

Analysis of Vibration Absorbers Using Flexible Multi-Body Dynamics

by

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ABSTRACT

Advancements in modern digital computers provide means for an accurate analysis of large-scale systems under dynamic loading conditions. The methods for the nonlinear analysis of these systems are based on the concept of replacing the actual system with an equivalent model made up from discrete bodies that have known elastic and inertia properties. The actual systems, in fact, form multibody systems consisting of interconnected rigid and deformable bodies, each of which may undergo large translational and rotational displacements. Examples of physical systems that can be modeled as multibody systems are machines, mechanisms, vehicles, robotic manipulators, and space structures.

There are many approaches to the finite-element simulation of large displacement problems in flexible multibody dynamics such as the floating reference frame formulation, different incremental approaches, or large rotation vector formulations. The disadvantage of all these approaches is the high nonlinear terms in the equations of motion since they use a local reference frame fixed to a flexible body. The recently introduced Absolute Nodal Coordinate Formulation (ANCF) is promotable from this point of view. In absolute nodal coordinate formulation, global displacement coordinates and global slopes are used to define the element nodal coordinates. Therefore, beam, plate and shell elements which have been established as non-isoparametric elements in classical finite element formulations become isoparametric elements with this new technique. As a result, exact modeling of the rigid body dynamics can be obtained using the element shape function and absolute nodal coordinates. Moreover, usage of this new method results in a constant system mass matrix and, therefore, the centrifugal and coriolis forces become zero.

In this dissertation the Beam-Tip Mass-Ball structure is treated as a flexible multibody dynamic system and the ANCF formulation is used to demonstrate the coupled nonlinear dynamics of a large deflection of a beam with an appendage consisting of a mass-ball system. It is also aimed to show the complete energy transfer between two modes

occurring when the beam frequency is twice the ball frequency, which is known as auto parametric vibration absorption in the literature. Results are discussed and compared with experimental and MSC.ADAMS findings.



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NUMENCLATURE

a	beam cross sectional area
A_{b1}	transformation matrix between the coordinate systems $x - y$ and $x_2 - y_2$
A_{b2}	transformation matrix between the coordinate systems $x - y$ and $x_3 - y_3$
A_t	transformation matrix between the coordinates $x - y$ and $x_1 - y_1$
B^j	boolean matrix of j^{th} element
C	damping matrix
$C(q, t)$	vector of holonomic constraint equations
$C_{q_{sys}}$	jacobian of the system constraint equations
$C_q(q, t)$	jacobian of the holonomic constraint equations
C_{qd}	jacobian of the constraint equations associated with the dependent coordinates
C_{qi}	jacobian of the constraint equations associated with the independent coordinates
C_R	coefficient of restitution
ds	infinitesimal arc length of the beam element
e	element nodal coordinates
E	young's modulus
F	external forces
$F_{c_{av}}$	average impact force
F_c	impact force between bodies
h	time step size
I	area moment of inertia of the beam element
J	jacobian matrix
J_b	mass moment of inertia of the ball
J_t	mass moment inertia of the tip mass
K_l	stiffness matrix due to longitudinal deformation
K_t	stiffness matrix due to bending deformation
l	beam element length
m	beam element mass
M	mass matrix
M_a	mass matrix of the element
m_b	ball weight
M_{rigid}	mass matrix of the rigid bodies
M_{sys}	system mass matrix
m_t	tip weight
n_m	N-R maximum number of iterations
P	linear momentum
q	generalized coordinates

$Q_{e_{rigid}}$	vector of generalized external forces of the rigid bodies
Q_c	generalized constraint forces
Q_{ch}	generalized constraint forces due to holonomic constraints
Q_{cnh}	generalized constraint forces due to non-holonomic constraints
q_d	vector of the dependent generalized coordinates
Q_{damp}	generalized viscous damping force
Q_e	generalized external forces
Q_f	vector of generalized distributed gravity forces
q_i	vector of the independent generalized coordinates
Q_k	generalized elastic forces
Q_l	generalized elastic forces due to the longitudinal deformation
Q_M	vector of generalized forces due to external moment
q_{rigid}	vector of generalized coordinates of the rigid bodies
Q_{SD}	generalized spring-damper forces
q_{sys}	system generalized coordinates
Q_{sys}	vector of system generalized forces
Q_t	generalized elastic forces due to bending deformation
R	distance between point A and point D in $x_2 - y_2$ coordinate system
r	global position vector of an arbitrary point
r_{c_b}	displacement vector of the contact point C on the ball
r_{c_t}	displacement vector of the contact point C on the tip mass
r_A	absolute displacement vector of the beam tip point A
r_b	radius of the ball
R_b	displacement vector of the center of mass of the ball
R_t	absolute displacement vector of the center of mass of the tip appendage
S	global shape function
s	arc length of the beam element
T	kinetic energy of the finite element
U	strain energy of the finite element
u	displacement vector
u_{c1}	displacement vector of point C relative to point A in $x_2 - y_2$ coordinate system
u_{c2}	displacement vector of point C relative to point D in $x_3 - y_3$ coordinate system
u_l	longitudinal deformation
U_l	strain energy due to longitudinal deformation
u_t	transverse deformation
U_t	strain energy due to bending
u_t	displacement vector of point A relative to point B
v_1	velocity vector of the body before impact
v_2	velocity vector of the body after impact

x	position of an arbitrary point on the element in the undeformed state
β	Newmark integration parameter
γ	Newmark integration parameter
$\mathbf{\Gamma}(\mathbf{q}, \mathbf{t})$	vector of non-holonomic constraint equations
$\mathbf{\Gamma}_{\dot{\mathbf{q}}}(\mathbf{q}, \dot{\mathbf{q}}, \mathbf{t})$	jacobian of the non-holonomic constraint equations
$\Delta \mathbf{P}$	linear momentum change
Δt	contact time
δW_e	virtual work of the applied external forces
δW_i	virtual work of the inertia forces of the body
δW_k	virtual work of the elastic forces due to deformation
ε	Newton-Raphson tolerance
ε_l	longitudinal strain
θ	angle between the coordinate systems $x - y$ and $x_2 - y_2$
κ	bending curvature
λ	vector of Lagrange multipliers
λ_{sys}	vector of system Lagrange multipliers
ξ	non-dimensional term defined as $\frac{x}{l}$
ϕ	ball rotation angle
ψ	beam tip rotation angle
ω	base excitation frequency
μ	vector of Lagrange multipliers
$(\ddot{})$	second derivative with respect to time
$(\dot{})$	derivative with respect to time

CHAPTER 1

LITERATURE SURVEY

1.1 Autoparametric Interaction and Vibration Absorbers

Many mechanical systems can be modeled as a beam with a lumped mass, such as a wing of an airplane with a mounted engine, a robot arm carrying a welding tool, or a traffic light. Understanding the dynamics of those systems having flexible and slender beams is of great importance in vibration analyses to prevent catastrophic failures of the structures. Therefore, there is an extensive amount of experimental and numerical work on the responses of beams in the nonlinear dynamics and vibration field.

Lauraa et al. (1974) is concerned with the determination of natural frequencies and modal shapes of a clamped-free beam which carries a finite mass at the free end for different concentrated mass/beam mass ratios. Saitoa et al. (1982) investigated the parametric vibration of a simply-supported horizontal beam and solved the non-linear equation of motion by using the one-mode approximation and applying Galerkin's method. Verma et al. (1978) analyzed cantilevered beams of variable cross-sections. Rao et al. (1989) studied large deflection problems of a uniform cantilever beam under a rotational distributed loading. Gurgoze (1986) and Liu et al. (1988) also worked on the vibrations of the constrained beams carrying a heavy tip body.

Resonance is an important field of study in vibrations since systems have large deflections due to the small excitations at the resonance frequencies, which may lead to failures. Autoparametric resonance is a special case of the internal resonance that occurs in non-linear, multi-degree of freedom systems. The autoparametric name comes from the source of the motion being the system itself rather than being an external loading as in the case of parametric vibration. Most of the research on modal interactions has concentrated on the autoparametric resonances in which the linear natural frequencies are commensurate or nearly commensurate. This type of resonance leads to an energy

transfer from higher modes to lower modes. Therefore, exciting a high frequency mode may produce a large amplitude response in a low frequency mode in the system. Autoparametric interaction may occur whenever the natural frequencies of the normal modes, ω_i , satisfy a linear algebraic relationship known as internal resonance condition (Ibrahim, 1985) . This condition can be defined as

$$\sum k_i \omega_i \quad (1.1)$$

where k_i are integers and the order of the internal resonance is the sum of the absolute k_i values (Ibrahim, 1995).

Therefore, autoparametric interaction may occur in a two degree of freedom system with quadratic non-linearity if

$$\omega_1 = \Omega, \omega_2 = 2\omega_1 \quad (1.2)$$

and

$$\omega_2 = \Omega, \omega_2 = 2\omega_1 \quad (1.3)$$

where ω_1, ω_2 and Ω are the lower mode, higher mode, and the excitation frequencies, respectively. A combination resonance with an external excitation may also occur if the condition

$$\Omega = \omega_1 + \omega_2 \text{ and } \omega_2 = 2\omega_1 \quad (1.4)$$

is satisfied.

Similarly, for a system having n^{th} order nonlinearity, the autoparametric resonance condition is given by

$$\omega_1 = \Omega, \omega_2 = n\omega_1 \quad (1.5)$$

and

$$\omega_2 = \Omega, \omega_2 = n\omega_1. \quad (1.6)$$

For more information on this field, one may refer to Nayfeh et al. (1979).

Autoparametric resonance, being one of the most interesting and challenging phenomena, has been studied by numerous authors including Ibrahim et al. (1975), Haddow et al. (1984), Hatwall et al. (1983), and Ibrahim et al. (1978). In addition, Nayfeh et al. (1994) and Nayfeh et al. (1995) studied energy transfers from high to low frequency modes in structures.

Understanding details of autoparametric resonance lead researchers to use the idea of mode interaction in vibration absorption. Therefore, various models of autoparametric vibration absorbers were studied. Haxton et al. (1972) studied a two degree of freedom system consisting of a main linear spring mass system with a beam tip mass attachment. Under harmonic excitation, they showed that the motion of the beam tip mass reduced the amplitude of vibration of the larger mass. Hatwal (1982) investigated a two degree of freedom system at moderately high excitation frequency. Nayfeh (1983) used the method of multiple scales for the solution of a two degree of freedom system having multi-frequency parametric excitation. Ashworth et al. (1986) studied the resonant behavior of general structures under single frequency harmonic excitation. Mustafa et al. (1994) and Mustafa et al. (1995) analyzed the dynamics of a large flexible column with a tip mass-pendulum arrangement theoretically and experimentally. Authors showed that the pendulum could be used as an effective autoparametric vibration absorber under periodic base excitation for columns. Cuvalci et al. (1996) studied the dynamic response of a beam-tip mass-pendulum system subjected to a periodic excitation. They solved the nonlinear equations of motion of the system in order to verify the autoparametric interaction between the modes. They also proved the existence of the energy transfer between the modes when the frequency of the beam is twice the frequency of the pendulum. Ertas et al. (2000) investigated the same system to understand the effectiveness of passive vibration absorbers on structures whose orientation varies. The same system is studied under random excitation by Cicek et al. (2002). Chakraborty et al. (1999) investigated the free and forced response of a nonlinear travelling beam having an intermediate guide. They showed that the presence of the guide reduces the vibration

level by avoiding resonance conditions. Tondl (1998) used an inverted pendulum as a passive vibration absorber and compared the results with traditional pendulum absorber results.

1.2 Flexible Multibody Formulations

Roots of the multibody dynamics go back to 1668 when Newton first introduced the dynamics of the particles. Later on, in 1776, Euler defined the rigid body dynamics that became the essential part of multibody systems analyses. For the modeling of the joints and constraints, he used the free body principle and the resultant equations of motion were named Newton-Euler equations. In 1743, d'Alembert proposed the virtual work method to find system equations. However, a systematic analysis of the constrained mechanical systems was proposed by Lagrange in 1788. Lagrange applied variational principles to the total kinetic and potential energy of the system to find the non-linear equations of motion in which constraints are treated explicitly as extra equations by using Lagrange Multipliers.

The first studies in multibody systems were on the dynamics of the rigid bodies which were related to gyro-dynamics, the mechanism theory, and biomechanics. A good review of this topic is given by Schiehlen (1997). One of the first formalisms is given by Hooker et al. (1965) in which he analyzed the satellites interconnected with spherical joints. Another formulation was published in 1967 by Roberson et al. (1967). Wittenburg (1977) wrote the first textbook on multibody dynamics in which he explained rigid body kinematics and dynamics as well as general multibody systems. In 1988, Nikravesh (1988) provided information about the computer-aided analysis of multibody systems in his textbook. Haug (1989) provided basic methods of the computer-aided kinematics and dynamics for spatial and planar systems. Many more authors provided text-books in the field of kinematic and dynamic simulations of multibody systems such as Roberson et al. (1988), Huston (1990), and Jalon et al. (1994).

Until now, discussed papers and textbooks were related to the multibody systems consisting of rigid bodies. However, in many applications, bodies undergo large deformations, which necessitate the modeling of the flexible bodies. Flexible multibody systems have attracted many researchers and several flexible multibody formulations have been established such as the floating frame of reference method, incremental finite element co-rotational method, and the large rotation vector method. Agrawal et al. (1985) proposed the component mode synthesis method in which each elastic component is identified by three sets of modes: rigid body, reference modes, and normal modes. Rigid body modes are used to describe the rigid body translation and large rotations of a body reference, reference modes are used to define a unique displacement field, and the normal modes are used to define the deformation relative to the body reference. An alternative formulation was proposed by Yoo et al. (1986) in which a lumped mass finite element structural analysis formulation is used to generate deformation modes. In the floating reference frame formulation, a mixed set of absolute and local deformation coordinates are used to define the configuration of the deformable body (Song, et al., 1980; Shabana, 1996; Shabana, 2005). This method became the most widely-used approach due to its straight-forward nature. However, the mass matrix, centrifugal, and coriolis forces appear to be highly nonlinear. The incremental finite element approach uses rotation angles as nodal variables, which lead to linearized kinematic equations. Therefore, models obtained by using incremental finite elements cannot describe the exact rigid body displacements (Rankin, et al., 1986). In order to solve this problem in the incremental finite element approach, a different approach called the large rotation vector formulation has been proposed. In this method, finite rotations are employed instead of infinitesimal rotations, which results in an exact modeling of the rigid body displacements (Simo, et al., 1986).

Most of the methods explained above suffer from highly nonlinear terms inside the mass matrix, centrifugal, and coriolis forces. Therefore, a new approach called the absolute nodal coordinate formulation (ANCF) was proposed for the solution of large deformation problems (Escalona, et al.; 1997, Escalona, et al.; 1998, Shabana, 2005). In this formulation, instead of the angle of rotations, absolute slopes are used as nodal variables.

The resulting mass matrix is constant and symmetric, and the vector of centrifugal and coriolis forces is identically equal to zero if one chooses the center of mass of the flexible body as the origin of the reference frame. Since absolute coordinates are used, formulations of joint constraints and generalized external forces become much simpler than the other formulations. Moreover, beams and plates that are treated as non-isoparametric elements in the classical finite element approach become isoparametric with the absolute nodal coordinate formulation. In this method, shape functions and the nodal coordinates are used to define the position of an arbitrary point on the flexible body. Therefore, depending on the system under investigation and element type, equations of the shape functions vary. Schwertassek et al. (1999) gives an informative paper related to the selection of the shape functions.

In the absolute nodal coordinate formulation, various methods are used to derive the generalized elastic forces of the flexible body. One of the approaches is called elastic line. In this approach, a local element coordinate system is used to describe the element deformation and the classical Euler-Bernoulli beam theory is used to find the strain energy. Two element frames can be used with this approach; the pinned and the tangent frames (Escalona, et al., 1997; Berzeri, et al., 1998; Berzeri, et al., 2001). In the pinned frame, one of the axes passes through the two nodes of the element, while the tangent frame is rigidly attached to one of the nodes of the element. Gerstmayr et al. (2008) stated that the normal forces obtained using the elastic line approach appear to be incorrect, and he proposed a modification for this method. The second approach is called continuum mechanics. This approach uses arc length to define the element deformation instead of using the local coordinate system. Berzeri et al. (2000) proposed four new force models having different degrees of complexity by using the continuum mechanics approach. The models account for the nonlinearities in the strain-displacement relationships, unlike the elastic line models. Sopanen et al. (2003) and Sugiyama et al. (2006) focused on the description of the elastic forces in three-dimensional beam elements. Two-dimensional shear deformable beam elements were studied by Omar et al. (2001) and Dufva et al. (2005). Shabana et al. (2008) investigated three different nonlinear constitutive models

for the elastic force calculations of the materials that are considered incompressible and nearly incompressible. These models were written as compressible Neo-Hookean, incompressible Neo-Hookean, and Mooney-Rivlin constitutive models. Cable elements were studied by Gerstmayr et al. (2006) and Sugiyama et al. (2003). In addition to the extensive work done on nonlinear cables and beams, Dmitrochenko et al. (2003), Yoo et al. (2004), Dmitrochenko (2008), and Matikainen (2009) studied plate and shell elements. A good review on different types of beam and plate elements can be found in Shabana (2008) and Yoo et al. (2006).

The issue of internal damping in the absolute nodal coordinate formulation was studied by Garcia-Vallejo et al. (2005). His model was based on linear viscoelasticity and did not show energy dissipation under rigid body motion. On the other hand, Lee et al. (2009) proposed a frequency-dependent generic external damping model and compared it with the proportional and quadratic damping models.

Many studies have been proposed for the numerical solutions of elastic bodies undergoing large deflections and displacements, but there have been few attempts to experimentally validate their numerical predictions. Yoo et al. did extensive work on the experimental validation of the absolute nodal coordinate formulation. The first paper comparing the results of ANCF simulations and experiments was written by Yoo et al. (2003). In the paper, they investigated free vibrations of a thin cantilever beam with end point weight under gravity and showed the accuracy of the ANCF method in large deformation problems by performing experiments and computer simulations (Yoo, et al., 2004). Moreover, the same authors worked on the analysis of a thin clamped plate that underwent large vibrations due to an attached end point weight and showed the validity of the absolute nodal coordinate formulation by the real experiments (Yoo, et al., 2004). In a recent study, Yoo et al. (2006) investigated the large deformation of a flexible beam having periodic base excitation. They modeled external damping of the system by adding proportional damping terms into the equations of motion of the system and compared the numerical results obtained by ANCF with the experimental findings.

Shabana (1998) proposed a procedure for the efficient implementation of the absolute nodal coordinate formulation for flexible multibody dynamics. This procedure utilized the fact that the mass matrix obtained by using ANCF is constant and symmetric. In this technique, Cholesky coordinates were introduced by defining a velocity transformation obtained from the decomposition of the mass matrix of the deformable body. The inertia matrix associated with the Cholesky coordinates is the identity matrix that allows an efficient solution to the equations of motion of the flexible multibody system. A demonstration of this method in flexible multibody applications was done by Yakoub et al. (1999).

The governing equations of motion of the flexible multibody systems are often modeled using algebraic constraint equations which lead to a combined set of ordinary differential and algebraic equations. These equation sets are called *Differential Algebraic Equations* (DAE's) in literature. Solutions to DAE's are not as straight-forward as ordinary differential equations (ODE's). Specialized numerical techniques have been developed for the solution of DAE's, and these techniques can be separated into two main classes: namely, State-Space Methods and Direct Integration Methods.

In state-space methods, the DAE problem is first reduced to a smaller dimension ODE problem; as a result, one can benefit from the extensive knowledge associated to ODE solvers. The first class of techniques eventually applied to the solution of DAE's was published by Gear (1971) for the solutions of ODE's. Since then, many researchers, such as Liang et al. (1987), Potra et al. (1991), Yen (1993) and Negrut et al. (2005), challenged the efficient and accurate solutions of DAE's using state-space methods. Moreover, state-spaced based implicit numerical integration methods for DAE's were proposed by Haug et al. (1997), Wu et al. (2000), and Negrut et al. (2003).

Although state-spaced techniques have the advantage of using the well-known ODE algorithms, they are computationally intensive, especially when dealing with large multibody systems that contain flexible bodies, contact and impact. Therefore, researchers focused on direct integration methods that are significantly faster and more

accurate. Gear et al. (1985) proposed an automatic integration method for the solution of the Euler-Lagrange equations with constraints. In this method, he reduced the index of the equation from three to two and used the variable step, variable-order backward differentiation formula (BDF) to solve the equations. A new time integration algorithm called The Generalized- α Method for solving structural dynamic problems was presented by Chung et al. (1993). In particular, it was shown that the new method achieved high frequency dissipation while minimizing unwanted low frequency dissipation. Yen et al. (1998) showed that the α methods introduced no additional oscillations and were stable and convergent. Hilbert et al. (1977) developed a space-time, finite-element method for classical elastodynamics, which employs the discontinuous Galerkin method in time. Later, various researchers implemented Hilber-Hughes-Taylor (HHT) into the solution of DAE's (Hussein, et al., 2008; Negrut, et al., 2007; Jay, et al., 2007). Negrut et al. (2007) proposed a new algorithm based on the HHT implicit method for the solution of index-3 differential algebraic equations. They also discussed the implementation details of the algorithm such as error estimation, integration step size control, and nonlinear system stopping criteria. A similar study was performed previously by Gavrea et al. (2005) in which the Newmark method was used for the solution of index-3 DAE's. Strategies on the implementation details and a formal proof for local convergence of the method were also presented in his study. A promising paper on the low-order numerical integration formulas for rigid and flexible multibody dynamics was written by Negrut et al. (2009). In the paper, authors discussed the HHT, Newmark, and BDF integration methods and compared several metrics such as order of convergence, energy preservation, velocity kinematic constraint drift, and efficiency. In a recent paper, Shabana et al. (2009) proposed a two-loop implicit sparse matrix numerical integration procedure that ensured the kinematic constraint equations were satisfied at the position, velocity, and acceleration levels.

From the 1960s, when first studies on multibody dynamics of rigid bodies were presented, to today, many researchers have worked on multibody dynamics. The development of the flexible body formulation and advancements in the computational technology opened

new doors to researchers in the multibody dynamic analysis field. Today, many scientists from different countries, as shown in Figure 1-1, are working on the multibody systems. As shown in Figure 1-2, theoretical and computational methods and flexible multibody systems have been hot topics. Finally, it is believed that this interest in the multibody systems will increase in future with the increasing technological demands.

Argentina				1		1		1			3
Australia						1			1		2
Austria	1							1		2	4
Belgium	1	1				2	1	1	2	1	9
Bulgaria			1			1					2
Canada	1	2	4	3	2	1	3	2	5	2	25
China						1			2	1	4
China-Taipei										1	1
Croatia					1						1
Czech Republic		1			1				2		4
Denmark	1	2	1			1		1	1	1	8
Finland					1				1		2
France		1	2	1	1	3	2	2		2	14
Germany	4	1	2	2	6	9	8	4		8	57
India							1			1	2
Iran					1						1
Israel	1									1	2
Italy		1		2	1	3	3	6	2	3	21
Japan					1			2			3
Korea	1	1		1		1		2	2		8
Mexico									1		1
Netherlands		1			2	2	1		1	1	8
Norway		1			1						2
Poland	1				1	3	1	1	3		10
Portugal	2				2	2	2	3	2		13
Romania								1			1
Russia			1		2		1	1		3	8
Saudi Arabia				1	1		1				3
South Africa						1					1
Spain	1			2		2	2		1	2	14
Sweden			1		2	1	1				5
Switzerland	1				1				1		3
UK	2				2		1	1		1	7
Ukraine					4						4
USA	3	4	5	2	7	6		3	6	5	49
Σ	20	16	17	15	40	41	36	36	45	36	302
Year	97	98	99	00	01	02	03	04	05	06	Σ

Fig. 2 Countries of main authors of papers published in the years 1997 to 2006. The left bold lines indicate the country with the largest number of papers in the year considered. The bold lower line indicates the year in which the country considered collected the largest number of papers

Figure 1-1: Countries of main authors of papers published in the years 1997 to 2006(Schiehlen, 2007)

Theoretical and computational methods	5	2	5	1		7	4		4	3	41
Flexible multibody systems	3	2	4	5	8	6	4	2	2	4	40
Large deformation phenomena							1	3	1	3	8
Parameter identification and data models					1		1	1		2	5
Contact and impact problems	1	2			1	2	3	5		1	21
Control and mechatronics		1	1		2	2	1	3	2	5	17
Nonholonomic systems						1	2		1		4
Multiphysics problems			1		1	3	4	1	5	1	16
Algorithms, integration codes, software	1	1	1	4	3	2		5	2	3	28
Simulation and virtual reality	1	1	2	2	1	3	2		2		14
Experiments and numerical verification				1		1	1	3		2	8
Optimization of mechanical systems	3	1				4	1	2	3		14
Machines and mechanisms	2	3	1	1	2	2	2		2	1	22
Vehicle dynamics	2	1		1	4	2	1	4	5		25
Robotics and walking machines	1	2	2		7	2	3	1	5	4	27
Biomechanical problems	1					4			2	2	9
Education in multibody dynamics									3		3
Σ	20	16	17	14	40	41	36	36	45	36	302
Year	97	98	99	00	01	02	03	04	05	06	Σ

Fig. 1 Research topics of papers published in the years 1997 to 2006. The *left bold lines* indicate the topic with the largest number of papers in the year considered. The *bold lower line* indicates the year in which the topic considered collected the largest number of papers

Figure 1-2: Research topics of papers published in the year 1997 to 2006(Schiehlen, 2007)

CHAPTER 2

RIGID BODY DYNAMICS

2.1 Introduction

In physics, rigid body dynamics is the study of the motion of rigid bodies. A rigid body is an idealization of a solid body in which the distance between any two points of the body remains constant at all times. In other words, a rigid body is undeformable regardless of the external forces applied on it.

Unlike particles, which have three degrees of freedom (translation in three directions), rigid bodies in space have three translational and three rotational degrees for a total of six degrees of freedom. Moreover, rigid bodies occupy space and have geometrical properties such as mass moment of inertia, area moment of inertia, center of mass, etc.

Therefore, rigid body dynamics is used extensively in multibody system analysis whenever the body rotational motion is important but the deformation has negligible effect on the motion of the system.

2.2 Rigid Body Kinematics

Throughout this section, position, velocity, and acceleration equations of planar rigid bodies will be explained. Detailed information about spatial body kinematics can be found in reference Wittenburg (2008).

If $X - Y$ is selected to be the global inertial frame of reference and $X^i - Y^i$ defines the body coordinate system rigidly attached to point O of the body i , then referring to Figure 2-1, the global position vector of an arbitrary point P on the body i can be written as

$$\mathbf{r}_p^i = \mathbf{R}^i + \mathbf{u}_p^i \quad (2.1)$$

where $\mathbf{R}^i$ is the global position vector of the origin O of the coordinate system $X^i - Y^i$ and $\mathbf{u}_p^i$ is the global position of the point P with respect to origin O .

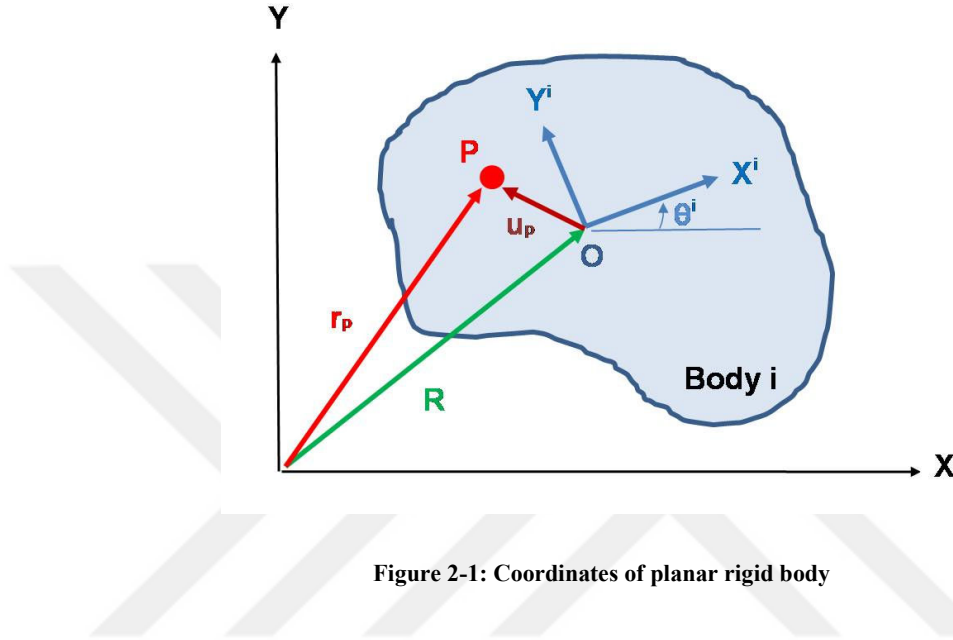


Figure 2-1: Coordinates of planar rigid body

If one defines the coordinates of point P in the body coordinate system $X^i - Y^i$ as $\overline{\mathbf{u}}_p^i$ then

$$\mathbf{u}_p^i = \mathbf{A}^i \overline{\mathbf{u}}_p^i \quad (2.2)$$

where $\mathbf{A}^i$ is the planar orthogonal transformation matrix defined as

$$\mathbf{A}^i = \begin{bmatrix} \cos \theta^i & -\sin \theta^i \\ \sin \theta^i & \cos \theta^i \end{bmatrix}. \quad (2.3)$$

Using equation 2.2, equation 2.1 can be re-written as

$$\mathbf{r}_p^i = \mathbf{R}^i + \mathbf{A}^i \overline{\mathbf{u}}_p^i. \quad (2.4)$$

Absolute velocity of the rigid body can be found by differentiating equation 2.4 with respect to time as

$$\dot{\mathbf{r}}_p^i = \dot{\mathbf{R}}^i + \dot{\mathbf{A}}^i \overline{\mathbf{u}}_p^i . \quad (2.5)$$

Using equation 2.3, the time derivative of the transformation matrix can be written as

$$\dot{\mathbf{A}}^i = \dot{\theta}^i \mathbf{A}_\theta^i \quad (2.6)$$

where the derivative of transformation matrix with respect to rotational coordinate θ^i is given by

$$\mathbf{A}_\theta^i = \begin{bmatrix} -\sin \theta^i & -\cos \theta^i \\ \cos \theta^i & -\sin \theta^i \end{bmatrix}. \quad (2.7)$$

Therefore equation 2.5 can be redefined as

$$\dot{\mathbf{r}}_p^i = \dot{\mathbf{R}}^i + \dot{\theta}^i \mathbf{A}_\theta^i \overline{\mathbf{u}}_p^i . \quad (2.8)$$

One can define the angular velocity vector $\boldsymbol{\omega}^i$ of body i as

$$\boldsymbol{\omega}^i = \dot{\theta}^i \mathbf{k} \quad (2.9)$$

where $\mathbf{k}$ is the unit vector along the axis of rotation that is perpendicular to the plane of motion.

