GEOMETRIC FORMULATION OF LAGRANGIAN MECHANICS AND SYMMETRIES

by

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ABSTRACT

GEOMETRIC FORMULATION OF LAGRANGIAN MECHANICS AND SYMMETRIES

We start the thesis by covering the geometric formulation of time-independent and time-dependent Lagrangian mechanics and the concept of dynamical symmetry. Then we give two applications. First we review and tie together the results given between alternative Lagrangians, constants of motion and symmetries using Cartan 2-forms. Then it is followed by theory of connections and geometric mechanics applied to the Dirac monopole problem.

ÖZET

SİMETRİ VE MEKANİĞİN GEOMETRİK FORMÜLASYONU

Bu teze, zamana bağımlı ve zamandan bağımsız Lagrange mekaniği ve simetri konseptinin geometrik formülasyonlarını açıklayarak başlıyoruz. Daha sonra bunların iki uygulamasını veriyoruz. Önce Cartan 2-formlarını kullanarak alternatif Lagrange fonksiyonları, dinamik sistem sabitleri ve simetriler arasındaki ilişkileri inceliyoruz. Bunu takiben bağlantı teorisi ve geometrik mekaniğinin Dirac monopol problemine uygulamasını veriyoruz.

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LIST OF SYMBOLS

$\mathbb{C}$	The field of complex numbers
$C^{\infty}(\mathcal{M})$	The set of infinitely differentiable real valued functions over the manifold $\mathcal{M}$
$\mathrm{d}\alpha$	The exterior derivative of the differential form α
$d\theta_L$	Cartan 2-form for the Lagrangian L
$\mathcal{E}$	Extended velocity space of a time-dependent dynamical system given as $\mathcal{TQ} \times \mathbb{R}$ with coordinates $(x^1, ..., x^n, v^1, ..., v^n, t)$
$\mathbb{I}$	Identity operator of the appropriate space
im(-)	Image of a map
ker(-)	Kernel of a map
L	Lagrangian function
$\mathcal{L}_X$	Lie derivative along the vector field X
$\mathcal{Q}$	Configuration space of the dynamical system with coordinates $(x^1, ..., x^n)$
$\mathbb{R}$	The field of real numbers
R_a	Right action by a group element a
S^n	The n-dimensional sphere
$\mathcal{TM}$	Tangent bundle of the manifold $\mathcal{M}$
$\mathcal{T}^*\mathcal{M}$	Cotangent bundle of the manifold $\mathcal{M}$
$\mathcal{TQ}$	Velocity space of the dynamical system with coordinates $(x^1, ..., x^n, v^1, ..., v^n)$
$\mathrm{tr}(-)$	The trace of a matrix
$U(n)$	The unitary group of degree n
$X^{(1)}$	Prolongation of a vector field on configuration space to the (extended) velocity space
$X \lrcorner \alpha$	Contraction of the differential form α with the vector field X
$\gamma(t)$	Trajectory of the dynamical system in the configuration space
$\tilde{\gamma}(t)$	Trajectory of the dynamical system in the velocity space

$\hat{\gamma}(t)$	Trajectory of the dynamical system in the extended velocity space
Γ	The vector field of the dynamical trajectory (one of the above depending on context)
$\Lambda^r(\mathcal{M})$	Set of r-forms over the manifold $\mathcal{M}$
$\chi(\mathcal{M})$	Set of vector fields over the manifold $\mathcal{M}$
$(-)^*$	Pullback operator
$(-)_*$	Pushforward operator
$\wedge$	The wedge product of differential forms
$-/-$	The quotient space

1. INTRODUCTION

1.1. Introduction

Until Newton, study of mechanics (except perhaps celestial mechanics) could not much enjoy the frutation of a relationship with mathematics. Unlike optics and celestial mechanics, mechanics was heavily under the influence of more philosophical arguments rather than mathematical and experimental treatment. Newton however has not only formulated laws of mechanics as differential equations but also developed the necessary tool of Calculus to study these systems. These laws were expressed in coordinates and seemingly changed form from coordinates to coordinates. With the work of Euler, Lagrange and Hamilton however (perhaps in an attempt to address Hilbert's 6th question) a more geometric Language of mechanics where laws were formulated in more invariant laws such as that of Lagrangians, Hamiltonians, Euler-Lagrange equations and Action principle were developed. In fact the concept of a Lagrangian and action principle has found applications in even modern branches of physics such as quantum field theory and seems to have more fundamental ties to physics rather than just being invariant descriptions of mechanics. Lagrangian and Hamiltonian formulation of mechanics also inevitably lead to treatment of physics with differential geometric tools leading to symplectic geometry. On the other hand thermodynamics with Boltzmann bore Ergodic Theory which lead to the branch of topological dynamics. More than these, geometric approaches to differential equations was initiated by the work of Lie and his results opened up tremendous amount of research areas in mathematics. With Lie's geometric treatment of differential equations and symmetries, geometry was inevitably being seated at the heart of physics. Finally with Einstein's special relativity it became quite clear that geometry was not only the language of classical physics but it was also the language of modern physics. Following that geometry has penetrated deep roots into physics in fields such as quantum field theory, gauge theory, string theory so much so that ideas from physical systems lead to mathematical discoveries. Much like the ancient Greeks were right about some atomic structures making up matter, they were also confirmed by these developments on their idea that geometry was the language of nature.

From classical to modern, physics is generally interested in forces and interactions that result from these forces. Newton's second law crudely tells us that classically (where there are no relativistic effects and wave character of particles is neglectable) the forces that regulate the motion of the particles are second order differential equations of position or equivalently first order differential equations of velocity. That is in fact pretty much the definition of force as given in the axiom 1 of chapter 1.2. In that section we will review the geometric formulation of Lagrangian mechanics and write some of the classical results such as Helmholtz conditions (which are conditions required to show the existence of Lagrangians describing dynamical systems) [1] in this geometric language.

In the second chapter we move to applications of this construction. In section 1 we talk about symmetries. It is well known that there is a relationship between a certain class of symmetries, so called Noether symmetries, and conserved quantities of a dynamical system due to Noether's Theorem proven by Emmy Noether in 1918 [2]. However Noether symmetries are not the only type of symmetries available. There is also a class of more general symmetries called dynamical symmetries. The relationship between dynamical symmetries, constants of motion and alternative Lagrangians were extensively studied around 1980-1984 by Prince, Crampin and Sarlet [6-8,10-13] using differential geometry. The aim of section 2.1 is to utilize the geometric formulation of mechanics given in the introduction to present these results.

Finally in section 2.2 we will talk about magnetic monopole. Magnetic monopoles, also called Dirac monopoles, are idealized single magnetic poles that are not yet observed in nature. They are sometimes idealized as the one end of an infinitely long and infinitesimally thin dipole rods. They are interesting in the sense that although their mechanics can be described using classical mechanics, the singular potential imposes a non-trivial topology on the problem. The U(1) bundle structure of phase freedom and a quantization condition on magnetic charges pop out of these topological obstructions (which are concepts that are usually seen in quantum mechanics and quantum field theory). We will formulate this problem using the geometric language of mechanics. We will see that a global description of this problem in the configuration space is not possible due to certain exactness requirements of 2-forms describing the mechanics. As we will explain the theory of connections, principal bundles and calculus of variations on the space of paths on the configuration space can be utilized to give us global action principles.

1.2. Geometric Formulation of Mechanics

We first need to fix some conventions regarding three types of spaces that describe the motion of particles or systems under forces. The configuration space for our system is denoted as $\mathcal{Q}$ which is usually $\mathbb{R}^n$ with coordinates $(x^1, x^2, ..., x^n)$. This space might contain information about the positions of many particles, not just one. So for the case of m particles each of which is in $\mathbb{R}^3$ the whole configuration space is $\mathbb{R}^{3m}$. The velocity space of our system, which also contains the tangent vectors to the configuration space, is denoted as $\mathcal{TQ} = \mathcal{T}\mathbb{R}^n$ with coordinates $(x^1, x^2, ..., x^n, v^1, v^2, ..., v^n)$. And finally the extended velocity space is denoted as $\mathcal{E}$ is given by $\mathcal{TQ} \times \mathbb{R}$ with coordinates $(x^1, x^2, ..., x^n, v^1, v^2, ..., v^n, t)$ where the last variable is time. Unless indices are required we will denote these respectively as (x, v) and (x, v, t). And finally if a function $f \in C^\infty(\mathcal{E})$ depends only on positions we will denote the function as $f(x)$ etc.

The study of mechanics is in fact study of the differential equations that regulate the position of our system in the configuration space. The simplest configuration curve is the physical trajectory of a single particle in $\mathbb{R}^3$. However physical trajectories are not the only type of trajectories we can have. Consider a set of n particles. Then we denote the position of this system in $\mathbb{R}^{3n}$ composed of n copies of $\mathbb{R}^3$ (one copy for each particle). So we also need to give some definitions regarding configuration and phase curves. Phase curves can crudely be seen as curves that contain both the configuration and the velocity of the each particle in our system and time as the final component. So for n particles the phase curve generally lies in $\mathbb{R}^{6n+1}$ with for instance coordinates $(x^1, y^1, z^1, v_x^1, v_y^1, v_z^1, ..., x^n, y^n, z^n, v_x^n, v_y^n, v_z^n, t)$. Denote the curve in the configuration space as $\gamma(t) : \mathbb{R} \to \mathcal{Q}$. Then we define the phase curves:

Definition 1.1. *Let $\gamma(t) : \mathbb{R} \to \mathcal{Q}$ (where $dim(\mathcal{Q}) = n$) be the configuration curve of our system with coordinates $\gamma(t) = (x^1(t), ..., x^n(t))$. Then the phase curve is defined as the curve $\hat{\gamma}(t) : \mathbb{R} \to \mathcal{TQ} \times \mathbb{R}$ with coordinates $\hat{\gamma}(t) = (q^1(t), ...q^n(t), v^1(t), ...v^n(t), t)$ satisfying two conditions:*

- $\pi \circ \hat{\gamma}(t) = \gamma(t)$ *or in coordinates* $x^i(t) = q^i(t)$ *where* $\pi : \mathcal{TQ} \to \mathcal{Q}$ *is the projection*
- $\frac{d\gamma^i(t)}{dt} = v^i(t) \forall t \in \mathbb{R}$

So from now on we denote our phase curve as $\hat{\gamma}(t) = (x(t), v(t), t)$. By an abuse of notation t will both be used as the parameter of curves in the extended phase space and the coordinates for the $\mathbb{R}$ part of the evolution space $\mathcal{E} = \mathcal{TM} \times \mathbb{R}$. Note that not every curve in $\mathcal{E}$ is a phase curve. So we have $\{$phase curves$\} \subset \{$curves in $\mathcal{E}\}$. We will generally be interested in this subset of curves. It is important to notice that only on these curves we will have $v^i(t) = \frac{dx^i}{dt}$.

Example 1.2. As a simple example we give the free falling particle under the gravitational force field $(0, 0, -g)$. It is phase curve will be in $\mathbb{R}^6 \times \mathbb{R}$ given as $(x_0, y_0, z_0 + v_0^z t - gt^2, v_0^x, v_0^y, v_0^z - gt, t)$. The specific curve can be determined by the initial conditions $v_0^z = v_0^y = v_0^x = 0$ and $x_0 = x', y_0 = y', z_0 = z'$.

And finally we start with the following axiom which is the seed of mechanics:

Axiom 1.3. *Newton's Second Law* There exists n functions $F^j(x, v, t) \in C^\infty(\mathcal{E})$ (or a function $F : \mathcal{E} \to \mathbb{R}^n$) with $j = 1, 2, ...n$ such that they determine the configuration of our system as a second order differential equation: $\frac{d^2\gamma^j}{dt^2} = F^j(x, v, t)|_{\gamma(t)}$ where $\gamma(t)$ is the trajectory of the system in the configuration system.

Here we take masses as unity for simplicity. Their inclusion in the calculations is merely carried out by multiplying the left side with a diagonal matrix whose diagonal entries are the masses of each particle. One remark is in order at this stage. It might be possible to formulate the above axiom in terms of an equality on 1-forms and vector fields on our configuration space where then $F^j(x, v, t)$ become the components of the tensor fields, and in fact we will do that. And so the solution curve for a mechanical system (with some force function $F : \mathcal{E} \to \mathbb{R}^n$) is a phase curve in $\mathcal{E}$ (so it satisfies definition 1.3) and whose projection to $\mathcal{Q}$ solves the differential equation in axiom 1 for the given force. Thus we have $\{\text{solutions}\} \subset \{\text{phase curves}\} \subset \{\text{curves in } \mathcal{E}\}$. Most of our equations will be valid on these solution curves (such as the Euler-Lagrange equation) and it is an important point to keep in mind.

We now give an equivalent formulation [3] for the conditions above in cartesian coordinates for solution curves:

Theorem 1.4. *Given a mechanical system with force $F : \mathcal{E} \to \mathbb{R}^n$, a curve $\hat{\gamma}(t)$ in $\mathcal{E}$ with tangent vector field X_p is a solution curve for this mechanical system iff its tangent vectors nullify the 1-forms in $\Lambda^1(\mathcal{E})$ given below:*

- $\theta^i = dx^i - v^i dt$
- $\psi^i = dv^i - F^i dt$

and satisfies $dt(X_p) = 1$

Proof. Let the tangent vector field to $\hat{\gamma}(t)$ be $(\frac{dx(t)^i}{dt}, \frac{dv(t)^i}{dt}, \tau(t))$ i=1,...,n. The last condition ensures $\tau(t) = 1$. Then it nullifies the forms above iff we have the equalities $\frac{dx(t)^i}{dt} = v(t)^i$ and $\frac{dv(t)^i}{dt} = F^i(x(t))$. So this curve satisfies definition 1.3 and has the properties described in Newton's second law. □

Hence any curve in $\mathcal{E}$ whose tangent vectors nullify the forms given above is in fact the phase curves that are the solutions of the system with force F. We will call them as either solutions or dynamical curves. This formulation is basically the geometric formulation of time-dependent Lagrangian mechanics. There are also specific names for the vector fields belonging to different kinds of curves. Vector fields of phase curves are called second order differential equation fields. Those vector fields that nullify the first 1-forms θ_i (and so are the vector fields to the solutions) are called dynamical vector fields (or Euler-Lagrange vector fields if they can be obtained from a Lagrangian, see section 1.3). Thus we have $\{\text{Euler-Lagrange fields}\} \subset \{\text{Second order differential}$

equation fields$\} \subset \chi(\mathcal{E})$. We will later on derive some other time-independent and time-dependent geometric formalisms using respectively symplectic forms and Cartan 2-forms.

Example 1.5. As an example to dynamical vector fields, consider the last example of free falling object. Then we have $\Gamma = \frac{\partial}{\partial t} - gt\frac{\partial}{\partial z} - g\frac{\partial}{\partial v^z}$ which indeed has the properties described above.

Now we given an inverse to Theorem 1.4.

Theorem 1.6. *Let γ be the solution curve (to a dynamical system with force $\mathcal{F}$) with the vector field $\Gamma = \frac{\partial}{\partial t} + v^i\frac{\partial}{\partial x^i} + F^i\frac{\partial}{\partial v^i}$. Let $\mathcal{S} \subset \bigwedge^1(\mathcal{E})$ be the subset of 1-forms that nullify Γ. Then if $\alpha \in \mathcal{S}$ we have $\alpha = a\psi^i + b\theta^i$.*

Proof. First note that θ^i, ψ^i and dt are 2n+1 independent 1-forms so at each point on the manifold they form a basis for the cotangent bundle at that point. Evidently then $\alpha = a_i\theta^i + b_i\psi^i + cdt$. But if $\alpha(\Gamma) = 0$ we have $a(v^i - v^i) + b(F^i - F^i) + c = 0 \Rightarrow c = 0$. So we have $\alpha = a^i\theta^i + b^i\psi^i$. During this proof it is essential to note that the decomposition $< \theta^i, \psi^i > \bigoplus < dt >$ is a well defined decomposition (i.e it is compatible with coordinate changes) for the tangent bundle. □

In this formulation it is very easy to show that Newton's equations are diffeomorphism covariant and also it is possible to derive Euler-Lagrange equations in a very constructive manner (where one is usually either given the Euler-Lagrange equations and shown that it is equivalent with Newton's equations of motions or it is shown that Euler-Lagrange equations are equivalent to D'Alembert's principle).

