A STUDY ON PERFECT AND REGULAR RINGS

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"Even miracles take a little time..." Cinderella

Dedicated to my parents with deep love and gratitude...

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Pınar Aydoğdu

ABSTRACT

One of the starting points of the historical developments of non-commutative rings and their modules is algebras over a field K. A K-algebra, its ideals and its modules are K-vector spaces. If a K-vector space is finite dimensional, then it satisfies some finiteness conditions. Emmy Noether worked on commutative rings with the ascending chain condition which are now called Noetherian rings. Emil Artin, who inspired by Noether's works, generalized Wedderburn's structure theorems on algebras to non-commutative rings with the descending chain condition, which are now called Artinian rings.

Regular rings were invented by von Neumann in the mid-1930's in order to provide an algebraic framework for studying the lattices of projections in the operator algebras. Von Neumann modelled this framework on the coordinatization of projective geometry.

Artinian and regular rings have been studied extensively and generalized in different ways by many authors (e.g. [7], [19], [23], [30], [43]). Some of these generalizations are perfect, semiperfect and semiregular rings.

The goal of this dissertation is to work on new concepts which are derived from Artinian and regular rings. Our aim is to characterize perfect and regular rings and some of their generalizations by considering some chain conditions as well as semiregular and semiperfect modules, and to investigate semiregular and semiperfect rings relative to some special ideals.

In the first chapter of this dissertation, we give the definitions of some basic notions and investigate some of their properties which are useful tools for our further studies, and we study the rings that are mentioned above.

In the second chapter, we introduce the notion of 'strong lifting submodule' which is a module theoretic version of the notion of strong lifting ideal. Strongly lifting ideals were studied by Nicholson and Zhou ([31]) to characterize semiregular and semiperfect rings relative to an ideal. Inspired by these works, we investigate strongly lifting submodules and obtain some new characterizations of semiregular and semiperfect modules relative to a projection-invariant submodule. In the last section of this chapter, we investigate rings over which every (projective) module M is $\tau(M)$-semiperfect for a preradical τ, and notice that this condition is one of the necessary and sufficient conditions for a ring to be (δ-)perfect.

The class of semiregular rings and the class of almost principally injective rings are contained in the class of generalized semiregular rings which are defined by Xiao and Tong ([42]). In the third chapter, we introduce generalized semiregular rings relative to an ideal, and investigate some of their properties. We also consider generalized semiregular rings relative to some special ideals such as the socle, the δ radical and the singular ideal of the ring.

The last chapter, Chapter 4, is concerned with chain conditions on non-summands. We investigate the properties of modules with chain conditions on non-summands by considering some module classes. We characterize Artinian and Noetherian modules in terms of these chain conditions. It is well known that a ring is right perfect if and only if it satisfies descending chain condition on cyclic left ideals. We deduce that a ring is right perfect if and only if it satisfies descending chain condition on non-summand cyclic left ideals and on summand left ideals. Moreover, if a ring satisfies descending chain condition on non-summand cyclic right ideals, then it is a semiregular ring with a left T-nilpotent Jacobson radical.
Keywords: (Semi)Perfect rings, (semi)regular rings, Artinian rings and modules, Noetherian rings and modules, injective modules, projective modules.
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TAM VE DÜZENLİ HALKALAR ÜZERİNE BİR ÇALIŞMA

Pınar Aydoğdu

ÖZET

Değişmeli olmayan halkaların ve bu halkalar üzerindeki modüllerin tarihi gelişimindeki başlangıç noktalarından biri bir K cismi üzerindeki cebirlerdir. Bu cebirler, bunların idealleri ve bu cebirler üzerindeki modüller K-vektör uzaylarıdır. Eğer bir K-vektör uzayı sonlu boyutlu ise bu modüller üzerinde bazı sonluluk özellikleri sağlanmaktadır. Emmy Noether, artan zincir koşulunu sağlayan değişmeli halkalar üzerinde çalışmıştır. Bu halkalar günümüzde, Noether halkalar olarak adlandırılmaktadır. Noether'in çalışmalarından etkilenen Emil Artin, Wedderburn'ün cebirler üzerindeki yapısal teoremlerini azalan zincir koşulunu sağlayan değişmeli olmayan halkalara genellemiştir. Literatürde bu halkalar, Artin halkalar olarak bilinmektedir.

Von Neumann düzenli halkalar, 1930'ların ortalarında, von Neumann tarafından operatör cebirlerin projeksiyon latislerini çalışmak için cebirsel bir çerçeve oluşturmak amacıyla tanımlanmıştır. Von Neumann'ın oluşturduğu bu cebirsel çerçeve projektif geometride kullanılmıştır.

Artin ve (von Neumann) düzenli halkalar pek çok yazarın ilgi odağı olmuş ve araştırmacılar tarafından farklı şekillerde genelleştirilmiştir (bkz. [7], [19], [23], [30], [43]). Tam (perfect), yarıtam (semiperfect) ve yarıdüzenli (semiregular) halkalar bu genellemelerin bazılarıdır.

Bu tezin amacı tam ve (von Neumann) düzenli halkalar ile bu halkaların bazı genellemelerini, bir altmodüle göre yarıdüzenli ya da yarıtam olan modüller ile bazı zincir koşulları göz önüne alınarak karakterize etmek; bir takım özel idealler ele alınarak tanımlanan yarıdüzenli ve yarıtam halkaları incelemektir.

Bu tezin birinci bölümü, çalışmalarımızda ihtiyaç duyulan bazı temel kavramların ve yukarıda söz konusu olan halka ile modüllerin tanıtılmasına ayrılmıştır. Bu çalışmada, R değişmeli olması gerekmeyen birimli ve birleşmeli bir halkayı temsil etmektedir. Modüller aksi belirtilmedikçe birimsel sağ R-modüller olacaktır.

Yarıdüzenli ve yarıtam halka kavramları, Yousif ve Zhou ([43]) tarafından, bir I ideali göz önüne alınarak I-yarıdüzenli ve I-yarıtam halkalara genelleştirilmiştir. Nicholson ve Zhou ([31]), I-yarıdüzenli ve I-yarıtam halkaları karakterize etmek amacıyla kuvvetli yükselten (strongly lifting) ideal (bkz. [25]) kavramı üzerinde çalışmışlardır. Nicholson ve Zhou'nun elde ettiği sonuçlara göre, bir R halkası I-yarıdüzenlidir (I-yarıtamdır) ancak ve ancak R/I halkası düzenlidir (yarıbasittir) ve I kuvvetli yükselten bir idealdir. I-yarıdüzenli ve I-yarıtam halka kavramları sağ-sol simetriktir.

Alkan ve Özcan ([1] ve [32]), bir modülün endomorfizmalar altında korunan bir U altmodülünü göz önüne alarak, I-yarıdüzenli ve I-yarıtam halkaların modül versiyonları olan U-yarıdüzenli ve U-yarıtam modülleri tanımlamışlar; I-yarıdüzenli ve I-yarıtam halkaların bazı özelliklerini modüllere taşımışlardır.

Tezin ikinci bölümünde, kuvvetli yükselten ideal tanımına paralel olarak kuvvetli yükselten altmodül tanımı verilmiş ve bu altmodüllerin bazı özellikleri incelenmiştir. U, bir M R-modülünün bir altmodülü olsun. $M/U = (A+U)/U \oplus (B+U)/U$ iken, $P \leq A$, $(A+U)/U = (P+U)/U$ ve $(B+U)/U = (Q+U)/U$ olacak şekilde $M = P \oplus Q$ ayrışımı varsa U altmodülüne *kuvvetli yükselten altmodül (strongly lifting submodule)* diyeceğiz. Bir R halkasının bir I ideali kuvvetli yükselten idealdir ancak ve ancak I, R sağ R-modülünün kuvvetli yükselten bir altmodülüdür. Yarı-projektif (self-projective) modüllerin dik toplananları kuvvetli yükselten altmodüllerdir. Ayrıca, yarı-projektif bir modülün sonlu değişim özelliğini (finite exchange property) sağlaması ile her altmodülünün kuvvetli yükselten olması denk ifadelerdir.

Bu bölümde, I-yarıtam halkaların karakterizasyonunda önemli bir rol oynayan I-yarıgüçlü ideal kavramı modüllere taşınmıştır. $A \nsubseteq U$ koşulunu sağlayan M'nin her A altmodülü için $B \leq A$ ve $B \nsubseteq U$ olacak şekilde M'nin bir B dik toplananı varsa M modülüne *U-yarıgüçlü (U-semipotent)* denir. M U-yarıgüçlü ve U, M'nin kuvvetli yükselten bir altmodülü ise M'ye *U-güçlü (U-potent)* denir. U izdüşümler altında korunan bir altmodül ve M U-yarıgüçlü bir modül ise M/U 0-güçlüdür. Bu ifadenin tersi U'nun kuvvetli yükselten bir altmodül olması durumunda doğrudur. M U-yarıgüçlü bir modül ise $N \nsubseteq U$ olan M'nin her N altmodülünün ayrıştırılamaz (indecomposable) olması ile yerel (local) olması denktir.

U-yarıgüçlü modüller yardımıyla U-yarıtam modüller karakterize edilmiştir. U, izdüşümler altında korunan bir altmodül olmak üzere, kuvvetli yükselten altmodüller göz önüne alınarak, U-yarıdüzenli ve U-yarıtam modüllerin yeni karakterizasyonları elde edilmiştir. M sonlu üretilmiş ve projektif bir R-modül ise, M U-yarıdüzenlidir ancak ve ancak M/U modülünün sonlu üretilmiş her altmodülü bir dik toplanandır ve U kuvvetli yükselten bir altmodüldür. M projektif bir R-modül ise, M U-yarıtamdır ancak ve ancak M/U yarıbasittir ve U kuvvetli yükselten bir altmodüldür. Bu bölümün son kısmında, τ bir preradikal olmak üzere, her (projektif) modülü $\tau()$-yarıtam olan halkalar incelenmiştir. Jacobson radikali Rad, Goldie torsion altmodülü Z_2 ve δ gibi bazı özel preradikaller göz önüne alınarak şu sonuçlar elde edilmiştir: Her R-modül M $Z_2(M)$-yarıtamdır ancak ve ancak R Z_2^r-yarıtam halkadır; her projektif R-modül M $\delta(M)$-yarıtamdır ancak ve ancak R δ-tam halkadır; bir R halkası Z_r-yarıtamdır ve Z_2^r injektiftir ancak ve ancak R halkası yarıtam ve sağ injektiftir.

Xiao ve Tong'un tanımlamış olduğu genelleştirilmiş yarıdüzenli halkalar (generalized semiregular rings), yarıdüzenli ve AP-injektif halkaları kapsamaktadır ([42]). Aynı çalışmada, söz konusu yazarlar, yarıdüzenli ve AP-injektif halkalar için bilinen bazı sonuçları genellemişlerdir. Tezin üçüncü bölümünde, bazı (özel) idealler göz önünde bulundurularak tanımlanan halka ve modüller, bu genellemelerden yararlanılarak, incelenmiştir.

F, bir M R-modülünün bir altmodülü olsun. [1]'e göre, bir M modülü F-yarıdüzenlidir ancak ve ancak her $m \in M$ için P projektif, $P \subseteq mR$ ve $Q \cap mR \subseteq F$ olacak şekilde $M = P \oplus Q$ ayrısımı vardır. Eğer F, M'nin endomorfizmaları altında korunan bir altmodül ise, M F-yarıdüzenlidir ancak ve ancak her $m \in M$ için P M'nin projektif dik toplananı ve $S \subseteq F$ olacak biçimde $mR = P \oplus S$ ayrışımı vardır. Üçüncü bölümde, yaklaşık yarıdüzenli modüller şu şekilde tanımlanmıştır: M bir R-modül, $S = End_R(M)$ ve F M'nin bir S-altmodülü olsun. Her $m \in M$ için, $P \subseteq Sm$ ve $Q \cap Sm \subseteq F$ olacak şekilde $l_M r_R(m) = P \oplus Q$ sol S-modül ayrışımı var ise M'ye *yaklaşık F-yarıdüzenli (almost F-semiregular)* denir. I, R halkasının bir ideali olsun. Bir R halkası bir sağ R-modül olarak yaklaşık I-yarıdüzenli ise, R'ye *sağ yaklaşık I-yarıdüzenli* halka denir. Yaklaşık I-yarıdüzenli halka kavramının sağ-sol simetrik olup olmadığı bilinmemektedir. $S = End_R(M)$ olmak üzere, ${}_SM$ F-yarıdüzenli bir modül ise, M_R yaklaşık F-yarıdüzenlidir. Bir APQ-injektif modül, her S-altmodülü F için, yaklaşık F-yarıdüzenlidir. Dahası, M_R APQ-injektiftir ancak ve ancak M_R yaklaşık 0-yarıdüzenlidir. Özel olarak, bir R halkasının AP-injektif olması için gerek ve yeter koşul R'nin sağ yaklaşık 0-yarıdüzenli olmasıdır. Sağ $J(R)$-yarıdüzenli halkalar, Xiao ve Tong'un tanımlamış olduğu genelleştirilmiş yarıdüzenli halkalardır.

Tezin bu bölümünde, F-yarıdüzenli modüllerle ilgili yeni karakterizasyonlar elde edilmiştir. Sağ yaklaşık I-yarıdüzenli halkaların ne zaman I-yarıdüzenli olabileceği sorusu üzerinde durulmuş ve [42]'de elde edilen bazı sonuçlar sağ yaklaşık I-yarıdüzenli halkalara genelleştirilmiştir. R halkasının bir e eşkare elemanı $eR = eRe$ ya da $ReR = R$ koşulunu sağlıyorsa R halkasının sağ yaklaşık I-yarıdüzenli olması eRe halkasının da sağ yaklaşık eIe-yarıdüzenli olmasını gerektirir. I, R halkasının bir ideali olmak üzere, $\mathbb{M}_n(R)$ matris halkası sağ yaklaşık $\mathbb{M}_n(I)$-yarıdüzenli ise, R de sağ yaklaşık I-yarıdüzenlidir. Alkan ve Özcan ([1, Sonuç 4.6]), projektif bir M R-modülü $Soc(M)$-yarıdüzenli ise M'nin düzenli olduğunu göstermişlerdir. Yaklaşık yarıdüzenli modüller için ise şu sonucun var olduğu gözlemlenmiştir: M_R yaklaşık $Soc({}_SM)$-yarıdüzenli ise, M yaklaşık yarıdüzenlidir; yani her $m \in M$ için $P \subseteq Sm$ ve $Q \cap Sm \ll {}_SM$ olacak şekilde $l_M r_R(m) = P \oplus Q$ sol S-modül ayrışımı vardır. $Rad({}_SM) \ll {}_SM$ ise, yaklaşık yarıdüzenli modüller $Rad({}_SM)$-yarıdüzenli olan modüllerdir. Bir R halkası için aşağıdaki gerektirmelerin var olduğu gözlemlenmiştir:

S_l-yarıdüzenli $\Rightarrow$ sağ yaklaşık S_l-yarıdüzenli $\Rightarrow$ sağ yaklaşık yarıdüzenli $\Rightarrow$ sağ yaklaşık δ_r-yarıdüzenli ve sağ yaklaşık δ_l-yarıdüzenli.

Z_r-yarıdüzenli $\Rightarrow$ sağ yaklaşık Z_r-yarıdüzenli $\Rightarrow$ sağ yaklaşık yarıdüzenli $\Rightarrow$ sağ yaklaşık δ_r-yarıdüzenli ve sağ yaklaşık δ_l-yarıdüzenli.

Bu gerektirmelerin terslerinin doğru olması gerekmediğine dair örnekler verilmiştir.

Bir $e \in R$ eşkare elemanı için $J(eRe) = eJ(R)e$ eşitliğinin var olduğu iyi bilinen bir sonuçtur. Aynı soru δ ideali için de göz önüne alınmış ve $eR = eRe$ koşulunu sağlayan bir eşkare eleman için bile benzer sonucun doğru olmadığı görülmüştür. Ancak, bir e eşkare elemanı $ReR = R$ koşulunu sağlıyorsa $\delta_l(eRe) = e\delta_l e$ eşitliği sağlanır. Sonuç olarak, R sağ yaklaçık δ_l-yarıdüzenli bir halka ve $ReR = R$ ise, eRe halkası $\delta_l(eRe)$-yarıdüzenlidir.

Bir modülün öz geniş altmodülleri ve sıfırdan farklı dar altmodülleri o modülün dik toplanan olmayan altmodülleridir. Bu durumda, eğer bir modül dik toplanan olmayan altmodüller üzerinde artan (azalan) zincir koşulunu sağlıyorsa, o zaman geniş ve dar altmodülleri üzerinde de artan (azalan) zincir koşulunu sağlar. Ancak, bu basit gözlemin tersinin her zaman doğru olmadığına dair değişmeli ve (von Neumann) düzenli bir halka örneği vardır (Örnek 4.2.5). Goodearl'ün [17, Önerme 3.6] sonucundan yararlanılarak geniş altmodülleri üzerinde artan zincir koşulunu sağlayan modüller karakterize edilmiştir. Buna göre, bir M R-modülünün geniş altmodülleri üzerinde artan zincir koşulunu sağlaması için gerek ve yeter koşul $M/Soc(M)$'nin Noether modül olmasıdır. Armendariz, bu sonucun bir duali olarak, bir M R-modülünün geniş altmodülleri üzerinde azalan zincir koşulunu sağlaması ile $M/Soc(M)$'nin Artin olmasının denk ifadeler olduğunu ispatlamıştır (bkz. [5, Önerme 1.1]). Varadarajan ([38, Önteorem 2.1]) ise, bir M R-modülünün dar altmodülleri üzerinde artan zincir koşulunu sağlaması ile $Rad(M)$'nin Noether olmasının denk olduğunu göstermiştir. Bu sonucun duali Al-Khazzi ve Smith ([2, Teorem 5]) tarafından ispatlanmıştır. Yani, bir M R-modülü dar altmodülleri üzerinde azalan zincir koşulunu sağlar ancak ve ancak $Rad(M)$ Artindir.

Tezin dördüncü ve son bölümünde, dik toplanan olmayan altmodüller üzerinde azalan ve artan zincir koşullarını sağlayan modüller, bazı modül sınıfları ele alınarak, detaylı olarak incelenmiştir. Dik toplanan olmayan altmodüller üzerinde artan ya da azalan zincir koşullarını sağlayan modüller yardımıyla Noether ve Artin modüllerle ilgili bazı karakterizasyonlar elde edilmiştir. Bir M R-modülünün Noether (Artin) ya da yarıbasit olması için gerek ve yeter koşul M'nin dik toplanan olmayan altmodüller üzerinde artan (azalan) zincir koşulunu sağlamasıdır. R bir sağ Noether halka ise, bir M R-modülü dik toplanan olmayan sonlu üretilmiş altmodüller üzerinde artan

zincir koşulunu sağlar ancak ve ancak M dik toplanan olmayan altmodüller üzerinde artan zincir koşulunu sağlar. Bir M modülü dik toplanan olmayan sonlu üretilmiş altmodüller üzerinde azalan zincir koşulunu sağlar ancak ve ancak M yerel Artin bir modüldür. Bir R halkasının sol tam olması ile devirli sağ idealler üzerinde azalan zincir koşulunu sağlamasının denk ifadeler oluşu iyi bilinen bir sonuçtur. Buradan, bir R halkası sol tamdır ancak ve ancak R, hem dik toplanan sağ idealler hem de dik toplanan olmayan devirli sağ idealler üzerinde azalan zincir koşulunu sağlar. Ayrıca, R halkası dik toplanan olmayan devirli sağ idealler üzerinde azalan zincir koşulunu sağlıyorsa R, Jacobson radikali sol T-üstelsıfır olan yarıdüzenli bir halkadır.
Anahtar Kelimeler: (Yarı)tam halkalar, (yarı)düzenli halkalar, Artin halkalar ve modüller, Noether halkalar ve modüller, injektif modüller, projektif modüller.
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Contents

NOTATION

R	A noncommutative and associative ring with identity
M_R	A unitary right R-module
$Mod-R$	The category of right R-modules
$\oplus_{i\in I}M_i$	Direct sum of R-modules M_i
$\Pi_{i\in I}M_i$	Direct product of R-modules M_i
$M^{(I)}$	$\oplus_{i\in I}M_i$, $M_i = M$
M^I	$\Pi_{i\in I}M_i$, $M_i = M$
$r_R(m)$	$\{r \in R \mid mr = 0\}$, the right annihilator of $m \in M$ in R
$l_M(a)$	$\{m \in M \mid ma = 0\}$, the left annihilator of $a \in R$ in M
$I \lhd R$	An ideal I of a ring R
$A \leq M$	A submodule A of an R-module M
$A \leq_e M$	An essential submodule A of an R-module M
$A \leq^{\oplus} M$	A direct summand A of an R-module M
$A \leq_c M$	A closed submodule A of an R-module M
$A \ll M$	A small submodule A of an R-module M
$A \ll_\delta M$	A δ-small submodule A of an R-module M
$Rad(M)$	Jacobson radical of an R-module M
$J(R)$	Jacobson radical of a ring R
$Soc(M)$	$\cap\{N \mid N \leq_e M\} = \oplus\{L \mid L$ is a simple submodule of $M\}$
S_r, S_l	$Soc(R_R)$, $Soc(_RR)$
$Z(M)$	$\{m \in M \mid r_R(m) \leq_e R\}$
Z_r, Z_l	$Z(R_R)$, $Z(_RR)$
$Z_2(M)$	$\{x \in M \mid$ there exists $I \leq_e R_R$ such that $xI \subseteq Z(M)\}$
Z_2^r	$Z_2(R_R)$
$\delta(M)$	$\cap\{K \leq M \mid M/K$ is a simple singular R-module$\}$
δ_r, δ_l	$\delta(R_R)$, $\delta(_RR)$
$Hom_R(M,N)$	The set of R-module homomorphisms from M to N
$End_R(M)$	The set of R-endomorphisms of M
$Ker(f)$	Kernel of an R-homomorphism f
$Im(f)$	Image of an R-homomorphism f

$\mathbb{Z}$	The set of integers
$\mathbb{Q}$	The set of rational numbers
$\mathbb{Z}_n$	The cyclic group $\mathbb{Z}/n\mathbb{Z}$
$\mathbb{Z}_{p^\infty}$	Prüfer p-group
$\mathbb{M}_n(R)$	The set of $n \times n$ matrices over a ring R

1 PRELIMINARIES

In this chapter, we shall give some basic notions and their properties which will be frequently used. Throughout this dissertation, R will stand for a noncommutative and associative ring with identity. Modules will be unital right R-modules unless otherwise stated.

1.1 Basic Notions

The basic notions and the basic properties mentioned in this section may be found in any standard text of Ring and Module Theory (e.g. [3], [18], [21]).

Definitions 1.1.1 A submodule N of an R-module M is called *essential in* M or M is called *an essential extension of* N if every nonzero submodule of M intersects N nontrivially, and we write $N \leq_e M$. If every nonzero submodule of an R-module is essential in M, then we say that M is *uniform*.

Proposition 1.1.2 *Let M be an R-module. Then the following statements hold:*

1) $K \leq_e M$ if and only if for each $0 \neq x \in M$ there exists an $r \in R$ such that $0 \neq rx \in K$.

2) Let $A \leq B \leq M$. Then $A \leq_e M$ if and only if $A \leq_e B \leq_e M$.

3) If $A \leq_e B \leq M$ and $A' \leq_e B' \leq M$, then $A \cap A' \leq_e B \cap B'$.

4) If $f : N \to M$ and $A \leq_e M$, then $f^{-1}(A) \leq_e N$.

5) If $\{A_\alpha\}_{\alpha \in I}$ is an independent family of submodules of M, and if $A_\alpha \leq_e B_\alpha \leq M$ for each α, then $\{B_\alpha\}_{\alpha \in I}$ is an independent family and $\oplus A_\alpha \leq_e \oplus B_\alpha$. Conversely, if $\oplus A_\alpha \leq_e \oplus B_\alpha$, then $A_\alpha \leq_e B_\alpha$ for each α.

Definitions 1.1.3 A submodule N of an R-module M is called a *complement* if there exists a submodule N' of M such that N is maximal with respect to the property that $N \cap N' = 0$. If N is a complement of a submodule of M, then we say that N is *closed in* M , and write $N \leq_c M$.

By Zorn's Lemma, every submodule of an R-module M has a complement in M.

Proposition 1.1.4 *Let M be an R-module and $N \leq M$. If N' is a complement of N in M, then $N \oplus N' \leq_e M$.*

Proposition 1.1.5 *The following statements are equivalent for a submodule C of an R-module M:*

1) C is a closed submodule of M.

2) C has no proper essential extension; i.e., if $C \leq_e N \leq M$, then $C = N$.

3) If $C \leq N \leq_e M$, then $N/C \leq_e M/C$.

4) If D is a complement of C in M, then C is a complement of D in M.