As it is shown in Shabana (2005), the absolute velocity of an arbitrary point P on the rigid body i can be written in terms of angular velocity vector, $\boldsymbol{\omega}^i$, as

$$\dot{\mathbf{r}}_p^i = \dot{\mathbf{R}}^i + \boldsymbol{\omega}^i \times \mathbf{A}^i \overline{\mathbf{u}}_p^i = \dot{\mathbf{R}}^i + \boldsymbol{\omega}^i \times \mathbf{u}_p^i . \quad (2.10)$$

Following a similar procedure, absolute acceleration vector can be found by taking the time derivative of the absolute velocity vector as

$$\ddot{\mathbf{r}}_p^i = \ddot{\mathbf{R}}^i + \dot{\theta}^i \dot{\mathbf{A}}_\theta^i \overline{\mathbf{u}}_p^i + \ddot{\theta}^i \mathbf{A}_\theta^i \overline{\mathbf{u}}_p^i . \quad (2.11)$$

Defining the angular acceleration vector of body i as

$$\boldsymbol{\alpha}^i = \ddot{\theta}^i \mathbf{k} , \quad (2.12)$$

the absolute acceleration vector of an arbitrary point P on rigid body i can be written as (Shabana, 2005)

$$\ddot{\mathbf{r}}_p^i = \ddot{\mathbf{R}}^i + \boldsymbol{\omega}^i \times (\boldsymbol{\omega}^i \times \mathbf{u}_p^i) + \boldsymbol{\alpha}^i \times \mathbf{u}_p^i. \quad (2.13)$$

2.3 Rigid Body Dynamics

Using the principal of virtual work, the dynamic equilibrium condition for rigid body i can be written as

$$\delta W_i^i = \delta W_e^i + \delta W_c^i \quad (2.14)$$

where the virtual work of externally applied forces δW_e^i can be defined in terms of the vector of generalized external forces Q_e^i and the vector of generalized coordinates of the rigid body i as

$$\delta W_e^i = \mathbf{Q}_e^{iT} \delta \mathbf{q}^i \quad (2.15)$$

and the vector $\mathbf{q}^i$ is given by

$$\mathbf{q}^i = \begin{pmatrix} \mathbf{R}^i \\ \theta^i \end{pmatrix} \quad (2.16)$$

where $\mathbf{R}^i$ is the position vector of the reference point and θ^i is the angular orientation of the body i as shown in Figure 2-2.

The virtual work of the joint constraint forces acting on the rigid body i can be written as

$$\delta W_c^i = \mathbf{Q}_c^{iT} \delta \mathbf{q}^i \quad (2.17)$$

where $\mathbf{Q}_c^i$ is the vector of the generalized constraint forces.

If the reference point O shown in Figure 2-2 is selected as the center of mass of the rigid body i , then the virtual work of the inertia forces is given by (Shabana, 2001)

$$\delta W_i^i = \ddot{\mathbf{q}}^{iT} \mathbf{M}^i \delta \mathbf{q}^i \quad (2.18)$$

where $\mathbf{M}^i$ is the diagonal mass matrix of the planar body i defined by

$$\mathbf{M}^i = \begin{bmatrix} m^i & 0 & 0 \\ 0 & m^i & 0 \\ 0 & 0 & J^i \end{bmatrix} \quad (2.19)$$

and m^i is the mass and, J^i is the mass moment of inertia of the rigid body i .

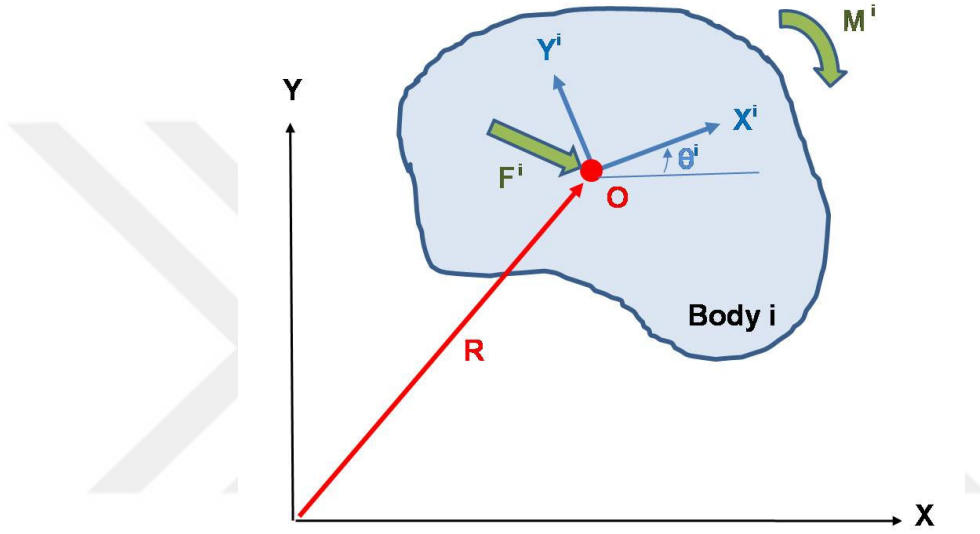


Figure 2-2: Planar Rigid body

Using equations 2.15, 2.17 and 2.18, the dynamic equilibrium condition can be written as

$$(\mathbf{M}^i \ddot{\mathbf{q}}^i - \mathbf{Q}_e^i - \mathbf{Q}_c^i)^T \delta \mathbf{q}^i = 0. \quad (2.20)$$

During the derivation of equation 2.20, constraint forces acting on the body i due to its interaction with other bodies in the system are included. Therefore, elements of the vector $\mathbf{q}^i$ can be treated as independent. Consequently, equation 2.20 can be re-written as

$$\mathbf{M}^i \ddot{\mathbf{q}}^i = \mathbf{Q}_e^i + \mathbf{Q}_c^i. \quad (2.21)$$

2.4 Systems of Rigid Bodies

Using equation 2.21, equations of motion of a multibody system consisting of multiple interconnected bodies as in Figure 2-3 can be defined as

$$\mathbf{M} \ddot{\mathbf{q}} = \mathbf{Q}_e + \mathbf{Q}_c \quad (2.22)$$

where $\mathbf{M}$ is the system mass matrix given by

$$\mathbf{M} = \begin{bmatrix} \mathbf{M}^1 & 0 & \dots & 0 \\ 0 & \mathbf{M}^2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mathbf{M}^{n_b} \end{bmatrix} \quad (2.23)$$

and n_b is the number of interconnected bodies.

The vector of system generalized coordinates $\mathbf{q}$, the vector of system generalized external forces $\mathbf{Q}_e$, and the vector of system generalized constraint forces $\mathbf{Q}_c$ are defined as

$$\mathbf{q} = \begin{bmatrix} q^1 \\ q^2 \\ \vdots \\ q^{n_b} \end{bmatrix}, \quad \mathbf{Q}_e = \begin{bmatrix} Q_e^1 \\ Q_e^2 \\ \vdots \\ Q_e^{n_b} \end{bmatrix}, \quad \mathbf{Q}_c = \begin{bmatrix} Q_c^1 \\ Q_c^2 \\ \vdots \\ Q_c^{n_b} \end{bmatrix}. \quad (2.24)$$

If the system has n_i constraints, then system constraint equations are given by

$$\mathbf{C}(\mathbf{q}, t) = \begin{bmatrix} C_1(\mathbf{q}, t) \\ C_2(\mathbf{q}, t) \\ \vdots \\ C_{n_i}(\mathbf{q}, t) \end{bmatrix} = 0 \quad (2.25)$$

and the jacobian of the constraints is given by

$$\mathbf{C}_q(\mathbf{q}, t) = \begin{bmatrix} \frac{\partial C_1(\mathbf{q}, t)}{\partial \mathbf{q}} \\ \frac{\partial C_2(\mathbf{q}, t)}{\partial \mathbf{q}} \\ \vdots \\ \frac{\partial C_{n_i}(\mathbf{q}, t)}{\partial \mathbf{q}} \end{bmatrix} = 0. \quad (2.26)$$

Using vector of Lagrange Multipliers λ , one can write generalized constraint forces Q_c as (Shabana, 2001)

$$Q_c = -C_q(q, t)^T \lambda \quad (2.27)$$

where $C_q(q, t)$ is given by equation 2.26 and λ is defined as

$$\lambda = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_{n_i} \end{bmatrix}. \quad (2.28)$$

Therefore, equation 2.22 can be written as

$$M \ddot{q} + C_q^T \lambda = Q_e \quad (2.29)$$

where M, q, C_q, λ, Q_e are given by equations 2.23, 2.24, 2.26, and 2.28, respectively.

Theoretical proof of equation 2.29 can be found in Shabana (2001).

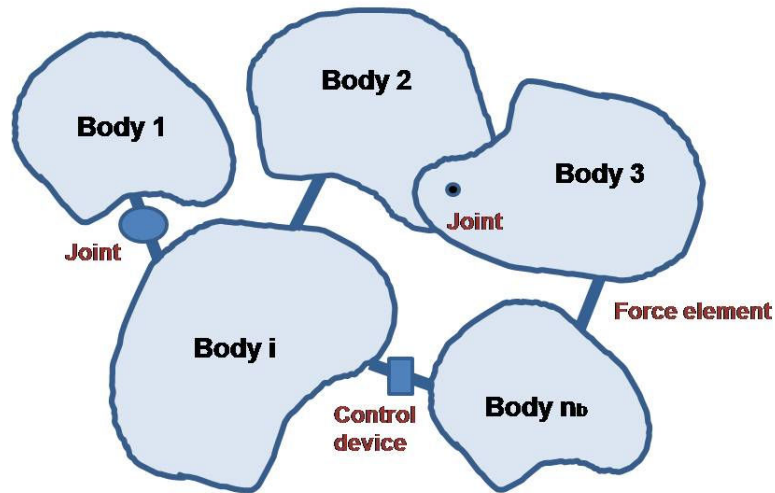


Figure 2-3: Multibody system

In equation 2.29, unknowns are the vector of accelerations $\ddot{\mathbf{q}}$ and the vector of Lagrange Multipliers $\boldsymbol{\lambda}$. If a system has n generalized coordinates and n_c constraint equations, then the total number of unknowns is $n + n_c$, which is more than the number of equations n . In order to solve the system, one needs to have n_c additional equations. These n_c additional equations are found using the constraint equations 2.25.

Taking derivatives of equations 2.25 once and twice with respect to time, one obtains

$$\mathbf{C}_q \dot{\mathbf{q}} = -\mathbf{C}_t \quad (2.30)$$

$$\mathbf{C}_q \ddot{\mathbf{q}} = \mathbf{Q}_d \quad (2.31)$$

where subscript t denotes the partial differentiation with respect to time and the vector $\mathbf{Q}_d$ is given by

$$\mathbf{Q}_d = -\mathbf{C}_{tt} - (\mathbf{C}_q \dot{\mathbf{q}})_q \dot{\mathbf{q}} - 2\mathbf{C}_{qt} \dot{\mathbf{q}}. \quad (2.32)$$

Combining equations 2.29 and 2.31, the constrained equations of motion of interconnected rigid bodies are defined as

$$\begin{bmatrix} \mathbf{M} & \mathbf{C}_q^T \\ \mathbf{C}_q & 0 \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{q}} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \mathbf{Q}_e \\ \mathbf{Q}_d \end{bmatrix}. \quad (2.33)$$

CHAPTER 3

FLEXIBLE BODY DYNAMICS

3.1 Introduction

Several techniques for the formulation of flexible multibody systems have been proposed such as the floating frame of reference method, the incremental finite element co-rotational method, and the large rotation vector method. Among these methods, the floating frame of reference formulation is the most widely used one. However, usage of this method is limited to small deformation cases. Recently, a new method called the absolute nodal coordinate formulation (ANCF) has been introduced by Escalona et al. (1997) and Escalona et al. (1998) for the analysis of large rotation and large deformation problems faced in flexible multibody dynamics. In the absolute nodal coordinate formulation (ANCF), global displacement coordinates and global slopes are used to define the element nodal coordinates. Therefore, beam, plate, and shell elements that have been established as non-isoparametric elements in classical finite element formulations become isoparametric elements with this new technique. As a result, the exact modeling of the rigid body dynamics can be obtained using the element shape function and absolute nodal coordinates. Moreover, the usage of this new method results in a constant system mass matrix and, therefore, the centrifugal and coriolis forces become zero. Another advantage of using the absolute nodal coordinate formulation is the simplicity of the generalized force terms and constraint equation terms in the system equation of motion due to the usage of absolute nodal coordinates.

Because of the above mentioned advantages of the absolute nodal coordinate formulation, this technique will be used in the solution of the system under investigation and the theory of this method will be explained in the remaining sections.

3.2 Displacement Field

In the finite element theory, the domain of the body is divided into regions called elements. Choosing small enough regions justifies the use of low-order polynomials to describe the displacement field of the element on the body. Therefore, the equation of motion of the element can be achieved by using these low order polynomials, and the standard finite element assembly techniques can be used to define the overall equations of motion of the flexible body.

Finite element approximations are based on the concept of separation of variables. In this theory, two sets of functions are defined. One set depends on the spatial coordinates and the other set depends only on time. By using the separation of variables, the displacement field of the element can be written as the product of spatial local coordinates and time dependent coefficients. One can select points on the element, called nodes, and assign variables such as displacements, rotations, and slopes as nodal coordinates. Using these nodal coordinate values in the reference state, one can write a set of algebraic equations and determine the time dependent coefficients in terms of the known nodal coordinates.

In the absolute nodal coordinate formulation, element nodal coordinates are defined in the inertial frame and a global shape function that has a complete set of rigid body modes is used together with these nodal coordinates in order to define the global position vector of an arbitrary point on the element.

In this dissertation, the dynamic system under investigation is assumed to be planar. Therefore, a planar beam element will be used through out the solution.

Referring to Figure 3-1, the global position vector $\mathbf{r}$ of an arbitrary point P on the element is defined in terms of the nodal coordinates and the element shape function as

$$\mathbf{r} = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = \mathbf{S}\mathbf{e} \quad (3.1)$$

where $\mathbf{S}$ is the global shape function and $\mathbf{e}$ is the vector of element nodal coordinates defined as

$$\mathbf{e} = [e_1 \ e_2 \ e_3 \ e_4 \ e_5 \ e_6 \ e_7 \ e_8]^T. \quad (3.2)$$

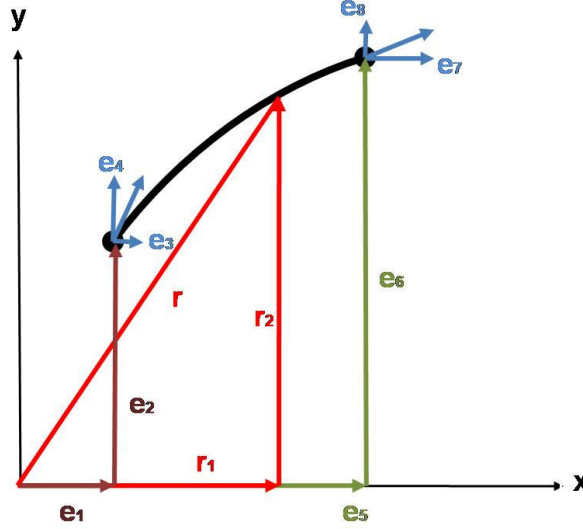


Figure 3-1: Planar beam element

The elements of the vector of nodal coordinates are defined as (Escalona, et al., 1997)

$$\begin{aligned} e_1 &= r_1(x=0), \quad e_2 = r_2(x=0), \quad e_3 = \frac{\partial r_1(x=0)}{\partial x}, \quad e_4 = \frac{\partial r_2(x=0)}{\partial x}, \\ e_5 &= r_1(x=l), \quad e_6 = r_2(x=l), \quad e_7 = \frac{\partial r_1(x=l)}{\partial x}, \quad \& \quad e_8 = \frac{\partial r_2(x=l)}{\partial x} \end{aligned} \quad (3.3)$$

where l is the beam element length and x is the axial coordinate that defines the position of an arbitrary point on the element in the undeformed state. e_1, e_2, e_5, e_6 are the absolute displacement coordinates and e_3, e_4, e_7, e_8 are the global slopes of the nodes.

Since one uses global coordinates, there is no reason to use different polynomials to interpolate the displacement components. Using a cubic polynomial for both components of the displacement, the element shape function can be defined as (Escalona, et al., 1997)

$$\mathbf{S} = \begin{bmatrix} s_1 & 0 & s_2 & 0 & s_3 & 0 & s_4 & 0 \\ 0 & s_1 & 0 & s_2 & 0 & s_3 & 0 & s_4 \end{bmatrix} \quad (3.4)$$

where

$$\begin{aligned} s_1 &= 1 - 3\xi^2 + 2\xi^3, & s_2 &= l(\xi - 2\xi^2 + \xi^3) \\ s_3 &= 3\xi^2 - 2\xi^3, & s_4 &= l(\xi^3 - \xi^2) \end{aligned} \quad (3.5)$$

and $\xi = \frac{x}{l}$.

The shape function presented above contains a complete set of rigid body modes that can define the arbitrary rigid body translational and rotational displacement, and the element defined by this shape function is an isoparametric element. Detailed discussion on this topic can be found in Shabana (2005).

3.3 Mass Matrix

Kinetic energy of the finite element can be written as

$$T = \frac{1}{2} \int \rho \dot{\mathbf{r}}^T \dot{\mathbf{r}} dV. \quad (3.6)$$

Using equation 3.1, kinetic energy can be defined as

$$T = \frac{1}{2} \dot{\mathbf{e}}^T \left(\int \rho \mathbf{S}^T \mathbf{S} dV \right) \dot{\mathbf{e}} = \frac{1}{2} \dot{\mathbf{e}}^T \mathbf{M}_a \dot{\mathbf{e}} \quad (3.7)$$

where V is the element volume, ρ is the mass density of the beam element material, and $\mathbf{M}_a$ is the mass matrix of the element. Using equation 3.7, the mass matrix of the element can be calculated as

$$\mathbf{M}_a = \int \rho \mathbf{S}^T \mathbf{S} dV$$

$$\mathbf{M}_a = \begin{bmatrix} \frac{13m}{35} & 0 & \frac{11lm}{210} & 0 & \frac{9m}{70} & 0 & -\frac{13lm}{420} & 0 \\ 0 & \frac{13m}{35} & 0 & \frac{11lm}{210} & 0 & \frac{9m}{70} & 0 & -\frac{13lm}{420} \\ \frac{11lm}{210} & 0 & \frac{l^2m}{105} & 0 & \frac{13lm}{420} & 0 & -\frac{l^2m}{140} & 0 \\ 0 & \frac{11lm}{210} & 0 & \frac{l^2m}{105} & 0 & \frac{13lm}{420} & 0 & -\frac{l^2m}{140} \\ \frac{9m}{70} & 0 & \frac{13lm}{420} & 0 & \frac{13m}{35} & 0 & -\frac{11lm}{210} & 0 \\ 0 & \frac{9m}{70} & 0 & \frac{13lm}{420} & 0 & \frac{13m}{35} & 0 & -\frac{11lm}{210} \\ -\frac{13lm}{420} & 0 & -\frac{l^2m}{140} & 0 & -\frac{11lm}{210} & 0 & \frac{l^2m}{105} & 0 \\ 0 & -\frac{13lm}{420} & 0 & -\frac{l^2m}{140} & 0 & -\frac{11lm}{210} & 0 & \frac{l^2m}{105} \end{bmatrix} \quad (3.8)$$

where m is the element mass and l is the element length. Equation 3.8 proves that the mass matrix of the element is symmetric and constant. Escalona et al. (1998) showed that the use of this mass matrix leads to the exact modeling of the rigid body inertia.

3.4 Generalized Elastic Forces

In order to develop the equations of motion of the beam element, generalized elastic forces, $\mathbf{Q}_k$, corresponding to the beam element have to be defined. If the strain energy of the element is U , then the vector of elastic forces is defined as

$$\mathbf{Q}_k = \left(\frac{\partial U}{\partial \mathbf{e}} \right)^T. \quad (3.9)$$

There are two main approaches to derive the strain energy of the element in the absolute nodal coordinate formulation studies. One of the approaches is called elastic line. In this approach, the local element coordinate system is used to describe the element deformation and the classical Euler-Bernoulli beam theory is used to find the strain energy. Two element frames can be used with this approach: the pinned and the tangent frames. In the pinned frame, one of the axes passes through the two nodes of the element, while the tangent frame is rigidly attached to the one of the nodes of the element. The second approach is called continuum mechanics. This approach uses arc length to define the element deformation instead of using the local coordinate system. For more

information on these approaches, one can refer to Escalona et al. (1997), Berzeri et al. (1998), and Berzeri et al. (2000).

3.4.1 *Elastic Line Approach*

Using the Euler-Bernoulli beam theory (Timoshenko, 1987), strain energy due to longitudinal and transverse deformations is defined as

$$U = \frac{1}{2} \int_0^l \left[E a \left(\frac{\partial u_l}{\partial x} \right)^2 + E I \left(\frac{\partial^2 u_t}{\partial x^2} \right)^2 \right] dx \quad (3.10)$$

where E is the young's modulus, a is the cross-sectional area, I is the area moment of inertia, u_l is the longitudinal deflection, and u_t is the transverse deflection of the beam element.

If P_1 in Figure 3-3 is the position of an arbitrary point in deformed configuration, then the displacement of P_1 relative to point O can be written as

$$\mathbf{u} = \mathbf{r}_p - \mathbf{r}_o = (\mathbf{S} - \mathbf{S}_0)\mathbf{e} \quad (3.11)$$

where $\mathbf{S}$ is the shape function defined in equation 3.4 and $\mathbf{S}_0$ is the shape function defined at point O .

Referring to Figure 3-2 and Figure 3-3, longitudinal and transverse deformations of the beam can be written as

$$\mathbf{u}_d = \begin{pmatrix} u_l \\ u_t \end{pmatrix} = \begin{pmatrix} \mathbf{u}^T \mathbf{t} - x \\ \mathbf{u}^T \mathbf{j} \end{pmatrix} \quad (3.12)$$

where x is the position of the point P_0 in undeformed state.

Using equation 3.12, derivatives of the longitudinal and transverse deformations in the strain energy equation can be found as

$$\frac{\partial u_l}{\partial x} = \frac{\partial (\mathbf{u}^T \mathbf{t} - x)}{\partial x} = \mathbf{e}^T \mathbf{S}'^T \mathbf{t} - 1 \quad (3.13)$$

$$\frac{\partial^2 u_t}{\partial x^2} = \frac{\partial^2 (u^T j)}{\partial x^2} = (S'' e)^T j \quad (3.14)$$

where

$$S' = \frac{\partial S}{\partial \xi}, \quad S'' = \frac{\partial^2 S}{\partial \xi^2}. \quad (3.15)$$

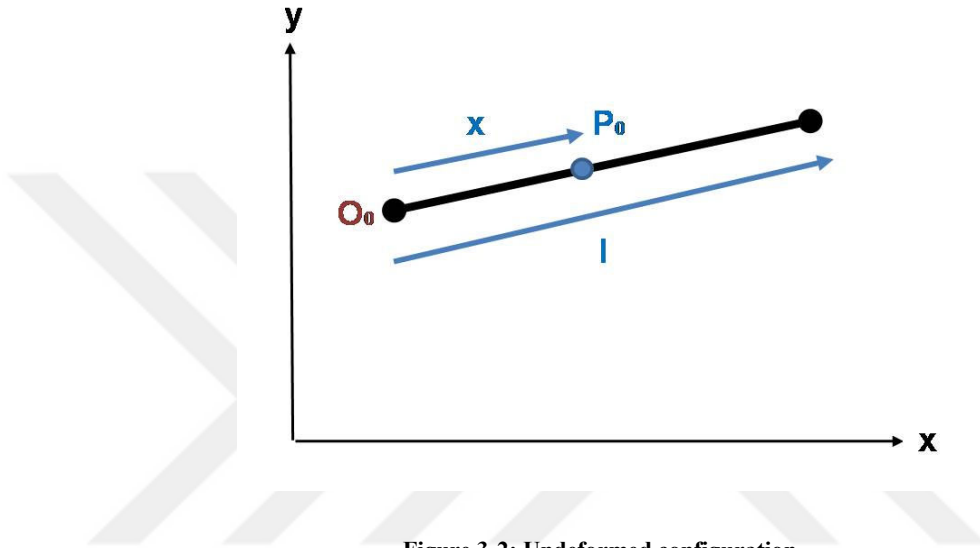


Figure 3-2: Undeformed configuration

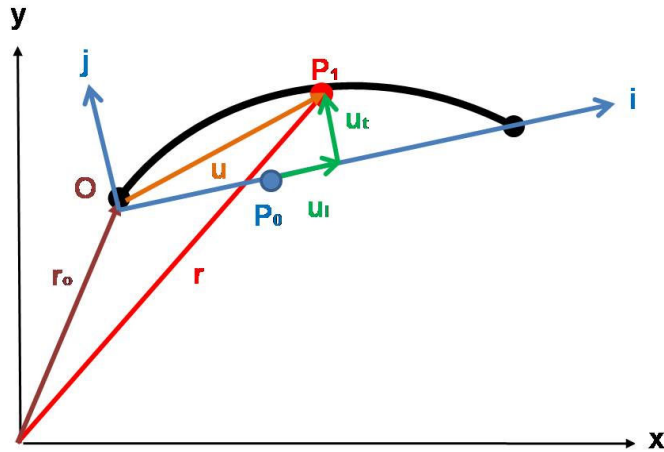


Figure 3-3: Deformed configuration

Putting equations 3.13 and 3.14 into strain energy equation 3.10 and arranging the terms results in (Berzeri, et al., 1998)

$$U = \frac{1}{2} (E a (l - 2 \mathbf{e}^T C_1 + \mathbf{e}^T C_2 \mathbf{e}) + E I \mathbf{e}^T C_3 \mathbf{e}) \quad (3.16)$$

where

$$C_1 = \int_0^1 \mathbf{S}'^T \mathbf{i} \, d\xi \quad (3.17)$$

$$C_2 = \frac{1}{l} \int_0^1 \mathbf{S}'^T \mathbf{i} \cdot \mathbf{i}^T \mathbf{S}' \, d\xi \quad (3.18)$$

$$C_3 = \frac{1}{l^3} \int_0^1 \mathbf{S}''^T \mathbf{j} \cdot \mathbf{j}^T \mathbf{S}'' \, d\xi. \quad (3.19)$$

Note that the unit vectors $\mathbf{i}$ and $\mathbf{j}$ depend on nodal coordinates and the selection of the local coordinate frame will affect their definitions.

If i_1, i_2 and j_1, j_2 are components of the unit vectors $\mathbf{i}$ and $\mathbf{j}$ in the inertial frame, respectively, and $\mathbf{S}_1$ and $\mathbf{S}_2$ are the 1st and 2nd rows of the shape function $\mathbf{S}$, then the vectors $\mathbf{i}$ and $\mathbf{j}$ and shape function $\mathbf{S}$ can be written in the following partitioned form

$$\mathbf{i} = \begin{pmatrix} i_1 \\ i_2 \end{pmatrix}, \quad \mathbf{j} = \begin{pmatrix} j_1 \\ j_2 \end{pmatrix}, \quad \mathbf{S} = \begin{pmatrix} S_1 \\ S_2 \end{pmatrix}. \quad (3.20)$$

Similarly, one can write derivatives in equation 3.15 as

$$\mathbf{S} = \begin{pmatrix} S_1 \\ S_2 \end{pmatrix}, \quad \mathbf{S}' = \begin{pmatrix} S'_1 \\ S'_2 \end{pmatrix}, \quad \mathbf{S}'' = \begin{pmatrix} S''_1 \\ S''_2 \end{pmatrix}. \quad (3.21)$$

Using equations 3.20 and 3.21 with equations 3.17, 3.18, and 3.19, one can find the strain energy equation (3.16) as (Escalona, et al., 1998)

$$U = \frac{1}{2} (E a l - 2 \mathbf{e}^T (i_1 A_1 + i_2 A_2) + \mathbf{e}^T (i_1^2 A_{11} + i_1 i_2 (A_{12} + A_{21}) + i_2^2 A_{22}) \mathbf{e} + \mathbf{e}^T (j_1^2 B_{11} + j_1 j_2 (B_{12} + B_{21}) + j_2^2 B_{22}) \mathbf{e}) \quad (3.22)$$

where

$$\begin{aligned}
A_{11} &= \frac{E a}{l} \int_0^l S_1'^T S_1' d\xi, & A_{12} &= \frac{E a}{l} \int_0^l S_1'^T S_2' d\xi, \\
A_{21} &= \frac{E a}{l} \int_0^l S_2'^T S_1' d\xi, & A_{22} &= \frac{E a}{l} \int_0^l S_2'^T S_2' d\xi, \\
B_{11} &= \frac{E I}{l^3} \int_0^l S_1''^T S_1'' d\xi, & B_{12} &= \frac{E I}{l^3} \int_0^l S_1''^T S_2'' d\xi, \\
B_{21} &= \frac{E I}{l^3} \int_0^l S_2''^T S_1'' d\xi, & B_{22} &= \frac{E I}{l^3} \int_0^l S_2''^T S_2'' d\xi, \\
A_1 &= E a \int_0^l S_1'^T d\xi, & A_2 &= E a \int_0^l S_2'^T d\xi.
\end{aligned} \tag{3.23}$$

Therefore, using equations 3.9, 3.22, and 3.23 and rearranging the terms, the elastic force Q_k can be found as (Escalona, et al., 1998)

$$\begin{aligned}
Q_k &= A_{11} \mathbf{e} i_1^2 + A_{22} \mathbf{e} i_2^2 + (A_{12} + A_{21}) \mathbf{e} i_1 i_2 - A_1 i_1 - A_2 i_2 + B_{11} \mathbf{e} j_1^2 + B_{22} \mathbf{e} j_2^2 + \\
&\quad (B_{12} + B_{21}) \mathbf{e} j_1 j_2 + \left(\mathbf{e}^T A_{11} \mathbf{e} i_1 + \frac{1}{2} \mathbf{e}^T (A_{12} + A_{21}) \mathbf{e} i_2 - A_1^T \mathbf{e} \right) \left(\frac{\partial i_1}{\partial \mathbf{e}} \right)^T + \\
&\quad \left(\mathbf{e}^T A_{22} \mathbf{e} i_2 + \frac{1}{2} \mathbf{e}^T (A_{12} + A_{21}) \mathbf{e} i_1 - A_2^T \mathbf{e} \right) \left(\frac{\partial i_2}{\partial \mathbf{e}} \right)^T + \left(\mathbf{e}^T B_{11} \mathbf{e} j_1 + \frac{1}{2} \mathbf{e}^T (B_{12} + \right. \\
&\quad \left. B_{21}) \mathbf{e} j_2 \right) \left(\frac{\partial j_1}{\partial \mathbf{e}} \right)^T + \left(\mathbf{e}^T B_{22} \mathbf{e} j_2 + \frac{1}{2} \mathbf{e}^T (B_{12} + B_{21}) \mathbf{e} j_1 \right) \left(\frac{\partial j_2}{\partial \mathbf{e}} \right)^T.
\end{aligned} \tag{3.24}$$

3.4.1.1 Pinned Frame

As stated previously, the pinned frame has one of its axes passing through the two nodes of the element. Referring to Figure 3-4, the following relationship holds between unit vectors $\mathbf{i}$ and $\mathbf{j}$ for the two dimensional problems

$$\begin{pmatrix} i_1 \\ i_2 \end{pmatrix} = \begin{pmatrix} j_2 \\ -j_1 \end{pmatrix}. \tag{3.25}$$

Therefore,

$$\begin{pmatrix} \frac{\partial i_1}{\partial \mathbf{e}} \\ \frac{\partial i_2}{\partial \mathbf{e}} \end{pmatrix} = \begin{pmatrix} \frac{\partial j_2}{\partial \mathbf{e}} \\ -\frac{\partial j_1}{\partial \mathbf{e}} \end{pmatrix}. \tag{3.26}$$