Theorem 1.7. *Newton's laws given above are diffeomorphism covariant.*

Proof. Using the construction in theorem 1.4, let the extended phase curve be $\hat{\gamma}(t)$ with tangent vectors field X_t. Let F be any diffeomorphism of $\mathcal{E}$. Then the equations of mechanics written as in theorem 1.4 in the new coordinates are:

$$F_*^{-1}\theta^i(F^*X_t) = F_* \circ F_*^{-1}\theta^i(X_t) = \theta^i(X_t)$$

$$F_*^{-1}\psi^i(F^*X_t) = F_* \circ F_*^{-1}\psi^i(X_t) = \psi^i(X_t) \qquad \square$$

Hence Newton's laws are globally valid. This is essentially the nature of differential equation systems expressed in terms of differential forms. Hence the diffeomorphism covariance property is an inheritance hidden in the first axiom which states that laws of mechanics are basically derived from differential equations. Before we derive Euler-Lagrange equations we give a definiton:

Definition 1.8. *The forces regulating the motion are called conservative if they can be expressed as the gradient of a position dependent potential function V(x):* $F^i(x) = \frac{-dV}{dx^j}\delta^{ij}$ *(we will from now on denote* $\frac{-dV}{dx^j}\delta^{ij}$ *as* $\frac{-dV}{dx^i}$ *since we work on Riemannian manifolds).*

Theorem 1.9. *For conservative forces, the Newton's equations in a coordinate system* (x, v, t) *are equivalent to Euler-Lagrange equations in any coordinates* (q, u, t) *where Euler-Lagrange equations are given as*

$$\frac{d}{dt}\left(\frac{\partial L}{\partial u^i}\right) = \frac{\partial L}{\partial q^i}$$

and the Lagrangian $L \in C^\infty(\mathcal{E})$ *is* $L = \sum_{i=1}^n \frac{(v^i)^2}{2} - V(x)$*.*

Proof. We will use a different notation just for this proof, to make the calculations more transparent. Let the extended phase curve be $\hat{\gamma}(t) = (\gamma(t), \tilde{\gamma}(t), t)$ with tangent vectors field X_t and F be a conservative force such that $F^i = -\frac{dV(x)}{dx^i}$ for a potential $V(x)$. Let $\hat{T}$ be the lifted diffeomorphism of the base manifold s.t that it is compatible with the structure of the tangent bundle (i.e a lift of coordinate transformation on $\mathcal{Q}$ s.t: $\hat{T}$: $(x, v, t) \to (T(x) = q(x), \tilde{T}(x, v) = u(x, v), t)$ which satisfies $u^i = \frac{\partial q^i}{\partial x^k}v^k$). Let the image of the curve under this diffeomorphism be $\hat{\beta}(t) = (T \circ \gamma(t), \tilde{T} \circ \tilde{\gamma}(t), t) = (\beta(t), \tilde{\beta}(t), t)$ with tangent vectors $X'_t = (\frac{d\beta^i}{dt}, \frac{d\tilde{\beta}^i}{dt}, 1)$ i=1,...,n. Then under the diffeomorphism we have:

$$(\hat{T}^{-1})^* \theta^i(X'_t) = 0 \Rightarrow \frac{\partial x^i}{\partial q^j}|_{\hat{\beta}(t)} \frac{d\beta^j}{dt}|_t = v^i \circ \tilde{T}^{-1} \circ \hat{\beta}(t) \Rightarrow \frac{d(x^i \circ \beta)}{dt}|_t = \tilde{\gamma}^i|_t$$

$$(\hat{T}^{-1})^* \psi^i(X'_t) = 0 \Rightarrow \frac{\partial v^i}{\partial q^j}|_{\hat{\beta}(t)} \frac{d\beta^j}{dt}|_t + \frac{\partial v^i}{\partial u^j}|_{\hat{\beta}(t)} \frac{d\tilde{\beta}^j}{dt}|_t = -\frac{\partial q^j}{\partial x^i}|_{T^{-1}\circ\hat{\beta}(t)} \frac{\partial V}{\partial q^j}|_{\hat{\beta}(t)}$$

$$\Rightarrow \frac{d(v^i \circ \hat{\beta})}{dt}|_t = -\frac{\partial q^j}{\partial x^i}|_{\hat{\gamma}(t)} \frac{\partial V}{\partial q^j}|_{\hat{\beta}(t)}$$

The first equation simply makes the new curve a phase curve. Looking at the second equation, if the Jacobian of the transformation $q(x)$ of the base manifold is J then the right side is simply J^T composed with a vector. Then $\frac{\partial q^j}{\partial x^i}|_{\hat{\gamma}(t)} = J^T_{ij}$ The inverse to J^T is $(J^{-1})^T$, so by multiplying both sides with $(J^{-1})^T_{ki} = \frac{\partial x^i}{\partial q^k}|_{\hat{\beta}(t)}$ we get:

$$\Sigma_i \frac{\partial x^i}{\partial q^k}|_{\hat{\beta}(t)} \frac{d(v^i \circ \hat{\beta})}{dt}|_t = -\delta_{kj} \frac{\partial V}{\partial q^j}|_{\hat{\beta}(t)}$$

But since under a diffeomorphism of the base manifold, the tangent manifold transforms as $u^i = \frac{\partial q^i}{\partial x^k} v^k$ we get $\frac{\partial q^i}{\partial x^k} = \frac{\partial u^i}{\partial v^k}$ and similarly $\frac{\partial x^k}{\partial q^i} = \frac{\partial v^k}{\partial u^i}$ given this is a diffeomorphism. Then putting back this in the equation

$$\Sigma_i \frac{\partial v^i}{\partial u^k}|_{\hat{\beta}(t)} \frac{d(v^i \circ \hat{\beta})}{dt}(t) = -\frac{\partial V}{\partial q^k}|_{\hat{\beta}(t)}$$

which is valid on $\hat{\beta}(t)$. Using the laws of one-dimensional calculus valid on the domain of the curve we have:

$$\Sigma_i \frac{d}{dt}[(\frac{\partial v^i}{\partial u^k} \circ \hat{\beta})(v^i \circ \hat{\beta})(t)] - (v^i \circ \hat{\beta})\frac{d}{dt}[\frac{\partial v^i}{\partial u^k} \circ \hat{\beta}(t)]$$

$$= \Sigma_i \frac{d}{dt}[\frac{\partial}{\partial u^k}(\frac{1}{2}(v^i)^2) \circ \hat{\beta}(t)] - (v^i \circ \hat{\beta})\frac{d}{dt}[\frac{\partial x^i}{\partial q^k} \circ \hat{\beta}(t)] = -\frac{\partial V}{\partial q^k}|_{\hat{\beta}(t)}$$

Now $[\frac{d}{dt}, \frac{\partial}{\partial q^k}]$ vanishes when it acts on x^i so the two derivatives commute and we arrive at:

$$\Sigma_i \frac{d}{dt}[\frac{\partial}{\partial u^k}(\frac{1}{2}(v^i)^2) \circ \hat{\beta}(t)] - \frac{\partial}{\partial q^k}[\frac{1}{2}(v^i)^2 \circ \hat{\beta}] = -\frac{\partial V}{\partial q^k}|_{\hat{\beta}(t)}$$

Note that kinetic energy of the system at point on the curve is given by $K = \Sigma_i(\frac{1}{2}(v^i)^2 \circ \hat{\beta})$ and that the potential is independent of velocity. Then we arrive at Euler-Lagrange equations:

$$\Sigma_i \frac{d}{dt}[\frac{\partial}{\partial u^k}((\frac{1}{2}(v^i)^2 - V(x))) \circ \hat{\beta}(t)] = \frac{\partial}{\partial q^k}[\frac{1}{2}(v^i)^2 - V] \circ \hat{\beta}(t)$$

where the calculations clearly portray the role of the dynamical trajectory in interpreting the Euler-Lagrange equations:

$$\frac{d}{dt}(\frac{\partial L}{\partial u^i} \circ \hat{\beta}(t)) = \frac{\partial L}{\partial q^i} \circ \hat{\beta}(t)$$

for all t □

Theorem 1.10. *Euler-Lagrange equations are diffeomorphism covariant.*

Proof. Given the Euler-Lagrange equations evaluated on the phase curves $\frac{d}{dt}(\frac{\partial L}{\partial u^i}) = \frac{\partial L}{\partial q^i} \Rightarrow \frac{\partial^2 L}{\partial u^i \partial q^j} u^j + \frac{\partial^2 L}{\partial u^i \partial u^j}\frac{\partial u^j}{\partial t} + \frac{\partial^2 L}{\partial u^i \partial t}$ (where partial derivative with respect to t is added for generality) one simply uses the chain rule for the transformation $(q, u, t) \to (x(q), v(q, u), t)$ to arrive at $\frac{d}{dt}(\frac{\partial L}{\partial v^i}) = \frac{\partial L}{\partial x^i}$. □

Thus we have shown that for position dependent conservative forces, Newton's equations are equivalent to Euler-Lagrange equations under arbitrary coordinate changes. We have made the distinction of position dependent conservative forces which is a common type of force seen in nature. However there is a very important type of velocity dependent force which is the electromagnetic force (with q=1) given as $F_i = [E_i(x,t) + (v \times B)_i(x,t)]$ where E and B are respectively the electric and magnetic fields such that $E_i(x,t) \in C^\infty(\mathbb{R}^n \times \mathbb{R})$ and $B_i(x,t) \in C^\infty(\mathbb{R}^n \times \mathbb{R})$. By a choice of special Lagrangian for this type of force the above theorem again becomes valid. As an example to a Lagrangian problem with velocity dependent potential, as an example we will now examine electromagnetic systems.

Example 1.11. Let the electromagnetic force be (q=1, m=1, c=1); $F^i = E(x,t)^i + (v \times B(x,t))^i$, s.t $E^i = -\frac{\partial \varphi}{\partial x^i} - \frac{\partial A_i}{\partial t}$ and $B_i = (\nabla \times A)_i$ for the electric potential $\varphi(x,t)$ and vector potential $A(x,t)$. Then we will show that the Lagrangian function $L = \sum_{i=1}^{n} \frac{(v^i)^2}{2} - \varphi(x,t) + A_i(x,t)v^i$ produces the equations of motion of a charged particle in an electromagnetic field through the Euler-Lagrange equations given above.

Let $(x(t), v(t), t)$ be the phase curve of the particle. Then forces regulating the motion are given by the Lorentz Law: $F^i = E^i + (v \times B)^i$. Then using the above Lagrangian $\frac{d}{dt}(\frac{\partial L}{\partial v_i}) = \frac{dv^i}{dt} + \frac{dA_i}{dt}$ and $\frac{\partial L}{\partial x^i} = -\frac{\partial \varphi}{\partial x_i} + \frac{\partial A_j}{\partial x^i} v^j$. Then Euler-Lagrange equations imply $\frac{dv^i}{dt} = -(\frac{dA^i}{dt} + \frac{\partial \varphi}{\partial x_i} - \frac{\partial A_j}{\partial x^i} v^j)$. Writing the time derivative explicitly one simply arrives at $\frac{dv^i}{dt} = F^i = E^i + (v \times B)^i$.

One remark can be made. A modified Euler-Lagrange equation for the electromagnetic field with the classical Lagrangian $L_0 = \sum_{i=1}^{n} \frac{(v^i)^2}{2} - \varphi(x,t)$ can also be written as:

$$\frac{d}{dt}(\frac{\partial L}{\partial v^i}) - \frac{\partial L}{\partial x^i} = B_{ik} v^k - \frac{\partial A_i}{\partial t} \text{ where } B_{ik} = \frac{\partial A_i}{\partial x^k} - \frac{\partial A_k}{\partial x^i}$$

In our case (for the Dirac monopole in chapter 2) where the vector potential will be independent of time, the last term $\frac{\partial A_i}{\partial t}$ vanishes. More details on this electromagnetic setup and its geometric manifestation is given in the second part of chapter 2. It is instructive to realize that the above construction can be applied to velocity dependent dissipative systems with suitable modifications.

It is also useful to make a comparison between change of coordinates and embedded submanifolds. We derived Euler-Lagrange equations from a change of coordinates on the geometric formulation of mechanics given in Theorem 1.4. This change of coordinates can also be seen as the restriction of the motion to a submanifold of $\mathcal{E}$ (see figure 1.1). For instance consider a system whose configuration is constrained to the unit sphere (of dimension $n-1$) in $\mathcal{Q}$. Then the velocity space of this system is this sphere with its tangent space. If this surface is given by holonomic constraints, it is

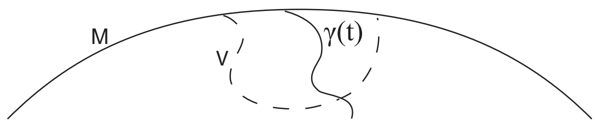

Figure 1.1. Restriction of motion to a submanifold

simply an embedded manifold. Therefore its coordinates can be given as slice charts which correspond to again first a local change of coordinates on the original charts and then the intersection of these new charts with the surface. Thus Euler-Lagrange equations are again valid for this submanifold and in fact just as would be expected the number of equations we have to solve decreases from n to n-1 (if this was a m-dimensional submanifold then the number of equations would be n-m), where the extra degrees of freedom disappear when the pullback of the embedding is applied.

1.3. Geometric Formulation of Lagrangian Mechanics

We now give an alternative formulation for time dependent Lagrangian formulation of mechanics based on 2-forms and using these definitions derive some results which will be used in the following sections.

Definition 1.12. *Given a Lagrangian for a dynamical system with coordinates* (x, v, t) *we define the 1-form* $\theta_L = Ldt + \frac{\partial L}{\partial v^i}\theta^i$*. Then Cartan 2-form for this dynamical system is defined as* $\omega_L = d\theta_L$*.*

Theorem 1.13. *If a Lagrangian is regular (that is* $\frac{\partial^2 L}{\partial v_i \partial v_j}$ *is a non-singular matrix at each point) then Cartan 2-form has a 1 dimensional kernel* Γ_p *at each* $p \in \hat{\gamma(t)} \subset \mathcal{E}$*. These vector fields are given in the form:*

$$a\frac{\partial}{\partial t} + av^i\frac{\partial}{\partial x^i} + aF^i\frac{\partial}{\partial v^i}$$

for $a, F^i \in C^\infty(\mathcal{E})$

Proof. Let $\Gamma = a\frac{\partial}{\partial t} + b^i\frac{\partial}{\partial x^i} + c^i\frac{\partial}{\partial v^i}$. The above equation requires $\Gamma\lrcorner d\theta_L = \Gamma\lrcorner(dL\wedge dt) + \Gamma\lrcorner(d(\frac{\partial L}{\partial v^i})\wedge\theta^i + \frac{\partial L}{\partial v^i}d\theta^i) = \Gamma(L)dt - a(dL) + \Gamma(\frac{\partial L}{\partial v^i})\theta^i) - (b^i - v^i a)d(\frac{\partial L}{\partial v^i}) + \frac{\partial L}{\partial v^i}(adv^i - cdt) = 0$. To find the components of this vector field we contract it by the basis vectors. First contracting it with $\frac{\partial}{\partial v^i}$ we get $-(b^i - v^i a)\frac{\partial^2 L}{\partial v^i \partial v^j} = 0$ which implies $b^i = v^i a$. Then contracting it with either $\frac{\partial}{\partial t}$ or $\frac{\partial}{\partial x^i}$ and using the above equality yields the identical result: $\Gamma(\frac{\partial L}{\partial v^i}) = a\frac{\partial L}{\partial x^i}$. Since this 2-form was evaluated on the dynamical curves of the system the right side by E-L equations is equal to $a\frac{d}{dt}(\frac{\partial L}{\partial v^i})$ which in turn means $\frac{dv^i}{dt}\frac{\partial L^2}{\partial v^i \partial v^j} = c^i\frac{\partial L^2}{\partial v^i \partial v^j}$. It follows from the regularity of Lagrangian that $c^i = \frac{dv^i}{dt} = F^i$. In fact we get no more conditions from this equality. The proof ends by noticing that the matrix of $(d\theta_L)_p$ is in fact fact has a non-singular $2n \times 2n$ upper block (see remark after Theorem 1.17) due to the regularity of Lagrangian. Then it follows that the kernel of $d\theta_L$ can be at most one dimensional and the computation above gives that vector space. □

Using Cartan 2-form one can give an equivalent definition of the Lagrangian mechanics.

