AN EFFICIENT ITERATIVE SOLUTION OF OSCILLATORY INTEGRALS CONTAINING THE PRODUCT OF BESSEL FUNCTIONS

by

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A thesis submitted to

The Graduate Institute of Sciences and Engineering

of

Fatih University

İn partial fulfillment of the requirements for the degree of

Master of Science

in

Mathematics

June 2011

Istanbul, Turkey

APPROVAL PAGE

I certify that this thesis satisfies all the requırements as a thesis for degree of Master of Science.

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This is to certify that I have read this thesis and that in my opinion it is fully adequate, in scope and quality, as a thesis for the degree of Master of Science.

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M. S. Thesis-Mathematics

June 2011

Supervisor: Assist. Prof. Dr. Ali SAHIN

ABSTRACT

The main purpose of this study is to examine oscillatory integrals which are containing the product of Bessel functions of the first kind. These types of integrals are appeared a result of the modeling of axisymmetric heat conduction in cracked nonhomogeneous medium. So, these integrals can be written as

$$\int_0^\infty D(r,s,\rho)\, J_m(r\rho)\, J_n(s\rho)\, d\rho \qquad m,n=0,1 \quad \text{and} \quad 0<r,s<1.$$

Function of $D(r,s,\rho)$ has some singularities like Cauchy and logarithmic type, besides $D(r,s,\rho)$ has oscillating property by means of Bessel functions. The solution of these types of integrals starts with the transformation of partial differential equation which has variable coefficient in cylindrical coordinate system to axisymmetric coordinate system because of independency of angular variable. Then thanks to Hankel integral transform the system turns to a system of ordinary differential equations. The solution of ordinary differential equation with the convenient boundary condition can be given as a system of singular integral equations with a square root singularity. Consequently, there is no closed form solution for the system of integral equations then solving this problem with appropriate numerical methods**Keywords:** Axisymmetric System, Singular Integrals, Numerical Analysis, Oscillatory Integrals

BESSEL FONKSİYONLARININ ÇARPIMLARINI İÇEREN SALINIMLI İNTEGRALLERİN VERİMLİ TEKRARLAMALI ÇÖZÜMLERİ

Zilal DİREK

Yüksek Lisans Tezi – Matematik

Haziran 2011

Tez Danışmanı: Yard. Doç. Dr. Ali ŞAHİN

ÖZ

Bu çalışmanın amacı

$$\int_0^\infty D(r,s,\rho)\, J_m(r\rho)\, J_n(s\rho)\, d\rho \qquad m,n=0,1 \quad \text{and} \quad 0<r,s<1,$$

şeklinde ifade edilen birinci tür Bessel fonksiyonlarının çarpımlarını içeren salınımlı integraller incelenecektir. $D(r,s,\rho)$ fonksiyonu Cauchy ve logaritmik türünde tekillikleri içerir ve ayrıca Bessel fonksiyonuyla birlikte salınım özelliklerine sahiptir. Bu tip salınım integralleri, eksenel simetrik koordinat sisteminde kompozit malzemelerdeki çatlak problemlerinin modelinde ortaya çıkar. Modelleme, silindirik koordinatlarda verilen değişken katsayılı kısmi türevli diferansiyel denklem sisteminin, açısal değişkenden bağımsız olması sebebiyle eksenel simetri koordinat sistemine dönüşmesiyle başlar. Sonrasında, Hankel integral dönüşümü kullanılarak kısmi türevli diferansiyel denklem sistemi, adi diferansiyel denklem sistemine dönüşür. Uygun sınır şartları altında adi diferansiyel denklem sisteminin çözümü, karekök tekillği içeren tekil integral denklemler sistemi olarak verilebilir. Bu tekil integrallerin kapalı formda bir çözümü olmadığından ötürü çözümler sayısal metodlarla bulunur.

Anahtar Kelimeler: Eksenel Simetri Sistemi, Tekil İntegraller, Sayısal Analiz, Düzensiz Salınımlı İntegraller.

DEDICATION

To my mother

ACKNOWLEDGEMENT

I express sincere appreciation to Assistant Prof. Ali Sahin for his guidance and insights throughout the research and his valuable suggestions and comments.

I express my thanks and appreciation to my family for their understanding, motivation and patience. Lastly, but in no sense the least I am thankful to all colleagues and friends who made my study at the university a memorable and valuable experience.

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CHAPTER 1

INTRODUCTION

Lots of various problems in the most important areas of mathematical physics and different technical problems are related with applications of Bessel functions. In the solution of problems of acoustic, radio physics, hydrodynamics, atomic and nuclear physics, it is naturally obtained different types of Bessel functions. Many of areas include wide range of different applications of Bessel functions. For example, heat conduction theory, dynamical and linked problems, problems are solving on spherical or cylindrical coordinates, different problems concerning the oscillations and problems on the concentration of the stress near cracks are some of them.

In addition to the theory and application, the numerical integration is one of the great and main subjects of mathematics. Therefore, the problem of numerical integration is a basic problem of numerical analysis. Some of numerical integration methods and, specifically, the quadrature method were examined. The earliest study about quadrature is done by Greek geometers, and then Arab mathematicians studied for quadrature in 9th century, and then,up to date, mathematicians explored quadrature method and its applications. There are also some studies for quadrature method mentioned in (Knigth and Newbey, 1970), (Hall, 1876), (Peterson, 1996) and (Crow, 1993)

In this study, it is considered a class of oscillating integrals containing the product of the first and zeroth order Bessel functions of the first kind given as

$$\int_0^{\infty} D(r,s,\rho)\, J_m(r\rho)\, J_n(s\rho)\, d\rho, \qquad m,n=0,1 \quad \text{and} \quad 0<r,s<1. \quad (1.1)$$

These types of integrals usually arise because of the part of modeling in linear elastic fracture mechanics, especially in composite materials in axisymmetric coordinate system. The integral contains singularities such as elliptic and logarithmic type and has an oscillating property together with Bessel functions. Due to no closed form solution for some special values of the integrand $D\ \ r,s,\rho$, it should be solved numerically.

The integrals given in (1.1) can be easily seen in fracture mechanics of composite materials such as an interface crack in axisymmetric medium. Due to the mixed boundary values at crack line, displacement in equilibrium equation can be represented by a singular integral equation which contains Fredholm type integral equation. Some of the best example related to the Equation (1.1) can be seen in the work done by Ozturk, (1993). He examined the stress intensity factors and the displacements on a penny-shaped crack in terms of the different loading conditions. In the formulation of the problem, he obtained the integrals given in (1.1). The same types of integrals can be seen also in the work done by Sahin, (1998) who examined the thermal stresses around an axisymmetric insulated barrier under constant heat flux. In his work, it is shown the elliptic and logarithmic singularities in the integrand as $s \to r$. Al-Borgi (2009) and his coworkers studied on thermal stresses and displacements under constant heat flux and different types of loading over a crack between a homogeneous material and a functionally graded material in an axisymmetric coordinate systems.

Because of the spherical and cylindrical coordinates, system of oscillatory integrals involving Bessel functions can be seen some problems in the areas of engineering (crack problems, magnetic field theory, and wave theory) and applied mathematics. Calculation of highly oscillatory integrals, especially in case of multivariable integrand, is difficult to solve a problem but the difficulty is overcome by means of developing computer technology and produced new algorithms.

In the past few years, new approaches such as Filon type (1982), Levin type (1996) and fixed-point calculation were improved for these types of problems. Filon (1928) brought a vision with the help of quadrature method for solution of oscillatory integrals and then in almost every area of about this topic have been studies. We come across studies on oscillatory integrals involving Bessel functions especially at magnetic field theory. Integrals defined at infinite intervals involving product of Bessel functions and their algorithm are examined by Kölbig (1995). He found the solution of integrals of Mellin integral transformation in terms of Gamma functions. In practical problems, we encountered with the integrals of involving product of Bessel functions which is defined at infinite interval are examined by Fabrikant (2003) and he obtained integrals of closed solutions for some special cases. This study, difficulty of well-known definite integrals of containing Bessel functions can be expressed as simple functions and the solution is founded through using representation of Bessel functions with integrals. The problem consists in integrating an arbitrary product of Bessel functions with an additional rational or exponential factor over a semi-infinite interval is done by Deun and Cools (2008). This study difficulty arises from the irregular oscillatory behavior and possible slow decay of the integrand, this problem is solved in two parts. First one finite part is computed using gauss-Legendre quadrature. Second one is infinite part, then infinite part is approximated using asymptotic expansions and this approximation is integrated exactly with the aid of the upper incomplete gamma function. In the case where a rational factor is present, this factor is first expanded in a Taylor series around infinity. A similar study is done by (Gebramiam, *et al.*, 2010). However, this study is solved integrals of products of spherical two Bessel functions multiplied by an exponential factor symbolically. The problem is first form into a related limit problem, which can be broken into two related sub problems involving exponential and exponential integral functions. In this study contain some special condition for preventing singularities for instance $\int_0^\infty e^{-x/u} x^n J_\nu(x) J_\mu(x)\, dx$ where $J_\nu(x)$ and $J_\mu(x)$ define spherical functions of integer orders, with $\nu \geq 0$, $\mu \geq 0$, $u > 0$ and n is an integer constant. From the asymptotic properties of spherical Bessel functions and convergence requirement as the argument x goes to zero, so at this study is obtained the constraint $n+\nu+\mu \geq 0$. Besides Xiang (2004) studied integral transformation of highly oscillatory function, Bessel functions. Filon-type method is used at this study. Xiang used quadrature method and took a function which is suitably smooth on 0,1 .

Calculation of oscillatory integrals is not possible with standard numerical analysis method like Simpson's rule or quadrature method. Integrals contain singularities besides features of oscillation. These singularities occur during calculation of elliptic integrals as logarithmic or during the variable approach to infinity. When it is scanned the current literature it can be seen that the solution of oscillatory integrals can be expressed as hypergeometric functions which are not easy to calculate. In our thesis, we calculated the oscillatory integrals containing a product of Bessel functions. Solution of these integrals can be divided into two parts analytical analysis and numerical analysis. First step is analytical analysis; at this point, we did analysis of oscillatory integrals and we examined what kinds of singularities are contained in integrand. Singularities of elliptic integrals (some of them is removable) are separated by analytical methods. Integrands which have feature of oscillation is passed through asymptotic analysis and defined having singularities at infinity. At this step, we took advantage of language of symbolic programming, MAPLE.

Second step is numerical analysis in which Gauss quadrature method is used for the solution of integrals. Integrands contain Bessel functions of the first kind which are defined intrinsically in FORTRAN programming language at version 11.1. The code for the algorithm of integrals has been written in FORTRAN programming language.

This study is organized in the six chapters. Starting with the introduction, it is mentioned some numerical integration methods in the second chapter. There are also given some analytical and numerical techniques how to handle singularities at infinity and shown to evaluate these types of integrals by quadrature method. Since Bessel functions and Chebyshev polynomials are used in this study it is given some information about their properties. Not only the first kind Bessel functions but also the other types are examined in details. The values of the special integrals related to Chebyshev polynomials are shown. In Chapter 4 and 5, the algorithms for calculation of the integrals given in (1.1) are obtained. Using the integral representation of the first kind Bessel functions and some new definitions, the iterative formula for each type of integrals is obtained. The conclusion of the study is given in the last chapter in which the summary of the iterative algorithms are given.

CHAPTER 2

NUMERICAL INTEGRATION

Numerical integration is the study of how the numerical value of an integral can be found. It is used by engineers and scientists to obtain approximate answers. The beginning of this subject is to be desired in former studies. A fine example of ancient numerical integration is the Greek quadrature of the circle by means of inscribed and circumscribed regular polygons. This process led Archimedes to an upper and lower bound of π. Over the centuries, particularly since the sixteenth century many methods of numerical integration have been devised. There are some methods for numerical integration as mentioned below: the goal of quadrature is to approximate the definite integral of $f(x)$ over the interval $[a,b]$ by evaluating $f(x)$ at a finite number of sample points. Composite trapezoidal and Simpson's rules are intuitive methods of finding the area under the curve $y=f(x)$ over $[a,b]$ by approximating that area with a series of trapezoids that lie above the intervals $[X_k, X_{k+1}]$. An approximation rule is used with step sizes h and $2h$; then an algebraic manipulation of the two answers is used to produce an improved answer. Each successive level of improvement increases the order of the error term from $O(h^{2N})$ to $O(h^{2N+2})$. This process, called Romberg integration. Lastly, Gauss-Legendre integration is another method to find the area under the curve $y=f(x)$, $-1\le x\le 1$. We have already seen that the trapezoidal rule is a method for finding the area under the curve and that it uses two-function evaluation at the end points $(-1, f(-1))$ and $(1, f(1))$. But, when the graph of $y=f(x)$ is concave down, the error in approximation is the entire region that lies between the curve and the line segment joining the points. If we can use nodes x_1 and x_2 lie inside the interval $[-1,1]$, the line through the two points $(X_1, f(X_1))$ and $(X_2, f(X_2))$ crosses the curve, and the area under the line more closely approximates the area under the curve. Thereby, the numerical integration is solving a problem of $\int_a^b f(x)dx$ using series expansion of integrand $f(x)$ at each particular point. In the series expansion of the integrand, it will be used an orthogonal functions. Numerical evaluation of some integrals cannot be evaluated easily due to singularities in the integrand. There are some methods to eliminate these singularities and then, using a suitable method the integral can be solved numerically very straightforward ways. In the following section, it will be introduced some methods (Kreyszıg, 2006), (Mathews and Fink, 2004) and (Devis and Rabinowitz, 1984).

2.1. TRAPEZOIDAL RULE

The formula of trapezoidal rule is given by

$$\int_a^b f(x)\,dx = \frac{h}{2}\left[f(x_0) + f(x_1)\right] - \frac{h^3}{12} f''(\xi) \ . \tag{2.1}$$

When function of f take positive value, the integral $\int_a^b f(x)\,dx$ equal to area as a trapezoid.

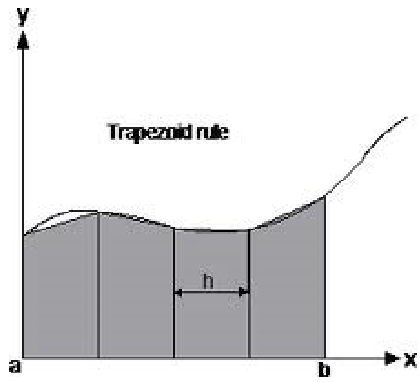

Figure 2.1 : Trapezoid rule

Trapezoid rule is very useful for function which has second order derivative is zero and this rule gives real value of integral.

2.2. SIMPSON'S RULE

Simpson's rule is used often for numerical integration because it combines simplicity of form with a rather accurate result. The major drawback of this method is the fact that it can only be applied to an even number of intervals, which can be an inconvenience in some cases.

2.2.1. Simpson's Formulas

A. For $n=2$ in $x_0<\xi<x_2$, (n is interval number for integral)

$$\int_{x_0}^{x_2} f(x)\,dx=\frac{h}{3}\left[f(x_0)+4f(x_1)+f(x_2)\right]-\frac{h^5}{90}f^{(4)}(\xi). \tag{2.2}$$

B. For $n=3$ in $x_0<\xi<x_3$,

$$\int_{x_0}^{x_3} f(x)\,dx=\frac{3h}{8}\left[f(x_0)+3f(x_1)+3f(x_2)+f(x_3)\right]-\frac{3h^5}{80}f^{(4)}(\xi). \tag{2.3}$$

C. For $n=4$ in $x_0<\xi<x_4$,

$$\int_{x_0}^{x_4} f(x)\,dx=\frac{2h}{45}\left(7f(x_0)+32f(x_1)+12f(x_2)\right.$$

$$\left.+32f(x_3)+7f(x_4)\right)-\frac{8h^7}{945}f^{(6)}(\xi). \tag{2.4}$$

2.3. ROMBERG INTEGRATION

Romberg integration is an iterative no adaptive scheme for automatic integration. It is considered automatic because the numbers of functional evaluations depend upon the behavior of the integrand over the entire interval of integration. It is no adaptive because it evaluates the integrand at a fixed set of points, not dependent upon the integrand itself.

If the function $f(x)$ is bounded and Riemann-integrable over the interval $[0,1]$ then the Romberg integration technique, which is an expanded version of the trapezoid rule that can be utilized in higher order equations, can be used. The definition of the trapezoid rule is

$$\int_a^b f(x)\,dx = \int_a^{x_1} f(x)\,dx + \ldots + \int_{x_{n-1}}^b f(x)\,dx,$$

$$\approx \frac{h}{2}\left[f(a)+f(x_1)\right]+\ldots+\frac{h}{2}\left[f(x_{n-1})+f(b)\right],$$

$$= \frac{h}{2}\left[f(a)+2f(x_1)+\ldots+2f(x_{n-1})+f(b)\right]. \qquad (2.5)$$

where

$$h = \frac{b-a}{n}$$

with n representing the number of subintervals the entire interval is divided up into.

2.3.1. Romberg Integration Formula

$$R_{1,1}=\frac{h_1}{2}\left[f(a)+f(b)\right]=\frac{b-a}{2}\left[f(a)+f(b)\right], \quad (2.6)$$

$$R_{2,1}=\frac{1}{2}\left[R_{1,1}+h_1\left(a+\frac{1}{2}h_1\right)\right], \quad (2.7)$$

$$\int_a^b f(x)\,dx=\frac{4R_{k,1}-R_{k-1,1}}{3}+\frac{b-a}{2160}\left[4h_k^4 f^{(4)}(\mu_k)-h_{k-1}^4 f^{(4)}(\mu_{k-1})\right],$$

$$=\frac{4R_{k,1}-R_{k-1,1}}{3}+O\left(h_k^4\right), \quad (2.8)$$

$$R_{i,j}=\frac{4^{j-1}R_{i,j-1}-R_{i-1,j-1}}{4^{j-1}-1}. \quad (2.9)$$

2.4. GAUSSIAN QUADRATURE

In numerical analysis, a quadrature rule is an approximation of the definite integral of a function, usually stated as a weighted sum of function values at specified points within the domain of integration. Quadrature rule is constructed to yield an exact result for polynomials of degree $2n-1$ or less by a suitable choice of the points x_i and weights w_i for $i=1,...,n$. The domain of integration for such a rule is conventionally taken as $[-1,1]$, so the rule is stated as

$$\int_{-1}^{1} f(x)\,dx \approx \sum_{i=1}^{n} w_i f(x_i). \quad (2.10)$$

Gaussian quadrature procedure as mentioned above arises accurate results if the function $f(x)$ is well approximated by a polynomial function within the range $[-1,1]$. The method is not suitable for functions with singularities. However, if the integrated functions can be written as $f(x) = w(x)g(x)$ where $g(x)$ is approximately polynomial, and $w(x)$ is known, then there are common weighting functions include

$$w(x) = (1-x^2)^{-1/2}, \quad \text{(Gauss-Chebyshev)},$$

$$w(x) = e^{-x^2}, \quad \text{(Gauss-Hermite)}.$$

2.4.1. Gauss Integration Formula

$$\int_a^b g(v)\,dv \approx \int_{-1}^{1} f(x)\,dx \approx \sum_{i=1}^{n} w_i f(x_i),$$

$$= \frac{b-a}{2}\int_{-1}^{1} f\left(\frac{(b-a)x+b+a}{2}\right)dx \tag{2.11}$$

where

$$x = -\frac{b+a}{b-a} + \frac{2v}{b-a} \quad \text{and} \quad v = \frac{a+b}{2} + \frac{b-a}{2}x, \quad dv = \frac{b-a}{2}dx. \tag{2.12}$$

Table 2.1: Gauss quadrature points and corresponding weight function values.

