STABILITY OF COVERS UNDER DIFFERENT RIGHTS STRUCTURES

A Master's Thesis

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Graduate School of Economics and Social Sciences
of
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I certify that I have read this thesis and have found that it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Arts in Economics.

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ABSTRACT

STABILITY OF COVERS UNDER DIFFERENT RIGHTS STRUCTURES

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A country's social welfare depends on firms' profits and consumers' surplus. Given unions of countries, a country's aim is to maximize its own social welfare when it decides to enter or exit a union. For examining unions, we use the notion of a cover as elaborated in Koray (2007).We utilize the findings of İlkılıç (2010) about the Cournot equilibrium in our setting to examine core stability and efficiency of covers of countries.We adapt different rights' structures based on; free exit, free entry, approved exit and approved entry introduced by Sertel (1992) to the context of covers, along with introducing some stronger structures and study how stability of covers varies when linkage costs are imposed upon countries.

Keywords: Social Welfare, Cover, Free Exit, Free Entry, Approved Exit, Approved Entry,Core Stability, Efficiency, Pareto Efficiency.

ÖZET

FARKLI HAKLAR YAPILARI ALTINDA ÖRTÜLERİN KARARLILIĞI

AKBULUT, Çiğdem

Yüksek Lisans, Ekonomi Bölümü

Tez Yöneticisi: Prof. Dr. Semih Koray

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Bir ülkenin sosyal refahı ülke içerisindeki firmaların kârlarına ve tüketici artığına bağlıdır. Verilen bir birlik yapısı altında, bir ülkenin bir birliğe katılma veya bir birlikten ayrılma kararları, ülkenin sosyal refahını en çoklaştırmak amacıyla alınmaktadır. Birlikleri incelemek için örtüleri Koray (2007)'de ele alındığı biçimiyle kullanıyoruz. "Ülke örtülerinin" çekirdek kararlılığı ve verimliliğini incelemek için, İlkılıç (2010) ın ele alınan bağlamda hesap edilmiş, Cournot dengesi bulgularından yararlanıyoruz. Sertel (1992) in serbest giriş, serbest çıkış, izinli giriş ve izinli çıkış temelinde tanımladığı haklar yapılarını örtü kavramına uyarlıyor ve daha güçlü başka bazı haklar yapılarını da tanımlıyoruz.Farklı haklar yapıları altında kararlı, Pareto verimli ve verimli örtüleri belirliyoruz ve ülkelerarası bağlantı kurmaya maliyet yüklenmesinin örtülerin kararlılığını nasıl etkilediğini inceliyoruz.

Anahtar Kelimeler: Sosyal Refah, Örtü, Sebest Giriş, Serbest Çıkış, İzinli Giriş, İzinli Çıkış, Çekirdek Kararlılığı, Verimlilik, Pareto Verimlilik.

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CHAPTER 1

INTRODUCTION

Network Theory is one of the main theories to understand social communication and economic relations. Specifically, in economics, network theory is used for improving our understanding of trade agreements, information sharing, political alliances, employer-employee relationships, professional collaborations, friendships and partnerships. Especially, in the last twenty years, there have been new developments in this area. New models such as the Coauthor Model, the Distance Based Model, and the Connections Model due to Jackson and Wolinsky (1996) are established to explain different kinds of network relations. As another example, networks are also used in modelling competitions and bargaining in markets and firms, Kranton and Minehart (1998), Corominas- Bosch (1999) and Rahmi Ilkilic (2010) did. In all these works, dıfferent notions of stability and efficiency are defined, and several allocation rules such as the Player Based Flexible Network Rule, the Linked Based Flexible Network Rule are introduced. There is, however, still some considerable space for further improvements in this area.

In network models, connections are formed bilaterally, in other words, they are represented as a link between two agents. On the other hand, not every relation needs to be bilateral. As an example, agreements among firms and nations like the Customs Union, NAFTA, the European Union or work groups of researchers reflect multilateral relations, rather than bilateral.

As one possible and motivating scenario, we may consider trade agreements among nations. Consider countries A, B and C. Without loss of generality, these countries are assumed to have the same demand and the same economic, technological structure. For simplicity, we take one kind of product which can be produced in all countries with the same technology. If A, B, and C come together and form a union, trade will be free among them. Note that, in a union among A, B, and C, for instance, C can also be in a union with other countries, say, D and E. However, a product which is produced in a country can only be sold in that country and the countries that are in a union with this country. For a country, the social welfare can be measured as the sum of the consumers' surplus, and the profit of the firm. Given the other countries' union structure, a country will decide to join a union, if that maximizes its own social welfare. Before we model the problem, we first consider what kind of concepts should be used for explaining a union structure in other words, multilateral links.

In order to deal with multilateral connections, first of all, it is not convenient to use networks where, links represent bilateral relation between two agents. Therefore, a network does not reflect the idea of "union". Secondly, we may use cooperative game with a partitional coalition structure. In other words, we may take each union as a coalition.

Definition 1 *Let* $|N| = n$ *and let* $\forall i \in N$ *,* $B_i \in 2^N \backslash \{\emptyset\}$*. A coalition structure is a partition* $B = \{B_1, \ldots, B_K\}$ *of the* n *players such that* $\bigcup B_k = N$ *and for all* $h \neq k$*,* $B_k \bigcap B_h = \emptyset$*.*

In this definition, the intersection of any two distinct coalitions must be empty. This means that, if a nation is in a union, then it cannot be in another union. However, in our model and in reality, a country can be in different unions. As an example, the USA is both in NAFTA (The North American Free Trade Agreement) with Canada and Mexico and in DR-CAFTA (Central America Free Trade Agreement) with Costa Rica, El Salvador, Guatemala, Honduras, and Nicaragua. Thus we need to have different structures to examine multilateral relations considering possible overlappings between different unions. For this purpose, "conference" structures and "cover" structures which are introduced by Myerson (1980) and Sertel (1992) respectively, can be used.

Definition 2 *A conference S is a set of two or more players (who might meet together to discuss their cooperative plans). A conference structure Q is any collection of conferences. Thus, $Q = \{S : S \subseteq N \text{ and } |S| \geq 2\}$.*

As we see, in "conference structure" a coalition has at least two members, so isolations are not allowed and also inclusions between two distinct coalitions are allowed. In most economic and trade agreements like the European Union, the Customs Union, once a set of countries agrees to have the same alliance and rules, then subsets are not allowed to make the some other alliance and rules. In other words, if A, B, C are in a union with a trade agreement then B and C are not allowed to make the some other trade agreement. Moreover a country can choose to be alone, in other words, it need not join a union. Therefore, in our study, since inclusions between any distinct coalitions are not allowed, and isolations are allowed, we use covers introduced in Sertel(1992) in our model. The definitions pertaining to covers are borrowed from Koray (2007).

Definition 3 *Given the set of players N, a hyperlink H is an element of $2^N \backslash \{\emptyset\}$. A subset C of 2^N is said to be a cover for N players if $\bigcup_{H \in C} H = N$ and $\nexists H, H' \in C$ such that $H \subsetneq H'$. We will denote the set of all covers for N by $\mathcal{C}^N$.*

As we notice, in cover structure a hyperlink (we will use hyperlink or hyperedge instead of a coalition in conference structure) may consist of one agent but inclusions between any distinct hyperlinks are not allowed.

Example 1 *Let* $N = \{1, 2, 3, 4\}$ *and* $C \in \mathcal{C}^N$, $C = \{12, 234\}$. *The cover* C *has two hyperlinks such that* 1 *and* 2 *are in a union and* 2, 3, 4 *are in a different union.*

In our research, we will investigate several questions. The first one is, given a cover structure, how much a firm in a country should produce so as to maximize its own profit. We will investigate whether this *Cournot* equilibrium is unique or not. Here, the unions that the country is in, are important to find the *Cournot* equilibrium for the profit- maximizing quantities of the firms. We will find the profit of the firm, and the consumers' surplus in a country whose sum will represent the social welfare of a country.

The second main question will be to determine whether a cover is core stable or not. For this, we will use k-stability and core stability concepts that Koray (2007) introduces for covers. For a country, to exit from or entry to a union may or may not require approval. Sertel (1992) introduced four possible membership rights in an abstract setting. In this aspect, either both exit and entry require approval, only one of them requires approval (i.e., approved entry- free exit or vice versa) , or none of them requires approval. Under approved entry condition, if in a union, at least one country's social welfare strictly decreases when another country joins, then this country will veto the entry of the new member. Similarly, under approved exit condition, if in a union, at least one country's social welfare strictly decreases when another country leaves, then this country will not approve the exit of the member. Stability in entry- exit conditions for hedonic games are defined and examined in Karakaya (2011). We will define four membership rights for covers, in addition to this, we will introduce and define strongly approved entry condition as well. We will investigate core stability notion in all cases respectively. We will investigate efficient and Pareto efficient covers.

While examining core stability under different exit-entry conditions, we will first consider entry without cost. However, in reality, countries pay cost such as forming institutions, applying the criteria of trade agreements, while entering a union. Similary, the incumbants in a union may incur a cost for the new comer. Therefore, we will do the same analysis considering entry cost.

As we mentioned before, we will assume that countries have the same demand and the same economic, technological structure. For simplicity, we will take one kind of product which can be produced in all countries with the same technology. In other words, countries will be symmetric. But in reality, they are not. Hence, as a further research, country's differences can be considered, and the same questions in terms of this difference can be answered.

In the literature, a similar research done about this subject is due to Ilkilic (2010). In his research, he models a bipartite network where links connect firms with markets. He looks at the Cournot game in which firms decide how much to sell at each market that they are connected to. He then considers the market analysis and examines the mergers and cartel formations. In Ilkilic (2010), he assumes that firms have convex quadratic costs and markets have affine inverse demand functions. Under these assumptions, he mainly finds that the Cournot game has a unique Nash equilibrium, and for the two firms the Cournot equilibrium differs from the no-merger situation if they share a market. For cartel solution, he establishes an algorithm to calculate the optimal cartel supply by each firm and consumption at each market.

We will use the assumptions of Ilkilic (2010) to model our problem.

CHAPTER 2

MODEL

Now let us model our scenario formally and state the problem. Countries in our scenario are considered as the agents, so we have n countries. The unions which they form are considered as hyperlinks.

We will use the assumptions of Ilkilic (2010) in our model so, firms have convex-quadratic costs and markets have affine inverse demand functions. Let $C \in \mathcal{C}^N$ be a cover, $H \in C$ be a hyperlink. Ilkilic (2010) assumes that given a quantity vector Q_C , the price at the country i is $p_i(Q_C) = \alpha_i - \beta_i c_i$ where $\alpha_i, \beta_i > 0$ and $c_i = q_{ii} + \sum\limits_{\substack{k \in N \setminus \{i\}\ st \\ \exists H \in C:\ i,k \in H}} q_{ik}$ is the total consumption at the country i.

Ilkilic (2010) assumes that for a firm j the total cost of production is $T_j(Q_C) = \frac{\gamma j}{2} s_j^2$ where $\gamma_j > 0$ and $s_j = q_{jj} + \sum\limits_{\substack{k \in N \setminus \{j\}\ st \\ \exists H \in C:\ k,j \in H}} q_{kj}$ is the total supply by a firm j. Then the profit function of a firm in country j is

$$\pi_j(Q_C) = \sum_{\substack{i \in N \setminus \{j\}\ st \\ \exists H \in C:\ i,j \in H}} (\alpha_i q_{ij} - \beta_i q_{ij} c_i) + \alpha_j q_{jj} - \beta_j q_{jj} c_j - \frac{\gamma j}{2} s_j^2.$$

Ilkilic (2010) assumes that q_{ij} is the supply of a firm j to the market i. Here, if i and j are in the same union (hyperlink) then, q_{ij} is the supply of a firm in the country j to the country i. Note that a firm in a country trades in its own country. If i and j are in the same union (i.e., if $\exists H \in C : i, j \in H$), then the best response of a firm in the country j supplying to the country i (so as to maximize its profit) is:

$$q_{ij}^* = \begin{cases} \frac{1}{2\beta_i+\gamma_j}(\alpha_i - \gamma_j \sum\limits_{\substack{t\in N\backslash\{i\}\ st \\ \exists H\in C:\ t,j\in H}} q_{tj} - \ \beta_i \sum\limits_{\substack{k\in N\backslash\{j\}\ st \\ \exists H\in C:\ i,k\in H}} q_{ik}) & \text{if } \frac{\partial \pi_j}{\partial q_{ij}} \geq 0 \\ 0 & \text{if } \frac{\partial \pi_j}{\partial q_{ij}} < 0 \end{cases}$$

or in other words,

$$q_{ij}^* = \begin{cases} \frac{\alpha_i - \gamma_j(s_j - q_{ij}) - \ \beta_i(c_i - q_{ij})}{2\beta_i+\gamma_j} & \text{if } \frac{\partial \pi_j}{\partial q_{ij}} \geq 0 \\ 0 & \text{if } \frac{\partial \pi_j}{\partial q_{ij}} < 0 \end{cases}$$

Note that as Ilkilic (2010) assumes, we will also assume $q_{ij} \geq 0$, $\forall i, j \in N$. For networks, Ilkilic (2010) shows that the game has a unique Cournot equilibrium. To do this, he constructs the problem as a Linear Complementarity Problem (LCP), and shows that the matrix in LCP is positive definite. Hence, he concludes that this LCP has the unique solution. Therefore, q_{ij}^* is the unique Cournot equilibrium for a firm in the country j supplying to the country i.

