
ess sup	Essential supremum

CHAPTER 1

INTRODUCTION

One of the fundamental problems of analysis is to approximate a given function f in some sense or other by functions having certain properties, and generally, by functions with better properties than f. It is to be expected that the better-behaved functions are to be constructed from the given f by some smoothing operation on f itself. The approximation of f by singular convolution integrals is of special interest.

1.1 Basic Definitions and Theorems

In this section, we give some basic definitions and useful theorems for the Fourier analysis which will be used throughout this thesis.

Definition 1.1 (Linear Space) *A nonempty set X is said to be a "linear space" (or a vector space) over a field $\mathbb{K}$, if it satisfies the following conditions:*

(i) $\forall x, y \in X,\ x + y \in X$

(ii) $\forall x, y \in X,\ x + y = y + x$

(iii) $\forall x, y, z \in X,\ x + (y + z) = (x + y) + z$

(iv) $\forall x \in X,\ \exists \theta \in X$ *such that* $x + \theta = \theta + x = x$

(v) $\forall x \in X,\ \exists \tilde{x} \in X$ *such that* $x + \tilde{x} = \theta$

(vi) $\forall x \in X$ *and* $\forall \alpha \in \mathbb{K},\ \alpha x \in X$

(vii) $\forall x \in X$ *and* $\forall \alpha, \beta \in \mathbb{K},\ \alpha(\beta x) = (\alpha\beta)x$

(viii) $\forall x \in X,\ 1x = x$

(ix) $\forall x \in X$ *and* $\forall \alpha \in \mathbb{K},\ \ \alpha(x+y) = \alpha x + \alpha y$

(x) $\forall x \in X$ *and* $\forall \alpha, \beta \in \mathbb{K},\ \ (\alpha+\beta)x = \alpha x + \beta x.$

Definition 1.2 (Linear Normed Space) *Let X be a linear space over a field $\mathbb{K}$. A function $\|.\| : X \longrightarrow \mathbb{R}^+$ satisfying the following conditions is said to be a "norm".*

(i) $\forall x \in X,\ \|x\| \geq 0$

(ii) $\forall x \in X,\ \|x\| = 0 \iff x = 0$

(iii) $\forall x \in X$ *and* $\forall \alpha \in \mathbb{K},\ \|\alpha x\| = |\alpha|\ \|x\|$

(iv) $\forall x, y \in X,\ \|x+y\| \leq \|x\| + \|y\|.$

A linear space on which a norm is defined is then called a "linear normed space".

Definition 1.3 (Linear Operator) *A linear operator T is an operator such that*

(i) *the domain $D(T)$ of T is a vector space and the range $R(T)$ lies in a vector space over the same field,*

(ii) *for each $x, y \in D(T)$ and scalar α,*

$$T(x+y) = Tx + Ty$$

and

$$T(\alpha x) = \alpha T x.$$

*(*Kreyszig, 1989*)*

Definition 1.4 (Integral Operator) *We can define an integral operator $T : C[0,1] \to C[0,1]$ by*

$$y = Tx \text{ , where } y(t) = \int_0^1 x(t)\, k(t,\tau)\, d\tau.$$

*Here k is a given function which is called the kernel of T and is assumed to be continuous on the closed square $G = J \times J$ in the $t\tau-$plane, where $J = [0, 1]$. This operator is linear. (*Kreyszig, 1989*)*

Definition 1.5 (Periodic Function) *A piecewise continuous function $f(x)$ in an interval $[a, b]$ is said to be periodic, if there exists a real positive number such that*

$$f(x + p) = f(x)$$

for all x, p is called the period of f and the smallest value of p is termed the fundamental period.
We always deal with $2\pi-$periodic functions. If f has period 2π, thus

$$f(x + 2\pi) = f(x)$$

for all x.

Definition 1.6 (L^p Space) *L^p is the set of functions which are Lebesgue integrable to the pth power over $\mathbb{R}$, for $1 \leq p < \infty$, and essentially bounded (bounded almost everywhere) on $\mathbb{R}$ if $p = \infty$. For $f \in L^p$*

$$\|f\|_p = \left\{ \int_{-\infty}^{\infty} |f(t)|^p \, dt \right\}^{\frac{1}{p}},$$

if $1 \leq p < \infty$ and in case $p = \infty$

$$\|f\|_\infty = ess \sup_{x \in \mathbb{R}} |f(t)|.$$

*Thus L^p consists precisely of those functions f for which the norm $\|f\|_p$ is finite number. (*Butzer, 1971*)*

Definition 1.7 $\left(L^1[a, b]\ \textbf{\textit{Space}}\right)$ *$L^1[a, b]$ is the set of functions which are Lebesgue*

integrable to the 1*th power over* $[a,b]$*. For* $f \in L^1[a,b]$

$$\|f\|_1 = \int_a^b |f(t)|\, dt.$$

*(*Butzer, 1971*)*

Definition 1.8 $\left(L^2[a,b]\ \textbf{\textit{Space}}\right)$ $L^2[a,b]$ *is the set of functions which are Lebesgue integrable to the* 2*th power over* $[a,b]$*. For* $f \in L^2[a,b]$

$$\|f\|_2 = \left(\int_a^b |f(t)|^2\, dt\right)^{\frac{1}{2}}.$$

*(*Butzer, 1971*)*

Definition 1.9 (Orthogonal and Orthonormal System) *A system of functions* $\varphi_m(x) \in L^2[a,b]$ *,* $(m = 1,2,...)$ *is said to be orthogonal on the interval* $[a,b]$*, if*

$$\int_a^b \varphi_m(x)\,\varphi_n(x)\, dx = 0,$$

where $m \neq n$*,* $(n = 1,2,...)$ *and*

$$\int_a^b \varphi_m^2(x)\, dx \neq 0.$$

The orthogonal system is said to be normal if

$$\int_a^b \varphi_m^2(x)\, dx = 1.$$

If the functions $\varphi_m(x) \in L^2[a,b]$ *,* $(m = 1,2,...)$ *are complex functions of the real variable x, then they are said to be orthogonal when*

$$\int_a^b \varphi_m(x)\,\bar{\varphi}_n(x)\, dx = 0,$$

where $m \neq n$, $(n = 1, 2, ...)$ *and*

$$\int_a^b |\varphi_m(x)|^2 \, dx \neq 0.$$

The system is normal if

$$\int_a^b |\varphi_m(x)|^2 \, dx = 1.$$

*(*Bary, 1964*)*

Lemma 1.10 *Let* f *is* $2\pi-$*periodic and integrable on any finite interval* $[a, b]$*,*

then

$$\int_a^b f(x)\,dx = \int_{a+2\pi}^{b+2\pi} f(x)\,dx = \int_{a-2\pi}^{b-2\pi} f(x)\,dx,$$

and

$$\int_{-\pi}^{\pi} f(x+a)\,dx = \int_{-\pi}^{\pi} f(x)\,dx = \int_{-\pi+a}^{\pi+a} f(x)\,dx.$$

Lemma 1.11 (Fatou's Lemma) *Let* $\{f_n\}_{n=1}^{\infty}$ *be a sequence of positive measurable functions defined on* $\mathbb{R}$ *. If*

$$\liminf_{n\to\infty} f_n(x) = f(x) \quad a.e.,$$

then

$$\int_{-\infty}^{\infty} f(u)\,du \leq \liminf_{n\to\infty} \int_{-\infty}^{\infty} f_n(u)\,du.$$

Theorem 1.12 (Lebesgue's Monotone Convergence Theorem) *Let* $\{f_n\}_{n=1}^{\infty}$ *be a monotonely increasing sequence of positive measurable functions defined on* $\mathbb{R}$ *. If*

$$\lim_{n\to\infty} f_n(x) = f(x) \quad a.e.,$$

then

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(u)\,du = \int_{-\infty}^{\infty} f(u)\,du.$$

This result is also known as Beppo Levi's theorem.

Theorem 1.13 (Lebesgue's Dominated Convergence Theorem) *Let* $\{f_n\}_{n=1}^{\infty} \in L^1(\mathbb{R})$, *and suppose that*

$$\lim_{n\to\infty} f_n(x) = f(x) \quad a.e.,$$

if there exists $g \in L^1(\mathbb{R})$ *such that*

$$|f_n(x)| \le g(x) \quad a.e. \quad for\ all\ n,$$

then $f \in L^1$ *and*

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(u)\,du = \int_{-\infty}^{\infty} f(u)\,du.$$

Theorem 1.14 (Fubini's Theorem) *Let* $x, y \in \mathbb{R}$, *and* $f(x,y)$ *be a (complex-valued) function of two (real) variables defined and measurable on the two-dimensional Euclidean-space* $\mathbb{R}^2$.

(i) *Suppose that* $f \in L^1\left(\mathbb{R}^2\right)$, *i.e., the double integral*

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dxdy$$

is absolutely convergent. Then, for almost all x, $f(x,y)$ *is absolutely integrable over* $\mathbb{R}$ *with respect to the variable* y, *i.e.*

$$f(x,.) \in L^1(\mathbb{R}) \quad a.e.$$

Moreover,

$$\int_{-\infty}^{\infty} f(.,y)\,dy \in L^1(\mathbb{R})$$

and

$$\int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty} f(x,y)\,dy\right\}dx = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dxdy.$$

(ii) *Suppose that*

$$\int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty}|f(x,y)|\,dy\right\}dx$$

exists as a finite number. Then $f \in L^1\left(\mathbb{R}^2\right)$ *and*

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dxdy = \int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty} f(x,y)\,dy\right\}dx = \int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty} f(x,y)\,dx\right\}dy.$$

The second part of Theorem 1.14 is also associated with the name Tonelli-Hobson.

Theorem 1.15 (Theorem of B. Levi) *Let* $\{f_n\}_{n=1}^{\infty} \in L^1[a,b]$. *If*

$$\sum_{n=1}^{\infty}\int_a^b |f_n(x)|\,dx < \infty,$$

then the series $\sum_{n=1}^{\infty} f_n(x)$ *converges absolutely almost everywhere on* (a,b) *to a finite number. If this sum is denoted by* $f(x)$, *then* $f \in L^1[a,b]$ *and*

$$\int_a^b f(x)\,dx = \sum_{n=1}^{\infty}\int_a^b f_n(x)\,dx.$$

Definition 1.16 (D-Point) *A point* $x \in \mathbb{R}$ *is called a D-point of the function* f *if*

$$\lim_{h\to 0}\frac{1}{h}\int_x^{x+h}[f(t)-f(x)]\,dt = 0. \tag{1.1}$$

*(*Bary, 1964*)*

D-points stand for differentiability; for the D-points of an integrable function f are precisely those points at which the indefinite integral of f is differentiable to the value $f(x)$.

Definition 1.17 (Lebesgue Point) *A point $x \in \mathbb{R}$ is called a Lebesgue point of the function f if*

$$\lim_{h\to 0} \frac{1}{h} \int_{x}^{x+h} |f(t) - f(x)|\, dt = 0. \tag{1.2}$$

*(*Bary, 1964*)*

Evidently, any Lebesgue point of f is a D-point of f but not conversely, and any point of continuity of an integrable function f is a Lebesgue point. Indeed; let $L(f)$ be the set of all Lebesgue points of $f \in L^1$ and $D(f)$ be the set of all points of all D-points of $f \in L^1$. Then

$$\left| \frac{1}{h} \int_{x}^{x+h} [f(t) - f(x)]\, dt \right| \leq \frac{1}{h} \int_{x}^{x+h} |f(t) - f(x)|\, dt.$$

The inequality shows that every Lebesgue point is also D-point. Thus,

$$L(f) \subset D(f).$$

Now, it's clear that any point, where $f(x)$ is continuous, is a Lebesgue point; in fact, if there exists any $\varepsilon > 0$ and the function $f(x)$ is continuous at the point x, it can be found a $\delta > 0$ such that

$$|f(t) - f(x)| < \varepsilon \quad \text{for} \quad |h| \leq \delta$$

and then

$$\frac{1}{h} \int_{x}^{x+h} |f(t) - f(x)|\, dt < \varepsilon,$$

or, we can write

$$\lim_{h\to 0} \frac{1}{h} \int_{x}^{x+h} |f(t) - f(x)|\, dt = 0.$$

This shows that any continuity point of the function $f \in L^1$ is also a Lebesgue point. Let $C(f)$ be the set of continuity points of $f \in L^1$. Then we can write

$$C(f) \subset L(f) \subset D(f).$$

Definition 1.18 (Convolutions on the Line Group) *Let $\mathbb{R}$ be the set of all real numbers, also called the real line or line group and let f, g be two (complex-valued) functions defined and measurable on $\mathbb{R}$. The expression*

$$(f * g) = \frac{1}{\sqrt[2]{2\pi}} \int_{-\infty}^{\infty} f(x-u)\, g(u)\, du \tag{1.3}$$

*is called the convolution (product) of f and g. (*Butzer, 1971*)*

Definition 1.19 (Convolutions on Periodic Functions) *Given $2\pi-$periodic integrable functions f on $\mathbb{R}$ and $\{K_n\}_{n=1}^{\infty} \in L^1(\mathbb{R})$, we define their convolution (product) $(f * K_n)$ on $[-\pi, \pi]$ by*

$$(f * K_n)(x) = \frac{1}{\sqrt[2]{2\pi}} \int_{-\pi}^{\pi} f(y)\, K_n(x-y)\, dy.$$

It is called a singular integral provided that the sequence $\{K_n\}_{n=1}^{\infty}$ is a (periodic) kernel with $\int_{-\pi}^{\pi} K_n(y)\, dy = 2\pi$ for each $n \in \mathbb{N}$. The above integral makes sense for each x, since the product of two integrable functions is also integrable. Also since the functions are periodic, we can change variables to see that

$$(f * K_n)(x) = \frac{1}{\sqrt[2]{2\pi}} \int_{-\pi}^{\pi} f(x-y)\, K_n(y)\, dy.$$

*(*Butzer, 1971*)*

Now let's show the properties of convolution.

Lemma 1.20 *Suppose that f, g and h are $2\pi-$ periodic integrable functions. Then*

(i) $f * (g+h) = (f * g) + (f * h)$.

(ii) $(cf) * g = c(f * g) = f * (cg)$ *for any* $c \in \mathbb{C}$.

(iii) $f * g = g * f$.

(iv) $(f * g) * h = f * (g * h)$.

(v) $f * g$ *is continuous.*

The first four items describe the algebraic properties of convolutions: linearity, commutativity and associativity. Property (**v**) exhibits an important principle: the convolution of $(f * g)$ is more regular than f or g. Here $(f * g)$ is continuous while f and g are merely (Riemann) integrable.

CHAPTER 2

FOURIER ANALYSIS

We use Fourier series to represent or approximate functions defined on a finite interval. Fourier series also have various specific properties of their own and we shall study some of them. Fourier series were first defined by Jean Baptiste Joseph Fourier (1768-1830) about 200 years ago. Fourier began this theory in 1804 by the work on 'Heat Equation'. He developed these series in order to solve certain problems in partial differential equation.

2.1 Solution of the Heat Equation

We will start our study by solving the heat equation[4].

$$\frac{\partial u}{\partial t} - \kappa \frac{\partial^2 u}{\partial x^2} = 0,$$

where $u(x,t)$ is a function of two variables and $\kappa > 0$. In the heat equation, x represents the position along the bar of length l measured from some origin, t represents time, $u(x,t)$ the temperature at position x, time t. The temperature $u(x,t)$ along the bar satisfies the heat equation

$$u_t - \kappa u_{xx} = 0, \quad 0 < x < l, \quad t > 0, \tag{2.1}$$

and boundary conditions

$$u(0,t) = 0, \quad u(l,t) = 0, \quad t > 0, \tag{2.2}$$

let the initial temperature along the bar at $t = 0$ be given by

$$u(x,0) = f(x), \quad 0 < x < l. \tag{2.3}$$

Assume that the heat equation has a solution of the form

$$u(x,t) = X(x)\,T(t), \quad X \neq 0, \quad T \neq 0.$$

Such a solution is called a seperation of variables solution. Substituting into the heat equation, we have

$$X(x)\,T'(t) = \kappa X''(x)\,T(t).$$

Dividing by $\kappa X(x)\,T(x)$, leads to

$$\frac{T'(t)}{\kappa T(t)} = \frac{X''(x)}{X(x)}.$$

Since the variables x, t appear on seperate sides of this equation, each side of this equality can only be equal to a constant, say λ. Then

$$T'(t) = \kappa\lambda T(t) \quad \text{and} \quad X''(x) = \lambda X(x).$$

$$T'(t) - \kappa\lambda T(t) = 0 \quad \text{and} \quad X''(x) - \lambda X(x) = 0.$$

These are ordinary differential equations with constant coefficient. λ is real but could be positive, negative or zero.