The unit vector $\mathbf{i}$ can be written for the pinned frame as (Berzeri, et al., 1998)

$$\mathbf{i} = \begin{pmatrix} i_1 \\ i_2 \end{pmatrix} = \frac{1}{d} \begin{pmatrix} e_5 - e_1 \\ e_6 - e_2 \end{pmatrix} \quad (3.27)$$

where d is the distance between the two nodes of the element and is defined as

$$d = \sqrt{(e_5 - e_1)^2 + (e_6 - e_2)^2} . \quad (3.28)$$

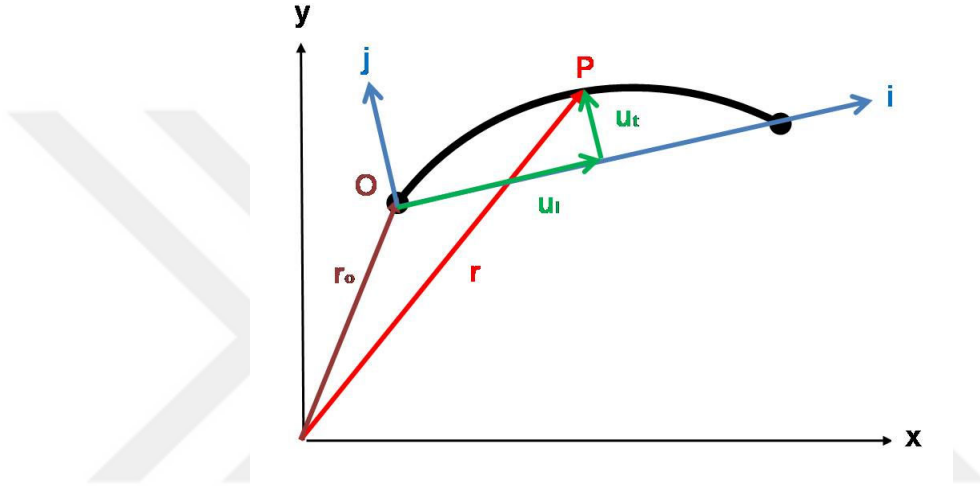


Figure 3-4: Pinned local coordinate system

Derivatives of i_1 and i_2 with respect to the nodal coordinates are written as (Berzeri, et al., 1998)

$$\left(\frac{\partial i_1}{\partial e} \right)^T = \frac{i_2^2}{d} \mathbf{I}_1 - \frac{i_1 i_2}{d} \mathbf{I}_2 \quad (3.29)$$

$$\left(\frac{\partial i_2}{\partial e} \right)^T = \frac{i_1^2}{d} \mathbf{I}_2 - \frac{i_1 i_2}{d} \mathbf{I}_1 \quad (3.30)$$

$$\mathbf{I}_1 = [-1 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0]^T, \ \mathbf{I}_2 = [0 \ -1 \ 0 \ 0 \ 0 \ -1 \ 0 \ 0]^T. \quad (3.31)$$

3.4.1.2 Tangent Frame

As shown in Figure 3-5, the tangent frame has its origin rigidly attached to the first node of the beam element. In this case, equations 3.25 and 3.26 still hold and the unit vector $\mathbf{i}$ and its derivative can be written in terms of nodal coordinates as (Berzeri, et al., 1998)

$$\mathbf{i} = \frac{1}{\sqrt{e_3^2 + e_4^2}} \begin{pmatrix} e_3 \\ e_4 \end{pmatrix} \quad (3.32)$$

$$\left(\frac{\partial \mathbf{i}_1}{\partial \mathbf{e}} \right)^T = \frac{e_4^2}{f^3} \mathbf{I}_3 - \frac{e_3 e_4}{f^3} \mathbf{I}_4 \quad (3.33)$$

$$\left(\frac{\partial \mathbf{i}_2}{\partial \mathbf{e}} \right)^T = \frac{e_3^2}{f^3} \mathbf{I}_4 - \frac{e_3 e_4}{f^3} \mathbf{I}_3 \quad (3.34)$$

where

$$\mathbf{I}_3 = [0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0]^T, \quad \mathbf{I}_4 = [0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 0]^T \quad (3.35)$$

$$f = \sqrt{e_3^2 + e_4^2}. \quad (3.36)$$

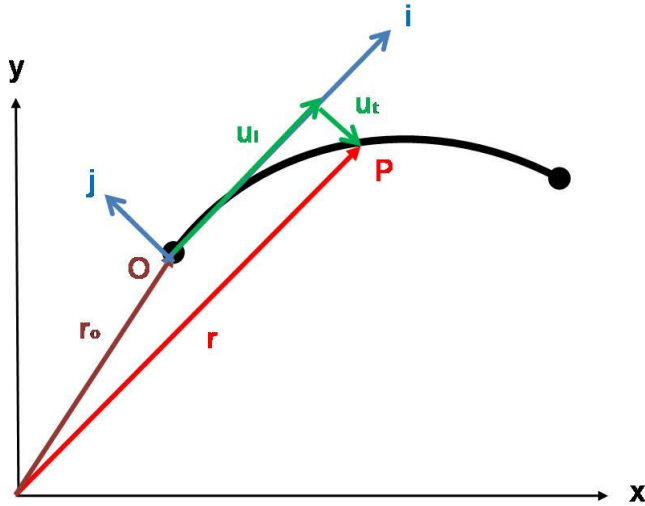


Figure 3-5: Tangent local coordinate system

3.4.2 Continuum Mechanics Approach

In this section, generalized elastic forces of the beam element will be found using the continuum mechanics approach. Throughout the derivation, the beam element cross-section is assumed to remain plane and perpendicular to the beam center line.

Referring to Figure 3-6, if $\mathbf{r}$ is the vector of the coordinates of an arbitrary point P on the beam element, x is the coordinate of that point in the undeformed state and s is the arc length, then, infinitesimal arc length of the beam element can be defined as

$$ds = \sqrt{\mathbf{r}'^T \mathbf{r}'} dx \quad (3.37)$$

where $\mathbf{r}' = \frac{d\mathbf{r}}{dx}$.

Using equation 3.37, the length of the beam after deformation can be found as

$$\int ds = \int_0^l \sqrt{\mathbf{r}'^T \mathbf{r}'} dx. \quad (3.38)$$

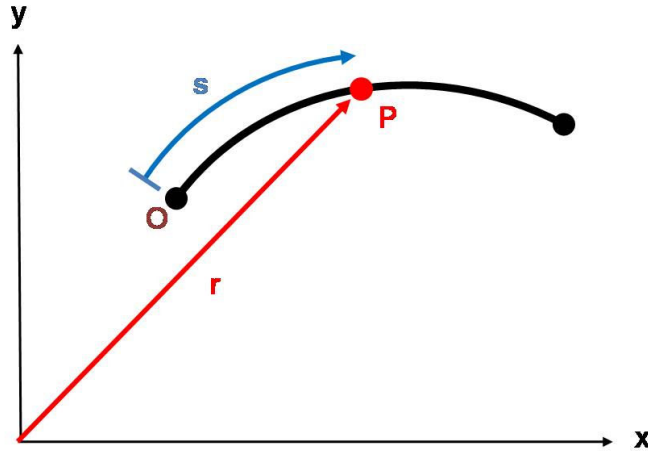


Figure 3-6: Deformed configuration

If ε_l is the longitudinal strain, then the following relationship between the deformed and undeformed length of an infinitesimal segment of the beam can be written as

$$ds^2 - dx^2 = 2 dx \varepsilon_l dx \quad (3.39)$$

which implies that (Berzeri, et al., 2000)

$$\varepsilon_l = \frac{1}{2} (\mathbf{r}'^T \mathbf{r}' - 1). \quad (3.40)$$

Using equation 3.1, equation 3.40 can be written as

$$\varepsilon_l = \frac{1}{2} (\mathbf{e}^T \mathbf{S}'^T \mathbf{S}' \mathbf{e} - 1) = \frac{1}{2} (\mathbf{e}^T \mathbf{S}_l \mathbf{e} - 1) \quad (3.41)$$

where $\mathbf{S}_l = \mathbf{S}'^T \mathbf{S}' = \frac{1}{l^2} \mathbf{S}_\xi^T \mathbf{S}_\xi$, $\mathbf{S}_\xi = \frac{d\mathbf{S}}{d\xi}$, and $\xi = x/l$.

If the material of the beam element is assumed to be isotropic, then strain energy due to the longitudinal deformation can be written as (Timoshenko, 1995)

$$U_l = \frac{1}{2} \int_0^l E a \varepsilon_l^2 dx \quad (3.42)$$

where E is young's modulus and a is beam cross-sectional area.

The vector of nodal coordinates, $\mathbf{e}$, can be written as a sum of two vectors proposed by Berzeri et al. (2000)

$$\mathbf{e} = \mathbf{e}_r + \mathbf{e}_f \quad (3.43)$$

where $\mathbf{e}_r$ represents an arbitrary rigid-body displacement and $\mathbf{e}_f$ is the result of the difference $\mathbf{e} - \mathbf{e}_r$. Using the $\mathbf{e}_r^T \mathbf{S}_l \mathbf{e}_r = 1$ identity and equation 3.41, the longitudinal strain can be defined as

$$\varepsilon_l = \frac{1}{2} (\mathbf{e} - \mathbf{e}_r)^T \mathbf{S}_l (\mathbf{e} + \mathbf{e}_r). \quad (3.44)$$

Since $\mathbf{e}_r$ is arbitrary, one may choose $\mathbf{e}_r = [0 \ 0 \ 1 \ 0 \ l \ 0 \ 1 \ 0]^T$ as proposed by Berzeri et al. (2000).

Using equations 3.42 and 3.44, vector $\mathbf{Q}_l$ of generalized elastic forces due to the longitudinal deformation can be written as

$$\mathbf{Q}_l = \left(\frac{\partial U_l}{\partial \mathbf{e}} \right)^T = \mathbf{K}_l \mathbf{e} . \quad (3.45)$$

Assuming E and a constant, the stiffness matrix $\mathbf{K}_l$ can be given as

$$\mathbf{K}_l = E a l \int_0^1 \varepsilon_l \mathbf{S}_l d\xi \quad (3.46)$$

which can be calculated as

$$\mathbf{K}_l = \begin{bmatrix} K_1 & 0 & K_2 & 0 & -K_1 & 0 & K_3 & 0 \\ 0 & K_1 & 0 & K_2 & 0 & -K_1 & 0 & K_3 \\ K_2 & 0 & K_4 & 0 & -K_2 & 0 & K_5 & 0 \\ 0 & K_2 & 0 & K_4 & 0 & -K_2 & 0 & K_5 \\ -K_1 & 0 & -K_2 & 0 & K_1 & 0 & -K_3 & 0 \\ 0 & -K_1 & 0 & -K_2 & 0 & K_1 & 0 & -K_3 \\ K_3 & 0 & K_5 & 0 & -K_3 & 0 & K_6 & 0 \\ 0 & K_3 & 0 & K_5 & 0 & -K_3 & 0 & K_6 \end{bmatrix} . \quad (3.47)$$

Entries for the stiffness matrix are given by

$$\begin{aligned} K_1 &= \frac{E a}{70 l^3} \left(72 ((e_1 - e_5)^2 + (e_2 - e_6)^2) \right. \\ &\quad + 18 ((e_1 - e_5)(e_3 + e_7) + (e_2 - e_6)(e_4 + e_8))l \\ &\quad \left. + 3 (-14 + e_3^2 + e_4^2 + e_7^2 + e_8^2)l^2 \right) \\ K_2 &= \frac{E a}{280 l^2} \left(36 e_1^2 + 36 e_2^2 + 36 e_5^2 + 36 e_6^2 - 24 e_3 e_5 l - 24 e_4 e_6 l - 14 l^2 - e_3^2 l^2 \right. \\ &\quad - e_4^2 l^2 + 2 e_3 e_7 l^2 + e_7^2 l^2 + 2 e_4 e_8 l^2 + e_8^2 l^2 + 24 e_1 (-3 e_5 + e_3 l) \\ &\quad \left. + 24 e_2 (-3 e_6 + e_4 l) \right) \\ K_3 &= \frac{E a}{280 l^2} \left(36 ((e_1 - e_5)^2 + (e_2 - e_6)^2) + 24 (e_1 e_7 - e_5 e_7 + (e_2 - e_6) e_8)l \right. \\ &\quad \left. + (-14 + e_3^2 + e_4^2 + 2 e_3 e_7 - e_7^2 + 2 e_4 e_8 - e_8^2)l^2 \right) \end{aligned}$$

$$\begin{aligned}
K_4 &= \frac{E a}{420 l} \left(18 e_1^2 + 18 e_2^2 + 18 e_5^2 + 18 e_6^2 + 3 e_3 e_5 l + 3 e_4 e_6 l - 3 e_5 e_7 l - 3 e_6 e_8 l \right. \\
&\quad \left. - 28 l^2 + 12 e_3^2 l^2 + 12 e_4^2 l^2 - 3 e_3 e_7 l^2 + e_7^2 l^2 - 3 e_4 e_8 l^2 + e_8^2 l^2 \right. \\
&\quad \left. - 3 e_1 (12 e_5 + (e_3 - e_7) l) - 3 e_2 (12 e_6 + e_4 l - e_8 l) \right) \\
K_5 &= \frac{E a}{840} \left(6 ((e_1 - e_5)(e_3 + e_7) + (e_2 - e_6)(e_4 + e_8)) \right. \\
&\quad \left. + (14 - 3 e_3^2 - 3 e_4^2 + 4 e_3 e_7 - 3 e_7^2 + 4 e_4 e_8 - 3 e_8^2) l \right) \\
K_6 &= \frac{E a}{420 l} \left(18 e_1^2 + 18 e_2^2 + 18 e_5^2 + 18 e_6^2 - 3 e_3 e_5 l - 3 e_4 e_6 l + 3 e_5 e_7 l + 3 e_6 e_8 l - \right. \\
&\quad \left. 28 l^2 + e_3^2 l^2 + e_4^2 l^2 - 3 e_3 e_7 l^2 + 12 e_7^2 l^2 - 3 e_4 e_8 l^2 + 12 e_8^2 l^2 - 3 e_1 (12 e_5 + \right. \\
&\quad \left. (-e_3 + e_7) l) - 3 e_2 (12 e_6 - e_4 l + e_8 l) \right).
\end{aligned} \tag{3.48}$$

The effect of bending can be introduced using equation (Timoshenko, 1995)

$$M = E I \kappa \tag{3.49}$$

where M is the bending moment, I is area moment of inertia, and κ is bending curvature which can be written as (Berzeri, et al., 2000)

$$\kappa = \left| \frac{d^2 r}{dx^2} \right| \tag{3.50}$$

Using equation 3.1, κ^2 can be calculated as

$$\kappa^2 = \mathbf{r}''^T \mathbf{r}'' = \mathbf{e}^T \mathbf{S}''^T \mathbf{S}'' \mathbf{e} . \tag{3.51}$$

Strain energy, U_t , due to bending can be written as (Timoshenko, 1995)

$$U_t = \frac{1}{2} \int_0^l E I \kappa^2 dx. \tag{3.52}$$

Using equations 3.51 and 3.52, strain energy due to bending can be written as

$$U_t = \frac{1}{2} \mathbf{e}^T \mathbf{K}_t \mathbf{e} \tag{3.53}$$

where the bending stiffness matrix is defined as

$$\mathbf{K}_t = \int_0^l E I \mathbf{S}''^T \mathbf{S}'' dx . \quad (3.54)$$

Assuming constant E and I , the stiffness matrix $\mathbf{K}_t$ can be written as

$$\mathbf{K}_t = \begin{bmatrix} \frac{12EI}{l^3} & 0 & \frac{6EI}{l^2} & 0 & -\frac{12EI}{l^3} & 0 & \frac{6EI}{l^2} & 0 \\ 0 & \frac{12EI}{l^3} & 0 & \frac{6EI}{l^2} & 0 & -\frac{12EI}{l^3} & 0 & \frac{6EI}{l^2} \\ \frac{6EI}{l^2} & 0 & \frac{4EI}{l} & 0 & -\frac{6EI}{l^2} & 0 & \frac{2EI}{l} & 0 \\ 0 & \frac{6EI}{l^2} & 0 & \frac{4EI}{l} & 0 & -\frac{6EI}{l^2} & 0 & \frac{2EI}{l} \\ -\frac{12EI}{l^3} & 0 & -\frac{6EI}{l^2} & 0 & \frac{12EI}{l^3} & 0 & -\frac{6EI}{l^2} & 0 \\ 0 & -\frac{12EI}{l^3} & 0 & -\frac{6EI}{l^2} & 0 & \frac{12EI}{l^3} & 0 & -\frac{6EI}{l^2} \\ \frac{6EI}{l^2} & 0 & \frac{2EI}{l} & 0 & -\frac{6EI}{l^2} & 0 & \frac{4EI}{l} & 0 \\ 0 & \frac{6EI}{l^2} & 0 & \frac{2EI}{l} & 0 & -\frac{6EI}{l^2} & 0 & \frac{4EI}{l} \end{bmatrix} . \quad (3.55)$$

The vector of generalized elastic forces due to the bending deformation, $\mathbf{Q}_t$, can be written as

$$\mathbf{Q}_t = \left(\frac{\partial U_t}{\partial \mathbf{e}} \right)^T = \mathbf{K}_t \mathbf{e} . \quad (3.56)$$

Finally, generalized elastic forces of the beam element, $\mathbf{Q}_k$, is the sum of the elastic forces due to longitudinal and bending deformations

$$\mathbf{Q}_k = \mathbf{K}_l \mathbf{e} + \mathbf{K}_t \mathbf{e} \quad (3.57)$$

where $\mathbf{K}_l$ and $\mathbf{K}_t$ are given by the equations 3.47 and 3.55, respectively.

3.5 Generalized Damping Force

In many applications of flexible multibody dynamics, magnitudes of the elastic and inertial forces are much higher than the magnitudes of damping forces. However, if the system has large deformations, including the damping force will increase the accuracy of the results obtained. External damping comes from aerodynamic and hydrodynamic drag

whereas internal damping is due to the energy dissipation between materials. For large deformation problems external damping becomes more important. There are two types of damping models widely used in flexible multibody problems: proportional damping and quadratic damping. Only proportional damping will be explained in this study because of the following reasons

- Ease of application, and
- Accuracy in large deformation analysis if 1st and 2nd modes of the system are under investigation as proposed by Yoo et al. (2003).

Viscous damping is proportional to the system velocity. Therefore, viscous damping force can be written as

$$\mathbf{Q}_{damp} = \mathbf{C} \dot{\mathbf{r}} \quad (3.58)$$

where $\mathbf{C}$ is the damping matrix defined as a combination of mass and stiffness matrix of the system

$$\mathbf{C} = \alpha \mathbf{M} + \beta \mathbf{K} . \quad (3.59)$$

Coefficients α and β can be written as (Lee, et al., 2009):

$$\alpha = \frac{2 \omega_1 \omega_2 (\zeta_1 \omega_2 - \zeta_2 \omega_1)}{\omega_2^2 - \omega_1^2} , \quad \beta = \frac{2 (\zeta_2 \omega_2 - \zeta_1 \omega_1)}{\omega_2^2 - \omega_1^2} \quad (3.60)$$

where ω_1, ω_2 and ζ_1, ζ_2 are frequencies and damping ratios for the first two modes of the system.

Referring to Figure 3-7, in low frequency excitation regions mass proportional damping is dominant, whereas in high frequency regions, stiffness proportional damping becomes dominant. For systems excited around their first natural frequency, the mass proportional damping model can be used for the sake of simplicity, and the damping matrix becomes

$$\mathbf{C} = \alpha \mathbf{M} . \quad (3.61)$$

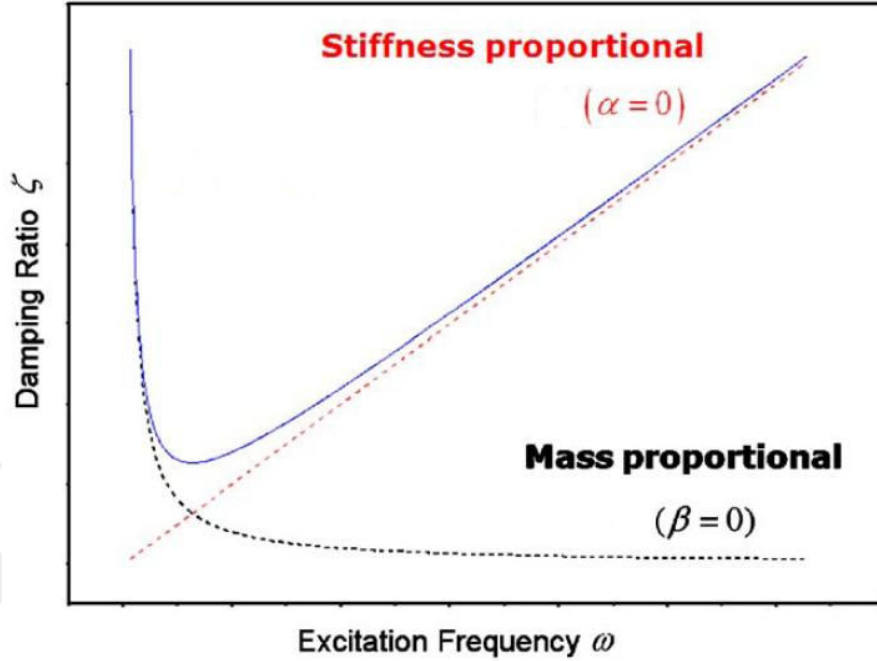


Figure 3-7: Characteristics of damping models (Lee, et al., 2009)

3.6 Generalized External Forces

Generalized external forces in multibody dynamics can be point forces, point moments, surface forces, or volume forces. In this section, different kinds of generalized external force definitions will be given.

3.6.1 Force Vector

Let $\mathbf{F}$ be an external force applied on an arbitrary point on the element. Then the virtual work done due to this external force can be defined as

$$\delta W_f = \mathbf{F}^T \delta \mathbf{r} \quad (3.62)$$

where $\delta \mathbf{r}$ is the virtual change in the position vector of the point of application of the force. Using equation 3.1, one can write the virtual work term as

$$\delta W_f = \mathbf{Q}_f^T \delta \mathbf{e} \quad (3.63)$$

where $\mathbf{Q}_f = \mathbf{S}^T \mathbf{F}$ is the vector of generalized forces associated with the element nodal coordinates.

As an example, virtual work due to the distributed gravity force of the beam element for the planar case can be found using equation 3.4 as

$$\int [0 \quad -\rho g] \mathbf{S} \delta \mathbf{e} \, dV = mg \left[0 \quad -\frac{1}{2} \quad 0 \quad -\frac{1}{12} \quad 0 \quad -\frac{1}{2} \quad 0 \quad \frac{1}{12} \right] \delta \mathbf{e} . \quad (3.64)$$

Therefore, using equation 3.63, the vector of generalized distributed gravity forces can be written as

$$\mathbf{Q}_f = mg \left[0 \quad -\frac{1}{2} \quad 0 \quad -\frac{1}{12} \quad 0 \quad -\frac{1}{2} \quad 0 \quad \frac{1}{12} \right]^T . \quad (3.65)$$

3.6.2 *Moment*

Virtual work due to the applied moment $\mathbf{M}$ acting on an arbitrary point on the element is given by

$$\delta W_M = \mathbf{M} \delta \alpha \quad (3.66)$$

where α is the angle of rotation of the cross-section. Using elements of the planar transformation matrix, the following identities can be found (Shabana, 2005):

$$\sin \alpha = d^{-\frac{1}{2}} \frac{\partial r_2}{\partial x} , \quad (3.67)$$

$$\cos \alpha = d^{-\frac{1}{2}} \frac{\partial r_1}{\partial x} , \text{ and} \quad (3.68)$$

$$d = \left(\frac{\partial r_1}{\partial x} \right)^2 + \left(\frac{\partial r_2}{\partial x} \right)^2 . \quad (3.69)$$

Using equations 3.67 and 3.68, it can be shown that (Shabana, 2005)

$$\delta\alpha = \frac{\frac{\partial r_1}{\partial x} \delta\left(\frac{\partial r_2}{\partial x}\right) - \frac{\partial r_2}{\partial x} \delta\left(\frac{\partial r_1}{\partial x}\right)}{d}. \quad (3.70)$$

Referring to Figure 3-1, if the concentrated moment M acts on the first node of the beam element then the generalized forces due to this moment are written as

$$\mathbf{Q}_M = \begin{bmatrix} 0 & 0 & -\frac{M e_4}{d} & \frac{M e_3}{d} & 0 & 0 & 0 & 0 \end{bmatrix}^T. \quad (3.71)$$

3.6.3 Spring-Damper Forces

If one has a spring-damper connection between the first nodes of the two beam elements and a is the connection node on the first element and b is the connection node in the second element, then the vector of generalized forces can be defined as

$$Q_{SD} = k \begin{pmatrix} e_1^a - e_1^b \\ e_2^a - e_2^b \end{pmatrix} + c \begin{pmatrix} \dot{e}_1^a - \dot{e}_1^b \\ \dot{e}_2^a - \dot{e}_2^b \end{pmatrix} \quad (3.72)$$

where k and c are the spring and damping coefficients.

3.7 Formulation of Constraints

3.7.1 Kinematic Constraints

Let $\mathbf{q} = [q_1 \ q_2 \ q_3 \ \cdots \ q_n]^T$ be the set of generalized coordinates of the flexible multibody system, where n is the number of coordinates. If n_c constraint equations where $n_c \leq n$ can be written in the form

$$\mathbf{C}(q_1, q_2, \dots, q_n, t) = \mathbf{C}(\mathbf{q}, t) = [\mathbf{C}_1(\mathbf{q}, t) \ \mathbf{C}_2(\mathbf{q}, t) \ \cdots \ \mathbf{C}_{n_c}(\mathbf{q}, t)]^T = 0 \quad (3.73)$$

then the set of independent constraint equations in 3.73 are said to be *holonomic*. If the time t in equation 3.73 does not appear explicitly then the constraints are said to be *scleronomic*. Otherwise, it is said to be *rheonomic*.

If the constraint equations have the form

$$\mathbf{F}(q_1, \dots, q_n, \dot{q}_1, \dots, \dot{q}_n, t) = \mathbf{F}(\mathbf{q}, \dot{\mathbf{q}}, t) = [\Gamma_1(\mathbf{q}, \dot{\mathbf{q}}, t) \quad \dots \quad \Gamma_{n_c}(\mathbf{q}, \dot{\mathbf{q}}, t)]^T = 0 \quad (3.74)$$

then the set of independent constraint equations in 3.74 are said to be *non-holonomic*. Non-holonomic constraints are said to be kinematic since they have velocity terms in their equations.

3.7.2 Joint Constraints

As stated previously, multibody systems consist of interconnected bodies and the connections between these bodies can be due to mechanical joints, control devices or force elements such as springs, dampers and actuators. In this section, the formulation of some of the joint constraints will be discussed.

Throughout the formulation, point P will denote the joint definition point, P^i will be the corresponding point on the body i , and P^j will be the corresponding point on body j . For spatial kinematic analysis, point P will have six degrees of freedom on each body. Three of these degrees of freedom will represent the translational motion of the body and three degrees of freedom will represent three independent rotational displacement. However, in planar case, point P will have two translational and one rotational degree(s) of freedom.

A *fixed joint* will restrict relative translational and rotational motions between the two bodies i and j . Therefore, the following joint constraints can be written for the fixed joint

$$\mathbf{r}_p^i = \mathbf{r}_p^j \quad (3.75)$$

$$\boldsymbol{\theta}^i - \boldsymbol{\theta}^j = \mathbf{c} \quad (3.76)$$

where $\mathbf{r}_p^i$ and $\mathbf{r}_p^j$ are the global position vectors, $\boldsymbol{\theta}^i$ and $\boldsymbol{\theta}^j$ are the global angular orientations of points P^i and P^j , respectively, and $\mathbf{c}$ is a constant vector.

The *spherical joint* shown in Figure 3-8a restricts translational motions and allows only the rotational ones. Corresponding kinematic conditions can be stated as

$$\mathbf{r}_p^i = \mathbf{r}_p^j . \quad (3.77)$$

The *cylindrical joint* shown in Figure 3-8b allows relative translational and rotational displacements between bodies i and j along the joint axis. Corresponding kinematic conditions can be written as

$$\mathbf{h}^i \times \mathbf{h}^j = 0 \quad (3.78)$$

$$\mathbf{h}^i \times \mathbf{s}^{ij} = 0 \quad (3.79)$$

where $\mathbf{h}^i$ and $\mathbf{h}^j$ are vectors drawn on bodies i and j along the joint axis and $\mathbf{s}^{ij}$ is the vector of variable magnitude that connects points P^i and P^j as shown in Figure 3-8b.

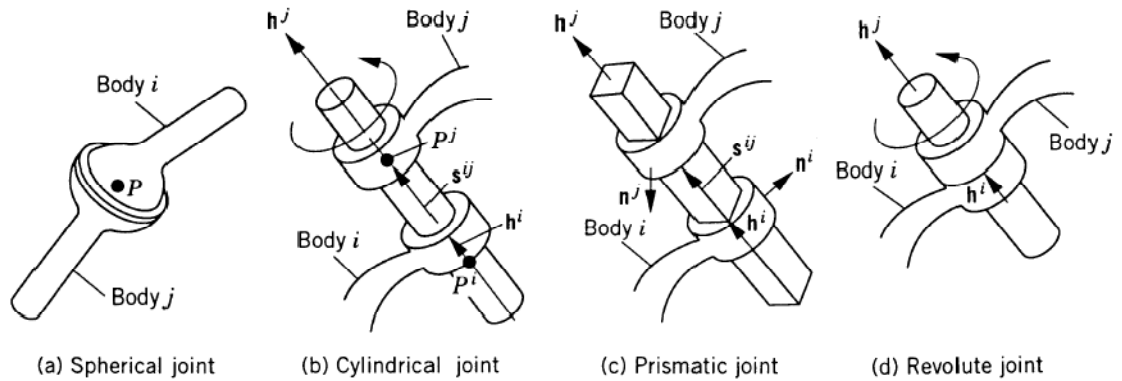


Figure 3-8: Mechanical joints (Shabana, 2001)

The *prismatic joint* shown in Figure 3-8c is a special case of the cylindrical joint in which the relative motion between the two bodies is restricted. If $\mathbf{n}^i$ and $\mathbf{n}^j$ are drawn perpendicular to the joint axis on bodies i and j , respectively as shown in Figure 3-8c, then these two vectors must be perpendicular at all times in order to eliminate relative rotation of the two bodies. Therefore, kinematic conditions for prismatic joint can be defined as

$$\mathbf{h}^i \times \mathbf{h}^j = 0 , \quad (3.80)$$

$$\mathbf{h}^i \times \mathbf{s}^{ij} = 0, \text{ and} \quad (3.81)$$

$$\mathbf{n}^{iT} \mathbf{n}^j = 0. \quad (3.82)$$

Similarly, the *revolute joint* is another special case of the cylindrical joint in which the relative translation between the two bodies is not allowed. Therefore, kinematic conditions for revolute joint shown in Figure 3-8d can be given by

$$\mathbf{h}^i \times \mathbf{h}^j = 0, \quad (3.83)$$

$$\mathbf{h}^i \times \mathbf{s}^{ij} = 0, \text{ and} \quad (3.84)$$

$$\mathbf{s}^{ijT} \mathbf{s}^{ij} = \mathbf{c} \quad (3.85)$$

where vectors $\mathbf{h}^i$, $\mathbf{h}^j$, $\mathbf{s}^{ij}$, and $\mathbf{c}$ are defined previously.

For more information about different types of mechanical joints, one can refer to Shabana (2001) and Parviz (1998).