Then the consumers' surplus $CS_i(Q_C)$ and social welfare $SW_i(Q_C)$ of the country i will be; $CS_i(Q_C) = \frac{\beta_i(c_i^*)^2}{2}$ where c_i^* denotes c_i in the equilibrium and $SW_i(Q_C) = \pi_i(Q_C) + CS_i(Q_C)$.Remember that we will assume that all countries have the same demand and the same economic, technological structure. Hence, we will assume that $\forall i, j \in N, \alpha_i = \alpha_j = \alpha, \beta_i = \beta_j = \beta$ and $\gamma_i = \gamma_j = \gamma$. Note that, if i and j are in the same union (hyperlink) then, q_{ij} is the supply of a firm in the country j to the country i.

CHAPTER 3

DEFINITIONS AND NOTATIONS

Koray (2007) defines the concepts of value and allocation function for covers as below.

Definition 4 *A function $v : \mathcal{C}^N \to \mathbb{R}$ is called a value function for $\mathcal{C}^N$ if $v(C) = 0$ whenever $|H| = 1$ for all $H \in C$. Given a value function $v : \mathcal{C}^N \to \mathbb{R}$, a function $Y : \mathcal{C}^N \to \mathbb{R}^N$ is called an allocation rule associated with v if, for any $C \in \mathcal{C}^N$, one has $v(C) = \sum_{i \in N} Y_i(C)$.*

Let $v : \mathcal{C}^N \to \mathbb{R}$ be a value function and Y an allocation rule associated with v.

Sertel (1992) introduced four possible membership rights in an abstract setting. In this aspect, either both exit and entry of a country to a union require approval, only one of them requires approval (i.e., approved entry-free exit or vice versa) , or none of them requires approval. In order to investigate the stability notion, we will consider these four membership rights and we will define two more membership rights. For this, we will utilize $k-$stability, core stability and core definitions that Koray (2007) introduces. Koray (2007) defines T- function on C, and we will use this concept in our definitions.

Definition 5 *Given a cover $C \in \mathcal{C}^N$,and $T \in 2^N \backslash \{\emptyset\}$, a function $f : C \to 2^N \backslash \{\emptyset\}$ is called a T-function on C if $\forall\ H \in C : f(H) \subset H$ and $H \backslash f(H) \subset T$.*

We will define obtainable covers under membership rights.Sertel (1992) introduces abbreviations for free exit (FX), free entry (FE), approved exit (AX) and approved entry (AE). We will use these abbreviations.

Definition 6 ***(Free Exit- Free Entry).*** *Given a value function $v \in V$ and an allocation rule Y associated with v, let $C \in \mathcal{C}^N$ and $T \in 2^N \backslash \{\emptyset\}$. A cover $C' \in \mathcal{C}^N$ is said to be obtainable from C via T relative to v and Y under free exit- free entry condition, if $C' \subset \{f(H) \bigcup P : H \in C, P \in 2^T\} \bigcup 2^T$ for some T-function f on C.Free exit- free entry condition is denoted by FX-FE.*

Note that under approved entry condition, if in a union, at least one country's social welfare strictly decreases when another country joins, then this country will veto the entry of the new member. Similarly, under approved exit condition, if in a union, at least one country's social welfare strictly decreases when another country leaves, then this country will not approve the exit of the member. Hence, in three definitions below, the second condition represents this situation.

Definition 7 ***(Free Exit- Approved Entry)*** *Given a value function $v \in V$ and an allocation rule Y associated with v, let $C,\ C' \in \mathcal{C}^N$ and $T \in 2^N \backslash \{\emptyset\}$. A cover $C' \in \mathcal{C}^N$ is said to be obtainable from C via T relative to v and Y under free exit- approved entry condition, if the following conditions hold ;*

1) $C' \subset \{f(H) \bigcup P : H \in C, P \in 2^T\} \bigcup 2^T$ for some T-function f on C,

2) $\forall H' \in C'$ such that $[H' \bigcap T \neq \emptyset$ and $\nexists H \in C : H' = H$ and $\exists H \in C : (H' \backslash H) \neq \emptyset,\ (H' \backslash H) \subset T]$ we have $\forall i \in H' : Y_i(C') \geq Y_i(C)$.

Free exit- approved entry condition is denoted by FX-AE.

Definition 8 ***(Approved Exit-Free Entry)*** *Given a value function $v \in V$ and an allocation rule Y associated with v, let $C,\ C' \in \mathcal{C}^N$ and $T \in 2^N \backslash \{\emptyset\}$. A cover $C' \in \mathcal{C}^N$ is said to be obtainable from C via T relative to v and Y under approved exit- free entry condition, if the following conditions hold ;*

1) $C' \subset \{f(H) \bigcup P : H \in C, P \in 2^T\} \bigcup 2^T$ *for some* T*-function* f *on* C*,*

2) $\forall H \in C$ *such that* $[H \bigcap T \neq \emptyset$ *and* $\nexists H' \in C' : H' = H$ *and* $\exists H' \in C' : (H \backslash H') \subset T]$ *we have* $\forall i \in H : Y_i(C') \geq Y_i(C)$*.*

Approved exit- free entry condition is denoted by AX-FE.

Definition 9 ***(Approved Exit-Approved Entry)*** *Given a value function* $v \in V$ *and an allocation rule* Y *associated with* v*, let* C*,* $C' \in \mathcal{C}^N$ *and* $T \in 2^N \backslash \{\emptyset\}$*. A cover* $C' \in \mathcal{C}^N$ *is said to be obtainable from* C *via* T *relative to* v *and* Y *under approved exit- approved entry condition, if the following conditions hold ;*

1) $C' \subset \{f(H) \bigcup P : H \in C, P \in 2^T\} \bigcup 2^T$*, for some* T*-function* f *on* C *,*

2) $\forall H \in C$ *such that* $[\nexists H' \in C' : H' = H]$ *we have* $\forall i \in H : Y_i(C') \geq Y_i(C)$*.*

Approved exit- approved entry condition is denoted by AX-AE.

So far we have considered the membership rights under non-transferable payoffs. On the other hand, approved exit of a country from a union and approved entrance of a country to a union can depend on the total social welfare of the countries in that union.Therefore, we will define membership rights under transferable payoffs.

Definition 10 ***(Free Exit-Free Entry with Transferable Payoffs)*** *Given a value function* $v \in V$ *and an allocation rule* Y *associated with* v*, let* $C \in \mathcal{C}^N$ *and* $T \in 2^N \backslash \{\emptyset\}$*. A cover* $C' \in \mathcal{C}^N$ *is said to be obtainable from* C *via* T *relative to* v *and* Y *under free exit- free entry with transferable payoffs condition, if* $C' \subset \{f(H) \bigcup P : H \in C, P \in 2^T\} \bigcup 2^T$ *for some* T*-function* f *on* C*.*

Note that under approved entry with transferable payoffs condition, if the total social welfare of the union strictly decreases when another country joins, then the union will veto the entry of the new member. Similarly, under approved exit with transferable payoffs condition, if the total social welfare of the union strictly decreases when another country leaves, then the union will not approve the exit of the member. Hence, in below three definitions, the second condition represents this situation.

Definition 11 ***(Free Exit-Approved Entry with Transferable Payoffs)*** *Given a value function $v \in V$ and an allocation rule Y associated with v, let C, $C' \in \mathcal{C}^N$ and $T \in 2^N \backslash \{\emptyset\}$. A cover $C' \in \mathcal{C}^N$ is said to be obtainable from C via T relative to v and Y under free exit- approved entry with transferable payoffs condition, if the following conditions hold ;*

1) $C' \subset \{f(H) \bigcup P : H \in C, P \in 2^T\} \bigcup 2^T$ for some T-function f on C,

2) $\forall H' \in C'$ such that $[H' \bigcap T \neq \emptyset$ and $\nexists H \in C : H' = H$ and $\exists H \in C : (H' \backslash H) \neq \emptyset$, $(H' \backslash H) \subset T]$ we have $\sum_{i \in H'} Y_i(C') \geq \sum_{i \in H'} Y_i(C)$.

Definition 12 ***(Approved Exit- Free Entry with Transferable Payoffs)*** *Given a value function $v \in V$ and an allocation rule Y associated with v, let C, $C' \in \mathcal{C}^N$ and $T \in 2^N \backslash \{\emptyset\}$. A cover $C' \in \mathcal{C}^N$ is said to be obtainable from C via T relative to v and Y under approved exit- free entry with transferable payoffs condition, if the following conditions hold ;*

1) $C' \subset \{f(H) \bigcup P : H \in C, P \in 2^T\} \bigcup 2^T$ for some T-function f on C,

2) $\forall H \in C$ such that $[H \bigcap T \neq \emptyset$ and $\nexists H' \in C' : H' = H$ and $\exists H' \in C' : (H \backslash H') \subset T]$ we have $\sum_{i \in H} Y_i(C') \geq \sum_{i \in H} Y_i(C)$.

Definition 13 ***(Approved Exit- Approved Entry with Transferable Payoffs)*** *Given a value function $v \in V$ and an allocation rule Y associated with v, let C, $C' \in \mathcal{C}^N$ and $T \in 2^N \backslash \{\emptyset\}$. A cover $C' \in \mathcal{C}^N$ is said to be obtainable from C via T relative to v and Y under approved exit- approved entry with transferable payoffs condition, if the following conditions hold ;*

1) $C' \subset \{f(H) \bigcup P : H \in C, P \in 2^T\} \bigcup 2^T$, for some T-function f on C , *2)* $\forall H \in C$ *such that* $[\ \nexists H' \in C' : H' = H]$ *we have* $\sum_{i \in H} Y_i(C') \geq \sum_{i \in H} Y_i(C)$.

Let $N = \{i, j, k\}$ and C, $C' \in \mathcal{C}^N$ such that $C = \{ij, jk\}$ and $C' = \{ij, jk, ik\}$. Let $T = \{i, k\}$ as the countries which deviate.When we pass from the cover C to the cover C', by definitions of the four membership rights (FX-FE, FX-AE, AX-FE and AX-AE), we do not consider the approval of the country j. However, in some cases, in order to pass from C to C', the approval of j is needed as i and k form a new union while they are still in a union with j in C'. Now we form the definition of free exit- strongly approved entry.

Definition 14 ***(Free Exit- Strongly Approved Entry)*** *Given a value function* $v \in V$ *and an allocation rule* Y *associated with* v*, let* C*,* $C' \in \mathcal{C}^N$ *and* $T \in 2^N \backslash \{\emptyset\}$*. A cover* $C' \in \mathcal{C}^N$ *is said to be obtainable from* C *via* T *relative to* v *and* Y *under strongly approved entry- free exit condition, if the following conditions hold ;*

1) $C' \subset \{f(H) \bigcup P : H \in C, P \in 2^T\} \bigcup 2^T$ *for some* T*-function* f *on* C*,*

2) $\forall H' \in C'$ *such that* $[H' \bigcap T \neq \emptyset$ *and* $\nexists H \in C : H' = H$ *and* $\exists H \in C :$ $(H' \backslash H) \neq \emptyset$*,* $(H' \backslash H) \subset T]$ *we have* $\forall i \in H' : Y_i(C') \geq Y_i(C)$*.*

3) $\forall H' \in C'$ *such that* $[H' \bigcap T \neq \emptyset$ *and* $\exists H \in C : H' \subset H$ *and* $\exists H'' \in C'$*,* $H' \neq H'' : (H' \bigcap T) \subset H'']$ *we have* $\forall i \in H' : Y_i(C') \geq Y_i(C)$*.*

Free Exit- Strongly Approved Entry condition is denoted by FX-SAE.

According to the definition, under free exit- strongly approved entry, countries can exit from their previous unions freely. However, if the deviating countries enter to other unions or form new unions without exiting from their previous unions, then the approval of the countries that are in unions with the deviating countries previously is required.

Similarly we will define approved exit- strongly approved entry condition.

Definition 15 ***(Approved Exit-Strongly Approved Entry)*** *Given a value function* $v \in V$ *and an allocation rule* Y *associated with* v*, let* C*,* $C' \in \mathcal{C}^N$ *and*

$T \in 2^N \backslash \{\emptyset\}$. *A cover* $C' \in \mathcal{C}^N$ *is said to be obtainable from* C *via* T *relative to* v *and* Y *under strongly approved entry- approved exit condition, if the following conditions hold ;*

1) $C' \subset \{f(H) \bigcup P : H \in C, P \in 2^T\} \bigcup 2^T$ *for some* T*-function* f *on* C,

2) $\forall H \in C$ *such that* $[\ \nexists H' \in C' : H' = H]$ *we have* $\forall i \in H : Y_i(C') \geq Y_i(C)$.

3) $\forall H' \in C'$ *such that* $[H' \bigcap T \neq \emptyset$ *and* $\exists H \in C : H' \subset H$ *and* $\exists H'' \in C'$, $H' \neq H'' : (H' \bigcap T) \subset H'']$ *we have* $\forall i \in H' : Y_i(C') \geq Y_i(C)$.

Approved Exit- Strongly Approved Entry condition is denoted by AX-SAE.

Now we will give k-stability, core stability and core definitions that Koray (2007) introduces.