$\int_{-1}^{1} f(x)dx = \sum_{i=1}^{n} w_{n,k} f(x_{n,k}) + E(f,n)$			
n	$x_{n,k}$	$w_{n,k}$	Error
2	0.5773502692 -0.5773502692	1 1	$\frac{f^{(iv)}(\xi)}{135}$
3	0.7745966692 -0.7745966692 0.0000000000	5/9 5/9 8/9	$\frac{f^{(6)}(\xi)}{15.750}$

4	0.8611363116 -0.8611363116 0.3399810436 -0.3399810436	0.3478548451 0.3478548451 0.6521451549 0.6521451549	$\dfrac{f^{(8)}(\xi)}{3472875}$
5	0.9061798459 -0.9061798459 0.5384693101 -0.5384693101 0.0000000000	0.2369266886 0.2369268851 0.4786286705 0.4786286705 0.5688888889	$\dfrac{f^{(10)}(\xi)}{1237732650}$
6	0.9324695142 -0.9324695142 0.6612093865 -0.6612093865 0.2386191861 -0.2386191861	0.1713244924 0.1713244924 0.3607615730 0.3607615730 0.4679139346 0.4679139346	$\dfrac{f^{(12)}(\xi)\,2^{13}[6!]^4}{[12!]^3\,13!}$

2.4.2. Change of Interval for Gaussian Quadrature

An interval over a,b must be changed into an interval over $-1,1$ before applying the Gaussian Quadrature rule. This change of interval can be done in the following way:

$$\int_a^b f\;x\;dx = \frac{b-a}{2}\int_{-1}^{1} f\left(\frac{b-a}{2}x + \frac{a+b}{2}\right)dx\,. \tag{2.13}$$

After applying the Gaussian Quadrature rule, the following approximation is:

$$\int_a^b f\;x\;dx \approx \frac{b-a}{2}\sum_{i=1}^{n} w_i f\left(\frac{b-a}{2}x_i + \frac{a+b}{2}\right). \tag{2.14}$$

Table 2.2: Weight functions of orthogonal polynomials along with definition interval

Interval	$w(x)$	**Orthogonal polynomials**
$[-1,1]$	1	Legendre Polynomials
$(-1,1)$	$(1-x)^{\alpha}(1+x)^{\beta}, \quad \alpha,\beta > -1$	Jacobi Polynomials
$(-1,1)$	$1/\sqrt{1-x^2}$	Chebyshev Polynomials (first kind)

$[-1,1]$	$\sqrt{1-x^2}$	Chebyshev Polynomials (second kind)
$[0,\infty)$	e^{-x}	Laguerre Polynomials
$(-\infty,\infty)$	e^{-x^2}	Hermite Polynomials

2.4.3. Error Estimates

The error of a Gaussian quadrature rule can be stated as follows

$$\int_a^b w(x) f(x)\,dx - \sum_{i=1}^{n} w_i f(x_i) = \frac{f^{(2n)}(\xi)}{(2n)!}(p_n, p_n) \qquad (2.15)$$

for some ξ in (a,b), where p_n is the orthogonal polynomial of degree n and where

$$(f,g) = \int_a^b w(x) f(x) g(x)\,dx. \qquad (2.16)$$

In the important special case of $w(x)=1$, we have the error estimate

$$\frac{(b-a)^{2n+1}(n!)^4}{(2n+1)[(2n)!]^3} f^{(2n)}(\xi), \qquad a<\xi<b. \qquad (2.17)$$

Since, it may be difficult to estimate the order $2n$ derivative, and furthermore the actual error may be much less than a bound established by the derivative. Another approach is to use two Gaussian quadrature rules of different orders, and to estimate the error as the difference between the two results.

2.5. QUADRATURE METHOD FOR INFINITE INTEGRALS

Consider the quadrature, (F.S. Action, 1990)

$$I(b,c) = \int_c^{\infty} \frac{dx}{1+bx^2+x^4}, \qquad c \geq 2. \tag{2.18}$$

This integral actually has a complicated analytic closed form but in the sequel we have as if it did not. Thus, we shall attempt to estimate its value, sometimes directly but sometimes by finding sufficiently close upper and lower bounds to serve as estimates.

2.5.1. Discarding the Small Terms

Since all terms in the denominator are positive, discarding any of them will yield an upper bound. A drastic discard of all but the x^4 term gives

$$I < \int_c^{\infty} \frac{dx}{x^4} = \frac{1}{3c^3} = 0.041667 = I_{11}. \tag{2.19}$$

A less drastic discard that of unity, also yields an integrable quantity. We have

$$I < \int_c^{\infty} \frac{dx}{x^2(x^2+b)} = \frac{1}{b}\left(\frac{1}{c} - \frac{1}{\sqrt{b}} \tan^{-1}\sqrt{\frac{b}{c}}\right) = 0.036352 = I_{12}. \tag{2.20}$$

This result, while a better bound than the previous one, suffers here by being expressed as the difference of two larger quantities. With our values $(1,2)$ for (b,c) we lose one significant figure and hence would probably prefer to replace the arctangent function by its series, thereby effecting the cancellation of the $1/c$ terms analytically.

2.5.2. Freezing the Small Terms

By replacing some of the denominator terms by their values at the lower limit we can often reduce the complexity of the integral sufficiently to permit analytic integration. By writing our integral as

$$\int_c^{\infty} \frac{dx}{x^4\left(1+{b}/{x^2}+{1}/{x^4}\right)} > \frac{1}{1+{b}/{c^2}+{1}/{c^4}}\int_c^{\infty}\frac{dx}{x^4}$$

$$= \frac{1}{1+{b}/{c^2}+{1}/{c^4}}.\frac{1}{3c^3} = 0.031746. \qquad (2.21)$$

We obtain a lower bound that now traps our tail between 0.0317 and 0.0417. We can also difference these analytic bounds to get an expression

$$\frac{b}{3c^5}\frac{1+1/bc^2}{1+b/c^2+1/c^4}. \qquad (2.22)$$

That could be used to determine how large c must be to guarantee any desired degree of closeness of these bounds. But these bounds are rather crude. We can do better. By writing our integral as

$$\int_c^{\infty} \frac{dx}{\left(x^4+1\right)\left(1+\dfrac{bx^2}{1+x^4}\right)} > \frac{1}{1+\dfrac{1+bc^2}{c^4}}\int_c^{\infty}\frac{dx}{x^4+1} = \left(\frac{17}{21}\right)(0.040689),$$

$$= 0.032939. \ (2.23)$$

and freezing the second factor at c we keep the denominator too large, thereby creating the lower bound shown. On the other hand, by simply freezing the bx^2 term at bc^2 we can create an upper bound. We get

$$I < \int_c^{\infty}\frac{dx}{x^4+bc^2+1} = \int_2^{\infty}\frac{dx}{x^4+5} = 0.037051. \qquad (2.24)$$

Both these bounds depend on our ability to integrate $\left(x^4+a^4\right)^{-1}$ analytically.

2.5.3. Expansion in a Series

We observe that the second factor in

$$\int_c^\infty \frac{1}{x^4} \cdot \frac{1}{1+\dfrac{1+bx^2}{x^4}}\, dx \tag{2.25}$$

is of the form $(1+\varepsilon)^{-1}$, which, ε being small, may be expanded by the binomial theorem to give

$$\int_c^\infty \frac{1}{x^4}\left[1 - \frac{1+bx^2}{x^4} + \left(\frac{1+bx^2}{x^4}\right)^2 - \ldots\right] dx$$

$$= \left(\frac{1}{3c^3} - \frac{1}{7c^7} + \frac{1}{11c^{11}} - \ldots\right) - b\left(\frac{1}{5c^5} - \frac{2}{9c^9} + \frac{3}{13c^{13}} - \ldots\right)$$

$$+ b^2\left(\frac{1}{7c^7} - \frac{3}{11c^{11}} + \ldots\right) - b^3\left(\frac{1}{9c^9} - \ldots\right) \tag{2.26}$$

where the coefficients are highly regular and easily discovered. There is no approximation here, so this technique may be used to give our integral to any precision we may have the patience to pursue.

2.5.4. More Sophisticated Bounds:

We can write our integral as

$$\int_c^\infty \frac{dx}{x^2 + b/2 \ ^2 + \ 1 - b^2/4} \tag{2.27}$$

and then simplify it by substituting

$$x = \sqrt{\frac{b}{2}} \tan\theta$$

to give

$$\sqrt{\frac{b}{2}}\int_{\theta_c}^{\pi/2}\frac{\sec^2\theta d\theta}{\frac{b^2}{4}\sec^4\theta+1-\frac{b^2}{4}}=\left(\frac{2}{b}\right)^{3/2}\int_{\theta_c}^{\pi/2}\frac{\cos^2\theta d\theta}{1+a\cos^4\theta}. \qquad (2.28)$$

Since $b=1$, we have

$$a=\frac{4}{b^2}-1=3.0 \quad \text{and} \quad \theta_c=\tan^{-1}\frac{c}{\sqrt{b/2}}=\tan^{-1}c\sqrt{2}.$$

For c larger than 2, θ_c is up near $\frac{1}{2}\pi$, the $\cos^4\theta$ term is small and we may decide how cavalierly we wish to treat it. If we neglect it entirely we get the upper bound

$$I_4<\left(\frac{2}{b}\right)^{3/2}\int_{\theta_c}^{\pi/2}\cos^2\theta d\theta=\left(\frac{2}{b}\right)^{3/2}\frac{1}{2}\left(\tan^{-1}\frac{\sqrt{b/2}}{c}-\frac{c\sqrt{b/2}}{c^2+b/2}\right)=0.36159. \qquad (2.29)$$

This is a little better than I_{12}.

2.5.5. Tail Estimation for Oscillatory Infinite Integrals

If we must evaluate

$$F\ b\ =\int_0^{\infty}\frac{dx}{x^2+b\cos x} \qquad 0<b\le 1, \qquad (2.30)$$

we have two types of difficulties:

1. $F(b)$ has a singularity as b approaches to zero.
2. We would prefer not to integrate numerically out to ∞ in x.

Here we shall look at the second difficulty, estimating the tail of the quadrature, even though it is usually the less serious of the two. Our problem is to evaluate

$$\int_c^{\infty}\frac{dx}{x^2+b\cos x} \qquad (2.31)$$

for b not too small, say unity. If we look at the graph of the integrand we see that it oscillates around the curve x^{-2} and that the oscillations decrease monotonically importance. We see immediately that the integral

$$\int_c^{\infty} \frac{dx}{x^2} = \frac{1}{c} \tag{2.32}$$

will be good approximation provided we begin at one of the points where $\cos c$ is ± 1 at $m\pi$, for then the two successive lobes of ignored area that occur over the next half-period will be of opposite sign and must partly cancel.

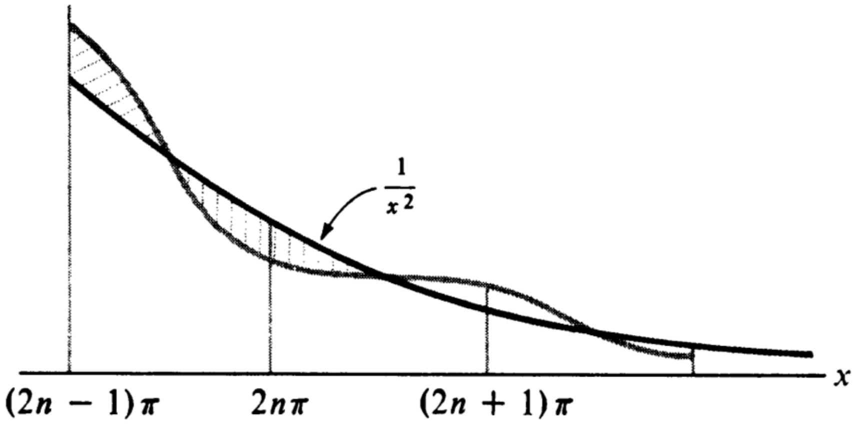

Figure 2.2: A graph showing the difference between x^{-2} and $\left(x^2 + b\cos x\right)^{-1}$.

Further, if we begin at x equal to $2n-1\ \pi$, use of the curve x^{-2} will result in an under estimate, whereas beginning at $2n\pi$ will produce an overestimate.

2.6. EXAMPLES OF GAUSSIAN QUADRATURE

We solved some example about Gaussian quadrature as mention (Bayram, 2009)

Example 2.1: Convert the integral $\int_{0.1}^{0.6} 2v^5\, dv$ to Gauss integral form.

Assume the integrand $g(v) = 2v^5$ and lower and upper limits as $a = 0.1,\ \ b = 0.6$, respectively. So,

$$v = \frac{a+b}{2} + \frac{b-a}{2}x = \frac{0.6+0.1}{2} + \frac{0.5}{2}x = 0.35 + 0.25x$$

and

$$dv = \frac{b-a}{2}dx = \frac{0.6-0.1}{2}dx = 0.25\,dx\,.$$

Then, we get

$$\int_{0.1}^{0.6} 2v^5\, dv = 0.25\int_{-1}^{1}\left(0.35+0.25x\right)^5 dx\,.$$

If we take $n = 1$ in the formula, that is $m = 2n - 1 = 1$, so

$$\int_{-1}^{1} f(x)\, dx \approx f\left(-\frac{\sqrt{3}}{3}\right)w_1 + f\left(\frac{\sqrt{3}}{3}\right)w_2$$

where $w_1 = w_2 = 1$. Finally, the result is

$$\int_{-1}^{1} (0.35 + 0.25x)^5\, dx \approx \frac{1}{4}\left(0.35 + 0.25\left(-\frac{\sqrt{3}}{3}\right)\right)^5 + \frac{1}{4}\left(0.35 + 0.25\left(\frac{\sqrt{3}}{3}\right)\right)^5$$
$$\approx 0.01494403$$

Example 2.2: Solve the integral $\int_{1}^{1.5} e^{-t^2}\, dt$ according to two-point Gauss integral.

Let us convert the integral into Gauss integral form as follows:

$$t=\frac{a+b}{2}+\frac{b-a}{2}x=\frac{1+1.5}{2}+\frac{0.5}{2}x=1.25+0.25x$$

and

$$dt=\frac{b-a}{2}dx=\frac{1.5-1}{2}dx=0.25\,dx\,.$$

Then, we get

$$\int_{1}^{1.5}e^{-t^2}\,dt=0.25\int_{-1}^{1}e^{-(1.25+0.25x)^2}\,dx=\frac{1}{4}\int_{-1}^{1}e^{-(5+x)^2/16}\,dx\,.$$

If we take $n=2$ in the formula, that is $m=2n-1=3$, so

$$\frac{1}{4}\int_{-1}^{1}e^{-(5+x)^2/16}\,dx\approx\frac{1}{4}\left[f(-0.7745966692)\,w_1+f(0)\,w_2+f(0.7745966692)\,w_3\right],$$

$$\frac{1}{4}\int_{-1}^{1}e^{-(5+x)^2/16}\,dx\approx\frac{1}{4}\left(f(-0.7745966692)\,\frac{5}{9}+f(0)\,\frac{8}{9}+f(0.7745966692)\,\frac{5}{9}\right),$$

$$\approx 0.1093641960\,.$$

CHAPTER 3

SPECIAL FUNCTIONS and POLYNOMIALS

3.1. BESSEL FUNCTIONS

Bessel functions are cylindrical functions that are used far and wide in various branches of science, technology and applied mathematics. These functions are involved in heat conduction theory, elasticity theory for spatial problems, oscillations and equilibrium of plates, the theory of shells, and for problems concerning stresses near the cracks. The functions were introduced to the mathematical world by German astronomer Friedrich Wilhelm Bessel in a memoir in the “Transactions of the Berlin Academy” during the year 1824. He derived the differential equation of the Bessel functions and also carried out the first organized study of general properties of its solutions.

3.2. BESSEL DIFFERENTIAL EQUATIONS

The process of separating the variables in problems whose solutions deal with the application of cylindrical and spherical coordinates results in the Bessel differential equation

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + \left(x^2 - v^2\right) y = 0. \tag{3.1}$$

where the value of v, that can be either integer or fractional value, represent the order of Bessel functions. This is a linear second order differential equation. The Bessel differential equation (3.1) has the solution of the form

$$y = x^{\alpha}\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+\alpha}$$

where $a_0 \neq 0$ and the power series $\sum_{n=0}^{\infty} a_n x^n$ converges for all x. In order to find this solution we must determine the equations of the first and second derivative, then we can get

$$(\alpha^2 - v^2)\, a_0 x^{\alpha} + \left[(\alpha+1)^2 - v^2\right] a_1 x^{\alpha+1} + \sum_{n=2}^{\infty} \left\{\left[(\alpha+n)^2 - v^2\right] a_n + a_{n-2}\right\} x^{n+\alpha} = 0.$$

When we solve this equation we can get the series solution of (3.1) as

$$y = a_0 x^{v}\left[1 - \frac{x^2}{2^2 2!\,(v+1)} + \frac{x^4}{2^4 2!\,(v+1)(v+2)} - \cdots\right] = a_0 x^{v}\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n} v!}{2^{2n} n!\,(v+n)!}\ .$$

By setting $a_0 = \dfrac{1}{2^v v!}$, a Bessel function of the first kind and of order v is achieved. It is denoted by $J_v(x)$ for $\forall v \in Z^{+}$, giving

$$J_v(x) = \frac{x^v}{2^v v!}\left[1 - \frac{x^2}{2^2 (v+1)} + \frac{x^4}{2!2^4 (v+1)(v+2)} - \cdots\right] = \sum_{n=0}^{\infty} (-1)^n \frac{(x/2)^{2n+v}}{n!\,(n+v)!}. \qquad (3.2)$$

For all values of the variable x and the index v the series converges absolutely and uniformly. The main properties of the first kind Bessel functions allow them to be defined as a series solution of a second order differential equation. Each term of this series can be individually differentiated and integrated.

The first kind Bessel functions (3.2) are oscillating functions asymptotically vanishing at infinity. The most useful first kind Bessel functions and the ones that are most closely studied here are $J_0(x)$ and $J_1(x)$ with their specific equations given as follows:

$$J_0(x)=\sum_{n=0}^{\infty}(-1)^n\frac{(x/2)^{2n}}{n!n!}\approx 1-\frac{(x/2)^2}{1!1!}+\frac{(x/2)^4}{2!2!}-\frac{(x/2)^6}{3!3!}+\cdots, \qquad (3.3)$$

$$J_1(x)=\sum_{n=0}^{\infty}(-1)^n\frac{(x/2)^{2n+1}}{n!(n+1)!}\approx \frac{x}{2}-\frac{(x/2)^3}{1!2!}+\frac{(x/2)^5}{2!3!}-\frac{(x/2)^7}{3!4!}+\cdots. \qquad (3.4)$$

Some interesting properties of these functions can be seen from their graphs below. Both of these functions

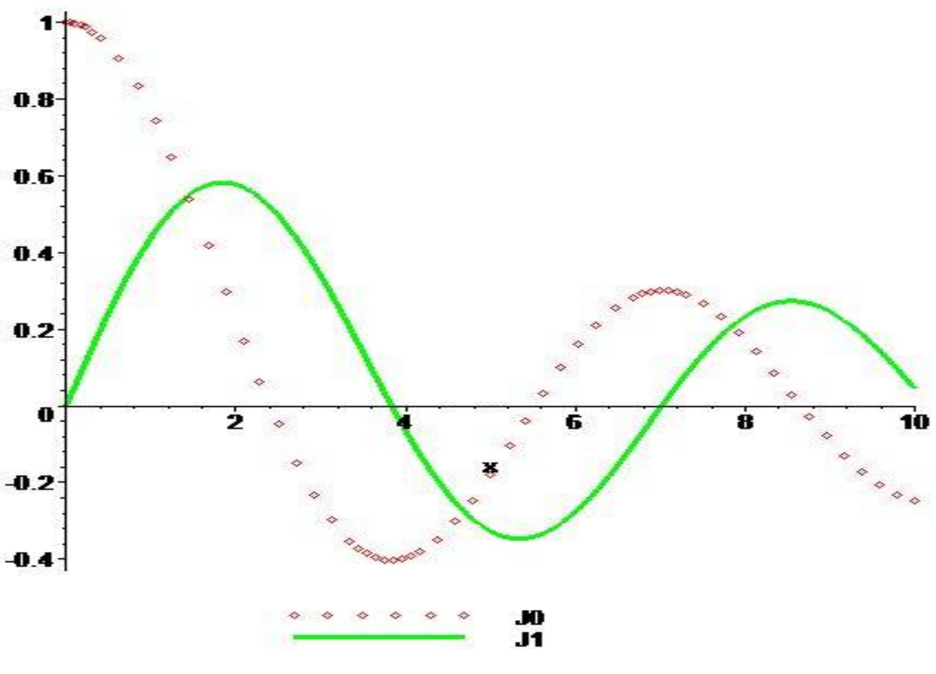

Figure 3.1: Bessel function J_0 and J_1

have an infinite number of positive zeros which are simple zeros because $J'_v(x_v)\neq 0$. It has been proven that for (3.3) a zero exists in all intervals of the length of π. It can be deduced, from the previous statement, that the first kind Bessel functions (3.2) have an infinite number of zeros $\forall n \in Z$ which forms a sequence tending to infinity. Their graphs strongly suggest the functions of sine and cosine, especially with the alternating manner of the occurrence of the zeros.