The additive group of all the homomorphisms from an R-module M to an R-module N (i.e., R-homomorphisms) is denoted by $Hom_R(M, N)$; R-endomorphisms on M is denoted by $End_R(M)$. An R-homomorphism f is called a *monomorphism* if it is one-to-one; f is called an *epimorphism* if it is onto; and f is called an *isomorphism* if it is both one-to-one and onto. A submodule N of an R-module M is said to be *fully-invariant* if $f(N) \leq N$ for any $f \in End_R(M)$.

A *simple module* is a nonzero module M in which the only submodules are 0 and M. If M is an R-module, then *the socle of* M is defined by $Soc(M) = \cap\{N|N \leq_e M\} = \oplus\{L|L$ is a simple submodule of $M\}$. $Soc(M)$ is a fully-invariant submodule of M. An R-module M is called *semisimple* if $Soc(M) = M$. If a module M does not contain any simple submodules, then we may assume that $Soc(M) = 0$. $S_r = Soc(R_R)$ and $S_l = Soc(_RR)$ are ideals of a ring R, and they need not be equal. For instance, if R is the ring of 2×2 upper triangular matrices over a field, then $S_r \neq S_l$. If R_R is semisimple, then R is called a *(right) semisimple* ring. R_R is semisimple if and only if $_RR$ is semisimple.

The next definition dualizes the notion of an essential extension:

Definitions 1.1.6 A submodule N of an R-module M is called a *small (superfluous) submodule* of M provided $K + N$ is a proper submodule of M whenever K is a proper submodule of M, and it is denoted by $N \ll M$. If every proper submodule of M is small, then M is called a *hollow module*.

Proposition 1.1.7 *Let M be an R-module. Then the following statements hold for $K \leq N \leq M$ and $H \leq M$.*

1) $N \ll M$ if and only if $K \ll M$ and $N/K \ll M/K$.

2) $H + K \ll M$ *if and only if* $H \ll M$ *and* $K \ll M$.

3) If $K \ll M$ *and* $f : M \to N$ *is an R-homomorphism, then* $f(K) \ll N$. *In particular, if* $K \ll M \leq N$, *then* $K \ll N$.

4) Let $K_1 \leq M_1 \leq M$, $K_2 \leq M_2 \leq M$ *and* $M = M_1 \oplus M_2$. *Then* $K_1 \oplus K_2 \ll M$ *if and only if* $K_1 \ll M_1$ *ve* $K_2 \ll M_2$.

5) If $K \leq L \leq M$, $K \ll M$ *and* $L \leq^{\oplus} M$, *then* $K \ll L$. *In particular, if* $K \ll M$ *and* $K \leq^{\oplus} M$, *then* $K = 0$.

The Jacobson radical $Rad(M)$ of an R-module M is the intersection of all maximal submodules of M, or equivalently is the sum of all small submodules of M. $Rad(M)$ is a fully-invariant submodule of M. The Jacobson radical of a ring R is denoted by $J(R)$, and it is an ideal of R.

Lemma 1.1.8 (Nakayama's Lemma) *The following statements are equivalent for an ideal* I *of a ring* R:

1) $I \leq J(R)$.

2) If $MI = M$, *then* $M = 0$ *for every finitely generated R-module* M.

3) $MI \ll M$ *for every finitely generated R-module* M.

A nonzero R-module M is said to be *indecomposable* if it is not a direct sum of two nonzero submodules; and M is called *local* if it has a largest proper submodule (namely $Rad(M)$). A local module is indecomposable.

Theorem 1.1.9 ([41]) *Let* M *be a nonzero R-module. Then* M *is local if and only if* M *is cyclic and hollow.*

Let I be an ideal of a ring R. I is said to be *prime* if, for ideals A and B of R, $A \subseteq I$ or $B \subseteq I$ whenever $AB \subseteq I$. A ring R is called a *prime ring* if the zero ideal is a prime ideal of R. An ideal of a ring is called *semiprime* if it is an intersection of prime ideals. R is said to be a *semiprime ring* if the zero ideal is semiprime. A ring with a zero Jacobson radical is a semiprime ring. In particular, semisimple rings and regular rings (see Section 1.4) are all semiprime.

Let M be an R-module. The set $ann_R(M) = \{r \in R | Mr = 0\}$ is the *annihilator* of M in R. M is called *faithful* if $ann_R(M) = 0$. If M is nonzero, then M is a faithful $R/ann_R(M)$-module. For any $m \in M$, the set $r_R(m) = \{r \in R | mr = 0\}$ is the annihilator of m in R, and it is a right ideal of R. For any $a \in R$, the set $l_M(a) = \{m \in M | ma = 0\}$ is the annihilator of a in M.

Definition 1.1.10 *Given any right R-module M, the* singular submodule *of M is the set*

$$Z(M) = \{m \in M \mid mI = 0 \textit{ for some essential right ideal } I \textit{ of } R \}.$$

Equivalently, $Z(M)$ is the set of those $m \in M$ for which the right ideal $r_R(m)$ is essential in R.

Lemma 1.1.11 *The following statements hold for an R-module M:*

1) $Z(M)S_r = 0$.

2) If $f : M \to N$ is an R-homomorphism, then $f(Z(M)) \leq Z(N)$.

3) If $M \leq N$, then $Z(M) = M \cap Z(N)$.

Corollary 1.1.12 *The following statements hold for a ring R:*

1) $Z_r = Z(R_R)$ is an ideal of R.

2) If $R \neq 0$, then $Z_r \neq R$.

3) Z_r does not contain any nonzero idempotent.

An R-module M is called a *singular module* provided $Z(M) = M$; and it is called a *nonsingular module* provided $Z(M) = 0$. Thus the ring R is a nonsingular right module if and only if $Z_r = 0$, and in this case R is called a *right nonsingular ring*. Likewise, we say that R is a *left nonsingular ring* if $Z_l = 0$. Right and left nonsingular rings are not equivalent (see [18, Exercise 1]).

Let M be an R-module and $N \leq M$. M/N is singular whenever $N \leq_e M$. The converse of this can easily fail; for example, let $M = \mathbb{Z}/2\mathbb{Z}$ and $N = 0$. M/N is a singular $\mathbb{Z}$-module but N is not an essential submodule of M. However, there are two special cases in which the converse does work.

Proposition 1.1.13 *Let M be nonsingular and let $N \leq M$. Then M/N is singular if and only if $N \leq_e M$.*

Note that the definition of projectivity will be given in Section 1.3.

Proposition 1.1.14 *Let P be a projective R-module and let $X \leq P$. Then P/X is singular if and only if $X \leq_e P$. In particular, if P is both projective and singular, then $P = 0$.*

The class of all nonsingular R-modules is closed under submodules, direct products, essential extensions and module extensions. The class of all singular R-modules is closed under submodules, factor modules and direct sums.

Proposition 1.1.15 *If M is any simple R-module, then M is either singular or projective, but not both.*

We easily obtain the following from Proposition 1.1.14.

Corollary 1.1.16 *If every cyclic right ideal of a ring R is projective, then R is a right nonsingular ring.*

Definition 1.1.17 Given any module M, define a submodule $Z_2(M)$ by the rule $\frac{Z_2(M)}{Z(M)} = Z(\frac{M}{Z(M)})$. $Z_2(M)$ is called the *Goldie torsion submodule* of M; and $Z_2^r = Z_2(R_R)$ is called the *Goldie torsion ideal* of a ring R.

Equivalently, the Goldie torsion submodule can be defined as follows:

$$Z_2(M) = \{x \in M \mid xI \subseteq Z(M) \text{ for some } I \leq_e R_R\}.$$

Lemma 1.1.18 *The following statements hold for an R-module M:*

1) If $N \leq M$, then $Z_2(N) = N \cap Z_2(M)$.

2) $Z(M) \leq_e Z_2(M)$.

3) $Soc(Z_2(M)) \subseteq Z(M)$.

4) $Z_2(M) \leq_c M$.

5) $M/Z_2(M)$ is nonsingular.

As a generalization of small submodules, δ-small submodules were introduced by Zhou ([46]). Various properties of δ-small submodules are given below. We refer the reader to [46] for all the unproved properties.

Definition 1.1.19 Let N be a submodule of an R-module M. N is said to be *δ-small* in M if $N + K \neq M$ for any proper submodule K of M with M/K singular. We use $N \ll_\delta M$ to indicate that N is a δ-small submodule of M.

Every small submodule or nonsingular semisimple submodule of M is δ-small in M. The δ-small submodules of a singular module are small submodules.

The next lemma explains how close the notion of δ-small submodules is to small submodules.

Lemma 1.1.20 *Let N be a submodule of an R-module M. Then the following statements are equivalent:*

1) $N \ll_\delta M$.

2) If $X + N = M$, then $M = X \oplus Y$ for a projective semisimple submodule Y with $Y \subseteq N$.

3) If $X + N = M$ with M/X Goldie torsion, then $X = M$.

Lemma 1.1.21 *Let M be an R-module. The following statements hold:*

1) For submodules N, K, L of M with $K \subseteq N$, we have

(a) $N \ll_\delta M$ if and only if $K \ll_\delta M$ and $N/K \ll_\delta M/K$.

(b) $N + L \ll_\delta M$ if and only if $N \ll_\delta M$ and $L \ll_\delta M$.

2) If $K \ll_\delta M$ and $f : M \to N$ is an R-homomorphism, then $f(K) \ll_\delta N$. In particular, if $K \ll_\delta M \subseteq N$, then $K \ll_\delta N$.

3) Let $K_1 \subseteq M_1 \subseteq M$, $K_2 \subseteq M_2 \subseteq M$ and $M = M_1 \oplus M_2$. Then $K_1 \oplus K_2 \ll_\delta M_1 \oplus M_2$ if and only $K_1 \ll_\delta M_1$ and $K_2 \ll_\delta M_2$.

The Jacobson radical $Rad(M)$ of an R-module M has the following characterizations:

$Rad(M) = \cap\{Ker(f) | f : M \to S$ is an R-homomorphism, S is a simple R-module$\}$

$= \cap\{K \leq M | K$ is a maximal submodule of $M\}$

$= \sum\{L \leq M | L \ll M\}$.

Parallel to the definition of the Jacobson radical, the submodule $\delta(M)$ of an R-module M is defined to be the set $\cap\{K \leq M | M/K$ is a simple singular R-module$\}$. Then $Rad(M) \subseteq \delta(M)$; and $Soc(M) \subseteq \delta(M)$ if M is a projective R-module.

Proposition 1.1.22 *Let M be an R-module. Each of the following sets is equal to $\delta(M)$.*

1) $M_1 = \cap\{Ker(f) | f : M \to S$ is an R-homomorphism , S is a simple singular R-module$\}$.

2) $M_2 = \sum\{L \leq M | L \ll_\delta M\}$.

Lemma 1.1.23 *Let M and N be R-modules.*

1) If $f : M \to N$ is an R-homomorphism, then $f(\delta(M)) \subseteq \delta(N)$. Therefore, $\delta(M)$ is a fully-invariant submodule of M and $M\delta(R_R) \subseteq \delta(M)$.

2) If $M = \oplus_{i\in I} M_i$, then $\delta(M) = \oplus_{i\in I}\delta(M_i)$.

3) If every proper submodule of M is contained in a maximal submodule of M, then $\delta(M)$ is the unique largest δ-small submodule of M.

Theorem 1.1.24 *Given a ring R, each of the following sets is equal to $\delta(R_R)$.*

1) $R_1 =$ the intersection of all essential maximal right ideals of R.

2) $R_2 =$ the unique largest δ-small right ideal of R.

3) $R_3 = \{x \in R | xR + K_R = R \Rightarrow K_R \leq^{\oplus} R_R\}$.

4) $R_4 = \cap\{P \lhd R | R/P$ has a faithful singular simple module $\}$.

5) $R_5 = \{x \in R | \forall y \in R, \exists$ a semisimple right ideal Y of R such that $(1 - xy)R \oplus Y = R_R\}$.

Corollary 1.1.25 *For a ring R, $\delta(R_R)/Soc(R_R) = J(R/Soc(R_R))$. In particular, $R = \delta(R_R)$ if and only if R is a semisimple ring.*

Note: $\delta(R_R)$ and $\delta({}_RR)$ need not be equal for a ring R. For instance, if R is the ring of 2×2 upper triangular matrices over a field, then $\delta(R_R) \neq \delta({}_RR)$.

1.2 Injective Modules and Some of Their Generalizations

In this section, we will introduce injective modules and give some basic properties of these modules. Also, we will deal with some generalizations of injective modules which will be used in the third chapter.

Definition 1.2.1 An R-module M is called *N-injective* (or *injective relative to N*), if every R-homomorphism from a submodule K of N to M can be lifted to an R-homomorphism from N to M. An R-module M is called an *injective module* if it is N-injective for every R-module N.

If an R-module M is M-injective, then it is called *quasi-injective.* A ring R is called *right self injective* if it is R-injective as a right R-module. By Baer's criterion, an R-module M is injective if and only if M is R-injective.

Direct products and direct summands of injective modules are injective. On the other hand, it is not true that every direct sum of injective modules is injective. Indeed, it is precisely the Noetherian rings (see Section 1.9) over which every direct sum of injectives is injective.

Proposition 1.2.2 ([24]) *Any (quasi-)injective R-module M satisfies the following two conditions:*

$(C1)$ *Every submodule of M is essential in a summand of M;*

$(C2)$ *If a submodule N of M is isomorphic to a summand of M, then N is a summand of M.*

An R-module M which satisfies $(C1)$ is also known as an *extending module* or a *CS module* in the literature.

Proposition 1.2.3 ([24]) *If an R-module M satisfies $(C2)$, then it satisfies the following condition:*

$(C3)$ *If M_1 and M_2 are summands of M such that $M_1 \cap M_2 = 0$, then $M_1 \oplus M_2$ is a summand of M.*

Definition 1.2.4 ([24]) An R-module M is called *continuous* if it satisfies $(C1)$ and $(C2)$; M is called *quasi-continuous* if it satisfies $(C1)$ and $(C3)$.

The following implications hold for an R-module:

Injective $\Rightarrow$ quasi-injective $\Rightarrow$ continuous $\Rightarrow$ quasi-continuous $\Rightarrow$ $(C1)$.

The Baer criterion for testing when an R-module M is injective naturally leads one to consider the case when one can extend any R-homomorphism from any cyclic right ideal of R to M to an R-homomorphism of R to M.

Definition 1.2.5 ([23]) An R-module M is said to be *principally injective*, or *P-injective* for short, if every map from any cyclic right ideal to M extends to a map of R to M.

An R-module M is P-injective if and only if $l_M r_R(a) = Ma$ for all $a \in R$, where l and r are the left and right annihilators, respectively. When the ring is P-injective as a right R-module, the ring is said to be a *right P-injective ring*.

Definition 1.2.6 ([35]) Given a module M_R, let $S = End_R(M)$. The module M is said to be *almost principally injective* (or *AP-injective* for short) if, for any $a \in R$, there exists an S-submodule X_a of M such that $l_M r_R(a) = Ma \oplus X_a$ as left S-modules.

If R_R is an AP-injective module, then R is called a *right AP-injective ring*. It immediately follows from the definitions that a P-injective module is AP-injective.

Definition 1.2.7 ([45]) Given a module M_R, let $S = End_R(M)$. The module M is called *almost principally quasi-injective* (or APQ-injective for short) if, for any $m \in M$, there exists an S-submodule X_m of M such that $l_M r_R(m) = Sm \oplus X_m$.

It is easy to observe that R_R is APQ-injective if and only if R_R is AP-injective.

1.3 Projective Modules

In this section, we refer the reader to [3] for the unproved properties of projective modules.

Definitions 1.3.1 Let M be an R-module. If N is an R-module, then M is *N-projective* (or *projective relative to N*) in case for each R-epimorphism $g : N \to K$ and each R-homomorphism $h : M \to K$ there is an R-homomorphism $f : M \to N$ such that $g \circ f = h$. An R-module P is called *projective* in case it is projective relative to every R-module. If an R-module M is M-projective, then M is called a *quasi-projective module*.

Direct sums and direct summands of projective modules are projective. A ring is a projective module over itself. Every free module is a projective module, too.

Proposition 1.3.2 *The following statements are equivalent for an R-module P:*

1) P is projective.

2) Every epimorphism $M \to P \to 0$ splits.

3) P is isomorphic to a direct summand of a free R-module.

We have the following useful result for quasi-projective modules.

Proposition 1.3.3 ([41, 41.14]) *Let M be a quasi-projective R-module. If $M = U + V$ and $U \leq^{\oplus} M$, then there exists a submodule V' of V such that $M = U \oplus V'$.*

A pair (P, p) is a *projective cover* of an R-module M in case P is a projective R-module and the epimorphism $p : P \to M$ has small kernel; i.e, $Ker(p) \ll P$. We may call P itself a projective cover of M.

Every projective R-module is a projective cover of itself. The $\mathbb{Z}$-modules $\mathbb{Z}_2$ and $\mathbb{Q}$ have no projective covers.

Proposition 1.3.4 *If $p_i : P_i \to M_i$ $(i = 1, \ldots, n)$ are projective covers, then $p = (\oplus_i p_i) : \oplus_i P_i \to \oplus_i M_i$ is a projective cover.*

The next lemma is The Fundamental Lemma for Projective Covers. One of its consequences is that if a module does have a projective cover, then it has (essentially) only one.

Lemma 1.3.5 *Suppose M_R has a projective cover $p : P \to M$. If Q_R is projective and $q : Q \to M$ is an epimorphism, then Q has a decomposition $Q = P \oplus P'$ such that*

1) $P \cong P'$;

2) $P'' \leq Ker(q)$;

3) $(q|_{P'}) : P' \to M$ is a projective cover for M.

Moreover, if $f : M_1 \to M_2$ is an isomorphism and if $p_1 : P_1 \to M_1$ and $p_2 : P_2 \to M_2$ are projective covers, then there is an isomorphism $f' : P_1 \to P_2$ such that $p_2 f' = f p_1$.

The Jacobson radical of a projective R-module is $Rad(P) = PJ(R)$. A similar result holds for $\delta(P)$ submodule as the next result shows.

Lemma 1.3.6 ([46]) *If P is a projective R-module, then $\delta(P) = P\delta(R_R)$ and $\delta(P)$ is the intersection of all essential maximal submodules of P.*

Definition 1.3.7 ([46]) A pair (P, p) is called a *projective δ-cover* of an R-module M if P is projective and p is an epimorphism of P onto M with $Ker(p) \ll_\delta P$.

The notion of projective δ-covers was introduced by Zhou ([46]) as a generalization of the notion of projective covers. Unlike projective covers, the projective δ-covers of a module are not unique up to isomorphism, but they differ by only a projective semisimple direct summand.

Every projective cover of an R-module M is a projective δ-cover. Some modules may not have projective δ-covers and some modules have projective δ-covers but no projective covers (see [46]).

Lemma 1.3.8 ([46]) *If $p_i : P_i \to M_i$ $(i = 1, \ldots, n)$ are projective δ-covers, then $p = (\oplus_i p_i) : \oplus_i P_i \to \oplus_i M_i$ is a projective δ-cover.*

Lemma 1.3.9 ([46]) *Suppose M_R has a projective δ-cover $p : P \to M$. If Q_R is projective and $q : Q \to M$ is an epimorphism, then there exist decompositions $P = A \oplus B$ and $Q = X \oplus Y$ such that*

1) $A \cong X$;

2) $p|_A : A \to M$ is a projective δ-cover;

3) $q|_X : X \to M$ is a projective δ-cover;

4) B is a projective semisimple module with $B \subseteq Ker(p)$ and $Y \subseteq Ker(q)$.

Lemma 1.3.10 ([46]) *Let P be a projective R-module and N a submodule of P. Then the following statements are equivalent:*

1) P/N has a projective δ-cover.

2) $P = P_1 \oplus P_2$ for some P_1 and P_2 with $P_1 \subseteq N$ and $P_2 \cap N \ll_\delta P$.

1.4 Regular and (I-)Semiregular Rings

This section is concerned with regular rings and some of their generalizations.

Lemma 1.4.1 ([39]) *For a ring R, the following statements are equivalent:*

1) For any $a \in R$, there exists $b \in R$ such that $aba = a$.

2) Every cyclic right (left) ideal is a direct summand of R.

3) Every finitely generated right (left) ideal is a direct summand of R.

Definition 1.4.2 An element a of a ring R is called *regular* if there exists $b \in R$ such that $a = aba$. A ring R is called *(von Neumann) regular* if it satisfies the equivalent conditions of Lemma 1.4.1.

Definition 1.4.3 Let I be a right or a left ideal of a ring R. We say that *idempotents can be lifted modulo I* if, whenever $a^2 - a \in I$, $a \in R$, there exists $e^2 = e \in R$ such that $e - a \in I$.

This notion is extended as follows:

Lemma 1.4.4 ([31]) *The following are equivalent for a right ideal T of a ring R:*

1) If $a^2 - a \in T$, there exists $e^2 = e \in aR$ such that $e - a \in T$.

2) If $a^2 - a \in T$, there exists $e^2 = e \in aRa$ such that $e - a \in T$.

3) If $a^2 - a \in T$, there exists $e^2 = e \in Ra$ such that $e - a \in T$.

Definition 1.4.5 ([25]) We say that a right ideal T of a ring R is *strongly lifting*, or that *idempotents lift strongly modulo T*, if the conditions in Lemma 1.4.4 are satisfied.

Let I be an ideal of a ring R. If idempotents lift modulo I, then I need not be a strongly lifting ideal. For instance, suppose that 0 and 1 are the only idempotents in R and R/I. If $I \neq R$, and $I \nsubseteq J(R)$, then idempotents lift modulo I, but not strongly. More specifically, one can consider $R = \mathbb{Z}$ and $I = p^k\mathbb{Z}$, where p is a prime number and $k \geq 1$ (see [31, Example 2]).

Lemma 1.4.6 ([31, Lemma 5]) *If idempotents can be lifted in R modulo $J(R)$, then they can be lifted strongly modulo every one-sided ideal contained in $J(R)$.*

Theorem 1.4.7 ([31, Theorem 10]) *The right socle S_r is strongly lifting in any ring.*

Proposition 1.4.8 ([31, Proposition 11]) *Let I be an ideal of a ring R and write $\overline{a} = a + I$ for $a \in R$. Assume that I is strongly lifting and that $\{\overline{a_1}, \ldots, \overline{a_n}\}$ are orthogonal idempotents in R/I.*

1) For each $n \geq 1$, there exist orthogonal idempotents $\{e_1, \ldots, e_n\}$ in R such that $\overline{e_i} = \overline{a_i}$ and $e_i \in a_iR$ for each $i = 1, 2, \ldots, n$.

2) If I contains no nonzero idempotent and $\overline{a_1} + \cdots + \overline{a_n} = \overline{1}$, we can choose orthogonal idempotents $e_i \in a_iR$ for each i such that $e_1 + \cdots + e_n = 1$.

Lemma 1.4.9 ([31, Lemma 26]) *Let I be an ideal of a ring R. The following are equivalent for a right ideal T of R:*

1) $T = eR \oplus S$, where $e^2 = e$ and $S \subseteq I$ is a right ideal.

2) There exists $e^2 = e \in T$ with $(1 - e)T \subseteq I$.

3) There exists $e^2 = e \in T$ with $T \cap (1 - e)R \subseteq I$.

Following [31], if I is an ideal of a ring R, we say that I *respects* a right ideal $T \subseteq R$ if the conditions in Lemma 1.4.9 are satisfied.

Lemma 1.4.10 ([31, Lemma 27]) *Let I be an ideal of a ring R and $a \in R$. Then I respects aR if and only if I respects Ra.*

Definition 1.4.11 ([31]) Let I be an ideal of a ring R. An element a of a ring R is called *I-semiregular* if I respects aR. The ring R is called an *I-semiregular ring* if every element is I-semiregular.

Theorem 1.4.12 ([31, Theorem 28]) *The following conditions are equivalent for an ideal I of a ring R:*

1) R is I-semiregular.

2) I respects every finitely generated right (left) ideal of R.

3) R/I is regular and idempotents lift strongly modulo I.

Hence, every ring R is R-semiregular, and the 0-semiregular rings are just the regular rings.

An I-semiregular ring R is called *semiregular* in case $I = J(R)$. Combining Theorem 1.4.12 with Lemma 1.4.6, one can observe that a ring R is semiregular if and only if $R/J(R)$ is regular and idempotents lift modulo $J(R)$. A well known result of Utumi ([37]) asserts that if R is a right continuous ring (i.e., the ring which satisfies $(C1)$ and $(C2)$), then R is semiregular and $J(R) = Z_r$. The converse of this problem was studied in [29] by Nicholson and Yousif. They showed that a ring R is semiregular and $J(R) = Z_r$ if and only if R is Z_r-semiregular (see [29, Theorem 2.4]), and called such rings *right weakly continuous*. It follows from Theorem 1.4.7 that a ring R is S_r-semiregular if and only if R/S_r is regular. It was shown in [1, Corollary 4.6] that S_r-semiregular rings are semiregular. But the converse need not be true (e.g. consider the ring $\mathbb{Z}_8$). δ_r-semiregular rings were defined by Zhou ([46]) as a generalization of a semiregular ring: A ring R is said to be δ_r-semiregular if R/δ_r is regular and idempotents lift modulo δ_r.