3.7.3 Generalized Constraint Forces

Let $\mathbf{q} = [q_1 \ q_2 \ q_3 \ \cdots \ q_n]^T$ be the set of generalized coordinates of the flexible body, where n is the number of coordinates. If n_h are the holonomic constraint equations where $n_h \leq n$ is written in the form

$$\mathbf{C}(q_1, q_2, \dots, q_n, t) = \mathbf{C}(\mathbf{q}, t) = [C_1(\mathbf{q}, t) \ C_2(\mathbf{q}, t) \ \cdots \ C_{n_h}(\mathbf{q}, t)]^T = 0 \quad (3.86)$$

and n_{nh} are the non-holonomic constraint equations where $n_{nh} \leq n$ is written in the form

$$\mathbf{\Gamma}(q_1, \dots, q_n, \dot{q}_1, \dots, \dot{q}_n, t) = \mathbf{\Gamma}(\mathbf{q}, \dot{\mathbf{q}}, t) = [\Gamma_1(\mathbf{q}, \dot{\mathbf{q}}, t) \ \cdots \ \Gamma_{n_c}(\mathbf{q}, \dot{\mathbf{q}}, t)]^T = 0 \quad (3.87)$$

then the jacobian of the constraint equations can be defined as

$$\mathbf{C}_q(\mathbf{q}, t) = \begin{bmatrix} \frac{\partial C_1(\mathbf{q}, t)}{\partial \mathbf{q}} \\ \frac{\partial C_2(\mathbf{q}, t)}{\partial \mathbf{q}} \\ \vdots \\ \frac{\partial C_{n_h}(\mathbf{q}, t)}{\partial \mathbf{q}} \end{bmatrix} = 0 \text{ and} \quad (3.88)$$

$$\mathbf{\Gamma}_{\dot{q}}(\mathbf{q}, \dot{\mathbf{q}}, t) = \begin{bmatrix} \frac{\partial \Gamma_1(\mathbf{q}, \dot{\mathbf{q}}, t)}{\partial \dot{\mathbf{q}}} \\ \frac{\partial \Gamma_2(\mathbf{q}, \dot{\mathbf{q}}, t)}{\partial \dot{\mathbf{q}}} \\ \vdots \\ \frac{\partial \Gamma_{n_{nh}}(\mathbf{q}, \dot{\mathbf{q}}, t)}{\partial \dot{\mathbf{q}}} \end{bmatrix} = 0. \quad (3.89)$$

Using the vector of Lagrange Multipliers, $\boldsymbol{\lambda}$, one can write generalized constraint forces due to holonomic constraints, $\mathbf{Q}_{ch}$, as (Shabana, 2001)

$$\mathbf{Q}_{ch} = -\mathbf{C}_q(\mathbf{q}, t)^T \boldsymbol{\lambda} \quad (3.90)$$

where $\mathbf{C}_q(\mathbf{q}, t)$ is given by equation 3.88 and $\boldsymbol{\lambda}$ is defined as

$$\boldsymbol{\lambda} = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_{n_h} \end{bmatrix}. \quad (3.91)$$

Similarly, the generalized constraint forces due to n_{nh} non-holonomic constraints can be defined as

$$\mathbf{Q}_{cnh} = -\mathbf{\Gamma}_{\dot{q}}(\mathbf{q}, \dot{\mathbf{q}}, t)^T \boldsymbol{\mu} \quad (3.92)$$

where $\mathbf{\Gamma}_{\dot{q}}(\mathbf{q}, \dot{\mathbf{q}}, t)$ is given by equation 3.89 and Lagrange Multipliers $\boldsymbol{\mu}$ is defined as

$$\boldsymbol{\mu} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_{n_{nh}} \end{bmatrix}. \quad (3.93)$$

Therefore, the generalized constraint equations of the system can be written as

$$\mathbf{Q}_c = -\mathbf{C}_q(\mathbf{q}, t)^T \boldsymbol{\lambda} - \boldsymbol{\Gamma}_{\dot{\mathbf{q}}}(\mathbf{q}, \dot{\mathbf{q}}, t)^T \boldsymbol{\mu}. \quad (3.94)$$

3.8 Equation of Motion of Flexible Body

The principle of virtual work for the unconstrained motion of the finite element j can be written as

$$\delta W_i = \delta W_k + \delta W_e \quad (3.95)$$

where δW_i is the virtual work of the inertia forces of the body, δW_k is the virtual work of the elastic forces due to deformation, and δW_e is the virtual work of the applied external forces.

If $\mathbf{F}^j$ is the external force acting on element j at the point P defined by the coordinate $\mathbf{x}_p^j$, then the virtual work of the element j due to the applied external forces can be defined using equation 3.1 as

$$\delta W_e^j = \mathbf{F}^{jT} \delta \mathbf{r}_p^j = \mathbf{F}^{jT} \mathbf{S}^j \delta \mathbf{e}^j = \mathbf{Q}_e^{jT} \delta \mathbf{e}^j \quad (3.96)$$

where $\mathbf{S}^j$ is shape function of element j defined at point $\mathbf{x}_p^j$, and $\mathbf{Q}_e^j = \mathbf{S}^{jT} \mathbf{F}^j$ is the vector of generalized external forces associated with the element nodal coordinates $\mathbf{e}^j$.

Similarly, the virtual work of the element j due to the inertia forces can be defined as (Shabana, 2005)

$$\delta W_i^j = (\mathbf{M}_a^j \ddot{\mathbf{e}}^j)^T \delta \mathbf{e}^j \quad (3.97)$$

and the virtual work of the element j due to generalized elastic forces can be given by

$$\delta W_k^j = -\mathbf{Q}_k^{jT} \delta \mathbf{e}^j \quad (3.98)$$

where $\mathbf{M}_a^j$ is the mass matrix of the element given by equation 3.8 and $\mathbf{Q}_k$ is the vector of generalized elastic forces given by equation 3.57.

Using equations 3.96, 3.97, and 3.98, the virtual work equation 3.95 can be obtained as

$$(\mathbf{M}_a^j \ddot{\mathbf{e}}^j + \mathbf{Q}_k^j - \mathbf{Q}_e^j)^T \delta \mathbf{e}^j = 0. \quad (3.99)$$

If the motion is unconstrained, the vector of element nodal coordinates $\mathbf{e}$ is independent and equation 3.99 can be written as

$$\mathbf{M}_a^j \ddot{\mathbf{e}}^j = \mathbf{Q}_e^j - \mathbf{Q}_k^j \quad (3.100)$$

which is the equations of motion of the finite element j .

Using classical finite element assembly techniques, equations of motion of the flexible body can be written as

$$\mathbf{M} \ddot{\mathbf{q}} = \mathbf{Q}_e - \mathbf{Q}_k \quad (3.101)$$

where $\mathbf{q} = [q_1 \ q_2 \ q_3 \ \cdots \ q_n]^T$ is the set of n generalized coordinates of the flexible body, $\mathbf{M}$ is the constant symmetric mass matrix, $\mathbf{Q}_k$ is the vector of body elastic forces and $\mathbf{Q}_e$ is the vector of the externally applied body forces such as gravity, magnetic, and other forces.

If $\mathbf{B}^j$ is a Boolean Matrix that defines the connectivity of the j^{th} element and n_e is the number of elements of the flexible body, then one can obtain the mass matrix and force vectors that appear in equation 3.101 from the mass matrices and force vectors of the finite elements forming the flexible body as

$$\mathbf{M} = \sum_{j=1}^{n_e} \mathbf{B}^{jT} \mathbf{M}_a^j \mathbf{B}^j, \quad (3.102)$$

$$\mathbf{Q}_k = \sum_{j=1}^{n_e} \mathbf{B}^{jT} \mathbf{Q}_k^j, \text{ and} \quad (3.103)$$

$$\mathbf{Q}_e = \sum_{j=1}^{n_e} \mathbf{B}^{jT} \mathbf{Q}_e^j. \quad (3.104)$$

If the flexible multibody system has holonomic and non-holonomic constraints, then using equation 3.94 for generalized constraint forces, the equation of motion of the flexible body can be written as

$$\mathbf{M} \ddot{\mathbf{q}} + \mathbf{C}_q(\mathbf{q}, t)^T \boldsymbol{\lambda} + \mathbf{\Gamma}_{\dot{\mathbf{q}}}(\mathbf{q}, \dot{\mathbf{q}}, t)^T \boldsymbol{\mu} = \mathbf{Q}_e - \mathbf{Q}_k \quad (3.105)$$

where $\mathbf{C}_q(\mathbf{q}, t)$, $\mathbf{\Gamma}_{\dot{\mathbf{q}}}(\mathbf{q}, \dot{\mathbf{q}}, t)$, $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ are given by equations 3.88, 3.89, 3.91, and 3.93, respectively.

In equation 3.105, the unknowns are the vector of accelerations $\ddot{\mathbf{e}}$ and the vectors of Lagrange Multipliers $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$. If the system has n generalized coordinates, n_h holonomic constraint equations and n_{nh} non-holonomic constraint equations, then the total number of unknowns is $n + n_h + n_{nh}$, which is more than the number of equations n . In order to be able to solve the system, one needs to have $n_h + n_{nh}$ additional equations. These $n_h + n_{nh}$ additional equations are found using the constraint equations 3.86 and 3.87.

Taking derivative of equations 3.86 once and twice with respect to time, one obtains

$$\mathbf{C}_q \dot{\mathbf{q}} = -\mathbf{C}_t \text{ and} \quad (3.106)$$

$$\mathbf{C}_q \ddot{\mathbf{q}} = \mathbf{Q}_d \quad (3.107)$$

where subscript t denotes partial differentiation with respect to time and the vector $\mathbf{Q}_d$ is given by

$$\mathbf{Q}_d = -\mathbf{C}_{tt} - (\mathbf{C}_q \dot{\mathbf{q}})_q \dot{\mathbf{q}} - 2\mathbf{C}_{qt} \dot{\mathbf{q}}. \quad (3.108)$$

Similarly, taking the derivative of equation 3.87 once with respect to time, one obtains

$$\mathbf{\Gamma}_{\dot{\mathbf{q}}} \ddot{\mathbf{q}} = \mathbf{Q}_f \quad (3.109)$$

where vector $\mathbf{Q}_f$ is given by

$$\mathbf{Q}_f = -\mathbf{\Gamma}_q \dot{\mathbf{q}} - \mathbf{\Gamma}_t. \quad (3.110)$$

Combining equations 3.105, 3.107, and 3.109, constrained equations of motion of the flexible body can be defined as

$$\begin{bmatrix} \mathbf{M} & \mathbf{C}_q^T & \mathbf{F}_{\dot{q}}^T \\ \mathbf{C}_q & 0 & 0 \\ \mathbf{F}_{\dot{q}} & 0 & 0 \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{q}} \\ \boldsymbol{\lambda} \\ \boldsymbol{\mu} \end{bmatrix} = \begin{bmatrix} \mathbf{Q}_e - \mathbf{Q}_k \\ \mathbf{Q}_d \\ \mathbf{Q}_f \end{bmatrix} \quad (3.111)$$

where $\mathbf{q} = [q_1 \ q_2 \ q_3 \ \cdots \ q_n]^T$ is the set of n generalized coordinates of the flexible body and $C_q, F_{\dot{q}}, \boldsymbol{\lambda}, \boldsymbol{\mu}, M, Q_k$, and Q_e are given by equations 3.88, 3.89, 3.91, 3.93, 3.102, 3.103, and 3.104, respectively.

For a flexible body having only holonomic constraints, equations of motion become

$$\begin{bmatrix} \mathbf{M} & \mathbf{C}_q^T \\ \mathbf{C}_q & 0 \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{q}} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \mathbf{Q} \\ \mathbf{Q}_d \end{bmatrix} \quad (3.112)$$

where $\mathbf{Q}$ is the vector of all forces including external, elastic, coriolis and centrifugal forces.

CHAPTER 4

IMPACT

Before explaining the details of impact dynamics between two planar rigid bodies, the definition of the coefficient of restitution (COR) of an object has to be defined. The COR is a fractional value representing the ratio of velocities after and before the impact. Theoretically, it is a number between 0 and 1. If the COR is 1, the collision is said to be perfectly elastic; if it is 0, the collision is said to be perfectly inelastic (i.e. the object stops at the surface with which it collides). Therefore, the mathematical definition of the COR for a one-dimensional case can be written as

$$C_R = \frac{v_f^j - v_f^i}{v_{in}^i - v_{in}^j} \quad (4.1)$$

where v_{in}^i, v_{in}^j are the scalar velocities of bodies i and j before impact and v_f^i, v_f^j are the scalar velocities after impact, respectively.

Using the conservation of momentum law, one can write

$$m^i v_1^i + m^j v_1^j = m^i v_2^i + m^j v_2^j \quad (4.2)$$

where m^i is the mass of rigid body i and m^j is the mass of rigid body j . Velocity vectors in equation 4.2 for planar motion can be given by

$$\mathbf{v}_1^i = \begin{pmatrix} v_{x1}^i \\ v_{y1}^i \end{pmatrix}, \quad \mathbf{v}_1^j = \begin{pmatrix} v_{x1}^j \\ v_{y1}^j \end{pmatrix}, \quad \mathbf{v}_2^i = \begin{pmatrix} v_{x2}^i \\ v_{y2}^i \end{pmatrix}, \quad \mathbf{v}_2^j = \begin{pmatrix} v_{x2}^j \\ v_{y2}^j \end{pmatrix} \quad (4.3)$$

where terms in equation 4.3 can be defined as

v_{x1}^i : Scalar velocity of body i in x-direction before impact,

v_{y1}^i : Scalar velocity of body i in y-direction before impact,

v_{x1}^j : Scalar velocity of body j in x-direction before impact,

v_{y1}^j : Scalar velocity of body j in y-direction before impact,

v_{x2}^i : Scalar velocity of body i in x-direction after impact,

v_{y2}^i : Scalar velocity of body i in y-direction after impact,

v_{x2}^j : Scalar velocity of body j in x-direction after impact, and

v_{y2}^j : Scalar velocity of body j in y-direction after impact.

Combining equations 4.1, 4.2, and 4.3, one can obtain velocities of the rigid bodies after impact in terms of the velocities before impact as

$$v_{x2}^i = \frac{m^i v_{x1}^i + m^j v_{x1}^j + m^j C_R (v_{x1}^j - v_{x1}^i)}{m^i + m^j}, \quad (4.4)$$

$$v_{y2}^i = \frac{m^i v_{y1}^i + m^j v_{y1}^j + m^j C_R (v_{y1}^j - v_{y1}^i)}{m^i + m^j}, \quad (4.5)$$

$$v_{x2}^j = \frac{m^i v_{x1}^i + m^j v_{x1}^j + m^i C_R (v_{x1}^i - v_{x1}^j)}{m^i + m^j}, \text{ and} \quad (4.6)$$

$$v_{y2}^j = \frac{m^i v_{y1}^i + m^j v_{y1}^j + m^i C_R (v_{y1}^i - v_{y1}^j)}{m^i + m^j}. \quad (4.7)$$

According to Newton's second law, the rate of change of the momentum of a particle is proportional to the resultant force acting on the particle and is in the direction of that force. Therefore, the impact force between bodies i and j can be defined as

$$\mathbf{F}_c = \frac{d\mathbf{P}}{dt} \quad (4.8)$$

where $\mathbf{P}$ is the linear momentum of one of the bodies.

Impact force acting on each body appears in pairs and follows Newton's third law. Therefore, the force acting on each body has the same magnitude yet different direction irrespective of the masses or sizes of the colliding bodies.

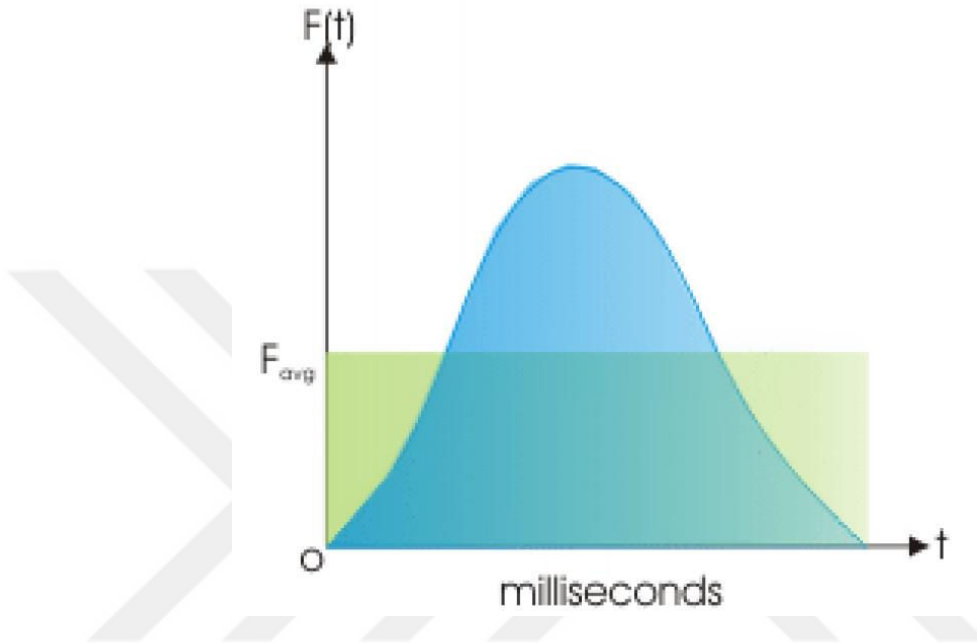


Figure 4-1: Average impact force (Singh)

Referring to Figure 4-1, the integral of force over the time of collision is graphically equal to the blue area under the curve. Therefore, one can have an average force during the time of collision that has the same area with the actual force-time curve. Using equation 4.8, the average force during collision can be defined as

$$F_{cav} = \frac{\Delta P}{\Delta t} \quad (4.9)$$

where Δt is the contact time and ΔP is the change in momentum of one of the colliding bodies given by

$$\Delta P = m^i(v_2^i - v_1^i) = m^j(v_2^j - v_1^j) . \quad (4.10)$$

CHAPTER 5

NUMERICAL SOLUTION

5.1 Introduction

Referring to the previous chapters, the equations of motion of a multibody system consisting of interconnected rigid and deformable bodies are a combined set of ordinary differential and algebraic equations. These kinds of equation sets are called *Differential Algebraic Equations* (DAE's) in literature. The solutions to DAE's are not as straight forward as ordinary differential equations. Specialized numerical techniques have been developed for the solution of DAE's, and these techniques can be separated into two main classes, namely, State-Space Methods and Direct Integration Methods.

In state-space methods, the DAE problem is first reduced to a smaller dimension ODE problem; as a result, one can benefit from the extensive knowledge associated to ODE solvers. However, these techniques are computationally intensive, especially when dealing with large multibody systems that contain flexible bodies, contact, and impact.

On the other hand, direct integration methods are significantly faster and accurate. Therefore, in this chapter, the direct integration approach based on the Wehage coordinate partitioning technique (Wehage, et al., 1982) and the Newmark method (Gavrea, et al., 2005) will be explained, and this approach will be used for the solution of DAE's of the multibody system under investigation.

The outline of the chapter is as follows. In section 2, the Wehage generalized coordinate partitioning technique will be described and the constraint equations will be written in terms of dependent and independent coordinates. In section 3, the Newton-Raphson algorithm for the solution of nonlinear algebraic equations will be explained. In section 4, the Newmark Method for the numerical integration of a set of second order differential equations will be given. Finally, in section 5, a two-loop sparse matrix numerical integration procedure will be proposed for the solution of DAE's.

5.2 Coordinate Partitioning

Multibody systems have kinematic constraints that describe mechanical joints as well as specified trajectories. Due to these constraint equations, not all the generalized coordinates are independent. Therefore, one has to identify dependent and independent coordinates and use the set of differential equations associated with the independent coordinates in order to find the independent variables. Dependent variables can then be determined by using the kinematic constraint equations.

The virtual change of the system generalized coordinates in equation 3.73 yields

$$\mathbf{C}_q \delta \mathbf{q} = \mathbf{0} \quad (5.1)$$

where $\mathbf{C}_q$ is the jacobian matrix defined in equation 3.88. $\mathbf{C}_q$ is an $n_c \times n$ matrix where n_c is the number of constraint equations and n is the number of generalized coordinates. Having n_c constraint equations means that the system has n_c dependent coordinates. As a result, the number of independent coordinates in the system is $n - n_c$. Therefore, one can write the vector of generalized coordinates in the following partitioned form

$$\mathbf{q} = \begin{pmatrix} \mathbf{q}_i \\ \mathbf{q}_d \end{pmatrix} \quad (5.2)$$

where $\mathbf{q}_i$ is the vector of independent coordinates and $\mathbf{q}_d$ is the vector of dependent coordinates. The dimension of the vector $\mathbf{q}_i$ is $n - n_c \times 1$ whereas the dimension of the vector $\mathbf{q}_d$ is $n_c \times 1$. At this point, one has to note that both the dependent and independent vectors of coordinates can have a set of reference and elastic coordinates. Using equation 5.2, one can redefine equation 5.1 as

$$\mathbf{C}_{q_i} \delta \mathbf{q}_i + \mathbf{C}_{q_d} \delta \mathbf{q}_d = \mathbf{0} \quad (5.3)$$

where $n_c \times n - n_c$ dimension $\mathbf{C}_{q_i}$ is the jacobian of the constraint equations associated with the independent coordinates and $n_c \times n_c$ dimension $\mathbf{C}_{q_d}$ is the jacobian of the constraint equations associated with the dependent coordinates.

Using equation 5.3, one can define the virtual change in the dependent coordinates in terms of the virtual change of the independent coordinates as

$$\delta \mathbf{q}_d = \mathbf{C}_{di} \delta \mathbf{q}_i \quad (5.4)$$

where $\mathbf{C}_{di}$ is an $n_c \times n - n_c$ matrix given by

$$\mathbf{C}_{di} = -\mathbf{C}_{qd}^{-1} \mathbf{C}_{qi} . \quad (5.5)$$

Equation 3.106 can be written in partitioned form as

$$\mathbf{C}_{qi} \dot{\mathbf{q}}_i + \mathbf{C}_{qd} \dot{\mathbf{q}}_d + \mathbf{C}_t = \mathbf{0} . \quad (5.6)$$

Using equation 5.6, one can get the dependent velocities in terms of the independent velocities as

$$\dot{\mathbf{q}}_d = \mathbf{C}_{di} \dot{\mathbf{q}}_i - \mathbf{C}_{qd}^{-1} \mathbf{C}_t . \quad (5.7)$$

Note that $\mathbf{C}_{qd}$ is $n_c \times n_c$ square matrix. Since the constraint equations are linearly independent, by choosing the independent coordinates appropriately, $\mathbf{C}_{qd}$ becomes non-singular and one can have its inverse.

5.3 Newton-Raphson Method

In this section, the most widely used multidimensional root finding method, Newton-Raphson, will be explained for the solution of nonlinear algebraic equations. This method has a good convergence property if one has a good initial guess.

Consider n nonlinear algebraic equations in n unknowns

$$\mathbf{F}(\mathbf{x}) = \mathbf{0} \quad (5.8)$$

where $\mathbf{x}$ denote the entire vector of the unknowns and $\mathbf{F}$ denote the entire vector of nonlinear algebraic equations.

Using the Taylor series in the neighborhood of $\mathbf{x}$, one can write function $\mathbf{F}$ as

$$\mathbf{F}(\mathbf{x} + \delta\mathbf{x}) = \mathbf{F}(\mathbf{x}) + \frac{\partial \mathbf{F}}{\partial \mathbf{x}} \delta\mathbf{x} + O(\delta\mathbf{x}^2). \quad (5.9)$$

The matrix of partial derivatives appearing in equation 5.9 is the Jacobian matrix $\mathbf{J}$ given by

$$\mathbf{J} = \frac{\partial \mathbf{F}}{\partial \mathbf{x}} = \begin{bmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} & \dots & \frac{\partial F_1}{\partial x_n} \\ \frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} & \dots & \frac{\partial F_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial F_n}{\partial x_1} & \frac{\partial F_n}{\partial x_2} & \dots & \frac{\partial F_n}{\partial x_n} \end{bmatrix}. \quad (5.10)$$

Combining equations 5.9 and 5.10, one can write

$$\mathbf{F}(\mathbf{x} + \delta\mathbf{x}) = \mathbf{F}(\mathbf{x}) + \mathbf{J} \delta\mathbf{x} + O(\delta\mathbf{x}^2). \quad (5.11)$$

Using equation 5.11, neglecting the higher order terms and setting $\mathbf{F}(\mathbf{x} + \delta\mathbf{x}) = 0$, one can obtain a set of linear equations for the corrections $\delta\mathbf{x}$ that move each function closer to zero as

$$\delta\mathbf{x} = -\mathbf{J}^{-1}\mathbf{F}. \quad (5.12)$$

Using the corrections, one can update the vector of variables as

$$\mathbf{x}_{new} = \mathbf{x}_{old} + \delta\mathbf{x}. \quad (5.13)$$

Using equation 5.12, equation 5.13 can be re-written as

$$\mathbf{x}_{new} = \mathbf{x}_{old} - \mathbf{J}^{-1}\mathbf{F} \quad (5.14)$$

where $\mathbf{x}_{old}$ denotes the previous values of the vector of unknowns $\mathbf{x}$ and $\mathbf{x}_{new}$ denotes the updated values of the vector of unknowns $\mathbf{x}$.

The Newton-Raphson method is an iterative root-finding method. If j and $j + 1$ are the iteration numbers then equation 5.14 can be written as

$$\mathbf{x}^{j+1} = \mathbf{x}^j - \mathbf{J}^{-1}(\mathbf{x}^j)\mathbf{F}(\mathbf{x}^j) \quad (5.15)$$

and the iterative sequence of values can be written as

$$\mathbf{x}^1 = \mathbf{x}^0 - \mathbf{J}^{-1}(\mathbf{x}^0)\mathbf{F}(\mathbf{x}^0)$$

$$\mathbf{x}^2 = \mathbf{x}^1 - \mathbf{J}^{-1}(\mathbf{x}^1)\mathbf{F}(\mathbf{x}^1)$$

$$\vdots$$

where the vector $\mathbf{x}^0$ is the initial guess vector for the n unknowns.

The computational procedure for Newton-Raphson Method is stated as follows

- i. Set iteration counter $j = 0$
- ii. Input initial guess vector $\mathbf{x}^0$, set the tolerance ε and maximum number of iterations n_m
- iii. Evaluate the matrix $\mathbf{J}^{-1}(\mathbf{x}^j)$ and the vector $\mathbf{F}(\mathbf{x}^j)$
- iv. Using equation 5.15, find $\mathbf{x}^{j+1}$
- v. If $\delta\mathbf{x} = -\mathbf{J}^{-1}(\mathbf{x}^j)\mathbf{F}(\mathbf{x}^j) \leq \varepsilon$ or $j = n_m$ terminate iteration; if not, increment j and repeat steps (iii) and (iv).

5.4 Newmark Integration Method

In 1959, Newmark proposed an integration method for the solution of structural dynamic problems (Newmark, 1959). From that point onward, many researchers modified and improved this technique and applied it to the dynamic analysis of many practical engineering problems. The reasons for the wide usage of the Newmark Method can be stated as

- i. Unlike state-space methods, the resulting second order ODE do not have to be reduced to first order, which results in a smaller dimension problem
- ii. It has good stability properties

- iii. By using the integration parameters, one can adjust the amount of damping introduced in the system.

In this section, the Newmark Method for the dynamic solution of the multibody systems will be explained, and it will be a modification of the originally proposed method since the dynamic equations are DAE's due to the kinematic constraints between the bodies. Furthermore, only holonomic case will be presented to keep the explanation simple, although the method can handle non-holonomic constraints.

Let $\mathbf{q} = [q_1 \ q_2 \ q_3 \ \cdots \ q_n]^T$ be the vector of generalized coordinates, $\dot{\mathbf{q}} = [\dot{q}_1 \ \dot{q}_2 \ \dot{q}_3 \ \cdots \ \dot{q}_n]^T$ be the vector of generalized velocities, and $\ddot{\mathbf{q}} = [\ddot{q}_1 \ \ddot{q}_2 \ \ddot{q}_3 \ \cdots \ \ddot{q}_n]^T$ be the vector of generalized accelerations of the multibody system. Then, using Newmark (1959), one can write the vectors $\mathbf{q}$ and $\dot{\mathbf{q}}$ as

$$\mathbf{q}_{n+1} = \mathbf{q}_n + h \dot{\mathbf{q}}_n + \frac{h^2}{2} ((1 - 2\beta) \ddot{\mathbf{q}}_n + 2\beta \ddot{\mathbf{q}}_{n+1}) \quad (5.16)$$

$$\dot{\mathbf{q}}_{n+1} = \dot{\mathbf{q}}_n + h ((1 - \gamma) \ddot{\mathbf{q}}_n + \gamma \ddot{\mathbf{q}}_{n+1}) \quad (5.17)$$

where h is the time step size, γ and β are the integration parameters and the subscripts n and $n + 1$ denote variables at time t_n and t_{n+1} , respectively.

Equations 5.16 and 5.17 are used to discretize the equations of motion of the multibody system given in equation 3.112 as

$$\begin{bmatrix} \mathbf{M} & \mathbf{C}_{\mathbf{q}_{n+1}}^T \\ \mathbf{C}_{\mathbf{q}_{n+1}} & 0 \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{q}}_{n+1} \\ \boldsymbol{\lambda}_{n+1} \end{bmatrix} = \begin{bmatrix} \mathbf{Q}_{n+1} \\ \mathbf{Q}_{d_{n+1}} \end{bmatrix}. \quad (5.18)$$

Note that based on equations 5.16 and 5.17, $\mathbf{q}_{n+1}$ and $\dot{\mathbf{q}}_{n+1}$ are functions of the vector $\ddot{\mathbf{q}}_{n+1}$, which in equation 5.18 becomes the unknown quantity that has to be computed with the Lagrange Multiplier vector $\boldsymbol{\lambda}_{n+1}$.

This method is implicit and A-Stable, i.e. stable in the whole left-hand plane provided (Hilbert, et al., 1977)

$$\gamma \geq \frac{1}{2} \quad , \quad \beta \geq \frac{\left(\gamma + \frac{1}{2}\right)^2}{4} . \quad (5.19)$$

For more information about the stability, accuracy, and implementation details, one can refer to Gavrea et al. (2005).

The computational procedure for the Modified Newmark Method is stated as follows

- i. Set counter $n = 0$
- ii. Input the time step size h and the integration parameters γ and β
- iii. Input initial guess vectors $\mathbf{q}_0$, $\dot{\mathbf{q}}_0$, $\ddot{\mathbf{q}}_0$, and $\boldsymbol{\lambda}_0$ at time $t = 0$
- iv. Insert equations 5.16 and 5.17 into equation 5.18 in order to form a set of nonlinear algebraic equations in terms of the unknowns $\ddot{\mathbf{q}}_{n+1}$ and $\boldsymbol{\lambda}_{n+1}$
- v. Using the Newton-Raphson Method and the values at time t_n , find the unknowns $\ddot{\mathbf{q}}_{n+1}$ and $\boldsymbol{\lambda}_{n+1}$
- vi. Using equations 5.16 and 5.17, find $\mathbf{q}_{n+1}$ and $\dot{\mathbf{q}}_{n+1}$
- vii. Increment n and repeat steps iv through vi until the desired simulation time is reached.