3.3. INTEGRAL REPRESENTATION

For numerous problems, integrals have variable limits that when taken through a process and given a specific interval can be expressed by cylindrical functions. This is true for Bessel functions whose Bessel's integral formula can be expressed as

$$J_v \; x = \frac{1}{\pi}\int_0^{\pi} \cos \; x\sin\theta - v\theta \; d\theta . \tag{3.5}$$

Now the following formulas can be established

$$J_{2v} \; x = \frac{1}{\pi}\int_0^{\pi} \cos \; x\sin\theta \; \cos \; 2v\theta \; d\theta . \tag{3.6}$$

$$J_{2v+1} \; x = \frac{1}{\pi}\int_0^{\pi} \sin \; x\sin\theta \; \sin \; 2v+1 \; \theta d\theta . \tag{3.7}$$

and by substituting $v = 0$ into (3.6) and (3.7) we can obtain the exact formula's for (3.3) and (3.4) as

$$J_0 \; x = \frac{1}{\pi}\int_0^{\pi} \cos \; x\sin\theta \; d\theta , \tag{3.8}$$

$$J_1 \; x = \frac{1}{\pi}\int_0^{\pi} \sin \; x\sin\theta \; \sin\theta d\theta . \tag{3.9}$$

This proves the above suggestion of sine's and cosine's involvement in the Bessel functions. To demonstrate further the close ties between, not only (3.3) and (3.4), but all first kind Bessel functions and trigonometric functions are the following equations:

$$\cos x = J_0(x) - 2J_2(x) + 2J_4(x) - \cdots$$

and

$$\sin x = 2J_1(x) - 2J_3(x) + 2J_5(x) - \cdots$$

3.3.1. Properties

We list below a few of the more useful properties of Bessel functions of order m, $m = 0,1,2,...$ (Watson, 1922).

$$J_{-m}(x) = (-1)^m J_m(x),$$

$$J_m(-x) = (-1)^m J_m(x),$$

$$J_m(0) = \begin{cases} 0, & m > 0 \\ 1, & m = 0 \end{cases},$$

$$\lim_{x \to 0^+} Y_m(x) = -\infty,$$

$$\frac{d}{dx}\left[x^{-v} J_v(x)\right] = -x^{-v} J_{v+1}(x).$$

It can be shown in a similar manner that

$$\frac{d}{dx}\left[x^{v} J_{v}(x)\right]=x^{v} J_{v-1}(x)$$

$$J_0'(x)=-J_1(x), \qquad Y_0'(x)=-Y_1(x).$$

If the values of variable x were to be imaginary then the result would be functions called modified Bessel functions and a system of solutions from the Bessel equation can be obtained from the following functions:

$$I_v(x)=e^{-\pi vi/2}J_v(ix), \tag{3.10}$$

$$K_v(x)=\frac{\pi i}{2}e^{\pi vi/2}H_v^{(1)}(ix). \tag{3.11}$$

Equation (3.10) arises when the differential equation

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(x^2-v^2)y=0 \tag{3.12}$$

occurs. In order to make this in the form of Bessel's equation you must replace x by $-it$ so that (3.12) becomes

$$t^2\frac{d^2}{dt^2}y+it\frac{d}{dt}y+(t^2+v^2)y=0 \tag{3.13}$$

Normalize (3.13) so that

$$y(x)=I_v(x)=i^{-v}J_v(ix) \tag{3.14}$$

which now can be rewritten as equation (3.10). By differentiation from the series definition of the first order Bessel function (3.2) we obtain

$$\frac{d}{dx}\left[x^{v} J_{v}(x)\right]=x^{v} J_{v-1}(x), \tag{3.15}$$

$$\frac{d}{dx}\left[x^{-v}J_v(x)\right]=-x^{-v}J_{v+1}(x). \tag{3.16}$$

Then by carrying out the differentiation on the left hand side of (3.15) and (3.16) we get

$$\frac{d}{dx}\left[J_v(x)\right]x^v+vx^{v-1}J_v(x)=x^vJ_{v-1}(x), \tag{3.17}$$

$$\frac{d}{dx}\left[J_v(x)\right]x^{-v}-vx^{-v-1}J_v(x)=-x^{-v}J_{v+1}(x). \tag{3.18}$$

Now by dividing both sides of (3.17) by x^v and dividing both sides of (3.18) by x^{-v}, it is deduced that

$$\frac{d}{dx}J_v(x)+\frac{v}{x}J_v(x)=J_{v-1}(x), \tag{3.19}$$

$$\frac{d}{dx}J_v(x)-\frac{v}{x}J_v(x)=-J_{v-1}(x). \tag{3.20}$$

Next, set $v=0$ into and $v=1$ and plug into equations (3.19) and (3.20), respectively, and two special cases are obtained that will be used in further proofs:

$$\frac{d}{dx}J_0(x)=-J_1(x), \tag{3.21}$$

$$\frac{d}{dx}J_1(x)=-\frac{J_1(x)}{x}+J_0(x). \tag{3.22}$$

Furthermore, by adding (3.19) and (3.20)

$$2\frac{d}{dx}J_v(x)-J_{v-1}(x)+J_{v+1}(x)=0 \tag{3.23}$$

obtain the relational equation

$$2\frac{d}{dx}J_v(x) = J_{v-1}(x) - J_{v+1}(x) \tag{3.24}$$

and by subtracting (3.19) from (3.20)

$$2\frac{v}{x}J_v(x) - J_{v-1}(x) - J_{v+1}(x) = 0 \tag{3.25}$$

another relational equation can be obtained

$$2\frac{v}{x}J_v(x) = J_{v-1}(x) + J_{v+1}(x) \tag{3.26}$$

Equations (3.24) and (3.26) are called recurrence relations, which are identities that relate the Bessel function for the Bessel function for different values of v, that satisfy the Bessel functions of the first kind.

3.3.2 Second and Third Kind Functions

Besides Bessel functions of the first kind two other series of solutions exists for the Bessel differential equation. Bessel functions of the second and third kind are also known as Neumann functions, $N_v(x)$, and Henkel functions $H_v^{(1)}(x)$ and $H_v^{(2)}(x)$, respectively. The second kind Neumann function is defined as

$$N_v(x) = \frac{\cos \pi v}{\sin \pi v}J_v(x) - \frac{1}{\sin \pi v}J_{-v}(x), \qquad v \notin Z. \tag{3.27}$$

The graph of the function is pictured below; it diverges at least logarithmically and does not have a finite solution at the origin.

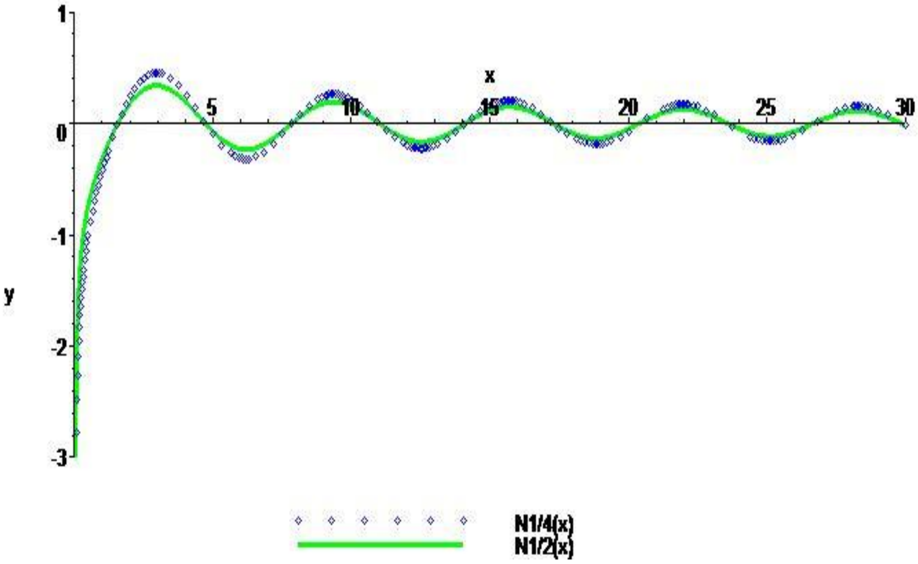

Figure 3.2: Graph of Neumann functions $N_{1/2}(x)$ and $N_{1/4}(x)$.

The third kind Henkel Functions $H_v^{(1)}(x)$ and $H_v^{(2)}(x)$, are defined as

$$H_v^{(1)}(x) = J_v(x) + iN_v(x) \tag{3.28}$$

and

$$H_v^{(2)}(x) = J_v(x) - iN_v(x). \tag{3.29}$$

The series expansions of these functions are

$$H_0^{(1)}(x) \approx i\frac{2}{\pi}\ln x + 1 + i\frac{2}{\pi}(y - \ln 2) + \cdots. \tag{3.30}$$

3.4. CHEBYSHEV POLYNOMIALS

Chebyshev polynomials have been most used in approximation work because a way to obtain a polynomial approximation is by truncating orthogonal polynomial expansions. The Chebyshev polynomials are known to be most historic of the various types of orthogonal polynomials, for the most part because they are related quite simply to trigonometric functions by Eq. (3.31). The type I polynomial and the most useful, is defined as

$$T_n(x) = \cos(n\theta) \quad \text{where} \quad \cos\theta = x, \quad -1 \le x \le 1.$$

(3.31)

From this definition and from common trigonometric functions Eqns. (3.32), (3.33) and (3.34) can be concluded

$$T_0(x) = 1,$$

(3.32)

$$T_1(x) = x,$$

(3.33)

$$T_2(x) = 2x^2 - 1.$$

(3.34)

Now the remaining Chebyshev polynomials may be determined by using the recurrence formula

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x). \tag{3.35}$$

which follows from the trigonometric identity

$$\cos(n+1)\theta + \cos(n-1)\theta = 2\cos\theta\cos(n\theta). \tag{3.36}$$

The general Chebyshev differential equation is written as

$$L(y) = (1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + \lambda^2 y = 0.$$

(3.37)

From (3.37) it can be directly taken that the Chebyshev polynomials are orthogonal over the interval $[-1,1]$ with respect to the weight factor $w(x) = (1-x^2)^{\pm 1/2}$ the orthogonality integral for Chebyshev polynomials of type I is defined as

$$\int_{-1}^{1} \frac{T_m(x)\,T_n(x)}{\sqrt{1-x^2}}\,dx = \begin{cases} 0 & , m \neq n \\ \pi/2 & , m = n \neq 0 \\ \pi & , m = n = 0 \end{cases} . \qquad (3.38)$$

The orthogonal properties of (3.38) can be used to expand an arbitrary function in the Chebyshev series

$$f(x) = \sum_{n=0}^{\infty} a_n T_n(x)$$

where

$$a_0 = \frac{1}{\pi}\int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}}\,dx \qquad \text{and} \qquad a_n = \frac{2}{\pi}\int_{-1}^{1} f(x)\frac{T_n(x)}{\sqrt{1-x^2}}\,dx .$$

Chebyshev series generally has a faster rate of convergence than the others.

CHAPTER 4

AN EFFICIENT ITERATIVE SOLUTION OF OSCILLATORY INTEGRALS CONTAINING THE PRODUCT OF BESSEL FUNCTIONS

4.1. INTRODUCTION

In this study, it is considered a class of oscillating integrals containing the product of Bessel functions of the first kind given as

$$\int_0^\infty D\left(r,s,\rho\right) J_m\left(r\rho\right) J_n\left(s\rho\right) d\rho, \quad m,n=0,1 \quad \text{and} \quad 0<r,s<1. \tag{4.1}$$

The function $D\left(r,s,\rho\right)$ has some singularities such as Cauchy and logarithmic types and also has an oscillating property together with Bessel functions. These types of oscillating integrals occur in modeling of fracture problems of composite materials or modeling of axisymmetric heat conduction in cracked nonhomogeneous medium in axisymmetric coordinate system. Modeling starts with the constitution equations that are the system of partial differential equations with variable coefficients in an axisymmetric coordinate system. Then, using the Hankel integral transform, it can be obtained a system of ordinary differential equations. Under the appropriate boundary conditions the solution of the system can be given as a system of singular integral equations with a Cauchy type singularity. Since there is no closed form solution for the system of integral equations then it should be solved numerically.

4.2. INTEGRATION OF THE PRODUCT OF BESSEL FUNCTIONS I

4.2.1. Iterative Method I.

In this section, the main idea is to represent the integral of the product of two Bessel functions of the first kind as a single integral form. For this purposes, it can be used the representation of the zeroth order Bessel function of the first kind, $J_0(\rho R)$, like

$$J_0(\rho R) = \sum_{n=0}^{\infty} J_n(r\rho) J_n(s\rho) \cos(n\phi) \tag{4.2}$$

where

$$R^2 = r^2 + s^2 - 2rs\cos\phi \tag{4.3}$$

from cosine theorem. Now, it can be obtained the representation of the product of two Bessel functions of the first kind by deriving Equation (4.2) as follows:
Rewriting the Equation (4.2) by separating the first term of the series and then differentiating both sides with respect to r can be obtained as

$$J_0(\rho R) = J_0(r\rho) J_0(s\rho) + 2\sum_{n=1}^{\infty} J_n(r\rho) J_n(s\rho) \cos(n\phi), \tag{4.4}$$

$$-\rho J_1(\rho R)\frac{r - s\cos\phi}{R} = -\rho J_1(r\rho) J_0(s\rho) + 2\sum_{n=1}^{\infty} \frac{d}{dr} J_n(r\rho) J_n(s\rho) \cos(n\phi) \tag{4.5}$$

where the following identities are used:

$$\frac{d}{dr} J_0(\rho R) = \frac{d}{dt} J_0(t) \frac{dt}{dR}\frac{dR}{dr},$$

$$\frac{d}{dr} J_0(\rho R) = -J_1(t)\,\rho\frac{1}{2}\left(r^2 + s^2 - 2rs\cos n\phi\right)^{-1/2}(2r - 2s\cos\phi),$$

$$\frac{d}{dr} J_0(\rho R) = -\rho J_1(\rho R)\frac{r - s\cos\phi}{R},$$

$$\frac{d}{dr}J_0\ r\rho\ =\frac{d}{dt}J_0\ t\ \frac{dt}{dr}=-\rho J_1\ r\rho\ .$$

Then, integrating both sides of (4.5) from 0 to π with respect to ϕ can be resulted as

$$\int_0^{\pi}-\rho J_1\ \rho R\ \frac{r-s\cos\phi}{R}d\phi=\int_0^{\pi}-\rho J_1\ r\rho\ J_0\ s\rho\ d\phi$$

$$+2\sum_{n=1}^{\infty}\frac{d}{dr}J_n\ r\rho\ J_n\ s\rho\ \int_0^{\pi}\cos\ n\phi\ d\phi\,. \quad (4.6)$$

Since the integral on the right-hand side is identically zero,

$$\int_0^{\pi}\cos\ n\phi\ d\phi=0, \qquad n=0,1,2,\ldots,$$

then Equation (4.6) can be reduced to

$$\int_0^{\pi}-\rho J_1\ \rho R\ \frac{r-s\cos\phi}{R}d\phi=-\rho J_1\ r\rho\ J_0\ s\rho\ \int_0^{\pi}d\phi=-\pi\rho J_1\ r\rho\ J_0\ s\rho\ .$$

As a result, product of two Bessel functions of the first kind can be written as

$$J_1\ r\rho\ J_0\ s\rho\ =\frac{1}{\pi}\int_0^{\pi}\left(\frac{r-s\cos\phi}{R}\right)J_1\ \rho R\ d\phi \qquad (4.7)$$

and the integral containing the product of Bessel functions given in (4.1) can be expressed in the form of

$$\int_0^{\infty}D\ \rho\ J_1\ r\rho\ J_0\ s\rho\ d\rho=\int_0^{\infty}D\ \rho\ d\rho\frac{1}{\pi}\int_0^{\pi}\frac{r-s\cos\phi}{R}J_1\ \rho R\ d\phi,$$

$$=\frac{1}{\pi}\int_0^{\pi}\left(\frac{r-s\cos\phi}{R}\right)d\phi\int_0^{\infty}D\ p\ J_1\ \rho R\ d\rho. \quad (4.8)$$

To simplify the expression in (4.8) can be defined the value of R given in (4.3) as follows:

$$R^2=r^2+s^2-2rs\ 1-2\sin^2\theta\ ,$$

$$R^2=r^2+s^2-2rs+4rs\sin^2\theta,$$

$$R^2 = \ r - s\ ^2 + 4rs\sin^2\theta .$$

Finally, the Equation (4.8) may be simplified as

$$\int_0^\infty D\ \rho\ J_0\ s\rho\ J_1\ r\rho\ d\rho = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{r - s + 2s\sin^2\theta}{R} d\theta \int_0^\infty D\ \rho\ J_1\ \rho R\ d\rho \qquad (4.9)$$

where $R^2 = \ r - s\ ^2 + 4rs\sin^2\theta$. In a similar manner, using Equation (4.4), it can be easily obtained that

$$J_0\ r\rho\ J_0\ s\rho\ = \frac{1}{\pi}\int_0^\pi J_0\ \rho R\ d\phi \qquad (4.10)$$

and then integrating both sides it can be obtained another product of two Bessel functions of the first kind as

$$\int_0^\infty D\ \rho\ J_0\ r\rho\ J_0\ s\rho\ d\rho = \frac{1}{\pi}\int_0^\pi \int_0^\infty D\ \rho\ J_0\ \rho R\ d\rho d\phi \qquad (4.11)$$

where $R^2 = r^2 + s^2 - 2rs\cos\phi$. Multiplying the both sides of Equation (4.4) by $\cos\phi$ and integrating from 0 to π with respect to ϕ, the Equation (4.4) can be reduced to

$$\pi J_1\ r\rho\ J_1\ s\rho\ = \int_0^\pi J_0\ \rho R\ \cos\ \phi\ d\phi$$

or

$$J_1\ r\rho\ J_1\ s\rho\ = \frac{2}{\pi}\int_0^{\pi/2} 1 - 2\sin^2\theta\ J_0\ R\rho\ d\theta . \qquad (4.12)$$

where $R^2 = \ r - s\ ^2 + 4rs\sin^2\theta$. Again, as it is done above, integrating from 0 to ∞ with respect to ρ, the last form of Equation (4.1) can be obtained as

$$\int_0^\infty D\ \rho\ J_1\ r\rho\ J_1\ s\rho\ d\rho = \frac{2}{\pi}\int_0^{\pi/2} 1 - 2\sin^2\theta\ d\theta \int_0^\infty D\ \rho\ J_0\ \rho R\ d\rho \qquad (4.13)$$

and defining a new variable like $t/R = \rho$, the Equation (4.13) can be reduced to

$$\int_0^\infty D \ \rho \ J_1 \ r\rho \ J_1 \ s\rho \ d\rho = \frac{2}{\pi}\int_0^{\pi/2}\left(\frac{1-2\sin^2\theta}{R}\right)d\theta\int_0^\infty D \ t/R \ J_0 \ t \ dt. \quad (4.14)$$

Finally, after some analysis, the integral in Equation (4.1) is turned into a double integral but the integrand is reduced to a single Bessel function of the first kind instead of their product. The next step is an evaluation of these single infinite integrals containing the product of Bessel functions of the first kind and a function $D \ \rho$ that is diminishing at infinity and well-defined at zero. For these purposes, in the following sections, integrands of these infinite integrals will be asymptotically examined and it will be obtained an iterative formula for each integral.