We have the following implications for a ring R:

$$S_r\text{-semiregular} \Rightarrow \text{semiregular} \Rightarrow \delta_r\text{-semiregular};$$
$$Z_r\text{-semiregular} \Rightarrow \text{semiregular} \Rightarrow \delta_r\text{-semiregular}.$$

1.5 Perfect and (I-)Semiperfect Rings

In this section, we introduce perfect rings and some of their generalizations.

Definition 1.5.1 A set I is said to be *right T-nilpotent* if, for any sequence $\{a_i | i \geq 1\}$ in I, there exists an integer $n \geq 1$ such that $a_n a_{n-1} \cdots a_1 = 0$; I is said to be *left T-nilpotent* if $a_1 a_2 \cdots a_{n-1} a_n = 0$.

Definition 1.5.2 A ring R is called *right perfect* if $R/J(R)$ is semisimple and $J(R)$ is right T-nilpotent.

Theorem 1.5.3 ([7], [10]) *The following statements are equivalent for a ring R:*

1) R is right perfect.

2) Every right R-module has a projective cover.

3) Every semisimple right R-module has a projective cover.

4) R satisfies descending chain condition on cyclic left ideals.

5) R satisfies descending chain condition on finitely generated left ideals.

Note: The notion of perfect rings is not left-right symmetric (see [30, Example B.36]).

Proposition 1.5.4 ([30]) *Let R be a right perfect ring. Then the following statements hold:*

1) eRe is a right perfect ring for any $e^2 = e \in R$.

2) $\mathbb{M}_n(R)$ is a right perfect ring for $n \geq 1$.

3) R/A is a right perfect ring for any ideal A of R.

A ring R is called *semipotent* if each one-sided ideal not contained in $J(R)$ contains a nonzero idempotent, and R is called *potent* if, in addition, idempotents lift modulo $J(R)$. In [31], the semipotent rings are generalized as follows:

Lemma 1.5.5 ([31, Lemma 19]) *Let I be an ideal of a ring R. The following are equivalent:*

1) If $T \nsubseteq I$ is a right (respectively, left) ideal there exists $e^2 = e \in T - I$.

2) If $a \notin I$ there exists $e^2 = e \in aR - I$ (respectively, $e \in Ra - I$).

3) If $a \notin I$ there exists $x \in R$ such that $xax = x \notin I$.

Definition 1.5.6 ([31]) If R is a ring and I is an ideal of R, we say that R is I-*semipotent* if the conditions in Lemma 1.5.5 are satisfied, and say that R is I-*potent* if it is I-semipotent and idempotents lift strongly modulo I.

Semipotent (potent) rings are just the $J(R)$-semipotent ($J(R)$-potent) rings. Every regular ring is 0-potent.

Theorem 1.5.7 ([31]) *Let I be an ideal of a ring R. The following are equivalent:*

1) R/I is semisimple and idempotents lift strongly modulo I.

2) I respects every right (respectively, left) ideal of R.

3) I respects countably generated right (respectively, left) ideal of R.

4) R is I-semipotent and I respects $\oplus_{i=1}^{\infty} e_i R$ (respectively, $\oplus_{i=1}^{\infty} Re_i$) for any orthogonal idempotents $e_i \in R$.

5) R is I-semipotent and contains no infinite orthogonal family of idempotents outside I.

6) R is I-semipotent and R/I is semisimple.

If I is an ideal of a ring R, the ring R is called *I-semiperfect* if it satisfies the equivalent conditions in Theorem 1.5.7; R is called *semiperfect* in case $I = J(R)$. Theorem 1.5.7, together with Lemma 1.4.6, gives that a ring R is semiperfect if and only if $R/J(R)$ is semisimple and idempotents lift modulo $J(R)$. By Theorem 1.4.7, we obtain that R is S_r-semiperfect if and only if R/S_r is semisimple. It is proved in [31, Corollary 37] that a ring R is Z_r-semiperfect is equivalent to the fact that R is a semiperfect ring and $J(R) = Z_r$. Zhou ([46]) defined δ_r-semiperfect rings as a generalization of semiperfect rings: A ring R is called *δ_r-semiperfect* if R/δ_r is semisimple and idempotents lift modulo δ_r.

S_r-semiregular rings are semiregular (see [1, Corollary 4.6]), but this is not the case for S_r-semiperfect rings; i.e., S_r-semiperfect rings need not be semiperfect (see [46, Example 4.1]).

It easily follows from the definitions that semiperfect rings are semiregular. But the converse is not true in general. For example, consider the ring $R = \prod_{i\in I} F_i$, where $F_i = F$ is a field. R is a regular ring but it is not semisimple since $Soc(R) = \oplus_{i\in I} F_i$.

As a generalization of right perfect rings, Zhou ([46]) introduced *right δ-perfect* rings. Following [46], a ring R is called *right δ-perfect* if every right R-module has a projective δ-cover. Notice that we use the notation 'δ-perfect' instead of 'δ_r-perfect'. Since there is no definition of a perfect ring relative to an ideal, we do not have a notion of δ_r-perfect rings.

Theorem 1.5.8 ([46, Theorem 3.8]) *The following conditions are equivalent for a ring R:*

1) R is a right δ-perfect ring.

2) Every semisimple right R-module has a projective δ-cover.

3) R is a δ_r-semiperfect ring and $\delta(M) \ll_\delta M$ for any module M.

4) R/S_r is a right perfect ring and idempotents lift modulo δ_r.

1.6 Generalized Semiregular Rings

In 2005, Xiao and Tong ([42]) introduced *generalized semiregular rings* as a generalization of semiregular and AP-injective rings. In this section, all the results are taken from the paper [42]. We will use the notation $l(a)$ (respectively, $r(a)$) for the left (respectively, right) annihilator of an element a of a ring R.

Definition 1.6.1 An element a of a ring R is called *right generalized semiregular* if there exixts two left ideals P, L of R such that $lr(a) = P \oplus L$, where $P \subseteq Ra$ and $Ra \cap L \ll R$. A ring R is called *right generalized semiregular* if each of its elements is right generalized semiregular. *Left generalized semiregular elements* and *left generalized semiregular rings* can be defined similarly.

Note: It is not known if the notion of right generalized semiregular rings is left-right symmetric.

It is known from [28] that a ring R is right P-injective if and only if for any $a \in R$, $lr(a) = Ra$. Thus, every right P-injective ring is right generalized semiregular. In particular, a right self-injective ring is right generalized semiregular.

Proposition 1.6.2 ([42, Proposition 1.2]) *If R is a right AP-injective or a semiregular ring, then R is right generalized semiregular.*

The following examples show that right generalized semiregular rings need not be semiregular or right AP-injective.

Example 1.6.3 ([42, Example 1.3(1)]) The *trivial extension* of R by a bimodule ${}_RV_R$ is the direct sum $T(R, V) = R \oplus V$ with multiplication $(r + v)(r' + v') = rr' + (rv' + vr')$. It is shown in [29] that the trivial extension $R = T(\mathbb{Z}, \mathbb{Q}/\mathbb{Z})$ is a commutative P-injective ring, but it is not semiregular since $R/J(R) \cong \mathbb{Z}$.

Note: If R is a right AP-injective ring, then $J(R) = Z(R_R)$ (see [35]).

Example 1.6.4 ([42, Example 1.3(2)]) Let $R = \begin{bmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{bmatrix}$. Then $J(R) = \begin{bmatrix} 0 & \mathbb{Z}_2 \\ 0 & 0 \end{bmatrix}$ and $Z(R_R) = Z({}_RR) = 0$. Since $J(R) \neq Z(R_R) = Z({}_RR)$, R is neither left nor right AP-injective. But $R/J(R) \cong \begin{bmatrix} \mathbb{Z}_2 & 0 \\ 0 & \mathbb{Z}_2 \end{bmatrix}$ is regular and idempotents lift modulo $J(R)$.

In [42], Xiao and Tong gave some sufficient conditions under which right generalized semiregular rings are semiregular.

Lemma 1.6.5 ([42, Lemma 1.4]) *If R is a right generalized semiregular ring and for every $a \in R$ there exists $e^2 = e \in R$ such that $r(a) = r(e)$, then R is semiregular.*

Due to Armendariz ([4]), a ring R is called right (left) PP if every cyclic right (left) ideal of R is projective. R is a right PP ring if and only if for every $a \in R$ there exists an idempotent $e \in R$ such that $r(a) = eR$.

Corollary 1.6.6 ([42, Corollary 1.7]) *Let R be a right PP ring. If R is a right generalized semiregular ring, then R is semiregular.*

Proposition 1.6.7 ([42, Proposition 1.8]) *If R is a right generalized semiregular ring, then $Z_r \subseteq J(R)$.*

Corollary 1.6.8 ([42, Corollary 1.9]) *If R is a semiregular ring, then $Z_r \subseteq J(R)$ and $Z_l \subseteq J(R)$.*

It is known that if R is a semiregular ring, then so is the ring eRe for any idempotent e of R (see [30, Corollary B.42]). Hence, if the $n \times n$ matrix ring $\mathbb{M}_n(R)$ over a ring R is semiregular for some $n \geq 1$, then so is R.

An idempotent element $e \in R$ is *left (respectively, right) semicentral* in R if $Re = eRe$ (respectively, $eR = eRe$) (see [8]).

Proposition 1.6.9 ([42, Proposition 1.11]) *Let R be a right generalized semiregular ring. If an idempotent e of R is right semicentral, then eRe is right generalized semiregular.*

Theorem 1.6.10 ([42, Theorem 1.15]) *Let e be an idempotent of R such that $ReR = R$. If R is a right generalized semiregular ring, then eRe is right generalized semiregular.*

Corollary 1.6.11 ([42, Corollary 1.16]) *Let R be a ring. If $\mathbb{M}_n(R)$ is right generalized semiregular for some $n \geq 1$, then so is R.*

1.7 Semiregular and Semiperfect Modules

Mares ([22]) called a projective module *semiperfect* if every homomorphic image has a projective cover, and showed that many of the properties of semiperfect rings can be extended to these modules. In [20], semiperfect modules were defined without the projectivity assumption on modules. In this dissertation, we will consider the latter definition of semiperfect modules.

An element x in an R-module M is called *regular* if $x(\alpha x) = x$ for some $\alpha \in Hom_R(M, R)$. Zelmanowitz ([44]) called a module *regular* if each of its elements is regular. He also proved that a module is regular if and only if every finitely generated (cyclic) submodule is a projective direct summand (see [44, Theorem 2.2]).

A class of semiregular modules was introduced by Nicholson ([26]). This class contains all regular and all semiperfect modules.

Definition 1.7.1 ([26]) An element x of an R-module M is called *semiregular* if there exists a decomposition $M = P \oplus Q$ with $P \subseteq xR$ projective and $xR \cap Q \ll M$. The module M is said to be a *semiregular module* if each of its elements is semiregular.

As the next result shows, Nicholson characterized the semiperfect modules among the projective semiregular ones.

Proposition 1.7.2 ([26, Proposition 1.19]) *A projective R-module M is semiperfect if and only if it is semiregular, $Rad(M) \ll M$ and $M/Rad(M)$ is semisimple.*

Dual to the continuous modules, *discrete modules* are defined as follows:

Definition 1.7.3 ([24]) Let M be an R-module.

($D1$) For every submodule A of M, there exists a decomposition $M = M_1 \oplus M_2$ such that $M_1 \leq A$ ve $A \cap M_2 \ll M$.

($D2$) For $A \leq M$, if M/A is isomorphic to a direct summand of M, then $A \leq^{\oplus} M$.

The module M is called *discrete* if it satisfies the conditions ($D1$) and ($D2$).

Proposition 1.7.4 ([24, Corollary 4.54]) *Let M be a projective R-module. Then M is discrete if and only if M is a direct sum of local submodules and $Rad(M) \ll M$.*

Proposition 1.7.5 ([24, Corollary 4.43]) *A projective module P is semiperfect if and only if P is discrete.*

1.8 The Exchange Property

Definition 1.8.1 ([14]) An R-module M is said to have the *exchange property* if for any module X and decompositions

$$X = M' \oplus Y = \oplus_{i \in I} N_i,$$

where $M' \cong M$, there exist submodules $N_i' \subseteq N_i$ for each i such that

$$X = M' \oplus (\oplus N_i').$$

If this condition holds for finite sets I (equivalently for $|I| = 2$) the module M is said to have the *finite exchange property*.

Every quasi-injective R-module has the exchange property (see [24, Theorem 1.21]).

Warfield ([40]) called a ring R an *exchange ring* if R_R has the (finite) exchange property. He verified that the definition was left-right symmetric and that a module has the finite exchange property if and only if its endomorphism ring is an exchange ring (see [40, Theorem 10]). He also showed that ([40, Theorem 3]) every semiregular ring is an exchange ring. Also, it was proved by Nicholson ([27, Corollary 2.4]) that a ring R is exchange if and only if $R/J(R)$ is exchange and idempotents lift modulo $J(R)$.

Theorem 1.8.2 ([27, Theorem 2.1]) *Let M be an R-module. Then $End_R(M)$ is an exchange ring if and only if M has the finite exchange property.*

Theorem 1.8.3 ([9, Theorem 3]) *A quasi-projective R-module M has the finite exchange property if and only if whenever $M = A + B$, there exists a decomposition $M = P \oplus Q$ such that $P \leq A$ and $Q \leq B$.*

Due to Nicholson ([27]), a ring R is called *clean* if every element of R is the sum of a unit and an idempotent. Every local ring is clean. Also, the ring of all $n \times n$

matrices over an algebraically closed field is clean. Moreover, a ring R is clean if and only if $R/J(R)$ is clean and idempotents lift modulo $J(R)$ (see [27]). Note that clean rings are exchange rings (see [27, Proposition 1.8]).

Following [11], a module is said to be *clean* if its endomorphism ring is clean. Discrete modules over any ring, quasi-projective modules over a right perfect ring and continuous modules are some examples of clean modules (see [11]).

1.9 Chain conditions

We refer the reader to [13], [15] and [21] for the unproved results in this section.

Let M be an R-module and $\mathcal{S}$ a non-empty collection of submodules of M ordered under inclusion. $\mathcal{S}$ is said to satisfy the *maximum (minimum)* condition if every subset of $\mathcal{S}$ has a maximal (minimal) element. $\mathcal{S}$ is said to satisfy the *ascending chain condition (acc)* if every chain $M_1 \subset M_2 \subset \dots$ with $M_i \in \mathcal{S}$ eventually stops; the *descending chain condition (dcc)* is analogously defined. In the case where $\mathcal{S}$ is the set of all submodules of M, modules satisfying the maximum condition are called *Noetherian*; modules satisfying the minimum condition are called *Artinian*. A ring R is called *right Artnian (right Noetherian)* if R_R is Artinian (Noetherian). A similar definition may be made on the left. R is Artinian (Noetherian) if it is both right and left Artinian (Noetherian). It is well known that an R-module M is Artinian (Noetherian) if and only if M has dcc (acc) on submodules; and that M is Noetherian if and only if every submodule of M is finitely generated. The Artinian and Noetherian properties are inherited by submodules and factor modules. Finitely generated modules over a right Artinian (right Noetherian) ring are Artinian (Noetherian), too.

Theorem 1.9.1 (Hopkins-Levitzki) *A ring R is right Artinian if and only if it is right Noetherian, $J(R)$ is nilpotent and $R/J(R)$ is semisimple.*

Corollary 1.9.2 *A right Artinian ring is right perfect.*

Theorem 1.9.3 *The following statements hold for an R-module M:*

1) M has acc (dcc) on essential submodules if and only if $M/Soc(M)$ is Noetherian (Artinian).

2) M has acc (dcc) on small submodules if and only if $Rad(M)$ is Noetherian (Artinian).

An R-module M is called *locally Artinian (locally Noetherian)* provided every finitely generated submodule of M is Artinian (Noetherian).

An R-module M is said to be *semiartinian* if every nonzero homomorphic image of M has an essential socle, or equivalently, if every nonzero homomorphic image of M has a nonzero socle. A ring R is *right semiartinian* if R_R is semiartinian. Clearly, every Artinian module is semiartinian and every right Artinian ring is right semiartinian. Submodules and factor modules of a semiartinian module are semiartinian, too.

Let M be an R-module. We say that M has *finite Goldie dimension* if M does not contain a direct sum of an infinite number of non-zero submodules. A ring R is said to have *finite right Goldie dimension* if R_R has finite Goldie dimension. Goldie dimension of a module is also known as *uniform dimension* in the literature. The next lemma gives the basic properties of modules of finite Goldie dimension.

Lemma 1.9.4 *Let M be a non-zero R-module.*

1) If M has finite Goldie dimension, then each non-zero submodule of M contains a uniform submodule, and there is a finite number of uniform submodules of M whose sum is direct and is an essential submodule of M.

2) Suppose that M has uniform submodules $U_1, \dots, U_n$ such that the sum $U_1 + \cdots + U_n$ is direct and is an essential submodule of M. Then M has finite Goldie dimension and the positive integer n is independent of the choice of the U_i. We call n the Goldie dimension *of M and denote it by $udim(M)$.*

A Noetherian or an Artinian R-module has finite Goldie dimension. If a semisimple R-module has finite Goldie dimension, then it s both Artinian and Noetherian.

Theorem 1.9.5 *Let M be a semisimple R-module. The following are equivalent:*

1) M is Artinian.

2) M is Noetherian.

3) M has finite Goldie dimension.

4) M is finitely generated.

2 SEMIREGULAR AND SEMIPERFECT MODULES RELATIVE TO A SUBMODULE

Semiregular and semiperfect rings were generalized to I-semiregular and I-semiperfect rings, for an ideal I of a ring R, by Yousif and Zhou ([43]). Nicholson and Zhou ([31]) studied on the concept of strongly lifting ideals to characterize I-semiregular and I-semiperfect rings. They proved that a ring R is I-semiregular (I-semiperfect) if and only if R/I is regular (semisimple) and I is strongly lifting. The notions of I-semiregular rings and I-semiperfect rings are left-right symmetric by Theorem 1.4.12 and Theorem 1.5.7.

In [1] and [32], U-semiregular and U-semiperfect modules are defined as module theoretic versions of I-semiregular and I-semiperfect rings by considering a fully-invariant submodule U of an R-module, and some properties of I-semiregular and I-semiperfect rings are generalized to modules.

In this chapter, we investigate strongly lifting submodules and U-semipotent modules for a submodule U of an R-module. We call a submodule U of a module M *strongly lifting* if whenever $M/U = (A+U)/U \oplus (B+U)/U$, then M has a decomposition $M = P \oplus Q$ such that $P \leq A$, $(A+U)/U = (P+U)/U$ and $(B+U)/U = (Q+U)/U$. We prove that an ideal I of a ring R is a strongly lifting ideal if and only if I is a strongly lifting submodule of R_R. M is called *U-semipotent* if for every submodule A of M such that $A \not\subseteq U$, there exists a summand B of M such that $B \leq A$ and $B \not\subseteq U$. We prove that if $U \leq M$ and M is U-semipotent, then for any submodule N of M with $N \not\subseteq U$, N is indecomposable if and only if N is local.

Moreover, we consider strongly lifting submodules to obtain a new characterization of U-semiregular and U-semiperfect modules for a projection-invariant submodule U. We prove that if M is a finitely generated and projective R-module, then M is U-semiregular if and only if every finitely generated submodule of M/U is a direct summand and U is strongly lifting. If M is a projective R-module, then M is U-semiperfect if and only if M/U is semisimple and U is strongly lifting.

Finally, we characterize rings over which every (projective) module M is $\tau(M)$-

semiperfect, where τ is a preradical. We consider some special preradicals such as Rad, Z_2 and δ. We prove that every right R-module M is $Z_2(M)$-semiperfect if and only if R is Z_2^r-semiperfect; every projective R-module M is $\delta(M)$-semiperfect if and only if R is right δ-perfect; and a ring R is Z_r-semiperfect and Z_2^r is injective if and only if R is semiperfect and right self-injective.

2.1 Strongly Lifting Submodules and Semipotent Modules Relative To A Submodule

Definition 2.1.1 Let U be a submodule of a module M. U is called a *strongly lifting submodule* of M if whenever $M/U = (A+U)/U \oplus (B+U)/U$, then M has a decomposition $M = P \oplus Q$ such that $P \leq A$, $(A+U)/U = (P+U)/U$ and $(B+U)/U = (Q+U)/U$.

Proposition 2.1.2 *Let I be an ideal of a ring R. Let $\overline{R} = R/I$ and $\overline{r} = r + I$ for any $r \in R$. The following statements are equivalent:*

(1) *I is a strongly lifting ideal.*

(2) *I is a strongly lifting submodule of R_R.*

Proof (1) $\Rightarrow$ (2) Let $\overline{R} = \overline{A} \oplus \overline{B}$. Let $\overline{1} = \overline{a} + \overline{b}$ where $a \in A$, $b \in B$. Then $\overline{a}$ and $\overline{b}$ are orthogonal idempotents. By Proposition 1.4.8, there exist orthogonal idempotents e_1 and e_2 in R such that $\overline{e_1} = \overline{a}$, $\overline{e_2} = \overline{b}$ and $e_1 \in aR$, $e_2 \in bR$. Then $R = e_1R \oplus (1-e_1)R$ and $e_1R \leq aR$, $\overline{e_1}\overline{R} = \overline{a}\overline{R} = \overline{A}$, $(\overline{1}-\overline{e_1})\overline{R} = (\overline{1}-\overline{a})\overline{R} = \overline{b}\overline{R} = \overline{B}$. Hence (2) holds.

(2) $\Rightarrow$ (1) Let $\overline{e}^2 = \overline{e} \in \overline{R}$. Then $\overline{R} = \overline{e}\overline{R} \oplus (\overline{1-e})\overline{R}$. By hypothesis, there is a decomposition $R = P \oplus Q$, where $P \leq eR$, $\overline{P} = \overline{e}\overline{R}$ and $\overline{Q} = (\overline{1-e})\overline{R}$. Then there exists an idempotent f in R such that $P = fR$ and $Q = (1-f)R$. Since $\overline{P} = \overline{fR} = \overline{e}\overline{R}$, $\overline{f} = \overline{ea}$, $\overline{e} = \overline{fb}$ for some $\overline{a}, \overline{b}$ in $\overline{R}$. This implies that $\overline{f}\overline{e} = \overline{e}$. Since $\overline{Q} = (\overline{1-f})\overline{R}$ and $\overline{f} = \overline{f}\overline{e} + \overline{f}(\overline{1-e})$, we have that $\overline{f} = \overline{e}$. Hence, I is strongly lifting. $\square$

Proposition 2.1.3 *Let M be a quasi-projective R-module and U a submodule of M. If U is a summand of M, then U is strongly lifting.*

Proof Let N be such that $M = U \oplus N$, and let $M/U = (A+U)/U \oplus (B+U)/U$. Consider the isomorphism $f : N \to M/U$. Then there exist submodules B_1 and B_2 of N such that $f(B_1) = (A+U)/U = (B_1+U)/U$, $f(B_2) = (B+U)/U = (B_2+U)/U$. Then $M/U = (B_1+U)/U \oplus (B_2+U)/U$. Since $B_1 \cap B_2 \leq (B_1+U) \cap (B_2+U) = U$, we have $B_1 \cap B_2 = 0$. Also, $N = B_1 + B_2$. Hence, $M = U \oplus N = U \oplus B_1 \oplus B_2$. Since $U \oplus B_1 = U + A$ is quasi-projective, there exists a submodule L of A such that $U \oplus B_1 = U \oplus L$ by Proposition 1.3.3. Thus, $M = U \oplus L \oplus B_2$, where $L \leq A$, $(L+U)/U = (A+U)/U$ and $(B_2+U)/U = (B+U)/U$, i.e., U is strongly lifting. $\square$

Nicholson ([27, Corollary 1.3]) showed that a ring R is exchange if and only if idempotents can be lifted modulo every right (left) ideal of R, which is also equivalent to saying that every right (left) ideal of R is strongly lifting (see [31, Theorem 4]). The next result shows that a similar fact holds for quasi-projective modules.

Theorem 2.1.4 *Let M be a quasi-projective R-module. Then the following statements are equivalent:*

(1) *M has the finite exchange property.*

(2) *Every submodule of M is strongly lifting.*

Proof (1) $\Rightarrow$ (2) Let $N \leq M$ and $M/N = (A+N)/N \oplus (B+N)/N$. Then $M = A + B + N$. By Theorem 1.8.3, there is a decomposition $M = P_1 \oplus P_2$ with $P_1 \leq A$, $P_2 \leq B + N$. Then $(P_1+N)/N = (A+N)/N$, $(P_2+N)/N = (B+N)/N$. Hence, N is strongly lifting.