Definition 16 *Let* $C \in \mathcal{C}^N$ *and* $k \in \{1, ..., n\}$, *and the exit- entry condition is given. We say that* C *is* k*-stable relative to* (v, Y) *under given exit- entry condition, if there is no* $T \in 2^N \backslash \{\emptyset\}$ *with* $|T| \leq k$ *such that* $\exists C' \in C^N$ *obtainable from* C *via* T *relative to* (v, Y) *under given exit- entry condition condition with* $\forall i \in T : Y_i(C') \geq Y_i(C)$ *and* $\exists j \in T : Y_j(C') > Y_j(C)$. C *is said to be strongly stable relative to* (v, Y) *under given exit- entry condition if* C *is* k*-stable relative to* (v, Y) *for all* $k \in \{1, ..., n\}$ *under given exit- entry condition.*

Definition 17 *Given a value function* $v \in V$ *, an allocation rule* Y *associated with* v, *and exit- entry condition, let* C, $C' \in \mathcal{C}^N$ *and* $T \in 2^N \backslash \{\emptyset\}$. *We say that* T *can improve upon* C *via* C' *relative to* (v, Y) *under given exit- entry condition if* C' *is obtainable from* C *via* T *relative to* (v, Y) *under given exit- entry condition with* $\forall i \in T : Y_i(C') \geq Y_i(C)$ *and* $\exists j \in T : Y_j(C') > Y_j(C)$.

Definition 18 *A cover* $C \in \mathcal{C}^N$ *is said to be core stable relative to* (v, Y) *under given exit- entry condition if there is no* $T \in 2^N \backslash \{\emptyset\}$ *such that* T *can improve upon* C *via some* $C' \in \mathcal{C}^N$ *relative to* (v, Y) *under given exit- entry condition.*

Note that core stability and strong stability are the same notions. But, core for cover characterizes the allocations for efficient covers such that no subset $S \subset N$ can deviate from the efficient cover under the allocation rule.

Definition 19 *Given a value function $v \in V$ and a cover $C \in \mathcal{C}^N$,an allocation $y \in \mathbb{R}^n$ for C is said to be core relative to $(N,\, v)$ if $\sum\limits_{i \in N} y_i \leq v(C)$ and $\forall S \subseteq N$: $\sum\limits_{i \in S} y_i \geq \hat{v}(C^S)$ where $\hat{v}(C^S) = \max_{C^{S\prime} \in \mathcal{C}^N} v(C^{S\prime})$ and $\forall S \subseteq N, C^S$ denotes a subset of $\mathcal{C}^N$ such that agents in $N \backslash S$ are isolated and agents in S are allowed to form any hyperlink.*

Koray (2007) introduces Pareto efficiency and efficiency for covers.

Definition 20 *Let $C \in \mathcal{C}^N$. We say that C is efficient relative to v if $v(C) = \max_{C' \in \mathcal{C}^N} v(C')$. Moreover, v is said to be Pareto efficient relative to (v, Y) if there is no $C' \in \mathcal{C}^N$ such that $\forall i \in N : Y_i(C') \geq Y_i(C)$ and $\exists j \in N : Y_j(C') > Y_j(C)$.*

Ilkilic (2010) models a bipartite network where links connect firms with markets. If we think that the firms in countries and the countries (markets) as in bipartite network, we reach the below result,

Claim 1 *Let $C \in \mathcal{C}^N$, and let $C' \in \mathcal{C}^N$. Assume that the bipartite graphs of the two covers are same. Then in the equilibrium, the social welfare of a country in the cover C is equal to its social welfare in the cover C'.*

Proof. Let $C \in \mathcal{C}^N$ and let $C' \in \mathcal{C}^N$. Assume that the bipartite graphs of the two covers are same. Let $i \in N$,and let $j \in N$, such that $\exists\ H \in C : i, j \in H$. Then since the bipartite graphs of the two covers are same, so $\exists\ H' \in C' : i, j \in H'$. Hence, by model assumptions, it follows that in the equilibrium, q^*_{ij} in C is equal to the q^*_{ij} in C' and $\pi_i(Q_C) = \pi_i(Q_{C'})$, $CS_i(Q_C) = CS_i(Q_{C'})$, so, $SW_i(Q_C) = SW_i(Q_{C'})$. Thus, in the equilibrium, the social welfare of a country in the cover C is equal to its social welfare in the cover C'. ■

Example 2 $N = \{1, 2, 3, 4\}$. *Let* $C \in \mathcal{C}^N$ *and let* $C' \in \mathcal{C}^N$ *such that,* $C = \{123, 4\}$ *and* $C' = \{12, 23, 13, 4\}$. *Then the bipartite graph structures of the two covers are same, so by Claim 1, in equilibrium, the social welfare of a country in the cover* C *is equal to its social welfare in the cover* C'.

Hence, we can treat the cover structure as a bipartite graph structure between firms in the countries and countries (markets) in our model. Therefore, we will modify complete cover definition as below.

Definition 21 *A cover* $C \in \mathcal{C}^N$ *is said to be complete-equivalent cover, if it is composed of only one hyperlink which contains all players or* $\forall\ i, j \in N\ ;\ \exists\ H \in C$ *such that* $i, j \in H$.

Definition 22 *A cover is said to be a single-centered star if the cover has at least two hyperlinks, there exist unique* $i \in N$, *such that* $\forall H, H' \in C$, $H \bigcap H' = \{i\}$ *and* $\forall H \in C, |H| = 2$. *In this case, we will call the unique element* i *as the center.A cover is said to be a multi-centered star if the cover has at least two hyperlinks, there exist* $S \in 2^N \backslash \{\emptyset, N\}$, *such that* $\forall H, H' \in C$, $H \bigcap H' = S$. *In this case, we will call* S *as the center.*

Definition 23 *Let* $C \in \mathcal{C}^N$. *An agent* i *is isolated if* $\exists! H \in C$ *such that* $i \in H$ *and* $|H| = 1$.

We define degree concept in covers as follows;

Definition 24 *Let* $C \in \mathcal{C}^N$. *Let* $i \in N$. *We define the degree of* i *in* C *as follows;* $\deg_i(C) = |\{j \in N \backslash \{i\} : \exists H \in C \text{ such that } i, j \in H\}|$

CHAPTER 4

RESULTS

4.1 Model without Linkage Cost

Proposition 1 *The social welfare of each country in the complete- equivalent cover in the equlilibrium is* $\frac{\alpha^2 n(2\beta+n\beta+n\gamma)}{2(\beta+\beta n+\gamma n)^2}$ *. Social welfare of each country in the complete-equivalent cover in equilibrium increases as the number of countries increases.*

Proof. Let $C \in \mathcal{C}^N$ be a complete-equivalent cover including n countries. Since all countries are identical, we have in equilibrium;

$\forall i, j \in N, q_{ij}^* = q_{ji}^* = q^* = \frac{\alpha}{(n+1)\beta+n\gamma}$

$\pi_i(Q_C) = \pi_j(Q_C) = \alpha n q^* - n^2\beta(q^*)^2 - \frac{\gamma}{2}n^2(q^*)^2$

$CS_i(Q_C) = \frac{\beta n^2(q^*)^2}{2}$ so we get,

$SW_i(Q_C) = \alpha n q^* - n^2(q^*)^2(\frac{\beta}{2} + \frac{\gamma}{2}) = \frac{\alpha^2 n(2\beta+n\beta+n\gamma)}{2(\beta+\beta n+\gamma n)^2}$

Now, the derivative of social welfare with respect to the number of countries, n, is $\frac{\partial(SW_i(Q_C))}{\partial n} = \frac{\alpha^2\beta^2}{(\beta+\beta n+\gamma n)^3}$ since we assume $\alpha, \beta, \gamma > 0$ so, $\frac{\partial(SW_i(Q_C))}{\partial n} > 0$, hence the social welfare of each country in the complete-equivalent cover in equilibrium increases as the number of countries increases. ■

Remark 1 *Let* $C \in \mathcal{C}^N$*. In the model, firms are profit maximizers. Hence, as Ilkilic (2010) presents for networks, given a cover* $C \in \mathcal{C}^N$*, the Cournot equilibrium for quantity levels can be written in the matrix form,* $-\alpha + D_C Q_C^* \geq 0$*. Ilkilic (2010) forms the matrix* D_C *for networks, and shows that* D_C *can be formed as* $D_C = R^T R$

where R has full rank. Hence, Ilkilic (2010) concludes that D_C is positive definite and so $\det(D_C) > 0$.

Before we show a monotonicity result, let us look at an example;

Example 3 *$N = \{i, j, k\}$. Let $C = \{ij, k\}$ and let $C' = \{ij, ik\}$. We will show that adding the hyperlink ik decreases the social welfare of j.In C, since there is no connection between i and k,so $q_{ik} = 0$ and $q_{ki} = 0$. In C', in equilibrium, $q_{ik} \geq 0$ and $q_{ki} \geq 0$.We will show that the increase in q_{ik} and q_{ki} will effect the quantity levels, q_{jj}, q_{ij}, q_{ji}, q_{ii} in equilibrium. Then we will show that $SW_j(Q_{C'}) < SW_j(Q_C)$. By Remark 1, the Cournot equilibrium for quantity levels can be written in the matrix form, $-\alpha + D_C Q_C^* \geq 0$ and $\det(D_C) > 0$.Here*

$$D_C = \begin{bmatrix} 2\beta+\gamma & \beta & \gamma & 0 \\ \beta & 2\beta+\gamma & 0 & \gamma \\ \gamma & 0 & 2\beta+\gamma & \beta \\ 0 & \gamma & \beta & 2\beta+\gamma \end{bmatrix} \text{ and } Q_C = \begin{bmatrix} q_{ii} \\ q_{ij} \\ q_{ji} \\ q_{jj} \end{bmatrix}$$

For the cover C, the Cournot equilibrium for quantity levels can be written in the equation form, such that, $F_l : \mathbb{R}^4 \to \mathbb{R}$ where $q_{ii}, q_{ij}, q_{ji}, q_{jj}$, are dependent variables, and q_{ik}, q_{ki} are independent variables, so,

$$F_1(q_{ii}, q_{ij}, q_{ji}, q_{jj}) = (2\beta+\gamma)q_{ii} - \alpha + \gamma(q_{ji} + q_{vi}) + \beta(q_{ij} + q_{iv}) = 0$$

$$F_2(q_{ii}, q_{ij}, q_{ji}, q_{jj}) = (2\beta+\gamma)q_{ij} - \alpha + \gamma q_{jj} + \beta(q_{ii} + q_{iv}) = 0$$

$$F_3(q_{ii}, q_{ij}, q_{ji}, q_{jj}) = (2\beta+\gamma)q_{ji} - \alpha + \gamma(q_{ii} + q_{vi}) + \beta q_{jj} = 0$$

$$F_4(q_{ii}, q_{ij}, q_{ji}, q_{jj}) = (2\beta+\gamma)q_{jj} - \alpha + \gamma q_{ij} + \beta q_{ji} = 0$$

Now, F_l is linear $\forall l \in \{1, ..., 4\}$ and $\frac{\partial(F_1,...,F_4)}{\partial(q_{ii},...,q_{jj})} = \det(D_C) > 0$.By Cramer's rule we have, given q_{tr} such that $t, r \in \{i, j\}$,

$$\frac{\partial q_{tr}}{\partial q_{ik}} = \frac{-\frac{\partial(F_1,F_2,F_3,F_4)}{\partial(q_{ii},...,q_{ik},...,q_{jj})}}{\frac{\partial(F_1,...,F_4)}{\partial(q_{ii},...,q_{jj})}} = \frac{-\left|\begin{bmatrix} \frac{\partial F_1}{\partial q_{ii}} & \cdots & \frac{\partial F_1}{\partial q_{ik}} & \cdots & \frac{\partial F_1}{\partial q_{jj}} \\ \vdots & & \vdots & & \vdots \\ \frac{\partial F_4}{\partial q_{ii}} & \cdots & \frac{\partial F_4}{\partial q_{ik}} & \cdots & \frac{\partial F_4}{\partial q_{jj}} \end{bmatrix}\right|}{\left|\begin{bmatrix} \frac{\partial F_1}{\partial q_{ii}} & \cdots & \frac{\partial F_1}{\partial q_{tr}} & \cdots & \frac{\partial F_1}{\partial q_{jj}} \\ \vdots & & \vdots & & \vdots \\ \frac{\partial F_e}{\partial q_{ii}} & \cdots & \frac{\partial F_4}{\partial q_{tr}} & \cdots & \frac{\partial F_4}{\partial q_{jj}} \end{bmatrix}\right|}$$

similarly, $\frac{\partial q_{tr}}{\partial q_{ki}}$ *can be written. Let* $\boldsymbol{x} = \begin{bmatrix} \frac{\partial F_1}{\partial q_{ik}} \\ \vdots \\ \frac{\partial F_4}{\partial q_{ik}} \end{bmatrix}_{(4x1)}$ *be the column vector, then,*

$\boldsymbol{x} = \begin{bmatrix} \beta \\ \beta \\ 0 \\ 0 \end{bmatrix}$ *similarly, let* $\boldsymbol{y} = \begin{bmatrix} \frac{\partial F_1}{\partial q_{ki}} \\ \vdots \\ \frac{\partial F_4}{\partial q_{ki}} \end{bmatrix}_{(4x1)}$ *be the column vector, then,* $\boldsymbol{y} = \begin{bmatrix} \gamma \\ 0 \\ \gamma \\ 0 \end{bmatrix}$ *Note that* $\alpha, \beta, \gamma > 0$*.Then,* $\frac{\partial q_{ii}}{\partial q_{ik}} = \frac{\partial q_{ij}}{\partial q_{ik}} = \frac{(-1)(3\beta^2+7\beta\gamma+2\gamma^2)}{3(\beta+2\gamma)(3\beta+2\gamma)} < 0$, $\frac{\partial q_{ji}}{\partial q_{ik}} = \frac{\partial q_{jj}}{\partial q_{ik}} = \frac{(\beta\gamma+2\gamma^2)}{3(\beta+2\gamma)(3\beta+2\gamma)} > 0$ *and* $\frac{\partial q_{ii}}{\partial q_{ki}} = \frac{\partial q_{ji}}{\partial q_{ki}} = \frac{(-1)(6\beta\gamma+6\gamma^2)}{3(\beta+2\gamma)(3\beta+2\gamma)} < 0$, $\frac{\partial q_{ij}}{\partial q_{ki}} = \frac{\partial q_{jj}}{\partial q_{ki}} = \frac{3\beta\gamma}{3(\beta+2\gamma)(3\beta+2\gamma)} > 0$*.Now, by total differentiation,* $dSW_j(Q_C) = \frac{\partial SW_j(Q_C)}{\partial q_{ik}} dq_{ik} + \frac{\partial SW_j(Q_C)}{\partial q_{ki}} dq_{ki}$*. Then,* $\frac{\partial SW_j(Q_C)}{\partial q_{ik}} = \frac{(-2)\alpha\beta(3\beta+\gamma)}{(3\beta+2\gamma)^2} < 0$ *and* $\frac{\partial SW_j(Q_C)}{\partial q_{ki}} = \frac{2\alpha\beta^2\gamma}{(\beta+2\gamma)(3\beta+2\gamma)^2} > 0$.