4.2.2. Evaluation of Infinite Integrals in the Form of $\int_0^\infty D \ \eta \ J_m \ \eta R \ d\eta$

Let us start with an asymptotic expansion of the integrand $D \ \eta$, that is named as $D_A \ \eta$, and the integral can be expressed as

$$\int_0^\infty D \ \eta \ J_m \ r\eta \ d\eta = \int_0^C D \ \eta \ J_m \ r\eta \ d\eta + \int_C^\infty D_A \ \eta \ J_m \ r\eta \ d\eta$$
$$+\int_C^\infty \left[D \ \eta - D_A \ \eta \right] J_m \ r\eta \ d\eta .$$

(4.15)

If the value of C in (4.15) can be chosen such that the difference $D \ \eta - D_A \ \eta$ is small enough, such as 10^{-30}, then the result of the last integral will be almost zero due to a diminishing function $J_m \ r\eta$ at infinity. Thus, the improper integral can be shown as

$$\int_0^{\infty} D\ \eta\ J_m\ r\eta\ d\eta = \int_0^{C} D\ \eta\ J_m\ r\eta\ d\eta + \int_C^{\infty} D_A\ \eta\ J_m\ r\eta\ d\eta .$$

(4.16)

The first integral in (4.16) can be evaluated using quadrature and for the second integral it will be obtained an iterative method by using integration by parts. Let us start with the derivative of properties the first and the zeroth order Bessel functions of the first kind, (Kronev, 2002), as follows:

$$\frac{d}{d\eta} J_1\ r\eta\ = -r\frac{J_1\ r\eta}{r\eta} + rJ_0\ r\eta\ ,$$

(4.17)

$$\frac{d}{d\eta} J_0\ r\eta\ = -rJ_1\ r\eta$$

(4.18)

and defining such new functions like

$$K_k = \int_C^{\infty} \frac{J_1\ \eta}{\eta^k} d\eta ,$$

(4.19)

$$L_k = \int_C^{\infty} \frac{J_0\ \eta}{\eta^k} d\eta ,$$

(4.20)

it can be obtained an iterative formula for each k using integration by parts. Defining that $u = J_1\ \eta$ and $dv = d\eta / \eta^k$ then the integral given in (4.19) can be expressed as

$$K_k = \int_C^{\infty} \frac{1}{\eta^k} J_1\ \eta\ d\eta = \frac{1}{1-k} \frac{J_1\ \eta}{\eta^{k-1}}\Bigg|_C^{\infty} - \frac{1}{1-k} \int_C^{\infty} \frac{1}{\eta^{k-1}} J_1'\ \eta\ d\eta . \qquad (4.21)$$

Using Equation (4.17) in (4.21), it can be obtained as

$$K_k = \frac{1}{1-k} \frac{J_1\ \eta}{\eta^{k-1}}\Bigg|_C^{\infty} - \frac{1}{1-k} \int_C^{\infty} \left(-r\frac{J_1\ \eta}{\eta^k} + r\frac{J_0\ \eta}{\eta^{k-1}} \right) d\eta . \qquad (4.22)$$

After some simplification, it can be easily obtained an iterative formula as follows:

$$K_k = \frac{1}{1-k}\frac{J_1\ \eta}{\eta^{k-1}}\Bigg|_C^{\infty} + \frac{1}{1-k}\int_C^{\infty}\frac{J_1\ \eta}{\eta^k}d\eta - \frac{1}{1-k}\int_C^{\infty}\frac{J_0\ \eta}{\eta^{k-1}}d\eta,$$

$$K_k = \frac{1}{1-k}\frac{J_1\ \eta}{\eta^{k-1}}\Bigg|_C^{\infty} + \frac{1}{1-k}K_k - \frac{r}{1-k}L_{k-1},$$

$$\left(1-\frac{1}{1-k}\right)K_k = \frac{1}{1-k}\frac{J_1\ \eta}{\eta^{k-1}}\Bigg|_C^{\infty} - \frac{r}{1-k}L_{k-1},$$

$$\frac{-k}{k-1}K_k = \frac{1}{k-1}\frac{J_1\ C}{C^{k-1}} + \frac{r}{k-1}L_{k-1},$$

$$K_k = \frac{1}{k}\frac{J_1\ C}{C^{k-1}} + \frac{r}{k}L_{k-1}$$

(4.23)

where $J_1\ \eta \to 0$, as $\eta \to \infty$. In a similar way, defining that $u = J_0\ \eta$ and $dv = d\eta / \eta^k$ then the integral given in (4.18) can be expressed as

$$L_k = \int_C^{\infty}\frac{1}{\eta^k}J_0\ \eta\ d\eta = \frac{1}{1-k}\frac{J_0\ \eta}{\eta^{k-1}}\Bigg|_C^{\infty} - \frac{1}{1-k}\int_C^{\infty}\frac{J_0'\ \eta}{\eta^{k-1}}d\eta. \qquad (4.24)$$

After some simplification, it can be easily obtained an iterative formula as follows:

$$L_k = \frac{1}{1-k}\frac{J_0\ \eta}{\eta^{k-1}}\Bigg|_C^{\infty} - \frac{1}{1-k}\int_C^{\infty} - \frac{J_1\ \eta}{\eta^{k-1}}d\eta,$$

$$L_k = \frac{1}{1-k}\frac{J_0\ \eta}{\eta^{k-1}}\Bigg|_C^{\infty} - \frac{1}{k-1}\int_C^{\infty} - \frac{J_1\ \eta}{\eta^{k-1}}d\eta,$$

$$L_k = \frac{1}{1-k}\frac{J_0\ \eta}{\eta^{k-1}}\Bigg|_C^{\infty} - \frac{1}{k-1}K_{k-1},$$

$$L_k = \frac{1}{k-1}\frac{J_0(C)}{C^{k-1}} - \frac{r}{k-1}K_{k-1}$$

(4.25)

where $J_0(\eta) \to 0$, as $\eta \to \infty$. Since the values of K_1 and L_1 defined in (4.19) and (4.20), respectively, and their iterative formula for k, $k = 2,3,4,\ldots$, are expressed (4.24) and (4.25), then one can easily obtain the result of the integral for $m=0$ and $m=1$ such that

$$\int_0^{\infty} D(\eta) J_m(r\eta)\, d\eta = \int_0^{C} D(\eta) J_m(r\eta)\, d\eta + \int_C^{\infty} D_A(\eta) J_m(r\eta)\, d\eta \tag{4.26}$$

where the asymptotical expansion of $D(\eta)$, that is $D_A(\eta)$, is given as

$$D_A(\eta) = \frac{a_1}{\eta} + \frac{a_2}{\eta^2} + \frac{a_3}{\eta^3} + \cdots + \frac{a_n}{\eta^n} + O\left(\frac{a_{n+1}}{\eta^{n+1}}\right).$$

(4.27)

4.3. INTEGRATION OF THE PRODUCT OF BESSEL FUNCTIONS II

4.3.1. Iterative Method II.

In this section, it will be started from the identity given by Watson, (1922), as

$$J_\nu(r\rho) J_\nu(s\rho) = \frac{\left(\frac{1}{2}rs\right)^{\nu}}{\Gamma\left(\nu+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)} \int_0^{\pi} \frac{J_\nu(R\rho)}{R^{\nu}} \rho^{\nu} \sin^{2\nu}\phi\, d\phi \tag{4.28}$$

where ν is an order of the Bessel function of the first kind and $\Gamma(n)$ is the gamma function. Using Equation (4.28) for $\nu = 0$ and $\nu = 1$, respectively, it can be written that

$$J_0 \; r\rho \; J_0 \; s\rho \; = \frac{1}{\pi}\int_0^{\pi} J_0 \; R\rho \; d\phi, \quad (4.29)$$

$$J_1 \; r\rho \; J_1 \; sp \; = \frac{rs}{\pi}\int_0^{\pi} \frac{J_1 \; R\rho}{R} \rho \sin^2 \phi d\phi. \quad (4.30)$$

To obtained the multiplication $J_1 \; r\rho \; J_0 \; s\rho$, Equation (4.29) should be differentiated with respect to r as follows:

$$\frac{d}{dr} \; J_0 \; r\rho \; J_0 \; s\rho \; = \frac{d}{dr}\left(\frac{1}{\pi}\int_0^{\pi} J_0 \; r\rho \; d\phi\right),$$

$$-\rho J_1 \; r\rho \; J_0 \; s\rho \; = \frac{1}{\pi}\int_0^{\pi} \frac{d}{dr} \rho J_0 \; R\rho \; d\phi,$$

$$= \frac{1}{\pi}\int_0^{\pi} \frac{d}{dR} J_0 \; R\rho \; \frac{dR}{dr} d\phi,$$

$$= \frac{1}{\pi}\int_0^{\pi} -\rho J_1 \; R\rho \; \frac{r - s\cos\phi}{R} d\phi,$$

$$J_1 \; r\rho \; J_0 \; s\rho \; = \frac{1}{\pi}\int_0^{\pi} J_1 \; R\rho \; \frac{r - s\cos\phi}{R} d\phi$$

(4.31)

where

$$R = \; r^2 + s^2 - 2rs\cos\phi \; ^{1/2},$$

(4.32)

$$\frac{dR}{dr} = \frac{1}{2} \; r^2 + s^2 - 2rs\cos\phi \; ^{-\frac{1}{2}} 2 \; r - s\cos\phi \; .$$

Similarly, it can be also obtained the product $J_0 \; r\rho \; J_1 \; s\rho$ by differentiating Equation (4.29) with respect to s as follows:

$$\frac{d}{ds} \; J_0 \; r\rho \; J_0 \; s\rho \; = \frac{d}{ds}\left(\frac{1}{\pi}\int_0^{\pi} J_0 \; r\rho \; d\phi\right),$$

$$-\rho J_0(r\rho) J_1(s\rho) = \frac{1}{\pi}\int_0^{\pi} \frac{d}{ds} J_0(R\rho)\, d\phi,$$

$$= \frac{1}{\pi}\int_0^{\pi} \frac{d}{dR} J_0(R\rho) \frac{dR}{ds} d\phi,$$

$$= \frac{1}{\pi}\int_0^{\pi} -\rho J_1(R\rho) \frac{s - r\cos\phi}{R} d\phi,$$

$$J_0(r\rho) J_1(s\rho) = \frac{1}{\pi}\int_0^{\pi} J_1(R\rho) \frac{s - r\cos\phi}{R} d\phi$$

(4.33)

where the value of R is given in (4.32).

4.3.2. Evaluation of Integrals in the Form of $\int_0^{\infty} D(\rho) J_m(r\rho) J_n(s\rho)\, d\rho$

Let us start with an asymptotic expansion of the integrand $D(\rho)$, that is named as $D_A(\rho)$, and the integral can be expressed as

$$\int_0^{\infty} D(\rho) J_m(r\rho) J_n(s\rho)\, d\rho = \int_0^{C} D(\rho) J_m(r\rho) J_n(s\rho)\, d\rho$$

$$+ \int_C^{\infty} D_A(\rho) J_m(r\rho) J_n(s\rho)\, d\rho + \int_C^{\infty} \left[D(\rho) - D_A(\rho)\right] J_m(r\rho) J_n(s\rho)\, d\rho. \quad (4.34)$$

If the value of C in (4.34) can be chosen such that the difference $D(\rho) - D_A(\rho)$ is small enough, such as 10^{-30}, then the result of the last integral will be almost zero due to both diminishing functions $J_m(r\rho)$ and $J_n(s\rho)$ at infinity. Thus, the improper integral can be shown as

$$\int_0^\infty D(\rho) J_m(r\rho) J_n(s\rho)\, d\rho = \int_0^C D(\rho) J_m(r\rho) J_n(s\rho)\, d\rho$$
$$+\int_C^\infty D_A(\rho) J_m(r\rho) J_n(s\rho)\, d\rho .$$

(4.35)

Since the function $D(\rho)$ is well-defined at zero then the first integral on the right in (4.35) can be evaluated using quadrature rule. On the other hand, $D_A(\rho)$ is an asymptotic expansion of $D(\rho)$ like

$$D_A(\rho) = \frac{a_1}{\rho} + \frac{a_2}{\rho^2} + \frac{a_3}{\rho^3} + \cdots + \frac{a_n}{\rho^n} + O\left(\frac{a_{n+1}}{\rho^{n+1}}\right),$$

(4.36)

for the second integral, it will be obtained an iterative method by using integration by parts. Let us define the following integrals for $k = 1, 2, 3, \ldots$ as

$$K_k(r,s) = \int_C^\infty \frac{1}{\rho^k} J_1(r\rho) J_1(s\rho)\, d\rho ,$$

(4.37)

$$M_k(r,s) = \int_C^\infty \frac{1}{\rho^k} J_0(r\rho) J_0(s\rho)\, d\rho ,$$

(4.38)

$$L_k(r,s) = \int_C^\infty \frac{1}{\rho^k} J_0(r\rho) J_1(s\rho)\, d\rho \qquad (4.39)$$

where it is obvious that

$$L_k(r,s) \neq L_k(s,r) .$$

(4.40)

Using the method of integration by parts, one can find an iterative method for the solution of $K_k(r,s)$, $M_k(r,s)$, $L_k(r,s)$ and $L_k(s,r)$ for each value of k.

Starting with the integral $K_k(r,s)$ and saying that $u = J_1(r\rho)J_1(s\rho)$ and $dv = {d\rho}/{\rho^k}$, the method of integration by parts gives us

$$K_k(r,s) = \frac{1}{1-k}\frac{J_1(r\rho)J_1(s\rho)}{\rho^{k-1}}\Bigg|_C^{\infty} - \frac{1}{1-k}\int_C^{\infty}\frac{1}{\rho^{k-1}}\frac{d}{d\rho}\left[J_1(r\rho)J_1(s\rho)\right]d\rho. \quad (4.41)$$

Let $r\rho = \theta$ and $s\rho = \psi$, then taking the derivative of $J_1(r\rho)J_1(s\rho)$ with respect to these variables, it can be obtained the following result:

$$\frac{d}{d\rho}\left[J_1(r\rho)J_1(s\rho)\right] = \frac{d}{d\theta}J_1(\theta)\frac{d\theta}{d\rho}J_1(\psi) + J_1(\theta)\frac{d}{d\psi}J_1(\psi)\frac{d\psi}{d\rho},$$

$$= r\,J_1'(\theta)J_1(\psi) + s\,J_1(\theta)J_1'(\psi),$$

$$= r\,J_1(\psi)\left(J_0(\theta) - \frac{1}{\theta}J_1(\theta)\right) + s\,J_1(\theta)\left(J_0(\psi) - \frac{1}{\psi}J_1(\psi)\right),$$

$$= r\,J_0(\theta)J_1(\psi) + s\,J_0(\psi)J_1(\theta) - \frac{r}{\theta}J_1(\theta)J_1(\psi) - \frac{s}{\psi}J_1(\theta)J_1(\psi),$$

$$= r\,J_0(r\rho)J_1(s\rho) + s\,J_0(s\rho)J_1(r\rho) - \frac{2}{\rho}J_1(r\rho)J_1(s\rho), \quad (4.42)$$

Now, substituting (4.42) into (4.41) will be resulted as

$$K_k(r,s) = \frac{1}{1-k}\frac{J_1(r\rho)J_1(s\rho)}{\rho^{k-1}}\Bigg|_C^{\infty} + \frac{2}{1-k}\int_C^{\infty}\frac{1}{\rho^k}J_1(r\rho)J_1(s\rho)d\rho$$

$$-\frac{r}{1-k}\int_C^{\infty}\frac{1}{\rho^{k-1}}J_0(r\rho)J_1(s\rho)\,d\rho - \frac{s}{1-k}\int_C^{\infty}\frac{1}{\rho^{k-1}}J_1(r\rho)J_0(s\rho)\,d\rho.$$

(4.43)

From the definition of $K_k(r,s)$ and $L_k(r,s)$ given in Equations (4.37) and (4.39), respectively, the Equation (4.43) can be simplified as follows:

$$K_k(r,s)=\frac{1}{1-k}\frac{J_1(r\rho)J_1(s\rho)}{\rho^{k-1}}\Bigg|_C^{\infty}+\frac{2}{1-k}K_k(r,s)-\frac{r}{1-k}L_{k-1}(r,s)-\frac{s}{1-k}L_{k-1}(s,r).$$

After substituting the upper and the lower bound values of ρ into the integrand such that $\rho\to\infty$ as $J_1(\rho)\to 0$, then it will be obtained as

$$\left(1-\frac{2}{1-k}\right)K_k(r,s)=\frac{1}{1-k}\frac{J_1(r\rho)J_1(s\rho)}{\rho^{k-1}}\Bigg|_C^{\infty}-\frac{r}{1-k}L_{k-1}(r,s)-\frac{s}{1-k}L_{k-1}(s,r),$$

$$\frac{k+1}{k-1}K_k(r,s)=\frac{1}{k-1}\frac{J_1(rC)J_1(sC)}{C^{k-1}}+\frac{r}{k-1}L_{k-1}(r,s)+\frac{s}{k-1}L_{k-1}(s,r).$$

Finally, for each $k\geq 2$, it can be obtained an iterative formula like

$$K_k(r,s)=\frac{1}{k+1}\frac{J_1(rC)J_1(sC)}{C^{k-1}}+\frac{r}{k+1}L_{k-1}(r,s)+\frac{s}{k+1}L_{k-1}(s,r). \qquad (4.44)$$

Now, using the same method above, it can be found an iterative method for $M_k(r,s)$ and saying that $u=J_0(r\rho)J_0(s\rho)$ and $dv={d\rho}/{\rho^k}$, the method of integration by parts gives us

$$M_k(r,s)=\frac{1}{1-k}\frac{J_0(r\rho)J_0(s\rho)}{\rho^{k-1}}\Bigg|_C^{\infty}-\frac{1}{1-k}\int_C^{\infty}\frac{1}{\rho^{k-1}}\frac{d}{d\rho}\left(J_0(r\rho)J_0(s\rho)\right)d\rho. \qquad (4.45)$$

Again, let $r\rho=\theta$ and $s\rho=\psi$, then taking the derivative of $J_0(r\rho)J_0(s\rho)$ with respect to these variables, it can be obtained the following result:

$$\frac{d}{d\rho}\left(J_0(r\rho)J_0(s\rho)\right)=\frac{d}{d\theta}J_0(\theta)\frac{d\theta}{d\rho}J_0(\psi)+J_0(\theta)\frac{d}{d\psi}J_0(\psi)\frac{d\psi}{d\rho},$$

$$=-rJ_1(\theta)J_0(\psi)-sJ_0(\theta)J_1(\psi),$$

$$= -r J_1(r\rho) J_0(s\rho) - s J_0(r\rho) J_1(s\rho).$$

(4.46)

Substituting the result above into (4.45), it can be written that

$$M_k(r,s) = \frac{1}{1-k} \frac{J_0(r\rho) J_0(r\rho)}{\rho^{k-1}} \Bigg|_C^{\infty}$$

$$+ \frac{r}{1-k} \int_C^{\infty} \frac{1}{\rho^{k-1}} J_1(r\rho) J_0(s\rho)\, d\rho + \frac{s}{1-k} \int_C^{\infty} \frac{1}{\rho^{k-1}} J_0(r\rho) J_1(s\rho)\, d\rho$$

and using the definition of $L_k(r,s)$ given in Equation (4.39), the Equation (4.45) can be simplified as follows:

$$M_k(r,s) = \frac{1}{1-k} \frac{J_0(r\rho) J_0(s\rho)}{\rho^{k-1}} \Bigg|_C^{\infty} + \frac{r}{1-k} L_{k-1}(s,r) + \frac{s}{1-k} L_{k-1}(r,s)$$

and as a limiting case $J_0(\rho) \to 0$ as $\rho \to \infty$, then for each $k \geq 2$, it can be obtained an iterative formula like

$$M_k(r,s) = \frac{1}{k-1} \frac{J_0(rC) J_0(sC)}{C^{k-1}} - \frac{r}{k-1} L_{k-1}(s,r) - \frac{s}{k-1} L_{k-1}(r,s).$$

(4.47)

Finally, the last integral $L_k(r,s)$ should be represented by an iterative formula as it was done above. In this case, it should not be forgotten that $L_k(r,s) \neq L_k(s,r)$ but the relation between them can be easily shown at the end. Let us apply the integration by part one more time to find an iterative formula. Let $r\rho = \theta$ and $s\rho = \psi$, then taking the derivative of $J_0(r\rho) J_1(s\rho)$ with respect to these variables, it can be obtained the following result:

$$L_k(r,s)=\frac{1}{1-k}\frac{J_0(r\rho)J_1(s\rho)}{\rho^{k-1}}\Bigg|_C^{\infty}-\frac{1}{1-k}\int_C^{\infty}\frac{1}{\rho^{k-1}}\frac{d}{d\rho}\left(J_0(r\rho)J_1(s\rho)\right)d\rho \tag{4.48}$$

where the derivative of $J_0(r\rho)J_1(s\rho)$ is

$$\frac{d}{d\rho}\left(J_0(r\rho)J_1(s\rho)\right)=-rJ_1(r\rho)J_1(s\rho)+sJ_0(r\rho)\left(-\frac{1}{s\rho}J_1(s\rho)+J_0(s\rho)\right),$$

$$=-rJ_1(r\rho)J_1(s\rho)+sJ_0(r\rho)J_0(s\rho)-\frac{1}{\rho}J_0(r\rho)J_1(s\rho).$$

Now, using the result of the derivative above and Equations (4.37) and (4.38), the integral $L_k(r,s)$ can be written as follows:

$$L_k(r,s)=\frac{1}{1-k}\frac{J_0(r\rho)J_1(s\rho)}{\rho^{k-1}}\Bigg|_C^{\infty}+\frac{1}{1-k}\int_C^{\infty}\frac{1}{\rho^{k}}J_0(r\rho)J_1(s\rho)\,d\rho$$

$$-\frac{s}{1-k}\int_C^{\infty}\frac{1}{\rho^{k-1}}J_0(r\rho)J_0(s\rho)\,d\rho+\frac{r}{1-k}\int_C^{\infty}\frac{1}{\rho^{k-1}}J_1(r\rho)J_1(s\rho)\,d\rho,$$

$$L_k(r,s)=\frac{1}{1-k}\frac{J_0(r\rho)J_1(s\rho)}{\rho^{k-1}}\Bigg|_C^{\infty}+\frac{1}{1-k}L_k(r,s)-\frac{s}{1-k}M_{k-1}(r,s)+\frac{r}{1-k}K_{k-1}(r,s),$$

$$\left(1-\frac{1}{1-k}\right)L_k(r,s)=\frac{1}{1-k}\frac{J_0(r\rho)J_1(s\rho)}{\rho^{k-1}}\Bigg|_C^{\infty}-\frac{s}{1-k}M_{k-1}(r,s)+\frac{r}{1-k}K_{k-1}(r,s).$$

After simplification the like terms and using the upper and the lower bound of integral, $J_0(\eta)\to 0$ and $J_1(\eta)\to 0$ as $\rho\to\infty$, then an iterative formula $L_k(r,s)$ for each $k\geq 2$ can be expressed as

$$L_k(r,s)=\frac{1}{k}\frac{J_0(rC)J_1(sC)}{C^{k-1}}+\frac{s}{k}M_{k-1}(r,s)-\frac{r}{k}K_{k-1}(r,s). \tag{4.49}$$

Exactly using the same idea, one can find an iterative formula for $L_k(s,r)$ like

$$L_k(s,r) = \frac{1}{k}\frac{J_0(sC)\,J_1(rC)}{C^{k-1}} + \frac{r}{k}M_{k-1}(r,s) - \frac{s}{k}K_{k-1}(r,s) \tag{4.50}$$

where

$$M_k(r,s) = M_k(s,r),$$

(4.51)

$$K_k(r,s) = K_k(s,r).$$

(4.52)

Now, let us evaluate each improper integral $K_1(r,s)$, $M_1(r,s)$, $L_1(r,s)$ and $L_1(s,r)$ using the product of the Bessel functions of the first kind given in Equations from (4.29) to (4.33).