(2) $\Rightarrow$ (1) Let $M = M_1 + M_2$ and $N = M_1 \cap M_2$. Then $M/N = M_1/N \oplus M_2/N$. By (2), there is a decomposition $M = P \oplus Q$ such that $P \leq M_1$, $(P+N)/N = M_1/N$, $(Q+N)/N = M_2/N$. Then $Q \leq M_2$. By Theorem 1.8.3, M has the finite exchange property. $\square$

Now we give the definition of a semipotent module relative to a submodule.

Definition 2.1.5 Let U be a submodule of an R-module M. The module M is called *U-semipotent* if, for every submodule A of M such that $A \not\subseteq U$, there exists a summand B of M such that $B \leq A$ and $B \not\subseteq U$. A ring R is called *semipotent* if R_R

is $J(R)$-semipotent. M is called *U-potent* if M is U-semipotent and U is a strongly lifting submodule of M.

The next example shows that there exists a U-semipotent module M where U is not strongly lifting.

Example 2.1.6 ([31, Example 23]) Consider the direct product $Q = \mathbb{Z} \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots$ of rings. Write $\overline{n} = n\overline{1}$ for all $n \in \mathbb{Z}$. Let R denote the following subring of Q:

$$R = \{(n, \overline{n_2}, \overline{n_3}, \cdots, \overline{n_k}, \overline{n}, \overline{n}, \ldots) | n, n_i \in \mathbb{Z}, k \geq 2\}.$$

$I = \{(2m, \overline{0}, \overline{0}, \ldots) | m \in \mathbb{Z}\}$ is an ideal of R. It was shown in [31] that R is an I-semipotent ring where idempotents lift modulo I but not strongly.

An R-module M is 0-potent if every non-zero submodule of M contains a non-zero summand of M. Every regular module is 0-potent. In fact, let M be a regular R-module and $0 \neq A \leq M$. Then there exists $0 \neq a \in A$ and $\alpha \in Hom_R(M, R)$ such that $a(\alpha a) = a$. This implies that aR is a non-zero summand of M in A.

On the other hand, an R-module M with zero radical and essential socle is 0-potent. In fact, let $0 \neq A \leq M$. Then A contains a simple submodule S. Since S is not small in M, then S is a summand of M.

Let I be an ideal of a ring R. It was shown in [31, Proposition 20] that if R is I-semipotent, then R/I is 0-potent, and that the converse holds if I is a strongly lifting ideal. We have the following module theoretic versions of these results.

Proposition 2.1.7 *Let M be an R-module and U a submodule of M. If M/U is 0-potent and U is strongly lifting, then M is U-potent.*

Proof Let A be a submodule of M such that $A \not\subseteq U$. Then $(A+U)/U$ is a non-zero submodule of the factor module M/U, so there exists a non-zero element $\overline{a} = a+U$ of $(A+U)/U$. Since M/U is 0-potent, there exists a non-zero summand B/U of M/U which is contained in $(aR+U)/U$. Hence, B/U is cyclic. Let $B/U = (bR+U)/U$, where $b \in B$. On the other hand, there exists $x \in A$ such that $\overline{b} = \overline{x}$. Then

$M/U = (xR+U)/U \oplus C/U$ for some $C \leq M$. Since U is strongly lifting, there is a decomposition $M = P \oplus Q$ such that $P \leq xR$, $(P+U)/U = (xR+U)/U$ and $(Q+U)/U = C/U$. Thus, P is a summand of M with $P \leq A$ and $P \nsubseteq U$. $\square$

Note that a submodule U of an R-module M is called *projection-invariant* if $\pi(U) \leq U$ for every projection π of M.

Lemma 2.1.8 ([16, Exercise 4.d, page 50]) *Let $M = M_1 \oplus M_2$ and U be any projection-invariant submodule of M. Then $U = (U \cap M_1) \oplus (U \cap M_2)$.*

Proposition 2.1.9 *Let U be a projection-invariant submodule of an R-module M. If M is U-semipotent, then M/U is 0-potent.*

Proof Let $0 \neq A/U \leq M/U$. Then $A \nsubseteq U$. By hypothesis, there exists a summand B of M such that $B \leq A$, $B \nsubseteq U$. Let B' be such that $M = B \oplus B'$. Since U is projection-invariant, $U = (B \cap U) \oplus (B' \cap U)$ by Lemma 2.1.8. This implies that $(B+U) \cap (B'+U) = [B + (B' \cap U)] \cap [B' + (B \cap U)] = U$. Hence, $(B+U)/U$ is a non-zero summand of M/U in A/U. $\square$

Proposition 2.1.10 *Let U be a submodule of an R-module M. If M is U-semipotent, then for every submodule N of M with $N \nsubseteq U$, N is $U \cap N$-semipotent.*

Proof Assume that M is U-semipotent. Let $N \leq M$ and $X \leq N$ be such that $X \nsubseteq U \cap N$. Then $X \nsubseteq U$. By assumption, there exits a summand Y of M such that $Y \leq X$ and $Y \nsubseteq U$. Then Y is a summand of N such that $Y \leq X$ and $Y \nsubseteq U \cap N$. Hence, N is $U \cap N$-semipotent. $\square$

Proposition 2.1.11 *If an R-module M is quasi-projective with the finite exchange property, then M is $Rad(M)$-semipotent.*

Proof Let $N \leq M$ be such that $N \nsubseteq Rad(M)$. Let $n \in N \setminus Rad(M)$. Then there exists a maximal submodule K of M such that $M = nR + K$. By Theorem 1.8.3, there is a decomposition $M = P \oplus Q$ such that $P \leq nR$ and $Q \leq K$. If $P \leq Rad(M)$, then $P \leq K$, and so $M = K$, a contradiction. Hence, $P \nsubseteq Rad(M)$, and so the proof is completed. $\square$

Proposition 2.1.12 *Let U be a submodule of an R-module M and assume that M is U-semipotent. Then the following statements are equivalent for a submodule N of M with $N \nsubseteq U$:*

(1) *N is indecomposable.*

(2) *For any submodule A of N with $A \nsubseteq U$, $A = N$.*

(3) *N is local.*

Proof (3) $\Rightarrow$ (1) It is obvious. (1) $\Rightarrow$ (2) Let $A \leq N$ with $A \nsubseteq U$. Then there exists a summand B of M such that $B \leq A$, $B \nsubseteq U$. So B is a summand of N. If $B = 0$, then $B \leq U$, a contradiction. Then $B = N$. This implies that $A = N$.

(2) $\Rightarrow$ (3) Since $N \nsubseteq U$, by (2) N is cyclic. Now let K be a proper submodule of N and $N = K + L$ for some L. We claim that $L = N$. Assume that $L \leq U$. If $K \leq U$, then $N \subseteq U$, a contradiction. If $K \nsubseteq U$, then $K = N$, again a contradiction. Hence, $L \nsubseteq U$ and so $L = N$. By Theorem 1.1.9, N is local. $\square$

Proposition 2.1.13 *If M is a $Rad(M)$-semipotent R-module, then every indecomposable summand N of M with $N \nsubseteq Rad(M)$ is local.*

Proof Let N be an indecomposable summand of M with $N \nsubseteq Rad(M)$. We claim that for every proper submodule K of N, $K \leq Rad(N)$. Let K be a proper submodule of N and assume that $K \nsubseteq Rad(N)$. Since $Rad(N) = N \cap Rad(M)$, $K \nsubseteq Rad(M)$. Since M is $Rad(M)$-semipotent, there exists a summand X of M such that $X \leq K$ and $X \nsubseteq Rad(M)$. Then X is a summand of N. Since N is indecomposable, we have that $X = N = K$, a contradiction. Hence N is local. $\square$

Proposition 2.1.14 *Let U be a projection-invariant submodule of an R-module M. If M is U-semipotent, then for any indecomposable summand $(A+U)/U$ of M/U, there exists a summand P of M such that $P \leq A$ and $(P+U)/U = (A+U)/U$.*

Proof Let $(A+U)/U$ be an indecomposable summand of M/U. Then $A \nsubseteq U$. Since M is U-semipotent, there exists a summand P of M such that $P \leq A$, $P \nsubseteq U$. Since U is projection-invariant, $(P+U)/U$ is a summand of M/U, and hence a summand of $(A+U)/U$. Since $(P+U)/U \neq 0$, then $(P+U)/U = (A+U)/U$. $\square$

2.2 Some Characterizations of U-semiregular and U-semiperfect Modules

Let U be a submodule of an R-module M. Following [1] and [32], the module M is called *U-semiperfect* (*U-semiregular*) if for any (cyclic) submodule N of M, there exists a decomposition $M = A \oplus B$ such that A is projective, $A \leq N$ and $N \cap B \leq U$. If U is a projection-invariant submodule of M, then this is equivalent to saying that for any (cyclic) submodule N of M, there exists a decomposition $N = A \oplus B$ such that A is a projective summand of M and $B \leq U$ (see [1, 32]). Clearly, U-semiperfect modules are U-semiregular. An R-module M is semiregular if and only if M is $Rad(M)$-semiregular. If M is projective and $Rad(M) \ll M$, then M is semiperfect if and only if M is $Rad(M)$-semiperfect.

Let U and N be any submodules of an R-module M. We say that U *respects* N if there exists a summand A of M contained in N such that $M = A \oplus B$ and $B \cap N \leq U$.

Lemma 2.2.1 *Let U be a projection-invariant submodule of an R-module M and N any submodule of M. Then the following statements are equivalent:*

(1) *U respects N.*

(2) *There exists a summand A of M contained in N such that $N = A \oplus B$ and $B \leq U$.*

(3) *There exists $\pi^2 = \pi$ in $End_R(M)$ with $\pi(M) \leq N$ such that $(1-\pi)(N) \leq U$.*

Proof By Lemma 2.1.8, it is obvious. □

If an R-module M satisfies $(D1)$, then for any submodule N of M, N has a decomposition $N = A \oplus B$, where $A \leq^{\oplus} M$ and $B \ll M$. Hence, $B \leq Rad(M)$. Therefore, $Rad(M)$ respects every submodule of M whenever M satisfies $(D1)$.

Let U be a submodule of an R-module M. It is obvious that if M is U-semiregular, then U respects every cyclic submodule of M. The converse holds if M is projective. It was shown in [1, Theorem 2.3] that an R-module M is U-semiregular if and only if for any finitely generated submodule N of M, there exists a decomposition $M = A \oplus B$ such that A is a projective summand of N and $B \cap N \subseteq U$. Using a similar technique, we show the following result:

Proposition 2.2.2 *Let U be a projection-invariant submodule of an R-module M. Then U respects every finitely generated submodule of M if and only if U respects every cyclic submodule of M.*

Proof Let N be a finitely generated submodule of M. Let $N = x_1R + \cdots + x_nR$, where $x_i \in N$ for $i = 1, \ldots, n$. We will use induction on the generating set of N. By assumption, we have $\beta : M \to x_nR$ such that $\beta^2 = \beta$ and $(1-\beta)(x_n) \subseteq U$. Consider $K = (1-\beta)x_1R + \cdots + (1-\beta)x_{n-1}R$. By induction, we may choose $\alpha : M \to K$ such that $\alpha^2 = \alpha$ and $(1-\alpha)(K) \subseteq U$. Define $\gamma = \beta + \alpha - \alpha\beta$. Then $\gamma^2 = \gamma$ and $\gamma(M) = \beta(M) \oplus \alpha(M)$ since $\beta\alpha = 0$. Since $N = K + x_nR$, we obtain $\gamma(M) = \beta(M) \oplus \alpha(M) \subseteq N$. Now take $n = a + x_nr$, where $a \in K$ and $r \in R$. Since U is projection-invariant, $(1-\gamma)(a + x_nr) = (1-\alpha)(a) + (1-\alpha)(1-\beta)(x_nr) \in U$. Hence, $(1-\gamma)(N) \subseteq U$. Thus, $M = \gamma(M) \oplus (1-\gamma)(M)$ is the desired decomposition.
□

Now we will give some necessary and sufficient conditions for an R-module M to be U-semiregular, where U is a projection-invariant submodule of M.

Theorem 2.2.3 *Let U be a projection-invariant submodule of an R-module M and $\overline{M} = M/U$. Consider the following conditions:*

(1) (*i*) *Every finitely generated submodule of $\overline{M}$ is a summand.*

(*ii*) *If $\overline{M} = \overline{A} \oplus \overline{B}$, where $\overline{A}$ is finitely generated, then there exists a decomposition $M = P \oplus Q$ such that $P \leq A$, $\overline{P} = \overline{A}$ and $\overline{Q} = \overline{B}$.*

(2) *U respects every finitely generated submodule of M.*

Then (1) $\Rightarrow$ (2)*; and* (2) $\Rightarrow$ (1) *if M is quasi-projective.*

Proof (1) $\Rightarrow$ (2) Let N be a finitely generated submodule of M. Then $\overline{M} = \overline{N} \oplus \overline{B}$ for some submodule $\overline{B}$. By hypothesis, $M = P \oplus Q$ such that $P \leq N$, $\overline{P} = \overline{N}$, $\overline{Q} = \overline{B}$. Since $N = P + (N \cap U)$ and $U = (U \cap P) \oplus (U \cap Q)$, we have that $Q \cap N \leq U$. So (2) follows.

(2) $\Rightarrow$ (1) (i) Let $X/U \leq M/U$ be finitely generated. Choose a finitely generated submodule N of M such that $X/U = (N+U)/U$. By (2), there is a decomposition $M = A \oplus B$ such that $A \leq N$ and $B \cap N \leq U$. Then $X/U = (A+U)/U$. Since

$U = (U \cap A) \oplus (U \cap B)$ and $(B+U) \cap (A+U) = (B+(U \cap A)) \cap (A+(U \cap B)) = U$, we get $\overline{A} \oplus \overline{B} = \overline{M}$. So $\overline{X}$ is a summand of $\overline{M}$.

For (ii), let $\overline{M} = \overline{A} \oplus \overline{B}$, where $\overline{A}$ is finitely generated. Let N be a finitely generated submodule of A such that $\overline{A} = \overline{N}$. Then $M = C \oplus D$ such that $C \leq N$ and $D \cap N \leq U$. Since $N = C \oplus (D \cap N)$, we obtain $M = (A+U)+B = (C+U)+B$. Since C is a summand of M and M is quasi-projective, there exists a summand Q of M such that $M = C \oplus Q$ and $Q \leq U + B$ (see Proposition 1.3.3). Now it can be seen that $C \leq A$, $\overline{C} = \overline{A}$ and $\overline{Q} = \overline{B}$. □

Corollary 2.2.4 *Let U be a projection-invariant submodule of a projective R-module M and $\overline{M} = M/U$. Then the following statements are equivalent:*

(1) *M is U-semiregular.*

(2) (*i*) *Every finitely generated submodule of $\overline{M}$ is a summand.*

(*ii*) *If $\overline{M} = \overline{A} \oplus \overline{B}$, where $\overline{A}$ is finitely generated, then there exists a decomposition $M = P \oplus Q$ such that $P \leq A$, $\overline{P} = \overline{A}$ and $\overline{Q} = \overline{B}$.*

In addition, if M is finitely generated, then they are equivalent to,

(3) (*i*) *Every finitely generated submodule of $\overline{M}$ is a summand.*

(*ii*) *U is strongly lifting.*

Corollary 2.2.5 *Let U be a submodule of an R-module M. If M is U-semiregular, then M is U-semipotent. If in addition, M is cyclic and quasi-projective, then M is U-potent.*

Proof Let A be a submodule of M with $A \not\subseteq U$. Let $a \in A \setminus U$. Then $M = X \oplus Y$, where $X \leq aR$ and $Y \cap aR \leq U$. This implies that $aR = X \oplus (Y \cap aR)$ and so $X \not\subseteq U$. Hence, M is U-semipotent. If M is finitely generated and quasi-projective, by the proof of (2) $\Rightarrow$ (1)(*ii*) in Theorem 2.2.3, U is strongly lifting. □

The next example shows that U-semipotent modules need not be U-semiregular even if M/U is regular.

Example 2.2.6 ([31, Example 52]) Let

$$R_1 = \{ \begin{bmatrix} n & m \\ 0 & n \end{bmatrix} : n, m \in \mathbb{Z}\} \text{ and } K = \{ \begin{bmatrix} 2n & m \\ 0 & 2n \end{bmatrix} : n, m \in \mathbb{Z}\},$$

and, for any natural number j, let

$$K_j = \{\begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix} : m \in j\mathbb{Z}\}.$$

Then R_1 is a ring and K, K_j are ideals of R_1. Let $Q = \prod_{i=1}^{\infty} F_i$, where each $F_i = \mathbb{Z}_2$, and $R_2 = \langle \oplus_{i=1}^{\infty} F_i, 1_Q \rangle$ be the subring of Q generated by $\oplus_{i=1}^{\infty} F_i$ and 1_Q. Set $T = R_1 \oplus R_2$. The subring $R = \langle K \oplus (\oplus_{i=1}^{\infty} F_i), 1_T \rangle$ of T is a Z_2^r-semipotent ring but it is not Z_2^r-semiregular. However, R/Z_2^r is regular.

Proposition 2.2.7 *Let U be a proper submodule of an indecomposable R-module M. Then the following statements are equivalent:*

(1) *U respects every finitely generated (cyclic) submodule of M.*

(2) *M is U-semipotent.*

(3) *M is local and $U = Rad(M)$.*

Proof (1) $\Rightarrow$ (2) It follows from the proof of Corollary 2.2.5.

(2) $\Rightarrow$ (3) By Proposition 2.1.12, M is local. Since $Rad(M)$ is maximal, then $U \leq Rad(M)$. Now let $x \in Rad(M) \setminus U$. Then there exists a summand B of M such that $B \leq xR$, $B \nsubseteq U$. Since $xR \ll M$, we have $B \ll M$. Then $B = 0$, a contradiction. Hence, $Rad(M) = U$.

(3) $\Rightarrow$ (1) Let N be a finitely generated (cyclic) submodule of M. If $N = M$, there is nothing to prove. Assume $N \neq M$. Then $N \leq Rad(M)$. Hence, the decomposition $M = 0 \oplus M$ completes the proof. □

Remark 2.2.8 In [1, Proposition 2.2], it was proved that for any fully-invariant submodule U of an R-module M, M is U-semiregular if and only if for any $x \in M$, there exists a regular element $y \in xR$ such that $x - y \in U$ and $xR = yR \oplus (x-y)R$. The same proof shows that the condition "$xR = yR \oplus (x-y)R$" is removable, even for a projection-invariant submodule U of M. We give below its proof for completeness. Also, it was proved in [1, Corollary 2.7] with some conditions that M is U-semiregular if and only if for all $x \in M$, there exists a regular element $y \in M$ such that $x - y \in U$.

Note the following lemma due to Nicholson ([26]).

Lemma 2.2.9 ([26, Lemma 1.1]) *Let M be an R-module and let $x \in M$ be a regular element. If $\alpha \in Hom(M, R)$ satisfies $x(\alpha x) = x$ and if we write $e = \alpha x$, then:*

(1) $e^2 = e$ *and* $x = xe$.

(2) $\alpha : xR \to eR$ *is an isomorphism so* xR *is projective.*

(3) $M = xR \oplus W$, *where* $W = \{w \in M | x(\alpha w) = 0\}$.

Theorem 2.2.10 *Let U be a projection-invariant submodule of an R-module M. Then the following statements are equivalent:*

(1) *M is U-semiregular .*

(2) *For any $x \in M$, there exists a regular element $y \in xR$ such that $x - y \in U$.*

Proof (1) $\Rightarrow$ (2) It follows from the Remark 2.2.8.

(2) $\Rightarrow$ (1) Let x and y be as in (2) and let $\alpha \in Hom_R(M, R)$ be such that $y(\alpha y) = y$. By Lemma 2.2.9, $M = yR \oplus W$, where $W = \{w \in M | y(\alpha w) = 0\}$ Hence, $xR = yR \oplus (xR \cap W)$. Let $\pi : M \to W$ be the projection map. Then $xR \cap W = \pi(xR \cap W) = \pi(xR) = \pi((x - y)R) \leq \pi(U) \leq U$. □

Now we consider U-semiperfect modules. If M is a U-semiperfect R-module, then U respects every submodule of M. If M is projective, then the converse is also true. We now give some characterizations of a U-semiperfect module which are analogous to that of a semiperfect ring relative to an ideal (see Theorem 1.5.7).

Theorem 2.2.11 *Let U be a projection-invariant submodule of an R-module M, $\overline{M} = M/U$ and $S = End_R(M)$. Consider the following conditions:*

(1) *$\overline{M}$ is semisimple and U is strongly lifting.*

(2) *U respects every submodule of M.*

(3) *U respects every countably generated submodule of M.*

(4) *M is U-semipotent and U respects $\oplus_{i=1}^{\infty} \pi_i(M)$ for any orthogonal idempotents $\pi_i \in S$.*

(5) *M is U-semipotent and there is no infinite orthogonal family of idempotents $\pi_i \in S$ such that $\pi_i(M) \not\subseteq U$.*

(6) *M is U-semipotent and $\overline{M}$ is semisimple.*

Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3), (5) $\Rightarrow$ (2) $\Rightarrow$ (6); (2) $\Rightarrow$ (1) *if M is quasi-projective;*

(3) $\Rightarrow$ (4) $\Rightarrow$ (5) *if M is finitely generated; and* (6) $\Rightarrow$ (1) *if M is finitely generated and quasi-projective.*

Proof (1) $\Rightarrow$ (2) Let N be a submodule of an R-module M. Since $\overline{M}$ is semisimple, there exists $B \leq M$ such that $U \leq B$ and $\overline{M} = \overline{N} \oplus \overline{B}$. By hypothesis, M has a decomposition $M = P \oplus Q$ such that $P \leq N$, $\overline{P} = \overline{N}$ and $\overline{Q} = \overline{B}$. Now we show that $Q \cap N \leq U$. Since $N = N \cap (N + U) = N \cap (P + U) = P + (N \cap U)$, then $Q \cap N = Q \cap (P + (N \cap U)) \leq Q \cap (P + (P \cap U) + (Q \cap U)) = Q \cap (P + (Q \cap U)) = (Q \cap U) + (Q \cap P) = Q \cap U \leq U$.

(2) $\Rightarrow$ (1) It follows from a proof similar to that of (2) $\Rightarrow$ (1) in Theorem 2.2.3.

(2) $\Rightarrow$ (3) It is obvious.

(3) $\Rightarrow$ (4) It follows from the proof of Corollary 2.2.5.

(4) $\Rightarrow$ (5) Assume that M is finitely generated. Let $\{\pi_i\}_{i=1}^{\infty}$ be a family of orthogonal idempotents in S such that $\pi_i(M) \nleq U$. By (4), $\oplus_{i=1}^{\infty}\pi_i(M) = A \oplus B$, where A is a summand of M and $B \leq U$. Since A is finitely generated, A is contained in $\oplus_{i=1}^{n}\pi_i(M)$ for some n. Then $\oplus_{i=1}^{\infty}\pi_i(M) = \oplus_{i=1}^{n}\pi_i(M) + B$. Let $k > n$ and $\pi_k(m) = \pi_1(m_1) + \cdots + \pi_n(m_n) + b$, where $m, m_i \in M$, $i = 1, \ldots n$ and $b \in B$. Then $\pi_k(m) = \pi_k(b)$. Since U is projection-invariant, $\pi_k(m) \in U$. Hence $\pi_k(M) \leq U$, a contradiction.

(5) $\Rightarrow$ (2) Assume (2) is not satisfied. By Lemma 2.2.1, there exists $N \leq M$ such that $N \cap (1 - \pi)(M) \nleq U$ for all $\pi^2 = \pi \in S$ with $\pi(M) \leq N$. Since $N \nleq U$, there exists a summand A_1 of M such that $A_1 \leq N$ and $A_1 \nleq U$. Let $M = A_1 \oplus B_1$ and let $\pi_1 : M \to A_1$ be the projection onto A_1 along B_1. Then $N = \pi_1(M) \oplus (N \cap B_1)$ and $N_1 = N \cap B_1 \nleq U$. Let A_2 be a summand of M such that $A_2 \leq N_1$ and $A_2 \nleq U$. If $M = A_2 \oplus B_2$ and $\alpha : M \to A_2$ is the projection onto A_2 along B_2, then $\pi_1\alpha = 0$. Let $\pi_2 = \alpha(1 - \pi_1)$. Then $\{\pi_1, \pi_2\}$ is an orthogonal set such that $\pi_i(M) \leq N$ for $i = 1, 2$. Since $\pi_2\alpha = \alpha$ and $\alpha(M) \nleq U$, we have $\pi_2(M) \nleq U$. Continuing the construction, suppose that $\{\pi_1, \ldots, \pi_n\}$ are orthogonal idempotents in S such that $\pi_i(M) \leq N$ and $\pi_i(M) \nleq U$ for $i = 1, \ldots n$. Let $\pi = \pi_1 + \cdots + \pi_n$. Then π is an idempotent, $\pi(M) \leq N$ and so $N \cap (1 - \pi)(M) \nleq U$. Let Y be a summand of M such that $Y \leq N \cap (1 - \pi)(M)$, $Y \nleq U$. If $M = Y \oplus Y'$ and $\beta : M \to Y$ is the projection onto Y along Y', then let $\pi_{n+1} = \beta(1 - \pi)$. This implies that $\{\pi, \pi_{n+1}\}$ is an orthogonal

set of idempotents in S such that $\pi(M) \nsubseteq U$, $\pi_{n+1}(M) \nsubseteq U$ since $\pi_{n+1}\beta = \beta$. Hence, $\{\pi_1, \ldots, \pi_n, \pi_{n+1}\}$ are orthogonal idempotents in S such that $\pi_i(M) \nsubseteq U$ for $i = 1, \ldots, n+1$. Thus, this process continues inductively to contradict (5).