Definition 1.14. *Given a dynamical system with Lagrangian L, the solution $\hat{\gamma}$ to this system are the family of curves whose tangent vector field Γ belong to the $ker(\omega_L)|_{\hat{\gamma}}$ and satisfies the normalization $dt(\Gamma) = 1$.*

The family of curves satisfying the conditions above has tangent vector field $\frac{\partial}{\partial t} + v^i\frac{\partial}{\partial x^i} + F^i\frac{\partial}{\partial v^i}$. This vector field vanishes on ψ_i and θ_i hence is the solution as required. The normalization condition is added to ensure the fact that time is an absolute homogenously flowing entity in non-relativistic mechanics. It also reduces the 1-dimensional vector space at each point to a single vector therefore giving us a vector field on $\mathcal{E}$ and thus a family of curves associated with it. The vector fields satisfying the above conditions are also called Euler-Lagrange fields for this dynamical system. This is called the geometric formalism of Lagrangian mechanics because if the equality $(\gamma)\lrcorner(d\theta_L) = 0$ is written in coordinates with Euler-Lagrange fields given, we get exactly Euler-Lagrange equations. From now on we denote the element of the $ker(\omega_L)$ with the property $dt(\gamma) = 1$ by Γ.

If the reader is acquainted with the symplectic geometry of time-independent mechanics with the canonical 2-form ω, then it is useful to note that here $(d\omega_L)$ plays the same rule except being singular (since it has a kernel). We now give an important convention. We will say that a $2n \times 2n$ matrix is maximal if it has a zero dimensional kernel and $(2n+1) \times (2n+1)$ matrix is maximal if it has one dimensional kernel (see section 3.1). It is useful to study the range of the Cartan 2-form:

Proposition 1.15. *Let L be a regular Lagrangian with the Cartan 2-form $d\theta_L$. Seeing $d\omega_L$ as a map from $\chi(\mathcal{E})$ to $\Lambda^1(\mathcal{E})$, let $im(d\omega_L) = \mathcal{R} \subset \chi(\mathcal{E})$. Then $\mathcal{R}$ is the set of 1-forms α s.t $\Gamma \lrcorner \alpha = 0$*

Proof. $\alpha \in \mathcal{R}$ if and only if there exists $X \in \chi(\mathcal{E})$ such that $X \lrcorner d\omega_L = \alpha$. Then $\Gamma \lrcorner \alpha = d\omega_L(\Gamma, X) = 0$. □

So the range of our Cartan 2-form is the set of 1-forms that nullify Γ. Since $ker(d\theta_L) = h\Gamma$ we have that $\mathcal{TE}_p / < \Gamma_p > \simeq \{\alpha_p \in \mathcal{R}\}$. Now we give a geometric form of Helmholtz conditions [4], which will be of great use to us in investigating alternative Lagrangians. But we first note; a second order differential equation field is a Euler-Lagrange field if there exists a Lagrangian and so a Cartan 2-form associated with it of the form $d\theta_L$ whose only kernel is $< \Gamma_p >$. And also a 2-form ω on an odd/even dimensional manifold will be called maximal when $\omega|_p$ evaluated at each point is a matrix with at most 1-dimensional/0-dimensional kernel.

Theorem 1.16. Helmholtz Conditions *Given a second order differential equation field $\Gamma = \frac{\partial}{\partial t} + v^i \frac{\partial}{\partial x^i} + F^i \frac{\partial}{\partial v_i}$ on $\mathcal{E}$ s.t $F^i \in C^\infty(\mathcal{E})$, Γ is an Euler-Lagrange field iff there exists a 2-form ω of maximal rank s.t*

- $\omega(V_i, V_j) = 0 \ \forall i, j$
- $d\omega = 0$
- $\Gamma \lrcorner \omega = 0$

where $V_i \in < \frac{\partial}{\partial v_i} >$ is an element from the vertical space at each point. In such a case we have $\omega = d\theta_L$ where L is the Lagrangian giving the Euler-Lagrange field Γ.

For the proof we refer the reader to [4]. Also maximality condition may be loosened for non-regular Lagrangian but then of course the kernel may be greater than 1-dimensional at each point. For other geometric versions equivalent to this the reader may also look at [5]. Thus here finding the maximal two-form satisfying the conditions above is equivalent to finding a Cartan 2-form for the system. We note here that the space $V_i =< \frac{\partial}{\partial v_i} >$ transforms well under coordinate changes. That is under coordinates changes vertical spaces are transformed to vertical spaces. We finally give the time-independent formulation of mechanics [6] which is based on a symplectic 2-form on $\mathcal{E}$.

Theorem 1.17. *Let L be a time-independent, regular Lagrangian. Let $\mathcal{TQ}$ be the velocity space of the dynamical system with coordinates (x,v). Define a 2-form for the system as $\omega_L = d(\frac{\partial L}{\partial v^i} dx^i)$. This is a symplectic 2-form for our system and more over the phase curves of our system are the curves with the tangent vector field Γ which satisfy*

$$\Gamma \lrcorner \omega_L = dE$$

where $E \in C^\infty(\mathcal{TQ})$ is given as $E = \frac{\partial L}{\partial v^i} v^i - L$

Proof. The proof simply proceeds by writing the above equation in coordinates. Let $\Gamma = a\frac{\partial}{\partial t} + b\frac{\partial}{\partial x^i} + c\frac{\partial}{\partial v^i}$. Then $\Gamma \lrcorner \omega_L = \frac{\partial^2 L}{\partial v^j \partial v^i}(-\frac{dx^i}{dt} dv^j + \frac{dv^j}{dt} dx^i) + \frac{\partial^2 L}{\partial x^j \partial v^i}(\frac{dx^j}{dt} dx^i - \frac{dx^i}{dt} dx^j)$ and $-dE = (\frac{\partial L}{\partial x^j} - v^j \frac{\partial^2 L}{\partial x^j \partial v^i}) dx^j - (v^i \frac{\partial^2 L}{\partial v^j \partial v^i}) dv^j$. Now evaluating these on the phase curves and equating the dx^i components one arrives at the Euler-Lagrange equations. □

One instructive exercise is to write ω_L and $d\theta_L$ in components to see how their maximality is associated to the regularness of the Lagrangian. We have:

$$\omega_L|_p = \left| \begin{array}{cc} (\frac{\partial^2 L}{\partial v^j \partial x^i})_{n\times n} & (\frac{\partial^2 L}{\partial v^j \partial v^i})_{n\times n} \\ (\frac{\partial^2 L}{\partial v^j \partial v^i})_{n\times n} & (0)_{n\times n} \end{array} \right|_p$$

$$d\theta_L|_p = \left| \begin{array}{ccc} (\frac{\partial^2 L}{\partial v^j \partial x^i})_{n\times n} & (\frac{\partial^2 L}{\partial v^j \partial v^i})_{n\times n} & (\frac{\partial E}{\partial x^i})_{1\times n} \\ (\frac{\partial^2 L}{\partial v^i \partial v^j})_{n\times n} & (0)_{n\times n} & (\frac{\partial E}{\partial v^i})_{1\times n}) \\ (-\frac{\partial E}{\partial x^i})_{n\times 1} & (-\frac{\partial E}{\partial v^i})_{n\times 1} & 0_{1\times 1} \end{array} \right|_p$$

Note that we no more have the time coordinate attached to $\mathcal{TQ}$ in the case of time-independent mechanics. Thus the main tool of time-independent Lagrangian (as well Hamiltonian mechanics) is symplectic geometry. It is also instructive to see two facts about this symplectic form. First of all if the differential is computed explicitly we have $\omega_L = d(\frac{\partial L}{\partial v^i}) \wedge dx^i$ whose pullback to the cotangent bundle $\mathcal{T}^*\mathcal{Q}$ of positions and momentum (x,p) is exactly equal to the canonical symplectic form $\omega = \sum_{i=1}^n dp^i \wedge dx^i$. The second fact is that the 2-form $\omega_L - dE \wedge dt$ is equal to the Cartan 2-form of time dependent Lagrangian mechanics (where in this case $(\omega_L)_p$ is to be seen as a $(2n + 1 \times 2n + 1)$ matrix with the upper $n \times n$ block equal to the $(\omega_L)_p$ of time independent mechanics and has zeros for the last row and column and $dE \wedge dt$ is seen as a $(2n + 1 \times 2n + 1)$ matrix with upper $2n \times 2n$ block being 0).

2. APPLICATIONS

2.1. Alternative Lagrangians, Symmetries and Constants of Motion

We first introduce the types of symmetries of dynamical systems. Since dynamical vector fields and cartan 2-forms totally describe a dynamical system, we will from now on denote dynamical systems as $(\Gamma, d\theta_L)$.

Definition 2.1. *Let $(\Gamma, d\theta_L)$ be a dynamical system and $\gamma(t)$ be its flows or the solutions. If φ is a diffeomorphism of $\mathcal{E}$, then it is a symmetry iff it takes solutions to solutions that is $\varphi \circ \gamma(t)$ is a solution.*

While we have given this definition for any diffeomorphism, we will mainly be interested in 1-parameter group of diffeomorphism and there are two equivalent definitions of symmetry for such diffeomorphisms;

Theorem 2.2. *Let the dynamical system be as above. If φ_ϵ is a 1-parameter group of diffeomorphisms of $\mathcal{E}$ generated by $X \in \chi(\mathcal{E})$ for $\epsilon \in \mathbb{R}$, then it is a symmetry iff $[X, \Gamma] = h\Gamma$ for $h \in C^\infty(\mathcal{E})$.*

Proof. ($\Leftarrow$) Let X satisfy $[X, \Gamma] = h\Gamma$. Then φ_ϵ is a symmetry iff $\varphi_\epsilon \circ \gamma(t)$ is a solution i.e its tangent vector field nullifies the 1-forms θ_i and ψ_i given in the last section. The tangent vector field to the new curve is given as $\Gamma^\epsilon = (\varphi_\epsilon)_* \Gamma$ which can be expanded as $\Gamma^\epsilon_p = \Gamma_p + \epsilon[X, \Gamma]_p + \frac{1}{2}\epsilon^2[X, [X, \Gamma]]_p + ... = \Gamma_p + h\epsilon\Gamma_p + \frac{1}{2}h^2\epsilon^2\Gamma_p + ...$ This obviously nullifies θ_i and ψ_i since Γ does.

($\Rightarrow$) Let φ_ϵ be a 1-parameter group of diffeomorphism such that it is a symmetry i.e $\varphi_\epsilon \circ \gamma(t)$ is a solution. Then $\Gamma^\epsilon = (\varphi_\epsilon)_* \Gamma$ nullifies θ_i and ψ_i. By theorem 1.13 we know then that Γ^ϵ_p belong to one dimensional vector space $< \Gamma_p >$ at each point p as $\Gamma^\epsilon_p = h(\epsilon)\Gamma_p$. Then we have $\Gamma^\epsilon_p = \Gamma_p + \epsilon[X, \Gamma]_p + \frac{1}{2}\epsilon^2[X, [X, \Gamma]]_p + ... = h(\epsilon)\Gamma_p$. Since this equality is valid for all ϵ in our domain, taking the derivative at $\epsilon = 0$ gives us the infinitesimal condition. □

Since we know Γ we can try to find restrictions on symmetries using the conditions above. Let $X = \tau\frac{\partial}{\partial t} + \beta^i \frac{\partial}{\partial x^i} + \eta^i \frac{\partial}{\partial v^i}$ be a symmetry. Then $[X, \Gamma] = h\Gamma \Rightarrow -\Gamma(\tau) = h$, $\eta^i - \Gamma(\beta^i) = hv^i$ and $X(F^i) - \Gamma(\eta^i) = hF^i$. The first two conditions are interesting in the sense that they determine η^i and we have $X = \tau\frac{\partial}{\partial t} + \beta^i \frac{\partial}{\partial x^i} + (\Gamma(\beta^i) - \Gamma(\tau)v^i)\frac{\partial}{\partial v^i}$. The last condition simplifies to the relation $X(F^i) = \Gamma^2(\beta^i) - \Gamma^2(\tau)v^i$.

Also looking at the above condition we see that in fact, every symmetry defines an equivalence class of symmetries. That is if Y is a symmetry so is $Y + a\Gamma$ and two symmetries are in the same class if they differ by a multiple of Γ. Thus from each class of symmetries we can find a representative such that $Y' \lrcorner dt = 0$ which can be formed by $Y' = Y - (Y \lrcorner dt)\Gamma$.

For the second equivalent condition, we define $\mathcal{I}_p$ for $p \in \mathcal{E}$ as the subspace over $C^\infty(\mathcal{E})$ spanned by the 1-forms θ^i, ψ^i defined in the previous section (i.e 1-forms given as $a\theta^i + b\psi^i$ for $a, b \in C^\infty(\mathcal{E})$, for more detailed discussion regarding generalized versions of this as differential ideals see [7]). Then;

Theorem 2.3. *Let the dynamical system be as above. If φ_ϵ is a 1-parameter group of diffeomorphisms of $\mathcal{E}$ generated by $X \in \chi(\mathcal{E})$, then it is a symmetry iff $\mathcal{L}_X\theta_i, \mathcal{L}_X\psi_i \in \mathcal{I}$.*

Proof. Let φ_ϵ be a 1-parameter group of diffeomorphisms generated by X. Then (by Theorem 1.6) $\mathcal{L}_X\theta_i, \mathcal{L}_X\psi_i \in \mathcal{I}$ iff $\mathcal{L}_X\theta_i(\Gamma) = 0$ and $\mathcal{L}_X\psi_i(\Gamma) = 0$ iff $\theta_i(\mathcal{L}_X\Gamma) = 0$ and $\psi_i(\mathcal{L}_X\Gamma) = 0$ (since for example $\mathcal{L}_X\theta_i(\Gamma) = X(\theta_i(\Gamma)) - \theta_i(\mathcal{L}_X\Gamma)$). But this is iff $\forall\epsilon$ $(\mathcal{L}_X\Gamma) = h\Gamma$ which is iff φ_ϵ is a symmetry (by Theorem 2.2). □

We will from now on mean by a symmetry, 1-parameter group of diffeomorphisms satisfying the conditions above. Then symmetries can be classified into two general classes. But before that we need to give the definition of a prolongation. Note that a vector field in $\mathcal{Q} \times \mathbb{R}$ (or only $\mathcal{Q}$) can be extended to a vector field in $\mathcal{E}$ in infinite ways. However for each given dynamical vector field Γ it is possible to define a unique extension called the prolongation of this vector field;

Definition 2.4. *Let $X \in \chi(\mathcal{Q} \times \mathbb{R})$ be a vector field with components $\tau\frac{\partial}{\partial t} + \beta^i \frac{\partial}{\partial x^i}$ where $\beta^i, \tau \in C^\infty(\mathcal{Q} \times \mathbb{R})$. Then its prolongation to $\mathcal{TE}$ is defined as the vector field:* $\tau\frac{\partial}{\partial t} + \beta^i \frac{\partial}{\partial x^i} + (\Gamma(\beta^i) - \Gamma(\tau)v^i)\frac{\partial}{\partial v^i}$

If X is a vector field in $\chi(\mathcal{Q})$ then the τ component is taken zero. Note that since time and position components does not depend on velocities, application of Γ to these components does not produce partial derivatives with respect to velocities. This condition is chosen so that in fact if X is any prolongation it automatically satisfies the $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial x^i}$ components of the equality given in Theorem 2.2 as shown by the calculation following the theorem.