4.3.3. Evaluation of $K_1(r,s) = \int_C^\infty \frac{1}{\rho} J_1(r\rho)\,J_1(s\rho)\,d\rho$

From Equation (4.30) by dividing both sides by ρ, it may be written that

$$\frac{J_1(r\rho)\,J_1(s\rho)}{\rho} = \frac{rs}{\pi}\int_0^\pi \frac{J_1(R\rho)}{R}\sin^2(\phi)\,d\phi$$

where $R^2 = r^2 + s^2 - 2rs\cos\phi$ as same as given in (4.32). Then, $K_1(r,s)$ can be expressed as a double integral of

$$K_1(r,s) = \int_C^\infty \frac{J_1(r\rho)\,J_1(s\rho)}{\rho}\,d\rho = \frac{rs}{\pi}\int_C^\infty \int_0^\pi \frac{J_1(R\rho)}{R}\sin^2(\phi)\,d\phi\,d\rho,$$

$$= \frac{rs}{\pi}\int_0^\pi \frac{\sin^2(\phi)}{R}\,d\phi\left(\int_C^\infty J_1(R\rho)\,d\rho\right),$$

(4.53)

The improper integral in (4.53) can be evaluated analytically defining $R\rho\big|_C^{\infty} = v\big|_{CR=X}^{\infty}$ as follows:

$$\int_C^{\infty} J_1 \;\; R\rho \;\; d\rho = \int_X^{\infty} J_1 \;\; v \;\; \frac{dv}{R} = \frac{1}{R} \;\; -J_0 \;\; v \;\; \Big|_X^{\infty} = \frac{1}{R} J_0 \;\; X \;\; = \frac{1}{R} J_0 \;\; CR \;\; . \qquad (4.54)$$

Then, substituting the result in (4.54) into (4.53) it can be found the value of improper integral $K_1 \;\; r, s$ in terms of a proper single integral like

$$K_1 \;\; r,s \;\; = \int_C^{\infty} \frac{J_1 \;\; r\rho \;\; J_1 \;\; s\rho}{\rho} d\rho = \frac{rs}{\pi} \int_0^{\pi} \frac{\sin^2 \;\; \phi}{R} J_0 \;\; CR \;\; d\phi \, . \qquad (4.55)$$

4.3.4. Evaluation of $M_1 \;\; r,s \;\; = \int_C^{\infty} \frac{1}{\rho} J_0 \;\; r\rho \;\; J_0 \;\; s\rho \;\; d\rho$

From Equation (4.29) by dividing both sides by ρ, it may be written that

$$\frac{J_0 \;\; r\rho \;\; J_0 \;\; s\rho}{\rho} = \frac{1}{\pi\rho} \int_0^{\pi} J_0 \;\; R\rho \;\; d\phi$$

where $R^2 = r^2 + s^2 - 2rs\cos\phi$ as same as given in (4.32). Then, $M_1 \;\; r,s$ can be expressed as a double integral of

$$M_1 \;\; r,s \;\; = \int_C^{\infty} \frac{J_0 \;\; r\rho \;\; J_0 \;\; s\rho}{\rho} d\rho = \frac{1}{\pi} \int_C^{\infty} \frac{1}{\rho} d\rho \int_0^{\pi} J_0 \;\; R\rho \;\; d\phi,$$

$$= \frac{1}{\pi} \int_0^{\pi} d\phi \left(\int_C^{\infty} \frac{J_0 \;\; R\rho}{\rho} d\rho \right).$$

(4.56)

The improper integral in (4.56) can be evaluated analytically defining $R\rho|_C^\infty = v|_{CR=X}^\infty$ as follows:

$$\int_C^\infty \frac{J_0(R\rho)}{\rho} d\rho = \int_X^\infty \frac{J_0(v)}{v} R \frac{dv}{R} = \int_X^\infty \frac{J_0(v)}{v} dv .$$

(4.57)

Now, using the identity, (Watson, 1922),

$$\int_X^\infty \frac{J_0(v)}{v} dv = -\gamma - \log\left(\frac{X}{2}\right) + \int_0^X \frac{1-J_0(u)}{u} du$$

$$= -\gamma - \log\left(\frac{X}{2}\right) - \sum_{k=1}^\infty \frac{(-1)^k}{2k(k!)^2}\left(\frac{x}{2}\right)^{2k}$$

(4.58)

and substituting the identity into (4.56) it can be replaced the improper integral with a single integral with a logarithmic function and an infinite series such as

$$M_1(r,s) = \int_C^\infty \frac{J_0(r\rho)J_0(s\rho)}{\rho} d\rho = \frac{1}{\pi}\int_0^\pi d\phi\left(-\gamma - \log\left(\frac{X}{2}\right) - \sum_{k=1}^\infty \frac{(-1)^k}{2k(k!)^2}\left(\frac{X}{2}\right)^{2k}\right),$$

$$M_1(r,s) = \int_C^\infty \frac{J_0(r\rho)J_0(s\rho)}{\rho} d\rho,$$

$$= -\gamma - \frac{1}{\pi}\int_0^\pi \left(\log\left(\frac{CR}{2}\right) + \sum_{k=1}^\infty \frac{(-1)^k}{2k(k!)^2}\left(\frac{CR}{2}\right)^{2k}\right) d\phi . \qquad (4.59)$$

When it is needed to evaluate these integrals numerically, the number of the terms in infinite series should be halted after specific numbers of the terms, like N that is heavily depended on the parameter C.

4.3.5. Evaluation of $L_1 \; r,s \; = \int_C^\infty \frac{1}{\rho} J_0 \; r\rho \; J_1 \; s\rho \; d\rho$

From Equation (4.33) by dividing both sides by ρ, it may be written that

$$\frac{J_0 \; r\rho \; J_1 \; s\rho}{\rho} = \frac{1}{\pi\rho}\int_0^\pi J_1 \; R\rho \; \frac{s - r\cos\phi}{R} d\phi$$

where $R^2 = r^2 + s^2 - 2rs\cos\phi$ as same as given in (4.32). Then, $L_1 \; r,s$ can be expressed as a double integral of

$$L_1 \; r,s \; = \int_C^\infty \frac{J_0 \; r\rho \; J_1 \; s\rho}{\rho} d\rho = \frac{1}{\pi}\int_A^\infty \int_0^\pi \frac{s - r\cos\phi}{R} \frac{J_1 \; R\rho}{\rho} d\phi \, d\rho,$$

$$= \frac{1}{\pi}\int_0^\pi \frac{s - r\cos\phi}{R} d\phi \int_C^\infty \frac{J_1 \; R\rho}{\rho} d\rho.$$

(4.60)

Using a change of variable $R\rho\big|_C^\infty = v\big|_{CR=X}^\infty$, the improper integral in (4.60) can be written as

$$\int_C^\infty \frac{J_1 \; R\rho}{\rho} d\rho = \int_X^\infty \frac{J_1 \; v}{v} dv$$

and from the identity given in (4.17), one can write that

$$\int_X^\infty \frac{J_1 \; v}{v} dv = \int_X^\infty \left(J_0 \; v \; - \frac{d}{dv} J_1 \; v \; \right) dv = \int_X^\infty J_0 \; v \; dv - J_1 \; v \; \big|_X^\infty,$$

$$= J_1 \; X \; + \int_X^\infty J_0 \; v \; dv. \qquad (4.61)$$

Finally, from the identity related to the evaluation of the zeroth order Bessel function of the first kind from zero to infinity, that is

$$\int_0^{\infty} J_0(v)\, dv = 1,$$

it can be written as

$$\int_X^{\infty} J_0(v)\, dv = 1 - \int_0^{X} J_0(v)\, dv. \tag{4.62}$$

Using the result given in (4.62), Equation (4.61) can be expressed as

$$\int_X^{\infty} \frac{J_1(v)}{v} dv = J_1(X) + 1 - \int_0^{X} J_0(v)\, dv,$$

and substituting the result into (4.60), $L_1(r,s)$ can be expressed as

$$\begin{aligned} L_1(r,s) &= \int_C^{\infty} \frac{J_0(r\rho) J_1(s\rho)}{\rho} d\rho \\ &= \frac{1}{\pi}\int_0^{\pi} \frac{s - r\cos\phi}{R}\left(1 + J_1(CR) - \int_0^{CR} J_0(v)\, dv\right) d\phi. \end{aligned} \tag{4.63}$$

Similarly, replacing r with s in (4.63) , it can be easily shown that

$$\begin{aligned} L_1(s,r) &= \int_C^{\infty} \frac{J_0(s\rho) J_1(r\rho)}{\rho} d\rho, \\ &= \frac{1}{\pi}\int_0^{\pi} \frac{r - s\cos\phi}{R}\left(1 + J_1(CR) - \int_0^{CR} J_0(v)\, dv\right) d\phi. \end{aligned} \tag{4.64}$$

CHAPTER 5

AN EFFICIENT ITERATIVE METHOD FOR INTEGRALS INVOLVING PRODUCT OF BESSEL AND EXPONENTIAL FUNCTIONS

In this section, it will be examined an improper integral involving the product of two Bessel functions of the first and the zeroth order and an exponential function such as in the form of

$$\int_0^{\infty} D\ \rho\ J_m\ r\rho\ J_n\ s\rho\ e^{-\alpha\rho} d\rho. \qquad m,n=0,1 \quad \text{and} \quad 0<r,s<1.$$

(5.1)

5.1. INTEGRATION OF THE PRODUCT OF BESSEL AND EXPONENTIAL FUNCTION I

5.1.1. Iterative Method I.

In this section, the main idea is to represent the integral of the product of two Bessel functions of the first kind as a single integral form. For this purposes, it can be used the representation of the zeroth order Bessel function of the first kind, $J_0\ \rho R$, like

$$J_0\ \rho R\ = \sum_{n=0}^{\infty} \in_n J_n\ r\rho\ J_n\ s\rho\ \cos n\phi$$

(5.2)

where

$$R^2 = r^2 + s^2 - 2rs\cos\phi$$

(5.3)

from cosine theorem. Now, it can be obtained the representation of the product of two Bessel functions of the first kind by deriving Equation (5.2) as follows:

Rewriting the Equation (5.2) by separating the first term of the series and then differentiating both sides with respect to r can be obtained as

$$J_0 \ \rho R \ = J_0 \ r\rho \ J_0 \ s\rho \ + 2\sum_{n=1}^{\infty} J_n \ r\rho \ J_n \ s\rho \ \cos \ n\phi \ , \tag{5.4}$$

$$-\rho J_1 \ \rho R \ \frac{r - s\cos\phi}{R} = -\rho J_1 \ r\rho \ J_0 \ \rho R \ + 2\sum_{n=1}^{\infty} \frac{d}{dr} J_n \ r\rho \ J_n \ s\rho \ \cos \ n\phi \tag{5.5}$$

where the following identities are used:

$$\frac{d}{dr} J_0 \ \rho R \ = -\rho J_1 \ \rho R \ \frac{r - s\cos\phi}{R},$$

$$\frac{d}{dr} J_0 \ r\rho \ = \frac{d}{dt} J_0 \ t \ \frac{dt}{dr} = -\rho J_1 \ r\rho \ .$$

Then, integrating both sides of (5.5) from 0 to π with respect to ϕ can be resulted as

$$\int_0^{\pi} -\rho J_1 \ \rho R \ \frac{r - s\cos\phi}{R} d\phi = \int_0^{\pi} -\rho J_1 \ r\rho \ J_0 \ s\rho \ d\phi$$

$$+ 2\sum_{n=1}^{\infty} \frac{d}{dr} J_n \ r\rho \ J_n \ s\rho \ \int_0^{\pi} \cos n\phi d\phi .$$

(5.6)

Since the integral on the right-hand side is identically zero,

$$\int_0^{\pi} \cos \ n\phi \ d\phi = 0, \qquad n = 1, 2, \ldots$$

then Equation (5.6) can be reduced to

$$\int_0^{\pi} -\rho J_1 \ \rho R \ \frac{r - s\cos\phi}{R} d\phi = -\rho J_1 \ r\rho \ J_0 \ s\rho \ \int_0^{\pi} d\phi = -\pi \rho J_1 \ r\rho \ J_0 \ s\rho \ .$$

As a result, product of two Bessel functions of the first kind can be written as

$$J_1(r\rho)J_0(s\rho)=\frac{1}{\pi}\int_0^{\pi}\left(\frac{r-s\cos\phi}{R}\right)J_1(\rho R)\,d\phi \tag{5.7}$$

and the integral containing the product of Bessel functions and exponential functions given in (5.1) can be expressed in the form of

$$\int_0^{\infty}D(\rho)J_1(r\rho)J_0(s\rho)e^{-2h\rho}d\rho=\int_0^{\infty}D(\rho)e^{-2h\rho}d\rho\frac{1}{\pi}\int_0^{\pi}\frac{r-s\cos\phi}{R}J_1(\rho R)\,d\phi$$

$$=\frac{1}{\pi}\int_0^{\pi}\left(\frac{r-s\cos\phi}{R}\right)d\phi\int_0^{\infty}D(p)J_1(\rho R)e^{-2h\rho}d\rho\,. \tag{5.8}$$

To simplify the expression in (5.8), the value of R, given in (5.3), can be defined as follows:

$$R^2=r^2+s^2-2rs(1-2\sin^2\theta),$$

$$R^2=r^2+s^2-2rs+4rs\sin^2\theta,$$

$$R^2=(r-s)^2+4rs\sin^2\theta$$

where $\phi=2\theta \Rightarrow d\phi=2d\theta$. Finally, the Equation (5.8) may be simplified as

$$\int_0^{\infty}D(\rho)J_0(s\rho)J_1(r\rho)e^{-2h\rho}d\rho$$

$$=\frac{2}{\pi}\int_0^{\pi/2}\frac{r-s+2s\sin^2\theta}{R}d\theta\int_0^{\infty}D(\rho)J_1(\rho R)e^{-2h\rho}d\rho\,. \tag{5.9}$$

In the evaluation of the integrals, it will be used the following identities such as

$$\int_0^{\infty}\rho^{v}J_0(\rho R)e^{-\alpha\rho}d\rho=\frac{(2R)^{v}\,\Gamma\left(v+\frac{1}{2}\right)}{\sqrt{\pi}\,(\alpha^2+R^2)^{v+1/2}},\qquad \mathrm{Re}\,v>-\frac{1}{2}, \tag{5.10}$$

$$\int_0^{\infty}\rho^{v+1}J_v(\rho R)e^{-\alpha\rho}d\rho=\frac{2\alpha(2R)^{v}\,\Gamma\left(v+\frac{3}{2}\right)}{\sqrt{\pi}\,(\alpha^2+R^2)^{v+3/2}},\qquad \mathrm{Re}\,v>-1, \tag{5.11}$$

$$\int_0^{\infty} J_v \;\; \rho R \;\; e^{-\alpha\rho}\,\frac{d\rho}{\rho}=\frac{\sqrt{\alpha^2+R^2}-\alpha \;\;^{v}}{vR^{v}}, \qquad\qquad \operatorname{Re} v>0. \qquad\qquad (5.12)$$

Using the identity in (5.10) for the value of $\nu=1$,

$$\int_0^{\infty}\rho J_1 \;\; \rho R \;\; e^{-\alpha\rho}d\rho=\frac{R}{\alpha^2+R^2 \;\; ^{3\!/\!2}}$$

where $\Gamma\left(1+\dfrac{1}{2}\right)=\dfrac{1}{2}\Gamma\left(\dfrac{1}{2}\right)=\dfrac{1}{2}\sqrt{\pi}$ then, it can be rewritten the Equation (5.9) for $D \;\; \rho \;\; =\rho$ as

$$\int_0^{\infty}\rho J_0 \;\; s\rho \;\; J_1 \;\; r\rho \;\; e^{-2h\rho}d\rho=\frac{2}{\pi}\int_0^{\pi\!/\!2}\frac{r-s+2s\sin^2\theta}{4h^2+R^2 \;\; ^{3\!/\!2}}d\theta, \qquad h\neq 0.$$

(5.13)

Similarly replacing s by r in the functions $J_0\left(r\rho\right)$ and $J_1\left(s\rho\right)$, getting the equation

$$\int_0^{\infty}\rho J_0 \;\; r\rho \;\; J_1 \;\; s\rho \;\; e^{-2h\rho}d\rho=\frac{2}{\pi}\int_0^{\pi\!/\!2}\frac{s-r+2s\sin^2\theta}{4h^2+R^2 \;\; ^{3\!/\!2}}d\theta, \qquad h\neq 0. \qquad (5.14)$$

From Equation (5.4) can be written as

$$\int_0^{\pi} J_0 \;\; \rho R \;\; d\phi=\pi J_0 \;\; r\rho \;\; J_0 \;\; s\rho \;\; \Rightarrow \;\; J_0 \;\; r\rho \;\; J_0 \;\; s\rho \;\; =\frac{1}{\pi}\int_0^{\pi} J_0 \;\; \rho R \;\; d\phi,$$

and then, getting the integration of (5.1) for the value of $\nu=0$ and $\mu=0$

$$\int_0^{\infty} D \;\; \rho \;\; J_0 \;\; r\rho \;\; J_0 \;\; s\rho \;\; e^{-2h\rho}d\rho=\frac{1}{\pi}\int_0^{\pi}d\phi\int_0^{\infty} D \;\; \rho \;\; J_0 \;\; \rho R \;\; e^{-2h\rho}d\rho,$$

so, using the identity of (5.11) for the value of $\nu=0$ and $D \;\; \rho \;\; =\rho$

$$\int_0^\infty \rho J_0(\rho R)\, e^{-2h\rho} d\rho = \frac{4h\Gamma\left(\frac{3}{2}\right)}{\sqrt{\pi}\,(4h^2+R^2)^{3/2}} = \frac{2h}{(4h^2+R^2)^{3/2}}.$$

Then,

$$\int_0^\infty \rho J_0(r\rho)\, J_0(s\rho)\, e^{-2h\rho} d\rho = \frac{1}{\pi}\int_0^{\pi/2} \frac{4h}{(4h^2+R^2)^{3/2}} d\theta \qquad (5.15)$$

providing that $\Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2}$. Again, from equation (5.4), by multiplying both sides by $\cos\phi$, it can be written as

$$\int_0^\pi J_0(\rho R) \cos(\phi)\, d\phi = \int_0^\pi J_0(r\rho)\, J_0(s\rho) \cos(\phi)\, d\phi + 2\int_0^\pi J_1(r\rho)\, J_1(s\rho) \cos^2(\phi)\, d\phi.$$

Since all the integrals are zero, except the second one because of cosine term, then

$$\int_0^\pi J_0(\rho R) \cos(\phi)\, d\phi = 2\int_0^\pi J_1(r\rho)\, J_1(s\rho) \cos^2(\phi)\, d\phi = \pi J_1(r\rho)\, J_1(s\rho). \qquad (5.16)$$