Now, first case: $\lambda > 0$. The solutions are

$$T(t) = Ce^{\kappa\lambda t}, \quad X(x) = Ae^{x\sqrt{\lambda}} + Be^{-x\sqrt{\lambda}},$$

where A, B, C are arbitrary constants. From the boundary conditions (2.2), we have

$$u(0,t) = 0, \quad u(l,t) = 0, \quad t > 0 \quad \text{if and only if} \quad X(0) = 0, \quad X(l) = 0.$$

That is,

$$0 = X(0) = A + B,$$

$$0 = X(l) = Ae^{l\sqrt{\lambda}} + Be^{-l\sqrt{\lambda}}.$$

This is an algebraic system in A and B. In order to have a nontrivial solution, we must have

$$e^{l\sqrt{\lambda}} - e^{-l\sqrt{\lambda}} = 0,$$

which holds only when $\lambda = 0$. There are no values of $\lambda > 0$.

Second case: For $\lambda = 0$,

$$T'(t) = 0 \quad \text{and} \quad X''(x) = 0,$$

hence

$$T(t) = C \quad \text{and} \quad X(x) = Ax + B.$$

Because of boundary conditions (2.2), we have

$$0 = X(0) = B, \quad 0 = X(l) = A.l,$$

and,

$$A = B = 0.$$

Then

$$u(x,t) = X(x)\,T(t) = 0.C,$$

$$u(x,t) = 0.$$

It is the trivial solution.

Third case: For $\lambda < 0$, let $\lambda = -\mu^2$ for some real μ. Then we have $\sqrt{\lambda} = i\mu$ and so

$$X''(x) + \mu^2 X(x) = 0.$$

The general solution of this equation is

$$X(x) = A\cos\mu x + B\sin\mu x.$$

From (2.2), we have

$$X(0) = A = 0$$

and

$$X(l) = B\sin\mu l = 0,$$

where $B \neq 0$, then

$$\sin\mu l = 0.$$

This equation has solutions $\mu l = n\pi,\ n = \pm1, \pm2, \ldots$, and hence $\mu = \frac{n\pi}{l}$, $n = \pm1, \pm2, \ldots$, and we can find

$$T_n(t) = C_n e^{-\frac{\kappa n^2\pi^2 t}{l^2}}, \quad X_n(x) = B_n \sin\left(\frac{n\pi x}{l}\right), \quad n = 1, 2 \ldots.$$

Therefore the nontrivial solutions of the heat equation satisfying boundary conditions are

$$u_n(x,t) = b_n e^{-\frac{\kappa n^2\pi^2 t}{l^2}} \sin\left(\frac{n\pi x}{l}\right), \quad n = 1, 2, \ldots \quad \text{where} \ \ b_n = C_n B_n.$$

By using superposition principle

$$u(x,t) \cong \sum_{n=1}^{\infty} b_n u_n(x,t) \cong \sum_{n=1}^{\infty} b_n e^{-\frac{\kappa n^2\pi^2 t}{l^2}} \sin\left(\frac{n\pi x}{l}\right),$$

which can satisfy the heat equation and the boundary conditions. For the initial condition (2.3) to be satisfied

$$u(x,0) = f(x) \cong \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right), \quad 0 < x < l. \tag{2.4}$$

Now consider the following boundary and initial value problem for the heat equation

$$u_t - \kappa u_{xx} = 0, \quad 0 < x < l, \quad t > 0$$

$$u(x,0) = f(x), \quad 0 < x < l,$$

the boundary conditions are

$$u_x(0,t) = 0, \quad u_x(l,t) = 0, \quad t > 0.$$

Nontrivial solutions of the heat equation satisfying boundary conditions are

$$u_n(x,t) = a_n e^{-\frac{\kappa n^2 \pi^2 t}{l^2}} \cos\left(\frac{n\pi x}{l}\right), \quad n = 0, 1, 2, \ldots.$$

By using superposition principle

$$u(x,t) \cong \sum_{n=0}^{\infty} a_n u_n(x,t) \cong \sum_{n=0}^{\infty} a_n e^{-\frac{\kappa n^2 \pi^2 t}{l^2}} \cos\left(\frac{n\pi x}{l}\right),$$

which can satisfy the heat equation and the boundary conditions. For the initial condition to be satisfied

$$u(x,0) = f(x) \cong \sum_{n=0}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right), \quad 0 < x < l. \tag{2.5}$$

Definition 2.1 *Let $f(x)$ be a real-valued piecewise continuous function defined on the interval $[-\pi, \pi]$. The series associated with f is called the Fourier series of f,*

$$f(x) \cong \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx] \tag{2.6}$$

also as a special the series (2.4) is called a 'Fourier sine series expansion' and the series (2.5) is also called a 'Fourier cosine series expansion',
where

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx, \quad n = 0, 1, 2, \ldots, \tag{2.7}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx, \quad n = 1, 2, \ldots. \tag{2.8}$$

These coefficients were found using trigonometric systems. Since trigonometric systems with factor $\frac{1}{\sqrt[2]{\pi}}$,

$$\frac{1}{\sqrt{2\pi}}, \frac{\cos x}{\sqrt{\pi}}, \frac{\sin x}{\sqrt{\pi}}, \frac{\cos 2x}{\sqrt{\pi}}, \frac{\sin 2x}{\sqrt{\pi}} \ldots \quad \text{defined on } [-\pi, \pi],$$

are orthonormal system.

Remark 2.2 *All of the orthonormal system is contained in* L^2

To find these formulas (2.7) and (2.8) for the coefficients a_n and b_n in terms of f, we start by assuming that the trigonometric series converges and has a continuous function $f(x)$ as its sum on the interval $[-\pi, \pi]$, that is,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx]. \tag{2.9}$$

If we integrate both sides of this equation and assume that it's permitable to integrate the series term by term, we get,

$$\begin{aligned}\int_{-\pi}^{\pi} f(x)\,dx &= \int_{-\pi}^{\pi} \frac{a_0}{2} dx + \int_{-\pi}^{\pi} \left\{ \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx] \right\} dx \\ &= 2\pi\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi} \cos nx dx + \sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi} \sin nx dx,\end{aligned}$$

but since n is integer,

$$\int_{-\pi}^{\pi} \cos nx dx = 0,$$

and similarly, sine is an odd function so,

$$\int_{-\pi}^{\pi} \sin nx dx = 0,$$

therefore, we have

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx.$$

To determine a_n for $n \geq 1$ we multiply both sides of the equation (2.9) by $\cos mx$ where m is an integer and $m \geq 1$ and integrate term-by-term from $-\pi$ to π.

$$\begin{aligned}
\int_{-\pi}^{\pi} f(x) \cos mx dx &= \int_{-\pi}^{\pi} \left\{ a_0 + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx] \right\} \cos mx dx \\
&= a_0 \int_{-\pi}^{\pi} \cos mx dx + \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi} \cos nx \cos mx dx \\
&\quad + \sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi} \sin nx \cos mx dx,
\end{aligned}$$

we have seen that the first integral is 0 and by definition of orthogonal system the third integral is also 0. So we get

$$\int_{-\pi}^{\pi} f(x) \cos mx dx = \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi} \cos nx \cos mx dx,$$

to find the coefficient a_n, we take $m = n$, because of the definition of orthonormal system, we have

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx, \quad n = 1, 2, 3....$$

Similarly, if we multiply both sides of the equation (2.9) by $\sin mx$ and integrate from $-\pi$ to π, we get

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx, \quad n = 1, 2, \ldots$$

Definition 2.3 *Let $f(x)$ be a complex-valued piecewise continuous function defined*

on the interval $[-\pi, \pi]$. *Then the series*

$$f(x) \cong \sum_{n=-\infty}^{\infty} c_n e^{inx} \tag{2.10}$$

where

$$c_n \cong \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx, \quad n = 0, \pm 1, \pm 2 \ldots \tag{2.11}$$

is called the complex Fourier series of f .

From the well-known Euler's identity,

$$e^{inx} = \cos nx + i \sin nx,$$

it follows that,

$$\cos nx = \frac{e^{inx} + e^{-inx}}{2}, \quad \sin nx = \frac{e^{inx} - e^{-inx}}{2i}$$

and as using the orthonormal system $\left\{ \frac{e^{inx}}{\sqrt[2]{2\pi}} \right\}$ for $n = \pm 1, \pm 2, \ldots$ in L^2 we can find the coefficients c_n. If we can write the series in (2.9) with respect to $\left\{ e^{inx} \right\}$, that is,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \left(\frac{e^{inx} + e^{-inx}}{2} \right) + b_n \left(\frac{e^{inx} - e^{-inx}}{2i} \right) \right],$$

we have the complex Fourier series in the form (2.10) where

$$c_0 = \frac{a_0}{2}, \quad c_n = \frac{a_n - ib_n}{2}, \quad c_{-n} = \frac{a_n + ib_n}{2}, \quad n = 1, 2, 3, \ldots.$$

Theorem 2.4 *Let* f *be* 2π*-periodic and Riemann integrable and complex-valued on* $[-\pi, \pi]$. *Then*

$$\sum_{n=-\infty}^{\infty} |c_n|^2 \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 \, dx. \tag{2.12}$$

Proof. Use $z\bar{z} = |z|^2$ for complex z, and consider the partial sum of complex Fourier series

$$S_N(f)(x) = \sum_{n=-N}^{N} c_n e^{inx}$$

$$
\begin{aligned}
0 \le \left| f(x) - \sum_{n=-N}^{N} c_n e^{inx} \right|^2 &= \left(f(x) - \sum_{n=-N}^{N} c_n e^{inx} \right) \overline{\left(f(x) - \sum_{m=-N}^{N} c_m e^{imx} \right)} \\
&= |f(x)|^2 - \sum_{n=-N}^{N} c_n e^{inx} \overline{f(x)} - \sum_{m=-N}^{N} \overline{c_m} e^{-imx} f(x) \\
&\quad + \sum_{n=-N}^{N} \sum_{m=-N}^{N} c_n \overline{c_m} e^{i(n-m)x}
\end{aligned}
$$

Dividing by 2π and integrating over $[-\pi, \pi]$,

$$
\begin{aligned}
0 &\le \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 \, dx - \sum_{n=-N}^{N} c_n \frac{1}{2\pi} \overline{\left(\int_{-\pi}^{\pi} e^{-inx} f(x) \, dx \right)} - \sum_{m=-N}^{N} \overline{c_m} \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-imx} f(x) \, dx \\
&\quad + \sum_{n=-N}^{N} \sum_{m=-N}^{N} c_n \overline{c_m} \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(n-m)x} dx \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 \, dx - \sum_{n=-N}^{N} c_n \overline{c_n} - \sum_{m=-N}^{N} c_m \overline{c_m} + \sum_{n=-N}^{N} c_n \overline{c_n} \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 \, dx - \sum_{n=-N}^{N} |c_n|^2
\end{aligned}
$$

for any N. Letting $N \to \infty$ leads to the result. □

It implies that the series $\sum_{n=-\infty}^{\infty} |c_n|^2$ converges where c_n are the Fourier coefficients of the Riemann integrable function f. Using the equations

$$
c_0 = \frac{a_0}{2}, \quad c_n = \frac{a_n - ib_n}{2}, \quad c_{-n} = \frac{a_n + ib_n}{2}, \quad n = 1, 2, \ldots .
$$

The inequality (2.12) can be written as

$$
\begin{aligned}
\sum_{n=-\infty}^{\infty} |c_n|^2 &= \frac{a_0^2}{4} + \sum_{n=1}^{\infty} \left| \frac{a_n - ib_n}{2} \right|^2 + \sum_{n=1}^{\infty} \left| \frac{a_n + ib_n}{2} \right|^2 \\
&= \frac{a_0^2}{4} + \frac{1}{2} \sum_{n=1}^{\infty} \left[|a_n|^2 + |b_n|^2 \right] \\
&\le \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 \, dx.
\end{aligned}
$$

This implies that the series $\sum_{n=1}^{\infty} |a_n|^2$, $\sum_{n=1}^{\infty} |b_n|^2$ also converge, where a_n, b_n are the Fourier cosine and sine coefficients of f. This inequality (2.12) is known as Bessel's inequality.

Corollary 2.5 *This theorem says that the general term of Fourier series tends to zero as n goes to ∞. It shows that the Fourier series can be convergent.*

Now, we will investigate the convergence of this series of the form (2.10). For this purpose we consider the N^{th} partial sum of the complex Fourier series of f,

$$S_N(f)(x) = \sum_{n=-N}^{N} c_n e^{inx}. \tag{2.13}$$

If the sequence (2.13) converges at a point x, we say that the Fourier series (2.10) converges at x. It would have been nice if such convergence did take place at every point x. Unfortunately, this is not the case. There are continuous functions for which the series (2.10) diverges for uncountably many x. Now we can proceed in two directions

(1) The addition conditions on f or

(2) Summability techniques.

In the first direction, we will see that if f is not only continuous but also differentiable, then the series (2.10) is convergent at every point x to the limit $f(x)$. More generally, this is true when f is not necessarily differentiable, but satisfies Lipschitz condition or is of bounded variation. In the second direction we will see that the Fourier series of any continuous function converges in the sense of Abel summation and Cesáro summation. From (2.13), we have

$$\begin{aligned} S_N(f)(x) &= \sum_{n=-N}^{N} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-iny} f(y)\, dy \right) e^{inx} \\ &= \int_{-\pi}^{\pi} \left(\frac{1}{2\pi} \sum_{n=-N}^{N} e^{in(x-y)} \right) f(y)\, dy. \end{aligned}$$

We define $D_N(x) = \frac{1}{2\pi} \sum_{n=-N}^{N} e^{inx}$. Then

$$S_N(f)(x) = \int_{-\pi}^{\pi} D_N(x-y)\, f(y)\, dy = \int_{-\pi}^{\pi} D_N(y)\, f(x-y)\, dy,$$

we find convolution integral or singular integral from partial sums of Fourier series, and we can write

$$(f * D_N)(x) = \int_{-\pi}^{\pi} D_N(x-y)\, f(y)\, dy = \int_{-\pi}^{\pi} D_N(y)\, f(x-y)\, dy.$$

So,

$$\begin{aligned} D_N(x) &= \frac{1}{2\pi} e^{-iNx} \sum_{n=0}^{2N} e^{inx} \\ &= \frac{1}{2\pi} e^{-iNx} \left(\frac{e^{i(2N+1)x} - 1}{e^{ix} - 1} \right) \\ &= \frac{1}{2\pi} \left(\frac{e^{i(N+1)x} - e^{-iNx}}{e^{ix} - 1} \right) \\ &= \frac{1}{2\pi} \left(\frac{e^{i\left(N+\frac{1}{2}\right)x} - e^{-i\left(N+\frac{1}{2}\right)x}}{e^{\frac{ix}{2}} - e^{-\frac{ix}{2}}} \right) \\ &= \frac{1}{2\pi} \frac{\sin\left(N+\frac{1}{2}\right)x}{\sin\left(\frac{x}{2}\right)}. \end{aligned}$$

Now let's show that D_N is a kernel:

Lemma 2.6

$$\int_{-\pi}^{0} D_N(x)\, dx = \int_{0}^{\pi} D_N(x)\, dx = \frac{1}{2}.$$

Proof.

$$\begin{aligned} D_N(x) &= \frac{1}{2\pi} + \frac{1}{2\pi} \sum_{\substack{n=-N \\ n \neq 0}}^{N} e^{inx} \\ &= \frac{1}{2\pi} + \frac{1}{\pi} \sum_{n=1}^{N} \cos nx, \end{aligned}$$

integrating both sides,

$$\begin{aligned}\int_{-\pi}^{0} D_N(x)\,dx = \int_{0}^{\pi} D_N(x)\,dx &= \int_{0}^{\pi}\left[\frac{1}{2\pi}+\frac{1}{\pi}\sum_{n=1}^{N}\cos nx\right]dx \\ &= \left[\frac{x}{2\pi}+\frac{1}{\pi}\sum_{n=1}^{N}\frac{\sin nx}{n}\right]_0^{\pi} \\ &= \frac{1}{2}.\end{aligned}$$

Also, we say that $D_N(x)$ is Stochastic Dirichlet kernel. □

Now we will investigate the properties of Dirichlet kernel,

(**i**)

$$\int_{-\pi}^{\pi} D_N(x)\,dx = 1.$$

It was proved in Lemma (2.6)

(**ii**) Since

$$\begin{aligned}D_N(-x) &= \frac{1}{2\pi}\frac{\sin\left(N+\frac{1}{2}\right)(-x)}{\sin\left(\frac{(-x)}{2}\right)} \\ &= \frac{1}{2\pi}\frac{\sin\left(N+\frac{1}{2}\right)x}{\sin\left(\frac{x}{2}\right)} \\ &= D_N(x).\end{aligned}$$

Hence,

$$D_N(-x) = D_N(x).$$

Dirichlet kernel is even.