Definition 2.5. *A Lie symmetry is a 1-parameter group of diffeomorphisms of the configuration space $\mathcal{Q}$ generated by a vector field $X \in \chi(\mathcal{Q} \times \mathbb{R})$ s.t its prolongation $X^{(1)}$ satisfies the conditions above. A dynamical symmetry is a 1-parameter group of diffeomorphisms of $\mathcal{E}$ generated by a vector field $Y \in \chi(\mathcal{E})$ s.t it satisfies the conditions above (From now on by a Lie symmetry generator, we will mean the prolonged version).*

The calculation after Theorem 2.2 shows that velocity components of a dynamical symmetry also satisfy the form given in the prolongation definition. However since now the time and position components also depend on velocities, application of the vector field Γ to these components also produce partial derivatives with respect to velocities.

Now, we give two particulary important subtypes of these symmetries:

Definition 2.6. *A Noether/Cartan symmetry is a Lie/Dynamical symmetry whose generator satisfies $\mathcal{L}_X d\theta_L = 0$.*

In fact any vector field satisfying the conditions above is automatically a symmetry generator. Consider such a X with the 1-parameter diffeomorphism φ. Then if Γ is a Euler-Lagrange field we have $d\theta_L(\varphi_*\Gamma) = \varphi^* d\theta_L(\Gamma) = d\theta_L(\Gamma)$ since $\mathcal{L}_X d\theta_L = 0$. This means that the flows of a Noether or Cartan symmetry exactly preserve the Cartan 2-form for the system. It is useful to note that since $\mathcal{L}_X d\theta_L = d(\mathcal{L}_X L) = 0$ we (at least locally) have $\mathcal{L}_X\theta_L = df$ for a real function on $\mathcal{E}$. For Noether and Cartan symmetries it is well known that they produce the following constants of motion (i.e real valued functions g such that $\Gamma(g) = 0$):

Theorem 2.7. *If X generates a Noether/Cartan symmetry such that $\mathcal{L}_X\theta_L = df$ for $f \in C^\infty(\mathcal{E})$ (if $\mathcal{E}$ is simply connected this is guaranteed) then $f - \theta_L(X)$ is a constant of motion.*

Proof. If $\mathcal{L}_X\theta_L = df$ then by Cartan's magic formula $\Gamma \lrcorner df = \Gamma \lrcorner d(\theta_L(X)) + d\theta_L(X, \Gamma) \Rightarrow \Gamma \lrcorner d(f - \theta_L(X)) = 0$ □

We will now move onto general results concerning Lie and Dynamical symmetries. Our first task is to build new Lagrangians using diffeomorphisms. Consider a diffeomorphism φ of the evolution space $\mathcal{E}$. We would like to establish the cases when these diffeomorphisms take the existing Cartan 2-form to other Cartan 2-forms for the same system, thus possibly giving us new Lagrangians (note that Noether/Cartan symmetries did not change the Cartan 2-form so they already trivially give the same Cartan 2-forms as diffeomorphism). This new 2-form, given as $\varphi^* d\theta_L$, must first of all satisfy the Helmholtz conditions to imply the existence of a new Lagrangian. It can easily be seen that closedness and maximal rank property are automatically satisfied and the other properties reduce to the following theorem [8]:

Theorem 2.8. *Let $(\Gamma, d\theta_L)$ be a dynamical system and $\varphi : \mathcal{E} \to \mathcal{E}$ a diffeomorphism. φ generates a new Cartan 2-form $\varphi^* d\theta_L$ iff φ is a symmetry of this system and φ maps the vertical subspaces of $\mathcal{TE}$ to a submanifold of $\mathcal{TE}$ on which $d\theta_L$ vanishes.*

Proof. ($\Rightarrow$) Given φ suppose it generates a new Cartan 2-form. Then $\Gamma \lrcorner \varphi^* d\theta_L = 0 \Rightarrow \varphi_*\Gamma \lrcorner d\theta_L = 0 \Rightarrow \varphi_*\Gamma = h\Gamma$ Since $\varphi_*\Gamma$ nullifies the Cartan 2-form, building the curves associated to this new vector field we see that φ takes solutions to solutions. The condition involving vertical subspaces follows from the fact that $\varphi^* d\theta_L(V_i, V_j) = d\theta_L(\varphi_* V_i, \varphi_* V_j) = 0$.

($\Leftarrow$) If we have a 2-form of maximal rank, then we can try to find the new Lagrangian by satisfying the Helmholtz conditions. Since φ is a symmetry, by definition it is a diffeomorphism. Then rank of $d\theta_L$ is equal to rank of $\varphi^* d\theta_L$ so this is the candidate Cartan 2-form. The vanishing of the new Cartan 2 form on vertical vector fields is automatically satisfied by the second assumption. It is also closed since pushforward commutes with exterior differentiation. Finally $\Gamma \lrcorner (\varphi^* d\theta_L) = \varphi_* \Gamma \lrcorner (d\theta_L) = h\Gamma \lrcorner (d\theta_L)$ since φ is a dynamical symmetry. So since Helmholtz conditions are satisfied, after this diffeomorphism we can find a new Lagrangian L' whose Cartan 2-form is $d\theta'_L = \varphi^* d\theta_L$. □

Thus we see that to satisfy the Helmholtz conditions, this diffeomorphism must be a symmetry. If we have one-parameter group of diffeomorphisms φ_s with a generator $X \in \mathcal{TE}$ then the following theorem can be proven [3, 8]:

Theorem 2.9. *Let $(\Gamma, d\theta_L)$ be a dynamical system and $\varphi_s : E \to E$ be one-parameter group of diffeomorphisms and a dynamical symmetry with generator X. Then (i). $\mathcal{L}_X \theta_L = \theta_{L'} + df$ and (ii) $d\theta_{L'} = \mathcal{L}_X d\theta_L$ satisfies all the properties of being a Cartan 2-form except maximality (i.e it gives at least a non-regular Lagrangian) iff $d\theta_L([X, V_i], V_j) + d\theta_L(V_i, [X, V_j]) = 0$ (vanishing condition). The new Lagrangian (possibly non-degenerate) is given as $L' = X(L) + \Gamma(f) + L\Gamma(\tau)$ (Remark: Lie symmetries and more generally any prolongation automatically satisfies the last condition)*

Proof. First note, $\Gamma \lrcorner \mathcal{L}_X d\theta_L = \mathcal{L}_X(\Gamma \lrcorner d\theta_L) - [X, \Gamma] \lrcorner d\theta_L = [X, \Gamma] \lrcorner d\theta'_L$. This is zero since X being a dynamical symmetry we have $[X, \Gamma] = h\Gamma$. Similarly $\mathcal{L}_X d\theta_L$ is closed since Lie derivative commutes with differential. Now $\mathcal{L}_X d\theta_L(V_i, V_j) = X(d\theta_L(V_i, V_j)) - d\theta_L([X, V_i], V_j) - d\theta_L(V_i, [X, V_j])$. By $d\theta_L$ being a Cartan 2-form, $\mathcal{L}_X d\theta_L(V_i, V_j) = 0$ iff $d\theta_L([X, V_i], V_j) + d\theta_L(V_i, [X, V_j]) = 0$. We will now solve $\mathcal{L}_X \theta_L = \theta'_L + df$ to finish the proof. Suppose $X = \tau \frac{\partial}{\partial t} + \beta^i \frac{\partial}{\partial x^i} + (\Gamma(\beta^i) - v^i \Gamma(\tau)) \frac{\partial}{\partial v^i}$ be a dynamical symmetry and $\Gamma = \frac{\partial}{\partial t} + v^i \frac{\partial}{\partial x^i} + \Lambda^i \frac{\partial}{\partial v^i}$. Then

$$\mathcal{L}_X \theta_L = X(L)dt + Ld\tau + X(\frac{\partial L}{\partial v^i} \theta_i) + \frac{\partial L}{\partial v^i} \mathcal{L}_X \theta^i$$

But

$$\mathcal{L}_X \theta^i = X \lrcorner d\theta^i + d(X \lrcorner \theta^i) = (\Gamma(\tau)v^i - \Gamma(\beta^i))dt + \tau dv^i + d(\beta^i - \tau v^i)$$

$$= [\frac{\partial \beta^i}{\partial x^j} + v^i \frac{\partial \tau}{\partial x^j}]\theta^j - \Lambda^j \frac{\partial \beta^i}{\partial v^j} + \Lambda^j \frac{\partial \tau}{\partial v^j} dt + \frac{\partial \beta^i}{\partial v^j} dv^j - \frac{\partial \tau}{\partial v^j} dv^j$$

And finally $Ld\tau$ can be written as

$$L(\Gamma(\tau)dt + \frac{\partial tau}{\partial x^i} dx^i + \frac{\partial \tau}{\partial v^i} - v^i \frac{\partial \tau}{\partial x^i} dx^i - \Gamma^i \frac{\partial \tau}{\partial v^i} dv^i)$$

Putting these back into the equation we get

$$\mathcal{L}_X \theta_L = [X(L) + \Gamma(\tau)]dt + [\frac{\partial L}{\partial v^i}(\frac{\partial \beta^i}{\partial x^j} - v^i \frac{\partial \tau}{\partial x^j}) + L\frac{\partial \tau}{\partial x^j} + X(\frac{\partial L}{\partial v^j})]\theta^j$$

$$+[L\frac{\partial \tau}{\partial v^j} + \frac{\partial L}{\partial x^i}(\frac{\partial \beta^i}{\partial v^j} - v^j \frac{\partial \tau}{\partial v^j})](dv^j - \Lambda^j dt)$$

On the other side of the equality we have

$$\theta'_L + df = L'dt + \frac{\partial L'}{\partial x^i}(dx^i - vdt) + \frac{\partial f}{\partial x^i} dx^i$$

$$+\frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial v^i} dv^i = [L' + \Gamma(f)]dt + (\frac{\partial L'}{\partial x^i} + \frac{\partial f}{\partial x^i})\theta^i + [\frac{\partial f}{\partial v^i}](dv^i - \Lambda^i dt)$$

Noting that (dt, θ^i, ψ^i) form an orthonormal basis and comparing the dt component of the equality we get

(i) $X(L) + L\Gamma(\tau) - \Gamma(f) = L'$

(ii) $\frac{\partial L}{\partial v^i}(\frac{\partial \beta^i}{\partial x^j} - v^i \frac{\partial \tau}{\partial x^j}) + L\frac{\partial \tau}{\partial x^j} + X(\frac{\partial L}{\partial v^j}) = \frac{\partial L'}{\partial x^j} + \frac{\partial f}{\partial x^j}$

(iii) $L\frac{\partial \tau}{\partial v^j} + \frac{\partial L}{\partial x^i}(\frac{\partial \beta^i}{\partial v^j} - v^j\frac{\partial \tau}{\partial v^j}) = \frac{\partial f}{\partial v^j}$.

Now the first condition is simply the definition of L'. The solvability of (ii) and (iii) both depends on the function f and so they must be compatible. It can be shown that the vanishing condition given above guarantees this [3, 8]. □

Thus this establishes the construction of an alternative Lagrangian from a dynamical or Lie symmetry if the given condition is satisfied.

Example 2.10. As an example we can look at the two-dimensional Kepler problem. The Lagrangian for the kepler problem is given as $L = \sum_i \frac{1}{2}(v^i)^2 + \frac{k}{r}$ where $r = (\sum_i (x^i)^2)^{\frac{1}{2}}$. The Euler-Lagrange equations produce $\frac{dv^i}{dt} = -\frac{kx^i}{r^{\frac{3}{2}}}$ and so

$$\Gamma = \frac{\partial}{\partial t} + v^i\frac{\partial}{\partial x^i} - \frac{kx^i}{r^{\frac{3}{2}}}\frac{\partial}{\partial v^i}$$

. Denoting $F^i = -\frac{kx^i}{r^{\frac{3}{2}}}$, the Cartan 2-form associated with the problem is

$$d\theta_L = d(Ldt - \frac{\partial L}{\partial v^i}\theta^i)$$

$$= \sum_i (F^i dx^i \wedge dt + v^i dv^i \wedge dt + dv^i \wedge \theta^i - v^i dv^i \wedge dt)$$

$$= F^i dx^i \wedge dt + dv^i \wedge dx^i - v^i dv^i \wedge dt$$

In terms of ψ and θ, $d\theta_L$ has the form $\sum_i \psi^i \wedge \theta^i$. It is known that this problem has the following symmetries (given as prolongations) [3] :

$$X^{(1)} = \frac{\partial}{\partial t}$$

$$Y^{(1)} = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} - v^y\frac{\partial}{\partial v^x} + v^x\frac{\partial}{\partial v^y}$$

$$Z^{(1)} = t\frac{\partial}{\partial t} + \frac{2}{3}x^i\frac{\partial}{\partial x^i} - \frac{1}{3}v^i\frac{\partial}{\partial v^i}$$

The first two are Noether symmetries since

$$\mathcal{L}_{X^{(1)}}d\theta_L = d(\sum_i(-F^i dx^i + v^i dv^i)) = \frac{3k}{r^{\frac{5}{2}}}(xydy \wedge dx + xydx \wedge dy) = 0$$

$$\mathcal{L}_{X^{(2)}}d\theta_L = d(-F^x ydt + F^y xdt) = 0$$

So their corresponding constants of motion can be obtained from $G = f - X \lrcorner d\theta_L$ and they are found to be:

$$E = \sum_i \frac{(v^i)^2}{2} - \frac{k}{x^i}(Energy)$$

$$L = xv^y - yv^x(Angular\ momentum)$$

The constant associated to the third symmetry along with the others is Runge-Lenz vector which is given as $w^i = x^i(\frac{dr}{dt})^2 - v^i(r \cdot \frac{dr}{dt}) - k\frac{x^i}{r}$. This third symmetry is not a Noether symmetry but since $[Z^{(1)}, V^i] = \frac{-1}{3}\frac{\partial}{\partial v^i}$ it satisfies the above condition by the vanishing of Cartan 2-form on vertical vector fields. Then we know that it can generate a new Lagrangian by the method above. We first need to compute $\mathcal{L}_X\theta_L$;

$$\mathcal{L}_X\theta_L = -F^i tdx^i + v^i tdv^i + \frac{2}{3}F^i x^i dt - \frac{2}{3}x^i dv^i + \frac{1}{3}(v^i)^2 dt - \frac{1}{3}v^i 2dx^i + d(Lt + \frac{2}{3}x^i v^i - v^i t)$$

$$= \frac{-1}{3}v^i dx^i + (L + \frac{2}{3}x^i F^i - \frac{2}{3}(v^i)^2)dt = \frac{1}{3}\theta_L = \theta_{L'} + df$$

So we have $df = 0$ and f is a constant. By simply noting $d\theta_{L'} = d\theta_L$ we have $L' = \frac{L}{3}$, so we get that the new Lagrangian is trivially related to the old one.