Take $dq_{ik} = q_{ik}^*$ *and* $dq_{ki} = q_{ki}^*$ *where* q_{ik}^* *and* q_{ki}^* *are Cournot equilibrium for the cover* $C', q_{ik}^* = \alpha(\beta+3\gamma)\backslash(4\beta^2+13\beta\gamma+7\gamma^2)$ *and* $q_{ki}^* = 4\alpha(\beta+2\gamma)\backslash(3(4\beta^2+13\beta\gamma+7\gamma^2))$*. So we get,* $dSW_j(Q_C) = \frac{(-2)\alpha^2\beta(\beta+\gamma)^2}{(3\beta+2\gamma)^2(4\beta^2+13\beta\gamma+7\gamma^2)} < 0$*.Hence,* $SW_j(Q_{C'}) < SW_j(Q_C)$.

As we see from the example, in two covers, the country j is in an union with the same country i. In cover C, the country i is only in a union with j, while in C' it is also in an union with k. We observe that the social welfare of the country j decreases when i is in a union with a different country, k.

Lemma 1 *Let* $C \in \mathcal{C}^N$*, and let* $C' \in \mathcal{C}^N$*. Let* $j \in N$*. Assume that* $\{j_k \in N : \exists H \in C$ *such that* $j_k, j \in H$ *and* $j \neq j_k\} = \{j_k \in N : \exists H' \in C'$ *such that* $j_k, j \in H'$ *and* $j \neq j_k\} = \{j_1, .., j_m\}$*.If* $\forall j_k \in \{j_1, .., j_m\}, |\{i \in N : \exists H \in C$ *such that* $j_k, i \in H\}| \leq |\{i \in N : \exists H' \in C'$ *such that* $j_k, i \in H'\}|$ *then, we have* $SW_j(Q_{C'}) \leq SW_j(Q_C)$*.*

Proof. Let $C \in \mathcal{C}^N$ and let $C' \in \mathcal{C}^N$. Let $j \in N$. Assume that $\{j_k \in N : \exists H \in C$ such that $j_k, j \in H$, and $j \neq j_k\} = \{j_k \in N : \exists H' \in C'$ such that $j_k, j \in H'$, and $j \neq j_k\} = \{j_1, .., j_m\}$.If $\forall j_k \in \{j_1, .., j_m\}$, $|\{i \in N : \exists H \in C$ such that $j_k, i \in H\}| = |\{i \in N : \exists H' \in C'$ such that $j_k, i \in H'\}|$ then we have $SW_j(Q_{C'}) = SW_j(Q_C)$.

Suppose $\exists j_k \in \{j_1, .., j_m\}$, such that $|\{i \in N : \exists H \in C \text{ such that } j_k, i \in H\}| < |\{i \in N : \exists H' \in C' \text{ such that } j_k, i \in H'\}|$.Without loss of generality, assume that, $\{i \in N : \exists H' \in C' \text{ such that } j_k, i \in H'\} = \{i \in N : \exists H \in C \text{ such that } j_k, i \in H\} \bigcup \{s\}$, where $s \in N$. In C, since there is no connection between j_k and s,so $q_{j_k s} = 0$ and $q_{s j_k} = 0$. In C', in equilibrium, $q_{j_k s} \geq 0$ and $q_{s j_k} \geq 0$.We will show that the increase in $q_{j_k s}$ and $q_{s j_k}$ will effect the quantity levels, $q_{j j_1}, q_{j_1 j}, \ldots, q_{j j_k}, q_{j_k j}, \ldots, q_{j j_m}$, $q_{j_m j}, q_{jj}$ in equilibrium. Then we will show that $SW_j(Q_{C'}) < SW_j(Q_C)$.

For the cover C, the Cournot equilibrium for quantity levels can be written in the equations form, such that, $F_l : \mathbb{R}^e \to \mathbb{R}$

where $e = |\{(t,r) : \exists H \in C \text{ such that } t, r \in H, \text{ where } t, r \in \{1, ...n\}\}|$. Hence, $\forall q_{tr}$ such that $\exists H \in C : t, r \in H$, where $t, r \in \{1, ...n\}$, we have

$$F_1(q_{11}, \ldots, q_{nn}) = (2\beta + \gamma) q_{11} - \alpha + \gamma \sum_{\substack{t \in N \setminus \{1\} \ st \\ \exists H \in C:\ t,1 \in H}} q_{t1} + \beta \sum_{\substack{k \in N \setminus \{1\} \ st \\ \exists H \in C:\ 1,k \in H}} q_{1k} = 0$$

$$\vdots \qquad\qquad\qquad\qquad \vdots$$

$$F_e(q_{11}, \ldots, q_{nn}) = (2\beta + \gamma) q_{nn} - \alpha + \gamma \sum_{\substack{t \in N \setminus \{n\} \ st \\ \exists H \in C:\ t,n \in H}} q_{tn} + \beta \sum_{\substack{k \in N \setminus \{n\} \ st \\ \exists H \in C:\ n,k \in H}} q_{nk} = 0$$

Now, F_l is linear $\forall l \in \{1, ...e\}$ and by Remark 1, $\frac{\partial(F_1, \ldots, F_e)}{\partial(q_{11}, \ldots, q_{nn})} = \det(D_C) > 0$, and $q_{j_k s}$, $q_{s j_k}$ are independent and all q_{tr} 's such that $\exists H \in C : t, r \in H$, where $t, r \in \{1, ...n\}$ are dependent variables. By Cramer's rule we have, given q_{tr} such that $\exists H \in C : t, r \in H$, where $t, r \in \{1, ...n\}$,

$$\frac{\partial q_{tr}}{\partial q_{j_k s}} = \frac{-\frac{\partial(F_1, \ldots, F_e)}{\partial(q_{11}, \ldots, q_{j_k s} \ldots, q_{nn})}}{\frac{\partial(F_1, \ldots, F_e)}{\partial(q_{11}, \ldots, q_{nn})}} = \frac{-\left| \begin{bmatrix} \frac{\partial F_1}{\partial q_{11}} & \cdots & \frac{\partial F_1}{\partial q_{j_k s}} & \cdots & \frac{\partial F_1}{\partial q_{nn}} \\ \vdots & & \vdots & & \vdots \\ \frac{\partial F_e}{\partial q_{11}} & \cdots & \frac{\partial Fe}{\partial q_{j_k s}} & \cdots & \frac{\partial F_e}{\partial q_{nn}} \end{bmatrix} \right|}{\left| \begin{bmatrix} \frac{\partial F_1}{\partial q_{11}} & \cdots & \frac{\partial F_1}{\partial q_{tr}} & \cdots & \frac{\partial F_1}{\partial q_{nn}} \\ \vdots & & \vdots & & \vdots \\ \frac{\partial F_e}{\partial q_{11}} & \cdots & \frac{\partial Fe}{\partial q_{tr}} & \cdots & \frac{\partial F_e}{\partial q_{nn}} \end{bmatrix} \right|}$$

similarly, $\frac{\partial q_{tr}}{\partial q_{s j_k}}$ is written.

Let $\boldsymbol{x} = \begin{bmatrix} \frac{\partial F_1}{\partial q_{j_k s}} \\ \vdots \\ \frac{\partial Fe}{\partial q_{j_k s}} \end{bmatrix}_{(ex1)}$ be the column vector, then, $\boldsymbol{x}_{l1} = \begin{cases} \beta, & \text{if } \frac{\partial F_l}{\partial q_{j_k s}} = \beta \\ \gamma, & \text{if } \frac{\partial F_l}{\partial q_{j_k s}} = \gamma \\ 0, & \text{if } \frac{\partial F_l}{\partial q_{j_k s}} = 0 \end{cases}$

similarly, let $\boldsymbol{y} = \begin{bmatrix} \frac{\partial F_1}{\partial q_{s j_k}} \\ \vdots \\ \frac{\partial Fe}{\partial q_{s j_k}} \end{bmatrix}_{(ex1)}$ be the column vector, then,

$$\boldsymbol{y}_{l1} = \begin{cases} \beta, & \text{if } \frac{\partial F_l}{\partial q_{s j_k}} = \beta \\ \gamma, & \text{if } \frac{\partial F_l}{\partial q_{s j_k}} = \gamma \\ 0, & \text{if } \frac{\partial F_l}{\partial q_{s j_k}} = 0 \end{cases}$$

Then, $\frac{\partial q_{tr}}{\partial q_{j_k s}} < 0$ if $t = j_k$ and $r \neq s$, $\frac{\partial q_{tr}}{\partial q_{j_k s}} < 0$ if $t \neq j_k$ and $r = s$, $\frac{\partial q_{tr}}{\partial q_{j_k s}} > 0$ if $t \neq j_k$ and $r \neq s$, where t, r, j_k and s are in a connected bipartite graph in C'.Similar reasoning holds for $\frac{\partial q_{tr}}{\partial q_{s j_k}}$. Now, by total differentiation, $dSW_j(Q_C) = \frac{\partial SW_j(Q_C)}{\partial q_{j_k s}} dq_{j_k s} + \frac{\partial SW_j(Q_C)}{\partial q_{s j_k}} dq_{s j_k}$. Here, $\frac{\partial SW_j(Q_C)}{\partial q_{j_k s}} < 0$, and $\frac{\partial SW_j(Q_C)}{\partial q_{s j_k}} > 0$. Take $dq_{j_k s} = q^*_{j_k s}$ and $dq_{s j_k} = q^*_{s j_k}$ where $q^*_{j_k s}$ and $q^*_{s j_k}$ are Cournot equilibrium for the cover C', so, we get, $dSW_j(Q_C) < 0$. Hence, $SW_j(Q_{C'}) < SW_j(Q_C)$. ■

Corollary 1 *Let $i \in N$ and $C \in \mathcal{C}^N$. The country i reaches its maximum social welfare if and only if it is the center of the single-centered star cover.*

Proof. Let $i \in N$ and $C \in \mathcal{C}^N$. If $|N| = 1$, it is trivial. If $|N| = 2$, then by Proposition 1, the result follows. Hence assume that $|N| \geq 3$. Assume that $C \in \mathcal{C}^N$ is a star cover, and i is the center of the star where all hyperlinks have only two countries.Then given $D = (3\beta^2 + 6\beta\gamma + 3\beta^2 n + 5\beta\gamma n + 4\gamma^2 n + 2\beta c n^2 + \gamma^2 n^2)$

$SW_i(Q_C) = (\alpha^2(16\beta^3 + 20\beta^2\gamma + 4\beta\gamma^2 - 2\beta^3 n + 52\beta^2\gamma n + 44\beta\gamma^2 n + 11\beta^3 n^2 + 6\beta^2\gamma n^2 + 45\beta\gamma^2 n^2 + 16\gamma^3 n^2 + 2\beta^3 n^3 + 20\beta^2\gamma n^3 + 4\beta\gamma^2 n^3 + 8\gamma^3 n^3 + \beta^2\gamma n^4 + 8\beta\gamma^2 n^4 + \gamma^3 n^4))\backslash(2.D^2)$ and $\forall j \in N\backslash\{i\}$ we have,

$SW_j(Q_C) = (\alpha^2(24\beta^3 + 24\beta^2\gamma + 4\beta\gamma^2 + 12\beta^3 n + 68\beta^2\gamma n + 44\beta\gamma^2 n + 6\beta^3 n^2 + 25\beta^2\gamma n^2 + 49\beta\gamma^2 n^2 + 16\gamma^3 n^2 + 8\beta^2\gamma n^3 + 12\beta\gamma^2 n^3 + 8\gamma^3 n^3 + 3\beta\gamma^2 n^4 + \gamma^3 n^4))\backslash(2.D^2)$

Now, $SW_i(Q_C) - SW_j(Q_C) = (\alpha^2\beta(-2 + n)(4\beta + 2\gamma + \beta n + 5\gamma n)(\beta + 2\beta n + \gamma n^2))\backslash(2.D^2)$

Since $n \geq 3$ and since in our model we assume that $\alpha > 0$, $\beta > 0$, $\gamma > 0$, so $\forall j \in N \backslash \{i\} : SW_i(Q_C) > SW_j(Q_C)$.

Thus the center i has strictly more social welfare than others'.