(**iii**) We have

$$\lim_{N\to\infty}\int_{-\pi}^{\pi}|D_N(x)|\,dx = +\infty \tag{2.14}$$

Proof. Obviously,

$$\begin{aligned}\int_{-\pi}^{\pi}|D_N(x)|\,dx &= \frac{1}{\pi}\int_0^{\pi}\left|\frac{\sin\left(\frac{(2N+1)x}{2}\right)}{\sin\left(\frac{x}{2}\right)}\right|dx\\ &= \frac{1}{\pi}\int_0^{\pi}\left|\sin Nx\cot\frac{x}{2}+\cos Nx\right|dx\\ &= \frac{1}{\pi}\int_0^{\pi}|\sin Nx|\cot\frac{x}{2}dx+0(1).\end{aligned}$$

Since the function $\left(\frac{1}{x}\right)-\cot x$ is bounded for $|x|\leq\frac{\pi}{2}$, we have

$$\begin{aligned}&= \frac{2}{\pi}\int_0^{\pi}\frac{|\sin Nx|}{x}dx+\frac{2}{\pi}\int_0^{\pi}|\sin Nx|\left\{\frac{1}{2}\cot\frac{x}{2}-\frac{1}{x}\right\}dx+0(1)\\ &= \frac{2}{\pi}\sum_{k=0}^{N-1}\int_{\frac{k\pi}{N}}^{\frac{(k+1)\pi}{N}}\frac{|\sin Nx|}{x}dx+0(1)\\ &= \frac{2}{\pi}\sum_{k=1}^{N-1}\int_0^{\frac{\pi}{N}}\frac{\sin Nx}{\left(\frac{k\pi}{N}+x\right)}dx+\frac{2}{\pi}\int_0^{\frac{\pi}{N}}\frac{\sin Nx}{x}dx+0(1)\\ &= \frac{2}{\pi}\int_0^{\frac{\pi}{N}}\sin Nx\left\{\sum_{k=1}^{N-1}\frac{1}{\left(\frac{k\pi}{N}+x\right)}\right\}dx+0(1).\end{aligned}$$

But in view of the inequality

$$\frac{N}{\pi}\left\{\frac{1}{2}+\frac{1}{3}+...+\frac{1}{N}\right\}\leq\sum_{k=1}^{N-1}\frac{1}{\left(\frac{k\pi}{N}+x\right)}\leq\frac{N}{\pi}\left\{1+\frac{1}{2}+\frac{1}{3}+...+\frac{1}{N-1}\right\},$$

valid for $0\leq x\leq\frac{\pi}{N}$, it follows that

$$\sum_{k=1}^{N-1}\frac{1}{\left(\frac{k\pi}{N}+x\right)}=\frac{N}{\pi}\{\log N+0(1)\}.$$

□

2.2 First Method: Addition of Condition

The function f (which is always assumed to be Riemann integrable) is defined on $[a, b]$. The first question one might ask is whether the partial sums of the Fourier series of f converge to f pointwise. That is, do we have

$$\lim_{N\to\infty} S_N(f)(x) = f(x)$$

for every x? We might ask the same question assuming that f is continuous and periodic. For a long time it was believed that under these additional assumptions the answer would be 'yes'. It was a surprise when Du bois-Reymond showed that there exists a continuous function whose Fourier series diverges at a point. Despite this negative result, we might ask what happens if we add more smoothness conditions on f; for example we might assume that f is continuously differentiable or twice continuously differentiable. We will see then that the Fourier series f converges to f uniformly (see [1]).

Theorem 2.7 *Let f be a piecewise smooth $2\pi-$periodic function on $\mathbb{R}$. Then*

$$\lim_{N\to\infty} S_N(f)(x) = \frac{1}{2}[f(x^-) + f(x^+)]$$

for every x. Hence

$$\lim_{N\to\infty} S_N(f)(x) = f(x)$$

for each point x of continuity of f.

Proof.

$$S_N(f)(x) - \frac{1}{2}[f(x+0) + f(x-0)] = \int_{-\pi}^{\pi} D_N(y)\, f(x-y)\, dy - \frac{1}{2}[f(x+0) + f(x-0)]$$

$$
\begin{aligned}
&= \int_0^{\pi} D_N(y)\left[f(x-y)-f(x-0)\right]dy + \int_{-\pi}^{0} D_N(y)\left[f(x-y)-f(x+0)\right]dy \\
&= \int_0^{\pi} D_N(y)\left[f(x-y)-f(x-0)\right]dy + \int_{\pi}^{0} D_N(-y)\left[f(x+y)-f(x+0)\right]d(-y) \\
&= \int_0^{\pi} D_N(y)\left[f(x-y)-f(x-0)+f(x+y)-f(x+0)\right]dy \\
&= \int_0^{\pi} \frac{1}{2\pi}\frac{\sin\left(N+\frac{1}{2}\right)y}{\sin\left(\frac{y}{2}\right)}\left[f(x-y)-f(x-0)+f(x+y)-f(x+0)\right]dy \\
&= \frac{1}{\pi}\int_0^{\pi} \sin\left[\left(N+\frac{1}{2}\right)y\right]\left(\frac{\frac{y}{2}}{\sin\frac{y}{2}}\right)\left[\frac{f(x-y)-f(x-0)}{y}+\frac{f(x+y)-f(x+0)}{y}\right]dy.
\end{aligned}
$$

For fixed x define

$$
g(y)=\left(\frac{\frac{y}{2}}{\sin\frac{y}{2}}\right)\left[\frac{f(x-y)-f(x-0)}{y}+\frac{f(x+y)-f(x+0)}{y}\right]
$$

which is an odd piecewise continuous function for $[-\pi,\pi]$ by the condition of piecewise smoothness on f.

$$
S_N(f)(x)-\frac{1}{2}\left[f(x+0)+f(x-0)\right]=\frac{1}{\pi}\int_0^{\pi}\sin\left[\left(N+\frac{1}{2}\right)y\right]g(y)\,dy
$$

$$
=\frac{1}{\pi}\int_0^{\pi}\sin(Ny)\left\{\cos\left(\frac{y}{2}\right)g(y)\right\}dy+\frac{1}{\pi}\int_0^{\pi}\cos(Ny)\left\{\sin\left(\frac{y}{2}\right)g(y)\right\}dy.
$$

The last two terms are the Fourier sine and cosine coefficients a_N, b_N of $\frac{1}{2}\cos\left(\frac{y}{2}\right)g(y)$ and $\frac{1}{2}\sin\left(\frac{y}{2}\right)g(y)$, respectively. By the Bessel's inequality

$$
S_N(f)(x)-\frac{1}{2}\left[f(x+0)+f(x-0)\right]=a_N+b_N\to 0,\text{ as } N\to\infty.
$$

The proof is complete. □

Corollary 2.8 *Let $f:\mathbb{R}\to\mathbb{C}$ be a 2π periodic and Riemann integrable function*

defined on $[-\pi, \pi]$*. Then the Fourier series of* f *converges to* f *in the* L^2*-norm. If*

$$\sum_{n=-\infty}^{\infty} |c_n|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 \, dx.$$

This equation is called Parseval's identity.

2.3 Second Method: Summability Techniques

We begin by taking ordinary averages of the partial sums, a technique which we will describe now in more details. Let us consider a series of complex numbers as;

$$c_0 + c_1 + c_2 + \ldots = \sum_{k=0}^{\infty} c_k.$$

We define the n^{th} partial sum S_n by

$$S_n = \sum_{k=0}^{n} c_k,$$

and say that the series converges to s if

$$\lim_{n \to \infty} S_n = s.$$

This is the most natural and most commonly used type of summability. There exists a whole series of methods permitting the 'sum' of an increasing series to be expressed [2]; these methods are called methods of series summation. The most generally used ones are the linear methods of summation which are devised on the following principle; let A be given matrix with an infinite number of rows and columns. Instead of considering the usual partial sums S_n of the series $\sum_{n=0}^{\infty} u_n$, we consider the quantities

$$\sigma_n = \sum_{k=0}^{\infty} a_{nk} S_k,$$

supposing that the series on the right-hand side of this equality converges ($n = 0, 1, 2, ...$); if

$$\lim_{n\to\infty} \sigma_n = S,$$

then the value S is called the sum of the series $\sum_{k=0}^{\infty} u_n$ and we say that the method, definable by a matrix, sums the series $\sum u_n$ to a value S. Methods defined in this way are called linear because if a series $\sum u_n$ is summable by such a method to the sum S, then the series $\sum Cu_n$, where C is a constant, is summable to CS and if a series $\sum v_n$ is summable to S_1, then the series $\sum (u_n + v_n)$ is summable to $S + S_1$. The method of summation is usually called regular, if any convergent series is summable by this method to the value S which is its sum in the classical meaning of the word, that is,

$$S = \lim_{n\to\infty} S_n.$$

Toeplitz proved that for the regularity of the linear method, definable by a matrix A, it is necessary and sufficient for the following three conditions to be fulfilled:

(**1**) $\lim_{n\to\infty} a_{nk} = 0 \quad (k = 0, 1, ...)$

(**2**) If $A_n = a_{n0} + a_{n1} + ... + a_{nk} + ...,$ then $\lim_{n\to\infty} A_n = 1.$

(**3**) If $K_n = \sum_{k=0}^{\infty} |a_{nk}|,$ then $K_n < C \quad (n = 1, 2, ...),$ where C is a constant.

These are usually known as Toeplitz conditions and the matrices satisfying them are known as T-matrices.

As the simplest example of the methods definable by T-matrices, we will consider the classical case, namely, the method of arithmetic means (or $(C, 1)$) introduced by Cesáro. Cesáro proposed ascribing a sum S to a divergent series [2].

2.3.1 Cesàro Summation

Consider the example of the series

$$1 - 1 + 1 - 1 + \ldots = \sum_{k=0}^{\infty} (-1)^k,$$

it's partial sums for the sequence $\{1, 0, 1, 0, \ldots\}$ which has no limit. Because this partial sum alternate evenly between 1 and 0, one might therefore suggest that $\frac{1}{2}$ is the limit of the sequence, and hence $\frac{1}{2}$ equals the sum of that particular series. We give a precise meaning to this by defining the average of the first N partial sums by

$$\sigma_N = \frac{s_0 + s_1 + \ldots + s_N}{N + 1}.$$

The quantity σ_N is called Cesàro sum of the sequence $\{s_k\}$ or Cesàro sum of the series $\sum_{k=0}^{\infty} c_k$. If σ_N converges to a limit σ as N tends to infinity, we say that the series $\sum c_k$ is Cesàro summable to σ. In the case of series of functions, we shall understand the limit in the sense of either pointwise or uniform converge. We will have no difficulty checking that in the above example, the series is Cesàro summable to $\frac{1}{2}$. Moreover, one can show that Cesàro summation is more inclusive process than convergence. In fact, if a series is convergent to s, then it is also Cesàro summable to the same limit s. We form Cesàro sum of the Fourier series, which by definition is

$$\sigma_N(f)(x) = \frac{S_0(f)(x) + \ldots + S_N(f)(x)}{N + 1}.$$

Now, from the partial sum of Fourier series we have,

$$S_N(f)(x) = (f * D_N)(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x - t) D_N(t)\, dt$$

we rewrite the Cesáro sum σ_N as

$$\begin{aligned}
\sigma_N(f)(x) &= \frac{1}{N+1}\sum_{k=0}^{N} S_k(f) \\
&= \frac{1}{N+1}\sum_{k=0}^{N}\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)\,D_k(t)\,dt\right] \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)\left[\frac{1}{N+1}\sum_{k=0}^{N} D_k(t)\right]dt \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)\,F_N(t)\,dt.
\end{aligned}$$

thus we can write,

$$\sigma_N(f)(x) = (f * F_N)(x),$$

where

$$F_N(x) = \frac{1}{N+1}\sum_{k=0}^{N} D_k(x) \tag{2.15}$$

or we can represent $F_N(x)$ by another form that is,

$$\begin{aligned}
F_N(x) &= \frac{1}{N+1}\sum_{k=0}^{N} D_k(x) \\
&= \frac{1}{N+1}\sum_{k=0}^{N}\frac{\sin\left[\left(k+\frac{1}{2}\right)x\right]}{\sin\frac{x}{2}} \\
&= \frac{1}{N+1}\frac{1}{\sin\frac{x}{2}}\left\{\sum_{k=0}^{N} e^{i\left(k+\frac{1}{2}\right)x}\right\} \\
&= \frac{1}{N+1}\frac{1}{\sin\frac{x}{2}}\left\{e^{\frac{ix}{2}}\frac{e^{i(N+1)x}-1}{e^{ix}-1}\right\} \\
&= \frac{1}{N+1}\frac{1}{\sin\frac{x}{2}}\left\{\frac{e^{i(N+1)x}-1}{e^{\frac{ix}{2}}-e^{-\frac{ix}{2}}}\right\} \\
&= \frac{1}{N+1}\frac{1-\cos\left[(N+1)x\right]}{2\sin^2\frac{x}{2}} \\
&= \frac{1}{N+1}\frac{\sin^2\left[(N+1)\frac{x}{2}\right]}{\sin^2\frac{x}{2}}.
\end{aligned} \tag{2.16}$$

We say that $F_N(x)$ is Fejér Kernel. Now let's see some properties of Fejér Kernel.

Lemma 2.9

(**i**) $F_N(x) = F_N(-x)$.

(**ii**) $F_N(x) \geq 0$ *for all* x.

(**iii**)

$$\int_{-\pi}^{\pi} F_N(x)\,dx = 1.$$

(**iv**) *For each fixed* $\delta > 0$,

$$\lim_{N\to\infty} \int_{\delta\leq|x|\leq\pi} F_N(x)\,dx = 0.$$

Proof.

(i)-(ii) From (2.16), (1) and (2) are clear.