If for a dynamical symmetry $X \in \mathcal{TE}$ the above condition is not satisfied, then it is still possible to obtain a new Lagrangian under certain conditions [9]. Let φ_ϵ be the 1-parameter group of flows of our symmetry. One first normalizes the symmetry by $X = X - dt(X)\Gamma$ so that $[X, \Gamma] = 0$ (i.e the symmetry has no $\frac{\partial}{\partial t}$ component). Given any solution γ among the family of solutions one then constructs the submanifold $\mathcal{M}$ in $\mathcal{E}$ of dimension 2 generated by the action of the 1-parameter group of flows on γ. Call the projection of $\mathcal{M}$ to $\mathcal{Q}$, $\mathcal{S}$. Then X is projected to this manifold with the resulting vector field Y. It is basically done by first evaluating X on $\mathcal{M}$ and then projecting it to $\mathcal{Q}$ by the pushforward of the tangent bundle projection map π i.e by deleting $\frac{\partial}{\partial v^i}$ components. Then it can be shown that the prolongation of Y, $Y^{(1)}$ satisfies $[Y^{(1)}, \Gamma] = 0$ on $\mathcal{M}$ so that it a symmetry of the trajectories there. More over since it is a prolongation it also satisfies the vanishing condition give above. Then $\mathcal{L}_(Y^{(1)})d\theta_L$ is a new Cartan 2-form for the solutions in $\mathcal{M}$ with the corresponding lagrangian given as before.

Now we move to building symmetries out of alternative Lagrangians. It is possible to construct four symmetries with two alternative Lagrangians for a given system [10]

.

Lemma 2.11. *Let $(\Gamma, d\theta_L)$ and $(\Gamma, d\theta_{L'})$ be identical dynamical systems (in the sense of having the same Euler-Lagrange fields) given by the two Lagrangians L, L'. If there exists a $Y = \tau\frac{\partial}{\partial t} + \beta^i\frac{\partial}{\partial x_i} + \eta^i\frac{\partial}{\partial v_i} \in \mathcal{TE}$ satisfying $\mathcal{L}_Y\theta_L = \theta'_L + df$ (and so also the condition $\mathcal{L}_Y d\theta_L = d\theta'_L$) then Y is a dynamical symmetry and Y satisfies:*

$$(\frac{\partial^2 L}{\partial v_i \partial v_j})(\beta^j - v^j\tau) = -\frac{\partial G}{\partial v^i}$$

$$\Gamma(G) = -L' \text{ with } G = f - Y \lrcorner \theta_L$$

Proof. We will show that $[Y, \Gamma] = h\Gamma$. $[Y, \Gamma] \lrcorner d\theta_L = \mathcal{L}_Y(\Gamma \lrcorner d\theta_L) + \Gamma \lrcorner \mathcal{L}_Y(d\theta_L) = \Gamma \lrcorner d\theta'_L = 0$. This characterizes Y as a dynamical symmetry. Consider now the equation

$\mathcal{L}_Y \theta_L = d(Y \lrcorner \theta_L) + Y \lrcorner d\theta_L = \theta_{L'} + df \Rightarrow Y \lrcorner d\theta_L = \theta_{L'} + d(f - Y \lrcorner \theta_L)$. Defining $G = f - Y \lrcorner \theta_L$ we have $\Gamma(G) = d\theta_L(\Gamma, Y) - \theta_{L'}(\Gamma) = -\theta_{L'}(\Gamma)$. But using $\theta_L = Ldt + \frac{\partial L}{\partial x^i}\theta^i$ we have $\Gamma(G) = -L'$. Also looking at the dv^i component of the equality $Y \lrcorner d\theta_L = \theta_{L'} + dG$ (i.e contracting it with $\frac{\partial}{\partial v^i}$) we have $(\frac{\partial^2 L}{\partial v_i \partial v_j})(\beta^j - v^j \tau) = -\frac{\partial G}{\partial v^i}$. □

Thus the aim is to find the vector field Y satisfying this equation by first finding the function G satisfying the second condition.

Theorem 2.12. *Given two alternative Lagrangians L,L' and G be a solution to the equation* $\Gamma(G) = -L'$. *Define* $\mu^i = \beta^i - v^i \tau$ *and set* $\eta^i = \Gamma(\beta^i) - u^i \Gamma(\tau)$ *Then* $Y = \tau\frac{\partial}{\partial t} + \mu^i \frac{\partial}{\partial x_i} + \eta^i \frac{\partial}{\partial u_i}$ *is a symmetry associated with these alternative Lagrangians through the equation* $\mathcal{L}_Y \theta_L = \theta_{L'} + df$.

Proof. Since $\Gamma \lrcorner (\theta'_L + dG) = L' + \Gamma(G) = 0$, by thereom 1.11 there exists a vector field $\tilde{Y} = \tilde{\tau}\frac{\partial}{\partial t} + \tilde{\beta}^i \frac{\partial}{\partial x_i} + \tilde{\eta}^i \frac{\partial}{\partial v_i}$ s.t $Y \lrcorner \theta_L = (\theta'_L + dG)$.

First of all, $\mathcal{L}_{\tilde{Y}} \theta_L = d(Y \lrcorner \theta_L) + \theta_{L'} + dG = \theta_{L'} + d(G + Y \lrcorner \theta_L)$. So by the lemma above $\tilde{Y}$ is a symmetry.

Also contracting this equality with $\frac{\partial}{\partial v^i}$ we obtain $(\frac{\partial^2 L}{\partial v_i \partial v_j})(\tilde{\beta}^j - v^j \tilde{\tau}) = -\frac{\partial G}{\partial v^i}$. By the lemma above then $(\tilde{\beta}^j - v^j \tilde{\tau}) = (\beta^j - v^j \tau)$. This also leads to the equation $(\Gamma(\tilde{\beta}^j) - \Lambda^j \tilde{\tau} - v^j \Gamma(\tilde{\tau})) = (\Gamma(\beta^j) - \Lambda^j \tau - v^j \Gamma(\tau))$

Since $\tilde{Y}$ is a dynamical symmetry we have $\tilde{Y} = \tilde{\tau}\frac{\partial}{\partial t} + \tilde{\beta}^i \frac{\partial}{\partial x_i} + \Gamma(\tilde{\beta}^i) - v^i \Gamma(\tilde{\tau})\frac{\partial}{\partial v_i}$. But then $Y = \tilde{Y} - (\tilde{\tau} - \tau)\Gamma = \tau\frac{\partial}{\partial t} + (\tilde{\beta}^i - \tilde{\tau}v^i + \tau v^i)\frac{\partial}{\partial x_i} + (\Gamma(\tilde{\beta}^i) - v^i\Gamma(\tilde{\tau}) - \tilde{\tau}\Lambda^i + \tau\Lambda^i)\frac{\partial}{\partial v_i} = \tau\frac{\partial}{\partial t} + (\beta^i - u^i\tau)\frac{\partial}{\partial x_i} + \Gamma(\beta^i - u^i\tau)\frac{\partial}{\partial u_i}$ by the equations derived above. □

Although we have shown the existence of symmetries due to alternative Lagrangians, solutions to the equation $\Gamma(G) = -L$ might not be unique. But even in that case two symmetries so obtained differ only by a Noether symmetry:

Theorem 2.13. *The symmetry obtained by two alternative Lagrangians with the method above differ by a Noether symmetry*

Proof. Consider G_1 and G_2 as two solutions to $\Gamma(G) = -L$ yielding two symmetry generators Y_1 and Y_2. Then by the equalities $Y_1 \lrcorner d\theta_L = dG_1$ and $Y_2 \lrcorner d\theta_L = dG_2$ we have $(Y_1 - Y_2) \lrcorner d\theta_L = d(G_1 - G_2) \Rightarrow \mathcal{L}_{(Y_1-Y_2)}\theta_L = d(G_1 - G_2 + (Y_1 - Y_2) \lrcorner \theta_L) \Rightarrow \mathcal{L}_{(Y_1-Y_2)} d\theta_L = 0$ which makes $Y_1 - Y_2$ a Noether symmetry by definition. □

Since it is also possible to pick L'=L, then out of two Lagrangians we can get four pairs of Lagrangians: $(L, L), (L, L'), (L', L'), (L', L)$. Using the method above we get four symmetries from these pairs. It should however be noted that symmetries constructed using these pairs need not be independent.

Corollary 2.14. *Given two Lagrangians L and L' we obtain four dynamical symmetries by* $\Gamma(G) = -L$ *and* $\Gamma(G') = -L$ *from the pairs* $(L, L), (L, L'), (L', L'), (L', L)$

Associated to these Lagrangians and symmetries we also have constants of motions:

Corollary 2.15. *Associated to the symmetries X and Y we have the constants of motion* $F = X \lrcorner Y \lrcorner d\theta_L$ *and* $F' = Y \lrcorner X \lrcorner d\theta_{L'}$*.*

Proof. $\Gamma(F) = (\mathcal{L}_\Gamma X) \lrcorner Y \lrcorner d\theta_L + X \lrcorner (\mathcal{L}_\Gamma Y) \lrcorner d\theta_L + X \lrcorner Y \lrcorner (\mathcal{L}_\Gamma d\theta_L)$

$$= h_1 \Gamma \lrcorner Y \lrcorner d\theta_L + X \lrcorner h_2 \Gamma \lrcorner d\theta_L = 0$$ □

Note that these constant of motion are locally given as $F = X \lrcorner Y \lrcorner d\theta_L = X \lrcorner (\mathcal{L}_Y \theta_L - d(Y \lrcorner \theta_L)) = X \lrcorner (\theta_{L'} + dH)$ where $dH = d(f + Y \lrcorner \theta_L)$.

Example 2.16. Too see how the procedure works, we examine the example given in [10] the following Lagrangians:

$$L_1 = \tfrac{1}{2}((v^1)^2 - (v^2)^2) - \tfrac{1}{2}a((x^1)^2 - (x^2)^2) - bx^1x^2$$

$$L_2 = v^1v^2 - ax^1x^2 - \tfrac{1}{2}((v^2)^2 - (v^1)^2)$$

produce the equations of motion $\frac{dv^1}{dt} = -ax^1 - bx^2$ and $\frac{dv^2}{dt} = -ax^2 + bx^1$. Now if we try to solve

$$\Gamma(G) = \frac{\partial G}{\partial t} + v^i\frac{\partial G}{\partial x^i} - (ax^1 + bx^2)\frac{\partial G}{\partial v^1} - (ax^2 - bx^1)\frac{\partial G}{\partial v^2} = -v^1v^2 + ax^1x^2 + \frac{1}{2}((v^2)^2 - ((v^1)^2))$$

Then a possible solution is seen to satisfy

$$\frac{\partial G}{\partial t} = 0, \quad \frac{\partial G}{\partial x^1} = v^2, \quad \frac{\partial G}{\partial x^2} = v^1, \quad \frac{\partial G}{\partial v^1} = -\frac{1}{2}x^2, \quad \frac{\partial G}{\partial v^2} = \frac{1}{2}x^1,$$

This is straightforward to integrate and it gives $G = -\frac{1}{2}(v^2x^1 + v^1x^2)$. Similarly G' can be calculated to be $G' = \frac{1}{2}(v^2x^2 - v^1x^1)$. Then solving the equations given in the theorem we have

$$\frac{\partial L}{\partial v^1 \partial v^1}\mu^1 = -\frac{\partial G}{\partial v^1} \Rightarrow \mu^1 = \frac{1}{2}x^2$$

$$\frac{\partial L}{\partial v^2 \partial v^2}\mu^2 = -\frac{\partial G}{\partial v^2} \Rightarrow \mu^1 = \frac{1}{2}x^1$$

Then if for instance $\beta^1 - v^1\tau = \frac{1}{2}x^2$ we have $\tau = 0$ and then $\eta^1 = \Gamma(\beta^1)$ and the symmetry is

$$X = \frac{1}{2}(x^2\frac{\partial}{\partial x^1} + x^1\frac{\partial}{\partial x^2} + v^2\frac{\partial}{\partial v^1} + v^1\frac{\partial}{\partial v^2})$$

Similiarly the other symmetry can be found as

$$Y = \frac{1}{2}(x^1\frac{\partial}{\partial x^1} - x^2\frac{\partial}{\partial x^2} + v^1\frac{\partial}{\partial v^1} - v^2\frac{\partial}{\partial v^2})$$

And the other two symmetries found by picking the same Lagrangian are trivial i.e $X' = -X$ and $Y' = Y$. Note that second symmetry is the scaled rotation generator. Finally we calculate one of the constants of motion associated:

$$d\theta_L = (-ax^1 - bx^2)dx^1 \wedge dt + (ax^2 + bx^1)dx^2 \wedge dt + v^1 dv^1 \wedge dt$$

$$-v^2 dv^2 \wedge dt + (dx^1 - v^1 dt) \wedge dv^1 + (dx^2 - v^1 dt) \wedge dv^2$$

and we have

$$d\theta_L(X, Y) = X \lrcorner \frac{1}{2}(v^2 dv^1 + v^1 dv^1) = 0$$

The other constant of motion is similarly trivial.

Example 2.17. We now apply this method to a real system i.e the 2D harmonic oscillator. Two Lagrangians for the 2D harmonic oscillator are given as:

$$L = \sum_i \frac{1}{2}(v^i)^2 - \frac{1}{2}(\omega x^i)^2$$

$$L' = v^1 v^2 - \omega^2 xy$$

with equations of motion and Euler-Lagrange field

$$\frac{dv^i}{dt} = -\omega^2 x^i$$

$$\Gamma = \frac{\partial}{\partial t} + v^i \frac{\partial}{\partial x^i} - \omega^2 x^i \frac{\partial}{\partial v^i}$$

Then solving $\Gamma(G) = -L'$ just as above we obtain the same conditions

$$\frac{\partial G}{\partial t} = 0, \quad \frac{\partial G}{\partial x^1} = v^2, \quad \frac{\partial G}{\partial x^2} = v^1, \quad \frac{\partial G}{\partial v^1} = x^2, \quad \frac{\partial G}{\partial v^2} = x^1,$$

so that $G = v^2x^1 + v^1x^2$. Then $\mu^1 = -x^2$, $\eta^1 = -\frac{dx^2}{dt}$ and $\mu^2 = -x^1$, $\eta^1 = -\frac{dx^2}{dt}$ and the symmetry is

$$X = -(x^2\frac{\partial}{\partial x^1} + x^1\frac{\partial}{\partial x^2} + v^2\frac{\partial}{\partial v^1} + v^1\frac{\partial}{\partial v^2})$$

This symmetry generates exponential homogenous scaling. The other solution is $G' = \frac{1}{2}(v^1x^1 + v^2x^2)$ with its symmetry being only half of the previous one. Thus again the associated constant of motion is zero.