Finally, it will be obtained

$$J_1(r\rho)\, J_1(s\rho) = \frac{1}{\pi}\int_0^\pi J_0(\rho R) \cos\phi\, d\phi = \frac{2}{\pi}\int_0^{\pi/2} (1-2\sin^2\theta)\, J_0(R\rho)\, d\theta$$

where $R^2 = (r-s)^2 + 4rs\sin^2\theta$ and $\phi = 2\theta$. Using the Equation (5.4) and integrating both sides from 0 to ∞

$$\int_0^\infty D(\rho)\, J_1(r\rho)\, J_1(s\rho)\, e^{-2h\rho} d\rho$$

$$= \frac{2}{\pi}\int_0^{\pi/2} (1-2\sin^2\theta)\, d\theta \int_0^\infty D(\rho)\, J_0(\rho R)\, e^{-2h\rho} d\rho. \qquad (5.17)$$

Applying the identity (5.11) one more time for the value of $\nu = 0$ and $D(\rho) = \rho$, it will be obtained that

$$\int_0^\infty \rho J_1(r\rho) J_1(s\rho) e^{-2h\rho} d\rho = \frac{2}{\pi}\int_0^{\pi/2} \frac{1-2\sin^2\theta}{R} d\theta \int_0^\infty \rho J_0(\rho R) e^{-2h\rho} d\rho$$

(5.18)

or

$$\int_0^\infty \rho J_1(r\rho) J_1(s\rho) e^{-2h\rho} d\rho = \frac{1}{\pi}\int_0^{\pi/2} \frac{4h(1-2\sin^2\theta)}{(4h^2+R^2)^{3/2}} d\theta. \quad (5.19)$$

5.1.2. Evaluation of an Infinite Integral in the Form of $\int_0^\infty D(\eta) J_m(\eta R) e^{-2h\eta} d\eta$.

Let us start with an asymptotic expansion of the integrand $D(\eta)$, that is named as $D_A(\eta)$, and the integral can be expressed as

$$\int_0^\infty D(\eta) J_m(r\eta) e^{-2h\eta} d\eta = \int_0^C D(\eta) J_m(r\eta) e^{-2h\eta} d\eta + \int_C^\infty D_A(\eta) J_m(r\eta) e^{-2h\eta} d\eta$$

$$+\int_C^\infty [D(\eta) - D_A(\eta)] J_m(r\eta) e^{-2h\eta} d\eta,$$

(5.20)

If the value of C in (5.20) can be chosen such that the difference $D(\eta) - D_A(\eta)$ is small enough, such as 10^{-30}, then the result of the last integral will be almost zero due to a diminishing function $J_m(r\eta)$ at infinity. Thus, the improper integral can be shown as

$$\int_0^\infty D(\eta) J_m(r\eta) e^{-2h\eta} d\eta = \int_0^C D(\eta) J_m(r\eta) e^{-2h\eta} d\eta$$

$$+\int_C^\infty D_A(\eta) J_m(r\eta) e^{-2h\eta} d\eta .$$

(5.21)

The first integral in (4.16) can be evaluated using quadrature and for the second integral it will be obtained an iterative method by using integration by parts. Let us start with the derivative of properties the first and the zeroth order Bessel functions of the first kind, (Kronev, 2002), as follows:

$$\frac{d}{d\eta} J_1(\eta) = -\frac{J_1(\eta)}{\eta} + J_0(\eta) ,$$

(5.22)

$$\frac{d}{d\eta} J_0(\eta) = -J_1(\eta) ,$$

(5.23)

$$\frac{d}{d\eta}\left(e^{-\rho\eta} J_1(\eta)\right) = -\rho e^{-\rho\eta} J_1(\eta) - \frac{e^{-\rho\eta}}{\eta} J_1(\eta) + e^{-\rho\eta} J_0(\eta) , \quad (5.24)$$

$$\frac{d}{d\eta}\left(e^{-\rho\eta} J_0(\eta)\right) = -\rho e^{-\rho\eta} J_0(\eta) - e^{-\rho\eta} J_1(\eta) . \quad (5.25)$$

and defining such new functions like

$$K_k(r) = \int_C^\infty \frac{e^{-h\eta}}{\eta^k} J_1(\eta) d\eta,$$

(5.26)

$$L_k(r) = \int_C^\infty \frac{e^{-h\eta}}{\eta^k} J_0(\eta) d\eta.$$

(5.27)

It can be obtained an iterative formula for each k using integration by parts. Defining that $u = J_1(\eta)$ and $dv = d\eta / \eta^k$ then the integral given in (4.19) can be expressed as

$$K_k(r) = \int_C^\infty \frac{e^{-h\eta}}{\eta^k} J_1(\eta)\, d\eta = \frac{1}{1-k} \frac{e^{-h\eta} J_1(\eta)}{\eta^{k-1}}\Bigg|_C^\infty - \frac{1}{1-k}\int_A^\infty \frac{1}{\eta^{k-1}} \left(e^{-h\eta} J_1(\eta)\right)' d\eta. \quad (5.28)$$

Using Equation (4.17) in (5.28), it can be obtained as

$$K_k(r) = \frac{1}{1-k} \frac{e^{-h\eta} J_1(\eta)}{\eta^{k-1}}\Bigg|_C^\infty$$

$$- \frac{1}{1-k}\int_C^\infty \frac{e^{-h\eta}}{\eta^{k-1}} \left(-he^{-h\eta} J_1(\eta) - \frac{e^{-h\eta}}{\eta} J_1(\eta) + e^{-h\eta} J_0(\eta) \right) d\eta. \qquad (5.29)$$

After some simplification, it can be easily obtained an iterative formula as follows:

$$K_k = \frac{1}{1-k} \frac{e^{-h\eta} J_1(\eta)}{\eta^{k-1}}\Bigg|_C^\infty + \frac{h}{1-k}\int_C^\infty \frac{e^{-h\eta}}{\eta^{k-1}} J_1(\eta)\, d\eta + \frac{1}{1-k}\int_C^\infty \frac{e^{-h\eta}}{\eta^{k-1}} \left(\frac{J_1(\eta)}{\eta} - J_0(\eta) \right) d\eta,$$

$$K_k = \frac{1}{1-k} \frac{e^{-h\eta} J_1(\eta)}{\eta^{k-1}}\Bigg|_C^\infty + \frac{h}{1-k} K_{k-1} + \frac{1}{1-k} K_k - \frac{1}{1-k} L_{k-1},$$

$$\left(1 - \frac{1}{k-1}\right) K_k = \frac{1}{1-k} \frac{e^{-h\eta} J_1(\eta)}{\eta^{k-1}}\Bigg|_C^\infty + \frac{h}{1-k} K_{k-1} - \frac{r}{1-k} L_{k-1},$$

$$K_k = -\frac{1}{k} \frac{e^{-h\eta} J_1(\eta)}{\eta^{k-1}}\Bigg|_C^\infty - \frac{h}{k} K_{k-1} + \frac{1}{k} L_{k-1},$$

$$K_k = \frac{1}{k} \frac{e^{-hC} J_1(C)}{C^{k-1}} - \frac{h}{k} K_{k-1} + \frac{1}{k} L_{k-1} \qquad (5.30)$$

where $J_1(\eta) \to 0$, as $\eta \to \infty$. In a similar way, defining that $u = J_0(\eta)$ and $dv = d\eta / \eta^k$ then the integral given in (4.20) can be expressed as

$$L_k = \int_C^\infty \frac{e^{-h\eta}}{\eta^k} J_0(\eta)\, d\eta = \frac{1}{1-k} \frac{e^{-h\eta} J_0(\eta)}{\eta^{k-1}}\Bigg|_C^\infty - \frac{1}{1-k}\int_C^\infty \frac{1}{\eta^{k-1}} \left(e^{-h\eta} J_0(r\eta)\right)' d\eta. \quad (5.31)$$

Using Equation(4.18) in (5.31), it can be obtained as

$$L_k = \frac{1}{1-k}\frac{e^{-h\eta}J_0(\eta)}{\eta^{k-1}}\Bigg|_C^{\infty} - \frac{1}{1-k}\int_C^{\infty}\frac{1}{\eta^{k-1}}\left(-he^{-h\eta}J_0(\eta) - e^{-h\eta}J_1(\eta)\right)d\eta . \qquad (5.32)$$

After some simplification, it can be easily obtained an iterative formula as follows:

$$L_k = \frac{1}{1-k}\frac{e^{-h\eta}J_0(\eta)}{\eta^{k-1}}\Bigg|_C^{\infty} + \frac{h}{1-k}\int_C^{\infty}\frac{e^{-h\eta}}{\eta^{k-1}}J_0(\eta)\,d\eta + \frac{1}{1-k}\int_C^{\infty}\frac{e^{-h\eta}}{\eta^{k-1}}J_1(\eta)\,d\eta ,$$

$$L_k = \frac{1}{1-k}\frac{e^{-h\eta}J_0(\eta)}{\eta^{k-1}}\Bigg|_C^{\infty} + \frac{h}{1-k}L_{k-1} + \frac{1}{1-k}K_{k-1},$$

$$L_k = -\frac{1}{1-k}\frac{e^{-hC}J_0(C)}{C^{k-1}} + \frac{h}{1-k}L_{k-1} + \frac{r}{1-k}K_{k-1} \qquad (5.33)$$

where $J_0(\eta) \to 0$, as $\eta \to \infty$. Since the values of K_1 and L_1 defined in (4.19) and (4.20), respectively, and their iterative formula for k, $k = 2,3,4,\ldots$, are expressed (5.30) and (5.33), then one can easily obtain the result of the integral for $m = 0$ and $m = 1$ such that

$$\int_0^{\infty} D(\eta) J_m(r\eta) e^{-2h\eta} d\eta = \int_0^{C} D(\eta) J_m(r\eta) e^{-2h\eta} d\eta$$

$$+\int_C^{\infty} D_A(\eta) J_m(r\eta) e^{-2h\eta} d\eta$$

(5.34)

where the asymptotical expansion of $D(\eta)$, that is $D_A(\eta)$, is given as

$$D_A(\eta) = \frac{a_1}{\eta} + \frac{a_2}{\eta^2} + \frac{a_3}{\eta^3} + \cdots + \frac{a_n}{\eta^n} + O\left(\frac{a_{n+1}}{\eta^{n+1}}\right).$$

(5.35)

5.1.3. Evaluation of $K_1 \; r \; = \int_C^{\infty} \frac{e^{-h\rho}}{\rho} J_1 \; r\rho \; d\rho$

It can be written that

$$\int_C^{\infty} \frac{e^{-h\rho}}{\rho} J_1 \; r\rho \; d\rho = \int_0^{\infty} \frac{e^{-h\rho}}{\rho} J_1 \; r\rho \; d\rho - \int_0^{C} \frac{e^{-h\rho}}{\rho} J_1 \; r\rho \; d\rho .$$

Also, from the closed form solution of

$$\int_C^{\infty} \frac{e^{-h\rho}}{\rho} J_1 \; r\rho \; d\rho = \frac{\sqrt{h^2 + r^2} - h}{r},$$

it can be simplified that

$$\int_C^{\infty} \frac{e^{-h\rho}}{\rho} J_1 \; r\rho \; d\rho = \frac{\sqrt{h^2 + r^2} - h}{r} - \int_0^{C} \frac{e^{-h\rho}}{\rho} J_1 \; r\rho \; d\rho ,$$

$$K_1 = \int_C^{\infty} \frac{e^{-h\rho}}{\rho} J_1 \; r\rho \; d\rho = \frac{\sqrt{h^2 + r^2} - h}{r} - \int_0^{C} \frac{e^{-h\rho}}{\rho} J_1 \; r\rho \; d\rho . \qquad (5.36)$$

5.1.4. Evaluation of $L_1 = \int_C^{\infty} \frac{e^{-h\rho}}{\rho} J_0 \; r\rho \; d\rho$

Using the following identity (Watson, 1922)

$$\int_0^{\infty} \; 1 - J_0 \; r\rho \; \frac{e^{-h\rho}}{\rho} d\rho = -\log\left(\frac{2h}{h + \sqrt{h^2 + r^2}} \right) \qquad (5.37)$$

and separating the integral (5.37) in two parts as

$$\int_0^{\infty} \; 1 - J_0 \; r\rho \; \frac{e^{-h\rho}}{\rho} d\rho = \int_0^{C} \; 1 - J_0 \; r\rho \; \frac{e^{-h\rho}}{\rho} d\rho + \int_C^{\infty} \; 1 - J_0 \; r\rho \; \frac{e^{-h\rho}}{\rho} d\rho , \quad (5.38)$$

the integral (5.38) can be written as follows:

$$\int_C^\infty \frac{J_0(r\rho)}{\rho} e^{-h\rho} d\rho = \int_C^\infty \frac{e^{-h\rho}}{\rho} d\rho$$

$$-\int_0^\infty \left(1 - J_0(r\rho)\right) \frac{e^{-h\rho}}{\rho} d\rho + \int_0^C \left(1 - J_0(r\rho)\right) \frac{e^{-h\rho}}{\rho} d\rho. \qquad (5.39)$$

Finally, the result can be given as

$$L_1 = \int_C^\infty \frac{e^{-h\rho}}{\rho} J_0(r\rho)\, d\rho = \mathrm{Ei}(1, hC)$$

$$+\log\left(\frac{2h}{h+\sqrt{h^2+r^2}}\right) + \int_0^C \left(1 - J_0(r\rho)\right) \frac{e^{-h\rho}}{\rho} d\rho. \qquad (5.40)$$

can be written an alternate method for evaluating L_1. Representing Bessel function $J_0(r\rho)$ as

$$J_0(r\rho) = \frac{1}{\pi}\int_0^\pi e^{ir\rho\cos\theta} d\theta$$

then the integral $L_1(r)$ may be expressed in the following form

$$L_1 = \int_C^\infty e^{-h\rho}\left(\frac{1}{\pi}\int_0^\infty e^{-ir\rho\cos\theta} d\theta\right)\frac{d\rho}{\rho} = \frac{1}{\pi}\int_0^\pi d\theta \int_C^\infty e^{-h\rho} e^{ir\rho\cos\theta}\frac{d\rho}{\rho},$$

$$L_1 = \frac{1}{\pi}\int_0^\pi d\theta \int_C^\infty \frac{e^{-(h-ir\cos\theta)\rho}}{\rho} d\rho = \frac{1}{\pi}\int_0^\pi \mathrm{Ei}\left((h\text{-}ir\cos\theta)C\right) d\theta$$

where is an exponential integral and shown in general form as

$$\mathrm{Ei}(n,z) = \int_z^\infty \frac{e^{-t}}{t^n} dt \qquad n = 1, 2, 3...$$

5.2 INTEGRATION OF THE PRODUCT OF BESSEL FUNCTIONS AND EXPONENTIAL FUNCTIONS II.

5.2.1. Iterative Method II.

In this section, it will be started from the identity given by Watson, (1922), as

$$J_{\nu}\;\; r\rho\;\; J_{\nu}\;\; s\rho\;\; = \frac{\left(\frac{1}{2} rs\right)^{\nu}}{\Gamma\left(\nu+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)} \int_0^{\pi} \frac{J_{\nu}\;\; R\rho}{R^{\nu}} \rho^{\nu} \sin^{2\nu}\;\; \phi\;\; d\phi \qquad (5.41)$$

where ν is an order of the Bessel function of the first kind and $\Gamma\;\; n$ is the gamma function. Using Equation (4.28) for $\nu=0$ and $\nu=1$, respectively, it can be written that

$$J_0\;\; r\rho\;\; J_0\;\; s\rho\;\; = \frac{1}{\pi}\int_0^{\pi} J_0\;\; R\rho\;\; d\phi\,,$$

(5.42)

$$J_1\;\; r\rho\;\; J_1\;\; sp\;\; = \frac{rs}{\pi}\int_0^{\pi} \frac{J_1\;\; R\rho}{R} \rho \sin^2 \phi\, d\phi\,. \qquad (5.43)$$

To obtain the multiplication $J_1\;\; r\rho\;\; J_0\;\; s\rho$, Equation (4.29) should be differentiated with respect to r as follows:

$$\frac{d}{dr}\;\; J_0\;\; r\rho\;\; J_0\;\; s\rho\;\;\; = \frac{d}{dr}\left(\frac{1}{\pi}\int_0^{\pi} J_0\;\; r\rho\;\; d\phi\right),$$

$$-\rho J_1\;\; r\rho\;\; J_0\;\; s\rho\;\; = \frac{1}{\pi}\int_0^{\pi} \frac{d}{dr} J_0\;\; R\rho\;\; d\phi\,,$$

$$= \frac{1}{\pi}\int_0^{\pi} \frac{d}{dR} J_0\;\; R\rho\;\; \frac{dR}{dr} d\phi\,,$$

$$= \frac{1}{\pi}\int_0^{\pi} -\rho J_1\;\; R\rho\;\; \frac{r - s\cos\phi}{R} d\phi\,,$$

$$J_1\;\; r\rho\;\; J_0\;\; s\rho\;\; = \frac{1}{\pi}\int_0^{\pi} J_1\;\; R\rho\;\; \frac{r - s\cos\phi}{R} d\phi \qquad (5.44)$$

where

$$R = \left(r^2 + s^2 - 2rs\cos\phi\right)^{1/2},$$

(5.45)

$$\frac{dR}{dr} = \frac{1}{2}\left(r^2 + s^2 - 2rs\cos\phi\right)^{-\frac{1}{2}} 2\left(r - s\cos\phi\right).$$

Similarly, it can be also obtained the product $J_0(r\rho)J_1(s\rho)$ by differentiating Equation (4.29) with respect to s as follows:

$$\frac{d}{ds}\left[J_0(r\rho)J_0(s\rho)\right] = \frac{d}{ds}\left(\frac{1}{\pi}\int_0^{\pi} J_0(r\rho)\, d\phi\right),$$

$$-\rho J_0(r\rho)J_1(s\rho) = \frac{1}{\pi}\int_0^{\pi} \frac{d}{ds} J_0(R\rho)\, d\phi,$$

$$= \frac{1}{\pi}\int_0^{\pi} \frac{d}{dR} J_0(R\rho) \frac{dR}{ds} d\phi,$$

$$= \frac{1}{\pi}\int_0^{\pi} -\rho J_1(R\rho) \frac{s - r\cos\phi}{R} d\phi,$$

$$J_0(r\rho)J_1(s\rho) = \frac{1}{\pi}\int_0^{\pi} J_1(R\rho) \frac{s - r\cos\phi}{R} d\phi$$

(5.46)

where the value of R is given in (4.32).

5.2.2. Evaluation of Integrals in the Form of $\int_0^{\infty} D(\rho) J_m(r\rho) J_n(s\rho) e^{-2h\rho}$

Let us start with an asymptotic expansion of the integrand $D(\rho)$, that is named as $D_A(\rho)$, and the integral can be expressed as

$$\int_0^{\infty} D(\rho) J_m(r\rho) J_n(s\rho) e^{-2h\rho} d\rho = \int_0^{C} D(\rho) J_m(r\rho) J_n(s\rho) e^{-2h\rho} d\rho$$

$$+\int_C^{\infty} D_A(\rho) J_m(r\rho) J_n(s\rho) e^{-2h\rho} d\rho$$

$$+\int_C^{\infty} \left[D(\rho) - D_A(\rho)\right] J_m(r\rho) J_n(s\rho) e^{-2h\rho} d\rho. \quad (5.47)$$

If the value of C in (5.47) can be chosen such that the difference $D(\rho) - D_A(\rho)$ is small enough, such as 10^{-30}, then the result of the last integral will be almost zero due to both diminishing functions $J_m(r\rho)$ and $J_n(s\rho)$ at infinity. Thus, the improper integral can be shown as

$$\int_0^{\infty} D(\rho) J_m(r\rho) J_n(s\rho) e^{-2h\rho} d\rho = \int_0^{C} D(\rho) J_m(r\rho) J_n(s\rho) e^{-2h\rho} d\rho$$

$$+\int_C^{\infty} D_A(\rho) J_m(r\rho) J_n(s\rho) e^{-2h\rho} d\rho.$$

(5.48)

Since the function $D(\rho)$ is well-defined at zero then the first integral on the right in (5.48) can be evaluated using quadrature rule. On the other hand, $D_A(\rho)$ is an asymptotic expansion of $D(\rho)$ like

$$D_A(\rho) = \frac{a_1}{\rho} + \frac{a_2}{\rho^2} + \frac{a_3}{\rho^3} + \cdots + \frac{a_n}{\rho^n} + O\left(\frac{a_{n+1}}{\rho^{n+1}}\right),$$

(5.49)

for the second integral, it will be obtained an iterative method by using integration by parts. Let us define the following integrals for $k = 1, 2, 3, \ldots$ as

$$K_k\ (r,s) = \int_C^\infty \frac{1}{\rho^k} J_1\ (r\rho)\ J_1\ (s\rho)\ e^{-2h\rho} d\rho\,, \tag{5.50}$$

$$M_k\ (r,s) = \int_C^\infty \frac{1}{\rho^k} J_0\ (r\rho)\ J_0\ (s\rho)\ e^{-2h\rho} d\rho\,, \tag{5.51}$$

$$L_k\ (r,s) = \int_C^\infty \frac{1}{\rho^k} J_0\ (r\rho)\ J_1\ (s\rho)\ e^{-2h\rho} d\rho\ . \tag{5.52}$$

where it is obvious that

$$L_k\ (r,s) \neq L_k\ (s,r)\ .$$

(5.53)

Using the method of integration by parts, one can find an iterative method for the solution of $K_k\ (r,s)$, $M_k\ (r,s)$, $L_k\ (r,s)$ and $L_k\ (s,r)$ for each value of k.