5.5 Two-Loop Sparse Matrix Numerical Integration Algorithm

In this section, a two-loop sparse matrix numerical integration method will be explained for the solution of the DAE's of constrained rigid and flexible multibody systems. The method will guarantee that the kinematic constraint equations are satisfied at the position, velocity, and acceleration levels. The computational algorithm based on the proposed method can be stated as follows

- i. Input flexible and rigid body data
- ii. Set counter $n = 0$
- iii. Input the time step size h and the integration parameters γ and β
- iv. Identify system dependent and independent coordinates

- v. Input initial displacement, velocity, acceleration and Lagrange Multiplier vectors of the system at time $t = 0$
- vi. Use the Newton-Raphson Method to solve the nonlinear kinematic constraint equation 3.73 for the dependent coordinates, $\mathbf{q}_{d_{n+1}}$, assuming that the independent coordinates are fixed
- vii. Using equations 5.16 and 5.17, find the displacement and velocity vectors $\mathbf{q}_{i_{n+1}}$ and $\dot{\mathbf{q}}_{i_{n+1}}$ of the independent coordinates in terms of the unknown acceleration vector $\ddot{\mathbf{q}}_{i_{n+1}}$
- viii. Using equation 5.7, find the vector of velocities, $\dot{\mathbf{q}}_{d_{n+1}}$, associated with the dependent coordinates in terms of $\ddot{\mathbf{q}}_{i_{n+1}}$. Note that $\dot{\mathbf{q}}_{i_{n+1}}$ defined previously in terms of $\ddot{\mathbf{q}}_{i_{n+1}}$
- ix. Combine the Lagrangian form of the equations of motion of the system given in 3.112 with the vectors $\mathbf{q}_{d_{n+1}}$, $\mathbf{q}_{i_{n+1}}$, $\dot{\mathbf{q}}_{i_{n+1}}$, and $\dot{\mathbf{q}}_{d_{n+1}}$ obtained in steps vi, vii, and viii, respectively
- x. Solve the set of equations obtained in step ix using the Newton-Raphson Method for the unknowns $\ddot{\mathbf{q}}_{i_{n+1}}$, $\ddot{\mathbf{q}}_{d_{n+1}}$, and λ_{n+1}
- xi. Using equations 5.16 and 5.17, find the displacement and velocity vectors $\mathbf{q}_{i_{n+1}}$ and $\dot{\mathbf{q}}_{i_{n+1}}$ of the independent coordinates
- xii. Using equation 5.7, find the vector of velocities, $\dot{\mathbf{q}}_{d_{n+1}}$, associated with the dependent coordinates
- xiii. Increment n and repeat steps vi through xii until the desired simulation time is reached.

The computational algorithm for the dynamic analysis of the multibody system can be seen in Figure 5-1.

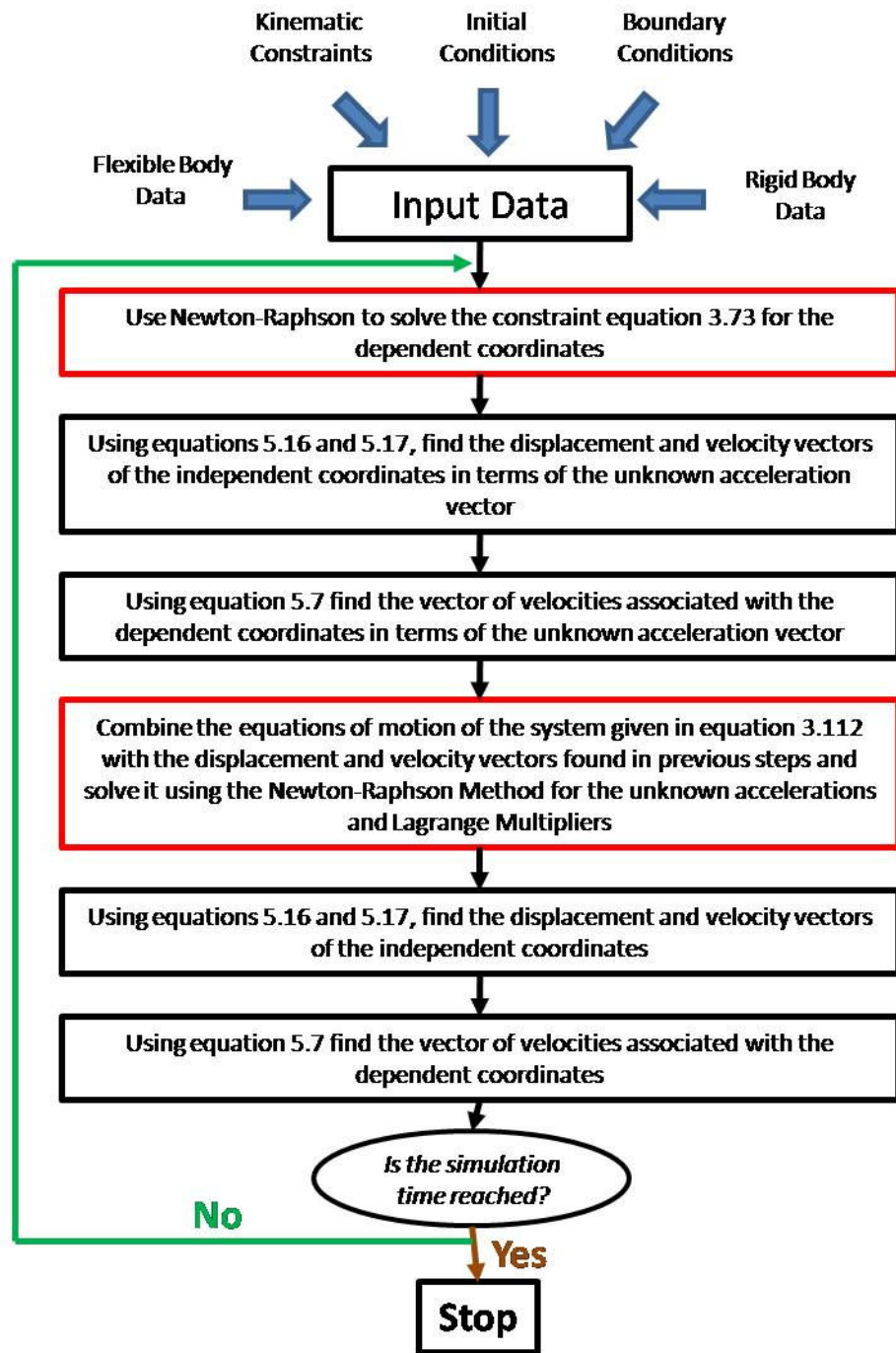


Figure 5-1: Computational Algorithm for Dynamic Analysis

CHAPTER 6

ADAMS

6.1 Introduction

Automatic Dynamic Analysis of Mechanical Systems (ADAMS) is a mechanical system analysis software that was implemented in the 1980s by researchers at the University of Michigan, Ann Arbor, Michigan. The original product, ADAMS/Solver, was an application that solves nonlinear equations. Later on, developers added ADAMS/View that allowed users to build, simulate, and examine results of their analysis in a single environment. Due to the success of the product in the market, researchers started to produce new add-on modules for the specific applications such as ADAMS/Car, ADAMS/Rail, ADAMS/Engine, ADAMS/Chassis, and ADAMS/Flex.

ADAMS multibody dynamics software enables engineers to easily create and test virtual prototypes of mechanical systems. Therefore, designers have full understanding of the system performance before actually building it, leading to better product designs in terms of performance, safety, and comfort and to a reduction in time consumed and costs.

The capabilities of ADAMS can be defined as

- Integrate mechanical components, pneumatics, hydraulics, electronics, and control systems technologies
- Helps engineers to study the dynamics of moving parts
- Identification of the loads and forces distributed throughout mechanical systems
- Improve and optimize the performance of designs

6.2 Flexible Body Theory in ADAMS

This section will give a brief overview of the theory used to implement flexible bodies in ADAMS.

During the analysis of the flexible bodies in ADAMS, body deformations relative to a local reference frame are assumed to be small while the local reference frame can undergo large, non-linear global motion.

Using finite element discretization, linear deformations of the nodes of the flexible body can be approximated as a linear combination of a smaller number of mode shapes as

$$\mathbf{u} = \Phi \mathbf{q} \quad (6.1)$$

where $\mathbf{u}$ is the vector of nodal deformations, $\mathbf{q}$ is the vector of modal coordinates, and Φ is the modal matrix that consists of the modes shapes of the flexible body.

Using Component Mode Synthesis (CMS), specifically the Craig-Bampton Method, equation 6.1 can be partitioned into boundary DOF, $\mathbf{u}_B$, and interior DOF, $\mathbf{u}_I$. The corresponding Craig-Bampton mode shapes are defined as

Constraint modes: Obtained by giving each boundary DOF a unit displacement while holding all other boundary DOF fixed; and

Fixed boundary normal modes: Obtained by fixing the boundary DOF and computing an eigensolution.

In order to have the frequency information of the constraint modes and disable the unwanted ones, such as rigid body modes, the Craig-Bampton modes are orthogonalized and equation 6.1 is redefined as

$$\mathbf{u} = \Phi^* \mathbf{q}^* \quad (6.2)$$

where, $\mathbf{q}^*$ is the vector of orthogonal modal coordinates and Φ^* is the orthogonalized Craig-Bampton modal matrix.

Using equation 6.2 with the flexible body dynamic equations, one can find equations of motion, mass matrix, stiffness matrix, damping matrix, and generalized force vectors of the flexible body. For detailed information on these topics, one can refer to reference MSC.Software (2003).

To sum up, ADAMS assigns a set of mode shapes to the flexible body and calculates the relative amplitude of each mode shapes during a time analysis. The principle of linear superposition is then used to combine the mode shapes at each time step to reproduce the total deformation of the flexible body.

For the time being, ADAMS cannot generate meshes and perform modal analysis to the bodies. Therefore, one should use third party finite element analysis programs to create a flexible body description file called the modal neutral file (MNF) as the foundation of a flexible body in ADAMS. The information in an MNF file includes:

- Geometry (location of nodes and node connectivity),
- Nodal mass and inertia,
- Mode shapes, and
- Generalized mass and stiffness for mode shapes.

The MNF is a binary file that contains a large amount of data that ADAMS needs to access frequently. In order to create MNF files, one can use finite element programs such as MSC.Nastran, MSC.Marc, ANSYS, ABAQUS, and I-DEAS. After creating the MNF file, required steps to build the flexible body model and run simulations are given in Figure 6-1.

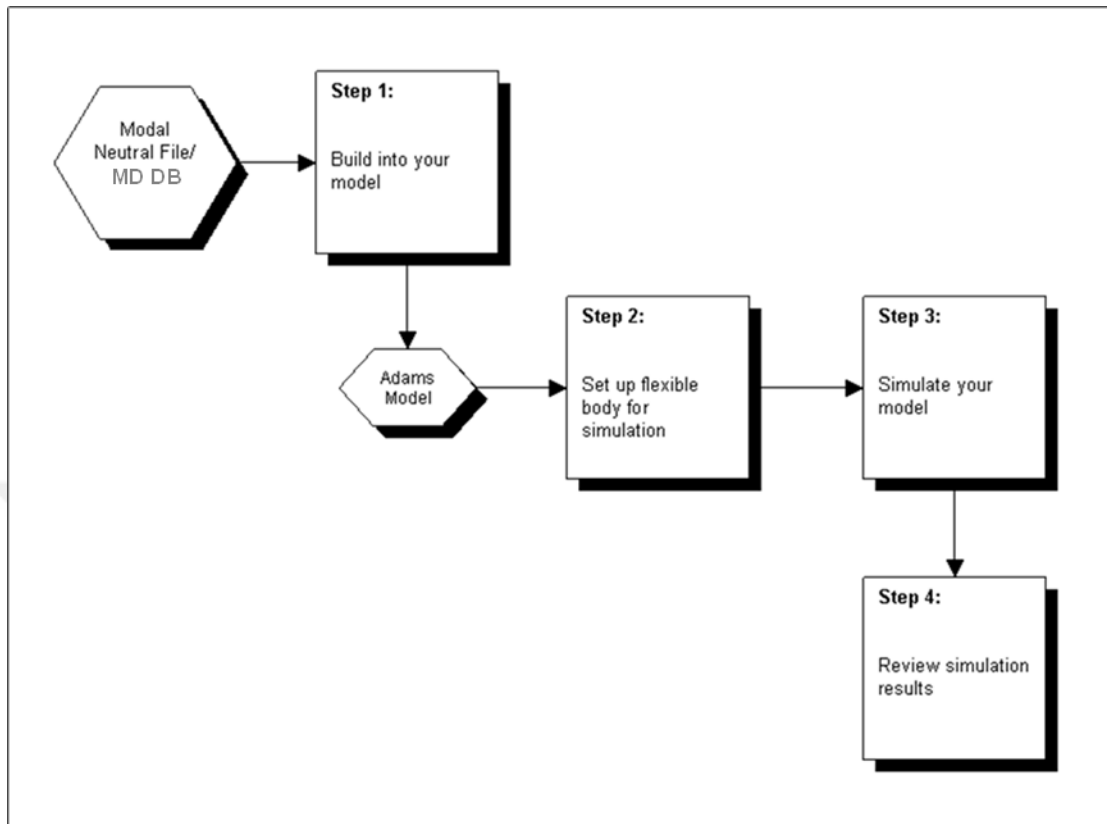


Figure 6-1: Steps in modeling flexible bodies in ADAMS (MSC.Software, 2003)

CHAPTER 7

MODELING BEAM-TIP MASS-BALL SYSTEMS

7.1 Coordinate Systems

Referring to Figure 7-1 and Figure 7-2, the following coordinate systems can be defined for the beam-tip mass-ball system

- $x - y$: Inertial coordinate system
- $x_1 - y_1$: Body coordinate system of the tip mass where origin is rigidly attached to the center of mass of the tip-mass
- $x_2 - y_2$: Ball coordinate system where origin attached to the end of the beam and the coordinate system rotates with the motion of the ball
- $x_3 - y_3$: Body coordinate system of the ball where origin is rigidly attached to the center of mass of the ball



Figure 7-1: Beam coordinate system

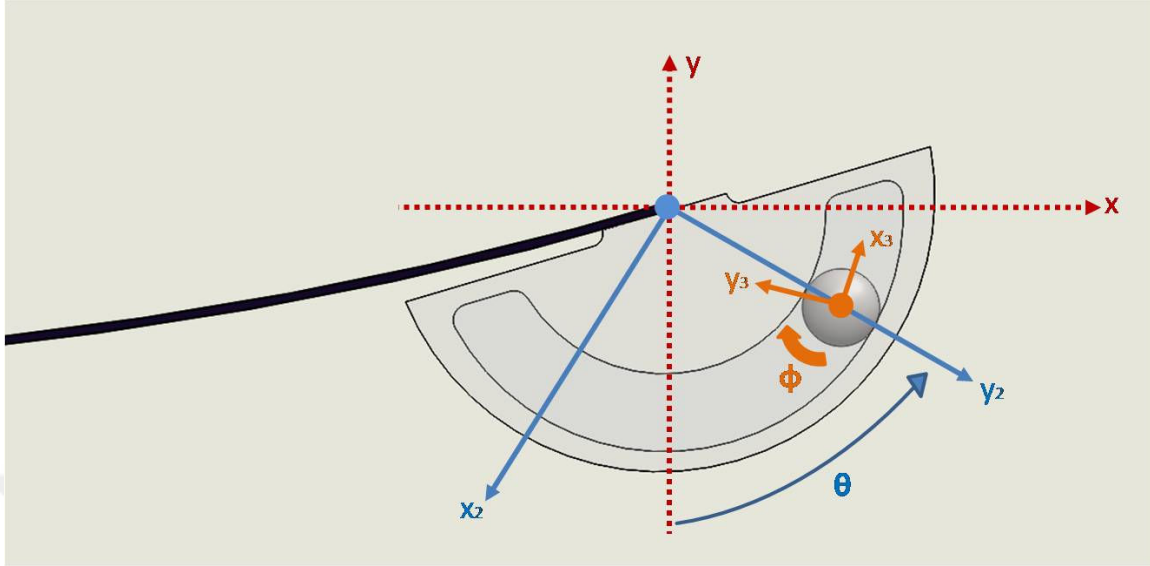


Figure 7-2: Tip Mass - Ball coordinate system

7.2 Generalized Coordinates

The beam-tip mass-ball system consists of three bodies among which the beam is assumed to be flexible, and the tip mass and ball are assumed to be rigid. The ANCF beam is modeled using three finite elements. Referring to Figure 7-3, the ANCF beam element has four nodes; each node has four degrees of freedom. Therefore, the total degree of freedom of the beam is 16. Table 1 shows the mapping from local to global node indices. Since the system is assumed to be planar, each rigid body has 3 degrees of freedom; as a result, the system has a total of 22 degrees of freedom.

Table 1: Mapping from local to global node indices

Finite Elements	Local Index of Degree of Freedom								Element d.o.f.
	1	2	3	4	5	6	7	8	
1	1	2	3	4	5	6	7	8	8
2	5	6	7	8	9	10	11	12	8
3	9	10	11	12	13	14	15	16	8

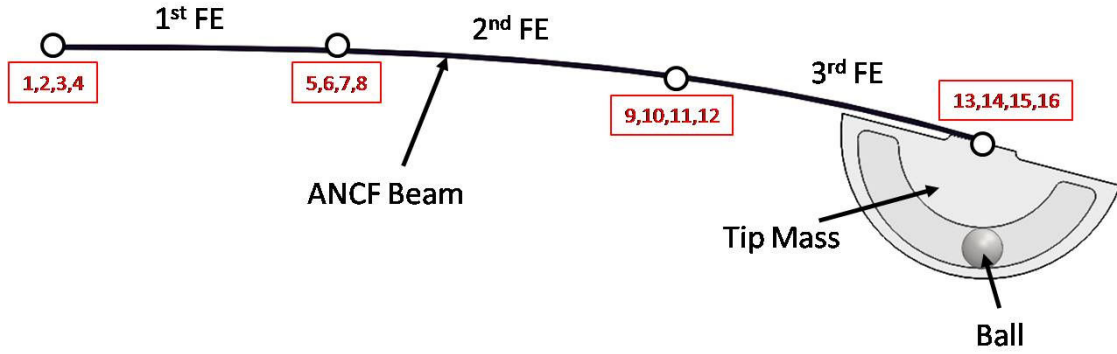


Figure 7-3: Global nodes

Therefore, the vector of system generalized coordinates can be defined as

$$\mathbf{q}_{sys} = [e_1 \ e_2 \ \cdots \ e_{15} \ e_{16} \ \phi \ R_{b_1} \ R_{b_2} \ R_{t_1} \ R_{t_2} \ \psi]^T \quad (7.1)$$

where $[e_1 \ e_2 \ \cdots \ e_{15} \ e_{16}]^T$ are beam absolute nodal coordinates, $[\phi \ R_{b_1} \ R_{b_2}]^T$ are ball rotational and translational coordinates, and $[R_{t_1} \ R_{t_2} \ \psi]^T$ are tip mass translational and rotational coordinates, respectively.

7.3 Kinematic Constraints

The connection between the free end of the beam and the tip mass is modeled using a fixed joint. Therefore, using equations 3.75 and 3.76 and referring to Figure 7-4, one can write the following constraint equations for the fixed joint between the two bodies

$$\mathbf{r}_A = \mathbf{R}_t + \mathbf{A}_t \mathbf{u}_t \quad (7.2)$$

$$\psi - \text{Arctan} \left[\frac{e_{16}}{e_{15}} \right] = 0. \quad (7.3)$$

$\mathbf{R}_t$ is the absolute displacement vector of the center of mass of the tip appendage given by

$$\mathbf{R}_t = \begin{pmatrix} R_{t1} \\ R_{t2} \end{pmatrix} \quad (7.4)$$

$\mathbf{r}_A$ is the absolute displacement vector of the beam free end A given by

$$\mathbf{r}_A = \begin{pmatrix} e_{13} \\ e_{14} \end{pmatrix} \quad (7.5)$$

and $\mathbf{A}_t$ is the transformation matrix between the coordinates $x - y$ and $x_1 - y_1$ defined as

$$\mathbf{A}_t = \begin{bmatrix} \cos \psi & -\sin \psi \\ \sin \psi & \cos \psi \end{bmatrix} \quad (7.6)$$

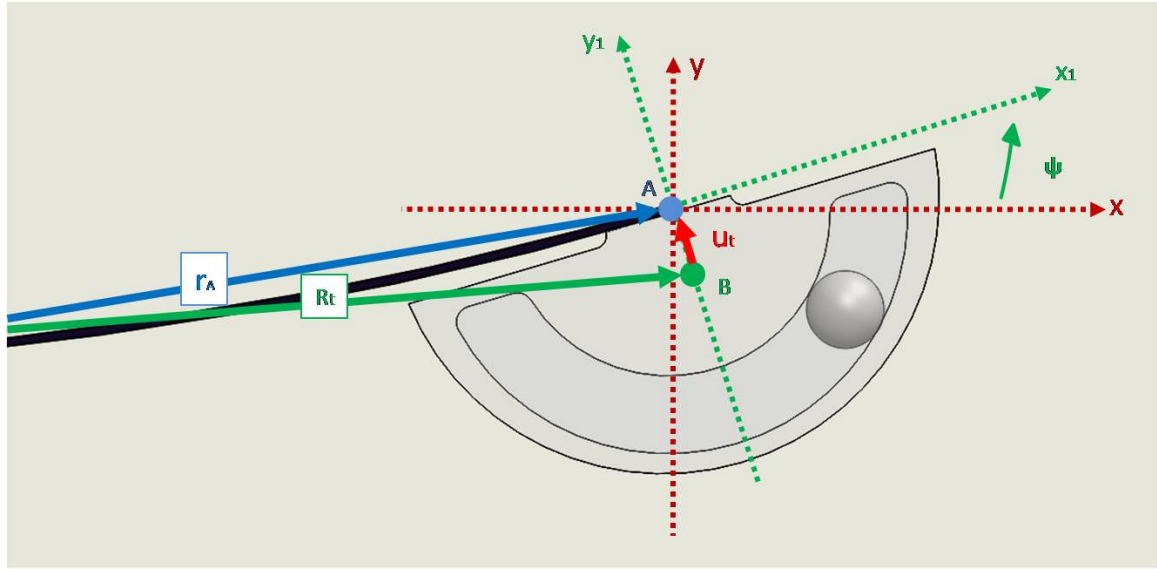


Figure 7-4: Tip Mass displacement vectors

$\mathbf{u}_t$ is the displacement vector of point A relative to point B in $x_1 - y_1$ coordinate system given by

$$\mathbf{u}_t = \begin{pmatrix} 0 \\ u_{t2} \end{pmatrix} \quad (7.7)$$

where u_{t2} is the distance between point A and point B. Therefore, constraint equations for the fixed joint can be written combining equations 7.2 and 7.3 as

$$e_{13} - R_{t_1} + u_{t_2} \sin \psi = 0 \quad (7.8)$$

$$e_{14} - R_{t_2} - u_{t_2} \cos \psi = 0 \quad (7.9)$$

$$\psi - \text{Arctan} \left[\frac{e_{16}}{e_{15}} \right] = 0 . \quad (7.10)$$

Constraint equations between the ball and the tip mass can be defined such that the velocity of the contact point C on the ball has to be equal to the velocity of the contact point C on the tip mass. Therefore, referring to Figure 7-5, one can write the following equations

$$\dot{\mathbf{r}}_{C_b} = \dot{\mathbf{r}}_{C_t} . \quad (7.11)$$

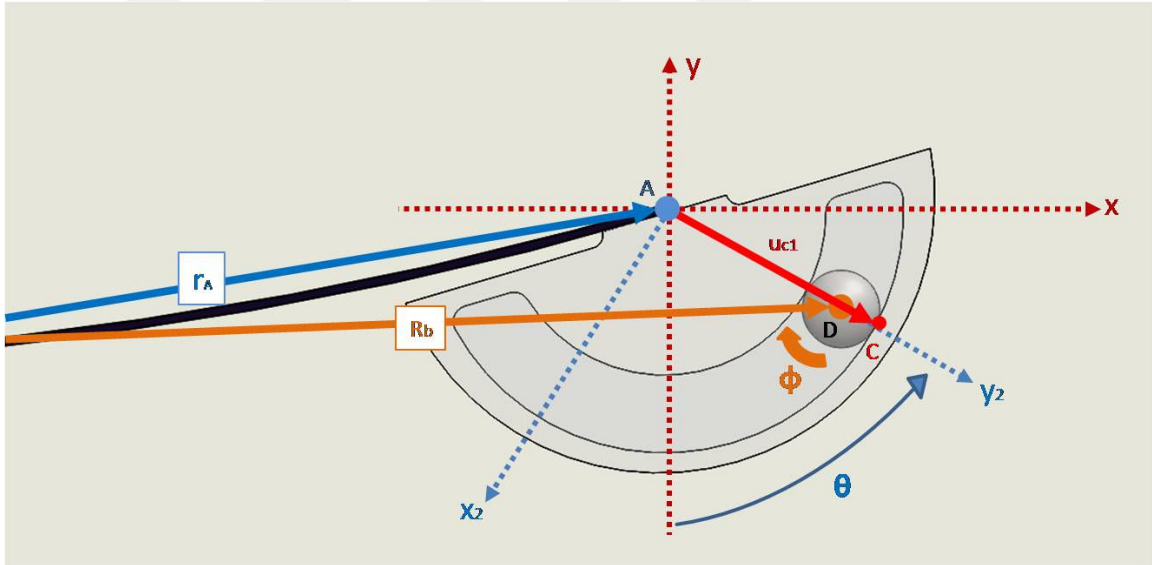


Figure 7-5: Ball displacement vectors

Using equation 2.5, $\dot{\mathbf{r}}_{C_b}$ and $\dot{\mathbf{r}}_{C_t}$ can be written as

$$\dot{\mathbf{r}}_{C_t} = \dot{\mathbf{r}}_A + \dot{\mathbf{A}}_{b1} \mathbf{u}_{c1} \quad (7.12)$$

$$\dot{\mathbf{r}}_{C_b} = \dot{\mathbf{R}}_b + \dot{\mathbf{A}}_{b2} \mathbf{u}_{c2} . \quad (7.13)$$

$\mathbf{u}_{c1}$ is the displacement vector of point C relative to point A in $x_2 - y_2$ coordinate system given by

$$\mathbf{u}_{c1} = \begin{pmatrix} 0 \\ R+r_b \end{pmatrix} \quad (7.14)$$

where R is the distance between point A and point D in $x_2 - y_2$ coordinate system and r_b is the radius of the ball.

$\mathbf{u}_{c2}$ is the displacement vector of point C relative to point D in $x_3 - y_3$ coordinate system given by

$$\mathbf{u}_{c2} = \begin{pmatrix} r_b \sin(\theta - \phi) \\ -r_b \cos(\theta - \phi) \end{pmatrix}. \quad (7.15)$$

$\mathbf{R}_b$ and $\dot{\mathbf{R}}_b$ are the absolute displacement and velocity vectors of the center of mass of the ball given by

$$\mathbf{R}_b = \begin{pmatrix} R_{b1} \\ R_{b2} \end{pmatrix}, \quad \dot{\mathbf{R}}_b = \begin{pmatrix} \dot{R}_{b1} \\ \dot{R}_{b2} \end{pmatrix}. \quad (7.16)$$

$\dot{\mathbf{r}}_A$ is the absolute velocity vector of the beam free end A given by

$$\dot{\mathbf{r}}_A = \begin{pmatrix} \dot{e}_{13} \\ \dot{e}_{14} \end{pmatrix}. \quad (7.17)$$

$\dot{\mathbf{A}}_{b1}$ is the time derivative of the transformation matrix $\mathbf{A}_{b1}$ between the coordinates $x - y$ and $x_2 - y_2$ defined as

$$\dot{\mathbf{A}}_{b1} = \begin{bmatrix} \dot{\theta} \sin \theta & \dot{\theta} \cos \theta \\ -\dot{\theta} \cos \theta & \dot{\theta} \sin \theta \end{bmatrix} \quad (7.18)$$

and $\dot{\mathbf{A}}_{b2}$ is the time derivative of the transformation matrix $\mathbf{A}_{b2}$ between the coordinates $x - y$ and $x_3 - y_3$ defined as

$$\dot{\mathbf{A}}_{b2} = \begin{bmatrix} \dot{\phi} \sin \phi & \dot{\phi} \cos \phi \\ -\dot{\phi} \cos \phi & \dot{\phi} \sin \phi \end{bmatrix}. \quad (7.19)$$

Therefore, equations 7.12 and 7.13 can be defined using above expressions as

$$\dot{\mathbf{r}}_{\mathbf{C}_t} = \begin{pmatrix} \dot{e}_{13} + (R + r_b) \cos \theta \dot{\theta} \\ \dot{e}_{14} + (R + r_b) \sin \theta \dot{\theta} \end{pmatrix} \quad (7.20)$$

$$\dot{\mathbf{r}}_{\mathbf{C}_b} = \begin{pmatrix} \dot{R}_{b_1} + r_b \cos \theta \dot{\theta} \\ \dot{R}_{b_2} + r_b \sin \theta \dot{\theta} \end{pmatrix}. \quad (7.21)$$

Using equations 7.11, 7.20, and 7.21, one can obtain the following constraint equations

$$\dot{R}_{b_1} + r_b \cos \theta \dot{\theta} = \dot{e}_{13} + (R + r_b) \cos \theta \dot{\theta} \quad (7.22)$$

$$\dot{R}_{b_2} + r_b \sin \theta \dot{\theta} = \dot{e}_{14} + (R + r_b) \sin \theta \dot{\theta}. \quad (7.23)$$

Note that the constraint equations 7.22 and 7.23 can be integrated in time, and the following holonomic constraint equations can be obtained

$$R_{b_1} = e_{13} + R \sin \theta \quad (7.24)$$

$$R_{b_2} = e_{14} - R \cos \theta. \quad (7.25)$$

Another constraint equation can be written using Figure 7-5 as

$$R(\theta - \psi) = \phi r_b \quad (7.26)$$

Using equation 7.26, θ can be found in terms of ψ and ϕ , and this expression can be inserted into the equations 7.24 and 7.25.

Finally, using equations 7.8, 7.9, 7.10, 7.24, 7.25, and 7.26, one can write the holonomic constraint equations of beam-tip mass-ball system as

$$\mathbf{C} = \left\{ \begin{array}{l} e_{13} - R_{t_1} + u_{t_2} \sin \psi = 0 \\ e_{14} - R_{t_2} - u_{t_2} \cos \psi = 0 \\ \psi - \text{Arctan} \left[\frac{e_{16}}{e_{15}} \right] = 0 \\ R_{b_1} - e_{13} - R \sin \left(\phi \frac{r_b}{R} + \psi \right) = 0 \\ R_{b_2} - e_{14} + R \cos \left(\phi \frac{r_b}{R} + \psi \right) = 0 \end{array} \right\}. \quad (7.27)$$

7.4 Identifying Dependent and Independent Coordinates

Referring to equation 7.27, the beam-tip mass-ball system has five constraint equations. Therefore, the system has five dependent coordinates and seventeen independent coordinates. Vectors of dependent and independent coordinates can be written using equation 7.1 as

$$\mathbf{q}_i = [e_1 \ e_2 \ \cdots \ e_{15} \ e_{16} \ \phi]^T \quad (7.28)$$

$$\mathbf{q}_d = [R_{b_1} \ R_{b_2} \ R_{t_1} \ R_{t_2} \ \psi]^T. \quad (7.29)$$

Therefore, using the constraint equations 7.27 and the vectors 7.28 and 7.29, one can define 5×17 matrix $\mathbf{C}_{q_i}$ and 5×5 matrix $\mathbf{C}_{q_d}$.

7.5 Impact Force

Referring to Figure 7-6, if $|\theta - \psi| \geq 65^\circ$, the system will have impact between the tip mass and the ball. The contact force, $\mathbf{F}_c$, between the bodies during the collision can be calculated using equations 4.9 and 4.10, where expressions in those equations can be given for the beam-tip mass-ball system as

$$m^i = m_t \ , \ m^j = m_b \quad (7.30)$$

$$\mathbf{v}_1^i = \begin{pmatrix} \dot{R}_{t_1} \\ \dot{R}_{t_2} \end{pmatrix}_t \ , \ \mathbf{v}_1^j = \begin{pmatrix} \dot{R}_{b_1} \\ \dot{R}_{b_2} \end{pmatrix}_t \quad (7.31)$$

where subscript t denotes the time of collision, and $\mathbf{v}_2^i$ and $\mathbf{v}_2^j$ can be found using equations 4.4, 4.5, 4.6, and 4.7.