Now we will prove that i has the maximum social welfare in the single-centered star cover than in other covers. Suppose contrary, suppose there exist a cover $C' \in \mathcal{C}^N$ such that i reaches its maximum social welfare. Then there are two cases;

Case 1: i is isolated in C'. Then, $SW_i(Q_{C'}) = \frac{\alpha^2(3\beta+\gamma)}{2(2\beta+\gamma)^2}$. Hence,

$SW_i(Q_C) - SW_i(Q_{C'}) = (\alpha^2\beta(-1+n)(-37\beta^4 + 25\beta^4 n - 27\beta^3\gamma + 4\beta^3\gamma n^3 - 4\gamma^4 - 9\gamma^4 n^2 + 5\gamma^4 n^3 - 11\beta^3\gamma n + 56\beta^3\gamma n^2 - 32\beta^2\gamma^2 n + 32\beta^2\gamma^2 n^2 - 15\beta\gamma^3 n^2 + 21\beta\gamma^3 n^3 + 32\beta^2\gamma^2 + 8\beta^4 n^2 + 24\beta^2\gamma^2 n^3))\backslash\ (2(2\beta+\gamma)^2.D^2)$

Since $n \geq 3$ and since in our model we assume that $\alpha > 0$, $\beta > 0$, $\gamma > 0$, so $SW_i(Q_C) - SW_i(Q_{C'}) > 0$. Hence, i cannot be isolated in C'.

Case 2: i is not isolated in C'. Then $\exists\ \{j_1, ..., j_{m-1}\} \subset N$ such that $\forall j_k \in \{j_1, ..., j_{m-1}\}, \exists H' \in C'$ such that $i, j_k \in H'$. Then by Lemma 1, for the country i, in order to reach its maximum social welfare, C' be such that,

$C' = \{ij_1, ij_2, ..., ij_m, H'_1, ..., H'_k\}$ where $H'_1, ..., H'_k$ are other hyperlinks in C'.

By Lemma 1, it should be $\forall j_k \in \{j_1, ..., j_{m-1}\}$, $j_k \notin H'_l$, $\forall l \in \{1, ..., k\}$ and $i \notin H'_l$, $\forall l \in \{1, ..., k\}$. But then, by above calculation,

$SW_i(Q_{C'}) = (\alpha^2(16\beta^3 + 20\beta^2\gamma + 4\beta\gamma^2 - 2\beta^3 m + 52\beta^2\gamma m + 44\beta\gamma^2 m + 11\beta^3 m^2 + 6\beta^2\gamma m^2 + 45\beta\gamma^2 m^2 + 16\gamma^3 m^2 + 2\beta^3 m^3 + 20\beta^2\gamma m^3 + 4\beta\gamma^2 m^3 + 8\gamma^3 m^3 + \beta^2\gamma m^4 + 8\beta\gamma^2 m^4 + \gamma^3 m^4))\backslash(2.P^2)$

where $P = (3\beta^2 + 6\beta\gamma + 3\beta^2 m + 5\beta\gamma m + 4\gamma^2 m + 2\beta c m^2 + \gamma^2 m^2)$

Since $m \geq 2$ and since in our model we assume that $\alpha > 0$, $\beta > 0$, $\gamma > 0$, so, $\frac{\partial SW_i(Q_{C'})}{\partial m} > 0$. Hence, $SW_i(Q_C) > SW_i(Q_{C'})$.Thus, contradiction. Hence, the country i reaches its maximum social welfare if it is the center of the single-centered star cover.

Now, assume that i has the maximum social welfare, we will prove that i is the center of the single-centered star cover.First of all, i cannot be isolated by case 1 of the first part of the proof. Hence, i is not isolated. Then by case 2 of the first part of the proof, it follows that, i reaches its maximum social welfare then it is the center of the single-centered star cover. Hence, the country i reaches its maximum social welfare if and only if it is the center of the single-centered star cover. ■

Given a cover $C \in \mathcal{C}^N$ and given two countries, i and j, assume that i and j are not in a union together. Moreover, assume that there exist a country, say, k, which is in a union of both of these countries. Then, if i and j decide to form a union between them without exiting from their recent unions, then their social welfare increases, and by Lemma 1, the social welfare of k decreases. Before we state and prove this observation, we will give an example,

Example 4 *$N = \{i, j, k\}$. Let $C = \{ik, kj\}$ and let $C' = \{ik, kj, ij\}$. We will show that adding the hyperlink ij increases the social welfare of i and j.In C, since there is no connection between i and j, so $q_{ij} = q_{ji} = 0$. In C', in equilibrium, $q_{ij} \geq 0$ and $q_{ji} \geq 0$.We will show that the increase in q_{ij} and q_{ji} will effect the quantity levels, $q_{ii}, q_{ik}, q_{ki}, q_{jj}, q_{jk}, q_{kj}, q_{kk}$ in equilibrium. Then we will show that $SW_j(Q_C) < SW_j(Q_{C'})$. Note that in this example, $SW_j(Q_C) = SW_i(Q_C)$ and $SW_j(Q_{C'}) = SW_i(Q_{C'})$.*

Given the cover $C \in \mathcal{C}^N$, Cournot equilibrium for quantity levels can be written in the matrix form, $-\alpha + D_C Q_C^* \geq 0$ where

$$D_C = \begin{bmatrix} 2\beta+\gamma & \beta & \gamma & 0 & 0 & 0 & 0 \\ \beta & 2\beta+\gamma & 0 & 0 & \gamma & 0 & \gamma \\ \gamma & 0 & 2\beta+\gamma & 0 & 0 & \beta & \beta \\ 0 & 0 & 0 & 2\beta+\gamma & \beta & \gamma & 0 \\ 0 & \gamma & 0 & \beta & 2\beta+\gamma & 0 & \gamma \\ 0 & 0 & \beta & \gamma & 0 & 2\beta+\gamma & \beta \\ 0 & \gamma & \beta & 0 & \gamma & \beta & 2\beta+\gamma \end{bmatrix} \quad \textit{and}$$

$$Q_C = \begin{bmatrix} q_{ii} \\ q_{ik} \\ q_{ki} \\ q_{jj} \\ q_{jk} \\ q_{kj} \\ q_{kk} \end{bmatrix}$$

For the cover C, the Cournot equilibrium for quantity levels can be written in the equations form, such that, $F_l : \mathbb{R}^7 \to \mathbb{R}$ where $q_{ii}, q_{ik}, q_{ki}, q_{jj}, q_{jk}, q_{kj}, q_{kk}$, are dependent variables, and q_{ij}, q_{ji} are independent variables, so,

$$F_1(q_{ii}, q_{ik}, q_{ki}, q_{jj}, q_{jk}, q_{kj}, q_{kk}) = (2\beta+\gamma)q_{ii} - \alpha + \gamma q_{ki} + \beta q_{ik} = 0$$

$$\vdots \qquad\qquad\qquad\qquad \vdots$$

$$F_7(q_{ii}, q_{ik}, q_{ki}, q_{jj}, q_{jk}, q_{kj}, q_{kk}) = (2\beta+\gamma)q_{kk} - \alpha + \gamma(q_{ik}+q_{jk}) + \beta(q_{ki}+q_{kj}) = 0$$

Now, F_l is linear $\forall l \in \{1, ..., 7\}$ and by Remark 1, $\frac{\partial(F_1,...,F_7)}{\partial(q_{ii},...,q_{kk})} = \det(D_C) > 0$.By Cramer's rule we have, given $q_{tr} \in \{q_{ii}, q_{ik}, q_{ki}, q_{jj}, q_{jk}, q_{kj}, q_{kk}\}$

$$\frac{\partial q_{tr}}{\partial q_{ij}} = \frac{-\frac{\partial(F_1,...,F_7)}{\partial(q_{ii},...,q_{ij},...,q_{kk})}}{\frac{\partial(F_1,...,F_7)}{\partial(q_{ii},...,q_{kk})}} = \frac{-\left|\begin{bmatrix} \frac{\partial F_1}{\partial q_{ii}} & \cdots & \frac{\partial F_1}{\partial q_{ij}} & \cdots & \frac{\partial F_1}{\partial q_{kk}} \\ \vdots & & \vdots & & \vdots \\ \frac{\partial F_7}{\partial q_{ii}} & \cdots & \frac{\partial F_7}{\partial q_{ij}} & \cdots & \frac{\partial F_7}{\partial q_{kk}} \end{bmatrix}\right|}{\left|\begin{bmatrix} \frac{\partial F_1}{\partial q_{ii}} & \cdots & \frac{\partial F_1}{\partial q_{tr}} & \cdots & \frac{\partial F_1}{\partial q_{kk}} \\ \vdots & & \vdots & & \vdots \\ \frac{\partial F_7}{\partial q_{ii}} & \cdots & \frac{\partial F_7}{\partial q_{tr}} & \cdots & \frac{\partial F_7}{\partial q_{kk}} \end{bmatrix}\right|}$$

similarly, $\frac{\partial q_{tr}}{\partial q_{ji}}$ can be written.

Let $\boldsymbol{x} = \begin{bmatrix} \frac{\partial F_1}{\partial q_{ij}} \\ \vdots \\ \frac{\partial F_7}{\partial q_{ij}} \end{bmatrix}_{(7x1)}$ *be the column vector, then,* $\boldsymbol{x} = \begin{bmatrix} \beta \\ \beta \\ 0 \\ \gamma \\ 0 \\ \gamma \\ 0 \end{bmatrix}$

similarly, let $\boldsymbol{y} = \begin{bmatrix} \frac{\partial F_1}{\partial q_{ji}} \\ \vdots \\ \frac{\partial F_7}{\partial q_{ji}} \end{bmatrix}_{(7x1)}$ *be the column vector, then,* $\boldsymbol{y} = \begin{bmatrix} \gamma \\ 0 \\ \gamma \\ \beta \\ \beta \\ 0 \\ 0 \end{bmatrix}$

Note that, $\alpha, \beta, \gamma > 0$, then,

$$\frac{\partial q_{ii}}{\partial q_{ij}} = \frac{\partial q_{jj}}{\partial q_{ji}} = \frac{(-1)(12\beta^3+55\beta^2\gamma+66\beta\gamma^2+27\gamma^3)}{(3(3\beta+5\gamma)(4\beta^2+13\beta\gamma+7\gamma^2))} < 0,$$

$\frac{\partial q_{ii}}{\partial q_{ji}} = \frac{\partial q_{jj}}{\partial q_{ij}} = \frac{(-1)(28\beta^2\gamma+84\beta\gamma^2+48\gamma^3)}{(3(3\beta+5\gamma)(4\beta^2+13\beta\gamma+7\gamma^2))} < 0,$

$\frac{\partial q_{ik}}{\partial q_{ij}} = \frac{\partial q_{jk}}{\partial q_{ji}} = \frac{(-1)(12\beta^3+55\beta^2\gamma+86\beta\gamma^2+39\gamma^3)}{(3(3\beta+5\gamma)(4\beta^2+13\beta\gamma+7\gamma^2))} < 0,$

$\frac{\partial q_{ik}}{\partial q_{ji}} = \frac{\partial q_{jk}}{\partial q_{ij}} = \frac{(20\beta^2\gamma+52\beta\gamma^2+24\gamma^3)}{(3(3\beta+5\gamma)(4\beta^2+13\beta\gamma+7\gamma^2))} > 0,$

$\frac{\partial q_{ki}}{\partial q_{ij}} = \frac{\partial q_{kj}}{\partial q_{ji}} = \frac{(5\beta^2\gamma+18\beta\gamma^2+9\gamma^3)}{((3\beta+5\gamma)(4\beta^2+13\beta\gamma+7\gamma^2))} > 0,$

$\frac{\partial q_{ki}}{\partial q_{ji}} = \frac{\partial q_{kj}}{\partial q_{ij}} = \frac{(-1)(11\beta^2\gamma+34\beta\gamma^2+19\gamma^3)}{((3\beta+5\gamma)(4\beta^2+13\beta\gamma+7\gamma^2))} < 0$ *and*

$\frac{\partial q_{kk}}{\partial q_{ji}} = \frac{\partial q_{kk}}{\partial q_{ij}} = \frac{5\beta\gamma+3\gamma^2}{3(4\beta^2+13\beta\gamma+7\gamma^2)} > 0.$

Now, by total differentiation, $dSW_j(Q_C) = \frac{\partial SW_j(Q_C)}{\partial q_{ij}} dq_{ij} + \frac{\partial SW_j(Q_C)}{\partial q_{ji}} dq_{ji}$

Then, $\frac{\partial SW_j(Q_C)}{\partial q_{ij}} = \frac{\alpha\beta(\beta+3\gamma)(144\beta^3+519\beta^2\gamma+498\beta\gamma^2+119\gamma^3)}{(9(3\beta+5\gamma)(4\beta^2+13\beta\gamma+7\gamma^2)^2)} > 0,$

and $\frac{\partial SW_j(Q_C)}{\partial q_{ji}} = \frac{(-1)(\alpha\beta\gamma(\beta+3\gamma)(45\beta^2+66\beta\gamma+49\gamma^2))}{(9(3\beta+5\gamma)(4\beta^2+13\beta\gamma+7\gamma^2)^2)} < 0$

Take $dq_{ij} = q_{ij}^*$ *and* $dq_{ji} = q_{ji}^*$ *where* q_{ij}^* *and* q_{ji}^* *are Cournot equilibrium for the cover* C', $q_{ij}^* = q_{ji}^* = \alpha \backslash (4\beta + 3\gamma)$.

Thus we get, $dSW_j(Q_C) = \frac{2\alpha^2\beta(\beta+3\gamma)(24\beta^2+39\beta\gamma+7\gamma^2)}{(9(4\beta+3\gamma)(4\beta^2+13\beta\gamma+7\gamma^2)^2)} > 0.$

Hence, $SW_j(Q_C) < SW_j(Q_{C'})$ *so,* $SW_i(Q_C) < SW_i(Q_{C'})$.