Starting with the integral $K_k\ (r,s)$ and saying that $u = J_1\ (r\rho)\ J_1\ (s\rho)\ e^{-2h\rho}$ and $dv = {d\rho}/{\rho^k}$, the method of integration by parts gives us

$$K_k\ (r,s) = \frac{1}{1-k} \frac{J_1\ (r\rho)\ J_1\ (s\rho)\ e^{-2h\rho}}{\rho^{k-1}}\Bigg|_C^\infty$$

$$- \frac{1}{1-k} \int_C^\infty \frac{1}{\rho^{k-1}} \frac{d}{d\rho} \left(J_1\ (r\rho)\ J_1\ (s\rho)\ e^{-2h\rho} \right) d\rho\,.$$

(5.54)

Let $r\rho = \theta$, $s\rho = \psi$ and $2h\rho = \xi$, then taking the derivative of $J_1\ (r\rho)\ J_1\ (s\rho)\ e^{-2h\rho}$ with respect to these variables, it can be obtained the following result:

$$\frac{d}{d\rho} \left(J_1\ (r\rho)\ J_1\ (s\rho)\ e^{-2h\rho} \right) = \frac{d}{d\theta} J_1\ (\theta) \frac{d\theta}{d\rho} J_1\ (\psi)\ e^{-\xi} + J_1\ (\theta) \frac{d}{d\psi} J_1\ (\psi) \frac{d\psi}{d\rho} e^{-\xi}$$

$$+ J_1\ (\theta)\ J_1\ (\psi) \frac{d}{d\xi} e^{-\xi} \frac{d\xi}{d\rho}\,,$$

$$= r J_1'(\theta) J_1(\psi) + s J_1(\theta) J_1'(\psi) - 2h J_1(\theta) J_1(\psi) e^{-\xi},$$

$$= r J_1(\psi)\left(J_0(\theta) - \frac{1}{\theta} J_1(\theta)\right) e^{-\xi} + s J_1(\theta)\left(J_0(\psi) - \frac{1}{\psi} J_1(\psi)\right) e^{-\xi}$$

$$- 2h J_1(\theta) J_1(\psi) e^{-\xi},$$

$$= r J_0(\theta) J_1(\psi) e^{-\xi} + s J_0(\psi) J_1(\theta) e^{-\xi} - \frac{r}{\theta} J_1(\theta) J_1(\psi) e^{-\xi}$$

$$- \frac{s}{\psi} J_1(\theta) J_1(\psi) e^{-\xi} - 2h J_1(\theta) J_1(\psi) e^{-\xi},$$

$$= r J_0(r\rho) J_1(s\rho) e^{-2h\rho} + s J_0(s\rho) J_1(r\rho) e^{-2h\rho}$$

$$- \frac{2}{\rho} J_1(r\rho) J_1(s\rho) e^{-2h\rho} - 2h J_1(r\rho) J_1(s\rho), \qquad (5.55)$$

Now, substituting (4.42) into (4.41) will be resulted as

$$K_k(r,s) = \frac{1}{1-k} \frac{J_1(r\rho) J_1(s\rho) e^{-2h\rho}}{\rho^{k-1}} \Bigg|_C^{\infty}$$

$$- \frac{r}{1-k} \int_C^{\infty} \frac{1}{\rho^{k-1}} J_0(r\rho) J_1(s\rho) e^{-2h\rho} d\rho - \frac{s}{1-k} \int_C^{\infty} \frac{1}{\rho^{k-1}} J_1(r\rho) J_0(s\rho) e^{-2h\rho} d\rho$$

$$+ \frac{2}{1-k} \int_C^{\infty} \frac{1}{\rho^{k}} J_1(r\rho) J_1(s\rho) e^{-2h\rho} d\rho + \frac{2h}{1-k} \int_C^{\infty} \frac{1}{\rho^{k-1}} J_1(r\rho) J_1(s\rho) e^{-2h\rho} d\rho. \qquad (5.56)$$

From the definition of $K_k(r,s)$ and $L_k(r,s)$ given in Equations (4.37) and (4.39), respectively, the Equation (4.43) can be simplified as follows:

$$K_k(r,s) = \frac{1}{1-k} \frac{J_1(r\rho) J_1(s\rho) e^{-2h\rho}}{\rho^{k-1}} \Bigg|_C^{\infty} - \frac{r}{1-k} L_{k-1}(r,s) - \frac{s}{1-k} L_{k-1}(s,r)$$

$$+ \frac{2}{1-k} K_k(r,s) + \frac{2h}{1-k} K_{k-1}(s,r).$$

After substituting the upper and the lower bound values of ρ into the integrand such that $\rho \to \infty$ as $J_1(\rho) \to 0$, then it will be obtained as

$$\left(1-\frac{2}{1-k}\right)K_k(r,s) = \frac{1}{1-k}\frac{J_1(r\rho)J_1(s\rho)e^{-2h\rho}}{\rho^{k-1}}\Bigg|_C^{\infty} - \frac{r}{1-k}L_{k-1}(r,s)$$

$$-\frac{s}{1-k}L_{k-1}(s,r) + \frac{2h}{1-k}K_{k-1}(s,r),$$

$$\frac{k+1}{k-1}K_k(r,s) = \frac{1}{k-1}\frac{J_1(rC)J_1(sC)e^{-2hC}}{C^{k-1}}$$

$$+\frac{r}{1-k}L_{k-1}(r,s) + \frac{s}{1-k}L_{k-1}(s,r) - \frac{2h}{1-k}K_{k-1}(r,s).$$

Finally, for each $k \geq 2$, it can be obtained an iterative formula like

$$K_k(r,s) = \frac{1}{k+1}\frac{J_1(rC)J_1(sC)e^{-2hC}}{C^{k-1}} + \frac{r}{k+1}L_{k-1}(r,s)$$

$$+\frac{s}{k+1}L_{k-1}(s,r) - \frac{2h}{k+1}K_{k-1}(s,r).$$

(5.57)

Now, using the same method above, it can be found an iterative method for $M_k(r,s)$ and saying that $u = J_0(r\rho)J_0(s\rho)e^{-2h\rho}$ and $dv = d\rho/\rho^k$, the method of integration by parts gives us

$$M_k(r,s) = \frac{1}{1-k}\frac{J_0(r\rho)J_0(s\rho)e^{-2h\rho}}{\rho^{k-1}}\Bigg|_C^{\infty}$$

$$-\frac{1}{1-k}\int_C^{\infty}\frac{1}{\rho^{k-1}}\frac{d}{d\rho}\left(J_0(r\rho)J_0(s\rho)e^{-2h\rho}\right)d\rho.$$

(5.58)

Again, let $r\rho=\theta$, $s\rho=\psi$ and $2h\rho=\xi$, then taking the derivative of $J_0\ \ r\rho\ \ J_0\ \ s\rho\ \ e^{-2h\rho}$ with respect to these variables, it can be obtained the following result:

$$\frac{d}{d\rho}\ \ J_0\ \ r\rho\ \ J_0\ \ s\rho\ \ e^{-2h\rho}\ \ =\frac{d}{d\theta}J_0\ \ \theta\ \ \frac{d\theta}{d\rho}J_0\ \ \psi\ \ e^{-\xi}+J_0\ \ \theta\ \ \frac{d}{d\psi}J_0\ \ \psi\ \ \frac{d\psi}{d\rho}e^{-\xi}$$

$$+J_0\ \ \theta\ \ J_0\ \ \psi\ \ \frac{d}{d\xi}e^{-\xi}\ \frac{d\xi}{d\rho},$$

$$=-r\,J_1\ \ \theta\ \ J_0\ \ \psi\ \ e^{-\xi}-s\,J_0\ \ \theta\ \ J_1\ \ \psi\ \ e^{-\xi}-2hJ_0\ \ \theta\ \ J_0\ \ \psi\ \ e^{-\xi},$$

$$=-r\,J_1\ \ r\rho\ \ J_0\ \ s\rho\ \ e^{-2h\rho}-s\,J_0\ \ r\rho\ \ J_1\ \ s\rho\ \ e^{-2h\rho}-2hJ_0\ \ r\rho\ \ J_0\ \ s\rho\ \ e^{-2h\rho}. \quad (5.59)$$

Substituting the result above into (4.45), it can be written that

$$M_k\ \ r,s\ \ =\frac{1}{1-k}\frac{J_0\ \ r\rho\ \ J_0\ \ s\rho\ \ e^{-2h\rho}}{\rho^{k-1}}\Bigg|_C^{\infty}+\frac{r}{1-k}\int_C^{\infty}\frac{1}{\rho^{k-1}}J_1\ \ r\rho\ \ J_0\ \ s\rho\ \ e^{-2h\rho}d\rho$$

$$+\frac{s}{1-k}\int_C^{\infty}\frac{1}{\rho^{k-1}}J_0\ \ r\rho\ \ J_1\ \ s\rho\ \ e^{-2h\rho}d\rho+\frac{2h}{1-k}\int_C^{\infty}\frac{1}{\rho^{k-1}}J_0\ \ r\rho\ \ J_0\ \ s\rho\ \ e^{-2h\rho}d\rho$$

and using the definition of $L_k\left(r,s\right)$ given in Equation (4.39), the Equation (4.45) can be simplified as follows:

$$M_k\ \ r,s\ \ =\frac{1}{1-k}\frac{J_0\ \ r\rho\ \ J_0\ \ s\rho\ \ e^{-2h\rho}}{\rho^{k-1}}\Bigg|_C^{\infty}$$

$$+\frac{r}{1-k}L_{k-1}\ \ s,r\ \ +\frac{s}{1-k}L_{k-1}\ \ r,s\ \ +\frac{2h}{1-k}M_{k-1}\ \ r,s$$

and as a limiting case $J_0\ \ \rho\ \ \to 0$ as $\rho\to\infty$, then for each $k\geq 2$, it can be obtained an iterative formula like

$$M_k\ \ r,s\ \ =\frac{1}{1-k}\frac{J_0\ \ rC\ \ J_0\ \ sC\ \ e^{-2hC}}{C^{k-1}}-\frac{r}{k-1}L_{k-1}\ \ s,r$$

$$-\frac{s}{k-1}L_{k-1}\ \ r,s\ \ -\frac{2h}{k-1}M_{k-1}\ \ r,s\ \ .$$

(5.60)

Finally, the last integral $L_k\ \ r,s$ should be represented by an iterative formula as it was done above. In this case, it should not be forgotten that $L_k\ \ r,s\ \neq L_k\ \ s,r$ but the relation between them can be easily shown at the end. Let us apply the integration by part one more time to find an iterative formula. Let $r\rho=\theta$ and $s\rho=\psi$, then taking the derivative of $J_0\ \ r\rho\ \ J_1\ \ s\rho$ with respect to these variables, it can be obtained the following result:

$$L_k\ \ r,s\ =\frac{1}{1-k}\frac{J_0\ \ r\rho\ \ J_1\ \ s\rho\ \ e^{-2h\rho}}{\rho^{k-1}}\Bigg|_C^{\infty}$$

$$-\frac{1}{1-k}\int_C^{\infty}\frac{1}{\rho^{k-1}}\frac{d}{d\rho}\ \ J_0\ \ r\rho\ \ J_1\ \ s\rho\ \ e^{-2h\rho}\ \ d\rho$$

(5.61)

where the derivative of $J_0\ \ r\rho\ \ J_1\ \ s\rho\ \ e^{-2h\rho}$ is

$$\frac{d}{d\rho}\ \ J_0\ \ r\rho\ \ J_1\ \ s\rho\ \ e^{-2h\rho}\ \ =-rJ_1\ \ r\rho\ \ J_1\ \ s\rho\ \ e^{-2h\rho}$$

$$+sJ_0\ \ r\rho\ \left(-\frac{1}{s\rho}J_1\ \ s\rho\ \ +J_0\ \ s\rho\ \right)e^{-2h\rho}-2hJ_0\ \ r\rho\ \ J_1\ \ s\rho\ \ e^{-2h\rho},$$

$$=-rJ_1\ \ r\rho\ \ J_1\ \ s\rho\ \ e^{-2h\rho}+sJ_0\ \ r\rho\ \ J_0\ \ s\rho\ \ e^{-2h\rho}$$

$$-\frac{1}{\rho}J_0\ \ r\rho\ \ J_1\ \ s\rho\ \ e^{-2h\rho}-2hJ_0\ \ r\rho\ \ J_1\ \ s\rho\ \ e^{-2h\rho}.$$

Now, using the result of the derivative above and Equations (4.37) and (4.38), the integral $L_k\ \ r,s$ can be written as follows:

$$L_k(r,s)=\frac{1}{1-k}\left.\frac{J_0(r\rho)J_1(s\rho)e^{-2h\rho}}{\rho^{k-1}}\right|_C^\infty$$

$$-\frac{s}{1-k}\int_C^\infty\frac{1}{\rho^{k-1}}J_0(r\rho)J_1(s\rho)e^{-2h\rho}d\rho+\frac{r}{1-k}\int_C^\infty\frac{1}{\rho^{k-1}}J_1(r\rho)J_1(s\rho)e^{-2h\rho}d\rho$$

$$+\frac{1}{1-k}\int_C^\infty\frac{1}{\rho^{k}}J_0(r\rho)J_1(s\rho)e^{-2h\rho}d\rho+\frac{2h}{1-k}\int_C^\infty\frac{1}{\rho^{k-1}}J_0(r\rho)J_1(s\rho)e^{-2h\rho}d\rho$$

$$L_k(r,s)=\frac{1}{1-k}\left.\frac{J_0(r\rho)J_1(s\rho)e^{-2h\rho}}{\rho^{k-1}}\right|_C^\infty$$

$$-\frac{s}{1-k}M_{k-1}(r,s)+\frac{r}{1-k}K_{k-1}(r,s)+\frac{1}{1-k}L_k(r,s)+\frac{2h}{1-k}L_{k-1}(r,s),$$

$$\left(1-\frac{1}{1-k}\right)L_k(r,s)=\frac{1}{1-k}\left.\frac{J_0(r\rho)J_1(s\rho)e^{-2h\rho}}{\rho^{k-1}}\right|_C^\infty$$

$$-\frac{s}{1-k}M_{k-1}(r,s)+\frac{r}{1-k}K_{k-1}(r,s)+\frac{2h}{1-k}L_{k-1}(r,s).$$

After simplification the like terms and using the upper and the lower bound of integral, $J_0(\eta)\to 0$ and $J_1(\eta)\to 0$ as $\rho\to\infty$, then an iterative formula $L_k(r,s)$ for each $k\geq 2$ can be expressed as

$$L_k(r,s)=\frac{1}{k}\frac{J_0(rC)J_1(sC)e^{-2hC}}{C^{k-1}}+\frac{s}{k}M_{k-1}(r,s)-\frac{r}{k}K_{k-1}(r,s)-\frac{2h}{k}L_{k-1}(r,s). \quad (5.62)$$

Exactly using the same idea, one can find an iterative formula for $L_k(s,r)$ like

$$L_k(s,r)=\frac{1}{k}\frac{J_0(rC)J_1(sC)e^{-2hC}}{C^{k-1}}+\frac{r}{k}M_{k-1}(r,s)-\frac{s}{k}K_{k-1}(r,s)-\frac{2h}{k}L_{k-1}(r,s) \quad (5.63)$$

where

$$M_k(r,s) = M_k(s,r),$$

(5.64)

$$K_k(r,s) = K_k(s,r).$$

(5.65)

Now, let us evaluate each improper integral $K_1(r,s)$, $M_1(r,s)$, $L_1(r,s)$ and $L_1(s,r)$ using the product of the Bessel functions of the first kind given in Equations from (4.29) to (4.33).

5.2.3. Evaluation of $K_1(r,s) = \int_C^\infty \frac{1}{\rho} J_1(r\rho) J_1(s\rho) e^{-2h\rho} d\rho$

From Equation (4.30) by dividing both sides by ρ, it may be written that

$$\frac{J_1(r\rho) J_1(s\rho)}{\rho} = \frac{rs}{\pi} \int_0^\pi \frac{J_1(R\rho)}{R} \sin^2\phi d\phi,$$

where $R^2 = r^2 + s^2 - 2rs\cos\phi$ as same as given in (4.32). Then, $K_1(r,s)$ can be expressed as double integral of

$$K_1(r,s) = \int_C^\infty \frac{J_1(r\rho) J_1(s\rho)}{\rho} e^{-2h\rho} d\rho = \int_C^\infty \frac{rs}{\pi} e^{-2h\rho} \int_0^\pi \frac{J_1(R\rho)}{R} \sin^2\phi d\phi d\rho$$

$$= \frac{rs}{\pi} \int_0^\pi \frac{\sin^2\phi}{R} \left(\int_C^\infty J_1(R\rho) d\rho e^{-2h\rho} \right) d\phi.$$

(5.66)

The improper integral in (5.66) can be written as

$$\int_A^\infty J_1(R\rho) e^{-2h\rho} d\rho = \int_0^\infty J_1(R\rho) e^{-2h\rho} d\rho - \int_0^A J_1(R\rho) e^{-2h\rho} d\rho, \qquad (5.67)$$

from the closed form of solution (5.67)

$$\int_0^{\infty} J_1 \;\; R\rho \;\; e^{-2h\rho} d\rho = \frac{1}{R}\left(1 - \frac{2h}{\sqrt{4h^2 + R^2}}\right)$$

then,

$$\int_A^{\infty} J_1 \;\; R\rho \;\; e^{-2h\rho} d\rho = \frac{1}{R}\left(1 - \frac{2h}{\sqrt{4h^2 + R^2}}\right) - \int_0^{A} J_1 \;\; R\rho \;\; e^{-2h\rho} d\rho. \qquad (5.68)$$

Substituting the result into (5.66)

$$K_1 \;\; r,s \;\; = \frac{rs}{\pi}\int_0^{\pi} \frac{\sin^2\phi}{R}\left(\frac{1}{R}\left(1 - \frac{2h}{\sqrt{4h^2 + R^2}}\right) - \int_0^{C} J_1 \;\; R\rho \;\; e^{-2h\rho} d\rho\right) d\phi.$$

As a result,

$$K_1 \;\; r,s \;\; = \int_C^{\infty} \frac{J_1 \;\; r\rho \;\; J_1 \;\; s\rho}{\rho} e^{-2h\rho} d\rho$$

$$= \frac{rs}{\pi}\int_0^{\pi} \frac{\sin^2\phi}{R}\left(\frac{1}{R}\left(1 - \frac{2h}{\sqrt{4h^2 + R^2}}\right) - \int_0^{C} J_1 \;\; R\rho \;\; e^{-2h\rho} d\rho\right) d\phi. \qquad (5.69)$$

5.2.4. Evaluation of $M_1(r,s) = \int_C^{\infty} \frac{1}{\rho} J_0 \;\; r\rho \;\; J_0 \;\; s\rho \;\; e^{-2h\rho} d\rho$

From Equation (4.29) by dividing both sides by ρ, it may be written that

$$\frac{J_0 \;\; r\rho \;\; J_0 \;\; s\rho}{\rho} = \frac{1}{\pi\rho}\int_0^{\pi} J_0 \;\; R\rho \;\; d\phi$$

where $R^2 = r^2 + s^2 - 2rs\cos\phi$ as same as given in (4.32). Then, $M_1 \;\; r,s$ can be expressed as double integral of

$$M_1(r,s) = \int_C^{\infty} \frac{J_0 \;\; r\rho \;\; J_0 \;\; s\rho}{\rho} e^{-2h\rho} d\rho = \frac{1}{\pi}\int_C^{\infty} \frac{1}{\rho} e^{-2h\rho} \int_0^{\pi} J_0 \;\; R\rho \;\; d\phi d\rho$$

$$= \frac{1}{\pi}\int_0^{\pi}\left(\int_C^{\infty}\frac{J_0(R\rho)}{\rho}e^{-2h\rho}d\rho\right)d\phi .$$