(iii) We will prove using (2.15)

$$F_N(x) = \frac{1}{N+1}\sum_{k=0}^{N} D_k(x),$$

use of the definition of the Dirichlet's Kernel yields

$$F_N(x) = \frac{1}{N+1}\sum_{k=0}^{N}\left[\frac{1}{2\pi}\sum_{m=-k}^{k} e^{imx}\right].$$

Integrating $F_N(x)$ then yields

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi} F_N(x)\,dx &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\frac{1}{N+1}\sum_{k=0}^{N}\sum_{m=-k}^{k} e^{imx}\right]dx \\ &= \frac{1}{N+1}\sum_{k=0}^{N}\sum_{m=-k}^{k}\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{imx}dx\right],\end{aligned}$$

when m is non-zero,

$$\int_{-\pi}^{\pi} e^{imx} dx = 0,$$

but when $m = 0$,

$$\int_{-\pi}^{\pi} e^{imx} dx = 2\pi,$$

thus,

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} F_N(x)\, dx = \frac{1}{N+1} \sum_{k=0}^{N} 1 = 1.$$

This is also Stochastic Kernel.

(iv) We use the Fejér kernel form at (2.15) . For $\delta \leq |x| \leq \pi$ we have

$$\frac{1}{\sin^2 \frac{x}{2}} \leq \frac{1}{\sin^2 \frac{\delta}{2}},$$

thus

$$0 \leq F_N(x) \leq \frac{1}{N+1} \frac{1}{\sin^2 \frac{\delta}{2}}, \quad \delta \leq |x| \leq \pi.$$

This uniformly converges to 0 as $n \to \infty$. □

Theorem 2.10 (Fejér Theorem) *Let $f : [-\pi, \pi] \to \mathbb{R}$ be a continuous function with $f(-\pi) = f(\pi)$. Then the Fourier series of f converges to f, uniformly.*

Proof.

$$\sigma_N(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-t)\, F_N(t)\, dt$$

by property (iii) of Lemma (2.9), we can then write

$$\begin{aligned} |\sigma_N(x) - f(x)| &\leq \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} [f(x-t) - f(x)]\, F_N(t)\, dt \right| \\ &\leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x-t) - f(x)|\, |F_N(t)|\, dt, \end{aligned}$$

by the non-negativity of the Fejér kernel, this reduces to

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x-t) - f(x)| F_N(t)\, dt. \tag{2.17}$$

A continuous function on $[-\pi, \pi]$ is uniformly continuous, that is, given $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|x - y| \leq \delta \rightarrow |f(x) - f(y)| \leq \varepsilon.$$

We now split our integral at (2.17) into two integrals,

$$= \left(\frac{1}{2\pi} \int_{|t|<\delta} |f(x-t) - f(x)| F_N(t)\, dt \right) + \left(\frac{1}{2\pi} \int_{\delta \leq |t| \leq \pi} |f(x-t) - f(x)| F_N(t)\, dt \right),$$

from the uniform continuity of f, the first integral is bounded above by

$$\begin{aligned} \frac{1}{2\pi} \int_{|t|<\delta} |f(x-t) - f(x)| F_N(t)\, dt &\leq \frac{1}{2\pi} \int_{|t|<\delta} \varepsilon F_N(t)\, dt \\ &\leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \varepsilon F_N(t)\, dt = \varepsilon. \end{aligned}$$

If we let $M = \max_{-\pi \leq t \leq \pi} |f(t)|$, then the second integral is bounded above by

$$\begin{aligned} \frac{1}{2\pi} \int_{\delta \leq |t| \leq \pi} |f(x-t) - f(x)| F_N(t)\, dt &\leq \frac{1}{2\pi} \int_{\delta \leq |t| \leq \pi} 2M F_N(t)\, dt \\ &= \frac{M}{\pi} \int_{\delta \leq |t| \leq \pi} F_N(t)\, dt, \end{aligned}$$

by property (*iv*) of Lemma(2.9), there exists an $N \in \mathbb{N}$ such that for all $n > N$

$$\int_{\delta \leq |t| \leq \pi} F_N(t)\, dt < \varepsilon . \frac{\pi}{M}.$$

Conclusively, for all $n > N$

$$|\sigma_N(x) - f(x)| \leq \varepsilon + \varepsilon = 2\varepsilon.$$

This completes the proof. □

2.3.2 Abel Summation

We will refer here to yet another classic and very important method of summation of Fourier series. It was first considered by Abel. Poisson applied this method of summation to Fourier series, therefore the given method when it is applied to trigonometric series is usually referred to as Poisson's method or the Abel-Poisson method. Abel's method is stronger than the method $(C, 1)$.

A series of complex numbers $\sum_{k=0}^{\infty} c_k$ is said to be Abel summable to s if for every $0 \leq r < 1$, the series

$$A(r) = \sum_{k=0}^{\infty} c_k r^k$$

converges, and

$$\lim_{r \to 1} A(r) = s.$$

The quantities $A(r)$ are called Abel sums of the series. One can prove that if the series converges to s, then it is Abel summable to s. Moreover, the method of Abel summation is even more powerful than the Cesáro method: when the series is Cesáro summation, it is always Abel summable to the same sum.

2.3.3 The Poisson Kernel and Dirichlet's Problem in the Unit Circle

To adapt Abel summability to the context of Fourier series, we define the Abel sum of the function $f(\theta)$

$$f(\theta) \sim \sum_{n=-\infty}^{\infty} a_n e^{in\theta}$$

by

$$A_r(f)(\theta) = \sum_{n=-\infty}^{\infty} r^{|n|} a_n e^{in\theta}.$$

Since the index n takes positive and negative values, it is natural to write $c_0 = a_0$ and $c_n = a_n e^{in\theta} + a_{-n} e^{-in\theta}$ for $n > 0$, so that the Abel sums of the Fourier series correspond to the definition given in the previous section for numerical series. We note that since f is integrable, $|a_n|$ is uniformly bounded in n, so that $A_r(f)$ converges absolutely and uniformly for each $0 \leq r < 1$. Just as in the case of Cesáro means, the key is that this Abel sum can be written as convolution

$$A_r(f)(\theta) = (f * P_r)(\theta),$$

where $P_r(\theta)$ is the Poisson kernel given by

$$P_r(\theta) = \sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta}.$$

In fact,

$$\begin{aligned} A_r(f)(\theta) &= \sum_{n=-\infty}^{\infty} r^{|n|} a_n e^{in\theta} \\ &= \sum_{n=-\infty}^{\infty} r^{|n|} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} f(\varphi) e^{-in\varphi} d\varphi \right) e^{in\theta} \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\varphi) \left(\sum_{n=-\infty}^{\infty} r^{|n|} e^{-in(\varphi-\theta)} \right) d\varphi, \end{aligned}$$

where the interchange of the integral and infinite sum is justified by the uniform convergence of the series. Now let's see some properties of Poisson kernel.

Lemma 2.11 *Let* $0 \leq r < 1$. *Then*

(i) *The Poisson kernel* $P_r(\theta)$ *with respect to independent variable* θ *is even function and has* $2\pi-$*period. Also,* $P_r(\theta) > 0$.

Indeed, we can find Poisson kernel with the solution of Dirichlet problem in the unit

circle as

$$P_r(\theta) = \frac{1 - r^2}{1 - 2r\cos\theta + r^2}.$$

Using this form,

$$\begin{aligned} 1 - 2r\cos\theta + r^2 &= 1 - 2r + r^2 + 2r - 2r\cos\theta \\ &= (1 - r)^2 + 4r\sin^2\frac{\theta}{2}. \end{aligned}$$

This equation shows clearly that (i) is satisfied by Poisson kernel.

(ii)

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta)\, d\theta = 1$$

We say that Poisson Kernel is also Stochastic kernel.

(iii)

$$\lim_{r\to 1} P_r(\theta) = 0, \quad \theta \neq 0.$$

if $\theta = 0$, *then*

$$P_r(\theta) = \frac{1 - r^2}{1 - 2r + r^2} = \frac{1 - r^2}{(1 - r)^2} = \frac{1 + r}{1 - r},$$

then

$$\lim_{r\to 1} P_r(0) = \infty,$$

Theorem 2.12 *For any summable* $f(x)$ *the series* $\sigma(f)$ *is summable almost everywhere by the Abel-Poisson method to this function* $f(x)$*; it is summable to* $\frac{1}{2}[f(x+0) + f(x-0)]$ *at any point of discontinuity of the first kind and to* $f(x)$ *at any point of continuity.*

The proof of this theorem is similar to that of the theorem of Fejér, so we omit it.

CHAPTER 3

APPROXIMATE IDENTITY

3.1 Representation of a Function by a Singular Integral at a Given Point

Let's take a look at some theorems and lemmas concerning convolutions of the function f with some kernels [6].

Theorem 3.1 (H. Lebesgue) *Let $K_1(t), K_2(t), K_3(t), \ldots$ be a sequence of measurable functions defined on $[a,b]$. If there exists a constant K such that*

$$|K_n(t)| < K \tag{3.1}$$

for all n and t, and if for every c ($a \leq c \leq b$), we have

$$\lim_{n\to\infty} \int_a^c K_n(t)\,dt = 0, \tag{3.2}$$

then the equality

$$\lim_{n\to\infty} \int_a^b f(t)\,K_n(t)\,dt = 0 \tag{3.3}$$

is valid for an arbitrary summable function $f(t)$ on $[a,b]$.

Proof. If $[\alpha,\beta]$ is a closed subinterval of $[a,b]$, then (3.2) implies that

$$\lim_{n\to\infty} \int_\alpha^\beta K_n(t)\,dt = 0. \tag{3.4}$$

Noting this, we now consider any continuous function $f(t)$ and for prescribed $\varepsilon > 0$, we subdivide $[a,b]$ by means of the points $x_0 = a < x_1 < \ldots < x_m = b$ into small

subintervals such that the oscillation of $f(t)$ is less than ε in each of them. Then

$$\int_a^b f(t) K_n(t)\, dt = \sum_{k=0}^{m-1} \int_{x_k}^{x_{k+1}} [f(t) - f(x_k)] K_n(t)\, dt + \sum_{k=0}^{m-1} f(x_k) \int_{x_k}^{x_{k+1}} K_n(t)\, dt. \qquad (3.5)$$

But

$$\left| \int_{x_k}^{x_{k+1}} [f(t) - f(x_k)] K_n(t)\, dt \right| \le K\varepsilon (x_{k+1} - x_k),$$

so that the first sum in (3.5) is not greater than $K\varepsilon(b-a)$. The second sum in (3.5) tends to zero as n increases, in virtue of (3.4) for $n > n_0$ it will be less than ε. For these n, we have

$$\left| \int_a^b f(t) K_n(t)\, dt \right| < \varepsilon [K(b-a) + 1],$$

so that (3.3) is proved for a continuous function $f(t)$.

Next, let $f(x)$ be a measurable bounded function:

$$|f(t)| \le M.$$

We take $\varepsilon > 0$, and using N. N. Luzin's theorem, find a continuous function $g(t)$ such that

$$mE(f \ne g) < \varepsilon, \quad |g(t)| \le M$$

then

$$\int_a^b f(t) K_n(t)\, dt = \int_a^b [f(t) - g(t)] K_n(t)\, dt + \int_a^b g(t) K_n(t)\, dt.$$

But

$$\left| \int_a^b [f(t) - g(t)] K_n(t)\, dt \right| = \left| \int_{E(f \ne g)} [f(t) - g(t)] K_n(t)\, dt \right| < 2KM\varepsilon,$$

where the integral

$$\int_a^b g K_n dt$$

tends to zero according to what has already been proved, and, for sufficiently large n, becomes less than ε. Hence, for these n, we have

$$\left| \int_a^b f(t) K_n(t) dt \right| < (2KM + 1) \varepsilon$$

which proves (3.3) for the case of a bounded mesaurable function.

Finally, let $f(t)$ be an arbitrary summable function. We take an $\varepsilon > 0$, and using the absolute continuity of the integral, we find a $\delta > 0$ such that for an arbitrary measurable set $e \subset [a, b]$ of measure $me < \delta$ we have

$$\int_e |f(t)| dt < \varepsilon.$$

This done, we find a measurable bounded function $g(t)$ such that

$$mE(f \neq g) < \delta.$$

We may assume that the function $g(t)$ vanishes on the set $E(f \neq g)$. Then

$$\int_a^b f(t) K_n(t) dt = \int_a^b [f(t) - g(t)] K_n(t) dt + \int_a^b g(t) K_n(t) dt.$$

But

$$\left| \int_a^b [f(t) - g(t)] K_n(t) dt \right| = \left| \int_{E(f \neq g)} f(t) K_n(t) dt \right| \leq K\varepsilon,$$

where the integral $\int_a^b gK_n dt$ will be less than ε for sufficiently large n; for such n, we have

$$\left| \int_a^b f(t) K_n(t) dt \right| < (K + 1) \varepsilon,$$

which proves the theorem. □

Theorem 3.2 (Riemann-Lebesgue) *For an arbitrary summable function f defined on* $[a, b]$,

$$\lim_{n\to\infty} \int_a^b f(t) \cos ntdt = \lim_{n\to\infty} \int_a^b f(t) \sin ntdt = 0.$$

In particular, the Fourier coefficients

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos ntdt, \qquad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin ntdt$$

of an arbitrary summable function tend to zero as $n \to \infty$. *If the relation* (3.3) *holds for every summable function* $f(t)$ *defined on* $[a, b]$, *then we shall say the sequence* $\{K_n(t)\}$ *converges weakly to zero as* $n \to \infty$.

In the following discussion, if it is not explicitly stated so otherwise, we shall assume that the kernel $K_n(t, x)$ is bounded for fixed n and x. Then the singular integral

$$f_n(x) = \int_a^b K_n(t, x)\, f(t)\, dt$$

is meaningful for an arbitrary summable function $f(t)$.

Theorem 3.3 (H. Lebesgue) *If for fixed* x ($a < x < b$) *and arbitrary* $\delta > 0$, *the kernel* $K_n(t, x)$ *converges weakly to zero in each of the intervals* $[a, x - \delta]$, $[x + \delta, b]$ *and, furthermore,*

$$\int_a^b |K_n(t, x)|\, dt < H(x),$$

where $H(x)$ *is independent of n, then the equality*

$$\lim_{n\to\infty} f_n(x) = f(x)$$

is valid for any summable function $f(t)$ *which is continuous at the point x.*

Proof. Since $K_n(t, x)$ is a kernel,

$$\lim_{n\to\infty}\int_a^b K_n(t,x)\,dt = 1,$$

and it will suffice to show that

$$\lim_{n\to\infty}\int_a^b [f(t)-f(x)]\,K_n(t,x)\,dt = 0.$$

For this purpose, having taken $\varepsilon > 0$ we find a $\delta > 0$ such that

$$|f(t)-f(x)| < \frac{\varepsilon}{3H(x)} \quad \text{for } |t-x| < \delta$$

Then for arbitrary n,

$$\left|\int_{x-\delta}^{x+\delta} [f(t)-f(x)]\,K_n(t,x)\,dt\right| < \frac{\varepsilon}{3}.$$

But each of the integrals

$$\int_a^{x-\delta} [f(t)-f(x)]\,K_n(t,x)\,dt, \quad \int_{x+\delta}^b [f(t)-f(x)]\,K_n(t,x)\,dt$$

tends to zero as $n \to \infty$, so that for $n > n_0$, each of them will be less than $\frac{\varepsilon}{3}$ in absolute value; for n, obviously

$$\left|\int_a^b [f(t)-f(x)]\,K_n(t,x)\,dt\right| < \varepsilon$$

which was to be proved. □

This theorem is relevant to the representation of a summable function at points of continuity, but in general a summable function does not have a single point of continuity, which, of course, diminishes the interest of this theorem.The question of representing a summable function at those points where this function is the derivative of its indefinite integral, or at Lebesgue points, is of great interest because, as we know, either of these sets of points fills almost the entire interval on which the function is

defined. We now go over to this question.

Lemma 3.4 (I. P. Natanson) *Let a summable function* $f(t)$ *possessing the property*

$$M = \sup_{0<h\leq b-a} \left\{ \frac{1}{h} \left| \int_a^{a+h} f(t)\, dt \right| \right\} < +\infty \tag{3.6}$$

be defined on the closed interval $[a, b]$.

For any nonnegative decreasing function $g(t)$*, defined and summable on* $[a, b]$*, the integral*

$$\int_a^b f(t)\, g(t)\, dt \tag{3.7}$$

exists and the inequality

$$\left| \int_a^b f(t)\, g(t)\, dt \right| \leq M \int_a^b g(t)\, dt \tag{3.8}$$

is valid.