As we see from the example, if we form a new union from indirectly connected countries, i and j, then, the social welfare of those countries increases. Now we will prove this observation as a lemma.

Lemma 2 *Let* $|N| \geq 3$. *Let* $C \in \mathcal{C}^N$ *and* $i, j, k \in N$.*Assume* $\exists H \in C$ *such that* $i, k \in H$ *and* $\exists \widetilde{H} \in C$ *such that* $j, k \in \widetilde{H}$ *but,* $\nexists H \in C$ *such that* $i, j \in H$. *If* $C' = \{H \in C, \forall H \in C\} \bigcup \{ij\}$, *then* $SW_i(Q_C) < SW_i(Q_{C'})$ *and* $SW_j(Q_C) < SW_j(Q_{C'})$.

Proof. Let $|N| \geq 3$. Let $C \in \mathcal{C}^N$ and $i, j, k \in N$. Assume $\exists H \in C$ such that $i, k \in H$ and $\exists \widetilde{H} \in C$ such that $j, k \in \widetilde{H}$ but, $\nexists H \in C$ such that $i, j \in H$. Let $C' = \{H \in C, \forall H \in C\} \bigcup \{ij\}$, then we will prove that $SW_i(Q_C) < SW_i(Q_{C'})$ and $SW_j(Q_C) < SW_j(Q_{C'})$.

In C, since there is no connection between i and j,so $q_{ji} = q_{ij} = 0$. In C', in equilibrium, $q_{ji} \geq 0$ and $q_{ij} \geq 0$.We will show that how the increase in q_{ji} and q_{ij}

will effect the quantity levels, q_{is}, q_{si} where $s \in N\backslash\{j\}$ in equilibrium. Then we will show that $SW_i(Q_C) < SW_i(Q_{C'})$.

For the cover C, the Cournot equilibrium for quantity levels can be written in the equations form, such that, $F_l : \mathbb{R}^e \to \mathbb{R}$ where

$e = |\{(t,r) : \exists H \in C \text{ such that } t,r \in H, \text{ where } t,r \in \{1,...n\}\}|$.

Hence, $\forall q_{tr}$ such that $\exists H \in C : t,r \in H$, where $t,r \in \{1,...n\}$, we have

$$F_1(q_{11},\dots,q_{nn}) = (2\beta+\gamma)q_{11} - \alpha + \gamma \sum_{\substack{t\in N\backslash\{1\}\ st \\ \exists H\in C:\ t,1\in H}} q_{t1} + \beta \sum_{\substack{k\in N\backslash\{1\}\ st \\ \exists H\in C:\ 1,k\in H}} q_{1k} = 0$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$F_e(q_{11},\dots,q_{nn}) = (2\beta+\gamma)q_{nn} - \alpha + \gamma \sum_{\substack{t\in N\backslash\{n\}\ st \\ \exists H\in C:\ t,n\in H}} q_{tn} + \beta \sum_{\substack{k\in N\backslash\{n\}\ st \\ \exists H\in C:\ n,k\in H}} q_{nk} = 0$$

Now, F_l is linear $\forall l \in \{1,...e\}$ and by Remark 1, $\frac{\partial(F_1,\dots,F_e)}{\partial(q_{11},\dots,q_{nn})} = \det(D_C) > 0$. Also, q_{ij}, q_{ji} are independent and all other q_{tr} 's such that $\exists H \in C : t,r \in H$, where $t,r \in \{1,...n\}$ are dependent variables. By Cramer's rule we have, given q_{tr} such that $\exists H \in C : t,r \in H$, where $t,r \in \{1,...n\}$, and $q_{tr} \neq q_{ij}$, $q_{tr} \neq q_{ji}$

$$\frac{\partial q_{tr}}{\partial q_{ij}} = \frac{-\frac{\partial(F_1,\dots,F_e)}{\partial(q_{11},\dots,q_{ij}\dots,q_{nn})}}{\frac{\partial(F_1,\dots,F_e)}{\partial(q_{11},\dots,q_{nn})}} = \frac{-\left|\begin{bmatrix} \frac{\partial F_1}{\partial q_{11}} & \cdots & \frac{\partial F_1}{\partial q_{ij}} & \cdots & \frac{\partial F_1}{\partial q_{nn}} \\ \vdots & & \vdots & & \vdots \\ \frac{\partial F_e}{\partial q_{11}} & \cdots & \frac{\partial F_e}{\partial q_{ij}} & \cdots & \frac{\partial F_e}{\partial q_{nn}} \end{bmatrix}\right|}{\left|\begin{bmatrix} \frac{\partial F_1}{\partial q_{11}} & \cdots & \frac{\partial F_1}{\partial q_{tr}} & \cdots & \frac{\partial F_1}{\partial q_{nn}} \\ \vdots & & \vdots & & \vdots \\ \frac{\partial F_e}{\partial q_{11}} & \cdots & \frac{\partial F_e}{\partial q_{tr}} & \cdots & \frac{\partial F_e}{\partial q_{nn}} \end{bmatrix}\right|}$$

similarly $\frac{\partial q_{tr}}{\partial q_{ji}}$ is written.

Let $\boldsymbol{x} = \begin{bmatrix} \frac{\partial F_1}{\partial q_{ij}} \\ \vdots \\ \frac{\partial F_e}{\partial q_{ij}} \end{bmatrix}_{(ex1)}$ be the column vector, then, $\boldsymbol{x}_{l1} = \begin{cases} \beta, & \text{if } \frac{\partial F_l}{\partial q_{ij}} = \beta \\ \gamma, & \text{if } \frac{\partial F_l}{\partial q_{ij}} = \gamma \\ 0, & \text{if } \frac{\partial F_l}{\partial q_{ij}} = 0 \end{cases}$

similarly, let $\boldsymbol{y} = \begin{bmatrix} \frac{\partial F_1}{\partial q_{ji}} \\ \vdots \\ \frac{\partial Fe}{\partial q_{ji}} \end{bmatrix}_{(ex1)}$ be the column vector, then, $\boldsymbol{y}_{l1} = \begin{cases} \beta, & \text{if } \frac{\partial F_l}{\partial q_{ji}} = \beta \\ \gamma, & \text{if } \frac{\partial F_l}{\partial q_{ji}} = \gamma \\ 0, & \text{if } \frac{\partial F_l}{\partial q_{ji}} = 0 \end{cases}$

Then, $\frac{\partial q_{tr}}{\partial q_{ij}} < 0$ if $t = i$ and $r \neq j$, $\frac{\partial q_{tr}}{\partial q_{ij}} < 0$ if $t \neq i$ and $r = j$, $\frac{\partial q_{tr}}{\partial q_{ij}} > 0$ if $t \neq i$ and $r \neq j$, where t, r, i and j are in a connected bipartite graph in C'.

Similar reasoning holds for $\frac{\partial q_{tr}}{\partial q_{ji}}$.

Now, by total differentiation, $dSW_i(Q_C) = \frac{\partial SW_i(Q_C)}{\partial q_{ij}} dq_{ij} + \frac{\partial SW_i(Q_C)}{\partial q_{ji}} dq_{ji}$

Here, $\frac{\partial SW_i(Q_C)}{\partial q_{ij}} < 0$, and $\frac{\partial SW_i(Q_C)}{\partial q_{ji}} > 0$. Take $dq_{ij} = q^*_{ij}$ and $dq_{ji} = q^*_{ji}$ where q^*_{ij} and q^*_{ji} are Cournot equilibrium for the cover C', so, we get, $dSW_i(Q_C) > 0$.

Hence, $SW_i(Q_{C'}) > SW_i(Q_C)$, and similarly, $SW_j(Q_{C'}) > SW_j(Q_C)$. ■

Lemma 3 *Let $C \in \mathcal{C}^N$.Let $|N| = n > 1$. Assume $\exists! d \in \{0, ...n-2\}$ such that $\forall i \in N$, $\deg_i(C) = d$. Then, complete-equivalent cover strictly increases the social welfare of all countries.*

Proof. Let $C \in \mathcal{C}^N$.Let $|N| > 1$. Assume $\exists! d \in \{0, ...n-2\}$ such that $\forall i \in N$, $\deg_i(C) = d$. Hence in equilibrium we have,$\forall i, j \in N : \exists H \in C$ such that $i, j \in H$, $q^*_{ii} = q^*_{ij} = q^*_{ji} = q^*_{jj} = \alpha \backslash (\; 2\beta + \gamma + \beta d + \gamma d)$ so, $SW_i(Q_C) = \frac{\alpha^2(1+d)(3\beta+\gamma+\beta d+\gamma d)}{2(2\beta+\gamma+\beta d+\gamma d)^2}$.

Let $C' \in \mathcal{C}^N$ be complete-equivalent cover, then by Proposation 1, we have $SW_i(Q_{C'}) = \frac{\alpha^2 n(2\beta+n\beta+n\gamma)}{2(\beta+\beta n+\gamma n)^2}$. Hence, we have $\forall i \in N$,

$SW_i(Q_{C'}) - SW_i(Q_C) = \frac{\alpha^2\beta^2(n-d-1)(3\beta+\gamma+\beta d+\gamma d+\beta n+\gamma n)}{2(2\beta+\gamma+\beta d+\gamma d)^2(\beta+\beta n+\gamma n)^2} > 0$ as $d < n-1$,and $\alpha, \beta, \gamma > 0$.

Hence,complete- equivalent cover strictly increases the social welfare of all countries. Note that, if $d = n-1$ then the cover is complete-equivalent cover. ■

In our model, the allocation rule and the value of a cover, is attained by the social welfare of each country. Hence, given $C \in \mathcal{C}^N$, we have $\forall i \in N$, $Y_i(C) = SW_i(Q_C) - \frac{\alpha^2(3\beta+\gamma)}{2(2\beta+\gamma)^2}$, here $\frac{\alpha^2(3\beta+\gamma)}{2(2\beta+\gamma)^2}$ represents the social welfare of a country when it is isolated. Therefore, $v(C) = 0$ whenever $|H| = 1$ for all $H \in C$ and given $C \in \mathcal{C}^N$, we have $v(C) = \sum_{i \in N} Y_i(C)$. However, in order to make calculations clear and easy, instead of dealing with the allocation rule, we will utilize social welfare concept in our proofs.

Now we will examine the core stable covers under given one of the four membership rights, free exit-free entry, free exit-approved entry, approved exit-free entry, or approved exit-approved entry.

Theorem 1 *Let (v, Y) is given above. A cover $C \in \mathcal{C}^N$ is core stable relative to (v, Y) under given one of the four membership rights, FX-FE, FX-AE, AX-FE, or AX-AE if and only if it is a complete-equivalent cover.*

Proof. First of all, if $|N| = 1$ the result follows trivially. If $|N| = 2$, the result follows from Proposition 1. Thus assume that $|N| \geq 3$.

Let $C \in \mathcal{C}^N$ is core stable relative to (v, Y) under given one of the four membership rights, FX-FE, FX-AE, AX-FE, or AX-AE. We will prove that $C \in \mathcal{C}^N$ is a complete-equivalent cover. Suppose contrary, then $\exists i, j \in N$ such that $\nexists H \in C$: $i, j \in H$. Then there are two cases,

Case 1: $\forall H, \overline{H} \in C$, we have $H \bigcap \overline{H} = \emptyset$. Then $\forall i \in N$, $\exists! H \in C$ such that $i \in H$. By Proposition 1, $SW_i(Q_C) = \frac{\alpha^2 m(2\beta + m\beta + m\gamma)}{2(\beta + \beta m + \gamma m)^2}$ where $|H| = m$.

Let $T = N$, and C' be a complete-equivalent cover. Then, $C' \subset \{f(H) \bigcup P : H \in C, P \in 2^T\} \bigcup 2^T$ for some T-function f on C, where $f(H) = H$,$\forall H \in C$.

Now, $\forall i \in N$ by Proposition 1, $SW_i(Q_{C'}) = \frac{\alpha^2 n(2\beta + n\beta + n\gamma)}{2(\beta + \beta n + \gamma n)^2}$. Since, $n > m$, so by Proposition 1, $SW_i(Q_{C'}) > SW_i(Q_C)$. Now, C' is obtainable from C via T relative to v and Y under given one of the four membership rights, FX-FE, FX-AE, AX-FE, or AX-AE and T can improve upon C via C' relative to (v, Y) under given exit- entry condition. Thus, the cover C is not core stable relative to (v, Y) . Contradiction. Therefore, $\exists H, \overline{H} \in C$, such that $H \bigcap \overline{H} \neq \emptyset$.

Case 2: $\exists H, \overline{H} \in C$, such that $H \bigcap \overline{H} \neq \emptyset$. By cover definition, $H \backslash (H \bigcap \overline{H}) \neq \emptyset$ and $\overline{H} \backslash (H \bigcap \overline{H}) \neq \emptyset$. Let $i \in H \backslash (H \bigcap \overline{H})$ and $j \in \overline{H} \backslash (H \bigcap \overline{H})$ and $k \in H \bigcap \overline{H}$.

Then, $i, k \in H$ and $j, k \in \overline{H}$.