(5.70)

Then using the following identity

$$\int_0^{\infty}\left(1-J_0(bx)\right)\frac{e^{-px}}{x}dx = -\log\left(\frac{2p}{p+\sqrt{p^2+b^2}}\right) \qquad (5.71)$$

and separating integral in (5.71) into two parts as

$$\int_0^{\infty}\left(1-J_0(bx)\right)\frac{e^{-px}}{x}dx = \int_0^{C}\left(1-J_0(bx)\right)\frac{e^{-px}}{x}dx + \int_C^{\infty}\left(1-J_0(bx)\right)\frac{e^{-px}}{x}dx . \qquad (5.72)$$

Equation (5.72) can be written as

$$\int_C^{\infty}\frac{J_0(bx)}{x}e^{-px}dx = \int_C^{\infty}\frac{e^{-px}}{x}dx - \int_0^{\infty}\left(1-J_0(bx)\right)\frac{e^{-px}}{x}dx + \int_0^{C}\left(1-J_0(bx)\right)\frac{e^{-px}}{x}dx ,$$

$$\int_0^{\infty}\frac{J_0(bx)}{x}e^{-px}dx = \mathrm{Ei}(1, pC) + \log\left(\frac{2p}{p+\sqrt{p^2+b^2}}\right) + \int_0^{C}\left(1-J_0(bx)\right)\frac{e^{-px}}{x}dx$$

and substituting the result of the identity into (5.70) it can be obtained as

$$M_1(r,s) = \mathrm{Ei}(1, 2hC)$$

$$+\frac{1}{\pi}\int_0^{\pi}\left[\log\left(\frac{4h}{2h+\sqrt{4h^2+R^2}}\right)+\int_0^{C}\left(1-J_0(R\rho)\frac{e^{-2h\rho}}{\rho}d\rho\right)\right]d\phi . \qquad (5.73)$$

5.2.5. Evaluation of $L_1(r,s) = \int_C^{\infty}\frac{1}{\rho}J_0(r\rho)J_1(s\rho)e^{-2h\rho}d\rho$

From Equation (4.33) by dividing both sides by ρ, it may be written that

$$\frac{J_0(r\rho)J_1(s\rho)}{\rho}=\frac{1}{\pi}\int_0^{\pi}\frac{s-r\cos\phi}{R}\frac{J_1(R\rho)}{\rho}d\phi,$$

where $R^2=r^2+s^2-2rs\cos\phi$ as same as given in (4.32). Then, $L_1(r,s)$ can be expressed as double integral of

$$L_1(r,s)=\int_C^{\infty}\frac{J_0(r\rho)J_1(s\rho)}{\rho}e^{-2h\rho}d\rho=\int_C^{\infty}\frac{1}{\pi}\int_0^{\pi}\frac{s-r\cos\phi}{R}\frac{J_1(R\rho)}{\rho}e^{-2h\rho}d\phi\, d\rho$$
$$=\frac{1}{\pi}\int_0^{\pi}\frac{s-r\cos\phi}{R}\left(\int_C^{\infty}\frac{J_1(R\rho)}{\rho}e^{-2h\rho}d\rho\right)d\phi.$$

(5.74)

The improper integral in (5.74) can be written as

$$\int_C^{\infty}\frac{J_1(R\rho)}{\rho}e^{-2h\rho}d\rho=\int_0^{\infty}\frac{J_1(R\rho)}{\rho}e^{-2h\rho}d\rho-\int_0^{C}\frac{J_1(R\rho)}{\rho}e^{-2h\rho}d\rho, \qquad (5.75)$$

from the closed form solution of (5.75),

$$\int_0^{\infty}\frac{J_1(R\rho)}{\rho}e^{-2h\rho}d\rho=\frac{\sqrt{4h^2+R^2}-2h}{R},$$

it can be written that

$$\int_C^{\infty}\frac{J_1(R\rho)}{\rho}e^{-2h\rho}d\rho=\frac{\sqrt{4h^2+R^2}-2h}{R}-\int_0^{C}\frac{J_1(R\rho)}{\rho}e^{-2h\rho}d\rho.$$

By substituting the result into (5.74), it can be easily shown that

$$L_1(r,s)=\frac{1}{\pi}\int_0^{\pi}\left(\frac{\sqrt{4h^2+R^2}-2h}{R}-\int_0^{C}\frac{J_1(R\rho)}{\rho}e^{-2h\rho}d\rho\right)\frac{s-r\cos\phi}{R}d\phi. \qquad (5.76)$$

5.2.6. Evaluation of $N_0(r,s)=\int_0^{\infty} J_0\ r\rho\ J_1\ s\rho\ e^{-2h\rho} d\rho$

From Equation (4.33) by multiplying both sides with $e^{-2h\rho}$ it can be written as

$$N_0(r,s)=\int_0^{\infty} J_0\ r\rho\ J_1\ s\rho\ e^{-2h\rho} d\rho=\frac{1}{\pi}\int_0^{\infty} e^{-2h\rho} d\rho \int_0^{\pi} \frac{s-r\cos\phi}{R} J_1\ R\rho\ d\phi$$

$$=\frac{1}{\pi}\int_0^{\pi} \frac{s-r\cos\phi}{R}\left(\int_0^{\infty} J_1\ R\rho\ e^{-2h\rho} d\rho\right) d\phi$$

and from the closed form solution of

$$\int_0^{\infty} J_1\ R\rho\ e^{-2h\rho} d\rho=\frac{1}{R}\left(1-\frac{2h}{\sqrt{4h^2+R^2}}\right),$$

It can be obtained that

$$N_0\ r,s\ =\int_0^{\infty} J_1\ R\rho\ e^{-2h\rho} d\rho=\frac{1}{\pi}\int_0^{\pi} \frac{s-r\cos\phi}{R^2}\left(1-\frac{2h}{\sqrt{4h^2+R^2}}\right) d\phi$$

where

$$R=\sqrt{r^2+s^2-2rs\cos\phi}\ .$$

5.2.7. Evaluation of $N_1(r,s)=\int_0^{\infty} J_0\ r\rho\ J_1\ s\rho\ \rho e^{-2h\rho} d\rho$

Similarly, from Equation (4.33) by multiplying both sides with $e^{-2h\rho}$, it can be found that

$$N_1(r,s)=\int_0^{\infty} J_0\ r\rho\ J_1\ s\rho\ e^{-2h\rho} \rho\, d\rho$$

$$=\frac{1}{\pi}\int_0^{\infty} e^{-2h\rho} \rho\, d\rho \int_0^{\pi} \frac{s-r\cos\phi}{R} J_1\ R\rho\ e^{-2h\rho} d\phi$$

$$=\frac{1}{\pi}\int_0^{\pi}\frac{s-r\cos\phi}{R}\left(\int_0^{\infty}J_1\;\;R\rho\;\;\rho e^{-2h\rho}d\rho\right)d\phi\,.$$

Then, using the identity such that

$$\int_0^{\infty}J_1\;\;R\rho\;\;e^{-2h\rho}\rho\,d\rho=\frac{R}{4h^2+R^2\;\;{}^{3/2}}\,,$$

It can be written that

$$N_1(r,s)=\int_0^{\infty}J_0\;\;r\rho\;\;J_1\;\;s\rho\;\;\rho e^{-2h\rho}d\rho=\frac{1}{\pi}\int_0^{\pi}\frac{s-r\cos\phi}{R}\left(\frac{R}{4h^2+R^2\;\;{}^{3/2}}\right)d\phi\,,$$

$$N_1(r,s)=\int_0^{\infty}J_0\;\;r\rho\;\;J_1\;\;s\rho\;\;\rho e^{-2h\rho}d\rho=\frac{1}{\pi}\int_0^{\pi}\frac{s-r\cos\phi}{4h^2+R^2\;\;{}^{3/2}}d\phi$$

where

$$R=\sqrt{r^2+s^2-2rs\cos\phi}\,.$$

CHAPTER 6

CONCLUSION

Examining the calculation of oscillatory integrals is very wide range area of numerical analysis. Many techniques either analytical or semi-analytical, which is name asymptotic approach, are used to evaluate oscillatory integrals. In this study, it is used semi-analytical method by expanding the integrand at infinity. After expanding the integrand at infinity, some types of singularities are occurred in the new integrand that is a series expansion of the original integrand. These types of singularities such as elliptic and logarithmic were eliminated by using some analytical techniques, and then the integrals were evaluated using a standard quadrature rule. Since the basic part of each integral is calculated then the result of the rest of the integrals can be found by an iterative method.

The following integral, as an example, is obtained from the solution of the thermal distribution around an insulated barrier in a semi-infinite composite material in axisymmetric coordinate system,

$$\int_0^{\infty} D\ \ r,s,\rho\ \ J_0\ \ r\rho\ \ J_1\ \ s\rho\ \ d\rho$$

(6.1)

where the function $D\ \ r,s,\rho$ is given as

$$D\ \ r,s,\rho\ \ = \frac{m_2 + m_1\ \ \rho}{-\rho - \rho\left(\dfrac{m_2 + m_1}{m_2 - m_1}\right)e^{-2m_2h} +\ \ e^{-2m_2h} - 1\ \ \ m_2 + m_1}$$

(6.2)

and m_i are the roots of the characteristic equation of the system and h is the distance between the barrier and the free surface of the material. As it is seen in Figure 6.1, due to integrand that contains the product of the Bessel functions of the first kind and the function $D(\rho, r, s)$, the integral is an oscillatory one.

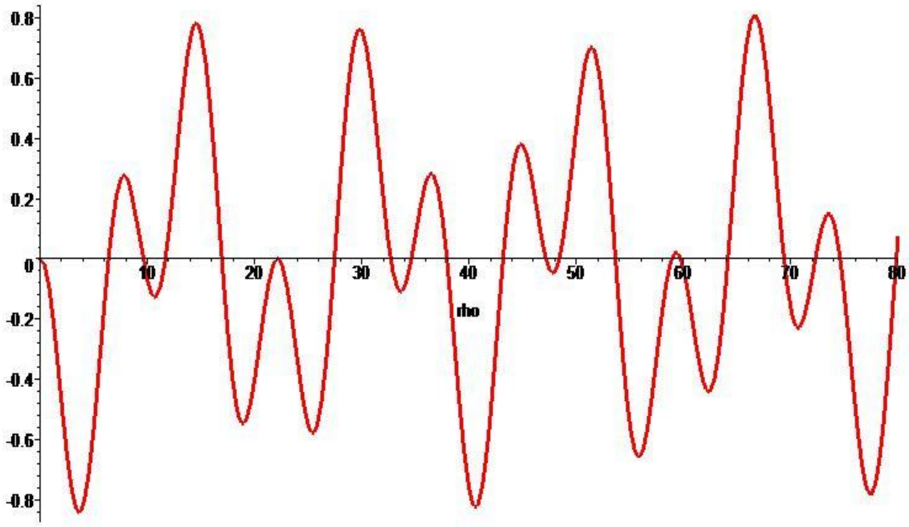

Figure 6.1: Graph of the integrand given in Equation (6.1).

In this study, integrals like in Equation (6.1) are evaluated first eliminating the singularities near the boundaries and some special points over the interval of integration, such as $r = s$ that are the parameters of the thermal problem. After that using the process given in Chapter 4, one can reduce the integral into the form of

$$\int_{C}^{\infty} \frac{J_m(r\rho)\, J_n(s\rho)}{\rho^k}\, d\rho$$

(6.3)

where C is dynamically determined with respect to the variables and parameters in function $D(r, s, \rho)$. Defining each integral in terms of the combination of m and n, one can find the following iterative formula for $k \geq 2$:

$$K_k(r,s) = \frac{1}{k+1}\frac{J_1(rC)\, J_1(sC)}{C^{k-1}} + \frac{1}{k+1}\left(rL_{k-1}(r,s) + sL_{k-1}(s,r) \right),$$

$$M_k(r,s) = \frac{1}{k-1}\frac{J_0(rC)\, J_0(sC)}{C^{k-1}} - \frac{1}{k-1}\left(rL_{k-1}(s,r) + sL_{k-1}(r,s) \right),$$

$$L_k(r,s) = \frac{1}{k}\frac{J_0(rC)J_1(sC)}{C^{k-1}} + \frac{1}{k}\left(sM_{k-1}(r,s) - rK_{k-1}(s,r)\right),$$

$$L_k(s,r) = \frac{1}{k}\frac{J_1(rC)J_0(sC)}{C^{k-1}} + \frac{1}{k}\left(rM_{k-1}(r,s) - sK_{k-1}(s,r)\right).$$

The initial values of integrals $K_1(r,s)$, $M_1(r,s)$, $L_1(r,s)$ and $L_1(s,r)$ are original parts of this study and their formulations were found analytically using the properties Bessel functions. These integrals can be represented by the following formula:

$$K_1(r,s) = \int_C^{\infty} \frac{J_1(r\rho)J_1(s\rho)}{\rho}\,d\rho = \frac{rs}{\pi}\int_0^{\pi} \frac{\sin^2\phi}{R^2} J_0(AR)\,d\phi,$$

$$M_1(r,s) = \int_C^{\infty} \frac{J_0(r\rho)J_0(s\rho)}{\rho}\,d\rho = \frac{1}{\pi}\int_0^{\pi}\left(\int_C^{\infty} \frac{J_0(R\rho)}{\rho}\,d\rho\right)d\phi,$$

$$L_1(r,s) = \int_C^{\infty} \frac{J_0(r\rho)J_1(s\rho)}{\rho}\,d\rho = \frac{1}{\pi}\int_0^{\pi} \frac{s - r\cos\phi}{R}\left(1 + J_1(CR) - \int_0^{CR} J_0(v)\,dv\right)d\phi,$$

$$L_1(s,r) = \int_C^{\infty} \frac{J_0(s\rho)J_1(r\rho)}{\rho}\,d\rho = \frac{1}{\pi}\int_0^{\pi} \frac{r - s\cos\phi}{R}\left(1 + J_1(CR) - \int_0^{CR} J_0(v)\,dv\right)d\phi$$

where $L_1(r,s) \neq L_1(s,r)$ due to the value of R that contains parameters r and s.

In a similar way, it is examined the oscillatory integral which is obtained in thermal distribution around an insulated barrier in composite materials in a semi-infinite axisymmetric coordinate system like

$$\int_0^{\infty} D(r,s,\rho)\,e^{-2h\rho} J_0(r\rho)J_1(s\rho)\,d\rho$$

(6.4)

where h is the distance between the barrier and the free surface of the material. In Figure 6.2, it is shown the shape of the oscillatory integrand in which the integrand is diminishing as $\rho \to \infty$. Using similar method as in the previous case, the singularities are eliminated and then the integrand is expanded asymptotically at infinity. The integral is separated into two parts: $\ 0,C\ $ and $\ C,\infty\ $. The first part is evaluated using standard numerical techniques such as quadrature rule and the second part,

$$\int_C^{\infty} e^{-2h\rho} \frac{J_m\ \ r\rho\ \ J_n\ \ s\rho}{\rho^k} d\rho\,, \tag{6.5}$$

is formulated as before. According to the value of parameter h, the oscillation of the integrand is changing. If the parameter $h<1$, then the oscillation is increasing. Otherwise, $h\geq 1$, the oscillation is decreasing and the integrand is quickly converging to the zero as $\rho\to\infty$. Defining each integral in terms of the combination of m and n, one can find the following iterative formula for $k\geq 2$:

$$K_k\ \ r,s\ \ = \frac{e^{-2hC}}{k+1}\frac{J_1\ \ rC\ \ J_1\ \ sC}{C^{k-1}} + \frac{r}{k+1}L_{k-1}\ \ r,s\ \ + \frac{s}{k+1}L_{k-1}\ \ s,r\ \ - \frac{2h}{k+1}K_{k-1}\ \ s,r\ \ ,$$

$$M_k\ \ r,s\ \ = \frac{e^{-2hC}}{1-k}\frac{J_0\ \ rC\ \ J_0\ \ sC}{C^{k-1}} - \frac{r}{k-1}L_{k-1}\ \ s,r\ \ - \frac{s}{k-1}L_{k-1}\ \ r,s\ \ - \frac{2h}{k-1}M_{k-1}\ \ r,s\ \ ,$$

$$L_k\ \ r,s\ \ = \frac{e^{-2hC}}{k}\frac{J_0\ \ rC\ \ J_1\ \ sC}{C^{k-1}} + \frac{s}{k}M_{k-1}\ \ r,s\ \ - \frac{r}{k}K_{k-1}\ \ r,s\ \ - \frac{2h}{k}L_{k-1}\ \ r,s\ \ ,$$

$$L_k\ \ s,r\ \ = \frac{e^{-2hC}}{k}\frac{J_0\ \ sC\ \ J_1\ \ rC}{C^{k-1}} + \frac{r}{k}M_{k-1}\ \ r,s\ \ - \frac{s}{k}K_{k-1}\ \ r,s\ \ - \frac{2h}{k}L_{k-1}\ \ s,r\ \ .$$

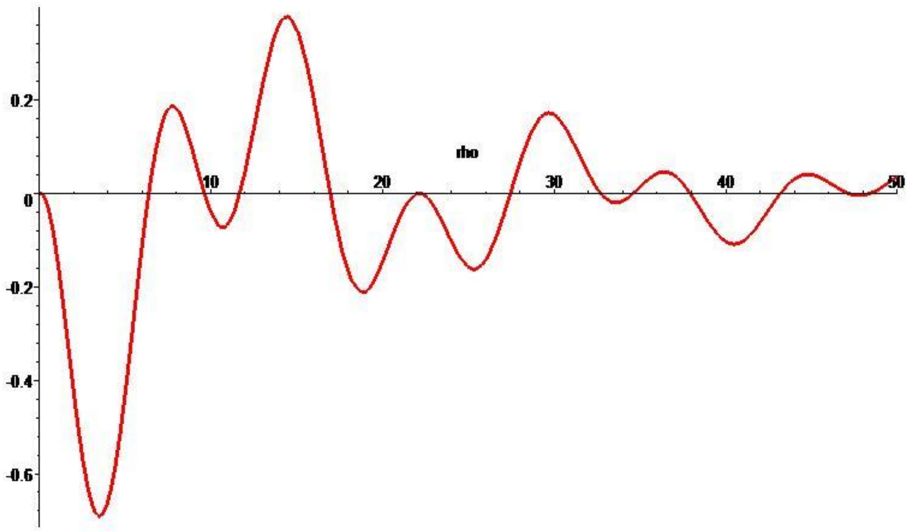

Figure 6.2: Graph of the integrand given in Equation (6.4).

The initial values of each integral for any value of parameter h can be given as follows:

$$K_1(r,s)=\int_C^{\infty}\frac{J_1(r\rho)\,J_1(s\rho)}{\rho}e^{-2h\rho}d\rho$$

$$=\frac{rs}{\pi}\int_0^{\pi}\frac{\sin^2\phi}{R^2}\left(\left(1-\frac{2h}{\sqrt{4h^2+R^2}}\right)-R\int_0^{C}J_1(R\rho)\,e^{-2h\rho}d\rho\right)d\phi,$$

$$M_1(r,s)=\int_C^{\infty}\frac{J_0(r\rho)\,J_0(s\rho)}{\rho}e^{-2h\rho}d\rho$$

$$=\mathrm{Ei}(1,2hC)+\frac{1}{\pi}\int_0^{\pi}\left[\log\left(\frac{4h}{2h+\sqrt{4h^2+R^2}}\right)+\int_0^{C}\left(1-J_0(R\rho)\right)\frac{e^{-2h\rho}}{\rho}d\rho\right]d\phi,$$

$$L_1(r,s)=\int_C^{\infty}\frac{J_0(r\rho)\,J_1(s\rho)}{\rho}e^{-2h\rho}d\rho$$

$$= \frac{1}{\pi} \int_0^{\pi} \left(\frac{\sqrt{4h^2 + R^2} - 2h}{R} - \int_0^C \frac{J_1(R\rho)}{\rho} e^{-2h\rho} d\rho \right) \frac{s - r\cos\phi}{R} d\phi \,,$$

$$L_1(s,r) = \int_C^{\infty} \frac{J_1(r\rho) J_0(s\rho)}{\rho} e^{-2h\rho} d\rho$$

$$= \frac{1}{\pi} \int_0^{\pi} \left(\frac{\sqrt{4h^2 + R^2} - 2h}{R} - \int_0^C \frac{J_1(R\rho)}{\rho} e^{-2h\rho} d\rho \right) \frac{r - s\cos\phi}{R} d\phi$$

where $L_1(r,s) \neq L_1(s,r)$ due to the value of R that contains parameters r and s.

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