(2) $\Rightarrow$ (6) By the proof of Corollary 2.2.5, M is U-semipotent, and by the proof of (2) $\Rightarrow$ (1)(i) in Theorem 2.2.3, $\overline{M}$ is semisimple.

(6) $\Rightarrow$ (1) Assume that M is finitely generated and quasi-projective. Let $\overline{M} = \overline{A} \oplus \overline{B}$. We will show that there exists a decomposition $M = P \oplus Q$ such that $P \leq A$, $\overline{P} = \overline{A}$, $\overline{Q} = \overline{B}$.

If $A \subseteq U$, then $\overline{M} = \overline{B}$ and hence $M = 0 \oplus M$ is the desired decomposition.

If $A \nsubseteq U$, then there exists a summand Y_1 of M such that $Y_1 \leq A$ and $Y_1 \nsubseteq U$. Let W_1 be such that $M = Y_1 \oplus W_1$. Then $A = Y_1 \oplus (A \cap W_1)$.

If $A \cap W_1 \subseteq U$, then $(A+U)/U = (Y_1+U)/U$. Also $M = A+B+U = Y_1+(A\cap W_1)+B+U = Y_1+B+U$. Since M is quasi-projective, there exists a submodule $X \subseteq B+U$ such that $M = Y_1 \oplus X$ by Theorem 1.3.3. Since $\overline{M} = \overline{A} \oplus \overline{X} = \overline{A} \oplus \overline{B}$, we have $\overline{X} = \overline{B}$. Thus, we obtain $M = Y_1 \oplus X$, $Y_1 \leq A$, $\overline{Y_1} = \overline{A}$ and $\overline{X} = \overline{B}$.

If $A \cap W_1 \nsubseteq U$, then there exists a summand Y_2 of M such that $Y_2 \leq A \cap W_1$, $Y_2 \nsubseteq U$. Let W_2 be such that $M = Y_2 \oplus W_2$. Then $W_1 = Y_2 \oplus (W_1 \cap W_2)$. So $M = Y_1 \oplus W_1 = Y_1 \oplus Y_2 \oplus (W_1 \cap W_2)$ implies that $A = Y_1 \oplus Y_2 \oplus (A \cap W_1 \cap W_2)$. This process produces strictly ascending chain $\overline{Y_1} \subset \overline{Y_1} \oplus \overline{Y_2} \subset \cdots \subset \overline{M}$. Since $\overline{M}$ is Noetherian, this process must stop so that $A \cap W_1 \cap \ldots \cap W_n \subseteq U$ for some positive integer n. Hence the proof is completed. $\square$

Corollary 2.2.12 *Let M be a projective R-module and U a projection-invariant submodule of M. The following statements are equivalent:*

(1) *M is U-semiperfect.*

(2) *M/U is semisimple and U is strongly lifting.*

Now we will characterize semiperfect modules. Recall that a projective module M with $Rad(M) \ll M$ is semiperfect if and only if $Rad(M)$ respects every submodule of M.

Theorem 2.2.13 *Let M be a projective R-module with $Rad(M) \ll M$ and let $S = End_R(M)$. Consider the following conditions:*

(1) *Every indecomposable summand of M is local and there is no infinite orthogonal family of idempotents $\pi_i \in S$ such that $\pi_i(M) \nsubseteq Rad(M)$.*

(2) *$End_R(M)$ is clean and there is no infinite orthogonal family of idempotents $\pi_i \in S$ such that $\pi_i(M) \nsubseteq Rad(M)$.*

(3) *M has the finite exchange property and there is no infinite orthogonal family of idempotents $\pi_i \in S$ such that $\pi_i(M) \nsubseteq Rad(M)$.*

(4) *M is semiperfect.*

Then (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Rightarrow$ (4)*. In addition,* (4) $\Rightarrow$ (1) *if M is finitely generated.*

Proof (1) $\Rightarrow$ (2) Since there is no infinite orthogonal family of idempotents $\pi_i \in S$ such that $\pi_i(M) \nsubseteq Rad(M)$, M is a finite direct sum of indecomposable submodules M_i such that $M_i \nsubseteq Rad(M)$. Then each M_i is local. By Proposition 1.7.4, M is discrete. Thus, $End_R(M)$ is clean (see Section 1.8).

(2) $\Rightarrow$ (3) Since $End_R(M)$ is clean, M has the finite exchange property by Theorem 1.8.2.

(3) $\Rightarrow$ (1) By Propositions 2.1.11 and 2.1.13, every indecomposable summand of M is local.

(1) $\Rightarrow$ (4) By Propositions 1.7.4 and 1.7.5, M is semiperfect.

(4) $\Rightarrow$ (1) Assume that M is finitely generated. By Theorem 2.2.11 and Proposition 2.1.13, (1) holds. □

A ring R is called *I-finite* if R has no infinite set of orthogonal idempotents.

By Theorems 2.2.11 and 2.2.13 we have the following corollary. For the equivalences (1)-(4) we refer the reader to [30]. The equivalences of (1), (5) and (6) are given in [12].

Corollary 2.2.14 *The following statements are equivalent for a ring R:*

(1) *R is semiperfect.*

(2) *R is semipotent and $R/J(R)$ is semisimple.*

(3) *R is semipotent and I-finite.*

(4) *Every primitive idempotent in R is local and R is I-finite.*

(5) *R is clean and I-finite.*

(6) *R is an exchange ring and I-finite.*

2.3 Rings Over Which Every (Projective) Module Is $\tau()$-semiperfect

A functor τ from Mod-R to itself is called a *preradical* on Mod-R if it satisfies the following properties:

i) $\tau(M)$ is a submodule of M for every left R-module M.

ii) If $f : M' \to M$ is a homomorphism in Mod-R, then $f(\tau(M')) \leq \tau(M)$ and $\tau(f)$ is the restriction of f to $\tau(M')$.

It was shown in [36] that any fully-invariant submodule defines a preradical.

In this section, we characterize rings R for which every projective R-module M is $\tau(M)$-semiperfect for some preradicals τ on R-Mod.

By definitions, every projective module M is $\tau(M)$-semiperfect if and only if for every projective module M, $\tau(M)$ respects every submodule of M.

Remark 2.3.1 It is well-known that a ring R is right perfect if and only if every projective right R-module is semiperfect (see [24, Theorem 4.41 and Corollary 4.43]). Also, if a projective module M is semiperfect, then M is $Rad(M)$-semiperfect. The converse is true if $Rad(M) \ll M$.

Theorem 2.3.2 *Let R be a ring. Then the following statements are equivalent:*

(1) *Every projective right R-module M is $Rad(M)$-semiperfect.*

(2) *R is right perfect.*

Proof (2) $\Rightarrow$ (1) It is obvious. (1) $\Rightarrow$ (2) It is enough to prove that for any projective R-module P, $Rad(P) \ll P$ by Remark 2.3.1. Let Y be a submodule of a projective module P such that $P = Rad(P) + Y$. By hypothesis, there is a decomposition $P = A \oplus B$, where $A \leq Y$ and $B \cap Y \leq Rad(P)$. Then $Y = A \oplus (B \cap Y)$ and so $P = Rad(P) + A$. Since A is a summand of P, there exists a submodule X of $Rad(P)$ such that $P = X \oplus A$ by Proposition 1.3.3. Then $Rad(X) = X \cap Rad(P) = X$. Since X is projective, $X = 0$. So $P = Y$. $\square$

For the singular submodule $Z(M)$ for a module M, the following theorem is given in [43, Proposition 3.3].

Theorem 2.3.3 *Let R be a ring. Then the following statements are equivalent:*

(1) *Every projective right R-module M is $Z(M)$-semiperfect.*

(2) *R is right perfect and $Z_r = J(R)$.*

There exists a right perfect ring R with $Z_r \neq J(R)$, for example the ring of 2×2 upper triangular matrices over a field. Hence, this ring does not satisfy (1) of Theorem 2.3.3.

Self-injective Artinian rings are known as *Quasi-Frobenius* (*QF-ring*, for short) in the literature. It is a well-known fact that R is a QF-ring if and only if every projective (injective) R-module is injective (projective). Also, it was proved in [32, Corollary 3.8] that R is a QF-ring if and only if every right R-module M is $Z(M)$-semiperfect.

Now we will deal with the Goldie torsion submodule. The next result was proven by Nicholson and Zhou in [31, Theorem 49].

Theorem 2.3.4 *Let R be a ring. The following statements are equivalent:*

(1) *R is Z_2^r-semiperfect.*

(2) *For any module M, $M = Z_2(M) \oplus X$, where X is semisimple.*

(3) *Every nonsingular R-module is injective.*

(4) *Every projective R-module M is $Z_2(M)$-semiperfect.*

In addition to Theorem 2.3.4 we may add the following characterization for Z_2^r-semiperfect rings.

Theorem 2.3.5 *R is a Z_2^r-semiperfect ring if and only if every R-module M is $Z_2(M)$-semiperfect.*

Proof The necessity is obvious. For the sufficiency, let M be an R-module and N a submodule of M. Then by the condition (2) in Theorem 2.3.4, $N = Z_2(N) \oplus X$ for some semisimple submodule X. Then X is nonsingular and projective. By the condition (3) in Theorem 2.3.4, X is injective and hence a projective summand of M. It follows that N has a decomposition $N = A \oplus B$ such that $A \leq^{\oplus} M$, A is projective and $B \leq Z_2(M)$. Thus, M is $Z_2(M)$-semiperfect. $\square$

Lemma 2.3.6 *If R is Z_2^r-semiperfect and Z_2^r is injective, then every finitely generated projective right R-module is injective. In particular, R is right self-injective.*

Proof Let P be a finitely generated projective R-module. Then P is a summand of a finitely generated free R-module. Since Z_2^r is injective, we have that $Z_2(P)$ is injective. Hence, $P = Z_2(P) \oplus X$ for some submodule X. On the other hand, $P/Z_2(P)$ is injective by Theorem 2.3.4. Then X is injective and so P is injective. $\square$

Theorem 2.3.7 *Let R be a ring. Then the following statements are equivalent:*

(1) *R is Z_r-semiperfect and Z_2^r is injective.*

(2) *R is Z_2^r-semiperfect, Z_2^r is injective and R is I-finite.*

(3) *R is semiperfect and right self-injective.*

Proof (1) $\Rightarrow$ (2) R is Z_r-semiperfect if and only if R is semiperfect and $J(R) = Z_r$ (see Section 1.5). Hence, (2) follows.

(2) $\Rightarrow$ (3) By Lemma 2.3.6, R is right self-injective. Since any right self-injective ring is clean (see Section 1.8), it follows from Corollary 2.2.14 that R is semiperfect.

(3) $\Rightarrow$ (1) Since R is right self-injective, $J(R) = Z_r$. Then R is Z_r-semiperfect. Since Z_2^r is closed in R, we have that Z_2^r is injective. $\square$

Now we will give some necessary and sufficient conditions for a ring to be QF.

Theorem 2.3.8 *Let R be a ring. Then the following statements are equivalent:*

(1) *R is a QF-ring.*

(2) *R is Z_2^r-semiperfect and for every projective right R-module P, $Z_2(P)$ is injective.*

(3) *R is Z_2^r-semiperfect, Z_2^r is injective and R is right Noetherian.*

Proof (1) $\Rightarrow$ (2) and (3) Since R is QF, R is semiperfect and $J(R) = Z_r \leq Z_2^r$. Then R is Z_2^r-semiperfect. Let P be a projective right R-module. Then P is injective. Since $Z_2(P)$ is closed in P, we have that $Z_2(P) \leq^{\oplus} P$. Hence, $Z_2(P)$ is injective.

(2) $\Rightarrow$ (1) Let P be a projective R-module. By hypothesis, $Z_2(P)$ is injective. Hence, there exists a submodule X of P such that $P = Z_2(P) \oplus X$. Since $P/Z_2(P)$ is nonsingular, X is injective by Theorem 2.3.4. Hence, P is injective.

(3) $\Rightarrow$ (1) Let P be a projective R-module. Then P is a summand of a free R-module $R^{(\Lambda)}$ for some index set Λ. Since R is right Noetherian, $Z_2(R^{(\Lambda)}) = Z_2(R_R)^{(\Lambda)}$ is injective. Hence, $Z_2(P)$ is injective. By a proof similar to that of (2) $\Rightarrow$ (1), P is injective. $\square$

Theorem 2.3.9 *Let R be a ring. Then the following statements are equivalent:*

(1) *Every projective right R-module M is $\delta(M)$-semiperfect.*

(2) *R is right δ-perfect.*

Proof (2) $\Rightarrow$ (1) Let R be a right δ-perfect ring. Then for any submodule N of a projective module P, P/N has a projective δ-cover. Hence, P is $\delta(P)$-semiperfect.

(1) $\Rightarrow$ (2) If every projective right R-module M is $\delta(M)$-semiperfect, then R is δ_r-semiperfect, and so idempotents lift modulo δ_r. By Theorem 1.5.8, it is enough to prove that $\overline{R} = R/S_r$ is right perfect. Since $J(\overline{R}) = \delta_r/S_r$, $\overline{R}/J(\overline{R})$ is semisimple.

Now we claim that, for every projective right R-module P, $\delta(P) \ll_\delta P$. Let P be a projective R-module and $P = \delta(P)+Y$, where P/Y is singular. By hypothesis, $P = A \oplus B$ such that $A \leq Y$ and $B \cap Y \leq \delta(P)$. Then $Y = A \oplus (B \cap Y)$ and so $P = \delta(P) + Y = \delta(P) + A$. Since A is a summand of P, there exists a submodule $X \leq \delta(P)$ such that $P = X \oplus A$ by Proposition 1.3.3. Since $\delta(X) = X \cap \delta(P) = X$, X is semisimple projective. Since P/Y is an epimorphic image of $P/A \cong X$, P/Y is projective. Because P/Y is singular, we have that $P = Y$. Hence $\delta(P) \ll_\delta P$.

Using the technique of [46, Theorem 3.7] and [3, Lemma 28.2], it can be seen that $J(\overline{R})$ is right T-nilpotent. We will give the proof for completeness.

Let $F \cong R^{(\aleph_0)}$ have a free basis $\{x_1, x_2, \ldots\}$. Let $a_1, a_2, \ldots$ be a sequence in δ_r and $G = \sum_{i=1}^{\infty}(x_i - x_{i+1}a_i)R$. Then $F = G + \delta(F)$. Since $\delta(F) \ll_\delta F$, $F = G \oplus Y$ for a semisimple submodule Y by Lemma 1.1.20. But then there exists a number n such that $Ra_{n+1}a_n \cdots a_1 = Ra_n \cdots a_1$ by [3, Lemma 28.2]. Then $a_n \cdots a_1 = ra_{n+1} \cdots a_1$ for some $r \in R$ and so $(1 - ra_{n+1})a_n \cdots a_1 = 0$. Hence, $a_n \cdots a_1 \in S_r$. Thus, $J(\overline{R}) = \delta_r/S_r$ is right T-nilpotent. $\square$

Özcan and Alkan proved in [32, Corollary 3.10] that R is semisimple if and only if every right R-module M is $\delta(M)$-semiperfect, if and only if every right R-module M is $\delta(M)$-semiregular.

The following results are also given in [32, Corollaries 2.24 and 3.5]: Every projective right R-module M is $Soc(M)$-semiperfect if and only if R is S_r-semiperfect. R is a QF-ring with $J(R)^2 = 0$ if and only if $J(R) \leq Z_r$ and every right R-module M is $Soc(M)$-semiperfect.

In addition, we note that for an ideal I of a ring R, R is I-semiperfect if and only if every finitely generated projective R-module M is MI-semiperfect (see [32, Corollary 2.11]).

3 A GENERALIZATION OF SEMIREGULAR AND ALMOST PRINCIPALLY INJECTIVE RINGS

Let M be an R-module and F a submodule of M_R. Recall that the module M is F-semiregular if and only if for any $m \in M$, there exists a decomposition $M = P \oplus Q$ such that P is projective, $P \subseteq mR$ and $Q \cap mR \subseteq F$. If F is a fully-invariant submodule of M_R, then M is F-semiregular if and only if for any $m \in M$, there exists a decomposition $mR = P \oplus S$ such that P is a projective summand of M and $S \subseteq F$.

In this chapter, we call a right R-module M *almost F-semiregular* if for any $m \in M$, there exists an S-module decomposition $l_M r_R(m) = P \oplus Q$ such that $P \subseteq Sm$ and $Q \cap Sm \subseteq F$, where $S = End_R(M)$ and F is a submodule of ${}_SM$. A ring R is called *right almost I-semiregular* for an ideal I of R if R_R is almost I-semiregular. If ${}_SM$ is F-semiregular, then M_R is almost F-semiregular. An APQ-injective module M_R is almost F-semiregular for any S-submodule F of M. Moreover,

$$M_R \text{ is } APQ\text{-injective} \Leftrightarrow M_R \text{ is almost 0-semiregular.}$$

Note that right almost $J(R)$-semiregular rings are called *right generalized semiregular rings* in [42].

We give a new characterization of F-semiregular modules by modifying the definition of almost F-semiregular modules. We give some conditions under which a right almost I-semiregular ring is I-semiregular. Some of the results in [42] are extended to a right almost I-semiregular ring R for an ideal I of R. We also prove that if R is a right almost I-semiregular ring, then eRe is a right almost eIe-semiregular ring for a right semicentral idempotent e of R (i.e., $eR = eRe$) or an idempotent e of R satisfying $ReR = R$. If the matrix ring $\mathbb{M}_n(R)$ is right almost $\mathbb{M}_n(I)$-semiregular for an ideal I of R, then R is right almost I-semiregular.

It was shown in [1, Corollary 4.6] that if an R-module M is projective and $Soc(M)$-semiregular, then M is semiregular. We are able to show that if M_R is almost $Soc({}_SM)$-semiregular, then M_R is almost semiregular, i.e., for any $m \in M$, there exists an S-module decomposition $l_M r_R(m) = P \oplus Q$ such that $P \subseteq Sm$ and $Q \cap Sm \ll {}_SM$. Note that almost semiregular R-modules are precisely $Rad({}_SM)$-semiregular R-modules whenever $Rad({}_SM) \ll {}_SM$.

We also consider right almost I-semiregular rings for some special ideals such as the socle, the singular ideal and the ideal δ. We prove that if R is right almost Z_r-semiregular, then R_R satisfies $(C2)$ and is almost semiregular.

We show that the following implications hold for a ring R:

S_l-semiregular $\Rightarrow$ right almost S_l-semiregular $\Rightarrow$ right almost semiregular $\Rightarrow$ right almost δ_r-semiregular and right almost δ_l-semiregular.

Z_r-semiregular $\Rightarrow$ right almost Z_r-semiregular $\Rightarrow$ right almost semiregular $\Rightarrow$ right almost δ_r-semiregular and right almost δ_l-semiregular.

Counterexamples to each of the inverse implications are given.

It is well known that $J(eRe) = eJ(R)e$ for any idempotent $e \in R$, but we observe with an example that $\delta_r(eRe) \neq e\delta_r e$ even if e is a right semicentral idempotent. However, if $e \in R$ is an idempotent with $ReR = R$, then $\delta_r(eRe) = e\delta_r e$. Consequently, if R is right almost δ_l-semiregular and $ReR = R$, then eRe is right almost $\delta_l(eRe)$-semiregular.

3.1 Almost Semiregular Modules

Definition 3.1.1 Let M be a right R-module, $S = End_R(M)$ and F a submodule of ${}_SM$. The module M_R is called *almost F-semiregular* if for any $m \in M$, there exists an S-module decomposition $l_M r_R(m) = P \oplus Q$ such that $P \subseteq Sm$ and $Q \cap Sm \subseteq F$. A ring R is called *right almost I-semiregular* for an ideal I of R if R_R is almost I-semiregular.

If a module M_R is APQ-injective, then M_R is almost F-semiregular for any submodule F of ${}_SM$, where $S = End_R(M)$. Moreover, M_R is almost 0-semiregular if and only if M_R is APQ-injective. In particular, a ring R is right almost 0-semiregular if and only if R is right AP-injective.

Proposition 3.1.2 *Let M be a right R-module, $S = End_R(M)$ and F any submodule of ${}_SM$. If ${}_SM$ is F-semiregular, then M_R is almost F-semiregular.*

Proof Let $m \in M$. Then there exists a decomposition ${}_SM = P \oplus Q$ such that $P \subseteq Sm$ and $Q \cap Sm \subseteq F$. Since $l_Mr_R(m) = l_Mr_R(m) \cap M$, by the modular law, we have $l_Mr_R(m) = P \oplus (l_Mr_R(m) \cap Q)$ and $(l_Mr_R(m) \cap Q) \cap Sm = Q \cap Sm \subseteq F$. Hence, M_R is almost F-semiregular. $\square$

Let M be an R-module and $S = End_R(M)$. It follows from Proposition 3.1.2 that if ${}_SM$ is semiregular, then M_R is almost $Rad({}_SM)$-semiregular. If R is an I-semiregular ring for an ideal I, then it is right and left almost I-semiregular, because the notion of I-semiregular rings is left-right symmetric.

When we take the summand P of $l_Mr_R(m)$ as a summand of M in Definition 3.1.1, we have the following result:

Theorem 3.1.3 *Let M be a right R-module and $S = End_R(M)$. If ${}_SM$ is projective and ${}_SF$ is a fully-invariant submodule of ${}_SM$, then the following are equivalent:*

(1) *${}_SM$ is F-semiregular.*

(2) *For any $m \in M$, there exists an S-module decomposition $l_Mr_R(m) = P \oplus Q$, where $P \subseteq Sm$, P is a summand of M and $Q \cap Sm \subseteq F$.*

Proof (1) $\Rightarrow$ (2) It follows from the proof of Proposition 3.1.2.

(2) $\Rightarrow$ (1) Let $m \in M$ and $l_Mr_R(m) = P \oplus Q$, where $P \subseteq Sm$, P is a summand of M and $Q \cap Sm \subseteq F$. Then $Sm = P \oplus (Q \cap Sm)$, where P is a projective summand of M and $Q \cap Sm \subseteq F$. Hence, ${}_SM$ is F-semiregular. $\square$

By Theorem 3.1.3, we obtain the following characterization of I-semiregular rings for an ideal I:

Corollary 3.1.4 *Let I be an ideal of a ring R. The following statements are equivalent:*

(1) *R is I-semiregular.*

(2) *For any $a \in R$, there exists a decomposition $l_Rr_R(a) = P \oplus Q$, where $P = Re \subseteq Ra$ for some $e^2 = e \in R$ and $Q \cap Ra \subseteq I$.*

(3) *For any $a \in R$, there exists a decomposition $r_Rl_R(a) = P \oplus Q$, where $P = eR \subseteq aR$ for some $e^2 = e \in R$ and $Q \cap aR \subseteq I$.*

Now we consider the module theoretic version of right generalized semiregular rings defined by Xiao and Tong ([42]).

Definition 3.1.5 Let M be a right R-module and $S = End_R(M)$. The module M is called *almost semiregular* if for any $m \in M$, there exists an S-module decomposition $l_M r_R(m) = P \oplus Q$ such that $P \subseteq Sm$ and $Q \cap Sm \ll M$. A ring R is called a *right almost semiregular* if R_R is almost semiregular.

Obviously, R is right almost $J(R)$-semiregular if and only if R is right almost semiregular. Semiregular or right AP-injective rings are right almost semiregular by Proposition 1.6.2.

Let M be a right R-module and $S = End_R(M)$. If ${}_SM$ is semiregular, then M_R is almost semiregular by a proof similar to that of Proposition 3.1.2. Moreover, if M_R is almost semiregular, then it is almost $Rad({}_SM)$-semiregular. The converse is true if $Rad({}_SM) \ll {}_SM$.