Let $T = \{i, j\}$ and $C' = C \bigcup \{ij\}$. Then, $C \subset \{f(H) \bigcup P : H \in C, P \in 2^T\} \bigcup 2^T$ for some T-function f on C, where $f(H) = H$,$\forall H \in C$. Since, $C' = C \bigcup \{ij\}$, so C' is obtainable from C via T relative to (v, Y) under given one of the four membership rights, FX-FE, FX-AE, AX-FE, or AX-AE. By Lemma 2, $SW_i(Q_{C'}) > SW_i(Q_C)$ and $SW_j(Q_{C'}) > SW_j(Q_C)$. Therefore, T can improve upon C via C' relative to (v, Y) under given exit- entry condition. Thus, the cover C is not core stable relative to (v, Y). Contradiction. Therefore, C is a complete- equivalent cover.

Conversely, assume that, $C \in \mathcal{C}^N$ is a complete-equivalent cover. We will prove that $C \in \mathcal{C}^N$ is core stable relative to (v, Y) under given one of the four membership rights, FX-FE, FX-AE, AX-FE, or AX-AE .

Suppose contrary, then $\exists T \in 2^N \backslash \{\emptyset\}$ and $\exists C' \in \mathcal{C}^N$,which is obtainable from C via T relative to v and Y under FX-FE, FX-AE, AX-FE, or AX-AE, such that T can improve upon C via C' relative to (v, Y) under FX-FE, FX-AE, AX-FE, or AX-AE. Hence, $\forall i \in T$, $SW_i(Q_C) \leq SW_i(Q_{C'})$ and $\exists j \in T$, $SW_j(Q_C) < SW_j(Q_{C'})$.

Now if $T = \{i\}$, then the only obtainable cover from C via T relative to (v, Y) under FX-FE, FX-AE, AX-FE, or AX-AE, is $C' = \{H, H'\}$ where $H = N \backslash \{i\}$ and $H' = \{i\}$. By Proposition 1, $SW_i(Q_{C'}) < SW_i(Q_C)$ since $1 < n$. Contradiction to the fact that $\forall i \in T$, $SW_i(Q_C) \leq SW_i(Q_{C'})$. Hence, $|T| \geq 2$.

Now let $i \in T$ and $\forall H \in C'$ such that $i \in H$, we have $\forall H' \in C'$, $H \bigcap H' = \emptyset$.Then given $|H| = m$, since $m < n$, by Proposition 1,$SW_i(Q_{C'}) < SW_i(Q_C)$. Contradiction to the fact that $\forall i \in T$, $SW_i(Q_C) \leq SW_i(Q_{C'})$. Thus, $\forall i \in T$ and $\forall H \in C'$such that $i \in H$, we have $\exists H' \in C'$, $H \bigcap H' \neq \emptyset$ where $H \neq H'$.

Now let $i \in T$ and assume $\exists! H \in C'$such that $i \in H$ and $\exists H' \in C'$ such that $H \bigcap H' \neq \emptyset$ where $H \neq H'$.Let C'' be a cover such that

$C'' = \{\overline{H} : \overline{H} \in C', \overline{H} \neq H$, and $\overline{H} \neq H'$ where $H' \in C'$ such that $H \bigcap H' \neq \emptyset\} \bigcup \{H' \backslash (H \bigcap H') : H' \in C'$ such that $H \bigcap H' \neq \emptyset$ where $H \neq H'\} \bigcup \{H\}$

By Lemma 1, $SW_i(Q_{C'}) < SW_i(Q_{C''})$. In C'', $\exists! \widetilde{H} \in C''$such that $i \in \widetilde{H}$, we have $\forall H'' \in C''$, $\widetilde{H} \bigcap H'' = \emptyset$. By above part of the proof, $SW_i(Q_{C''}) < SW_i(Q_C)$. Hence, by transitivity, $SW_i(Q_{C'}) < SW_i(Q_C)$.Contradiction to the fact that $\forall i \in T$, $SW_i(Q_C) \leq SW_i(Q_{C'})$.

Hence, $\forall i \in T$, $\exists$ $H, H' \in C'$such that $i \in H \bigcap H'$, where $H \neq H'$. $\forall i \in T$, define $\forall i \in T$, $\deg_i(C') \geq 2$. Now assume that, $\exists! \overline{d}$ such that $\forall i \in T$, $\deg_i(C') = \overline{d}$. Note that, by the definition of T-function f on C, and since C' is not a complete-equivalent cover, $\overline{d} < n - 1$. Then by Lemma 3, $\forall i \in T$, $SW_i(Q_{C'}) < SW_i(Q_C)$. Contradiction to the fact that $\forall i \in T$, $SW_i(Q_C) \leq SW_i(Q_{C'})$.

Thus, assume that $\nexists! \overline{d}$ such that $\forall i \in T$, $\deg_i(C') = \overline{d}$. Then $\exists t \in T$ such that $\deg_t(C') = \min\{\deg_i(C') : i \in T\}$. Now let C'' be a cover such that $\forall i \in T$, $\deg_i(C'') = \deg_t(C')$. Then by Lemma 1, $SW_t(Q_{C'}) \leq SW_t(Q_{C''})$. Note that, by the definition of T-function f on C, $\deg_t(C')$, and since C' is not a complete cover, $\deg_t(C') < n - 1$. Then by Lemma 3, $SW_t(Q_{C''}) < SW_t(Q_C)$. Hence, for $t \in T$, $SW_t(Q_{C'}) < SW_t(Q_C)$.Contradiction to the fact that $\forall i \in T$, $SW_i(Q_C) \leq SW_i(Q_{C'})$.

Thus, $\nexists T \in 2^N \backslash \{\emptyset\}$ and $\nexists C' \in \mathcal{C}^N$,which is obtainable from C via T relative to v and Y under FX-FE, FX-AE, AX-FE, or AX-AE, is such that T can improve upon C via C' relative to (v, Y) under FX-FE, FX-AE, AX-FE, or AX-AE. Hence, $C \in \mathcal{C}^N$ is core stable relative to (v, Y) under FX-FE, FX-AE, AX-FE, or AX-AE.

■

Now we find core stable covers relative to (v, Y) under free exit- strongly approved entry condition.

Proposition 2 *Let $v \in V$, be any value function and Y allocation rule associated with the value function v. Let $C \in \mathcal{C}^N$ be a core stable cover relative to (v, Y) under FX-FE. Then it is core stable cover relative to (v, Y) under FX-SAE or AX-SAE.*

Proof. Let $v \in V$, be any value function and Y allocation rule associated with the value function v. Let $C \in C^N$ be a core stable cover relative to (v, Y) under FX-FE. Thus, given any $T \in 2^N \backslash \{\emptyset\}$, $\forall C' \in \mathcal{C}^N$ where $C' \subset \{f(H) \bigcup P : H \in C, P \in 2^T\} \bigcup 2^T$ for some $T-$function f on C, T does not improve upon C via C' relative to (v, Y). Thus, $\nexists T \in 2^N \backslash \{\emptyset\}$ and $\nexists C' \in \mathcal{C}^N$,which is obtainable from C via T relative to v and Y under FX-SAE or AX-SAE, is such that T can improve upon C via C' relative to (v, Y) under FX-SAE or AX-SAE. Hence, $C \in \mathcal{C}^N$ is core stable relative to (v, Y) under FX-SAE or AX-SAE. ■

Theorem 2 *Complete-equivalent cover is core stable under free exit- strongly approved entry. For $|N| \leq 3$, it is the only core stable cover under FX-SAE.*

Proof. First of all, we will prove that complete-equivalent cover is core stable relative to (v, Y) under FX-SAE. It follows from Theorem 1, and Proposition 2.

Now we will prove that for $|N| \leq 3$, it is the only core stable cover.

For $|N| = 1$ and $|N| = 2$, the result is trivial.

For $|N| = 3$, and $N = \{i, j, k\}$, by Proposition 1, we only consider the cover, $C' = \{ij, jk\}$. For $T = \{i, k\}$, and $C'' = \{ik, j\}$, by Lemma 2, and by the definition of FX-SAE, T can improve upon C' via C'' relative to (v, Y) under FX-SAE. Hence, $C' \in \mathcal{C}^N$ is not core stable relative to (v, Y) under FX-SAE. Therefore, for $|N| \leq 3$, complete-equivalent cover is the only core stable cover relative to (v, Y) under FX-SAE. ■

Remark 2 *For $|N| \geq 4$, and for some parameters β and γ, there are other covers which are core stable relative to (v, Y) under FX-SAE.*

Example 5 *Consider $|N| = 4$, and $N = \{i, j, k, l\}$ by Proposition 1,by above case, we only need to consider these covers or equilvalents of them;*

Let $C' = \{ij, jk, kl\}$. For $T = \{i, l\}$, and $C'' = \{il, jk\}$, by Lemma 2, and by the definition, T can improve upon C' via C'' relative to (v, Y) under FX-SAE. Hence, $C' \in \mathcal{C}^N$ is not core stable relative to (v, Y) under FX-SAE.

Let $C' = \{ij, jk, kl, il\}$. For $T = N$, and $C'' = \{iljk\}$, by Lemma 3, T can improve upon C' via C'' relative to relative to (v, Y) under FX-SAE. Hence, $C' \in \mathcal{C}^N$ is not core stable relative to (v, Y) under FX-SAE.

Let $C' = \{ij, jkl\}$. For $T = \{i, k, l\}$, and $C'' = \{ikl, j\}$, by Lemma 1, and by the definition, T can improve upon C' via C'' relative to (v, Y) under FX-SAE. Hence, $C' \in \mathcal{C}^N$ is not core stable relative to (v, Y) under FX-SAE.

Let $C' = \{ij, ik, il\}$. For $T = \{k, l\}$, and $C'' = \{ij, kl\}$, by Lemma 2, and by the definition of free exit- strongly approved entry condition, T can improve upon C' via C'' relative to (v, Y) under FX-SAE. Hence, $C' \in \mathcal{C}^N$ is not core stable relative to (v, Y) under FX-SAE.

Let $C = \{ijk, jkl\}$. For $T = \{i, l\}$, and $C' = \{il, jk\}$, by direct calculations we have

$$SW_i(Q_{C'}) - SW_i(Q_C) = \frac{\alpha^2\beta(6\beta^4 - 183\beta^3\gamma - 920\beta^2\gamma^2 + 685\beta\gamma^3 + 988\gamma^4)}{6(3\beta + 2\gamma)^2(10\beta^2 + 51\beta\gamma + 35\gamma^2)^2}$$

similar result is valid for the country l.Now, C is not core stable if $\alpha > 0$, $\gamma = \beta$. By direct calculations we have,$SW_i(Q_{C'}) - SW_i(Q_C) = \frac{\alpha^2}{2400\beta} > 0$.

Now we will prove that C is core stable relative to (v, Y) under FX-SAE if $\alpha > 0$, $\gamma = (1\backslash 2)\beta$.

Suppose contrary, then $\exists T \in 2^N \backslash \{\emptyset\}$ and $\exists C' \in \mathcal{C}^N$,which is obtainable from C via T relative to (v, Y) under FX-SAE, such that T can improve upon C via C' relative to (v, Y) under FX-SAE. Hence, $\forall t \in T$, $SW_t(Q_{C'}) \geq SW_t(Q_C)$. Trivially $|T| \neq 1$.

If $T = \{i, l\}$ then we only need to consider $C' = \{il, jk\}$, since $SW_i(Q_{C'}) - SW_i(Q_C) = \frac{-1345\alpha^2}{1503792\beta} < 0$. Hence,$T$ cannot improve upon C via C' relative to (v, Y)

under FX-SAE. Thus, $T \subsetneq \{i, l\}$. *But then* $\forall T \in 2^N \backslash \{\emptyset\}$ *such that* $j \in T$ *(or* $k \in T$*) by Lemma 1,2 and 3,* $\nexists C' \in \mathcal{C}^N$ *,which is obtainable from* C *via* T *relative to* v *and* Y *under FX-SAE, such that* T *can improve upon* C *via* C' *relative to* (v, Y) *under FX-SAE. Thus,* C *is core stable relative to* (v, Y) *under FX-SAE*

Now, we will find the core stable covers under approved exit- strongly approved entry.

Corollary 2 *Given* $(v,\, Y)$ *in the model, complete-equivalent cover is core stable relative to* $(v,\, Y)$ *under AX-SAE.*

Proof. Proof follows from Theorem 1, and Proposition 2. ■

Theorem 3 *Given* $(v,\, Y)$ *in the model, star cover is core stable relative to* $(v,\, Y)$ *under strongly approved exit-approved entry condition.*

Proof. Let $C \in \mathcal{C}^N$ be a star cover.Then, the cover has at least two hyperlinks, $\exists S \in 2^N \backslash \{\emptyset, N\}$, *such that* $\forall H, H' \in C$, $H \bigcap H' = S$. We will prove that C is core stable relative to (v, Y) under AX-SAE.

Suppose contrary, then $\exists T \in 2^N \backslash \{\emptyset\}$ and $\exists C' \in \mathcal{C}^N$,which is obtainable from C via T relative to v and Y under AX-SAE, such that T can improve upon C via C' relative to (v, Y) under AX-SAE. Hence, $\forall i \in T$, $SW_i(Q_C) \leq SW_i(Q_{C'})$ and $\exists j \in T$, $SW_j(Q_C) < SW_j(Q_{C'})$. Now, by the definition of AX-SAE and since C is a star, so $\forall T \in 2^N \backslash \{\emptyset\}$ and $\forall C' \in \mathcal{C}^N$,which is obtainable from C via T relative to v and Y under AX-SAE,$\forall k \in S$, $SW_k(Q_C) \leq SW_k(Q_{C'})$ should be. If $|S| = 1$, by Corollary 1, $\forall C' \in \mathcal{C}^N$ we have $SW_k(Q_C) > SW_k(Q_{C'})$, if $|S| \neq 1$ by Lemma 1,2 and 3, $\exists k \in S$ such that $SW_k(Q_C) > SW_k(Q_{C'})$. Contradiction. Hence, $\nexists T \in 2^N \backslash \{\emptyset\}$ and $\nexists C' \in \mathcal{C}^N$,which is obtainable from C via T relative to v and Y under AX-SAE, such that T can improve upon C via C' relative to (v, Y) under AX-SAE. Thus, star cover is core stable relative to (v, Y) under AX-SAE. ■

Now we will consider the efficient, Pareto efficient covers under the membership rights.