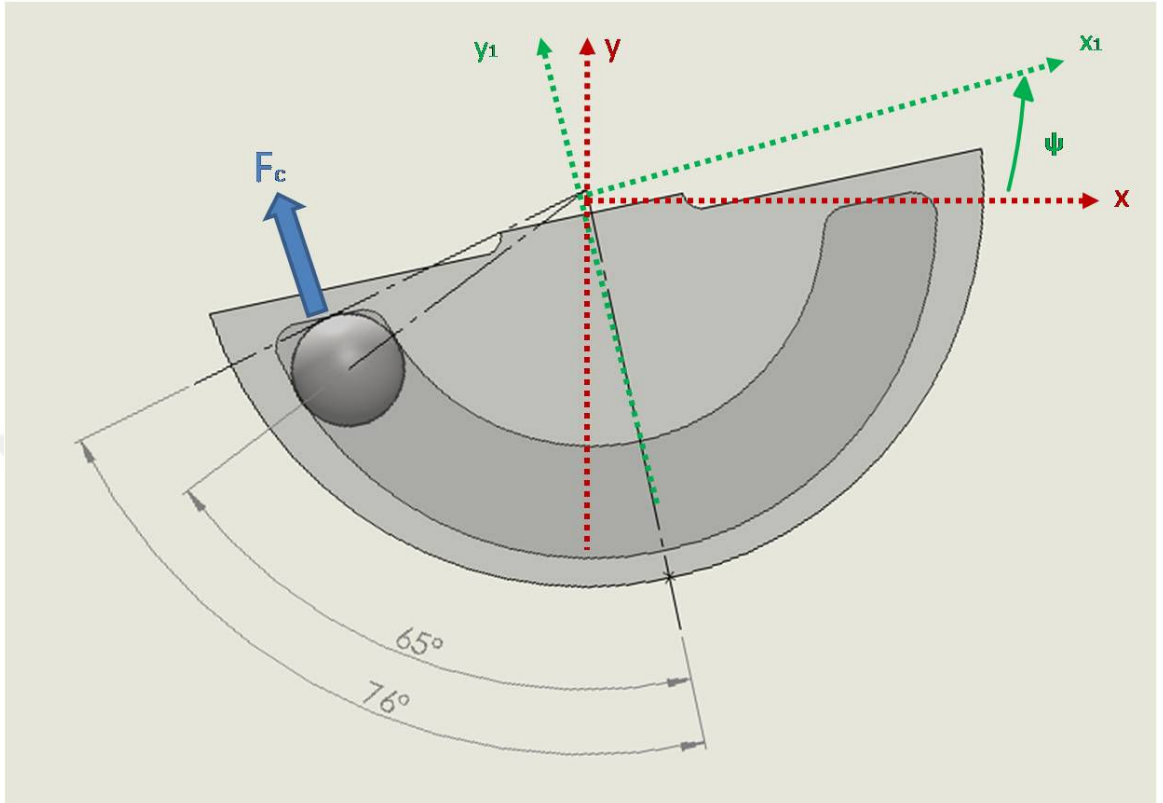


Figure 7-6: Impact Condition

7.6 Rigid Body Equations

The system has two rigid bodies, namely, the tip mass and the ball. Therefore, using equations 2.23 and 2.24, the mass matrix, generalized external forces, and generalized displacement vectors for the rigid bodies can be written as

$$\mathbf{q}_{rigid} = [\phi \quad R_{b_1} \quad R_{b_2} \quad R_{t_1} \quad R_{t_2} \quad \psi]^T \quad (7.32)$$

$$\mathbf{M}_{rigid} = \begin{bmatrix} J_b & 0 & 0 & 0 & 0 & 0 \\ 0 & m_b & 0 & 0 & 0 & 0 \\ 0 & 0 & m_b & 0 & 0 & 0 \\ 0 & 0 & 0 & J_t & 0 & 0 \\ 0 & 0 & 0 & 0 & m_t & 0 \\ 0 & 0 & 0 & 0 & 0 & m_t \end{bmatrix} \quad (7.33)$$

$$\mathbf{Q}_{e_{rigid}} = \begin{bmatrix} 0 \\ F_c \sin \psi \\ -m_b g - F_c \cos \psi \\ -F_c \sin \psi \\ -m_t g + F_c \cos \psi \\ F_c l_c \end{bmatrix}. \quad (7.34)$$

7.7 System Assembly

7.7.1 System Mass Matrix

The system mass matrix can be obtained using classical finite element assembly procedures as

$$\mathbf{M}_{sys} = \sum_{j=1}^3 \mathbf{B}^j T \mathbf{M}_a^j \mathbf{B}^j + \mathbf{B}^r T \mathbf{M}_{rigid} \mathbf{B}^r \quad (7.35)$$

where $\mathbf{M}_a^j$ for each of the three beam elements is given by equation 3.8, $\mathbf{M}_{rigid}$ is given in equation 7.33, and the Boolean Matrices can be defined as

$$\mathbf{B}^1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (7.36)$$

$$\mathbf{B}^2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (7.37)$$

$$\mathbf{B}^3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (7.38)$$

$$\mathbf{B}^r = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \quad (7.39)$$

7.7.2 System Generalized Forces

The system generalized forces including generalized external forces, beam elastic forces, and impact forces can be defined using Boolean Matrices as

$$\mathbf{Q}_{sys} = \sum_{j=1}^3 \mathbf{B}^{jT} \mathbf{Q}_e^j + \mathbf{B}^{rT} \mathbf{Q}_{e_{rigid}} - \sum_{j=1}^3 \mathbf{B}^{jT} \mathbf{Q}_k^j \quad (7.40)$$

where $\mathbf{Q}_e^j$ for each of the beam elements is given by equation 3.65 since only external force acting on the system is the gravity force, $\mathbf{Q}_k^j$ for each beam element is given by equation 3.57, $\mathbf{Q}_{e_{rigid}}$ is given in equation 7.34, and the Boolean Matrices are defined previously in equations 7.36, 7.37, 7.38, and 7.39.

7.7.3 System Generalized Constraint Forces

The jacobian of the system constraint equations can be defined as

$$\mathbf{C}_{q_{sys}} = \frac{\partial \mathbf{C}}{\partial \mathbf{q}_{sys}} = \begin{bmatrix} \frac{\partial C_1}{\partial e_1} & \frac{\partial C_1}{\partial e_2} & \dots & \frac{\partial C_1}{\partial \psi} \\ \frac{\partial C_2}{\partial e_1} & \frac{\partial C_2}{\partial e_2} & \dots & \frac{\partial C_2}{\partial \psi} \\ \frac{\partial C_3}{\partial e_1} & \frac{\partial C_3}{\partial e_2} & \dots & \frac{\partial C_3}{\partial \psi} \\ \frac{\partial C_4}{\partial e_1} & \frac{\partial C_4}{\partial e_2} & \dots & \frac{\partial C_4}{\partial \psi} \\ \frac{\partial C_5}{\partial e_1} & \frac{\partial C_5}{\partial e_2} & \dots & \frac{\partial C_5}{\partial \psi} \end{bmatrix} \quad (7.41)$$

where C_1, C_2, C_3, C_4 , and C_5 are the 1st, 2nd, 3rd, 4th, and 5th rows of the system constraint equations vector given by 7.27, and the vector of system generalized coordinates $\mathbf{q}_{sys}$ is given in equation 7.1.

Vector $\mathbf{Q}_{d_{sys}}$ can be found using equation 3.108 as

$$\mathbf{Q}_{d_{sys}} = -\mathbf{C}_{tt} - \left(\mathbf{C}_{q_{sys}} \dot{\mathbf{q}}_{sys} \right)_{q_{sys}} \dot{\mathbf{q}}_{sys} - 2\mathbf{C}_{q_{sys}t} \dot{\mathbf{q}}_{sys}. \quad (7.42)$$

7.8 System Equations of Motion

The system equations of motion in augmented form can be written as

$$\begin{bmatrix} \mathbf{M}_{sys} & \mathbf{C}_{q_{sys}}^T \\ \mathbf{C}_{q_{sys}} & 0 \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{q}}_{sys} \\ \boldsymbol{\lambda}_{sys} \end{bmatrix} = \begin{bmatrix} \mathbf{Q}_{sys} \\ \mathbf{Q}_{d_{sys}} \end{bmatrix} \quad (7.43)$$

where $\mathbf{M}_{sys}$, $\mathbf{Q}_{sys}$, $\mathbf{C}_{q_{sys}}$, and $\mathbf{Q}_{d_{sys}}$ are given by equations 7.35, 7.40, 7.41, and 7.42, respectively.

Since the system has five constraint equations, the number of Lagrange Multipliers of the system, $\boldsymbol{\lambda}_{sys}$, can be defined as

$$\lambda_{sys} = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \end{bmatrix}. \quad (7.44)$$

7.9 Boundary Conditions

The system has a sinusoidal excitation at the left end point of the beam, and the slope vector at the left end of the beam also has to be horizontal since the beam is clamped at that end. Therefore, the boundary conditions for the left end of the beam can be written as

$$\begin{cases} e_1 = 0 \\ e_2 = X \sin \omega t \\ e_4 = 0 \end{cases} \quad (7.45)$$

where X is the amplitude of the excitation and ω is the excitation frequency.

7.10 Numerical Integration

7.10.1 System Parameters

Numerical integration parameters, rigid body parameters and flexible body parameters are given in Table 2, Table 3, and Table 4, respectively.

Table 2: Numerical integration parameters

Newmark Parameters
$\gamma = 0.7$
$\beta = 0.36$
$h = 0.003 \text{ sec}$
Newton-Raphson Parameters
$\varepsilon = 10^{-5}$
$n_m = 100$

Table 3: Rigid body data

Tip Mass
$J_t = 1.871 \times 10^{-5} \text{ kg} \cdot \text{m}^2$
$m_t = 0.1154 \text{ kg}$
$u_{t_2} = 0.0204 \text{ m}$
Ball
$J_b = 3.15 \times 10^{-7} \text{ kg} \cdot \text{m}^2$
$m_b = 0.014 \text{ kg}$
$r_b = 0.0075 \text{ m}$
$R = 0.0395 \text{ m}$

Table 4: Flexible beam data

Flexible Beam
$l = \frac{0.365}{3} \text{ m}$
$m = \frac{0.116}{3} \text{ kg}$
$a = 4.06 \times 10^{-5} \text{ m}^2$
$E = 200 \text{ Gpa}$
$I = 8.47 \times 10^{-12} \text{ m}^4$

7.10.2 Initial Conditions

One can perform static analysis to the beam-tip mass-ball system to find the initial $\mathbf{q}_{sys}$ values. Also, the initial velocity, acceleration, and Lagrange Multiplier vectors can be approximated and the following initial conditions can be defined for the system

$$\mathbf{q}_{sysint} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0.12163 \\ -0.00248 \\ 0.99928 \\ -0.03796 \\ 0.24315 \\ -0.00848 \\ 0.9983 \\ -0.05821 \\ 0.36458 \\ -0.01605 \\ 0.99794 \\ -0.06416 \\ -0.33815 \\ 0.36458 \\ -0.05555 \\ 0.36327 \\ -0.03641 \\ -0.06421 \end{bmatrix}, \quad \dot{\mathbf{q}}_{sysint} = \begin{bmatrix} 0 \\ \mathbf{X}\omega \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \ddot{\mathbf{q}}_{sysint} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -g \\ 0 \\ 0 \\ 0 \\ 0 \\ -g \\ 0 \\ 0 \\ 0 \\ -g \\ 0 \\ 0 \\ 0 \\ 0 \\ -g \\ 0 \\ -g \\ 0 \end{bmatrix}, \quad \lambda_{sysint} = \begin{bmatrix} 0 \\ -\mathbf{m}_b \mathbf{g} \\ 0 \\ -\mathbf{m}_t \mathbf{g} \\ 0 \end{bmatrix}. \quad (7.46)$$

CHAPTER 8

RESULTS & DISCUSSION

8.1 One-Ball Case

The first case for the numerical investigation consists of the beam-tip mass with a one-ball arrangement. System parameters given in Table 3 and Table 4 are used during the numerical solution, and detailed system dynamics, including frequency response curves, time history curves, FFT curves, phase plane curves, and energy curves, are plotted for various base excitation frequencies. For each of the numerical analysis, the frequency of the beam-tip mass system is set to 4.13 Hz, and in order to maintain the condition of autoparametric interaction, the frequency of the ball is tuned to one half of the beam-tip mass system frequency (i.e., it is set to 2.065 Hz).

Figure 8-1 shows the frequency response curves of the beam and the ball for the forcing amplitude of 1mm peak-to-peak. To create this figure, a numerical analysis at the base excitation frequencies 3.50 Hz, 3.70 Hz, 3.80 Hz, 3.85 Hz, 3.90 Hz, 3.95 Hz, 4.00 Hz, 4.05 Hz, 4.10 Hz, 4.13 Hz, 4.25 Hz, 4.40 Hz, 4.50 Hz, and 4.70 Hz was performed. At each excitation frequency, transients were allowed to die out before the amplitude of the response was recorded. In other words, steady-state response amplitudes were used in order to create frequency response curves. A detailed investigation of the Figure 8-1 reveals a strong autoparametric interaction between the beam and the ball when the forcing frequency reaches 3.95 Hz and the first jump phenomenon is observed. At this point, point A on the beam response curve and point C on the ball response curve, the ball starts to oscillate and the beam's response starts to decrease. The oscillation of the ball and the decrease in the beam response continue until points B and D where the frequency of the beam is 4.13 Hz. Therefore, points A-B on the beam response curve and points C-D on the ball response curve are very important since they define the autoparametric region of the energy exchange. From Figure 8-1, it is evident that the

amplitude of the beam is decreased and the energy is transferred to the ball when the primary resonance case is reached.

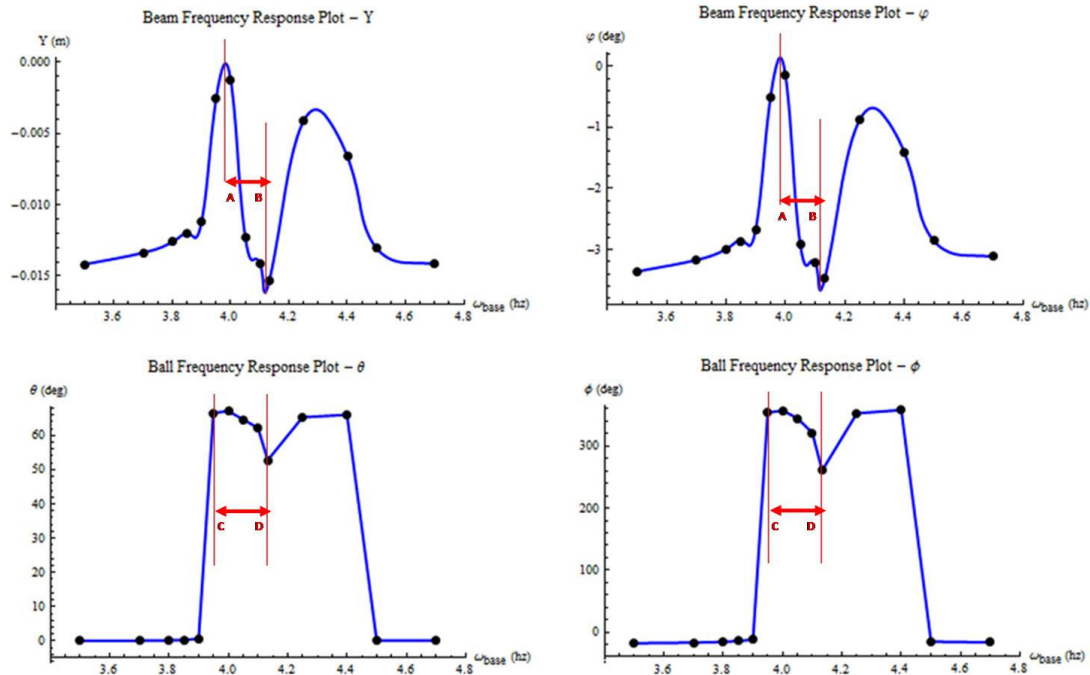


Figure 8-1: Frequency response curves for an excitation amplitude of 1mm

The frequency response curve of a previously studied similar system (Cuvalci, et al., 1996) that consists of beam-tip mass and pendulum attachment is shown in Figure 8-2. Comparing Figure 8-1 with Figure 8-2, one can observe the similarities between the response patterns such that the second system (the ball for this case, the pendulum for the reference case) has a sudden amplitude increase at the start of the autoparametric region whereas the beam response starts to decrease. Moreover, the second peak observed in the beam response for both cases is due to the beating phenomena and will be explained later in this section.

Figure 8-3 shows a recently done experimental study on the beam-tip mass-ball system by Landis (2009) with slightly different system parameters. A comparison of this curve with the numerical findings in Figure 8-1 reveals a good qualitative agreement between

frequency response curves, especially ball response curves. One can easily see similar patterns between these response curves.

For each of the numerical analyses performed to obtain Figure 8-1, the detailed system dynamics, including impact details, beam-ball trajectories, and FFT of the base excitation, beam, and ball, are given. In addition to this data, additional system information such as time histories of the system angles, translational displacements, phase plane curves, system velocity curves, beam tip translational acceleration curves, and system energy curves are given for some special points on the frequency response curve. Those important frequency values are as follows: 3.70 Hz is the frequency before the autoparametric region, 4.00 Hz is the frequency inside the autoparametric region, 4.13 Hz is the frequency at which complete energy transfer occurs, 4.25 Hz is the frequency where beating phenomena is observed, and 4.50 Hz is the frequency after the autoparametric region.

Before having detailed discussions on the active absorber cases, one can refer to Figure 8-4, which shows the system dynamics when the ball is locked. For the passive absorber case, the ball is locked inside the housing track and the system is excited at its natural frequency. Figure 8-5 shows the phase plane curve of the system where a periodic response can be seen. As shown in Figure 8-6, the FFT of the system has a single peak at 4.13 Hz, which is the natural frequency of the system.

Figure 8-7 to Figure 8-27 shows system dynamics for the forcing frequencies before the first jump phenomenon. Common observations for the forcing frequency range 3.50 Hz to 3.90 Hz can be stated as

- The impact force during the total simulation time is zero; in other words, impact does not exist between the ball and the tip mass for these frequency ranges. This result can also be seen by looking to the time history curves of the beam and the ball in which the beam has oscillation whereas the ball does not have any of significance.

- The second most common observation is obtained by investigating the phase plane curves. Although numerical values on the horizontal and vertical axis can change slightly for each frequency value, all phase plane curves have the same pattern. As an example, the phase plane curves for the forcing frequency of 3.70 Hz are shown in Figure 8-13. At this point, it is important to note that these curves are obtained for the full simulation time. Therefore, curves start from the static equilibrium position. After passing the transient response region, they go to steady state where darker circular patterns on the curves are due to responses at steady state. Beam y vs $\dot{y}$ and ball ϕ vs $\dot{\phi}$ curves prove that the responses are periodic. The noninteractive motion between the beam and the ball can be verified by looking to θ vs ψ and ϕ vs ψ curves in Figure 8-13. From these curves, it is clear that both the beam and the ball frequency ratios are one.

Investigating the FFT curves for the frequency range 3.50 Hz to 3.90 Hz shows that the dominant frequency information is due to the base excitation. Figure 8-9 and Figure 8-17 shows frequency information for 3.50 Hz and 3.70 Hz, respectively. In both figures, two peaks are observed in which one of them is the base excitation frequency, which is dominant, and the other is the beam-mass frequency. Since the ball is not oscillating, the ball frequency information in these figures cannot be observed.

The FFT curves for 3.80 Hz, 3.85 Hz, and 3.90 Hz are shown in Figure 8-21, Figure 8-24, and Figure 8-27, respectively. One can see the frequency information of the ball due to the slight ball oscillations as compared to the previous cases. However, frequency information of the base excitation still dominates since the system is not in autoparametric region.

Figure 8-28 through Figure 8-30 show the dynamics of the beam and the ball at the start of the autoparametric region, namely 3.95 Hz. As it can be seen from time history curves, the ball starts to oscillate significantly; therefore, one can see the frequency of the ball in Figure 8-30. Also, referring to Figure 8-28, the impact force between the ball and the tip mass can be observed.

The dynamics of the beam and the ball at the forcing frequency of 4.00 Hz within the autoparametric region is given in Figure 8-31 through Figure 8-39. Figure 8-38 shows that the ball frequency is one-half of the beam frequency. Also, one should note that the frequency information of the beam can be seen inside the FFT curve of the ball. The coupling effect of the ball on the beam can be seen by looking to θ vs ψ and ϕ vs ψ curves in Figure 8-34. Referring to the energy curves in Figure 8-39, one can see that the dominant energy term for the beam is the strain energy, for the ball is the kinetic energy, and for the tip mass is the potential energy.

Figure 8-40 through Figure 8-45 show the system dynamics for the forcing frequency of 4.05 Hz and 4.10 Hz. Numerical results show similar patterns with 4.00 Hz since all of them are within the autoparametric region.

Among the numerical analysis performed for the one-ball case, the most important results are given in Figure 8-46 through Figure 8-58 for the forcing frequency of 4.13 Hz where complete energy transfer from the beam to the ball takes place. From Figure 8-54, one can see that the beam frequency (4.13 Hz) is twice the ball frequency (2.065 Hz) and the forcing frequency is equal to the beam frequency (4.13 Hz), which is the condition for the complete autoparametric interaction. Coupling between the modes of the beam and the ball can also be verified by looking to the phase plane curves of the system given in Figure 8-49. Note that these curves are obtained for the full simulation time. In order to clarify the existence of the coupling effect, one can refer to Figure 8-53 in which phase plane curves are obtained using steady state data from one cycle to the next. The loop shown in this figure is the evidence for one to two frequency relationships between the beam and the ball.

For further proof of the energy transfer, one can refer to Figure 8-57, which shows the kinetic energies of the beam and the ball during the full simulation time. From this figure, one can see that the beam kinetic energy diminishes as time passes; however, the ball kinetic energy still exists. A similar pattern can be seen in Figure 8-50 where time history plots of the beam and the ball are shown.

Detailed system energy curves given in Figure 8-55 through Figure 8-58 will be compared with ADAMS data found in section 8.2 in order to validate the numerical results found by the absolute nodal coordinate formulation.

Referring to Figure 8-84, which compares the transverse displacement of the beam tip for active and passive absorber cases, and Figure 8-85 that compares the rotational angle of the ball for active and passive absorber cases, one can claim the ball as a suitable autoparametric vibration absorber under periodic excitation.

Looking to Figure 8-61 and Figure 8-63, time history curves of the beam and the ball for the forcing frequency of 4.25 Hz show an interesting phenomenon called *beating* in vibration literature. Beating can be expressed as a special case in which the amplitude of the vibrations periodically vary when the forcing frequency is very close to the frequency of the free vibrations of the system. Beating is undesirable in absorber systems because energy can be transferred from the secondary system (ball) back to the primary system (beam-tip mass) as it can be seen from system energy curves in Figure 8-68. At this point, it is noteworthy to mention that Landis (2009) experimentally observed beating at 4.25 Hz for the beam-tip mass-ball system having slightly different input parameters.

As it can be seen from the FFT of the ball in Figure 8-66, besides the peak value at 2.065 Hz that corresponds to the ball frequency, a peak with a value around 1.9 Hz has been observed that is different than the excitation frequency (4.25 Hz) and the beam frequency (4.13 Hz). The source of this peak can be attributed to the beating phenomenon. Therefore, a beating computation analysis will be performed to figure out and confirm this conclusion.

Figure 8-67 shows the time history curve of the ball during a complete beating cycle. Referring to this figure, the beating period can be computed using the start and end time of the cycle as

$$BeatingPeriod = 26.1 - 22.2 = 3.9 \text{ sec} . \quad (8.1)$$

Similarly, the oscillation period can be computed as

$$OscillationPeriod = \frac{BeatingPeriod}{8} = 0.4875 \text{ sec} \quad (8.2)$$

where 8 is the number of oscillations during one complete beating cycle.

Moreover, equations for the beating period and the oscillation period in terms of the beating frequencies ω_1 and ω_2 can be written as (Mayoof, 2009)

$$BeatingPeriod = \frac{2\pi}{\omega_1 - \omega_2} \quad (8.3)$$

$$OscillationPeriod = \frac{4\pi}{\omega_1 + \omega_2} . \quad (8.4)$$

Combining equations 8.1, 8.2, 8.3, and 8.4 and solving for the unknown frequencies, one can obtain

$$\omega_1 = 13.82 \frac{rad}{s} = 2.2 \text{ Hz}$$

$$\omega_2 = 11.94 \frac{rad}{s} = 1.9 \text{ Hz} .$$

Therefore, it is confirmed that the additional peaks seen around 1.9 Hz and 2.2 Hz in the ball FFT in Figure 8-66 are due to beating.

Similarly, investigating the beam FFT curve in Figure 8-66, one can see two major peaks corresponding to the excitation frequency (4.25 Hz) and the beam natural frequency (4.13 Hz) and two minor peaks corresponding to integer multiples (x2) of the beating frequencies ω_1 and ω_2 .

Figure 8-72 to Figure 8-83 show system dynamics for the forcing frequencies after autoparametric region, namely, for 4.50 Hz and 4.70 Hz. These figures show similar patterns with the ones obtained before autoparametric region. It is observed that the impact does not exist between the ball and the tip mass for these frequency ranges. Furthermore, time history curves reveal that the oscillations of the ball and the beam are considerably lower than the ones obtained inside autoparametric region. Phase plane curves also have similar patterns. Investigating Figure 8-75 for the forcing frequency of

4.50 Hz, beam y vs $\dot{y}$ and ball ϕ vs $\dot{\phi}$ curves prove that the responses are periodic, and θ vs ψ and ϕ vs ψ curves verify non-interactive motion between the beam and the ball. Moreover, from these curves, it is clear that both the beam and the ball frequency ratios are one.

Figure 8-79 and Figure 8-83 show FFT information for frequencies 4.50 Hz and 4.70 Hz, respectively. In both figures, two peaks are observed where one of them is the base excitation frequency, which is dominant, and the other one is the beam-mass frequency. Since the ball is not oscillating in these cases, ball frequency information cannot be observed.

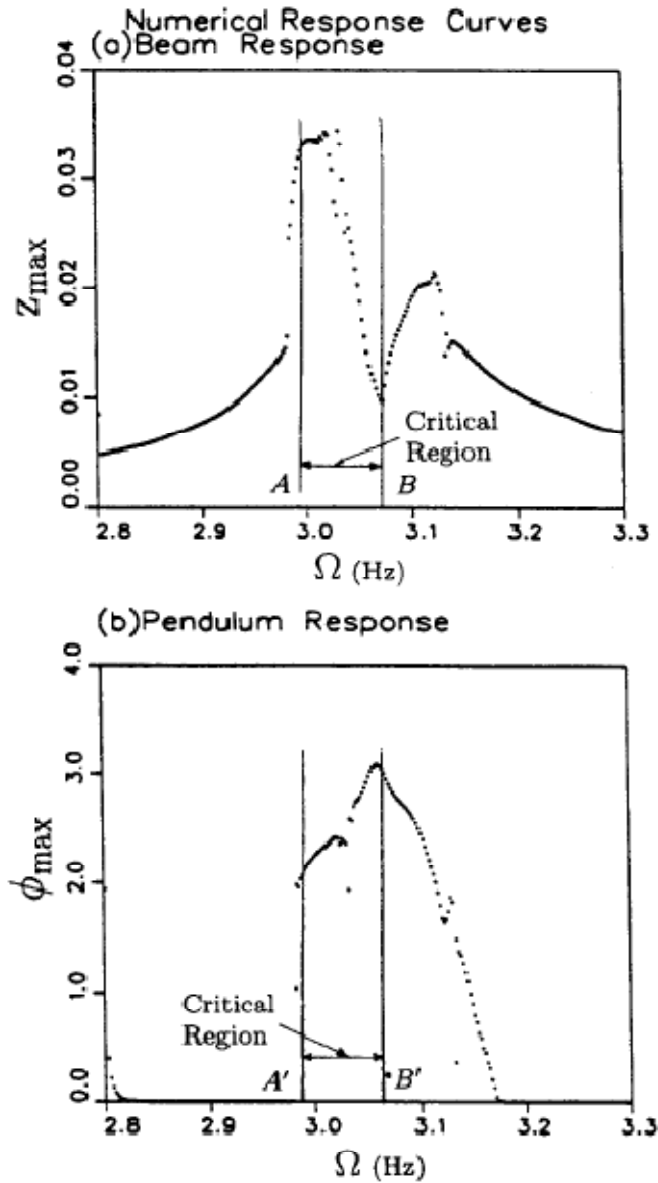


Fig. 5 Frequency response curves for $f = 0.000125$ m, $m_p/m \approx 1/4$, $c_b = 0.10999$ kg/s, $c_p = 0.02182$ kg/s, and $EI = 1.373974$ Nm²

Figure 8-2: Beam-tip mass-pendulum system frequency response curves (Cuvalci, et al., 1996)

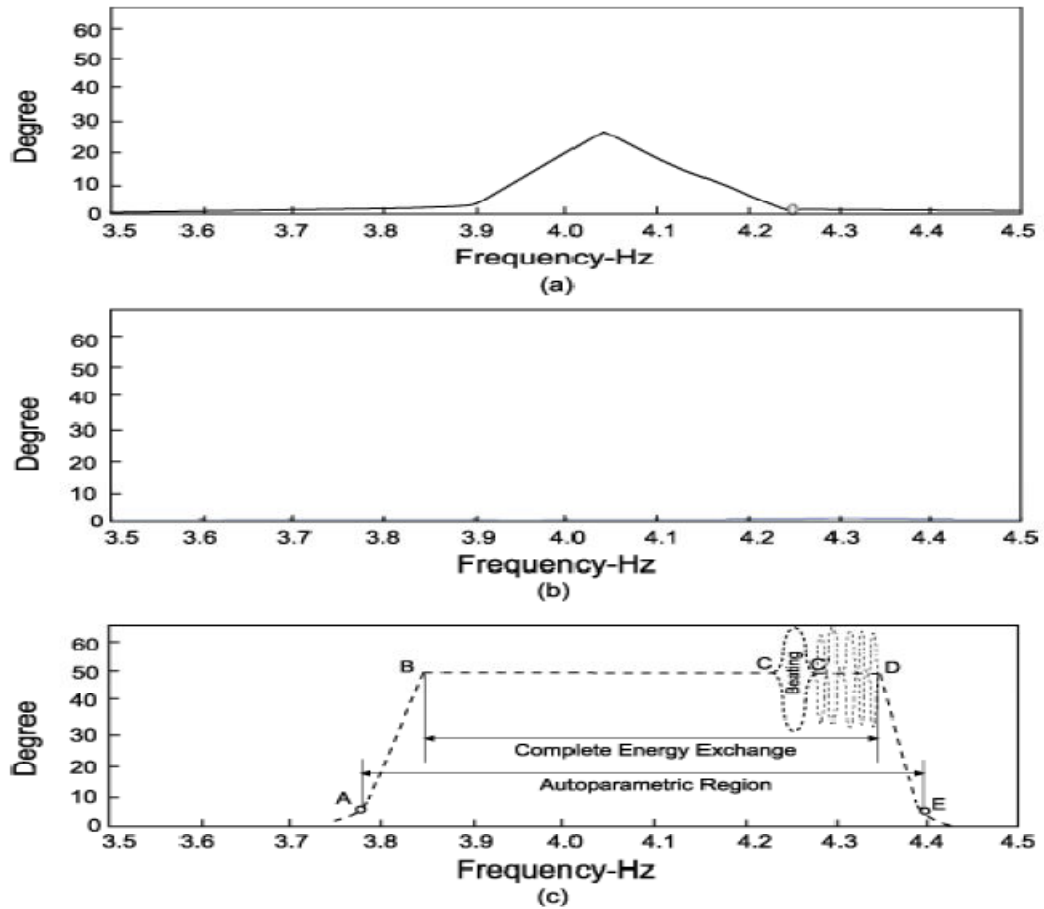


Figure 5-3 Experimental frequency response curves. (a) without absorber, (b) beam-mass response with absorber, (c) free-pendulum response for $f_0 = 1.88$ mm and