Finally we give the relationship between alternative Lagrangians and constants of motion [11]. In theorem 2.7 we showed how to construct constants of motion associated to Noether or Cartan symmetries. The association of a closed form of a constant of motion to two alternative Lagrangians (and so to arbitrary dynamical symmetries) is a bit more complex. We need the following theorem:

Theorem 2.18. *Let $(\Gamma, d\theta_L)$ be a dynamical system and S a (1,1) tensor such that $\mathcal{L}_\Gamma S = 0$ then tr(S^n) are constant among the flows of Γ (for any n).*

Proof. Letting (x^i) denote the coordinates on $\mathcal{E}$, $tr(S)(p) = S(dx^i, \frac{\partial}{\partial x^i})|_p \Rightarrow \mathcal{L}_\Gamma(tr(S)) = (\mathcal{L}_\Gamma S)(dx^i, \frac{\partial}{\partial x^i})|_p + S(\frac{\partial \Gamma_i}{\partial x^k}dx^k, \frac{\partial}{\partial x^i})|_p - S(dx^i, \frac{\partial \Gamma_k}{\partial x^i}\frac{\partial}{\partial x^k}) = S(\frac{\partial \Gamma_k}{\partial x^i}dx^i, \frac{\partial}{\partial x^k})|_p - S(dx^i, \frac{\partial \Gamma_k}{\partial x^i}\frac{\partial}{\partial x^k}) = 0$. So $tr(S)$ is a constant of motion. Similiarly since $\mathcal{L}_\Gamma S^n = (\mathcal{L}_\Gamma S) \circ S^{n-1} + S \circ (\mathcal{L}_\Gamma S) \circ S^{n-2} + ... = 0$ since $\mathcal{L}_\Gamma S = 0$ and by the same calculation above we have $\Gamma(tr(S^n)) = 0$ □

Then if we have two Cartan 2-forms $d\theta_L$ and $d\theta'_L$ arising from two Lagrangians, we have the following result;

Theorem 2.19. *The (1,1) tensor Λ uniquely defined for any vector field $X \in \chi(\mathcal{E})$ as; $(\Lambda(X)) \lrcorner d\theta_L = X \lrcorner d\theta_{L'}$ and s.t $\Lambda(X) \lrcorner dt = 0$ satisfies the property in theorem 2.18 (hence giving constants of motion).*

Proof. First we will show that Λ is indeed well defined. Since $\Gamma \lrcorner X \lrcorner d\theta_{L'} = 0$ there exists a vector field Y s.t $Y \lrcorner d\theta_L = X \lrcorner d\theta_{L'}$. Note however this Y is arbitrary up to addition of a multiple of Γ since it is in the kernal of $d\theta_L$. So to satisfy the last condition for Λ we set $Y' = Y - (Y \lrcorner dt)$ so that we have $Y'(t) = 0$. Thus we have $\Lambda(X) = Y'$ for a unique Y'.

We will now show it satisfies the properties of being a tensor. Suppose $(\Lambda(X)) \lrcorner d\theta_L = X \lrcorner d\theta_{L'}$ and $(\Lambda(Y)) \lrcorner d\theta_L = Y \lrcorner d\theta_{L'}$ s.t $\Lambda(X) = X'$ and $\Lambda(Y) = Y'$. Now $(X'+Y') \lrcorner d\theta_L = (X+Y) \lrcorner d\theta_{L'} \Rightarrow \Lambda(X+Y) = (X'+Y') = \Lambda(X) + \Lambda(Y)$. Also if $(\Lambda(fX)) = Y$ then $Y \lrcorner d\theta_L = fX \lrcorner d\theta_{L'} \Rightarrow (iff \neq 0)\frac{Y}{f} \lrcorner d\theta_L = X \lrcorner d\theta_{L'}$. Then $f\Lambda(X) = f\frac{Y}{f} = Y$.

Now we will show that $\mathcal{L}_\Gamma \Lambda = 0$. For this consider $\mathcal{L}_\Gamma(\Lambda(X) \lrcorner d\theta_L) = \mathcal{L}_\Gamma(X \lrcorner d\theta'_L)$ Now using $\mathcal{L}_\Gamma d\theta_L = \mathcal{L}_\Gamma d\theta_{L'} = 0$ we have $\mathcal{L}_\Gamma(\Lambda(X)) \lrcorner d\theta_L = \mathcal{L}_\Gamma(X) \lrcorner d\theta_{L'}$ But by definition of Λ this is equal to $\Lambda(\mathcal{L}_\Gamma X) \lrcorner d\theta_L$. So $\mathcal{L}_\Gamma(\Lambda(X)) \lrcorner d\theta_L = (\mathcal{L}_\Gamma \Lambda) X \lrcorner d\theta_L + \Lambda(\mathcal{L}_\Gamma X) \lrcorner d\theta_L = \Lambda(\mathcal{L}_\Gamma X) \lrcorner d\theta_L \Rightarrow (\mathcal{L}_\Gamma \Lambda) X \lrcorner d\theta_L = 0 \Rightarrow (\mathcal{L}_\Gamma \Lambda) X = f\Gamma$. Since now $dt(\Gamma) = 1$ we have $\beta(X) = (\mathcal{L}_\Gamma \Lambda)(X) \lrcorner dt$. But we have $(\mathcal{L}_\Gamma \Lambda)(X) \lrcorner dt = \mathcal{L}_\Gamma(\Lambda(X) \lrcorner dt) - \Lambda(X) \lrcorner (\mathcal{L}_\Gamma dt)$ Using $\Lambda(X) \lrcorner dt = 0$ and $\mathcal{L}_\Gamma dt = 0$ we have $(\mathcal{L}_\Gamma \Lambda)(X) = 0$ for all X. Now using theorem 2.18 we have our result. □

Hence we have shown that we can associate constants of motions and dynamical symmetries to alternative Lagrangians as well as we can associate alternative Lagrangians (and therefore constants of motion) to dynamical symmetries. It is also known that constants of motion produce Cartan and Noether symmetries thus linking the cycle. Starting from dynamical symmetries we can build alternative Lagrangians and from alternative Lagrangians we can build constants of motion. Starting from alternative Lagrangians we can build symmetries and constants of motions. Finally starting from constants of motion we can build symmetries and so alternative Lagrangians. We thus have; given a dynamical system $(\Gamma, d\theta_L)$, there are dynamical symmetries iff there are alternative Lagrangians iff there are constants of motion. However this statement is not as useful as it sounds since as we have seen in examples alternative Lagrangians are always possible by either gauge transformations and scaling and constants of motion are always available as simply constants.

Example 2.20. We again use harmonic oscillator to demonstrate the procedure. We first need to calculate the Cartan 2-forms for each Lagrangian. They are given as:

$$d\theta_L = F^i dx^i \wedge dt + dv^i \wedge dx^i - v^i dv^i \wedge dt$$

$$d\theta'_L = F^1 dx^2 \wedge dt + F^2 dx^1 \wedge dt - v^2 dv^1 \wedge dt - v^1 dv^2 \wedge dt$$

Then we calculate the effect of Λ on each basis vector and get:

$$\Lambda(\frac{\partial}{\partial x^1}) = \frac{\partial}{\partial x^2} \quad \Lambda(\frac{\partial}{\partial x2}) = \frac{\partial}{\partial x^1} \quad \Lambda(\frac{\partial}{\partial v^1}) = -\frac{\partial}{\partial v^2} \quad \Lambda(\frac{\partial}{\partial v^2}) = -\frac{\partial}{\partial v^1}$$
$$\Lambda(\frac{\partial}{\partial t}) = v^2 \frac{\partial}{\partial x^1} + v^1 \frac{\partial}{\partial x^2} + F^2 \frac{\partial}{\partial F^1} + F^1 \frac{\partial}{\partial x^2}$$

Then the matrix of Λ at each point is calculated as

$$\Lambda|_p = \left| \begin{matrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ v^2 & v^1 & F^2 & F^1 & 0 \end{matrix} \right|_p$$

The trace of this matrix and any of its powers are independent of v^2, v^1, F^2, F^1 since the last column is zero. So we again get trivial constants of motion.

2.2. Geometric Formulation of Dirac Monopole Problem

One could, using the geometric formulation of mechanics, also produce the action functionals in a geometric setting. This formulation can even be applied to certain problems where the forces are velocity dependent, such as those of electromagnetic nature. But there are certain exactness requirements for a global formulation of the action principle in such cases. Dirac monopole problem is a case where global exactness is not possible in the configuration space $\mathcal{Q}$. However in this chapter we will explore a natural principle bundle structure on $\mathcal{Q}$, the U(1) bundle which removes this problem hence making it possible to write globally valid action principles [12]. Also defining an appropriate connection we arrive at the topological quantization scheme [13] where it is seen that magnetic charge, if it exists, should be quantized. This is essentially a time-independent problem so we use the symplectic formalism of time independent mechanics on $\mathcal{TQ}$.

We first describe the geometric formulation of action principle. It is known that if a Lagrangian exists for a dynamical system, then the dynamical trajectory of a mechanical system is the path on which the variational derivative of the functional $\Phi[\gamma] = \int_\gamma L dt$ is zero, i.e the path that extremizes the functional.

Theorem 2.21. Hamilton's Action Principle *Given a dynamical system* $(\Gamma, d\theta_L)$*, a path* $\gamma(t)$ *in* $\mathcal{Q}$ *(with two conditions* $\gamma(t_1)$ *and* $\gamma(t_2)$ *known) is the trajectory of the system in the configuration iff it extremizes the action functional*

$$\Phi[\gamma] = \int_\gamma L(q, u, t) dt$$

Proof. Given a small variation to the trajectory the new trajectory is $\gamma(t)+\delta\gamma(t)$ where $\delta\gamma(t) = \epsilon\eta(t)$ for a small real number ϵ. The variation also satisfies $\delta\gamma(t_1) = \delta\gamma(t_2) = 0$. Then the lift of the trajectory to the tangent bundle (such that it is still a phase curve see definition 1.1) is $(\gamma+\delta\gamma, \frac{d\gamma}{dt}+\frac{d\delta\gamma}{dt})$. The functional derivative of the action functional is calculated as:

$$\delta\Phi = \int_\gamma \frac{d}{dt}[\frac{\partial L}{\partial u^i} - \frac{\partial L}{\partial q^i}]\delta\gamma^i dt$$

The functional derivative is zero iff $\delta\Phi$ is zero for all $\delta\gamma$ which is true iff $\frac{d}{dt}[\frac{\partial L}{\partial u^i} - \frac{\partial L}{\partial q^i}]$ everywhere on the path of the integration which is iff (by theorem 1.9) the path we integrate over is the trajectory of the system. □

We also have the following result:

Proposition 2.22. *If an exact 1-form df is added to Ldt the equations of motion calculated from the action principle remain invariant*

Proof. We have; $\Phi[\gamma] = \int_\gamma [L(q,u,t) + \frac{df}{dt}]dt$. Then if we denote the coordinates as $\xi = (x^i, v^i, t)$,

$$\delta\Phi = \int_\gamma \frac{d}{dt}[\frac{\partial L}{\partial u^i} - \frac{\partial L}{\partial q^i}]\delta\gamma^i dt + \frac{d}{dt}[\frac{\partial f}{\partial \xi^i}]\delta\xi^i dt + \frac{\partial f}{\partial \xi^i}\frac{d}{dt}[\delta\xi^i]dt$$

But the last term vanishes after an integration by parts and we have the same equations of motion. □

The Hamiltonian analogue of this is given by the Legendre transformation: $\Phi[\zeta] = \int_\zeta \mathcal{A} - Hdt$ where $\mathcal{A} = p_i dx^i \in \Lambda^1(T^*Q)$ is called the canonical 1-form and the symplectic form is given as $\omega = -d\mathcal{A} \in \Lambda^2(T^*Q)$. Here the trajectory ζ is the trajectory of the system in T^*Q. Application of variational principles to this action functional produces the Hamilton's equations of motion: $\frac{\partial H}{\partial p_i} = \frac{dq^i}{dt}$ and $\frac{\partial H}{\partial q^i} = -\frac{dp_i}{dt}$ where the time derivatives are taken on the trajectory ζ. In this chapter we will also use Hamilton's equation of motion. We accept that Hamilton's equations of motion are equivalent to Lagrange's equations of motion which was in chapter 1 shown to be equivalent to Newton's equations of motion in a geometric setting. For the momentum coordinates of the phase space we use lower indices as p_i.

In any electromagnetic problem with electric potential $\varphi = 0$ and magnetic field $\vec{B}$, if the magnetic field is derivable as the curl of a vector potential that is $B = \nabla \times \vec{A}$, then the generalized momentum is $p'_i = \frac{p_i - eA_i}{2m}$ with the Hamiltonian $H = \frac{(\vec{p}-e\vec{A}).(\vec{p}-e\vec{A})}{2m}$. Considering the magnetic field two form $B =^* (B_i dx^i)$ (where $*$ is the Hodge star dual), the geometric condition that B is derivable from a vector potential is equivalent to $B = dA$ where $A = A_i dx^i$ which is equivalent to the condition above. For instance in three dimensions for the magnetic field two form we have $B = B_x dy \wedge dz + B_y dz \wedge dx + B_z dx \wedge dy$. Note here that B_i and A_i are components of the magnetic field and vector potential. If a vector potential for this problem exists we have $B_{ik} = \frac{\partial A_i}{\partial x^k} - \frac{\partial A_k}{\partial x^i}$.

Theorem 2.23. *Let A be the canonical 1-form defined as $\mathbf{A} = p_i dx^i + A_i dx^i$ where $A_i dx^i$ is the vector potential 1-form and B be the magnetic field 2-form for an electromagnetic system with Hamiltonian H. Then the action functional $\Phi[\zeta] = \int_\zeta \mathbf{A} - H dt$ produces Hamilton's equations of motion for the system.*

Proof. First we show that indeed the action functional given above gives Hamilton's equations of motion. Let ζ be the path and $\delta\zeta$ be the variation. The coordinates of the phase space are given as $\xi = (x, p)$. Then

$$\delta\psi = \int\limits_{\zeta+\delta\zeta} \mathbf{A}_\mu(\xi + \delta\xi) d(\xi + \delta\xi)^\mu - \int\limits_{t_1}^{t_2} H(\xi + \delta\xi) dt + \int\limits_{\zeta} \mathbf{A} - \int H dt$$

$$= \int\limits_{t_1}^{t_2} [\mathbf{A}_\mu \frac{d}{dt}(\delta\xi^\mu) - \delta\xi^\upsilon \frac{\partial \mathbf{A}_\mu}{\partial \xi^\upsilon} p^\mu] dt - \int\limits_{t_1}^{t_2} \delta\xi^\mu \frac{\partial H}{\partial \xi^\mu} dt$$

$$= \int\limits_{t_1}^{t_2} \delta\xi^\mu [(d\mathbf{A})_{\mu\nu} \frac{d\xi^\upsilon}{dt} + \frac{\partial H}{\partial \xi^\mu}] dt = 0 \text{ for all } \delta\xi \text{ where } (d\mathbf{A})_{\mu\nu} = \omega_{\mu\nu}.$$

Then this is equivalent to Hamilton's equations of motion on the dynamical curve $x(t)$ are $\omega_{\mu\nu} \frac{d\xi^\nu}{dt} = \frac{\partial H}{\partial \xi^\mu}$ where it is easy to see that

$$\omega = \begin{vmatrix} (B_{jk})_{n\times n} & -\mathbb{I}_{n\times n} \\ \mathbb{I}_{n\times n} & 0_{n\times n} \end{vmatrix}$$

So this is explicitly $\frac{dx_j}{dt} = \frac{\partial H}{\partial p_j}$ and $\frac{dp_j}{dt} = -\frac{\partial H}{\partial q_j} + B_{jk} \frac{\partial H}{\partial p_k}$. Then by the Legendre transformation $L(x, v) = v^j p_j(x, v) - H(x, p)$ we have $\frac{\partial H}{\partial x^j} = -\frac{\partial L}{\partial x^j}$ and $p_j = -\frac{\partial L}{\partial v^j}$. Using these to write the Euler-Lagrange equations we have $\frac{d}{dt} \frac{\partial L}{\partial v^i} - \frac{\partial L}{\partial x^i} = \frac{dp_i}{dt} + \frac{\partial H}{\partial x^j} = B_{jk} v^k$. This is equivalent to the modified Euler-Lagrange equations given for the electromagnetic system in the remark following example 1.11 (note we have $\frac{\partial A_j}{\partial t} = 0$) where since we used the classical Hamiltonian $H = \sum_i \frac{1}{2}(v^i)^2 + V$ the Lagrangian is also the classical one $L = \sum_i \frac{1}{2}(v^i)^2 - V$ □

Note that the construction of this action relied on the existence of a 1-form A such that $A = dB$ i.e the exactness of the magnetic field 2-form.In the case of Dirac monopole problem the above method is not directly applicable due to a singularity in the magnetic field form. In this problem a magnetic monopole is fixed in origin and we look at the dynamics of a charged particle under the influence of the magnetic field created by the monopole. The magnetic field form is given in cartesian coordinates as $B = \frac{eg}{r^3}(x_1 dx^2 \wedge dx^3 - x_2 dx^3 \wedge dx^1 + x_3 dx^1 \wedge dx^2)$(where we have explicitly denoted magnetic and electric charge as we will later on explore quantization) which is closed but not exact. Note that we can only define this form in $\mathbb{R}^3 \setminus \{0\}$ due to the singularity at the origin.