Proof. To clarify the conditions of the Lemma(3.4), we note that the case where $g(a) = +\infty$ is not excluded. If, however, $g(a) < +\infty$, the function $g(t)$ is bounded and integral (3.7) exists as an ordinary Lebesgue integral. Going over to the proof of the lemma, we note that, without loss of generality, we may assume $g(b) = 0$. In fact, if this were not so, we would simply introduce the function $g^*(t)$ defined by the equalities

$$\begin{aligned} g^*(t) &= g(t), \text{ if } a \leq t < b \\ g^*(t) &= 0, \quad \text{if } t = b \end{aligned}$$

to replace $g(t)$.

Having proved the theorem for $g^*(t)$, we could then replace $g^*(t)$ by $g(t)$ everywhere because this would have no effect on the value of the integrals involved. Thus, we shall assume $g(b) = 0$. Let $a < \alpha < b$. On the closed interval $[\alpha, b]$, the function

$g(t)$ is bounded and the integral

$$\int_a^b f(t)\,g(t)\,dt \tag{3.9}$$

certainly exists. If we set

$$F(t) = \int_a^t f(u)\,du,$$

integral (3.9) can be expressed in the form of Stieltjes integral

$$\int_a^b f(t)\,g(t)\,dt = \int_a^b g(t)\,dF(t),$$

from which, after integrating by parts, we find

$$\int_a^b f(t)\,g(t)\,dt = -F(a)\,g(a) + \int_a^b F(t)\,d\left[-g(t)\right].$$

But, by (3.6), we have that

$$|F(t)| \le M(t-a), \tag{3.10}$$

and since $g(t)$ is decreases,

$$g(\alpha)(\alpha - a) \le \int_a^\alpha g(t)\,dt \tag{3.11}$$

This means that

$$|F(a)\,g(a)| \le M\int_a^\alpha g(t)\,dt.$$

On the other hand, the function $-g(t)$ increases. This and (3.10) imply that

$$\left|\int_a^b F(t)\,d\left[-g(t)\right]\right| \le M\int_a^b (t-a)\,d\left[-g(t)\right].$$

We transform the integral in the right member by using the formula for integration by parts:

$$\int_a^b (t-a)\, d\left[-g(t)\right] = g(\alpha)(\alpha - a) + \int_a^b g(t)\, dt.$$

From this, together with (3.11), it follows that

$$\left| \int_a^b (t-a)\, d\left[-g(t)\right] \right| \leq \int_a^b g(t)\, dt.$$

Combining the above, we obtain

$$\left| \int_a^b f(t)\, g(t)\, dt \right| \leq M \left\{ \int_a^\alpha g(t)\, dt + \int_\alpha^b g(t)\, dt \right\}. \tag{3.12}$$

Although this inequality was established under the assumption that $g(b) = 0$, it also remains valid without this hypothesis as was already explained. This means we can replace the limit b by β, where $\alpha < \beta < b$. But then, letting α and β tend to a, we see that

$$\lim \int_\alpha^\beta f(t)\, g(t)\, dt = 0,$$

which proves the existence of the integral (3.7). Finally, we pass to the limit as $\alpha \to a$ in (3.12) and obtain (3.8); this completes the proof of the lemma. □

Theorem 3.5 (P. I. Romanovsky) *Suppose the kernel $K_n(t, x)$ is positive and possesses the following property:*

For fixed n and x, the kernel $K_n(t, x)$, as a function of t only, increases on the closed interval $[a, x]$ and decreases on the closed interval $[x, b]$. Then, for an arbitrary summable function $f(t)$ which is the derivative of its indefinite integral at the point x, we have

$$\lim_{n\to\infty} \int_a^b f(t)\, K_n(t, x)\, dt = f(x) \tag{3.13}$$

Proof. Since $K_n(t,x)$ is a kernel, it is sufficient to verify that

$$\lim_{n\to\infty}\int_a^b [f(t)-f(x)]\,K_n(t,x)\,dt = 0 \tag{3.14}$$

Splitting the last integral into two integrals extending over the closed intervals $[a,x]$ and $[x,b]$, we shall consider only the second of them since the first one is investigated analogously. We take $\varepsilon > 0$ and find a $\delta > 0$ such that for $0 < h \le \delta$,

$$\left|\frac{1}{h}\int_x^{x+h}[f(t)-f(x)]\,dt\right| < \varepsilon,$$

which is possible since $f(t)$ is the derivative of its indefinite integral at the point $t = x$. Then according to the preceding Lemma(3.4),

$$\left|\int_x^{x+\delta}[f(t)-f(x)]\,K_n(t,x)\,dt\right| \le \varepsilon\int_x^{x+\delta}K_n(t,x)\,dt \le \varepsilon\int_a^b K_n(t,x)\,dt.$$

since $K_n(t,x)$ is a kernel,

$$\lim_{n\to\infty}\int_a^b K_n(t,x)\,dt = 1.$$

This quantity is bounded since it has a finite limit. This means that there exists a constant $K(x)$ such that

$$\int_a^b K_n(t,x)\,dt < K(x).$$

Hence,

$$\left|\int_x^{x+\delta}[f(t)-f(x)]\,K_n(t,x)\,dt\right| < \varepsilon K(x). \tag{3.15}$$

On the other hand, if $x+\delta \le t \le b$,

$$K_n(t,x) \le K_n(x+\delta,x) \le \frac{1}{\delta}\int_x^{x+\delta}K_n(t,x)\,dt < \frac{K(x)}{\delta}.$$

This means the functions $K_n(t) = K_n(t,x)$ are uniformly bounded on the closed interval $[x+\delta, b]$, and Lebesgue's theorem is satisfied. But the second condition of this theorem is also satisfied for these functions because $K_n(t,x)$ is a kernel. Hence $K_n(t,x)$ converges weakly to zero on the closed interval $[x+\delta, b]$ and for sufficiently large n,

$$\left| \int_{x+\delta}^{b} [f(t) - f(x)] K_n(t,x)\, dt \right| < \varepsilon.$$

For n,

$$\left| \int_{x}^{b} [f(t) - f(x)] K_n(t,x)\, dt \right| < \varepsilon [K(x) + 1].$$

so that

$$\lim_{n\to\infty} \int_{x}^{b} [f(t) - f(x)] K_n(t,x)\, dt = 0. \tag{3.16}$$

This completes the proof of the theorem. □

Theorem 3.6 (D . K. Faddayev) *If for every n, the kernel $K_n(t,x)$ has a convex majorant $\Psi_n(t,x)$ such that*

$$\int_{a}^{b} \Psi_n(t,x)\, dt < K(x) < +\infty,$$

where $K(x)$ depends only on x, then (3.13) *is valid for arbitrary $f(t) \in L$, having the point $t = x$ as a Lebesgue point.*

Proof. Here, also, it will be sufficient to prove equality (3.16). Taking an $\varepsilon > 0$, we find a $\delta > 0$ such that for $0 < h \le \delta$,

$$\frac{1}{h} \int_{x}^{x+h} |f(t) - f(x)|\, dt < \varepsilon.$$

According to the Lemma(3.4),

$$\left| \int_{x}^{x+\delta} \{f(t) - f(x)\} K_n(t,x)\, dt \right| \le \int_{x}^{x+h} |f(t) - f(x)| \Psi_n(t,x)\, dt \le \varepsilon \int_{x}^{x+h} \Psi_n(t,x)\, dt < \varepsilon K(x).$$

On the other hand, on the closed interval $[x+\delta, b]$, the sequence $\{K_n(t)\} = \{K_n(t,x)\}$ converges weakly to zero because, for $t \in [x+\delta, b]$,

$$|K_n(t,x)| \le \Psi_n(t,x) \le \Psi_n(x+\delta, x) \le \frac{1}{\delta}\int_x^{x+\delta} \Psi_n(t,x)\,dt < \frac{K(x)}{\delta}.$$

Noting this, we complete the proof of Faddeyev's theorem exactly as above. □

3.2 Approximate Identity

Definition 3.7 ***(a) Periodic case:***

A family of kernels $\{K_n(x)\}_{n=1}^{\infty}$ on the $[-\pi, \pi]$ is said to be a family of Approximate Identity if it satisfies the following properties:

(i) *For all $n \ge 1$,*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(x)\,dx = 1.$$

(ii) *There exists $M > 0$ such that for all $n \ge 1$,*

$$\int_{-\pi}^{\pi} |K_n(x)|\,dx \le M.$$

(iii) *For every $\delta > 0$,*

$$\int_{\delta \le |x| \le \pi} |K_n(x)|\,dx \to 0, \quad as\ n \to \infty,$$

*or instead of **(iii)**, we shall often assume the property*

$$\lim_{n\to\infty}\left[\sup_{\delta \le |x| \le \pi} |K_n(x)|\right] = 0.$$

If the functions K_n happen to be positive, they usually have the familiar bell-shaped graph: the area under the curve $y = K_n(y)$ is equal to 2π, whereby, for increasing n, the peak at $y = 0$ becomes higher and narrower in such a way that the area under the curve

near $y = 0$ comes out equal to 2π.

Definition 3.8 ***(b) On the real line:***

A kernel $\{K_n(x)\}$ *is called an Approximate Identity on the real line if there is some constant* $M > 0$ *with*

$$\|K_n(x)\|_1 \leq M$$

$$\lim_{n\to\infty} \int_{\delta\leq|x|} |K_n(x)|\, dx = 0, \quad \delta > 0$$

or

$$\lim_{n\to\infty} \left[\sup_{\delta\leq|x|} |K_n(x)|\right] = 0, \quad \delta > 0.$$

The name Approximate Identity is justified by the fact that for $f \in C[-\pi, \pi]$ the sequence $\{f * K_n\}$ tends uniformly to f for $n \to \infty$. Indeed, in view of the properties of $\{K_n(x)\}$, the magnitude of the convolution integral $(f * K_n)$ for large n essentially depends upon the value of its integrand near $y = 0$ (f being bounded). Since $f(x - y)$ is then near to $f(x)$ (f being continuous),

$$\begin{aligned}(f * K_n)(x) &= f(x)\left(\frac{1}{2\pi}\right)\int_{-\delta}^{\delta} K_n(y)\, dy \\ &= f(x),\end{aligned}$$

for large n. One of the important features of the convolution integral $(f * K_n)$ is that the 'best' properties of each of its factors are inherited by the product itself. This is due to the translation-invariance and commutativity of convolutions.

Theorem 3.9 *Let* $\{K_n\}_{n=1}^{\infty}$ *be a family of Approximate Identity kernels, and* f *be an integrable function on the interval* $[-\pi, \pi]$*, then*

$$\lim_{n\to\infty} (f * K_n)(x) = f(x)$$

wherever f *is continuous at x. If* f *is continuous everywhere then the above limit is uniform.*

Proof. We will prove

$$\lim_{n\to\infty} |(f * K_n)(x) - f(x)| = 0.$$

$$|(f * K_n)(x) - f(x)| = \left| \int_{-\pi}^{\pi} f(t) K_n(t-x)\, dt - f(x) \right|$$

$$\leq \left| \int_{-\pi}^{\pi} f(t) K_n(t-x)\, dt - \int_{-\pi}^{\pi} f(x) K_n(t-x)\, dt \right| + |f(x)| \left| \int_{-\pi}^{\pi} K_n(t-x)\, dt - 1 \right|$$

Since

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} K_n(t-x)\, dt = 1,$$

the last term on the right side of the inequality tends to zero as $n \to \infty$. We can eliminate the last term , thus we have

$$= \left| \int_{-\pi}^{\pi} f(t) K_n(t-x)\, dt - \int_{-\pi}^{\pi} f(x) K_n(t-x)\, dt \right|$$

or we can write

$$= \left| \int_{-\pi}^{\pi} [f(t) - f(x)] K_n(t-x)\, dt \right|$$

$$\leq \int_{-\pi}^{\pi} |f(t) - f(x)|\, |K_n(t-x)|\, dt.$$

Since f is continuous at x, for any $\varepsilon > 0$ choose δ so that $|t-x| < \delta$ implies $|f(t) - f(x)| < \varepsilon$

$$= \int_{x-\delta}^{x+\delta} |f(t) - f(x)|\, |K_n(t-x)|\, dt + \int_{\delta \leq |t-x|} |f(t) - f(x)|\, |K_n(t-x)|\, dt$$

$$\leq \varepsilon \int_{x-\delta}^{x+\delta} |K_n(t-x)|\, dt + \int_{\delta \leq |t-x|} |f(t) - f(x)|\, |K_n(t-x)|\, dt$$

$$\leq \varepsilon \int_{-\pi}^{\pi} |K_n(t-x)|\, dt + \int_{\delta \leq |t-x|} |f(t)|\, |K_n(t-x)|\, dt + |f(x)| \int_{\delta \leq |t-x|} |K_n(t-x)|\, dt,$$

let $\max_{x\in[-\pi,\pi]} |f(t)| = M$, we can write

$$\leq \varepsilon \int_{-\pi}^{\pi} |K_n(t-x)|\,dt + M \int_{\delta\leq|t-x|} |K_n(t-x)|\,dt + |f(x)| \int_{\delta\leq|t-x|} |K_n(t-x)|\,dt,$$

by property of Approximate Identity,

$$\lim_{n\to\infty} \int_{\delta<|t-x|} |K_n(t-x)|\,dt = 0,$$

finally we have,

$$\lim_{n\to\infty} |(f * K_n)(x) - f(x)| \leq \varepsilon.$$

The proof is complete. □

If $\varepsilon > 0$ and f is continuous at x, choose δ so that $|y| < \delta$ implies $|f(x-y) - f(x)| < \varepsilon$. Then, by the first property of Approximate Identity kernels, we can write

$$(f * K_n)(x) - f(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} K_n(y)\, f(x-y)\,dy - f(x)$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} K_n(y)\,[f(x-y) - f(x)]\,dy.$$

Hence,

$$\begin{aligned} |(f * K_n)(x) - f(x)| &= \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} K_n(y)\,[f(x-y) - f(x)]\,dy \right| \\ &\leq \frac{1}{2\pi} \int_{|y|<\delta} |K_n(y)|\,|f(x-y) - f(x)|\,dy \\ &\quad + \frac{1}{2\pi} \int_{\delta\leq|y|\leq\pi} |K_n(y)|\,|f(x-y) - f(x)|\,dy \\ &\leq \frac{\varepsilon}{2\pi} \int_{-\pi}^{\pi} |K_n(y)|\,dy + \frac{2B}{2\pi} \int_{\delta\leq|y|\leq\pi} |K_n(y)|\,dy, \end{aligned}$$

where B is a bound for f. The first term is bounded by $\frac{\varepsilon M}{2\pi}$ because of the second property of Approximate Identity. By the third property we see that for all large n, the second term will be less than ε. Therefore, some constant $c > 0$ and all large n we have

$$|(f * K_n)(x) - f(x)| \leq c\varepsilon,$$

thereby proving the first assertion in the theorem. If f is continuous everywhere, then it is uniformly continuous, and δ can be chosen independent of x. This provides the desired conclusion that $f * K_n \to f$ uniformly.