The following result generalizes Lemma 1.6.5.

Proposition 3.1.6 *Let I be an ideal of a ring R. If R is right almost I-semiregular and there exists $e^2 = e \in R$ such that $r_R(a) = r_R(e)$ for any $a \in R$, then R is I-semiregular.*

Proof Let $a \in R$. Then there exists a decomposition $l_R r_R(a) = P \oplus Q$ such that $P \subseteq Ra$ and $Q \cap Ra \subseteq I$ as left ideals. Since $r_R(a) = r_R(e)$ for some $e^2 = e \in R$, $Re = P \oplus Q$ and $a = ae$. Let $e = p + q$, where $p = ra \in P$ and $q \in Q$. Then $a = ae = ara + aq$ and $ra = rara + raq$. Since $ra - rara = raq \in P \cap Q = 0$, ra is an idempotent. Also, we have $a(1 - ra) = a - ara = aq \in Q \cap Ra \subseteq I$. Hence, R is I-semiregular. $\square$

Corollary 3.1.7 *If $l_R r_R(a)$ is a summand of R for any $a \in R$ and R is right almost I-semiregular for an ideal I, then R is I-semiregular.*

Proof Let $a \in R$. By hypothesis $l_R r_R(a) = Re$ for some idempotent e. Then $r_R(a) = r_R l_R r_R(a) = r_R(e)$ and the claim holds by Proposition 3.1.6. $\square$

Recall that a ring R is a right PP-ring if and only if for any $a \in R$, $r_R(a) = eR$ for some idempotent $e \in R$. Hence, we have the following result:

Corollary 3.1.8 *Let R be a right PP-ring. If R is a right almost I-semiregular ring for an ideal I, then R is I-semiregular.*

Nicholson and Zhou proved in [31, Proposition 41] that if R is an I-semiregular ring for an ideal I, then eRe is eIe-semiregular for any idempotent e of R. We consider this property for almost I-semiregular rings. Using the techniques of Proposition 1.6.9 and Theorem 1.6.10 we obtain the following two results:

Theorem 3.1.9 *If R is a right almost I-semiregular ring for an ideal I and e is a right semicentral idempotent of R, then eRe is a right almost eIe-semiregular ring.*

Proof Let $a \in eRe$. Then there is a decomposition $l_R r_R(a) = P \oplus Q$ such that $P \subseteq Ra$ and $Q \cap Ra \subseteq I$. We claim that $l_{eRe} r_{eRe}(a) = eP \oplus eQ$. Take any $y \in eP \subseteq ePe$, where $y = ey_1$, $y_1 \in P \subseteq l_R r_R(a)$. For any $x \in r_{eRe}(a) \subseteq r_R(a)$, $y_1 x = 0$ which gives $yx = ey_1 x = 0$. Hence, $eP \subseteq l_{eRe} r_{eRe}(a)$. Similarly, $eQ \subseteq l_{eRe} r_{eRe}(a)$. Now take $x \in l_{eRe} r_{eRe}(a)$. Then for any $y \in r_R(a)$, we obtain $aeye = aye = 0$ since $a \in eRe$. So $xeye = 0$. Since $x, y \in eRe$ and e is right semicentral, we have $xy = xey = xeye = 0$. Thus, $l_{eRe} r_{eRe}(a) \subseteq l_R r_R(a)$. Write $x = p + q$, where $p \in P$ and $q \in Q$. Then $x = ex = ep + eq \in eP + eQ$. Since $eP \cap eQ \subseteq P \cap Q = 0$, we obtain $l_{eRe} r_{eRe}(a) = P \oplus Q$.

We also have $eP \subseteq eRa = eRea$ and $eQ \cap eRea \subseteq e(eQ \cap eRea)e$. Hence, $eQ \cap eRea \subseteq Q \cap Ra \subseteq I$ implies that $eQ \cap eRea \subseteq eIe$. $\square$

Theorem 3.1.10 *Let e be an idempotent of R such that $ReR = R$. If R is a right almost I-semiregular ring for an ideal I, then eRe is a right almost eIe-semiregular ring.*

Proof Let $a \in eRe$. Then there exists a decomposition $l_R r_R(a) = P \oplus Q$, where $P \subseteq Ra$ and $Q \cap Ra \subseteq I$. We will show that $l_{eRe} r_{eRe}(a) = ePe \oplus eQe$. Since $1 - e \in r_R(a)$, we observe that $q(1-e) = 0$ for any $q \in Q$, which implies $Q = Qe$. Similarly, $P = Pe$. Hence, $ePe \cap eQe = 0$. Since $P = Pe$ and $Q = Qe$, we have $ePe \subseteq l_{eRe}r_{eRe}(a)$ and $eQe \subseteq l_{eRe}r_{eRe}(a)$. Now take $x \in l_{eRe}r_{eRe}(a)$. Write $1 = \sum_{i=1}^{n} a_i e b_i$ for some $a_i, b_i \in R$. For any $y \in r_R(a)$, we get $aeya_ie = aya_ie = 0$ for each i. Then $xeya_ie = 0$ for each i, which gives $xy = xey = xey \sum_{i=1}^{n} a_i e b_i = 0$ since $x \in eRe$. It follows that $l_{eRe}r_{eRe}(a) \subseteq l_R r_R(a)$. Let $x = p+q$, where $p \in P$ and $q \in Q$. Hence, $x = exe = epe + eqe \in ePe + eQe$. Thus, $l_{eRe}r_{eRe}(a) = ePe \oplus eQe$. Also, we have $ePe \subseteq (eRe)a$ and $eQe \cap (eRe)a \subseteq e(Q \cap Ra)e \subseteq eIe$. □

Proposition 3.1.11 *Let S be a right almost I-semiregular ring for an ideal I of S. If $\varphi : S \to R$ is a ring isomorphism, then R is a right almost $\varphi(I)$-semiregular ring.*

Proof Let $a \in R$. Then there is a decomposition $l_S r_S(\varphi^{-1}(a)) = P \oplus Q$ such that $P \subseteq S\varphi^{-1}(a)$ and $Q \cap S\varphi^{-1}(a) \subseteq I$. If $x \in l_R r_R(a)$, then $\varphi^{-1}(x) \in l_S r_S(\varphi^{-1}(a))$. Then we obtain a decomposition $l_R r_R(a) = \varphi(P) \oplus \varphi(Q)$, where $\varphi(P) \subseteq Ra$ and $\varphi(Q) \cap Ra \subseteq \varphi(I)$. Hence, R is a right almost $\varphi(I)$-semiregular ring. □

The following result generalizes Corollary 1.6.11.

Corollary 3.1.12 *Let I be an ideal of a ring R and let $n \geq 1$. If $\mathbb{M}_n(R)$ is right almost $\mathbb{M}_n(I)$-semiregular, then R is right almost I-semiregular.*

Proof Let $S = \mathbb{M}_n(R)$. Then $Se_{11}S = S$ and $R \cong e_{11}Se_{11}$, where e_{11} is the $n \times n$ matrix whose $(1,1)$-entry is 1, others are 0. By Theorem 3.1.10, $e_{11}Se_{11}$ is right almost $e_{11}\mathbb{M}_n(I)e_{11}$-semiregular. Let $\varphi : e_{11}Se_{11} \to R$ be the isomorphism. Since $\varphi(e_{11}\mathbb{M}_n(I)e_{11}) = I$, R is right almost I-semiregular by Proposition 3.1.11. □

3.2 Almost Semiregular Rings Relative To Some Special Ideals

In this section, we will consider right almost semiregular rings relative to some special ideals, i.e., we will deal with right almost I-semiregular rings, where $I = S_r$ or $I = \delta_r$ or $I = Z_r$.

We begin with some examples.

Example 3.2.1 *There exists a right AP-injective ring R that is not semiregular. Hence, there exists a right almost I-semiregular ring R that is not I-semiregular for ideals $I = J(R)$ or Z_r or S_r.*

Proof Consider the trivial extension $R = T(\mathbb{Z}, \mathbb{Q}/\mathbb{Z})$. R is a commutative P-injective ring with $J(R) = Z_r$ ([29]). Hence, R is an almost I-semiregular ring for any ideal I of R. Since $R/J(R) \cong \mathbb{Z}$, the ring R is not semiregular whence R is not Z_r-semiregular (see Section 1.4). If R was S_r-semiregular, it would be semiregular (see Section 1.4). Thus, R is not S_r-semiregular, either. $\square$

Example 3.2.2 *There exists a right almost S_r-semiregular ring R that is not S_r-semiregular.*

Proof Let $R = \mathbb{Z}_8$. Since R is a self-injective ring, it is almost I-semiregular for any ideal I of R. But since R/S_r is not regular, R is not S_r-semiregular (see Section 1.4). $\square$

Xiao and Tong ([42, Example 4.8]) showed that $\mathbb{Z}$ is not a right almost semiregular ring. Hence, Example 3.2.1 shows that the class of right almost semiregular rings is not closed under homomorphic images.

Alkan and Özcan proved in [1] that if M_R is a projective $Soc(M_R)$-semiregular module, then M_R is semiregular. We have the following result for almost semiregular modules:

Proposition 3.2.3 *Let M be a right R-module and $S = End_R(M)$. If M_R is almost $Soc(_SM)$-semiregular, then M_R is almost semiregular.*

Proof Let $m \in M$. Then there exists a decomposition $l_M r_R(m) = A \oplus B$ such that $A \subseteq Sm$ and $B \cap Sm \subseteq Soc(_SM)$. By the modular law, $Sm = A \oplus (B \cap Sm)$. Then $B \cap Sm$ is a finite direct sum of simple S-submodules. If every simple submodule of $B \cap Sm$ is in $Rad(_SM)$, then $B \cap Sm \ll M$ and hence M_R is almost semiregular. Assume that there exists a simple submodule S_1 of $B \cap Sm$ such that $S_1 \nsubseteq Rad(_SM)$.

Then S_1 is a summand of M and hence a summand of B. Let L_1 be such that $B = S_1 \oplus L_1$. Then $l_M r_R(m) = A \oplus S_1 \oplus L_1$.

Similarly, $L_1 \cap Sm$ is a finite direct sum of simple submodules. If every simple submodule of $L_1 \cap Sm$ is in $Rad(_SM)$, then M_R is almost semiregular. Assume that there exists a simple submodule S_2 of $L_1 \cap Sm$ such that $S_2 \nsubseteq Rad(_SM)$. Then S_2 is a summand of M and so there exists a submodule L_2 such that $L_1 = S_2 \oplus L_2$. It follows that $l_M r_R(m) = A \oplus S_1 \oplus S_2 \oplus L_2$. This process produces a strictly descending chain $B \cap Sm \supset L_1 \cap Sm \supset L_2 \cap Sm \ldots$. Since $B \cap Sm$ is semisimple and finitely generated, it is Artinian. Hence, this process must stop so that $L_n \cap Sm \subseteq Rad(_SM)$ for some positive integer n. Hence, $l_M r_R(m) = (A \oplus S_1 \oplus \ldots \oplus S_n) \oplus L_n$, where $A \oplus S_1 \oplus \ldots \oplus S_n \leq Sm$ and $L_n \cap Sm \ll M$. Thus, M_R is almost semiregular. $\square$

Corollary 3.2.4 *If R is right almost S_l-semiregular, then R is right almost semiregular.*

The converse of Corollary 3.2.4 is not true in general as the next example shows.

Example 3.2.5 *There exists a right almost semiregular ring that is not right almost S_l-semiregular (S_r-semiregular).*

Proof Let $R = \mathbb{Z}_{(p)}$ be the localization of the ring of integers $\mathbb{Z}$ at a prime p. Since R is a local ring, it is semiregular whence it is right almost semiregular. We claim that the ring R is not right almost S_l-semiregular. Take a non-zero element a in $J(R)$. Since a is non-zero, we have $l_R r_R(a) = R$. Because R is indecomposable as a left R-module, the only decomposition is $l_R r_R(a) = R = R \oplus 0$. Because a is non-unit in R, we have $Ra \neq R$. On the other hand, if R was right almost S_l-semiregular, then we would have $Ra \subseteq S_l$ by the definition of the almost S_l-semiregularity. But this is a contradiction since $S_l = 0$. $\square$

If R is a right almost S_l-semiregular ring, then the ring R need not be semiregular, because right AP-injective rings need not be semiregular (see Example 3.2.1).

Lemma 3.2.6 ([15, Lemma 18.4]) *Let M be an R-module. If A is summand of M, then $(A + Soc(M))/Soc(M)$ is a summand of $M/Soc(M)$.*

Recall from Section 1.4 that R is S_l-semiregular if and only if R/S_l is a regular ring. If R is right almost S_l-semiregular, then $(Ra + S_l)/S_l$ is a summand of $(l_R r_R(a) + S_l)/S_l$ for any $a \in R$ by Lemma 3.2.6.

Nicholson and Yousif ([29]) showed that if R is an I-semiregular ring for an ideal I of R, then $J(R) \subseteq I$ and $Z_r \subseteq I$. In particular, if R is a S_l-semiregular ring, then $J(R) \subseteq S_l$ and $Z_r \subseteq S_l$. On the other hand, $J(R)$ or Z_r need not be contained in S_l if R is right almost S_l-semiregular (see Example 3.2.2).

We know from Proposition 1.6.7 that if R is a right almost semiregular ring, then $Z_r \subseteq J(R)$. Hence, if R is right almost S_l-semiregular, then $Z_r \subseteq J(R)$.

Because of the fact that $S_l \subseteq \delta_l$, R being right almost S_l-semiregular implies that R is right almost δ_l-semiregular. Also, if R is δ_l-semiregular, then $Z_r \subseteq \delta_l$. We have the following result for right almost δ_l-semiregular rings:

Proposition 3.2.7 *If R is right almost δ_l-semiregular and R/S_l is a projective right R-module, then $Z_r \subseteq \delta_l$.*

Proof Let $a \in Z_r$. If $a \notin \delta_l$, then there exists an essential maximal left ideal N of R such that $a \notin N$. Then $R = Ra + N$. Write $1 = ya + n$, where $y \in R$ and $n \in N$. Since Z_r is an ideal and $R \neq Z_r$, we have $n \neq 0$. Because $r_R(ya) \cap r_R(n) = 0$ and $ya \in Z_r$, we obtain that $r_R(n) = 0$. By hypothesis, $R = l_R r_R(n) = P \oplus Q$, where $P = Re \subseteq Rn$ for some $e^2 = e \in R$ and $Q \cap Rn \subseteq \delta_l$.

Let $\overline{R} = R/S_l$. If $\overline{R} = 0$, then R is semisimple and $Z_r = 0 \subseteq \delta_l = R$. Assume that $\overline{R} \neq 0$. If $\overline{e} = \overline{1}$, then $\overline{R}\overline{n} = \overline{N} = \overline{R}$. Since $S_l \subseteq N$, we have $N = R$, which is a contradiction. Hence, $\overline{e} \neq \overline{1}$. Since $r_R(ya) \leq_e R$, $\overline{R}/\overline{r_R(ya)} \cong R/(r_R(ya) + S_l)$ is a singular right R-module. This implies that $\overline{r_R(ya)} \leq_e \overline{R}$, because $\overline{R}$ is a projective right R-module. Since $\overline{r_R(ya)} \subseteq r_{\overline{R}}(\overline{ya})$, we have that $r_{\overline{R}}(\overline{ya}) \leq_e \overline{R}$.

Now $(\overline{1} - \overline{e})\overline{R} \cap r_{\overline{R}}(\overline{ya}) \neq 0$. Let $0 \neq (\overline{1} - \overline{e})\overline{r} \in (\overline{1} - \overline{e})\overline{R} \cap r_{\overline{R}}(\overline{ya})$. Let $n = se + t$, where $s \in R$ and $t \in Q$. Then $t = n - se \in Q \cap Rn \subseteq \delta_l$ and $\overline{t} \in \delta_l/S_l = J(R/S_l)$. So $\overline{1} - \overline{t}$ is unit in $\overline{R}$. Also, we have $\overline{n}(\overline{1} - \overline{e})\overline{r} = (\overline{1} - \overline{ya})(\overline{1} - \overline{e})\overline{r} = (\overline{1} - \overline{e})\overline{r}$ and $\overline{n}(\overline{1} - \overline{e})\overline{r} = (\overline{se} + \overline{t})(\overline{1} - \overline{e})\overline{r} = \overline{t}(\overline{1} - \overline{e})\overline{r}$. Then $(\overline{1} - \overline{t})(\overline{1} - \overline{e})\overline{r} = \overline{0}$. Hence, $(\overline{1} - \overline{e})\overline{r} = \overline{0}$, a contradiction. $\square$

Proposition 3.2.8 *If R is right almost δ_l-semiregular, R/S_l is a projective right R-module and $S_l \subseteq Z_l$, then $Z_r \subseteq J(R)$.*

Proof It follows from a proof similar to that of Proposition 3.2.7. $\square$

Example 3.2.9 *There exists a right almost δ_l (or δ_r)-semiregular ring that is not right almost semiregular.*

Proof ([26, Example 2.15]) Let F be a field and $I = \begin{bmatrix} F & F \\ 0 & F \end{bmatrix}$. Consider the ring

$$R = \{(x_1, x_2, \ldots, x_n, x, x, \ldots) \mid n \in \mathbb{N},\ x_i \in M_2(F),\ x \in I\}.$$

Zhou showed that R is a δ_r-semiregular (δ_l-semiregular) ring but it is not semiregular ([46, Example 4.3]). Since every nonzero one-sided ideal contains a nonzero idempotent, we have $Z_r = Z_l = J(R) = 0$.

Now we claim that R_R does not satisfy $(C2)$ condition. Take the element $\alpha = (x, x, \ldots)$ of R and the idempotent $g = (e, e, \ldots)$, where $x = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $e = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Then $\alpha R \cong gR$. One can observe that the idempotents in αR is of the form $f = (f_1, f_2, \ldots, f_n, 0, 0, \ldots)$, where $f_i = 0$ or $f_i = \begin{bmatrix} 1 & d \\ 0 & 0 \end{bmatrix}$, $d \in F$ for $i = 1, 2, \ldots, n$. Hence, $fR \neq \alpha R$ for each idempotent $f \in \alpha R$. Thus, R_R does not satisfy $(C2)$. By Theorem 3.2.16 below, R is not right almost semiregular. $\square$

It is well known that $J(eRe) = eJ(R)e$ for any idempotent e of R. We consider this property for δ_r which will be used in the forthcoming corollary. Recall that

$$\begin{aligned} \delta_r &= \{x \in R : \forall y \in R, \exists \text{a semisimple right ideal}\, Y \text{ of}\, R \ni R_R = (1 - xy)R \oplus Y\} \\ &= \bigcap\{\text{ideals}\, P \,\text{of}\, R \,:\, R/P \,\text{has a faithful singular simple module}\} \end{aligned}$$

.

Theorem 3.2.10 *Let e be an idempotent of R such that $ReR = R$. Then $\delta_l(eRe) = e\delta_l e$.*

Proof It is known that if e is an idempotent such that $ReR = R$, then the category of left R-modules, R-Mod, and the category of left eRe-modules, eRe-Mod, are Morita equivalent (see [21]) under the functors given by

$$\begin{aligned} \mathcal{F}: \quad & R\text{-}Mod \longrightarrow eRe\text{-}Mod, & \mathcal{G}: \quad & eRe\text{-}Mod \longrightarrow R\text{-}Mod \\ & M \longmapsto eM & & T \longmapsto Re \otimes_{eRe} T. \end{aligned}$$

We know from Corollary 1.1.25 that $\delta_l = R$ if and only if R is semisimple. Therefore, if $\delta_l = R$, then R is semisimple and so is eRe. This gives that $\delta_l(eRe) = eRe = e\delta_l e$.

Now assume that $\delta_l \neq R$. Let P be an ideal of R such that R/P has a faithful singular simple module N. Denote $\overline{R} = R/P$. Since $\overline{R}\overline{e}\overline{R} = \overline{R}$, the categories $\overline{R}$-Mod and $\overline{e}\overline{R}\overline{e}$-Mod are Morita equivalent. It is known that being faithful ([21, 18.47 and 18.30]), being a singular module ([18, p. 34]) and being a simple module ([30, Corollary A.8]) are Morita invariant properties. Therefore, $\overline{e}N$ is a faithful singular simple $\overline{e}\overline{R}\overline{e}$-module. Since $\overline{e}\overline{R}\overline{e} \cong eRe/ePe$, we have that $\delta_l(eRe) \subseteq ePe \subseteq P$. This holds for any ideal P such that R/P has a faithful singular simple module. Thus, $\delta_l(eRe) \subseteq e\delta_l e$.

For the reverse inclusion, let $a \in \delta_l$. Then $Reae \ll_\delta R$. Now we claim that $eRe(eae) \ll_\delta eRe$. Let K be a left ideal of eRe such that $eRe = eRe(eae) + K$. Write $e = ereae + k$, where $r \in R$ and $k \in K$. This implies that $1 = e + (1 - e) = ereae + k + (1 - e) \in Reae + RK + R(1 - e)$ and so $R = Reae + RK + R(1 - e)$. Since $Reae \ll_\delta R$, there exists a semisimple projective left ideal Y of R such that $Y \subseteq Reae$ and $R = Y \oplus [RK + R(1 - e)]$ by Lemma 1.1.20. Hence, we obtain that $eRe = eYe + (eRe)K = eY + K$. Since $Y \cap RK = 0$, we have that $eY \cap K = 0$. But since $ReR = R$, eY is a semisimple projective left eRe-module. So $eRe = eY \oplus K$, $eY \subseteq eRe(eae)$ and eY is a semisimple projective eRe-module. By Lemma 1.1.20, $eRe(eae) \ll_\delta eRe$. Thus, $e\delta_l e \subseteq \delta_l(eRe)$. □

Corollary 3.2.11 *Let e be an idempotent of R such that $ReR = R$. If R is right almost δ_l-semiregular, then eRe is right almost $\delta_l(eRe)$-semiregular.*

Proof It follows from Theorems 3.2.10 and 3.1.10. □

Now we will consider right semicentral idempotents.

Theorem 3.2.12 *If e is a right semicentral idempotent of R, then $e\delta_l e \subseteq \delta_l(eRe)$ and $\delta_r(eRe) \subseteq e\delta_r e$.*

Proof Let $a \in \delta_l$. Since δ_l is an ideal, $eae \in \delta_l$. By Theorem 1.1.24, there exists a semisimple left ideal Y of R such that ${}_RR = R(1-eae) \oplus Y$. Let $1 = x(1-eae)+y$, where $x \in R$ and $y \in Y$. Then $e = ex(1-eae)e + eye = exe(e-eae) + eye$ and so $eRe = eRe(e-eae) + eYe$. Since e is right semicentral, this sum is direct. Now we claim that eYe is a semisimple eRe-module. Let $Y = \oplus_{i=1}^{n} S_i$, where S_i is a simple left R-module, for $i = 1, 2, \ldots, n$. Since e is right semicentral, $eYe = \oplus_{i=1}^{n} eS_ie$. Let $S_1 = Rs$ for some $s \in R$. Then $eS_1e = eRse = eRe(ese) \cong eRe/l_{eRe}(ese)$. Let K be a left ideal of eRe such that $l_{eRe}(ese) \subset K$. Then there exists $k \in K$ such that $k \notin l_{eRe}(ese)$. Since $l_{eRe}(ese) = l_{eRe}(es) = l_R(es) \cap eRe$, we have $k \notin l_R(es)$. Then $kes \neq 0$. But since $l_R(s)$ is maximal in R, we have that $l_R(s) + Rke = R$. Let $1 = x+yke$, where $x \in l_R(s)$ and $y \in R$. Then $e = ex+eyek$. Since $xs = 0$, we have $exese = 0$. Then $ex \in l_{eRe}(ese) \subset K$, so $ex \in K$. It follows that $e \in K$. Hence, we show that $l_{eRe}(ese)$ is a maximal left ideal of eRe. So eS_1e is simple. This proves that eYe is semisimple.

Now $eRe = eRe(e-eae) \oplus eYe$ with eYe semisimple. Since a is any element in δ_l, we have that $e\delta_l e \subseteq \delta_l(eRe)$.

For the other inclusion, let P be an ideal of R and V be a faithful singular simple right R/P-module. Then Ve is an eRe-module. If $Ve = 0$, then $\delta_r(eRe) \subseteq eRe \subseteq P$.

Assume that $Ve \neq 0$. First note that V is a simple singular R-module, because $V \cong \frac{R/P}{K/P}$ for some essential maximal right ideal K/P of R/P. It follows that $V \cong R/K$ as right R-modules and K is an essential maximal right ideal of R. Hence, V is a simple singular R-module. Since V is a simple R-module, Ve is a simple eRe-module. We claim that Ve is a singular eRe-module. Let ve be the generator of Ve. To show that $r_{eRe}(ve) = r_R(v) \cap eRe$ is an essential right ideal of eRe, let $0 \neq exe \in eRe$. Since $ex \neq 0$ and $r_R(v)$ is essential in R, there exists $t \in R$ such that $0 \neq ext \in r_R(v)$. Because e is right semicentral, we have $0 \neq ext = exte \in r_{eRe}(ve)$. Hence, Ve is a singular simple eRe-module. Now, $V\delta_r(eRe) = Ve\delta_r(eRe) = 0$ by the definition of δ_r. Since V is a faithful R/P–module, we have that $\delta_r(eRe) \subseteq P$.