Theorem 2.24. *The magnetic potential 2-form for the Dirac monopole problem in* $\mathbb{R}^3 \setminus \{0\}$ *is not exact*

Proof. Consider the embedding of a sphere in $\mathbb{R}^3 \setminus \{0\}$ $\mathbf{i} : S \hookrightarrow \mathbb{R}^3 \setminus \{0\}$. The magnetic field two form in polar coordinates is $B = egsin(\theta) d\theta \wedge d\varphi$. Then the sphere is a compact manifold without boundary in $\mathbb{R}^3$ $\{0\}$. The integral is

$$\int_0^{\pi} d\theta \int_0^{2\pi} d\varphi sin\theta = -4\pi eg$$

But Stoke's theorem implies that the integral of every exact form over a compact manifold without boundary is zero i.e if there exists a compact manifold without boundary such that the integral of a form over that surface is non-zero, then that form is not exact. □

Removing a "Dirac string" is a known method to make this form locally exact. In this method the dynamics is defined in the space $\mathbb{R}^3 \setminus \{z - axis\}$ where then a 1-form

A can be calculated such that $A = dB$ (see below). For a topological approach to this, note that the origin is removed to avoid the singularity in the magnetic field. So our domain is no more star shaped and we no longer can get exactness of our closed form B by Poincaré's theorem. However it is possible to remove the positive or negative z axis (or any other) completely which removes both the singularity and keeps the star shape property of our domain and Poincaré's theorem is applicable in this case. So this is a topological proof that removing a Dirac string makes B exact by Poincaré's theorem. But when we remove a Dirac string we can define our dynamics using at least two coordinate charts, where on each a half-axis is removed. Note also that, if one tries to calculate a magnetic potential form by Ad-hoc calculation as $egsin(\theta)d\theta \wedge d\varphi = dA$, it also comes with a singularity at positive z-axis; $A = df - egcos(\theta)d\varphi$ where df is a closed form. Depending on the choice of df A becomes singular at certain axes. For instance for df = $\pm d\varphi$ we have $A = \frac{1}{r}\frac{xdy-ydx}{\pm\sqrt{(x^2+y^2+z^2)}+z}$ and A is singular at positive and negative z-axis, making it necessary to define the dynamics locally. However for instance these two forms agree on overlaps up to a total differential factor which is $2d\varphi$. Such terms do not alter action functionals and thus are gauge transformations. This gauge transformation introduced in an ad-hoc manner can be put into the form of a transition function on an appropriate principal bundle resulting in global dynamics rather than local.

To realize the idea explained above, we will use the machinery developed in Appendix 3.2. Classically for the monopole problem two A forms are defined as above for domains with north and south Dirac strings removed where the base manifold is taken to be sphere (where they are multiplied by $\frac{i}{2}$ to make them lie algebra valued 1-forms so our corresponding magnetic field 2-form is assumed to be multiplied by $\frac{i}{2}$, since in calculating the variational equations we only use form components in fact this generalization does not have an effect on equations of motion);

$$\mathbf{A}^+ = \frac{i}{2} \otimes (1 - cos(\theta^+))d\varphi^+$$

$$\mathbf{A}^- = \frac{i}{2} \otimes (-1 - cos(\theta^-))d\varphi^-$$

These 1-forms are lie-algebra valued where the lie algebra is $\mathfrak{u}(1)$ whose generator is i. They differ by a total differential making them connections on the base manifold. Also their differential gives the same magnetic field 2-form (or the curvature), but they do not agree on the overlaps (for instance polar coordinates). So they are local connection forms. We will define our coordinates using two charts. The first chart U^+ is the $\mathbb{R}^3 \setminus 0$ in polar coordinates with the positive z-axis removed given by the local coordinates $(r^+, \theta^+, \varphi^+)$. The second chart U^2 is the $\mathbb{R}^3 \setminus \{0\}$ in polar coordinates with the negative z-axis removed with the local coordinates $(r^-, \theta^-, \varphi^-)$. The transitions in the overlaps are simply $(r^+ = r^-, \theta^+ = \theta^-, \varphi^+ = \varphi^-)$. Accepting that a U(1) principal bundle can be constructed over this space [12], it is then completely by the transition function in the overlaps. We define the transition function as $\eta^+ = \eta^- - i\varphi$ where e^η is the local coordinates of the U(1) bundle over each chart. These transitions functions trivially satisfy the cocycle conditions. Now we define the corresponding 1-forms over the U(1) bundle as $\mathcal{A} = \pi^* A + d\eta$ so in coordinates:

$$\mathcal{A}^+ = i \otimes d\eta^+ + \frac{i}{2} \otimes (1 - cos(\theta^+))d\varphi^+ \in \Lambda^1(\pi^{-1}(U^+))$$

$$\mathcal{A}^- = i \otimes d\eta^- + \frac{i}{2} \otimes (-1 - cos(\theta^-))d\varphi^- \in \Lambda^1(\pi^{-1}(U^-))$$

Theorem 2.25. *The $\mathcal{A}$ form given above is globally defined and are connection 1-forms on the U(1) bundle over the configuration space $\mathbb{R}^3 \setminus \{0\}$.*

Proof. We will first look at the kernels of the 1-forms. Let $H_p = a\frac{\partial}{\partial r} + b\frac{\partial}{\partial \theta} + c\frac{\partial}{\partial \varphi} + d\frac{\partial}{\partial \eta} \in \mathcal{T}_p\mathcal{E}$. Then if $A(H_p) = 0$ we have $i(d + b(1 - cos(\theta))) = 0$. So $H_p = a\frac{\partial}{\partial r} + b\frac{\partial}{\partial \theta} + c\frac{\partial}{\partial \varphi} + b(cos(\theta) - 1)\frac{\partial}{\partial \eta}$. The vertical vector fields are given by $f\frac{\partial}{\partial \eta}$. At each point the kernel is three dimensional and the vertical vector field is one dimensional. More over their intersection requires $b = 0 \Rightarrow f = 0$. So indeed they are direct sum decomposition of $\mathcal{T}_p\mathcal{E}$. Moreover $\mathcal{L}_{\frac{\partial}{\partial \eta}}\mathcal{A} = 0$. So we have the smooth distribution satisfying the requirements given in definition 3.4. They are globally defined since under the transition functions $\eta^+ = \eta^- - i\varphi$ the 1-forms transform as $d\eta^+ = d\eta^- - id\varphi$ and in the overlaps $\mathcal{A}^+ = \mathcal{A}^-$. □

And it is seen that the curvature form defined on the principal bundle by $\mathcal{B} = d\mathcal{A}$ is in fact the pullback of the magnetic field 2-form B on $\mathcal{Q}$ to the principal bundle P by the projection map. We see that it is not possible to construct a globally defined A in $\mathcal{Q}$ such that $B = dA$ however in the principal bundle we have a globally defined connection such that $\mathcal{B} = d\mathcal{A}$. This is a specific example to Theorem 3.11. Also defining a constant section σ over U^+ and U^- we get an example to Theorem 3.6. The quantization comes into play once we realize that magnetic 2-form is in fact a curvature form. The generalized Gauss-Bonnet theorem tells us that the integral of the curvature form over any 2-surface in a Riemannian manifold is $2\pi n$ where n is a characteristic dependent on the surface [14]. Now taking the surface to be a 2-sphere in U(1) dependent on the coordinates θ, φ we get the quantization rule $eg = \frac{n}{2}$.

Now we move the action principle on the U(1) bundle. It was shown that exactness of B is required if we are to construct an action functional for the system in the classical way. But in fact the globally exact connection we have just defined can be used to build up an equivalent action principle.

Theorem 2.26. *An equivalent action principle on U(1) bundle for the electromagnetic problem is*

$$\Phi(\gamma') = \int_{t_1}^{t_2} [\mathbb{A} - \pi^* H] dt$$

where $\mathbb{A} = p_i dx^i + \mathcal{A}$

Proof. Simply applying the variational methods as in Theorem 2.23 will give us the same equations of motions since $\pi^* H$ has the same components as before and only total differential are added to our vector potential 1-forms which are gauge transformations. □

Now we will explore another method which will pave the way to understanding the U(1) nature of our principal bundle that we just built. It will also have ties to path integral formulations. In the cases where an exact form B is not available, it is possible to write an equivalent action functional on an area piece (rather than a curve) on $\mathcal{Q}$ using the 2-form B instead of A. Fix a point q_0 with coordinates $(q_0)^j$ in the configuration space. Given a trajectory $\gamma : q^j(t)$ for $t_1 < t < t_2$ and a continuous family of curves, $\beta^j(t,\sigma)$ for each $0 < \sigma < 1$ such that $\beta^j(0,t) = q_0^j$ and $\beta^j(1,t) = q^j(t)$ sweep the surface $\mathcal{S}$, then we have

Theorem 2.27. *Define the functional given below:*

$$\Phi[\mathcal{S}] = \int_{\mathcal{S}} B + \int_{\gamma} L dt$$

where S is the area swept by the contraction of the trajectory γ (given above) to a single point not on the trajectory and B is the magnetic field 2-form. If $\mathcal{S}$ is chosen such that under the variation of this functional leaving the points $t_1, t_2, \beta(0,t_1), \beta(0,t_2)$ and the edges $\beta(\sigma,t_1), \beta(\sigma,t_2)$ fixed (see figure 2.1) the value of the functional is unchanged then γ is the dynamical trajectory of our system and so this defines an equivalent action principle.

Proof. For the local coordinates of $\mathcal{S}$ we use (σ, t) with $\beta(t,1) = x(t), \beta(t,0) = x_0$ and $\delta\beta^j(t_1) = \delta\beta^j(t_2) = 0$

$$\Phi[\mathcal{S}] = \int_{\mathcal{S}} B + \int_{t_1}^{t_2} L dt = \int_{t_1}^{t_2} dt \int_0^1 d\sigma B_{jk}(\beta) \frac{\partial \beta^j}{\partial \sigma} \frac{\partial \beta^k}{\partial t} + \int_{t_1}^{t_2} L dt$$

Then

$$\delta\Phi = \int_{t_1}^{t_2} dt \int_0^1 d\sigma [\frac{\partial B_{jk}}{\partial \beta^i} \delta\beta^i \frac{\partial \beta^j}{\partial \sigma} \frac{\partial \beta^k}{\partial t} + B_{jk} \frac{\partial \beta^j}{\partial \sigma} \frac{\partial \delta\beta^k}{\partial t} + B_{jk} \frac{\partial \delta\beta^j}{\partial t} \frac{\partial \sigma^k}{\partial t}] + \int_{t_1}^{t_2} [\frac{\partial L}{\partial \xi^i}] \delta\xi^i dt$$

$$= \int_{t_1}^{t_2} dt \int_0^1 d\sigma [\frac{\partial B_{jk}}{\partial \beta^i} \delta\beta^i \frac{\partial \beta^j}{\partial \sigma} \frac{\partial \beta^k}{\partial t} - \frac{\partial}{\partial t}(B_{jk} \frac{\partial \beta^j}{\partial \sigma}) \delta\beta^k - \frac{\partial}{\partial \sigma}(B_{jk} \frac{\partial \beta k}{\partial t \sigma}) \delta\beta^j]$$

$$+ \int_0^1 d\sigma (B_{jk} \frac{\partial \beta^j}{\partial \sigma}) \delta\beta^k |_{t_1}^{t_2} + \int_{t_1}^{t_2} dt (B_{jk} \frac{\partial \beta k}{\partial t}) \delta\beta^j |_{\sigma=0}^{\sigma=1} + \int_{t_1}^{t_2} [\frac{\partial L}{\partial \xi^i}] \delta\xi^i dt$$

using

$\beta(t, 1) = x(t), \beta(t, 0) = x_0$ and $\delta\beta^j(t_1) = \delta\beta^j(t_2) = 0$

$$= \int_{t_1}^{t_2} dt \int_0^1 d\sigma [\frac{\partial B_{jk}}{\partial \beta^i} \delta\beta^i \frac{\partial \beta^j}{\partial \sigma} \frac{\partial \beta^k}{\partial t} - \frac{\partial}{\partial t}(B_{jk} \frac{\partial \beta^j}{\partial \sigma}) \delta\beta^k - \frac{\partial}{\partial \sigma}(B_{jk} \frac{\partial \beta k}{\partial \sigma}) \delta\beta^j]$$

$$+ \int_{t_1}^{t_2} dt B_{jk} v^k \delta x^j + \int_{t_1}^{t_2} [\frac{\partial L}{\partial \xi^i}] \delta\xi^i dt = 0$$

Then equating the coefficients of $\delta\beta^i$and$\delta\xi^i = (\delta x^i, \delta v^i)$to zero we have two equations:

I. $\frac{\partial B_{jk}}{\partial \beta^i} \frac{\partial \beta^j}{\partial \sigma} \frac{\partial \beta^k}{\partial t} \delta\beta^i - \frac{\partial B_{jk}}{\partial \beta^i} \frac{\partial \beta^j}{\partial t} \frac{\partial \beta^k}{\partial \sigma} \delta\beta^k - \frac{\partial B_{jk}}{\partial \beta^i} \frac{\partial \beta^i}{\partial \sigma} \frac{\partial \beta^k}{\partial t} \delta\beta^j$

$= \delta\beta^i \frac{\partial \beta^j}{\partial \sigma} \frac{\partial \beta^k}{\partial t} [\partial_i B_{jk} + \partial_k B_{ij} + \partial_j B_{ki}] = 0$

II. $\int_{t_1}^{t_2} dt [B_{jk} v^k \delta x^j + [\frac{\partial L}{\partial \xi^i}] \delta\xi^i] = 0$

The first equation is automatically satisfied by B being closed. The second equa-

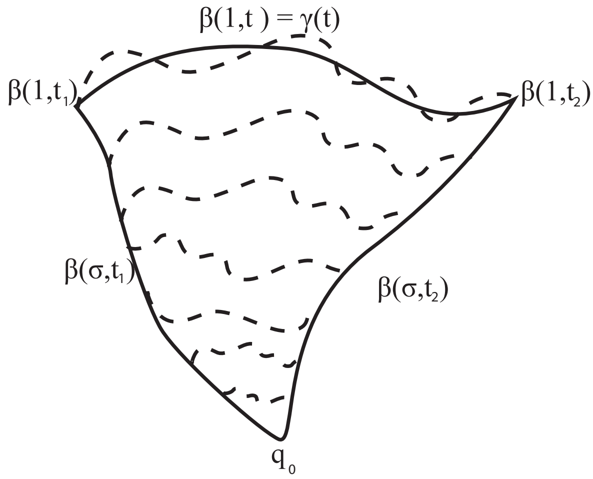

Figure 2.1. The area piece and its variation in Theorem 2.25

tion leads to the Euler-Lagrange equations given in Theorem 2.23:

$$\frac{d}{dt}\frac{\partial L}{\partial v^i} - \frac{\partial L}{\partial x^i} = B_{ik}v^k$$

□

It is important to note that in this setup the area S becomes a trajectory in $\mathcal{F}$; the space of paths in $\mathcal{Q}$. It is also possible to build the Hamilton-Jacobi formalism of this action where it reduces to normal Hamilton-Jacobi equations if the action principle is restricted to dynamical trajectory. Thus upon writing the action over the space of paths in configuration space, we no longer need B to be exact.