Figure 8-3: Beam-tip mass-ball system frequency response curves (Landis, 2009)

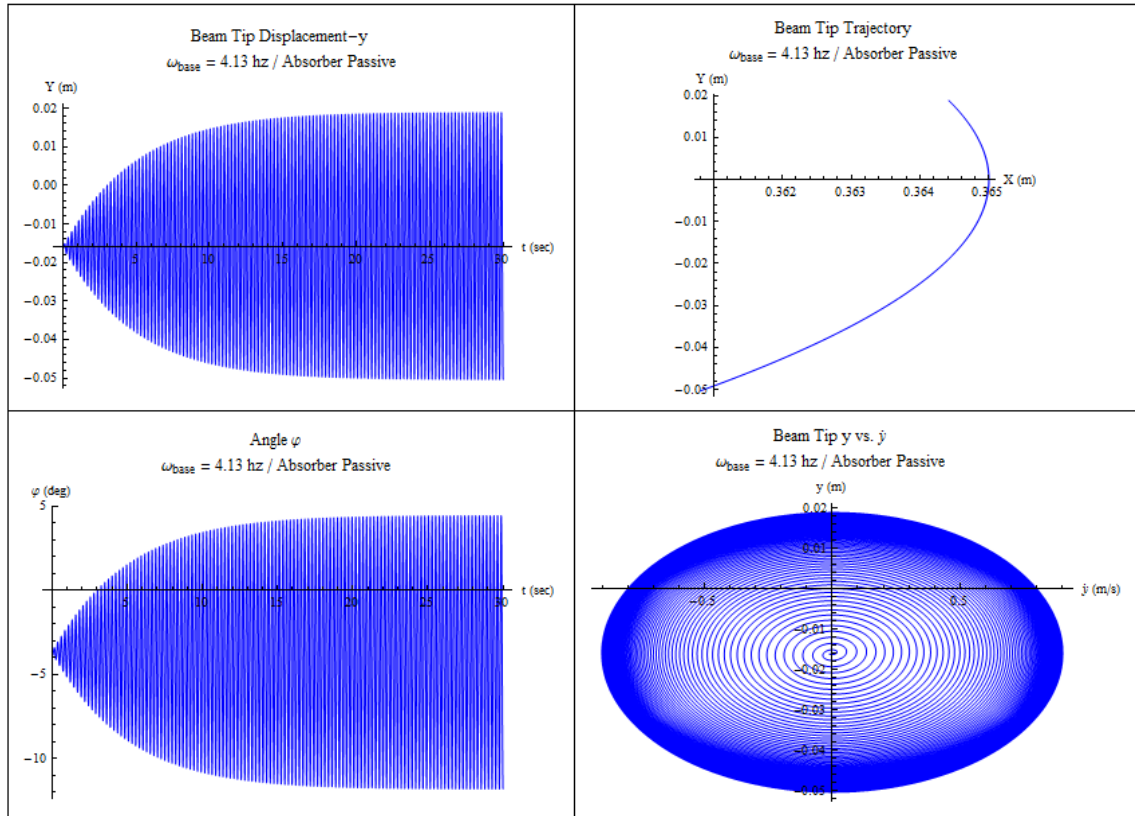


Figure 8-4: Ball locked / detailed dynamics of beam for the forcing frequency of 4.13 Hz

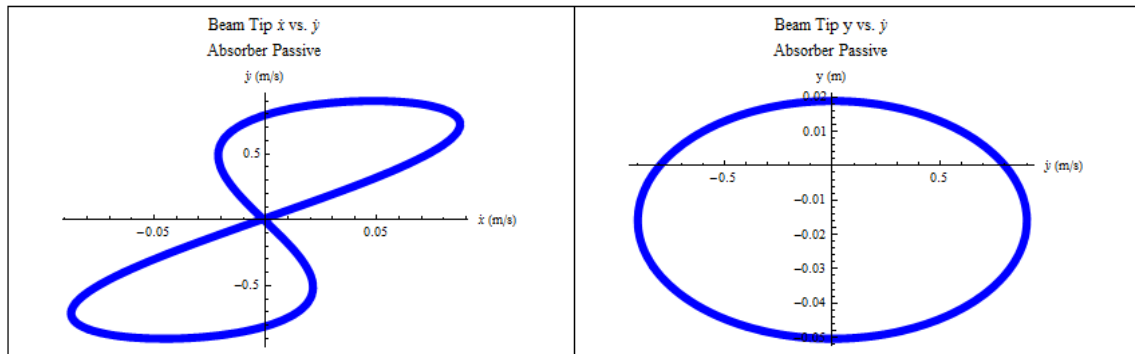


Figure 8-5: Ball locked / phase plane curves for the forcing frequency of 4.13 Hz at steady-state

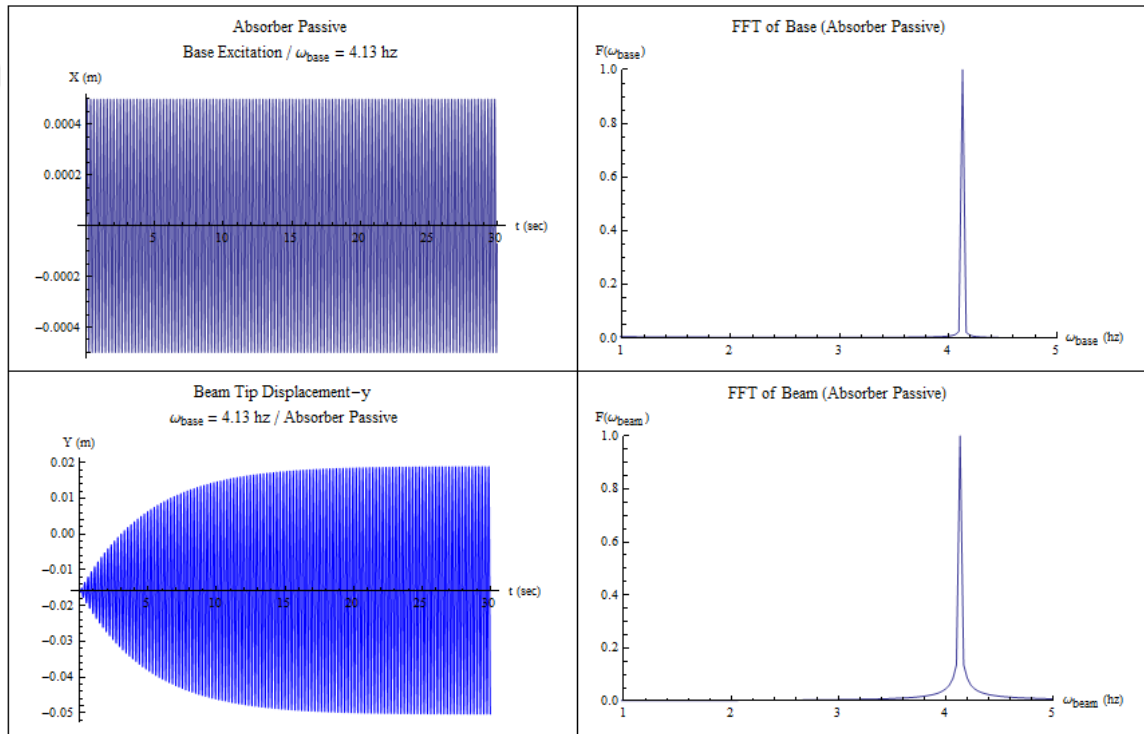


Figure 8-6: Absorber passive / FFT of the system for the forcing frequency of 4.13 Hz

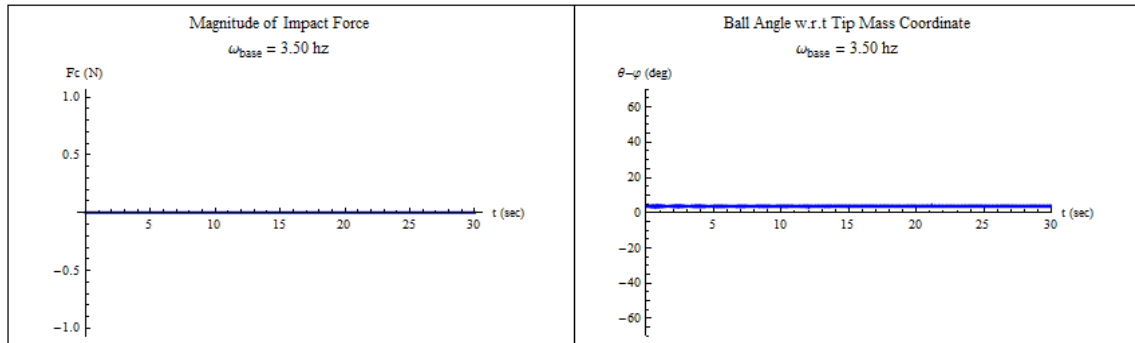


Figure 8-7: Impact details for the forcing frequency of 3.50 Hz

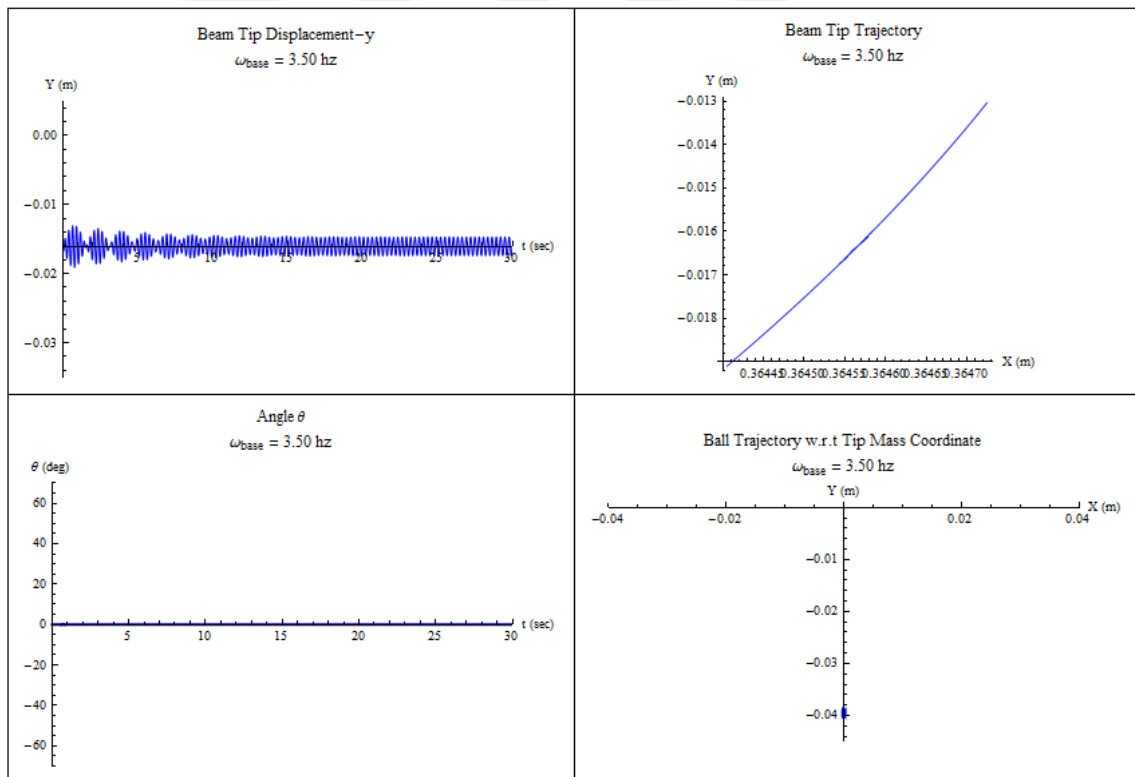


Figure 8-8: System trajectories for the forcing frequency of 3.50 Hz

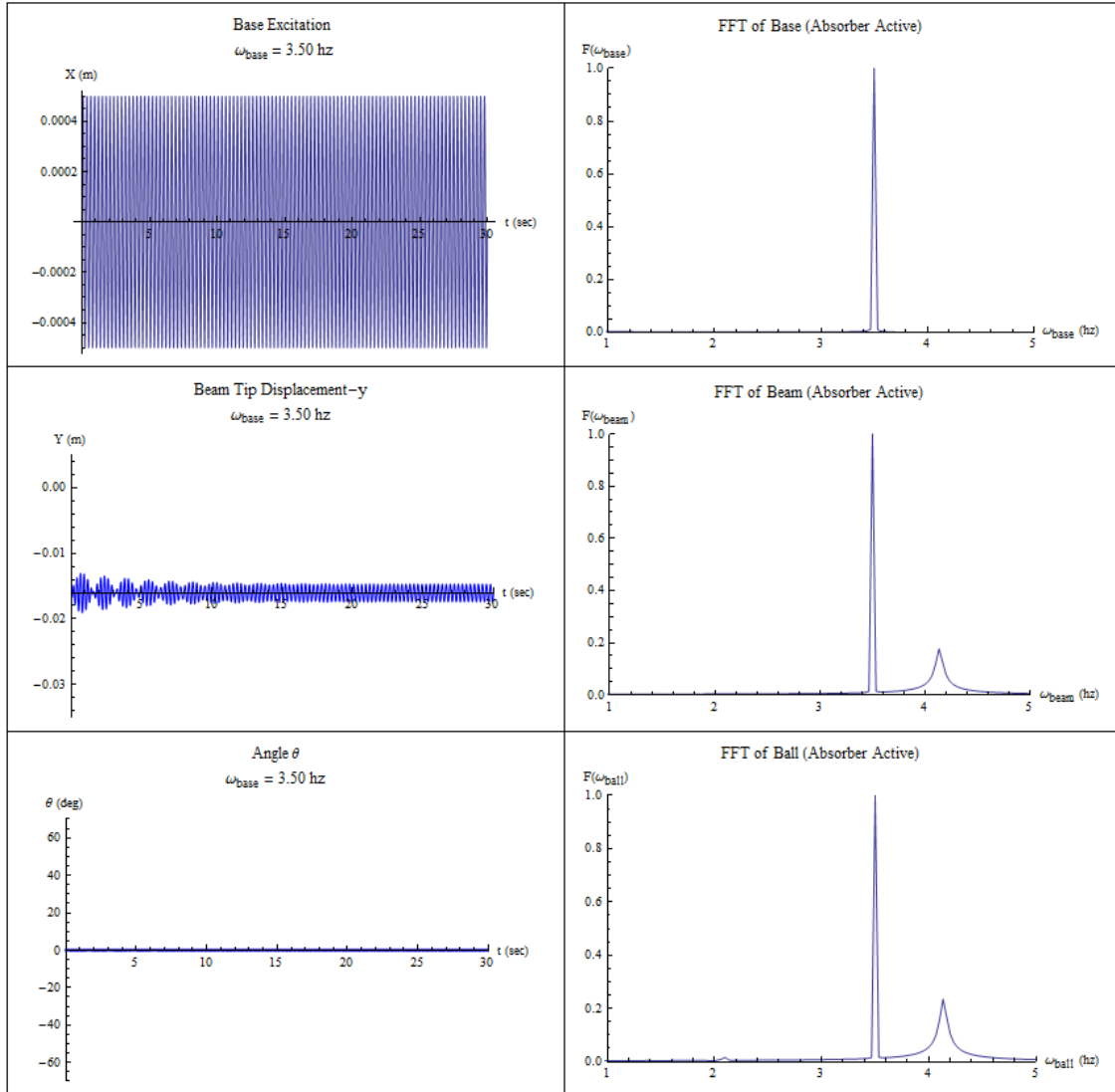


Figure 8-9: FFT of the system for the forcing frequency of 3.50 Hz

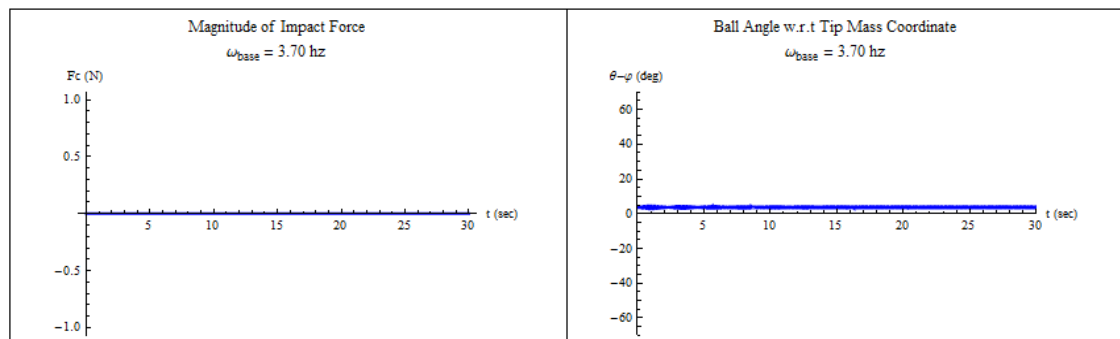


Figure 8-10: Impact details for the forcing frequency of 3.70 Hz

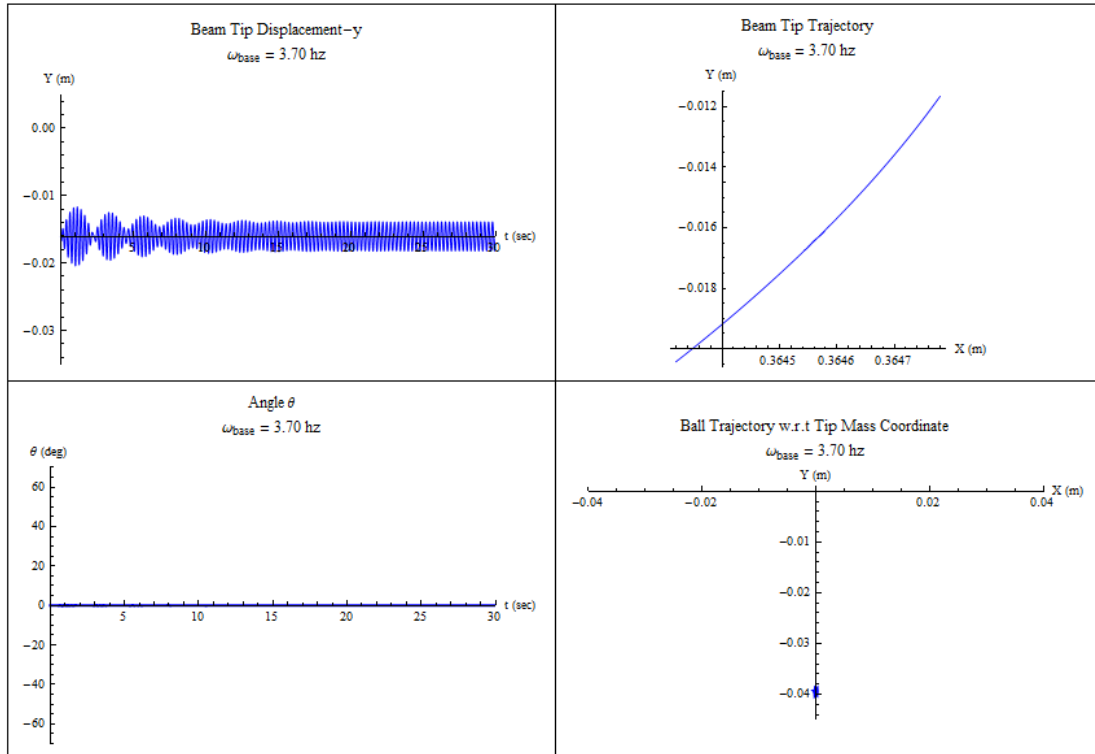


Figure 8-11: System trajectories for the forcing frequency of 3.70 Hz

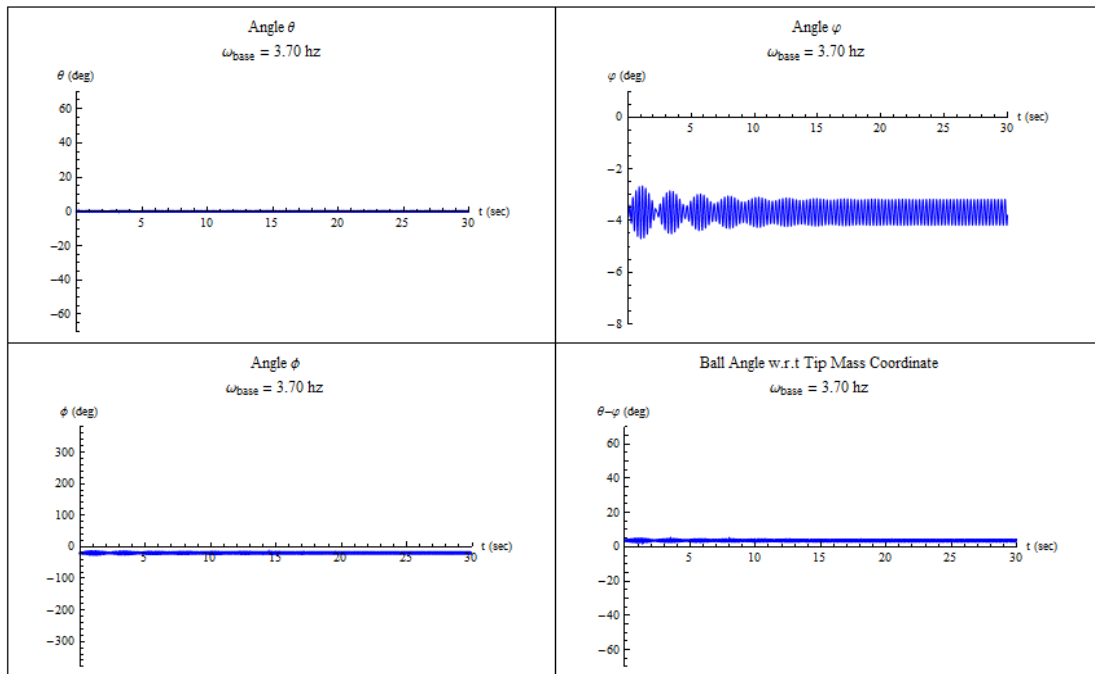


Figure 8-12: Time histories of the system angles for the forcing frequency of 3.70 Hz

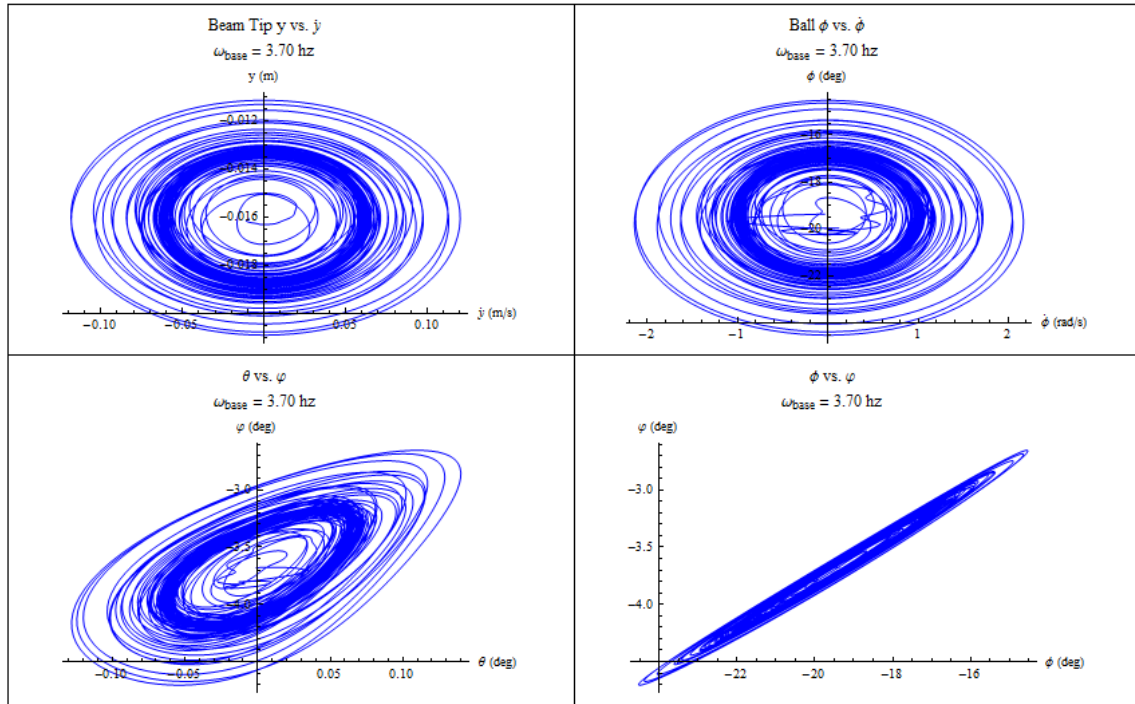


Figure 8-13: Phase plane curves for the forcing frequency of 3.70 Hz

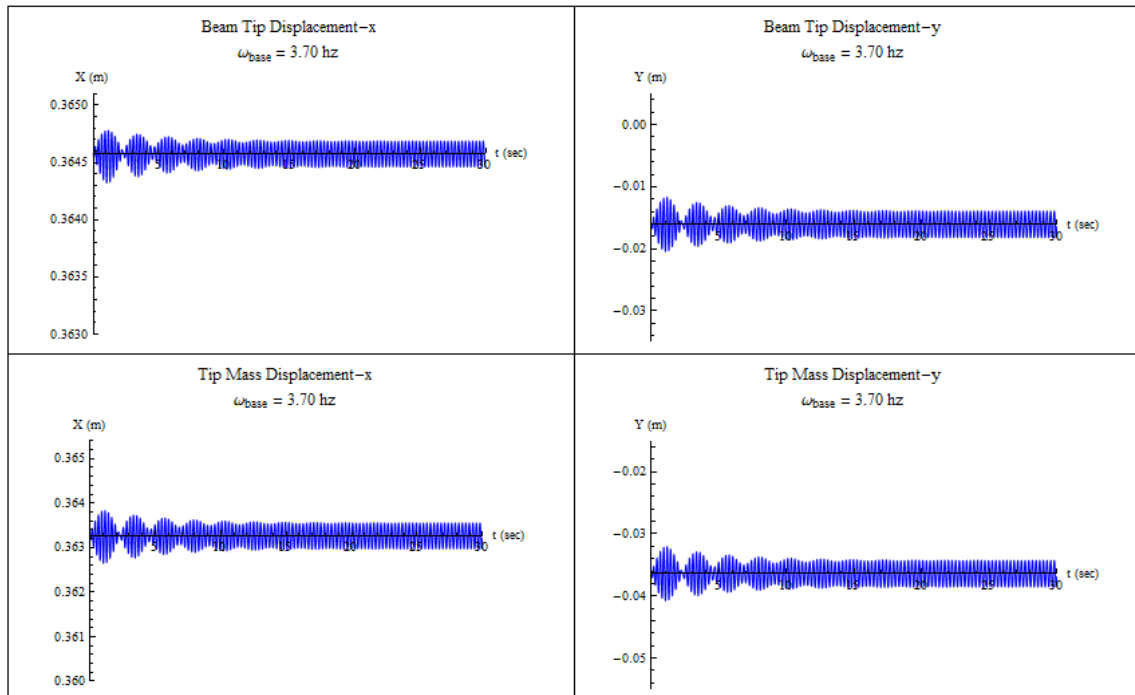


Figure 8-14: Time histories of the system translational displacements for the forcing frequency of 3.70 Hz