Therefore $\delta_r(eRe) \subseteq P$ for each ideal P of R such that R/P has a faithful singular simple module. So $\delta_r(eRe) \subseteq \delta_r$ and hence $\delta_r(eRe) \subseteq e\delta_r e$. $\square$

Corollary 3.2.13 *Let e be a right semicentral idempotent of R. If R is right almost δ_l-semiregular, then eRe is right almost $\delta_l(eRe)$-semiregular.*

Proof It follows from Theorems 3.2.12 and 3.1.9. $\square$

The following example shows that the equality $e\delta_l e = \delta_l(eRe)$ does not hold even if e is a right semicentral idempotent.

Example 3.2.14 *There exists a right semicentral idempotent $e \in R$ such that $e\delta_l e \subset \delta_l(eRe)$.*

Proof Let R be the ring of 2×2 upper triangular matrices over a field F and $e = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$. Then $eR = eRe$ and $e\delta_l e = 0$, where δ_l is the first row of R. Since eRe is a semisimple projective left eRe-module, $\delta_l(eRe) = eRe$. $\square$

We have the following results about right almost Z_r-semiregular (Z_l-semiregular) rings:

Lemma 3.2.15 ([35, Lemma 2.12]) *Let $a, b \in R$ such that $aR \cong bR = eR$, where $e^2 = e \in R$. Then there exists an idempotent $f \in R$ such that $af = a$ and $r_R(a) = r_R(f)$.*

Theorem 3.2.16 *Let I be an ideal of a ring R. If R is right almost I-semiregular and $I \subseteq Z_r$, then R_R satisfies $(C2)$.*

Proof Let $a \in R$ such that $aR \cong eR$, where $e^2 = e \in R$. By Lemma 3.2.15, there exists an idempotent $f \in R$ such that $a = af$ and $r_R(a) = r_R(f)$. By the proof of Proposition 3.1.6, there exists an idempotent $h \in R$ such that $h \in Ra$ and $a(1-h) \in I$. By Lemma 1.4.10, there exists an idempotent $g \in R$ such that $g \in aR$ and $(1-g)a \in I$. Then $aR = gR \oplus S$, where $S = (1-g)aR \subseteq I$. By assumption, S is a singular right R–module. Since aR is projective, we have that $S = 0$. Thus, $aR = gR$. $\square$

Corollary 3.2.17 *If R is right almost Z_r-semiregular, then R_R satisfies $(C2)$.*

We know from [30, Lemma 2.3] that if R_R satisfies $(C2)$, then $Z_r \subseteq J(R)$. Hence, we have the following result:

Corollary 3.2.18 *If R is right almost Z_r-semiregular, then R is right almost semiregular.*

The following two examples show that the converse of Corollary 3.2.18 is not true in general.

Example 3.2.19 *There is an Artinian ring R such that R is Z_l-semiregular but not right almost Z_r-semiregular.*

Proof Let $R = \begin{bmatrix} \mathbb{Z}_4 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{bmatrix}$. Then

$$S_r = \begin{bmatrix} 2\mathbb{Z}_4 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{bmatrix}, S_l = \begin{bmatrix} 2\mathbb{Z}_4 & \mathbb{Z}_2 \\ 0 & 0 \end{bmatrix},$$

$$Z_r = l_R(S_r) = \begin{bmatrix} 2\mathbb{Z}_4 & 0 \\ 0 & 0 \end{bmatrix}, Z_l = r_R(S_l) = \begin{bmatrix} 2\mathbb{Z}_4 & \mathbb{Z}_2 \\ 0 & 0 \end{bmatrix}.$$

The ring R is Z_l-semiregular but it is not Z_r-semiregular ([31, Example 40]). Now we claim that R is not right almost Z_r-semiregular. Let $a = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ in R. Then $Ra = \begin{bmatrix} 0 & \mathbb{Z}_2 \\ 0 & 0 \end{bmatrix}$ and $l_R r_R(a) = \begin{bmatrix} 0 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{bmatrix}$. If R is right almost Z_r-semiregular, then there is a decomposition $l_R r_R(a) = P \oplus Q$, where $P \subseteq Ra$ and $Q \cap Ra \subseteq Z_r$. Since $Ra \cap Z_r = 0$, $Q \cap Ra = 0$. This implies that $Ra = P$ is a summand of $l_R r_R(a)$ which is a contradiction. Hence, R is not right almost Z_r-semiregular. $\square$

Example 3.2.20 Let R be the ring of 2×2 upper triangular matrices over a field F. The ring R is Artinian but R_R does not satisfy $(C2)$ ([30, Example 1.20]). Hence, R is right almost semiregular but not right almost Z_r-semiregular.

A well known result of Utumi asserts that if R_R is continuous, then it is semiregular and $Z_r = J(R)$ (see Section 1.4). This fact together with Corollary 3.2.17 gives us the following result:

Proposition 3.2.21 *A ring R is right almost Z_r-semiregular and R_R satisfies $(C1)$ if and only if R is right continuous.*

In [1, Corollary 3.5], it was proved that a finitely generated projective module M is continuous if and only if M is $Z(M)$-semiregular and M satisfies $(C1)$. Hence, Proposition 3.2.21 generalizes this result in the ring case.

Proposition 3.2.22 *If R is right almost $Z_l \cap \delta_l$-semiregular, then it is right almost semiregular.*

Proof Let $a \in R$. Then there exists a decomposition $l_R r_R(a) = P \oplus Q$ such that $P \subseteq Ra$ and $Q \cap Ra \subseteq Z_l \cap \delta_l$. We claim that $Q \cap Ra \subseteq J(R)$. Let $x \in Q \cap Ra$. To see that $x \in J(R)$, we must show that $1 - yx$ is left invertible in R for any $y \in R$. Let $u = 1 - yx$, where $y \in R$. Since $x \in \delta_l$, there exists a semisimple left ideal Y of R such that $R(1 - yx) \oplus Y = R$ by Theorem 1.1.24. Let $\varphi : R \to Y$ be the projection. Then $\varphi(Q \cap Ra) \subseteq \varphi(Z_l) \subseteq Z(Y) = 0$, and so $Ryx \subseteq Q \cap Ra \subseteq Ker\varphi = R(1 - yx)$. Since $R = Ryx + R(1 - yx)$, we have that $R = R(1 - yx)$. Hence, $x \in J(R)$ and $Q \cap Ra \ll R$. $\square$

Proposition 3.2.23 *If R is right almost I-semiregular for an ideal I such that $J(R) \cap I = 0$, then $J(R) \subseteq Z_r$.*

Proof Let $a \in J(R)$ and assume that $a \notin Z_r$. Then there exists a nonzero right ideal K of R such that $r_R(a) \cap K = 0$. Take $s \in K$ such that $as \neq 0$. Let $0 \neq u \in asR$. By hypothesis, there is a decomposition $l_R r_R(u) = P \oplus Q$, where $P \subseteq Ru$, $Q \cap Ru \subseteq I$. Without loss of generality we can assume that $u = as$. Then it can be seen that $r_R(as) = r_R(s)$. Then $l_R r_R(as) = l_R r_R(s) = P \oplus Q$. Write $s = das + x$, where $d \in R$ and $x \in Q$. Then $(1 - da)s = x$ and so $u = as = a(1 - da)^{-1}x \in J(R) \cap (Q \cap Ru) \subseteq J(R) \cap I = 0$, a contradiction. Hence, $a \in Z_r$. $\square$

Corollary 3.2.24 *If R is right almost S_l-semiregular and R/S_l is a projective right R-module, then $J(R) = Z_r$ and R is right almost Z_r-semiregular.*

Proof Since S_l is a summand of R, $J(R) \cap S_l = Rad(S_l) = 0$. By Proposition 3.2.23, $J(R) \subseteq Z_r$. By Corollary 3.2.4, R is right almost semiregular. By Proposition 1.6.7, $Z_r \subseteq J(R)$. Hence, $J(R) = Z_r$ and R is right almost Z_r-semiregular. □

The following example shows that the assumption "$J(R) \cap I = 0$" in Proposition 3.2.23 is not removable in case $I = Z_l$.

Example 3.2.25 Let R be the ring in Example 3.2.19. R is a right almost Z_l-semiregular ring. Since $J(R) = \begin{bmatrix} 2\mathbb{Z}_4 & \mathbb{Z}_2 \\ 0 & 0 \end{bmatrix}$, we obtain $J(R) \cap Z_l \neq 0$ and $J(R) \nsubseteq Z_r$.

4 CHAIN CONDITIONS ON NON-SUMMANDS

Let M be an R-module. By a *non-summand* of the module M we mean a submodule K which is not a direct summand of M. Among the non-summands of M we could mention proper essential submodules and non-zero small submodules. This chapter is concerned with the study of ascending and descending chain conditions (respectively, acc and dcc) on certain non-summands.

Goodearl ([17, Proposition 3.6]) proved that an R-module M satisfies acc on essential submodules if and only if $M/\mathrm{Soc}(M)$ is Noetherian. Goodearl's result has a dual due to Armendariz ([5, Proposition 1.1]) who proved that a module M satisfies dcc on essential submodules if and only if $M/\mathrm{Soc}(M)$ is Artinian. The results of Goodearl and Armendariz can also be found at [15, 5.15]. Varadarajan ([38, Lemma 2.1]) proved that a module M satisfies acc on small submodules if and only if $\mathrm{Rad}(M)$ is Noetherian; Al-Khazzi and Smith ([2, Theorem 5]) proved that a module M satisfies dcc on small submodules if and only if $\mathrm{Rad}(M)$ is Artinian. We shall give an example of a commutative von Neumann regular ring R such that R satisfies acc and dcc on essential ideals and on small ideals but R satisfies neither acc nor dcc on non-summands.

In this chapter, modules satisfying ascending or descending chain conditions on non-summand submodules belongs to some particular classes $\mathcal{X}$, such as the class of all R-modules, finitely generated, finite dimensional and cyclic modules, are considered. It is proved that a module M satisfies acc (respectively, dcc) on non-summands if and only if M is semisimple or Noetherian (respectively, Artinian). Over a right Noetherian ring R, a right R-module M satisfies acc on finitely generated non-summands if and only if M satisfies acc on non-summands; a right R-module M satisfies dcc on finitely generated non-summands if and only if M is locally Artinian. Moreover, if a ring R satisfies dcc on cyclic non-summand right ideals, then R is a semiregular ring such that the Jacobson radical $J(R)$ is left T-nilpotent.

4.1 Module Classes

Let R be a ring. By a *class* $\mathcal{X}$ of R-modules we mean a collection of R-modules which contains a zero module and which is closed under isomorphisms. If a module belongs to $\mathcal{X}$, then we say that it is an *$\mathcal{X}$-module*. By an *$\mathcal{X}$-submodule* (respectively, *$\mathcal{X}$-summand*, *$\mathcal{X}$-non-summand*) we mean an $\mathcal{X}$-module which is also a submodule (respectively, summand, non-summand) of M. This section is concerned with chain conditions on $\mathcal{X}$-non-summands of a module. It is clear that every semisimple R-module satisfies both acc and dcc on $\mathcal{X}$-non-summands and that every Noetherian (respectively, Artinian) R-module satisfies acc (respectively, dcc) on $\mathcal{X}$-non-summands. Note the following elementary fact:

Proposition 4.1.1 *An R-module M satisfies acc (respectively, dcc) on $\mathcal{X}$-submodules if and only if M satisfies acc (respectively, dcc) both on $\mathcal{X}$-summands and on $\mathcal{X}$-non-summands.*

Recall that a module is Noetherian if and only if it satisfies acc on finitely generated submodules. Thus Proposition 4.1.1 shows that a module M is Noetherian if and only if M satisfies acc both on finitely generated summands and on finitely generated non-summands. Note too that every finite dimensional module satisfies acc and dcc on summands so that we have the following immediate corollary to Proposition 4.1.1.

Corollary 4.1.2 *A finite dimensional module satisfies acc (respectively, dcc) on $\mathcal{X}$-non-summands if and only if M satisfies acc (respectively, dcc) on $\mathcal{X}$-submodules.*

Lemma 4.1.3 *Let M be a module which satisfies acc (respectively, dcc) on $\mathcal{X}$-non-summands. Then every submodule of M satisfies acc (respectively, dcc) on $\mathcal{X}$-non-summands.*

Proof Let N be any submodule of M. If K is an $\mathcal{X}$-non-summand of N then K is an $\mathcal{X}$-non-summand of M. The result follows. $\square$

Lemma 4.1.4 *Let $\mathcal{X}$ be a class of R-modules which is closed under extensions and let N be an $\mathcal{X}$-submodule of an R-module M. Suppose that M satisfies acc (respectively, dcc) on $\mathcal{X}$-non-summands. Then M/N satisfies acc (respectively, dcc) on $\mathcal{X}$-non-summands.*

Proof Let K be a submodule of M containing N such that K/N is an $\mathcal{X}$-non-summand of M/N. Then K is an $\mathcal{X}$-submodule of M because $\mathcal{X}$ is closed under extensions. Moreover, K is a non-summand of M. Thus K is an $\mathcal{X}$-non-summand of M. The result follows. □

We shall see in Section 4.2 that if N is a submodule of an R-module M such that the modules N and M/N both satisfy acc (respectively, dcc) on non-summands then M need not satisfy acc (respectively, dcc) on non-summands. Indeed, more is true. We shall give an example of R-modules A_1 and A_2 which both satisfy acc on non-summands such that the module $A_1 \oplus A_2$ does not satisfy acc on non-summands and also an example of R-modules B_1 and B_2 which both satisfy dcc on non-summands but $B_1 \oplus B_2$ does not satisfy dcc on non-summands. However, in some situations the direct sum of modules with acc (respectively, dcc) on $\mathcal{X}$-non-summands also has the same property. For example, note the following result:

Lemma 4.1.5 *Let $\mathcal{X}$ be a class of R-modules such that, for each non-zero $\mathcal{X}$-module X, every non-zero submodule of X contains a non-zero $\mathcal{X}$-submodule. Let an R-module $M = M_1 \oplus M_2$ be a direct sum of submodules M_i $(i = 1,2)$ such that M_1 contains no non-zero $\mathcal{X}$-submodule and M_2 satisfies acc (respectively, dcc) on $\mathcal{X}$-non-summands. Then M satisfies acc (respectively, dcc) on $\mathcal{X}$-non-summands.*

Proof Let L be an $\mathcal{X}$-non-summand of M. By hypothesis, $L \cap M_1 = 0$. Let $\pi : M \to M_2$ denote the canonical projection. Then $\pi(L) \cong L$ so that $\pi(L)$ is an $\mathcal{X}$-submodule of M_2. Next note that $M_1 \oplus L = M_1 \oplus \pi(L)$ so that $\pi(L)$ is a non-summand of M and hence also of M_2, because L is a non-summand of M. Let $L_1 \subseteq L_2 \subseteq \dots$ be any ascending chain of $\mathcal{X}$-non-summands of M. Then $\pi(L_1) \subseteq \pi(L_2) \subseteq \dots$ is an ascending chain of $\mathcal{X}$-non-summands of M_2. Suppose that there exists a positive integer n such that $\pi(L_n) = \pi(L_{n+1}) = \dots$. Then $M_1 \oplus L_n = M_1 \oplus L_{n+1} = \dots$ and hence $L_n = L_{n+1} = \dots$. Thus if M_2 satisfies acc on $\mathcal{X}$-non-summands then so too does M. A similar result give the proof for descending chains. $\square$

In particular, Lemma 4.1.5 applies to classes $\mathcal{X}$ which are closed under taking submodules. However, it applies more widely. For example, the class of finitely generated R-modules is not closed under taking submodules (if R is not right Noetherian) but satisfies the property of Lemma 4.1.5.

Lemma 4.1.6 *Let $\mathcal{X}$ be a class of modules closed under finite direct sums. Let M be a module which satisfies acc (respectively, dcc) on $\mathcal{X}$-non-summands. Let L and N be submodules of M such that $L \cap N = 0$. Then L satisfies acc (respectively, dcc) on $\mathcal{X}$-submodules or every $\mathcal{X}$-submodule of N is a direct summand of M and hence also of N.*

Proof Suppose that M satisfies acc on $\mathcal{X}$-non-summands. By Lemma 4.1.3, the module $L \oplus N$ also satisfies acc on $\mathcal{X}$-non-summands. Suppose there exists an $\mathcal{X}$-submodule K of N which is not a direct summand of M. Let $H_1 \subseteq H_2 \subseteq \dots$ be any ascending chain of $\mathcal{X}$-submodules of L. For each $i \geq 1$, $H_i \cap K = 0$ and $H_i \oplus K$ is an $\mathcal{X}$-non-summand of M (otherwise, K is a direct summand of M). Thus $H_1 \oplus K \subseteq H_2 \oplus K \subseteq \dots$ is an ascending chain of $\mathcal{X}$-non-summands of M and, by hypothesis, $H_n \oplus K = H_{n+1} \oplus K = \dots$ for some positive integer n. It follows that $H_n = H_{n+1} = \dots$. Thus L satisfies acc on $\mathcal{X}$-submodules. The proof for descending chains is similar. $\square$

Theorem 4.1.7 *Let $\mathcal{X}$ be a class of R-modules which is closed under finite direct sums and under taking direct summands. Then an R-module M satisfies acc on $\mathcal{X}$-non-summands if and only if, for every $\mathcal{X}$-non-summand N of M, M satisfies acc on $\mathcal{X}$-submodules which contain N.*

Proof The sufficiency is clear. Conversely, suppose that M satisfies acc on $\mathcal{X}$-non-summands. Let L be any $\mathcal{X}$-submodule of M such that there exists a properly ascending chain $L = L_1 \subset L_2 \subset \dots$ of $\mathcal{X}$-submodules of M. By hypothesis, there exists a positive integer n such that L_n is a direct summand of M. Let N be a submodule of M such that $M = L_n \oplus N$. For each $i \geq n$, $L_i = L_n \oplus (L_i \cap N)$. By hypothesis, $L_n \cap N \subset L_{n+1} \cap N \subset \ldots$ is a properly ascending chain of $\mathcal{X}$-submodules of N. By Lemma 4.1.6, L is a direct summand of M. The result follows. $\square$

Corollary 4.1.8 *Let $\mathcal{X}$ be a class of R-modules which is closed under extensions and also under taking homomorphic images. Then an R-module M satisfies acc on $\mathcal{X}$-non-summands if and only if M/N satisfies acc on $\mathcal{X}$-submodules for every $\mathcal{X}$-non-summand N of M.*

Proof Suppose first that M satisfies acc on $\mathcal{X}$-non-summands. Let N be any $\mathcal{X}$-non-summand of M. Let $\bar{L}_1 \subseteq \bar{L}_2 \subseteq \ldots$ be any ascending chain of $\mathcal{X}$-submodules of M/N. For each $i \geq 1$, $\bar{L}_i = L_i/N$ for some submodule L_i of M containing N. By hypothesis, L_i is an $\mathcal{X}$-submodule of M for all $i \geq 1$. By Theorem 4.1.7, $L_n = L_{n+1} = \ldots$ and hence $\bar{L}_n = \bar{L}_{n+1} = \ldots$ for some positive integer n. Thus M/N satisfies acc on $\mathcal{X}$-submodules.

Conversely, suppose that M/N satisfies acc on $\mathcal{X}$-submodules for each $\mathcal{X}$-non-summand N of M. Let L be any $\mathcal{X}$-non-summand of M and let $H_1 \subseteq H_2 \subseteq \ldots$ be any ascending chain of $\mathcal{X}$-submodules of M such that $L \subseteq H_1$. Then $H_1/L \subseteq H_2/L \subseteq \ldots$ is an ascending chain of $\mathcal{X}$-submodules of M/L. There exists a positive integer k such that $H_k/L = H_{k+1}/L = \ldots$ and hence $H_k = H_{k+1} = \ldots$. By Theorem 4.1.7, M satisfies acc on $\mathcal{X}$-non-summands. $\square$

The next result is a companion theorem to Theorem 4.1.7.

Theorem 4.1.9 *Let $\mathcal{X}$ be a class of R-modules which is closed under finite direct sums and under taking direct summands. Then an R-module M satisfies dcc on $\mathcal{X}$-non-summands if and only if every $\mathcal{X}$-non-summand of M satisfies dcc on $\mathcal{X}$-submodules.*

Proof The sufficiency is clear. Conversely, suppose that M satisfies dcc on $\mathcal{X}$-non-summands. Let N be any $\mathcal{X}$-non-summand of M. Suppose that N does not satisfy dcc on $\mathcal{X}$-submodules and let $N_1 \supset N_2 \supset \ldots$ be a properly descending chain of $\mathcal{X}$-submodules of N. By hypothesis, there exists a positive integer k such that

N_k is a direct summand of M. Let L be a submodule of M such that $M = N_k \oplus L$. Now $N_k \supset N_{k+1} \supset \dots$ is a properly descending chain of $\mathcal{X}$-submodules of N_k so that, by Lemma 4.1.6, every $\mathcal{X}$-submodule of L is a direct summand of L. However, $N = N_k \oplus (N \cap L)$ gives that $N \cap L$ is a direct summand of L and hence N is a direct summand of M, a contradiction. The result follows. $\square$

4.2 The Class of R-modules

Let R be any ring. In this section, we consider modules with ascending or descending chain conditions on $\mathcal{X}$-non-summands, where $\mathcal{X} = Mod - R$. Lemma 4.1.6 has the following immediate consequence:

Lemma 4.2.1 *Let M be a module which satisfies acc (respectively, dcc) on non-summands and let L and N be submodules of M such that $L \cap N = 0$. Then L is Noetherian (respectively, Artinian) or N is semisimple.*

Now let R be a right Noetherian ring which is not semiprime Artinian and let U be any non-finitely generated semisimple R-module. Then the R-modules R and U both satisfy acc on non-summands, but Lemma 4.2.1 shows that the module $R \oplus U$ does not satisfy acc on non-summands. In the same way, if R is right Artinian (but not semiprime) then the R-modules R and U both satisfy dcc on non-summands but the module $R \oplus U$ does not satisfy dcc on non-summands by Lemma 4.2.1.

Theorem 4.2.2 *An R-module M satisfies acc on non-summands if and only if M is semisimple or Noetherian.*

Proof The necessity is clear. For the sufficiency assume that M satisfies acc on non-summands. Since M satisfies acc on essential submodules, $M/\mathrm{Soc}(M)$ is Noetherian (see Section 1.9). If $\mathrm{Soc}(M)$ is finitely generated, then M is Noetherian. Suppose that $\mathrm{Soc}(M)$ is not finitely generated. Then $\mathrm{Soc}(M) = S_1 \oplus S_2$ for some non-finitely generated submodules S_1, S_2. Because S_2 is not Noetherian, $M = S_1 \oplus L$ for some submodule L of M by Lemma 4.1.6. But S_1 not being Noetherian gives that L is semisimple by Lemma 4.2.1. Hence, if $\mathrm{Soc}(M)$ is not finitely generated then M is semisimple. $\square$

Theorem 4.2.2 has the following analogue. The proof is rather similar but we give it for completeness.

Theorem 4.2.3 *A module M satisfies dcc on non-summands if and only if M is semisimple or Artinian.*

Proof The necessity is clear. For the sufficiency assume that M satisfies dcc on non-summands. Since M satisfies dcc on essential submodules, $M/Soc(M)$ is Artinian (see Section 1.9). If $\mathrm{Soc}(M)$ is finitely generated, then M is Artinian. On the other hand, if $\mathrm{Soc}(M)$ is not finitely generated then M is semisimple by the proof of Theorem 4.2.2. □

Theorems 4.2.2 and 4.2.3 have the following immediate consequence which also give a new characterization of right Noetherian and right Artinian rings.

Corollary 4.2.4 *For any ring R, a finitely generated R-module M satisfies acc (respectively, dcc) on non-summands if and only if M is Noetherian (respectively, Artinian).*

In particular, for a ring R, if R_R satisfies dcc on non-summands, then R_R satisfies acc on non-summands.

Now we give an example to show that there exist modules with acc (respectively, dcc) on essential and on small submodules but which do not have acc (respectively, dcc) on non-summands.