As mentioned before we now explore an idea presented in [12]. We have seen that defining the action principle over either the U(1) bundle or area pieces on the configuration space rather than paths resolves the problem of exactness of B. The problem stems from the fact that our potential forces our configuration space to have a non-trivial topology of $H^2(C) \neq 0$. That is not every two surface is contractible. In such cases the area form $sin(\theta)d\theta \wedge d\varphi$ is an invariant for classifying the surfaces. For instance the integral of this 2-form over the 2-sphere enclosing origin is 4π. To demonstrate the relationship between U(1) bundle and the non-triviality of the path space one has to pass to the wave function formalism as such non-trivial topologies are axiomatized to have effect on only non-classical systems. But instead of the classical wave function picture, wave functionals depending on the choice of paths are adopted motivated by the calculations above. We now present the equivalence relation on the space of paths $\mathcal{F}$ given in [12]. In that paper it is shown that the quotient space of $\mathcal{F}$ under this equivalence is a U(1) bundle.

Fix a point x_0 and let $\mathcal{P}$ be the space of paths starting at that point and ending some other point x. Define the projection map $\pi : \mathcal{P} \Rightarrow \mathcal{Q}$ such that if γ is a path ending at the point x, then $\pi(\gamma) = x$. Thus $\mathcal{F} = \bigcup_{x \in \mathcal{Q}} \pi^{-1}(x)$. Now for two paths γ and γ' that start at x_0 and end at x, let $S(\gamma, \gamma')$ be any surface bounded by these paths and define the integral $A(\gamma, \gamma') = exp(\int_{S(\gamma,\gamma')} B)$ where B is. Then we have $\gamma \sim \gamma'$ if they start and end at the same point and if $A(\gamma, \gamma') = 0$. Suppose γ, γ' and γ'' are homotopic paths. Then we have $A(\gamma, \gamma) = 0$, $A(\gamma, \gamma') = -A(\gamma', \gamma)$ and $A(\gamma, \gamma') + A(\gamma', \gamma'')$

Now using this idea of space of paths $\mathcal{F}$, we study a mathematical motivation to this heuristic introduction of U(1) bundle. It is shown in [12] that by building a natural relation $\backsim$ on $\mathcal{F}$, then $\mathcal{F}/\sim$ becomes a principal U(1) bundle over $\mathcal{Q}$. So the construction of wave-function formalism as functionals defined on paths in $\mathcal{F}/\sim$ naturally leads to a U(1) bundle structure and definition of wave functions up to a phase difference. This construction starts with picking a 2-form B. Now given two paths γ, γ' (i.e points in $\mathcal{F}$) and a surface Ξ bounded by the two, let the surface be parameterized by the local coordinates $\beta^j(t, \sigma)$ for $t_1 < t < t_2$ and $0 < \sigma < 1$ one defines the quantity

$$A(\gamma, \gamma') = \int_{\Xi} B = \frac{1}{2} \int_{t_1}^{t_2} dt \int_0^1 d\sigma B_{jk} \frac{\partial \beta^j}{\partial \sigma} \frac{\partial \beta^k}{\partial t}$$

Now this quantity will denote a phase difference between paths depending on the topology of the surrounding manifold. However we expect that this quantity is invariant of the surface bounding the two paths. So it should first of all remain invariant under small deformations leaving the borders unchanged i.e

$$A(\gamma,\gamma') = \int_{t_1}^{t_2} dt \int_0^1 d\sigma B_{jk}(\beta)\frac{\partial\beta^j}{\partial\sigma}\frac{\partial\beta^k}{\partial t} \text{ Then}$$

$$\delta A = \int_{t_1}^{t_2} dt \int_0^1 d\sigma[\frac{\partial B_{jk}}{\partial\beta^i}\delta\beta^i\frac{\partial\beta^j}{\partial\sigma}\frac{\partial\beta^k}{\partial t} + B_{jk}\frac{\partial\beta^j}{\partial\sigma}\frac{\partial\delta\beta^k}{\partial t} + B_{jk}\frac{\partial\delta\beta^j}{\partial\sigma}\frac{\partial\beta^k}{\partial t}]$$

$$= \int_{t_1}^{t_2} dt \int_0^1 d\sigma dB_{jk}\delta\beta^j\delta\beta^k = 0 \text{ (see Theorem 2.27)}$$

$$\Rightarrow dB = 0$$

So we require this form to be closed for physical consistency. Our choice for this form will evidently be the magnetic field 2-form which is closed and not exact so that there is a S^2 centered around the origin on which the integral of B is 4π and of course any other surface continuously deformable to S^2. This also evident from the fact that B is in fact the area area form on the sphere. Now since the Hamilton-Jacobi theory mentioned in the previous parts regarding extremization over surfaces depended not only on the configuration but also the paths that lead to the configuration, we consider the generalization of this to wave functionals i.e wave functions $\psi(x,\gamma)$ that depend on paths as well configuration. In this setup the path γ is considered to start at a fixed x_0 and end at x. Then in [12] following consistency condition is introduced:

$$\psi(x,\gamma) = exp[\tfrac{i}{h}A(\gamma,\gamma')]\psi(x,\gamma')$$

That is changes in γ leaving q fixed should be a phase factor so that $|\psi(x,\gamma)|^2$ is independent of path since it is the probability distribution for finding the state ψ at q. The factor $exp[\frac{i}{h}A(\gamma,\gamma')]$ is generally introduced in path integral formulations on topologically non-trivial spaces motivated by such results as Berry phase and Aharonov-Bohm effect. It has the motivation that it should be a universal quantity dependent on the topology of the space alone and not the states ψ. It has ties to homotopy theory where fundamental groups describing the topology of the space are characterized by an invariant integral described as above. We finally implicitly require that for two paths starting and ending at the same points, there is always a surface bounded by the two. And since we are working in a space greater than 2, we have that the paths are homotopic to each other. In this case we have the following results [12]:

- $A(\gamma, \gamma') = -A(\gamma', \gamma)$
- $A(\gamma, \gamma'') = A(\gamma, \gamma') + A(\gamma', \gamma'')$

where γ, γ' and γ'' are paths that start and end at the same points. Now defining the equivalence relation on $\mathcal{F}$ as, $\gamma \sim \gamma'$ if they start and end at the same points and $A(\gamma, \gamma') = 0$. By the properties above this is indeed a equivalence relation. So in fact through $A(\gamma, \gamma')$ wave functional and phase differences depend on equivalence classes $\{B\}, \{B'\}$. Hence rather than working in the huge and unwieldy space $\mathcal{F}$, it is much more convenient to work in $\mathcal{F} \backslash \sim$. In [12] it is shown that this space is equivalent to a U(1) bundle on $\mathcal{Q}$ and that the transition functions between local trivialization of the bundle are simply phase differences; hence wave functionals differ from each other up to a phase difference.

APPENDIX A: SYMPLECTIC GEOMETRY

We first give an introduction to finite dimensional symplectic manifolds and geometry, assuming knowledge of basic rudiments of manifold theory.

Definition A.1. *A symplectic manifold is smooth manifold* $(\mathcal{M}^n, \omega)$ *equipped with a non-degenarete, closed 2-form* ω*.*

Then this 2-form takes the form of a $n \times n$ nonsingular matrix at each $p \in \mathcal{M}$ denoted as ω_p. Actually there is a requirement that the manifold should have even dimension should it be equipped with such a form. By the definition of differential forms, $(\omega_p)_{ij}$ $= -(\omega_p)_{ji}$. Then suppose that the manifold is of odd dimension $2n+1$ i.e the matrix ω_p is a $(2n+1) \times (2n+1)$ matrix that is non-singular by assumption. So $|\omega_p| \neq 0$ but $\omega_p = -\omega_p^T$ and this implies $|\omega_p| = |\omega_p^T| = |-\omega_p| = -1^{2n+1}|\omega_p| \Rightarrow |\omega_p| = 0$ and this contradicts the assumption that ω is nonsingular. So from now on we take that $\mathcal{M}$ has dimension 2n. This property makes tangent bundles $(T\mathcal{M})$ and cotangent bundles $(T^*\mathcal{M})$ likely candidates for symplectic manifolds. In fact it is possible to equip $T^*\mathcal{M}$ with such a form canonically using only geometric structures that belong to $T^*\mathcal{M}$ while one requires differentials of certain functions to equip $T\mathcal{M}$ with such a geometry.

Theorem A.2. (Darboux) *Let* $(\mathcal{M}^n, \omega)$ *be a symplectic manifold. Then* $\forall p \in \mathcal{M}$*, there are smooth coordinates* $(x^1, x^2, ...x^n, p^1, p^2, ...p^n)$ *such that the symplectic form locally can be given as:* $\omega = \sum_{i=1}^n dp^i \wedge dx^i$*.*

Note that these local coordinates do not necessarily form a compatible atlas. However in the case of $T^*\mathcal{M}$, it can be shown that there exists a globally defined symplectic form given as above with the use of local coordinates of $T^*\mathcal{M}$ [15].

It is useful to realize that same structure can only be defined using vector spaces, where a vector space becomes a symplectic vector space whenever it is equipped with a non-degenerate alternating 2-tensor. This leads to study of symplectic linear algebra, whose results can then be applied to geometry of symplectic manifolds since ω_p acts on the vector spaces $T_p\mathcal{M}$

Note that this 2-form can be seen as a map between $T\mathcal{M}$ and $T^*\mathcal{M}$. That is when it is contracted with a tangent vector, it produces a cotangent vector by the assignment $\omega : X \mapsto X \lrcorner \omega | X \lrcorner \omega : Y \mapsto \omega(X, Y)$. By virtue of ω_p being nonsingular, then ω_p^* becomes a isomorphism between $T\mathcal{M}$ and $T^*\mathcal{M}$. That is, it becomes an isomorphism between a vector space and its dual.

APPENDIX B: PRINCIPAL BUNDLES AND CONNECTIONS

For completeness sake, in this chapter we will give some knowledge concerning principal bundles and connections on principle bundles which will be used in the section 2.2.

Definition B.1. *A principal G-bundle P is a fiber bundle with the projection map* $\pi : P \to M$ *(where M is the base manifold and G is the structure Lie group) equipped with a continuous action* $P \times G \to P$ *which is free,transitive and onto on each fiber F. We denote a principal bundle by P(M,G) or P.*

We denote the right action by $a \in G$ as R_a. Due to properties of the G-action we can essentially say that fibers are identical to the group G, though one has to choose an identity for the fiber to give it a group structure.

Two common examples for a principal bundle are the U(1) bundle over S^n (or over suitable base manifolds) with structure group $G = U(1)$ and the frame bundle. We now define Ehresman connections:

Definition B.2. *Let P(M,G) be a principal bundle. For each* $u \in P$ *denote* V_u *as the vertical subspace of* $\mathcal{T}_u P$ *which tangent to the fiber at u (i.e it lies in the tangent space of* $\mathcal{T}_u P$*). Then an Ehresmann connection on P(M,G) is a smooth distribution* H_u *(called the horizontal subspace) such that:*

- $\mathcal{T}_u P = H_u \oplus V_u \forall u \in P$
- $(R_a)_* H_u = H_{ua}$ *where* $a \in G, u \in P$

Note that first condition means each vector field can be decomposed into horizontal and vertical components and the second condition means horizontal fibers are preserved under the pushforward of right action. It is also possible to define connections using projection 1-forms [16]:

Theorem B.3. *Let V_u be the vertical subspace of $\mathcal{T}_u P$ and let α be a 1-form such that:*

- $\alpha_u^2 = \alpha_u$
- $im(\alpha_u) = V_u$

$\forall u \in P$. *Then α is called a connection 1-form and it defines an Ehresmann connection by $ker(\alpha_u) = H_u$.*

For the existence and uniqueness of connections on principal bundles, we refer the readers to [17]. We now have the following theorem [18]:

Theorem B.4. *Let P(M,G) be a principle bundle with Lie group G and Lie algebra $\mathfrak{g}$. If $\{U_i\}$ is an open covering of M with local sections $\{\sigma_i\}$ and $\{A_i\}$ are locally defined $\mathfrak{g}$ valued 1-forms on M then there exists a connection ω on P(M,G) such that $(\sigma_i)^*\omega = A_i$*

These $\{A_i\}$ are called local connections for the base manifold and since $\{U_i\}$ is just a covering, not an atlas, they do not necessarily transform well in the overlaps. Note that by the theorem ω is globally defined on all of P(M,G) and this brings the following restriction:

Theorem B.5. *Let $\{A_i\}$ and $\{\sigma_i\}$ be defined as above, if $\exists$ i,j s.t $U_i \cap U_j \neq \emptyset$ then in the overlap $A_j = t_{ij}^{-1} A_i t_{ij} + t_{ij} dt_{ij}$ where $t_{ij} \in G$ are the transition functions that relate the sections in the overlaps: $\sigma_j = \sigma_i t_{ij}$*

We finally explain curvature. Note first that any $\mathfrak{g}$ (dim n) valued n-form α can be written as $\alpha = \sum_{i=1}^{n} \alpha^i \otimes e_i$ where e_i is a basis for $\mathfrak{g}$ and α^i is a basis for the n-forms. Then the exterior derivative of α is given as $d\alpha = \sum_{i=1}^{n} (d\alpha^i) \otimes e_i$. We define covariant derivative using Ehresmann connections.

Definition B.6. *Let $\alpha \in \Lambda^n(M) \otimes \mathfrak{g}$ be a Lie algebra valued n-form. For any vector fields $X_1, X_2, ..., X_n$ (denoting their horizontal components as $X_1^H, ..., X_n^H$), the covariant derivative of α is defined as:*

$$D\alpha(X_1, X_2, ..., X_n) = d\alpha(X_1^H, ..., X_n^H)$$

Then;

Definition B.7. *Curvature 2-form Ω of a principal bundle is the covariant derivative of the connection 1-form ω*

We have the following very useful result:

Theorem B.8. *Curvature form satisfies the following relation:*

$$\Omega(X, Y) = d\omega(X, Y) + [\omega(X), \omega(Y)]$$

This has the consequence that if the tangent space of our manifold supports orthogonal frames then curvature form is simply $d\omega(X, Y)$ i.e it is exact. Now if M is a Riemannian surface then it is possible to find an open cover $\{U_i\}$ of M where each cover supports an orthonormal frame. Then it can be shown that the curvature form is globally defined on M by $\Omega = dA_i$ [14]. However since A_i does not necessarily transform well in the overlaps, this curvature form is not globally exact on M. But we have the following important result (for which an example with physical significance is given in section 2.2) [14]

Theorem B.9. *Let M be a Riemannian surface and P(M,G) a principal U(1) bundle over M. If A_i are local connection 1-forms on M with globally defined but locally exact curvature form $\Omega = dA_i$, then the curvature form defined on P(M,G) as $\Theta = (\pi)^* dA_i = d((\pi)^* A_i) = d\omega$ is globally exact. In this setup where e^η is the local coordinates of the the connection 1-form on P(M,G) becomes $\mathcal{A} = \pi^* A_i + d\eta$ which is globally defined.*

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