Recall that

$$S_N(f)(x) = (f * D_N)(x)$$

where

$$D_N(x) = \sum_{n=-N}^{N} e^{inx}$$

is the Dirichlet kernel. Theorem 2.3 would imply that the Fourier series of f converges to $f(x)$ wherever f is continuous at x. Unfortunately this is not the case. Indeed, an estimate shows that D_N violates the second property; more precisely one has

$$\int_{-\pi}^{\pi} |D_N(x)|\, dx \geq c \log N, \quad \text{as } N \to \infty.$$

However, we should note that the formula for D_N as a sum of exponentials immediately gives

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} D_N(x)\, dx = 1,$$

so the first property of Approximate Identity is actually verified. The fact that the mean value of D_N is 1, while the integral of its absolute value is large, is a result of cancellations. The function $D_N(x)$ takes on positive and negative values and oscillates very rapidly as N gets large. This observation suggests that the pointwise convergence of Fourier series is intricate and may even fail at points of continuity. Since a Fourier series may fail to converge at individual points, we are led to try to overcome this

failure by interpreting the limit

$$\lim_{N\to\infty} S_N(f) = f \quad \text{in different sense}$$

Theorem 3.10 *We suppose that $K_n(x)$ is a kernel that is Approximate Identity and $f \in L_1(-\infty,\infty)$. If $x = x_0$ is a continuity point of the function f, then*

$$\lim_{n\to\infty} A_n(f, x_0) = f(x_0)$$

where

$$A_n(f,x) = \int_{-\infty}^{\infty} f(t)\, K_n(t-x)\, dt, \quad -\infty < x < \infty.$$

Proof. Let's write

$$|A_n(f,x_0) - f(x_0)| \le \int_{-\infty}^{\infty} |f(x_0+t) - f(x_0)|\, K_n(t)\, dt.$$

Since $x = x_0$ is a continuity point of the function f, for all $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|t| < \delta \quad then \quad |f(x_0+t) - f(x_0)| < \varepsilon. \tag{3.17}$$

$$\begin{aligned} |A_n(f,x_0) - f(x_0)| &\le \int_{-\infty}^{\delta} |f(x_0+t) - f(x_0)|\, K_n(t)\, dt \\ &\quad + \int_{-\delta}^{\delta} |f(x_0+t) - f(x_0)|\, K_n(t)\, dt \\ &\quad + \int_{\delta}^{\infty} |f(x_0+t) - f(x_0)|\, K_n(t)\, dt \\ &\le i_n' + i_n'' + i_n'''. \end{aligned}$$

$K_n(t)$ is a non-negative function and

$$\int_{-\infty}^{\infty} K_n(t)\, dt = 1$$

then

$$i_n'' < \varepsilon \int_{-\delta}^{\delta} K_n(t)\,dt < \varepsilon.$$

Since $K_n(t)$ is even function

$$\begin{aligned}
i_n' + i_n''' &= \int_{\delta}^{\infty} |f(x_0 - t) - f(x_0)|\, K_n(t)\,dt + \int_{\delta}^{\infty} |f(x_0 + t) - f(x_0)|\, K_n(t)\,dt \\
&\leq \int_{\delta}^{\infty} |f(x_0 - t)|\, K_n(t)\,dt + |f(x_0)| \int_{\delta}^{\infty} K_n(t)\,dt \\
&\quad + \int_{\delta}^{\infty} |f(x_0 + t)|\, K_n(t)\,dt + |f(x_0)| \int_{\delta}^{\infty} K_n(t)\,dt \\
&\leq \sup_{|t|\geq\delta} K_n(t) \int_{\delta}^{\infty} |f(x_0 - t)|\,dt + \sup_{|t|\geq\delta} K_n(t) \int_{\delta}^{\infty} |f(x_0 + t)|\,dt \\
&\quad + 2\,|f(x_0)| \int_{\delta}^{\infty} K_n(t)\,dt \\
&\leq 2\,\|f\|_1 \sup_{|t|\geq\delta} K_n(t) + 2\,|f(x_0)| \int_{\delta}^{\infty} K_n(t)\,dt.
\end{aligned}$$

According to the Approximate Identity as $n \to \infty$, the right side of this inequality is zero. The proof is complete. □

Theorem 3.11 *We assume that $K_n(t)$ is a $2\pi-$periodic kernel that is Approximate Identity and $f \in L_1(-\pi, \pi)$. If $f(x)$ is a continuous function at the point $x = x_0$, then*

$$\lim_{n\to\infty} B_n(f, x_0) = f(x_0),$$

where

$$\lim_{n\to\infty} B_n(f, x) = \int_{-\pi}^{\pi} f(t)\, K_n(x - t)\,dt.$$

Proof. Let's choose again a number δ satisfying (3.17),that is,

$$|t| < \delta \quad then \quad |f(x_0 + t) - f(x_0)| < \varepsilon.$$

$$
\begin{aligned}
|B_n(f,x_0)-f(x_0)| &\leq \int_{-\pi}^{-\delta} |f(x_0+t)-f(x_0)|\,K_n(t)\,dt \\
&\quad + \int_{-\delta}^{\delta} |f(x_0+t)-f(x_0)|\,K_n(t)\,dt \\
&\quad + \int_{\delta}^{\pi} |f(x_0+t)-f(x_0)|\,K_n(t)\,dt \\
&\leq I_n' + I_n'' + I_n''',
\end{aligned}
$$

and

$$
\begin{aligned}
I_n'' &< \varepsilon \int_{-\delta}^{\delta} K_n(t)\,dt \\
&< \varepsilon \int_{-\pi}^{\pi} K_n(t)\,dt \\
&= \varepsilon,
\end{aligned}
$$

and also $K_n(t)$ is even function

$$
\begin{aligned}
I_n' + I_n''' &= \int_{\delta}^{\pi} \left(|f(x_0-t)-f(x_0)| + |f(x_0+t)-f(x_0)|\right) K_n(t)\,dt \\
&\leq \left(2|f(x_0)| + 2\|f\|_1\right)\left(\sup_{|t|\geq\delta} K_n(t)\right),
\end{aligned}
$$

and by definition of Approximate Identity as $n \to \infty$, the right side of this inequality is zero. The proof is complete. □

Theorem 3.12 *Let $K_n(t)$ be a monotone decreasing kernel that is Approximate Identity on $[0,\infty]$. Then for any Lebesgue point of the function $f(x) \in L_p(-\infty,\infty)$*

$$
\lim_{n\to\infty} A_n(f,x) = f(x)
$$

where

$$
A_n(f,x) = \int_{-\infty}^{\infty} f(t)\,K_n(x-t)\,dt, \quad -\infty < x < \infty.
$$

Proof.

$$A_n(f,x) - f(x) = \int_0^\infty [f(x+t) + f(x-t) - 2f(x)] K_n(t)\, dt,$$

and then

$$|A_n(f,x) - f(x)| \leq \int_0^\infty |f(x+t) + f(x-t) - 2f(x)| K_n(t)\, dt.$$

We assume that for $1 < p < \infty$, p' is the conjugate of p, that is, $\frac{1}{p} + \frac{1}{p'} = 1$ where $f \in L_p(-\infty, \infty)$. Using Hölder Inequality,

$$\begin{aligned}
&|A_n(f,x) - f(x)| \\
&\quad \leq \int_0^\infty |f(x+t) + f(x-t) - 2f(x)| K_n^{\frac{1}{p}}(t) K_n^{\frac{1}{p'}}(t)\, dt \\
&\quad \leq \left(\int_0^\infty |f(x+t) + f(x-t) - 2f(x)|^p K_n(t)\, dt \right)^{\frac{1}{p}} \left(\int_{-\infty}^\infty K_n(t)\, dt \right)^{\frac{1}{p'}} \\
&\quad = \left(\int_0^\infty |f(x+t) + f(x-t) - 2f(x)|^p K_n(t)\, dt \right)^{\frac{1}{p}}.
\end{aligned}$$

or we can write

$$|A_n(f,x) - f(x)|^p \leq \int_0^\infty |f(x+t) + f(x-t) - 2f(x)|^p K_n(t)\, dt. \tag{3.18}$$

Now, let's define a function as

$$F(t) = \int_0^t |f(x+\xi) + f(x-\xi) - 2f(x)|^p\, d\xi. \tag{3.19}$$

Clearly,

$$dF(t) = |f(x+t) + f(x-t) - 2f(x)|^p\, dt. \tag{3.20}$$

Assume that x is a Lebesgue point of the function $f(x)$. By definition of Lebesgue point at x,

$$\lim_{h\to 0}\left(\frac{1}{h}\int_0^h |f(x+t)+f(x-t)-2f(x)|^p\,dt\right)^{\frac{1}{p}}=0.$$

Using the function $F(t)$in (3.19)

$$\lim_{h\to 0}\left(\frac{1}{h}F(h)\right)^{\frac{1}{p}}=0$$

or

$$\lim_{h\to 0}\frac{1}{h}F(h)=0.$$

That is, for every $\varepsilon>0$ there exists a $\delta>0$ such that

$$F(h)<\varepsilon h,\quad h\le\delta. \tag{3.21}$$

So,

$$\begin{aligned}|A_n(f,x)-f(x)|^p &\le \int_0^{\delta}|f(x+t)+f(x-t)-2f(x)|^p\,K_n(t)\,dt\\ &\quad+\int_{\delta}^{\infty}|f(x+t)+f(x-t)-2f(x)|^p\,K_n(t)\,dt\\ &= m_n'+m_n''. \end{aligned}\tag{3.22}$$

Let's show that

$$\lim_{n\to\infty}m_n'=\lim_{n\to\infty}m_n''=0.$$

Firstly using (3.20),

$$m_n'=\int_0^{\delta}K_n(t)\,dF(t)$$

by using integration by parts

$$\begin{aligned} m_n^{'} &= [K_n(t)\,F(t)]_0^{\delta} - \int_0^{\delta} F(t)\,dK_n(t) \\ &= K_n(\delta)\,F(\delta) + \int_0^{\delta} F(t)\,d[-K_n(t)]. \end{aligned}$$

Since $K_n(t)$ is monotone decreasing on $[0,\infty)$, $-K_n(t)$ is monotone increasing. When (3.21)is applied to the function F under and outside the integral then we get,

$$m_n^{'} \leq \varepsilon\delta K_n(\delta) + \varepsilon \int_0^{\delta} t d[-K_n(t)],$$

and

$$m_n^{'} \leq \varepsilon \int_0^{\delta} K_n(t)\,dt < \varepsilon \int_{-\infty}^{\infty} K_n(t)\,dt = \varepsilon$$

or for any n

$$m_n^{'} < \varepsilon. \tag{3.23}$$

Now, consider $m_n^{''}$.For $a > 0$ and $b > 0$

$$(a+b)^p \leq 2^p (a^p + b^p)$$

then

$$\begin{aligned} |f(x+t) + f(x-t) - 2f(x)|^p &\leq 2^p (|f(x+t) + f(x-t)|^p + 2^p |f(x)|^p) \\ &\leq 2^p (2^p |f(x+t)|^p + 2^p |f(x-t)|^p + 2^p |f(x)|^p) \\ &= 2^{2p} (|f(x+t)|^p + |f(x-t)|^p + |f(x)|^p). \end{aligned}$$

Also, since $K_n(t)$ is monotone decreasing on $[0,\infty)$

$$\begin{aligned} m_n'' &\leq K_n(\delta)\, 2^{2p}\left(\int_{\delta}^{\infty} |f(x+t)|^p\, dt + \int_{\delta}^{\infty} |f(x-t)|^p\, dt\right) + 2^{2p}\, |f(x)|^p \int_{\delta}^{\infty} K_n(t)\, dt \\ &\leq 2^{2p+1}\, \|f\|_p\, K_n(\delta) + 2^{2p}\, |f(x)|^p \int_{\delta}^{\infty} K_n(t)\, dt. \end{aligned}$$

By definition of Aproximate Identity

$$\lim_{n\to\infty} K_n(\delta) = 0, \qquad \lim_{n\to\infty} \int_{\delta}^{\infty} K_n(t)\, dt = 0,$$

then

$$\lim_{n\to\infty} m_n'' = 0. \tag{3.24}$$

Thus, if we use (3.23),(3.24) in (3.22) we have

$$\lim_{n\to\infty} |A_n(f,x) - f(x)|^p = 0.$$

The proof is complete. □

Theorem 3.13 *Let $K_n(t)$ be a $2\pi-$periodic and monotone decreasing kernel that is Approximate Identity on $[0,\pi]$.Then for any Lebesgue point of the function $f(x) \in L_p(-\pi,\pi)$.*

$$\lim_{n\to\infty} B_n(f,x) = f(x)$$

where

$$B_n(f,x) = \int_{-\pi}^{\pi} f(t)\, K_n(x-t)\, dt, \quad -\pi < x < \pi.$$

Theorem 3.14 *Let $K_n(t)$ be a monotone decreasing kernel that is Approximate Identity on $[0,\infty]$.Then for any D-point of the function $f(x) \in L_1(-\infty,\infty)$*

$$\lim_{n\to\infty} A_n(f,x) = f(x)$$

where

$$A_n(f,x) = \int_{-\infty}^{\infty} f(t)\,K_n(x-t)\,dt, \quad -\infty < x < \infty.$$

Proof. Since $K_n(t)$ is even function

$$\begin{aligned} A_n(f,x) &= \int_{-\infty}^{\infty} f(x+t)\,K_n(t)\,dt \\ &= \int_{-\infty}^{0} f(x+t)\,K_n(t)\,dt + \int_{0}^{\infty} f(x+t)\,K_n(t)\,dt \\ &= \int_{0}^{\infty} [f(x+t) + f(x-t)]\,K_n(t)\,dt. \end{aligned} \tag{3.25}$$

Also,by the definition of Aproximate Identity

$$2\int_{0}^{\infty} K_n(t)\,dt = 1,$$

by multiplying both sides of this equation with $f(x)$

$$2f(x)\int_{0}^{\infty} K_n(t)\,dt = f(x),$$

or we can write

$$f(x) = \int_{0}^{\infty} 2f(x)\,K_n(t)\,dt,$$

and subtracting this function from both sides of the (3.25), then we have

$$A_n(f,x) - f(x) = \int_{0}^{\infty} [f(x+t) + f(x-t) - 2f(x)]\,K_n(t)\,dt. \tag{3.26}$$

Now, let x be a D-point of the function $f(x)$. By definition of D-point

$$\lim_{h\to 0}\frac{1}{h}\int_{0}^{h} [f(x+t) + f(x-t) - 2f(x)]\,dt = 0,$$

that is, for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\left| \int_0^h [f(x+t) + f(x-t) - 2f(x)]\, dt \right| < \varepsilon h. \tag{3.27}$$

Then let's write the integral at the right side of the formula (3.26) as the sum of two integrals, that is,

$$\begin{aligned} A_n(f,x) - f(x) &= \int_0^{\delta} [f(x+t) + f(x-t) - 2f(x)]\, K_n(t)\, dt \\ &\quad + \int_{\delta}^{\infty} [f(x+t) + f(x-t) - 2f(x)]\, K_n(t)\, dt \\ &= r_n' + r_n''. \end{aligned} \tag{3.28}$$

Firstly, let's consider r_n' and define a function as $F(t)$

$$F(t) = \int_0^t [f(x+\xi) + f(x-\xi) - 2f(x)]\, d\xi. \tag{3.29}$$

According to the inequality (3.27)

$$|F(t)| < \varepsilon t, \quad t \le \delta \tag{3.30}$$

and also as seen from formula (3.29)

$$dF(t) = f(x+t) + f(x-t) - 2f(x).$$

By using integration by parts

$$\begin{aligned} r_n^{'} &= \int_0^{\delta} K_n(t)\, dF(t) \\ &= [K_n(t)\, F(t)]_0^{\delta} - \int_0^{\delta} F(t)\, dK_n(t) \\ &= K_n(\delta)\, F(\delta) + \int_0^{\delta} F(t)\, d[-K_n(t)]. \end{aligned}$$

Since $K_n(t)$ is a monotone decreasing kernel, $-K_n(t)$ is monotone increasing and using inequality (3.30), we have

$$\left|r_n^{'}\right| \leq \varepsilon\delta K_n(\delta) + \delta \int_0^{\delta} td(-K_n(t)),$$

by integration by parts, again

$$\left|r_n^{'}\right| \leq \varepsilon\delta K_n(\delta) + \varepsilon\delta K_n(\delta) + \varepsilon \int_0^{\delta} K_n(t)\, dt.$$

Since $K_n(t)$ is a monotone decreasing

$$\int_0^{\delta} K_n(t)\, dt \geq \delta K_n(\delta),$$

and since

$$\int_{-\infty}^{\infty} K_n(t)\, dt = 1,$$

$$\left|r_n'\right| \leq 3\varepsilon \int_{-\infty}^{\infty} K_n(t)\, dt = 3\varepsilon. \tag{3.31}$$

Now, consider r_n'',

$$\begin{aligned}
\left|r_n''\right| &\leq \int_{\delta}^{\infty} |f(x+t)+f(x-t)-2f(x)|\, K_n(t)\, dt \\
&\leq K_n(\delta) \int_{\delta}^{\infty} |f(x+t)+f(x-t)|\, dt + 2|f(x)| \int_{\delta}^{\infty} K_n(t)\, dt \\
&\leq 2\|f\|_1 K_n(\delta) + 2|f(x)| \int_{\delta}^{\infty} K_n(t)\, dt,
\end{aligned}$$

and by definition of Approximate Identity,

$$\lim_{n\to\infty} r_n'' = 0. \tag{3.32}$$

By applying formulas (3.31), (3.32) into (7.28), then the proof is complete. □

Theorem 3.15 *Let $K_n(t)$ be a $2\pi-$periodic and monotone decreasing kernel that is Approximate Identity on $[0,\pi]$ .Then for every D- point of the function $f(x) \in L_1(-\pi,\pi)$.*

$$\lim_{n\to\infty} B_n(f,x) = f(x)$$

where

$$B_n(f,x) = \int_{-\pi}^{\pi} f(t)\, K_n(x-t)\, dt, \quad -\pi < x < \pi.$$

CHAPTER 4

ON CONVERGENCE OF CONVOLUTION TYPE SINGULAR INTEGRAL OPERATORS DEPENDING ON TWO PARAMETERS

In this chapter we will study a recent paper due to Karslı, 2007 which is deal with the theory of approximation of singular integral operators[11]. In papers [7], [10], [12] studied the pointwise convergence of integrable functions in $L_1(-\pi,\pi)$, by a two-parameter family of convolution type singular integral operators of the form

$$U(f;x,\lambda)=\int_{-\pi}^{\pi} f(t)\,K(t-x,\lambda)\,dt \quad , \quad x\in(-\pi,\pi).$$

They considered four cases where x_0 is a continuity point, a Lebesgue point, a generalized Lebesgue point and a $\mu-$generalized Lebesgue point of f.