Example 4.2.5 *Let K be any field and let S be the commutative ring which is the direct product of a countably infinite number of copies of K, that is, $S = \prod_{i=1}^{\infty} K_i$, where $K_i = K$ for all $i \geq 1$. Let R denote the subring of S consisting of all elements $\{k_i\}$ such that $k_i \in K\,(i \in I)$ and $k_n = k_{n+1} = \dots$ for some positive integer n. Then R is a commutative von Neumann regular ring which satisfies acc and dcc on essential ideals and on small ideals but satisfies neither acc nor dcc on non-summand ideals.*

Proof It is clear that R is a von Neumann regular ring. Thus, $J(R) = 0$ and trivially R satisfies acc and dcc on small ideals. Moreover, S_r is the set of elements $\{k_i\}$ of R such that, for some positive integer n, $k_i = 0$ for all $i \geq n$. Thus, R/S_r is isomorphic to K and R satisfies acc and dcc on essential ideals. By Corollary 4.2.4, R does not satisfy acc on non-summand ideals and also does not satisfy dcc on non-summand ideals. □

4.3 The Class of Finitely Generated and Finite Dimensional Modules

In this section, we let $\mathcal{X}$ denote the class of finitely generated R-modules. Clearly, regular modules satisfy both acc and dcc on finitely generated non-summands.

Lemma 4.3.1 *Let M be an R-module. If every cyclic submodule of M is a finite dimensional direct summand, then M is semisimple.*

Proof Let $M \neq 0$ and let $0 \neq m \in M$. Because mR is finite dimensional, there exist a positive integer n and non-zero indecomposable submodules $L_i\,(1 \leq i \leq n)$ of mR such that $mR = L_1 \oplus \cdots \oplus L_n$. Let $1 \leq i \leq n$ and let $0 \neq x \in L_i$. By hypothesis, xR is a direct summand of M, and hence also of L_i so that $L_i = xR$. It follows that L_i is simple for all $1 \leq i \leq n$. Therefore, mR is semisimple for all $m \in M$. It follows that M is semisimple. □

Theorem 4.3.2 *The following statements are equivalent for an R-module M:*

(i) *M satisfies acc on finitely generated non-summands.*

(ii) *M/L is Noetherian for every finitely generated non-summand L of M.*

(iii) *For every non-finitely generated submodule N of M, every finitely generated submodule of N is a direct summand of M.*

Proof (i) $\Leftrightarrow$ (ii) By Corollary 4.1.8.

(ii) $\Rightarrow$ (iii) Let N be any non-finitely generated submodule of M. Let L be any finitely generated submodule of N. If L is not a direct summand of M, then M/L

is Noetherian by (ii). Hence, N is finitely generated, a contradiction. Thus, every finitely generated submodule L of N is a direct summand of M.

(iii) $\Rightarrow$ (ii) Let H be any finitely generated non-summand of M. By (iii), every submodule of M containing H is finitely generated, and hence M/H is Noetherian. $\square$

Corollary 4.3.3 *Let R be a right Noetherian ring. Then the following statements are equivalent for an R-module M:*

(i) *M satisfies acc on non-summands.*

(ii) *M satisfies acc on finitely generated non-summands.*

(iii) *M is semisimple or Noetherian.*

Proof The implications (i) $\Rightarrow$ (ii) and (iii) $\Rightarrow$ (i) are obvious. For (ii) $\Rightarrow$ (iii), assume that the module M is not Noetherian. Then M is not finitely generated since R is right Noetherian. By Theorem 4.3.2(iii), every finitely generated submodule of M is a direct summand. It follows from Lemma 4.3.1 that M is semisimple. $\square$

Corollary 4.3.4 *Let M be an R-module which satisfies acc on finitely generated non-summands. Then M is Noetherian or M contains an essential submodule N such that every finitely generated submodule of N is a direct summand of M.*

Proof If M is finite dimensional then M is Noetherian by Corollary 4.1.2. Suppose that M is not finite dimensional. Let a submodule $L = L_1 \oplus L_2 \oplus \dots$ be a direct sum of non-zero submodules $L_i\,(i \geq 1)$ of M. Let K be a complement of L in M and let $N = L \oplus K$. Then N is an essential submodule of M. Suppose that H is any finitely generated submodule of N. Note that $H \subseteq L_1 \oplus \cdots \oplus L_n \oplus K$ for some positive integer n, and hence $L_{n+1} \oplus L_{n+2} \oplus \dots$ embeds in M/H. Thus, M/H is not Noetherian so that H is a direct summand of M by Theorem 4.3.2. $\square$

Next we aim to give an example of a module which satisfies acc on finitely generated non-summands but which is neither Noetherian nor regular. First we state a well known lemma whose proof we shall include for completeness.

Lemma 4.3.5 *Let N be a finitely generated submodule of an R-module M such that every cyclic submodule of N is a direct summand of M. Then N is a direct summand of M.*

Proof There exist a positive integer k and elements $x_i \in N\,(1 \le i \le k)$ such that $N = x_1R + \cdots + x_kR$. If $k = 1$, then there is nothing to prove. Suppose that $k \ge 1$. There exists a submodule L of M such that $M = x_1R \oplus L$. Then $N = x_1R \oplus (N \cap L)$. If $\pi : N \to N \cap L$ is the canonical projection, then $N \cap L$ is generated by the (k-1) elements $\pi(x_2), \ldots, \pi(x_k)$. By induction, $N \cap L$ is a direct summand of M, and hence also of L. It follows that N is a direct summand of M. □

Example 4.3.6 *Let D be a commutative Noetherian domain with field of fractions $K \neq D$. Let T be the subring of the ring R of Example 4.2.5 consisting of all elements $\{k_i\}$ of R such that, for some positive integer n, $k_i \in D$ for all $i \ge n$. Then T is a commutative ring such that the T-module T satisfies acc on finitely generated non-summands but T is not Noetherian nor regular.*

Proof Note that $\mathrm{Soc}(T) = \mathrm{Soc}(R)$. Let L be any finitely generated non-summand of T_T. By Lemma 4.3.5, there exists $x \in L$ such that xT is not a direct summand of T. Then $x = \{k_i\}$, where $k_n = k_{n+1} = \ldots$ and k_n is a non-zero element of D, for some positive integer n. Then xT contains all elements of T of the form $\{h_i\}$, where $h_i = 0$ for all $1 \le i \le n-1$ and for all $i \ge m$ for some integer $m \ge n+1$. It follows that $\mathrm{Soc}(T)/(xT \cap \mathrm{Soc}(R))$ is Noetherian. But $T/\mathrm{Soc}(T) \cong D$ so that $T/\mathrm{Soc}(T)$ is Noetherian. This implies that T/xT, and hence also T/L, is Noetherian. By Theorem 4.3.2, T_T satisfies acc on finitely generated non-summands. T is not Noetherian because $\mathrm{Soc}(T)$ is not finitely generated. Also, if a is any non-zero non-unit element of D and s is the element $\{k_i\}$ of T with $k_i = a$ for all $i \ge 1$ then sT is not a direct summand of T so that T_T is not regular. □

We now consider modules which satisfy dcc on finitely generated non-summands.

Remark 4.3.7 An R-module M satisfies dcc on finitely generated submodules if and only if M satisfies dcc on cyclic submodules, and in this case M is semiartinian (see [41, 31.8]).

Theorem 4.3.8 *Let M be an R-module. Every finitely generated submodule of M is a direct summand of M if and only if M satisfies dcc on finitely generated non-summands and every simple submodule is a direct summand of M.*

Proof The necessity is clear. Conversely, suppose that M satisfies the stated conditions. Suppose further that M contains a finitely generated non-summand. Let L be a minimal finitely generated non-summand of M. Note that $L \neq 0$. Let x be any non-zero element of L. Suppose that $L \neq xR$. It follows that xR is a direct summand of M so that $M = xR \oplus N$ for some submodule N of M. Now $L = xR \oplus (L \cap N)$, and hence $L \cap N$ is a finitely generated submodule of M. Clearly, $L \neq L \cap N$ and this implies that $L \cap N$ is a direct summand of M, and hence also of N, giving the contradiction that L is a direct summand of M. Thus, $L = xR$ for every non-zero element x of L. It follows that L is a simple module, a contradiction. □

Theorem 4.1.9 gives the following result without further proof.

Theorem 4.3.9 *An R-module M satisfies dcc on finitely generated non-summands if and only if every finitely generated non-summand of M satisfies dcc on finitely generated submodules.*

Compare the next result with Corollary 4.3.4.

Proposition 4.3.10 *Let M be a module which satisfies dcc on finitely generated non-summands. Then every finitely generated submodule of M is a direct summand of M or M has essential socle.*

Proof Suppose that the module M has a finitely generated non-summand. By Theorem 4.3.8, M contains a simple submodule L which is not a direct summand. Let H be a complement of L in M so that $L \oplus H$ is an essential submodule of M. By Lemma 4.1.6, H satisfies dcc on finitely generated submodules, and hence H has essential socle by Remark 4.3.7. It follows that M has essential socle. □

Clearly, locally Artinian modules satisfy dcc on finitely generated submodules. Compare the next result with Corollary 4.3.3.

Theorem 4.3.11 *Let R be a right Noetherian ring. Then an R-module M satisfies dcc on finitely generated non-summands if and only if M is locally Artinian.*

Proof The sufficiency is clear. Conversely, suppose that M satisfies dcc on finitely generated non-summands. Let N be any finitely generated submodule of M. If $N \subseteq \mathrm{Soc}M$, then N is Artinian. Suppose that $N \nsubseteq \mathrm{Soc}M$. Then there exists a maximal submodule L of N such that $N \cap \mathrm{Soc}M \subseteq L$. Suppose that L is not Artinian and let $L_1 \supset L_2 \supset \dots$ be any properly descending chain of submodules of L. Because N is finitely generated, so too is L_i for each $i \geq 1$, and hence L_k is a direct summand of M for some positive integer k. There exists a submodule H of M such that $M = L_k \oplus H$. By Lemma 4.1.6, every finitely generated submodule of H is a direct summand of H. Let $0 \neq h \in H$. Every submodule of hR is finitely generated, because R is right Noetherian, and hence is a direct summand of hR. Thus hR is semisimple for every non-zero $h \in H$. It follows that H is semisimple and thus $H \subseteq \mathrm{Soc}M$. It follows that $M = L_k + \mathrm{Soc}M$, and hence $N = L_k + (N \cap \mathrm{Soc}M) \subseteq L$, a contradiction. Thus, L is Artinian. So N is Artinian, too. It follows that M is locally Artinian. □

The condition that R is right Noetherian cannot be removed in Theorem 4.3.11.

Example 4.3.12 *Let R be a commutative ring with unique maximal ideal J such that $J^2 = 0$ and J is not finitely generated. Then J is a non-finitely generated semisimple R-module. The R-module R is not Artinian and hence is not locally Artinian. Let A be a finitely generated non-summand of R. Then $A \neq R$ so that $A \subseteq J$. Hence, A is Artinian because A is semisimple. Thus, the R-module R satisfies dcc on finitely generated non-summands but is not locally Artinian and is not regular.*

To be specific, let V be an infinite dimensional vector space over a field F. Consider the ring $R = \{\begin{bmatrix} x & v \\ 0 & x \end{bmatrix} | x \in F, v \in V\}$ and $J = \begin{bmatrix} 0 & V \\ 0 & 0 \end{bmatrix}$. Then R and J satisfy the stated conditions.

Remark 4.3.13 If N is a fully-invariant submodule of an R-module M, then $N = (N \cap M_1) \oplus (N \cap M_2)$, and hence $M/N = ((M_1 + N)/N) \oplus ((M_2 + N)/N)$ for all submodules M_1 and M_2 of M such that $M = M_1 \oplus M_2$.

Compare the following with Lemma 4.1.4.

Lemma 4.3.14 *If an R-module M satisfies dcc on finitely generated non-summands then so too does every factor module M/N, where N is a fully-invariant submodule of M.*

Proof Let N be a nonzero fully-invariant submodule of M. Let $\overline{M} = M/N$ and let $\overline{K} \subseteq \overline{L}$ be finitely generated submodules of $\overline{M}$. There exist positive integers s, t and elements x_i, y_j in $M\,(1 \leq i \leq s, 1 \leq j \leq t)$ such that $\overline{L} = (x_1 + N)R + \cdots + (x_s + N)R$ and $\overline{K} = (y_1 + N)R + \cdots + (y_t + N)R$. For each $1 \leq j \leq t$ there exist elements $r_{ij} \in R\,(1 \leq i \leq s)$ and $u_j \in N$ such that $y_j = \sum_{i=1}^{s} x_i r_{ij} + u_j$. Let $z_j = \sum_{i=1}^{s} x_i r_{ij}\,(1 \leq j \leq t)$. Then $\overline{K} = (z_1 + N)R + \cdots + (z_t + N)R$ and $z_1R + \cdots + z_tR \subseteq x_1R + \cdots + x_sR$.

Now let $\overline{L}_1 \supseteq \overline{L}_2 \supseteq \ldots$ be any descending chain of finitely generated non-summands of $\overline{M}$. By the above remarks, we can suppose without loss of generality that $\overline{L}_i = (L_i + N)/N\,(i \geq 1)$ for some descending chain $L_1 \supseteq L_2 \supseteq \ldots$ of finitely generated submodules of M. Remark 4.3.13 shows that L_i is a non-summand of M for each $i \geq 1$. By hypothesis, there exists a positive integer k such that $L_k = L_{k+1} = \ldots$, and hence $\overline{L}_k = \overline{L}_{k+1} = \ldots$. □

Proposition 4.3.15 *If M satisfies dcc on finitely generated non-summands, then there exists a semiartinian submodule S of M such that every finitely generated submodule of M/S is a direct summand.*

Proof Let $0 = S_0 \subseteq S_1 \subseteq \cdots \subseteq S_\alpha \subseteq S_{\alpha+1} \subseteq \ldots$ be the socle series of M, where for each ordinal $\alpha \geq 0$, $S_{\alpha+1}/S_\alpha = Soc(M/S_\alpha)$ and $S_\alpha = \bigcup_{0 \leq \beta < \alpha} S_\beta$ when α is a limit ordinal. Note that S_α is a fully-invariant submodule of M for each ordinal $\alpha \geq 0$. Because M is a set, there must exists an ordinal $\rho \geq 0$ such that $S_\rho = S_{\rho+1}$, and hence M/S_ρ has zero socle. Note that S_ρ is semiartinian.

Now suppose that M satisfies dcc on finitely generated non-summands. By Lemma 4.3.14, M/S_ρ satisfies dcc on finitely generated non-summands. Finally, by Proposition 4.3.10, every finitely generated submodule of M/S_ρ is a direct summand.
□

We have the following result for the class of finite dimensional modules.

Theorem 4.3.16 *Let M be a module such that every non-zero submodule contains a uniform submodule. Then M satisfies acc on finite dimensional non-summands if and only if M is Noetherian or every uniform submodule of M is a direct summand.*

Proof ($\Leftarrow$) Suppose M is not Noetherian. Let L be a finite dimensional submodule of M. Suppose $L \neq 0$. Let $U \leq L$ and U be uniform. Then $M = U \oplus U'$ for some $U' \leq M$. Then $L = U \oplus (L \cap U')$, where $udim(L \cap U') < udim(L)$. By induction, $L \cap U'$ is a direct summand of U' so that L is a direct summand of M.

($\Rightarrow$) Suppose M satisfies acc on finite dimensional non-summands. Suppose M contains a (non-zero) finite dimensional non-summand. We shall show that M is Noetherian. Let H be a maximal finite dimensional non-summand of M. Then $M \neq H$ because H is a non-summand of M. Suppose that H is not essential in M. Then $H \cap L = 0$ for some non-zero submodule L. By hypothesis, L contains a uniform submodule U. Then $H \oplus U$ is finite dimensional and hence, by the choice of H, a direct summand of M. This implies that H is a direct summand of M, a contradiction. Thus, H is essential in M, and so M is finite dimensional. Every submodule of M is also finite dimensional. This gives that M satisfies acc on non-summands. By Theorem 4.2.2, M is Noetherian or semisimple. But since M is finite dimensional, it is Noetherian. $\square$

4.4 The Class of Cyclic Modules

A non-empty subset I of a ring R *acts t-nilpotently* on an R-module M if, for every sequence $a_1, a_2 \dots$ of elements in I and every $m \in M$, we have $ma_1a_2 \cdots a_{i-1}a_i = 0$ for some $i \in \mathbb{N}$ (depending on m) (see [41]). The set I is called *left T-nilpotent* if it acts t-nilpotently on R_R (see Section 1.5). Recall from Section 1.5 that a ring R is left perfect if and only if $J(R)$ is left T-nilpotent and $R/J(R)$ is semisimple, and this occurs if and only if R satisfies dcc on cyclic right ideals. By Proposition 4.1.1, we have the following:

Proposition 4.4.1 *A ring R is left perfect if and only if R satisfies dcc both on summands and cyclic non-summand right ideals.*

Example 4.2.5 shows that there exists a commutative ring satisfying dcc on finitely generated (cyclic) non-summands but which is not perfect.

Proposition 4.4.2 *Let M be an R-module which satisfies dcc on cyclic non-summands. Then $J(R)$ acts t-nilpotently on M.*

Proof Let $a_1, a_2, \dots$ be a sequence of elements in $J(R)$ and $m \in M$. Consider the descending chain

$$ma_1R \supseteq ma_1a_2R \supseteq ma_1a_2a_3R \supseteq \dots .$$

Assume that there exists a k such that $ma_1a_2 \dots a_kR$ is a direct summand of M. Since $ma_1a_2 \dots a_kR \subseteq mJ(R) \subseteq RadM$, $ma_1a_2 \dots a_kR$ is small and a direct summand of M. This implies that $ma_1a_2 \dots a_k = 0$. Now we assume that for every k, $ma_1a_2 \dots a_kR$ is not a direct summand of M. By hypothesis, there exists i such that $ma_1a_2 \dots a_iR = ma_1a_2 \dots a_{i+1}R \subseteq ma_1a_2 \dots a_iJ(R)$. By Nakayama's Lemma, we have $ma_1a_2 \dots a_i = 0$. □

Proposition 4.4.3 *Let M be an R-module satisfying dcc on cyclic non-summands and let $S = End_R(M)$. Suppose that the module M is a faithful right R-module and a finitely generated left S-module. Then $J(R)$ is left T-nilpotent.*

Proof Since M_R is faithful and ${}_SM$ is finitely generated, it follows that R_R embeds in M_R^k for some positive integer k (see [41, 15.3 and 15.4]). By Proposition 4.4.2, $J(R)$ acts t-nilpotently on M_R^k. Hence, $J(R)$ is left T-nilpotent. □

Corollary 4.4.4 *If R is a ring satisfying dcc on cyclic non-summand right ideals, then $J(R)$ is left T-nilpotent.*

Remark 4.4.5 If $a \in R$ and $a - aba$ is regular for some b in R, then a is regular. For, there exists c in R such that $a - aba = (a - aba)c(a - aba)$, and thus $a = ada$, where $d = b + (1 - ba)c(1 - ab)$.

Theorem 4.4.6 *Let R be a ring which satisfies dcc on cyclic non-summand right ideals. Then R is a semiregular ring such that $J(R)$ is left T-nilpotent.*

Proof By Corollary 4.4.4, $J(R)$ is left T-nilpotent. To prove that R is semiregular, without loss of generality, we may assume that $J(R) = 0$ (adapt the proof of Lemma 4.3.14). Let $0 \neq a \in R$. There exists a maximal right ideal M_1 of R such that $a \notin M_1$. Then $1 = ar + b$ for some $r \in R, b \in M_1$. It follows that $a_1 = a - ara = ba \in M_1$. Now suppose that $a_1 \neq 0$. By the same argument, there exist a maximal right ideal M_2 and elements $r_1 \in R, b_1 \in M_2$ such that $a_2 = a_1 - a_1 r_1 a_1 = b_1 a_1 \in M_1 \cap M_2$. If $a_2 \neq 0$, then repeat the argument. This gives a sequence of elements $a = a_0, a_1, a_2, \ldots$ of R and a sequence of maximal right ideals $M_1, M_2, \ldots$ of R such that, for each $i \geq 0$, $a_{i+1} = a_i - a_i r_i a_i$ for some $r_i \in R$ and $a_i \in M_1 \cap \cdots \cap M_i$, $a_i \notin M_{i+1}$. Thus, we obtain a strictly descending chain $a_0 R \supset a_1 R \supset \ldots$. By hypothesis, there exists a positive integer n such that $a_n R$ is a direct summand of R_R. There exists an idempotent e in R such that $a_n R = eR$. It can easily be shown that a_n is regular. By Remark 4.4.5, a is regular, too. It follows that every element of R is regular and so R is von Neumann regular. $\square$

The converse of Theorem 4.4.6 need not be true. Note the following fact:

Lemma 4.4.7 *Let R be a ring. Let e be an idempotent in R such that $eR + J(R)$ is a direct summand of R_R. Then $J(R) \subseteq eR$.*

Proof Note that

$$eR+J(R) = (eR+J(R))\cap[eR\oplus(1-e)R] = eR\oplus[(1-e)R\cap(eR+J(R))] = eR\oplus(1-e)J(R).$$

It follows that $(1-e)J(R) = fR$ for some idempotent f in R. But $f \in J(R)$ so that $f = 0$. Hence, $(1-e)J(R) = 0$ and $J(R) \subseteq eR$. $\square$

Example 4.4.8 *Let K be any field, let $K_i = K\,(i \geq 1)$ and let the ring $S = \prod_{i\geq 1} K_i$. Let U denote the simple ideal of S consisting of all elements in S of the form (k,0,0,...) with $k \in K$. Let R denote the trivial extension of S by U. Then R is a commutative semiregular ring with simple Jacobson radical $J(R)$ such that $J(R)^2 = 0$ but R does not satisfy dcc on cyclic non-summand ideals.*

Proof The ring R consists of all elements of the form (s, u), with $s \in S$ and $u \in U$, with addition and multiplication defined by

$$(s, u) + (s', u') = (s + s', u + u') \text{ and } (s, u)(s', u') = (ss', su' + s'u)$$

for all $s, s' \in S$ and $u, u' \in U$. It is well known that R is a commutative ring. Moreover, the set J of all elements of R of the form $(0, u)$ with $u \in U$ is an ideal of R such that $R/J \cong S$ so that R/J is von Neumann regular and $J^2 = 0$. Thus, J is the Jacobson radical of R. Because U is a simple S-module, J is a simple R-module. Let $f_0 = (1, 0, 0, ...)$ and for each $i \geq 1$ let f_i denote the element $(0, 0, ..., 0, 1, 1, 1, ...)$ of S with nth component 1 for all $n \geq i + 1$. Let $e_i = (f_i, 0) \in R$. Note that $U = Sf_0$ and for each $i \geq 1$, f_i is an idempotent of S such that $f_i f_0 = 0$. Further note that $Sf_1 \supset Sf_2 \supset \ldots$. Let $i \geq 1$. Then $R(f_i, f_0)$ is a cyclic ideal of R such that $R(f_i, f_0) = Re_i + J$. Because e_i is an idempotent in R such that $Re_i = \{(sf_i, 0) : s \in S\}$, $J \nsubseteq Re_i$. Lemma 4.4.7 gives that $R(f_i, f_0)$ is not a direct summand of R_R for each $i \geq 1$. Moreover, $R(f_1, f_0) \supset R(f_2, f_0) \supset \ldots$. Thus, R does not satisfy dcc on cyclic non-summand ideals. $\square$

OPEN QUESTIONS

Question 1 It is known that the right (left) socle of any ring is a strongly lifting ideal (see Theorem 1.4.7). Is the socle of any module a strongly lifting submodule?

Question 2 It is known that the notion of I-semiregular rings is right-left symmetric for any ideal I (see Theorem 1.4.12). Is this also true for almost I-semiregular rings?

Question 3 A ring R satisfies dcc on cyclic right ideals if and only if R satisfies dcc on finitely generated right ideals, if and only if R is a left perfect ring (see Theorem 1.5.3). Hence, we have the following implications for a ring R:

R is left perfect $\Rightarrow$ R satisfies dcc on finitely generated non-summand right ideals $\Rightarrow$ R satisfies dcc on cyclic non-summand right ideals.

We know from Example 4.2.5 that if a ring satisfies dcc on finitely generated (cyclic) non-summand right ideals, then the ring need not be left perfect. So the following question arises:

Does a ring which satisfies dcc on cyclic non-summand right ideals, also satisfy dcc on finitely generated non-summand right ideals?

Question 4 We proved that if M is a finite dimensional module, then M satisfies acc on cyclic submodules if and only if it satisfies acc on cyclic non-summands. Characterize modules which satisfy acc on cyclic non-summands.

The following implication is known for a ring R:

R satisfies dcc on cyclic right ideals ($\Leftrightarrow$ R is left perfect) $\Rightarrow$ R satisfies acc on cyclic left ideals (see [21, p. 230]).

It is also known that a ring satisfies acc on right (left) summands if and only if it satisfies dcc on left (right) summands ([21, Proposition 6.59]).

If a ring satisfies dcc (acc) on cyclic non-summand right ideals, then does it satisfy acc (dcc) on cyclic non-summand left ideals?

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