For networks, Jackson (2003) notes that, a network is efficient relative to a value function if it is Pareto efficient relative to the value function and for all allocation rules .Thus, as Jackson (2003) states for networks *"Efficiency is the more natural notion in situations where there is some freedom to reallocate value through transfers, while Pareto efficiency might be more reasonable in contexts where the allocation rule is fixed (and we are not able or willing to make further transfers or to make interpersonal comparisons of utility)."*

For covers, the same reasoning is valid.

Remark 3 *Note that, given any* (v, Y) *and given a membership right, if a cover is core stable relative to* (v, Y) *then it is Pareto efficient relative to* (v, Y)*. Thus, complete-equivalent cover is Pareto efficient relative to* (v, Y) *under all membership rights. However, star cover is Pareto efficient relative to* (v, Y) *but it is not core stable under some rights structures. Similarly, complete- equivalent cover is the efficient cover but star is not efficient.*

4.2 Model with Linkage Cost

So far in our analysis we assume that the entry to a union does not require cost. However, in reality, countries pay cost when they enter to a union and the countries in that union pay cost for the new comer. For instance, Turkey is a candidate for European Union. In order to join European Union, a new ministery, European Union Ministery, has been established, and some cost has been spent for controlling whether the trade products satisfy EU criteria or not. Similarly, EU countries has sent some funds to Turkey in order Turkey to be ready for EU. Hence, both sides pay cost.

Let $i \in N$ and let $C_n \in \mathcal{C}^N$ be complete- equivalent cover. Let $C_1 = \{i\}$ and let $C_n = \{i_1 i_2 ... i_n\}$. Define; $\forall n > 1, \forall k < n$,

$f_{1,n} = SW_i(Q_{C_n}) - SW_i(Q_{C_1})$ where $n > 1$.

$f_{k,n} = [SW_i(Q_{C_n}) - SW_i(Q_{C_k})] \backslash (n-k)$ where $n > k$.

Then we have $\forall t, p \geq 1$ and $r, y \geq 2$,

$f_{p,r} < f_{t,y}$ if and only if one of them holds; $\begin{cases} [p = t = 1 \text{ and } r < y] \text{ or} \\ [t < p] \text{ or} \\ [t = p \neq 1 \text{ and } r > y] \end{cases}$

Hence, $|N| = n$ we have,

$$f_{(n-1),n} < f_{(n-2),n} < ... < f_{2,4} < f_{2,3} < f_{1,2} < ... < f_{1,n} \qquad (0)$$

Note that $f_{p,r} > 0, \forall p, r \geq 1$. We also have;

$$f_{1,3} = f_{1,2} + f_{2,3}$$

$$f_{1,4} = f_{1,2} + f_{2,3} + f_{3,4} = f_{1,3} + f_{3,4}$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$f_{1,n} = f_{1,(n-1)} + f_{(n-1),n} \qquad (1)$$

Define M as constant cost and m as the variable cost. Assume $m \leq M$. If a country is isolated, in order to join a union, H, it pays $M + |H| .m$, similarly in that union, the countries pay m as a cost for the new comer. If a country is not isolated, in order to join a union, it will only pay cost for the countries which had not connection with it previously. Similarly, in that union, the countries which had not connection with the new comer at past, pay cost. Hence, while entrance requires cost, exit will be costless. Given $C \in \mathcal{C}^N, i \in N$, define social welfare of a country i in the "cost" case as $\overline{SW}_i(Q_C)$.Then,

$$\overline{SW}_i(Q_C) = \begin{cases} SW_i(Q_C) & \text{if } \deg_i(C) = 0 \\ SW_i(Q_C) - M - \deg_i(C).m & \text{otherwise} \end{cases}$$

Then the value function, $\overline{v} \in V$ and the associated allocation rule, $\overline{Y}$ is attained by the social welfare of each country. Hence, given $C \in \mathcal{C}^N$, we have $\forall i \in N$, $\overline{Y}_i(C) = \overline{SW}_i(Q_C) - \frac{\alpha^2(3\beta+\gamma)}{2(2\beta+\gamma)^2}$, here $\frac{\alpha^2(3\beta+\gamma)}{2(2\beta+\gamma)^2}$ represents the social welfare of a country when it is isolated. Therefore, $\overline{v}(C) = 0$ whenever $|H| = 1$ for all $H \in C$ and given $C \in \mathcal{C}^N$, we have $\overline{v}(C) = \sum_{i \in N} \overline{Y}_i(C)$.

We will prove an observation as a claim:

Claim 2 *Let $C \in \mathcal{C}^N$, $|N| = n \geq 3$. If $M \leq f_{2,3}$ and $m \leq f_{2,n}$, then $\forall k \in \{2, 3, ..., n\}$ we have $M + (k-1)m < f_{1,k}$.*

Proof. Let $C \in \mathcal{C}^N$ and let $|N| = n \geq 3$. Let $M \leq f_{2,3}$ and $m \leq f_{2,n}$. We will prove that given $n \geq 3, \forall k \in \{2, 3, ..., n\}$ we have $M + (k-1)m < f_{1,k}$.Now we know that $M + (k-1)m \leq f_{2,3} + (k-1)f_{2,n}$.Hence, given $n \geq 3, \forall k \in \{2, 3, ..., n\}$ we will prove that $f_{2,3} + (k-1)f_{2,n} < f_{1,k}$.

Let $n = 3$ then for $k = 2$, by (0) and (1), we have, $f_{2,3} + f_{2,3} < f_{1,2}$. For $k = 3$, we have $f_{2,3} + 2.f_{2,3} < f_{1,2} + f_{2,3} = f_{1,3}$.

Given arbitrary $n \geq 3$, we will do induction on k.

If $k = 2$ then, $f_{2,3} + f_{2,n} < f_{2,3} + f_{2,3} < f_{1,2}$. Assume by induction, the assumption is true $\forall k \leq (n-1)$. Then we will prove for $k = n$. Now for $k = n$, we know that

$f_{2,3} + f_{2,n}(n-1) < f_{2,3} + f_{2,(n-1)}(n-2) + f_{2,n}$ since $f_{2,n} < f_{2,(n-1)}$. By induction assumption, we also know that;

$f_{2,3} + f_{2,(n-1)}(n-2) < f_{1,(n-1)}$. We want to prove that; $f_{2,3} + f_{2,n}(n-1) < f_{1,n}$.Now,

$$[f_{1,n} - (n-1).f_{2,n}] - [f_{1,(n-1)} - (n-2)f_{2,(n-1)}] =$$
$$\frac{\alpha^2\beta^2(\beta+\gamma)(3\beta^2-2\beta\gamma+2\beta\gamma n^2-2\gamma^2+\gamma^2n^2+7\beta^2n+10\beta\gamma n+3\gamma^2n+\beta^2n^2)}{2(3\beta+2\gamma)^2(\beta+\beta n+\gamma n)^2(-\gamma+\beta n+\gamma n)^2} > 0.$$

Thus, $f_{2,3} + f_{2,n}(n-1) < f_{1,n}$. Therefore, given $n \geq 3, M \leq f_{2,3}$, and $m \leq f_{2,n}$ we have $M + (k-1)m < f_{1,k}, \forall k \in \{2, 3, ..., n\}$. ■

Note that denote core stable as "CS", and not core stable as "NCS". Below two examples, core stability is considered as core stability relative to $(\overline{v}, \overline{Y})$.

Now we will make the analysis for $|N| = 3$.

Let $N = \{i, j, k\}$ and let $C_1 = \{i, j, k\}$, $C_2 = \{ij, k\}, C_3 = \{ij, ik\}$ and $C_4 = \{ijk\}$. Then;

$f_1 = SW_i(C_2) - SW_i(C_1) = \frac{\alpha^2\beta^2(5\beta+3\gamma)}{2(2\beta+\gamma)^2(3\beta+2\gamma)^2} = f_{12}$

$f_2 = SW_i(C_3) - SW_i(C_1) = \frac{2\alpha^2\beta(55\beta^4+276\beta^3\gamma+436\beta^2\gamma^2+216\beta\gamma^3+25\gamma^4)}{9(2\beta+\gamma)^2(4\beta^2+13\beta\gamma+7\gamma^2)^2}$

$f_3 = SW_i(C_4) - SW_i(C_1) = \frac{2\alpha^2\beta^2(3\beta+2\gamma)}{2(2\beta+\gamma)^2(4\beta+3\gamma)^2} = f_{13}$

$f_4 = SW_i(C_3) - SW_i(C_2) = \frac{\alpha^2\beta(315\beta^4+1551\beta^3\gamma+2593\beta^2\gamma^2+1733\beta\gamma^3+400\gamma^4)}{18(3\beta+2\gamma)^2(4\beta^2+13\beta\gamma+7\gamma^2)^2}$

$f_5 = SW_i(C_4) - SW_i(C_2) = \frac{\alpha^2\beta^2(7\beta+5\gamma)}{2(3\beta+2\gamma)^2(4\beta+3\gamma)^2} = f_{23}$

$f_6 = SW_j(C_3) - SW_j(C_1) = \frac{\alpha^2\beta(24\beta^4+180\beta^3\gamma+355\beta^2\gamma^2+70\beta\gamma^3-53\gamma^4)}{18(2\beta+\gamma)^2(4\beta^2+13\beta\gamma+7\gamma^2)^2}$

$f_7 = SW_j(C_4) - SW_j(C_3) = \frac{\alpha^2\beta(336\beta^4+1896\beta^3\gamma+3413\beta^2\gamma^2+2262\beta\gamma^3+477\gamma^4)}{18(4\beta+3\gamma)^2(4\beta^2+13\beta\gamma+7\gamma^2)^2}$

Then we have; $f_5 < f_1 < f_3 < f_4 < f_2$ and $f_6 < f_7 < f_4$ and $f_5 < f_7$ and $f_6 < f_1$. Now, we know by direct calculations, the core stable covers are;

Result 1: If $M \leq f_6$ and so $m \leq f_6$, or if $M \leq f_5$ and so $m \leq f_5$ then, $M + m < f_{12}$ and $M + 2m < f_{13}$. Hence, C_3 is CS under AX-SAE and C_4 is CS under all rights structures.

Result 2: If $f_6 < M < f_7$ and so $m < f_7$ then there are two cases;

Case 1: If $M + m < f_{12}$ and $M + 2m < f_{13}$ then, C_3 is CS under AX-SAE and C_4 is CS under all structures.

Case 2: If $M + m > f_{12}$ and $M + 2m > f_{13}$ then, C_1 is CS under AX-AE,FX-AE,AX-SAE, & FX-SAE, C_3 is CS under AX-SAE.

Result 3: If $f_7 \leq M < f_1$ and $f_5 \leq m < f_1$ then, $M + m > f_{12}$ and $M + 2m > f_{13}$ hence,C_1 is CS under AX-AE,FX-AE,AX-SAE, & FX-SAE, C_3 is CS under AX-SAE.

Result 4:$f_1 \leq M$ and so $m \leq f_1$, then for some β and γ, we have $M + 2m > f_2$, in this case C_1 is CS under all rights structures.

Now, let $C_4 = \{ijk\}$ be given. Suppose that, M and m are such that $M + m > f_{12}$

and $M + 2m > f_{13}$. By the model, we have, $\forall i \in N, \overline{SW}_i(C_4) < \overline{SW}_i(C_1)$. On the other hand, when a country exits from a union, it cannot get the cost, which was spent for entrance, back. We will examine this situation for $|N| = 3$. Since, for C_2, we always have $SW_i(C_2) > SW_i(C_1)$ so, we only need to consider two covers.

1) Let $C_3 = \{ij, ik\}$ be given, then, if $m \leq f_6$ or $m \leq f_5$ then C_3 is CS under AX-SAE, but if $m \geq f_7$ then C_3 is CS under all rights structures.

2) Let $C_4 = \{ijk\}$ be given, and suppose $M + m > f_{12}$ and $M + 2m > f_{13}$, then C_4 is CS under all rights structures.

Note that, when there is a linkage cost, core stable covers differ according to α, β, γ and $|N|$ under different rights structures.

CHAPTER 5

CONCLUSION

In our research, we investigated several questions about stability of covers under different rights structures. In order to model our problem, we use the assumptions and findings of Ilkilic (2010) to calculate the social welfare of a country in a given cover structure. Sertel (1992) introduced four possible membership rights in an abstract setting. In this aspect, either both exit and entry require approval, only one of them requires approval (i.e., approved entry- free exit or vice versa). We define four membership rights for covers, in addition to this, we introduce and define strongly approved entry condition for covers. We investigated core stability notion in all cases respectively. We investigated efficient and Pareto Efficient covers.While examining core stability and exit-entry conditions, we first considered entry without linkage cost.Then, we did the same analysis considering entry cost. We concluded that core stable covers differ under different rights structures.

As we mention before, we assume that countries have the same demand and the same economic, technological structure. For simplicity, we take one kind of product which can be produced in all countries with the same technology. In other words, countries are symmetric. But in reality, it is not. Hence, as a further research, country's differences can be considered, and the same questions in terms of this difference can be answered. Besides, as a further research, transferable payoffs can be considered.

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