In this study, the author, investigates the pointwise convergence of $T(f;x,\lambda)$ to $f(x_0)$ in $L_1(a,b)$, by another two-parameter family of convolution type singular integral operators of the form

$$T(f;x,\lambda)=\int_{a}^{b} f(t)\,K(t-x,\lambda)\,dt \quad , \quad x\in(a,b),\ \lambda\in\Lambda\subset R \tag{4.1}$$

We note that taking $\lambda\in N$, $x=x_0$, and $K\geq 0$ this two-parameters family is reduced to the single parameter sequence in [5] and [6].

First, we shall give conditions which provide the operator's existence.

Definition 4.1 (**Class** $\mathcal{A}$).*We take a family* $\mathcal{K}=(K_\lambda)_{\lambda\in\Lambda}$ *of functions* $K(t,\lambda):R\times\Lambda\rightarrow R$. *We will say that the function* $K(t,\lambda)$ *belongs to class* $\mathcal{A}$, *if the following conditions hold:*

a) *As function of* t, $K(t,\lambda)$ *is defined on* $(-\infty,\infty)$ *and integrable for each fixed* $\lambda \in \Lambda$ *(* Λ *is a given set of numbers with an accumulation point* λ_0*).*

b) $\lim\limits_{(x,\lambda)\to(x_0,\lambda_0)} \int\limits_{-\infty}^{\infty} K(t-x,\lambda)\,dt = 1.$

c) $\lim\limits_{\lambda\to\lambda_0} \int\limits_{|t|>\delta} |K(t,\lambda)|dt = 0,$ *for every* $\delta > 0.$

d) $\lim\limits_{\lambda\to\lambda_0} [\sup\limits_{|t|\geq\delta} |K(t,\lambda)|\,] = 0,$ *for every* $\delta > 0.$

e) *There exists a* $\delta_0 > 0$ *such that* $|K(t,\lambda)|$ *is non-decreasing on* $(-\delta_0, 0]$ *and non-increasing on* $[0,\delta_0)$ *as a function of* t*, for each* $\lambda \in \Lambda$.

4.1 Existence of the Operator

The proofs of the theorems are based on the following theorem.

Theorem 4.2 *Let* $1 \leq p < \infty$*. We assume that the kernel function* $K(t,\lambda)$ *satisfies a*)*. If* $f \in L_p < a,b >$*, then the operators, defined in (4.1),* $T(f) \in L_p < a,b >$ *and*

$$\|T(f)\|_{L_p<a,b>} \leq \|K(.,\lambda)\|_{L_1(-\infty,\infty)} \|f\|_{L_p<a,b>}$$

for every $\lambda \in \Lambda$*. This implies that* $T(f;x,\lambda)$ *defines a continuous transformation over* $L_p< a,b >$*, where* $< a,b >$ *is an arbitrary interval in* $\mathbb{R}$*.*

Proof. This kind of existence theorem is also valid in general functional spaces (see e.g. [7]).

We denote $\tilde{f} \in L_1(R)$ as

$$\tilde{f}(t) := \begin{cases} f(t) & , \quad t \in < a,b > \\ 0 & , \quad t \notin < a,b > \end{cases} . \tag{4.2}$$

□

4.2 Main Results

In the present section, we assume that $< a, b >$ is an arbitrary finite interval in $\mathbb{R}$, such as $[a, b]$, $[a, b)$, $(a, b]$ or (a, b).

Definition 4.3 **($\mu-$Generalized- Lebesgue Point)** *Let x_0 be a $\mu-$generalized- Lebesgue point of function $f(x) \in L_1 < a, b >$, i.e., for some x_0 satisfy the condition*

$$\lim_{h\to 0^+} \frac{1}{\mu(h)} \int_0^h |f(x_0 + t) - f(x_0)|\, dt = 0,$$

where $\mu(t)$ defined on $[0, b - a]$, increasing, absolutely continuous and $\mu(0) = 0$.

Theorem 4.4 *Suppose that the function $K(t, \lambda)$ belongs to class $\mathcal{A}$. Let x_0 be a $\mu-$generalized- Lebesgue point of function $f(x) \in L_1 < a, b >$, i.e., for some x_0 satisfy the condition*

$$\lim_{h\to 0^+} \frac{1}{\mu(h)} \int_0^h |f(x_0 + t) - f(x_0)|\, dt = 0, \tag{4.3}$$

where $\mu(t)$ defined on $[0, b - a]$, increasing, absolutely continuous and $\mu(0) = 0$. If (x, λ) tends to (x_0, λ_0) on any planar set Z on which the function

$$\int_{x_0-\delta}^{x_0+\delta} |K(t - x, \lambda)|\, \mu'(|t - x_0|)dt + 2\,|K(0, \lambda)|\, \mu(|x - x_0|),\ 0 < \delta < \delta_0 \tag{4.4}$$

is bounded, then

$$\lim_{(x,\lambda)\to(x_0,\lambda_0)} |T(f; x, \lambda) - f(x_0)| = 0. \tag{4.5}$$

Proof. Suppose that

$$x_0 + \delta < b,\ \ x_0 - \delta > a \ \text{ and } \ 0 < x_0 - x < \frac{\delta}{2}, \tag{4.6}$$

for any $0 < \delta < \delta_0$. Set $|I(x,\lambda)| = |T(f;x,\lambda) - f(x_0)|$. According to condition *b*) and (4.2), we shall write

$$\begin{aligned}
|I(x,\lambda)| &= \left| \int_a^b f(t)K(t-x,\lambda)dt - f(x_0) \right| \\
&= \left| \int_{-\infty}^{\infty} \tilde{f}(t)K(t-x,\lambda)dt - f(x_0) \right| \\
&= \left| \int_{-\infty}^{\infty} \tilde{f}(t)K(t-x,\lambda)dt - \tilde{f}(x_0)\int_{-\infty}^{\infty} K(t-x,\lambda)dt + \tilde{f}(x_0)\int_{-\infty}^{\infty} K(t-x,\lambda)dt - f(x_0) \right|
\end{aligned}$$

Also according to (4.2) and (4.6), we have $\tilde{f}(x_0) = f(x_0)$. Hence from (4.3), we can divide the last equality as follows:

$$\begin{aligned}
|I(x,\lambda)| &\le \int_{-\infty}^{\infty} \left| \tilde{f}(t) - \tilde{f}(x_0) \right| |K(t-x,\lambda)|\,dt + |f(x_0)| \left| \int_{-\infty}^{\infty} K(t-x,\lambda)dt - 1 \right| \\
&= \left\{ \int_{-\infty}^{a} + \int_{a}^{x_0-\delta} + \int_{x_0-\delta}^{x_0} + \int_{x_0}^{x_0+\delta} + \int_{x_0+\delta}^{b} + \int_{b}^{\infty} \right\} \left| \tilde{f}(t) - \tilde{f}(x_0) \right| |K(t-x,\lambda)|\,dt \\
&\quad + |f(x_0)| \left| \int_{-\infty}^{\infty} K(t-x,\lambda)dt - 1 \right| \\
&= I_0(x,\lambda) + I_1(x,\lambda) + I_2(x,\lambda) + I_3(x,\lambda) + I_4(x,\lambda) + I_5(x,\lambda) + I_6(x,\lambda).
\end{aligned}$$

Since

$$|I(x,\lambda)| \le I_0(x,\lambda) + I_1(x,\lambda) + I_2(x,\lambda) + I_3(x,\lambda) + I_4(x,\lambda) + I_5(x,\lambda) + I_6(x,\lambda),$$

it is sufficient to show that the first four terms on the right hand side of the last inequality tends to zero as $(x,\lambda) \to (x_0,\lambda_0)$ on Z.

Firstly we consider $I_1(x,\lambda)$ and $I_4(x,\lambda)$.

From condition *e*) and (4.2), it is seen that

$$
\begin{aligned}
I_1(x,\lambda) &= \int_a^{x_0-\delta} |f(t)-f(x_0)|\,|K(t-x,\lambda)|\,dt \\
&\le \sup_{t\in<a,x_0-\delta]} |K(t-x,\lambda)| \int_a^{x_0-\delta} |f(t)-f(x_0)|\,dt.
\end{aligned}
$$

Since $x_0 - x < \frac{\delta}{2}$

$$
\begin{aligned}
I_1(x,\lambda) &\le \sup_{|u|>\frac{\delta}{2}} |K(u,\lambda)| \int_a^{b} |f(t)-f(x_0)|\,dt \\
&= \sup_{|u|>\frac{\delta}{2}} |K(u,\lambda)| \left\{ \|f\|_{L_1(a,b)} + |f(x_0)|\,(b-a) \right\}. \qquad (4.7)
\end{aligned}
$$

In the same way, we find that

$$
\begin{aligned}
I_4(x,\lambda) &= \int_{x_0+\delta}^{b} |f(t)-f(x_0)|\,|K(t-x,\lambda)|\,dt \\
&\le \sup_{|u|>\frac{\delta}{2}} |K(u,\lambda)| \int_{x_0+\delta}^{b} |f(t)-f(x_0)|\,dt \\
&\le \sup_{|u|>\frac{\delta}{2}} |K(u,\lambda)| \left\{ \|f\|_{L_1(a,b)} + |f(x_0)|\,(b-a) \right\}. \qquad (4.8)
\end{aligned}
$$

Next we consider $I_2(x,\lambda)$.

From (4.3) and (4.2), for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$
\int_{x_0-h}^{x_0} |f(t)-f(x_0)|\,dt < \varepsilon\,\mu(h) \qquad (4.9)
$$

and

$$
\int_{x_0}^{x_0+h} |f(t)-f(x_0)|\,dt < \varepsilon\,\mu(h) \qquad (4.10)
$$

for all $0 < h \le \delta$. Let

$$
F(t) := \int_t^{x_0} |f(y)-f(x_0)|\,dy, \qquad (4.11)
$$

then

$$
dF(t) = -|f(t)-f(x_0)|\,dt. \qquad (4.12)
$$

From (4.9) and (4.11), for all t satisfying the condition $x_0 - t \leq \delta$, we have

$$F(t) \leq \varepsilon\, \mu(x_0 - t). \tag{4.13}$$

Hence, now we can evaluate $I_2(x, \lambda)$.

Evaluation of $I_2(x, \lambda)$ can be done in the following way:

$$I_2(x, \lambda) = \int_{x_0-\delta}^{x_0} |f(t) - f(x_0)|\, |K(t - x, \lambda)|\, dt$$

By (4.11) and (4.12), we can rewrite $I_2(x, \lambda)$ as

$$|I_2(x, \lambda)| = \left| -\int_{x_0-\delta}^{x_0} |K(t - x, \lambda)|\, dF(t) \right|.$$

Integration by parts give us,

$$\begin{aligned} |I_2(x, \lambda)| &= \left| -F(x_0 - \delta)\, |K(x_0 - \delta - x, \lambda)| + \int_{x_0-\delta}^{x_0} F(t)\, \frac{\partial}{\partial t} |K(t - x, \lambda)|\, dt \right| \\ &\leq |F(x_0 - \delta)|\, |K(x_0 - \delta - x, \lambda)| + \int_{x_0-\delta}^{x_0} |F(t)|\, \frac{\partial}{\partial t} |K(t - x, \lambda)|\, dt. \end{aligned}$$

It follows that from (4.13),

$$|I_2(x, \lambda)| \leq \varepsilon \mu(\delta)\, |K(x_0 - \delta - x, \lambda)| + \varepsilon \int_{x_0-\delta}^{x_0} \mu(x_0 - t) \frac{\partial}{\partial t} |K(t - x, \lambda)|\, dt.$$

Integration by parts again, then we get

$$\begin{aligned} |I_2(x, \lambda)| &\leq \varepsilon \mu(\delta)\, |K(x_0 - \delta - x, \lambda)| + \varepsilon \{-\mu(\delta)\, |K(x_0 - \delta - x, \lambda)| \\ &\qquad - \int_{x_0-\delta}^{x_0} \mu'(x_0 - t)\, |K(t - x, \lambda)|\, dt \Bigg\} \\ &= \varepsilon \int_{x_0-\delta}^{x_0} \mu'(x_0 - t)\, |K(t - x, \lambda)|\, dt. \end{aligned}$$

Let

$$I_{2,1}(x,\lambda) := \int_{x_0-x-\delta}^{x_0-x} \mu'(x_0-x-t)\,|K(t,\lambda)|\;dt. \tag{4.14}$$

Here we note that, if a function f is monotone on $[a,b]$ then

$$V[f;a;b] = \bigvee_a^b(f) = |f(b)-f(a)|\,.$$

From *e*) and (4.6), for (4.14) we obtain the following estimate:

$$\begin{aligned}
I_{2,1}(x,\lambda) &= \int_{x_0-x-\delta}^{x_0-x} \left\{ \bigvee_{x_0-x-\delta}^{t} |K(s,\lambda)| + |K(x_0-\delta,\lambda)| \right\} \mu'(x_0-x-t)dt \\
&= \int_{x_0-x-\delta}^{0} \left[\bigvee_{x_0-x-\delta}^{t} |K(s,\lambda)| \right] \mu'(x_0-x-t)dt \\
&\quad + \int_{0}^{x_0-x} \left[\bigvee_{x_0-x-\delta}^{t} |K(s,\lambda)| \right] \mu'(x_0-x-t)dt \\
&\quad + |K(x_0-\delta,\lambda)| \int_{x_0-x-\delta}^{x_0-x} \mu'(x_0-x-t)dt \\
&= \int_{x_0-x-\delta}^{0} [\,|K(t,\lambda)| - |K(x_0-\delta-x,\lambda)|\,]\,\mu'(x_0-x-t)dt \\
&\quad + \int_{0}^{x_0-x} \{|K(0,\lambda)| - |K(x_0-\delta-x,\lambda)|\}\,\mu'(x_0-x-t)dt \\
&\quad + \int_{0}^{x_0-x} \{|K(0,\lambda)| - |K(t,\lambda)|\}\,\mu'(x_0-x-t)dt \\
&\quad + |K(x_0-\delta,\lambda)| \int_{x_0-x-\delta}^{x_0-x} \mu'(x_0-x-t)dt
\end{aligned}$$

$$= \int_{x_0-x-\delta}^{0} |K(t,\lambda)|\,\mu'(x_0-x-t)dt + 2\,|K(0,\lambda)| \int_{0}^{x_0-x} \mu'(x_0-x-t)dt$$
$$- |K(x_0-\delta-x,\lambda)| \left\{ \int_{x_0-x-\delta}^{0} \mu'(x_0-x-t)dt + \int_{0}^{x_0-x} \mu'(x_0-x-t)dt \right\}$$
$$- \int_{0}^{x_0-x} |K(t,\lambda)|\,\mu'(x_0-x-t)dt + |K(x_0-\delta-x,\lambda)| \int_{x_0-\delta}^{x_0} \mu'(x_0-t)dt$$

$$\leq \int_{x_0-x-\delta}^{x_0-x} |K(t,\lambda)|\,\mu'(x_0-x-t)dt + 2\,|K(0,\lambda)| \int_{0}^{x_0-x} \mu'(x_0-x-t)dt$$
$$+ |K(x_0-\delta-x,\lambda)| \left\{ \int_{x_0-\delta}^{x_0} \mu'(x_0-t)dt - \int_{x_0-\delta}^{x_0} \mu'(x_0-t)dt \right\}$$

$$= \int_{x_0-x-\delta}^{x_0-x} |K(t,\lambda)|\,\mu'(x_0-x-t)dt + 2\,|K(0,\lambda)|\,\mu(|x-x_0|)$$
$$= \int_{x_0-\delta}^{x_0} |K(t-x,\lambda)|\,\mu'(x_0-t)dt + 2\,|K(0,\lambda)|\,\mu(|x-x_0|). \tag{4.15}$$

Consequently by (4.15),

$$|I_2(x,\lambda)| \leq \varepsilon \left\{ \int_{x_0-x-\delta}^{x_0-x} |K(t,\lambda)|\,\mu'(x_0-t)dt + 2\,|K(0,\lambda)|\,\mu(|x-x_0|) \right\}. \tag{4.16}$$

We can use similar method for evaluating $I_3(x,\lambda)$.

Let

$$G(t) := \int_{x_0}^{t} |f(y) - f(x_0)|\,dy. \tag{4.17}$$

Then

$$dG(t) = |f(t) - f(x_0)|\,dt. \tag{4.18}$$

Moreover, for all t satisfying the condition $t - x_0 \leq \delta$, we have from (4.10) and (4.17)

$$G(t) \leq \varepsilon\ \mu(t - x_0). \tag{4.19}$$

According to (4.17) and (4.18), we can rewrite $I_3(x, \lambda)$ as

$$|I_3(x, \lambda)| = \left| \int_{x_0}^{x_0+\delta} |K(t - x, \lambda)|\, dG(t) \right|$$

Integration by parts, we find that

$$\begin{aligned} |I_3(x, \lambda)| &= \left| G(x_0 + \delta)\ |K(x_0 + \delta - x, \lambda)| + \int_{x_0}^{x_0+\delta} G(t)\, \frac{\partial}{\partial t} \left(-|K(t - x, \lambda)|\right)\, dt \right| \\ &\leq |G(x_0 + \delta)|\, |K(x_0 + \delta - x, \lambda)| + \int_{x_0}^{x_0+\delta} |G(t)|\, \frac{\partial}{\partial t} \left(-|K(t - x, \lambda)|\right) dt. \end{aligned}$$

While $t \in [x_0, x_0 + \delta)$, according to condition e) and (4.6), $\frac{\partial}{\partial t}\left(-|K(t - x, \lambda)|\right)$is positive. Therefore, from (4.19) we can write the following inequality.

$$|I_3(x, \lambda)| \leq \varepsilon \mu(\delta)\ |K(x_0 + \delta - x, \lambda)| + \varepsilon \int_{x_0}^{x_0+\delta} \mu\,(t - x_0) \frac{\partial}{\partial t} \left(-|K(t - x, \lambda)|\right) dt. \tag{4.20}$$

If we using integration by parts, the second term of the right hand side in (4.20), then

$$\begin{aligned} |I_3(x, \lambda)| &\leq \varepsilon \mu(\delta)\ |K(x_0 + \delta - x, \lambda)| + \varepsilon \left\{ -\mu(\delta)\, |K(x_0 + \delta - x, \lambda)| \right. \\ &\qquad \left. + \int_{x_0}^{x_0+\delta} \mu'(t - x_0)\, |K(t - x, \lambda)|\, dt \right\} \\ &= \varepsilon \int_{x_0}^{x_0+\delta} \mu'(t - x_0)\, |K(t - x, \lambda)|\ dt. \end{aligned}$$

Here we denote

$$I_{3,1}(x, \lambda) := \int_{x_0-x}^{x_0-x+\delta} \mu'(t + x - x_0)\, |K(t, \lambda)|\, dt. \tag{4.21}$$

While $t \in [x_0, x_0 + \delta)$, according to condition e) and (4.6), for (4.21) we obtain the following estimate:

$$\begin{aligned} I_{3,1}(x,\lambda) &= \int_{x_0}^{x_0+\delta} \left\{ \bigvee_{t}^{x_0+\delta} |K(s-x,\lambda)| + |K(x_0+\delta-x,\lambda)| \right\} \mu'(t-x_0)dt \\ &= \int_{x_0}^{x_0+\delta} \left[\bigvee_{t}^{x_0+\delta} |K(s-x,\lambda)| \right] \mu'(t-x_0)dt \\ &\quad + |K(x_0+\delta-x,\lambda)| \int_{x_0}^{x_0+\delta} \mu'(t-x_0)dt \end{aligned}$$

$$\begin{aligned} &= \int_{x_0}^{x_0+\delta} [\, |K(t-x,\lambda)| - |K(x_0+\delta-x,\lambda)| \,] \mu'(t-x_0)dt \\ &\quad + |K(x_0+\delta-x,\lambda)| \int_{x_0}^{x_0+\delta} \mu'(t-x_0)dt \\ &= \int_{x_0}^{x_0+\delta} |K(t-x,\lambda)| \, \mu'(t-x_0)dt. \end{aligned} \tag{4.22}$$

From (4.22), we find that

$$|I_3(x,\lambda)| \le \varepsilon \int_{x_0}^{x_0+\delta} |K(t-x,\lambda)| \, \mu'(t-x_0)dt. \tag{4.23}$$

Finally, let us consider the integrals $I_0(x,\lambda)$ and $I_5(x,\lambda)$. According to (4.2) we write

$$\begin{aligned} I_0(x,\lambda) + I_5(x,\lambda) &= \int_{t \notin <a,b>} \left| \tilde{f}(t) - f(x_0) \right| |K(t-x,\lambda)| \, dt \\ &= \int_{t \notin <a,b>} |f(x_0)| \, |K(t-x,\lambda)| \, dt \\ &= |f(x_0)| \int_{t \notin <a,b>} |K(t-x,\lambda)| \, dt. \end{aligned}$$

By (4.6), when $t \notin [a, b]$, we have $t < a$ or $t > b$. If $t < a$ then

$$t - x < a - x < x_0 - x - \delta < -\frac{\delta}{2} < 0,$$

while if $t > b$

$$t - x > b - x > x_0 - x + \delta > \delta > 0.$$

This implies that, for any $\delta > 0$

$$\int\limits_{t \notin <a,b>} |K(t-x,\lambda)|\, dt \leq \int\limits_{|t-x|>\frac{\delta}{2}} |K(t-x,\lambda)|\, dt = \int\limits_{|u|>\frac{\delta}{2}} |K(u,\lambda)|\, du.$$

Thus we get

$$I_0(x,\lambda) + I_5(x,\lambda) \leq |f(x_0)| \int\limits_{|u|>\frac{\delta}{2}} |K(u,\lambda)|\, du, \tag{4.24}$$

which in view of condition c) tends to zero as $(x, \lambda) \to (x_0, \lambda_0)$.

Combining (4.7), (4.8), (4.16), (4.23) and (4.24), we get

$$\begin{aligned} |I(x,\lambda)| \quad \leq \quad & 2 \sup_{|u|>\frac{\delta}{2}} |K(u,\lambda)| \left\{ \|f\|_{L_1(a,b)} + |f(x_0)|\,(b-a) \right\} \\ & + \varepsilon \int\limits_{x_0}^{x_0+\delta} |K(t-x,\lambda)|\, \mu'(t-x_0) dt \\ & + \varepsilon \left\{ \int\limits_{x_0-x-\delta}^{x_0-x} |K(t,\lambda)|\, \mu'(t-x_0-x-t) dt + 2\,|K(0,\lambda)|\, \mu(|x-x_0|) \right\} \\ & + |f(x_0)| \left| \int\limits_a^b K(t-x,\lambda) dt - 1 \right| \end{aligned}$$

$$\leq \quad 2\sup_{|u|>\frac{\delta}{2}} |K(u,\lambda)| \left\{ \|f\|_{L_1(a,b)} + |f(x_0)|\,(b-a) \right\}$$
$$+\varepsilon \left\{ \int_{x_0-\delta}^{x_0+\delta} |K(t-x,\lambda)|\,\mu'(|t-x_0|)dt + 2\,|K(0,\lambda)|\,\mu(|x-x_0|) \right\}$$
$$+ |f(x_0)| \left| \int_a^b K(t-x,\lambda)dt - 1 \right|.$$

This inequality is shown to be also valid for $-\frac{\delta}{2} < x_0 - x < 0$.

Therefore our theorem now follows, in view of conditions *b*), *c*), *d*) and (4.4).

$$\lim_{(x,\lambda)\to(x_0,\lambda_0)} T(f;x,\lambda) = f(x_0)$$

and this proves (4.5).□

If we consider the case $< a, b >= \mathbb{R}$, we have the following theorem .

Theorem 4.5 *Suppose that the function $K(t,\lambda)$ belongs to class $\mathcal{A}$. If (x,λ) tends to (x_0,λ_0) on any planar set Z on which the function*

$$\int_{x_0-\delta}^{x_0+\delta} |K(t-x,\lambda)|\,\mu'(|t-x_0|)dt + 2\,|K(0,\lambda)|\,\mu(|x-x_0|),\ 0<\delta<\delta_0$$

is bounded. Then at each point x_0 for which (4.3) *holds, we have for $f(x) \in L_1(-\infty,\infty)$*

$$\lim_{(x,\lambda)\to(x_0,\lambda_0)} |T(f;x,\lambda) - f(x_0)| = 0.$$

Proof. We can divide $|I(x,\lambda)| := |T(f;x,\lambda) - f(x_0)|$ four parts as follows:

$$|I(x,\lambda)| \quad \leq \quad \left\{ \int_{-\infty}^{x_0-\delta} + \int_{x_0-\delta}^{x_0} + \int_{x_0}^{x_0+\delta} + \int_{x_0+\delta}^{\infty} \right\} |f(t) - f(x_0)|\,|K(t-x,\lambda)|\,dt$$
$$+ |f(x_0)| \left| \int_{-\infty}^{\infty} K(t-x,\lambda)dt - 1 \right|$$
$$= \quad I_1(x,\lambda) + I_2(x,\lambda) + I_3(x,\lambda) + I_4(x,\lambda) + I_6(x,\lambda).$$

Analogously to the proof of Theorem 4.1 we can show that $I_2(x,\lambda) \to 0$, $I_3(x,\lambda) \to 0$ and $I_6(x,\lambda) \to 0$ as $(x,\lambda) \to (x_0,\lambda_0)$. Hence, it is sufficient to show that

$$\lim_{(x,\lambda)\to(x_0,\lambda_0)} I_1(x,\lambda) = 0$$

and

$$\lim_{(x,\lambda)\to(x_0,\lambda_0)} I_4(x,\lambda) = 0.$$

According to (4.6), when $t \notin [x_0-\delta, x_0+\delta]$, we have $t < x_0-\delta$ or $t > x_0+\delta$. If $t < x_0-\delta$ then

$$t - x < x_0 - x - \delta < -\frac{\delta}{2} < 0,$$

while if $t > x_0 + \delta$

$$t - x > x_0 - x + \delta > \delta > \frac{\delta}{2} > 0.$$

Therefore, we can write $I_1(x,\lambda) + I_4(x,\lambda)$ as follows:

$$\begin{aligned}
I_1(x,\lambda) + I_4(x,\lambda) &= \int\limits_{t\notin[x_0-\delta,x_0+\delta]} |f(t) - f(x_0)|\,|K(t-x,\lambda)|\,dt \\
&\le \int\limits_{t\notin[x_0-\delta,x_0+\delta]} |f(t)|\,|K(t-x,\lambda)|\,dt + |f(x_0)| \int\limits_{t\notin[x_0-\delta,x_0+\delta]} |K(t-x,\lambda)|\,dt \\
&\le \sup_{|u|>\frac{\delta}{2}} |K(u,\lambda)| \int\limits_{|t-x|>\frac{\delta}{2}} |f(t)|\,dt + |f(x_0)| \int\limits_{|u|>\frac{\delta}{2}} |K(u,\lambda)|\,du \\
&\le \sup_{|u|>\frac{\delta}{2}} |K(u,\lambda)|\,\|f\|_{L_1(-\infty,\infty)} + |f(x_0)| \int\limits_{|u|>\frac{\delta}{2}} |K(u,\lambda)|\,du,
\end{aligned}$$

for any $\delta > 0$. Hence from *c*) and *d*), $I_1(x,\lambda) + I_4(x,\lambda)$ tends to zero as $(x,\lambda) \to (x_0,\lambda_0)$.

Thus, the proof of the theorem is completed.□

Example. We consider the function

$$K(t,\lambda)=\begin{cases} 2\lambda & , \quad t\in[0,\frac{1}{\lambda}] \\ -\lambda & , \quad t\in[-\frac{1}{\lambda},0) \\ 0 & , \quad t\notin[-\frac{1}{\lambda},\frac{1}{\lambda}] \end{cases}, \tag{4.25}$$

where $\Lambda=[1,\infty)$ is a set of indices with natural topology and $\lambda_0=\infty$ is an accumulation point of Λ in this topology.

From (4.25), one has

$$\int_R K(t-x,\lambda)dt=-\int_{[-\frac{1}{\lambda},0)}\lambda dt+\int_{[0,\frac{1}{\lambda}]}2\lambda dt=1<\infty.$$

Furthermore, it is easy to see that

$$\lim_{\lambda\to\infty}\int_{|t|\geq\delta}|K(t,\lambda)|dt=0,$$

and

$$\lim_{\lambda\to\infty}[\sup_{|t|\geq\delta}|K(t,\lambda)|]=0,$$

for every $\delta>0$.

In addition, by (4.25) $|K(t,\lambda)|$ is non-decreasing on $(-\infty,0]$ and non-increasing on $[0,\infty)$ as a function of t, for each $\lambda\in\Lambda$.

This implies that the kernel function $K(t,\lambda)$ belongs to **Class** $\mathcal{A}$.

We assume that $\mu(t) = t$. If we use (4.25) in Theorem 4.1 and Theorem 4.2, we find

$$\int_{x_0-\delta}^{x_0+\delta} |K(t-x,\lambda)|\,\mu'(|t-x_0|)dt + 2\,|K(0,\lambda)|\,\mu(|x-x_0|)$$

$$= \int_{x_0-\delta}^{x_0+\delta} |K(t-x,\lambda)|\;dt + 4\,\lambda\;|x-x_0|\,.$$

The first part of the right hand side of this equality is finite. Furthermore

$$\lim_{(x,\lambda)\to(x_0,\infty)} \lambda\;|x-x_0| = M < \infty,$$

if and only if the rates of convergence of $\lambda \to \infty$ and $x \to x_0$ are equivalent.

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