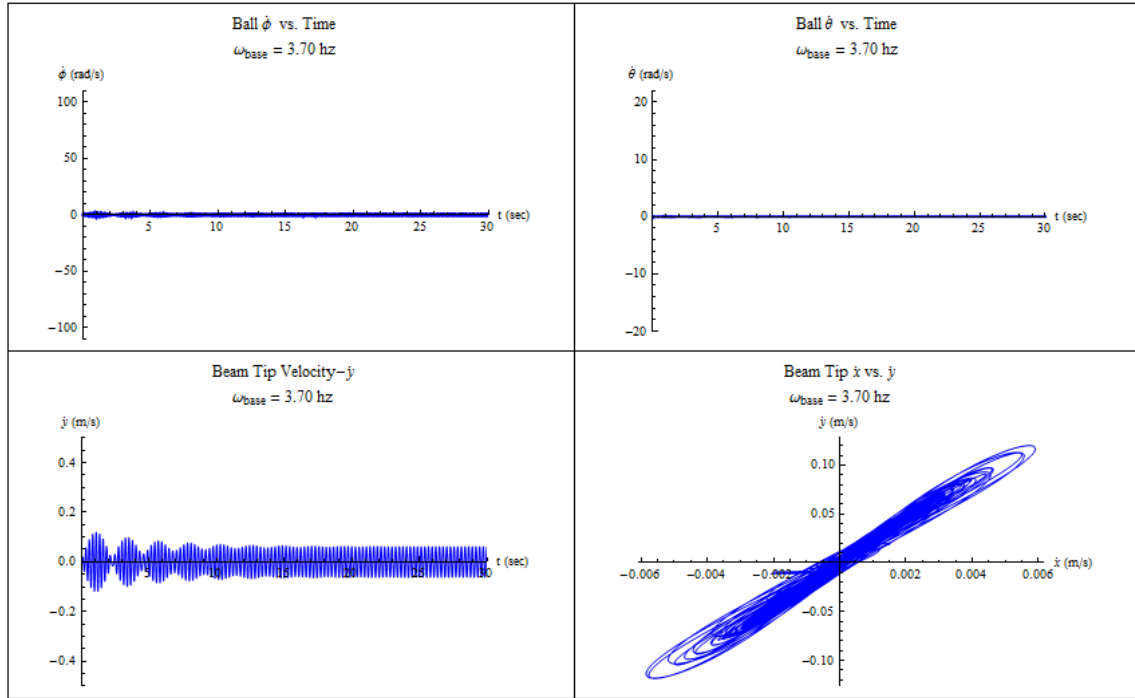


Figure 8-15: System velocity curves for the forcing frequency of 3.70 Hz

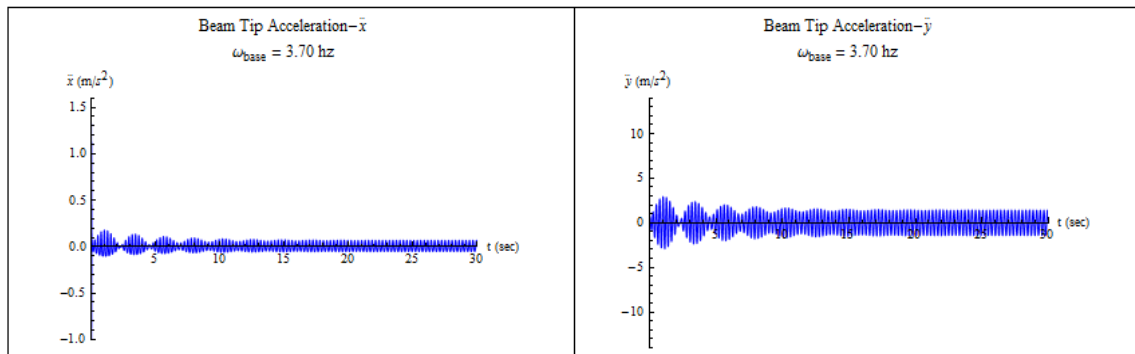


Figure 8-16: Beam tip acceleration for the forcing frequency of 3.70 Hz

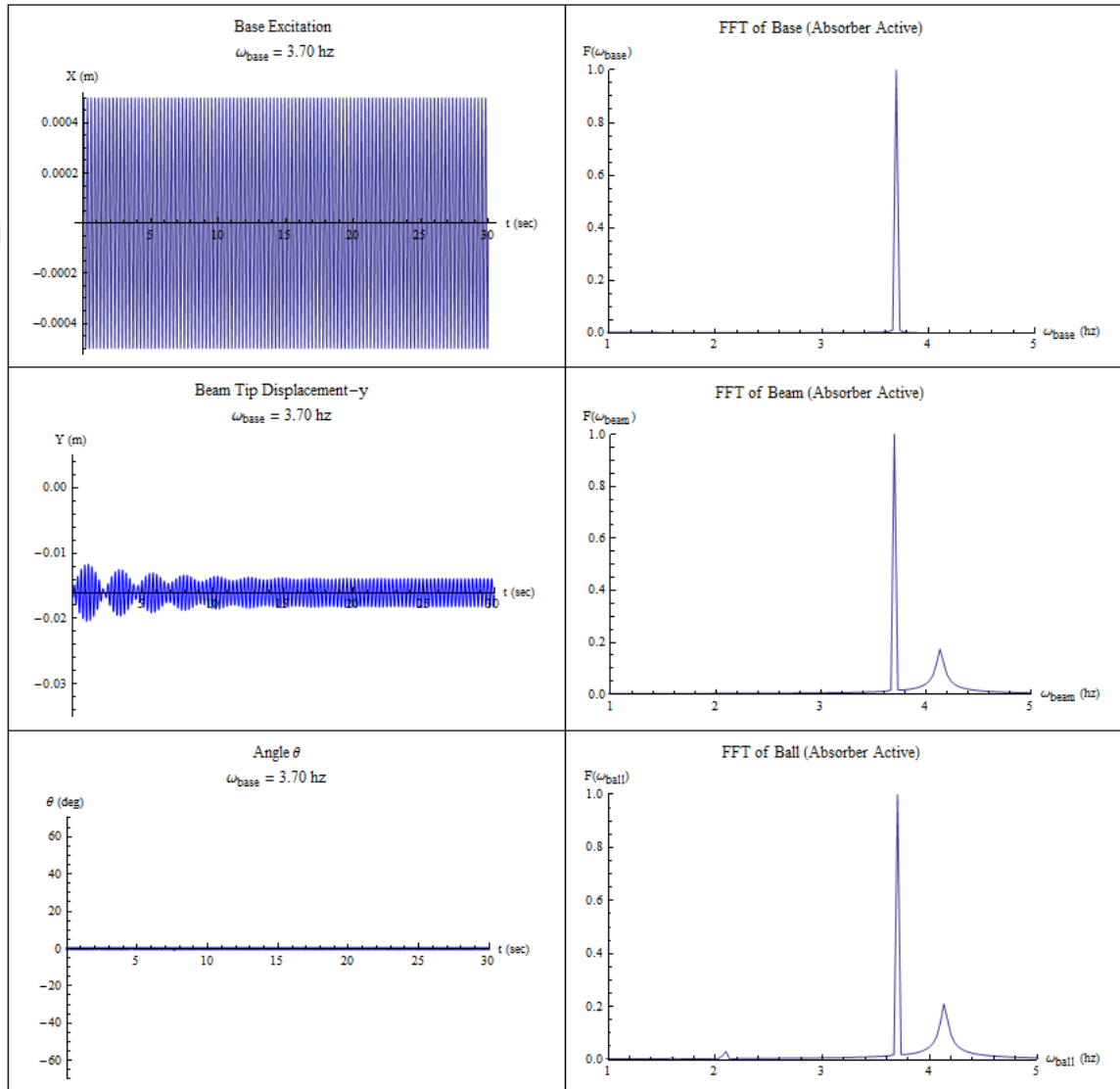


Figure 8-17: FFT of the system for the forcing frequency of 3.70 Hz

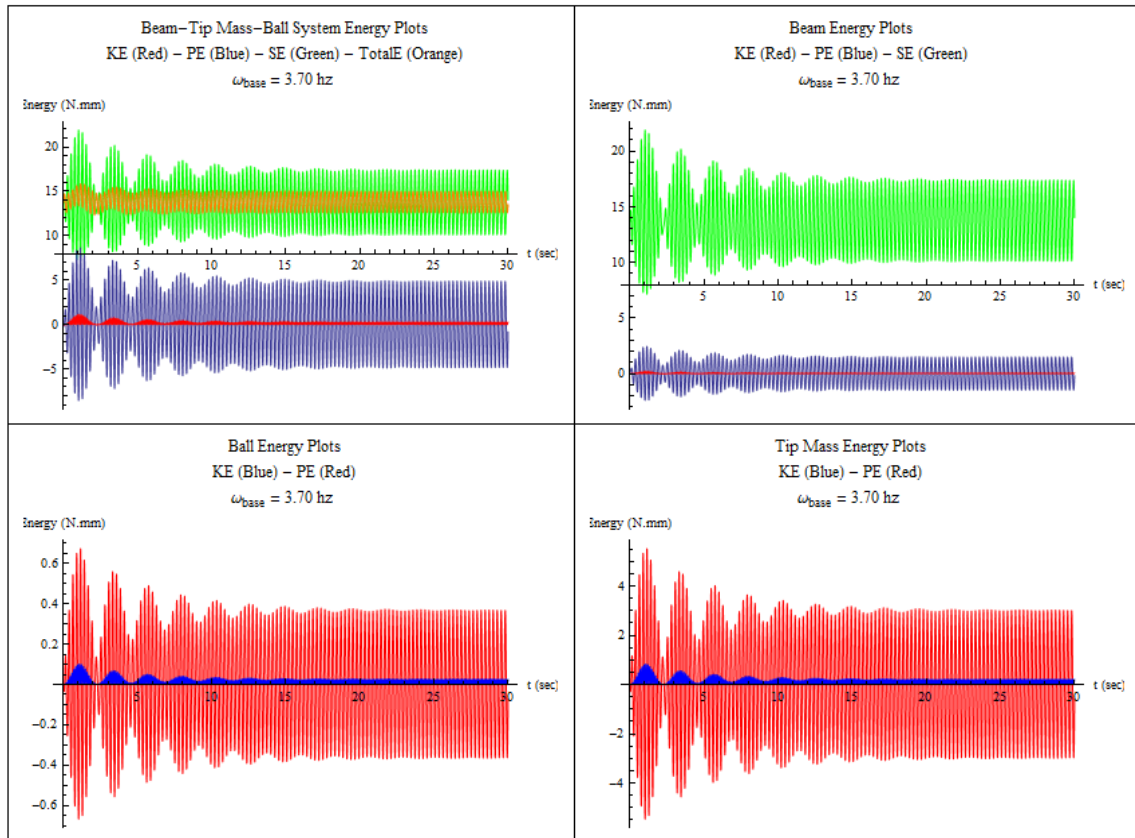


Figure 8-18: System energy curves for the forcing frequency of 3.70 Hz

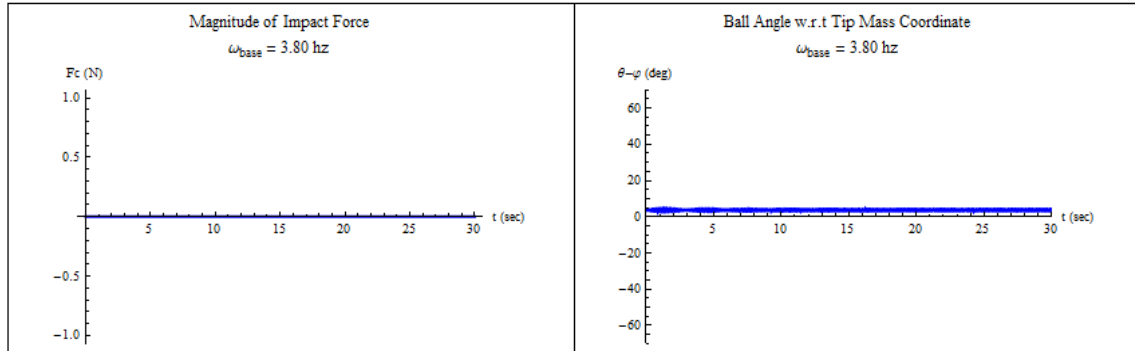


Figure 8-19: Impact details for the forcing frequency of 3.80 Hz

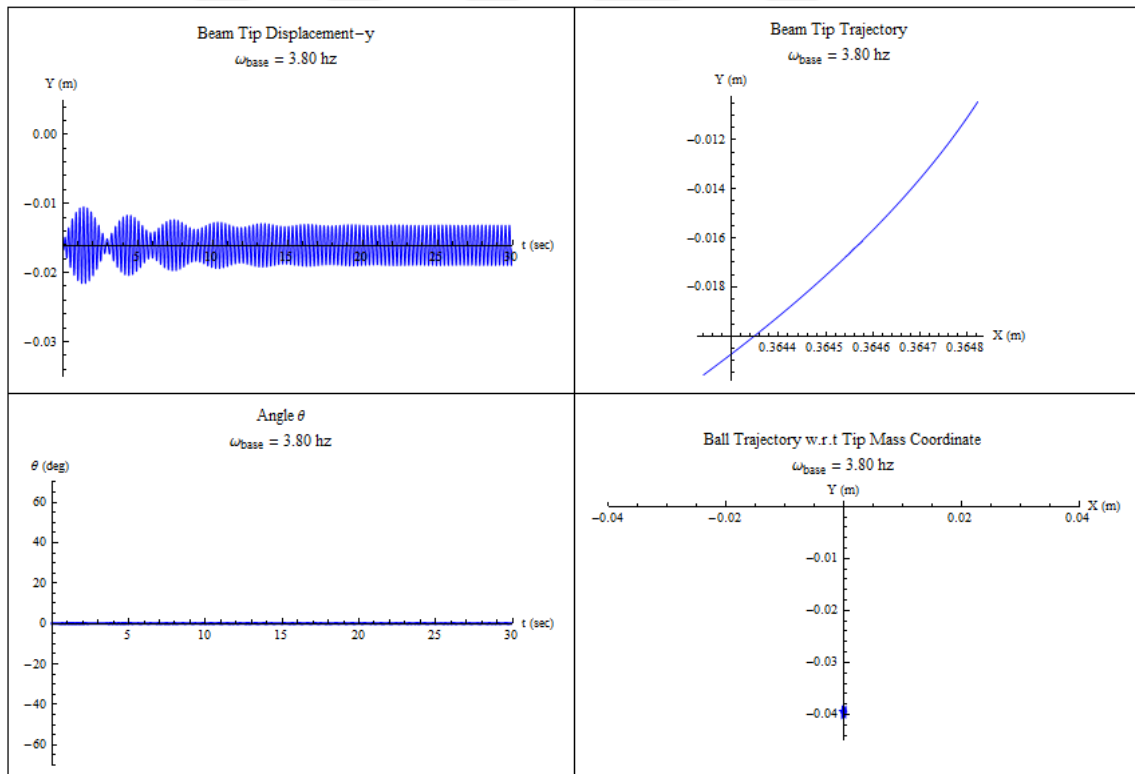


Figure 8-20: System trajectories for the forcing frequency of 3.80 Hz

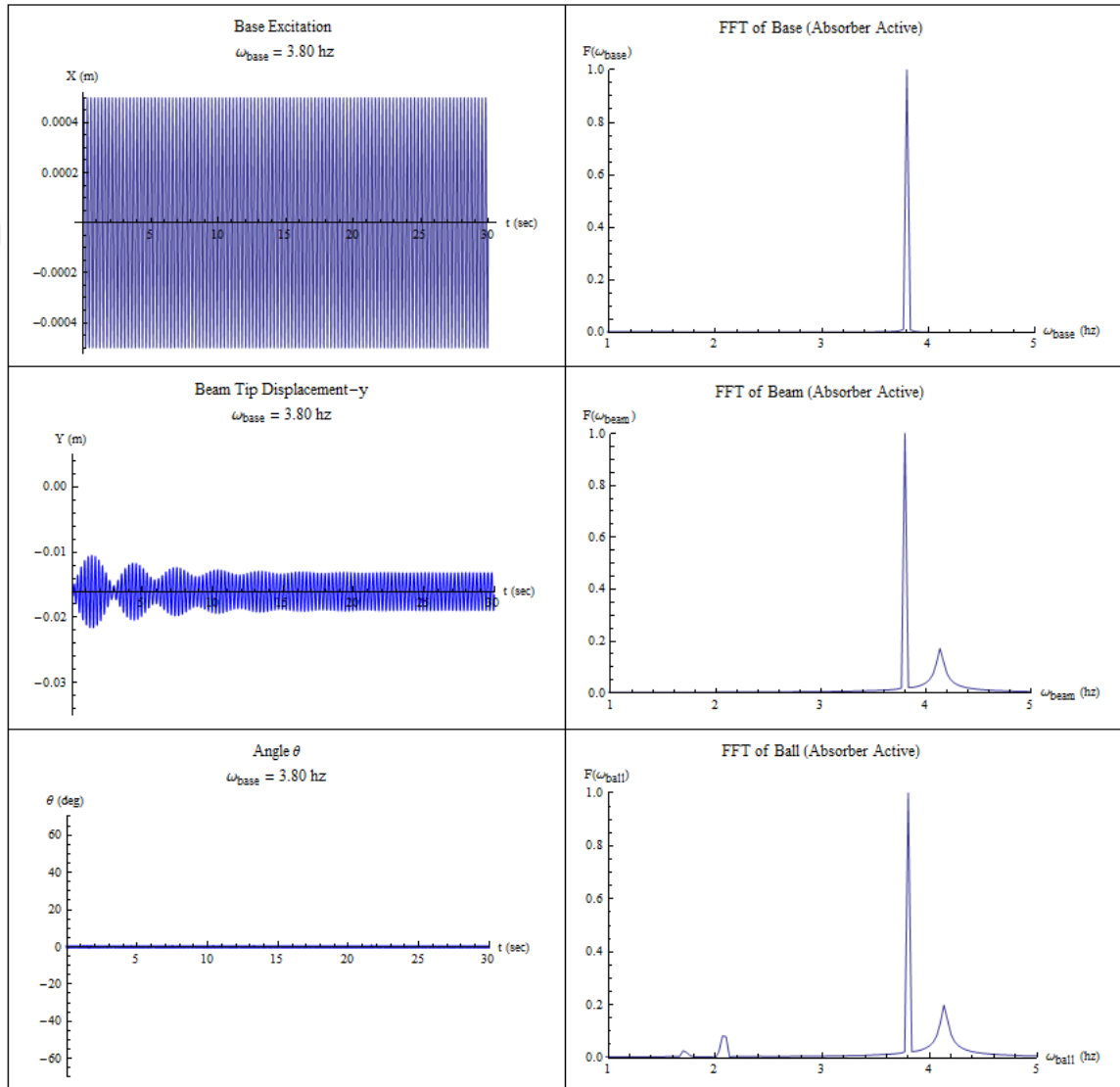


Figure 8-21: FFT of the system for the forcing frequency of 3.80 Hz

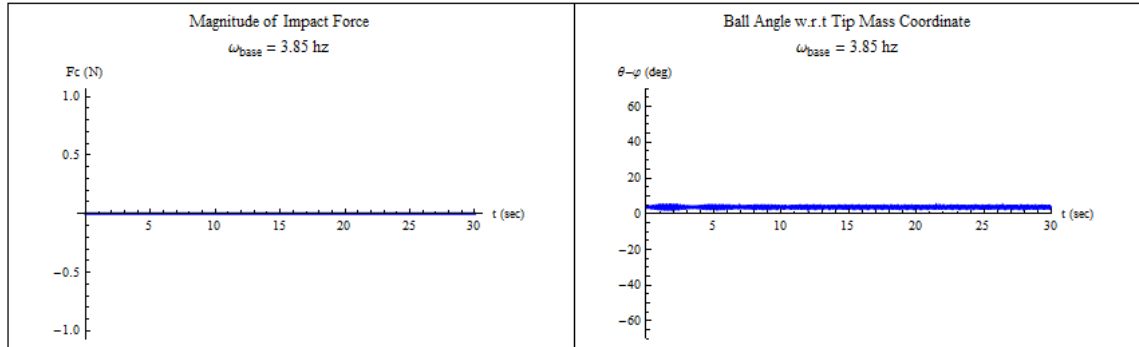


Figure 8-22: Impact details for the forcing frequency of 3.85 Hz

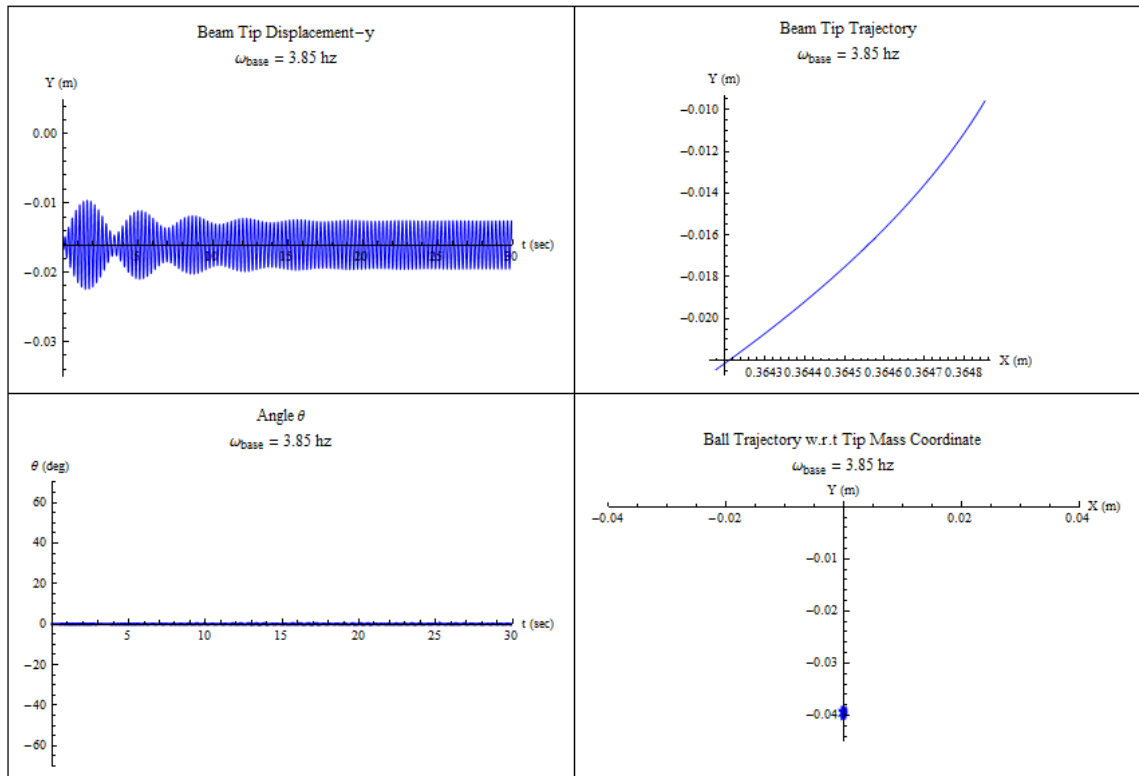


Figure 8-23: System trajectories for the forcing frequency of 3.85 Hz

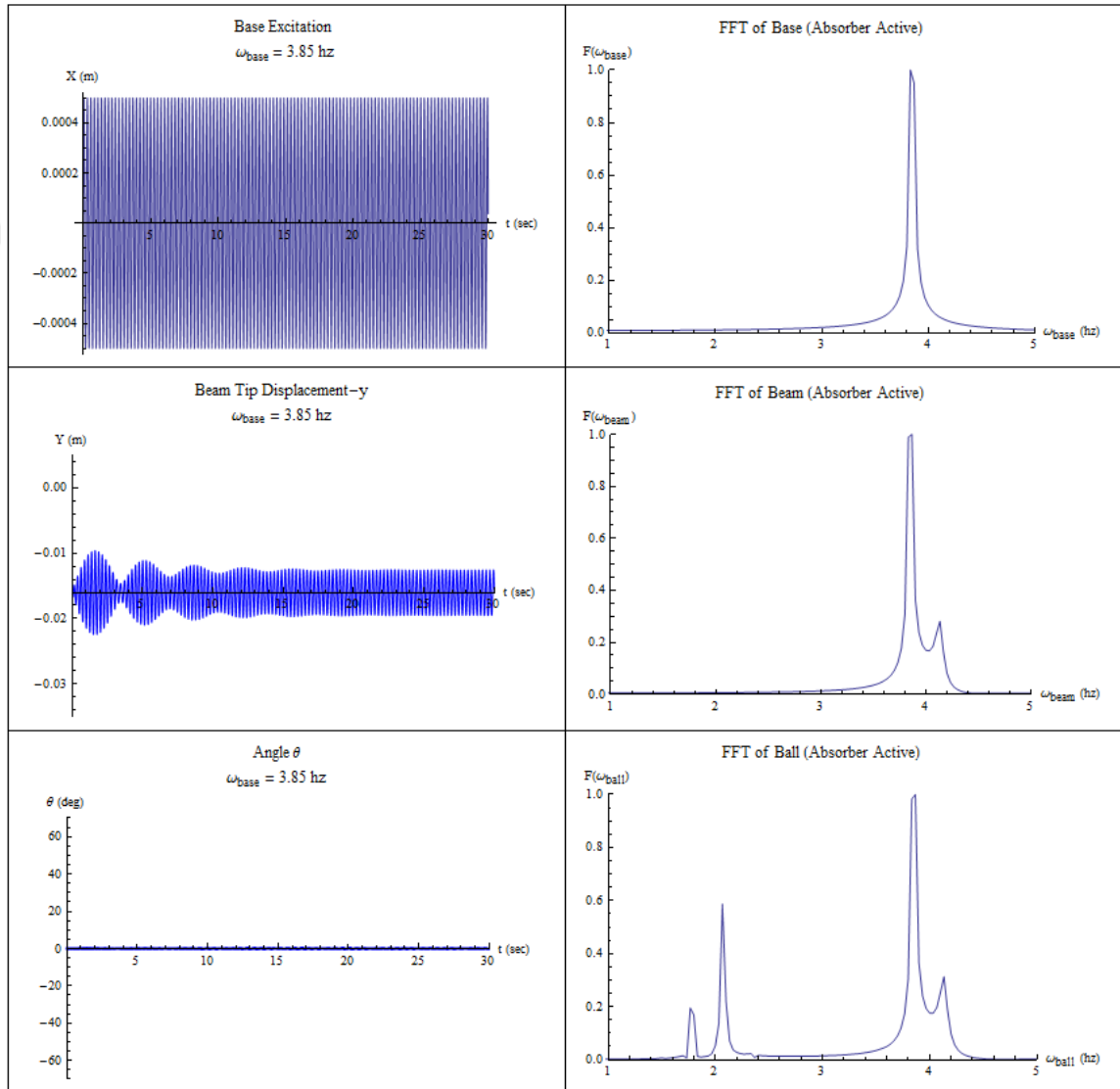


Figure 8-24: FFT of the system for the forcing frequency of 3.85 Hz

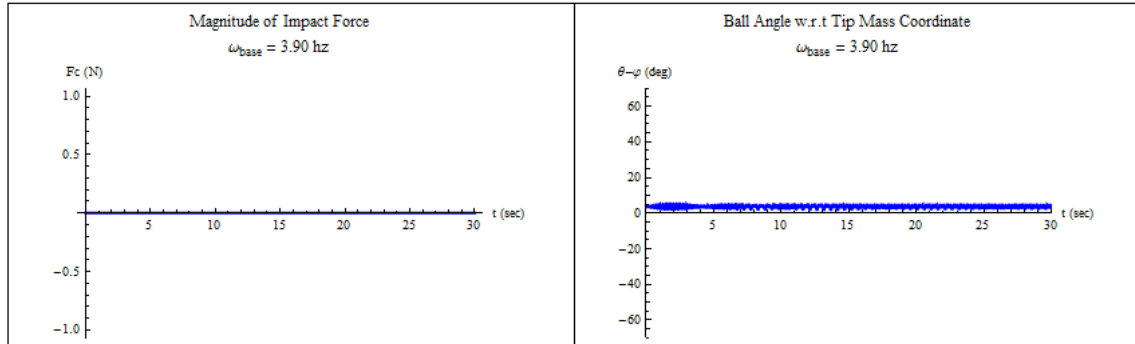


Figure 8-25: Impact details for the forcing frequency of 3.90 Hz

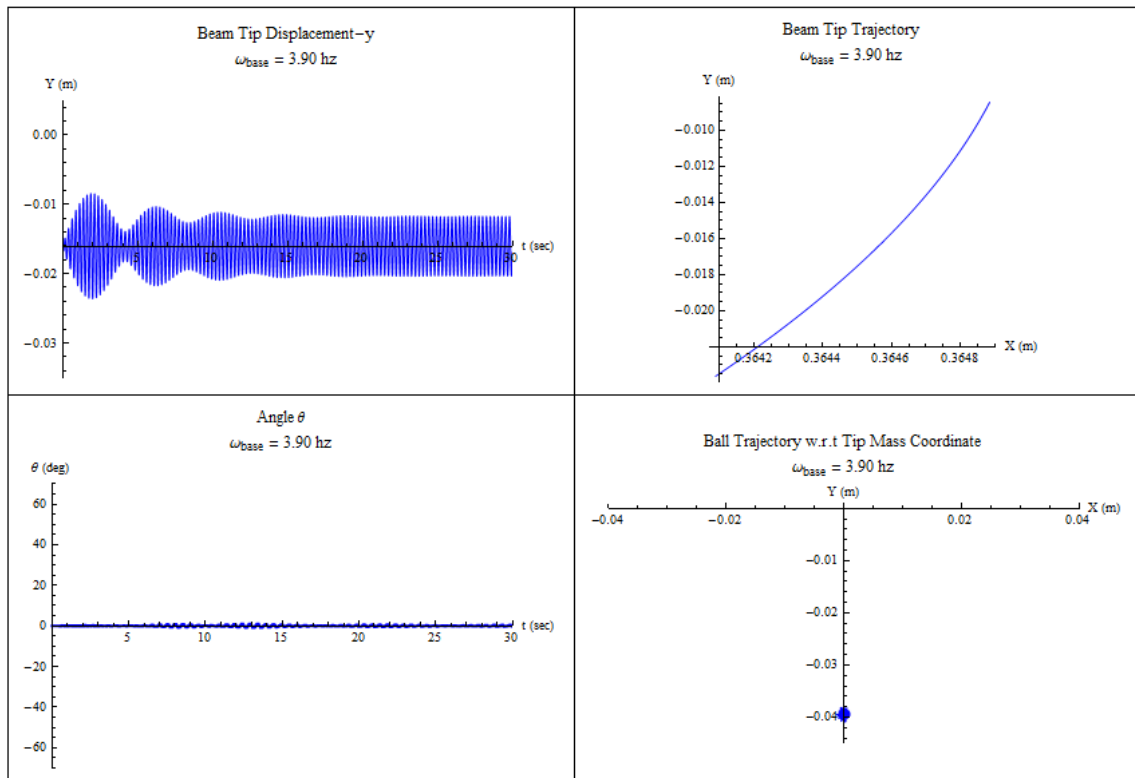


Figure 8-26: System trajectories for the forcing frequency of 3.90 Hz

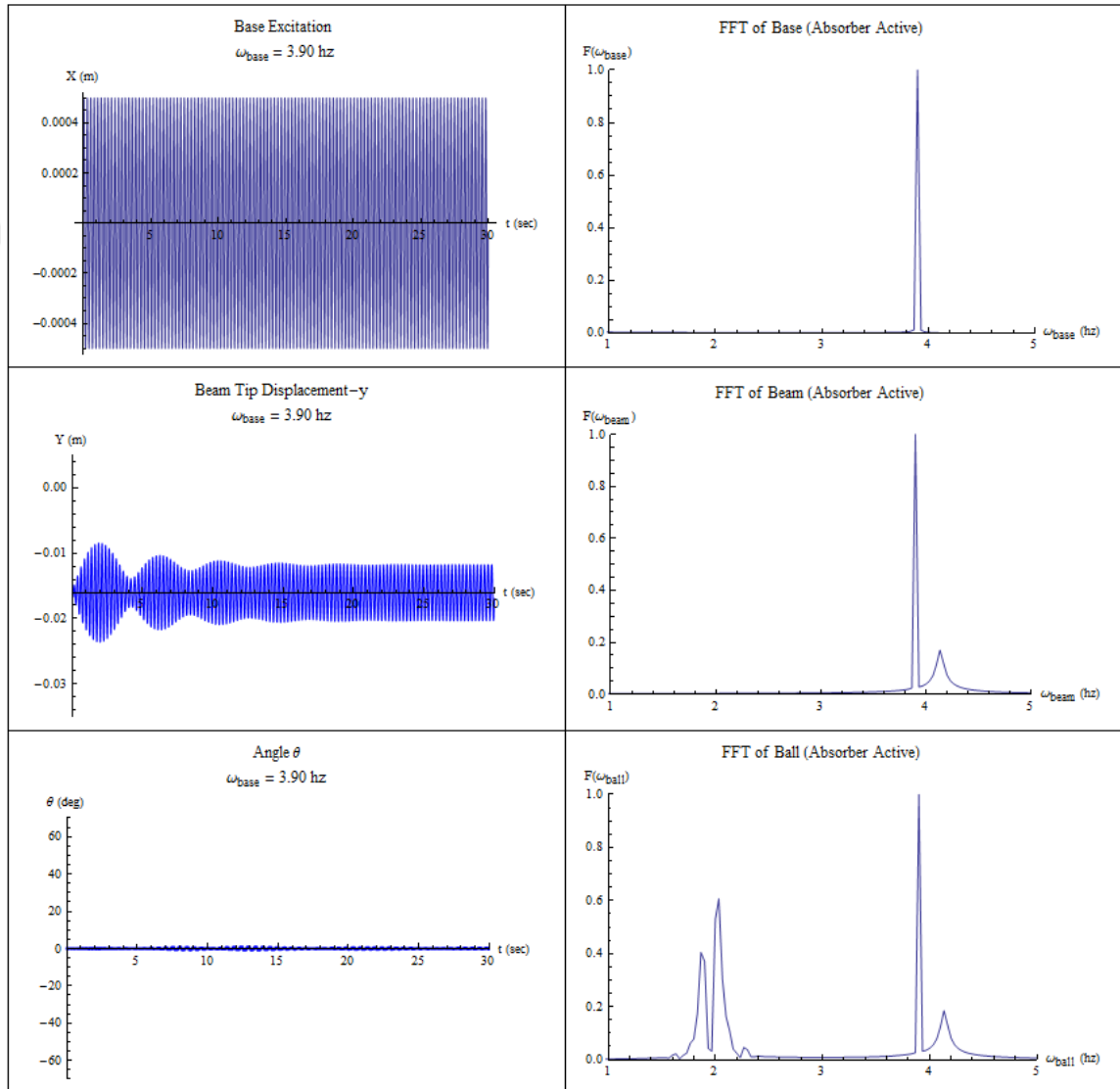


Figure 8-27: FFT of the system for the forcing frequency of 3.90 Hz

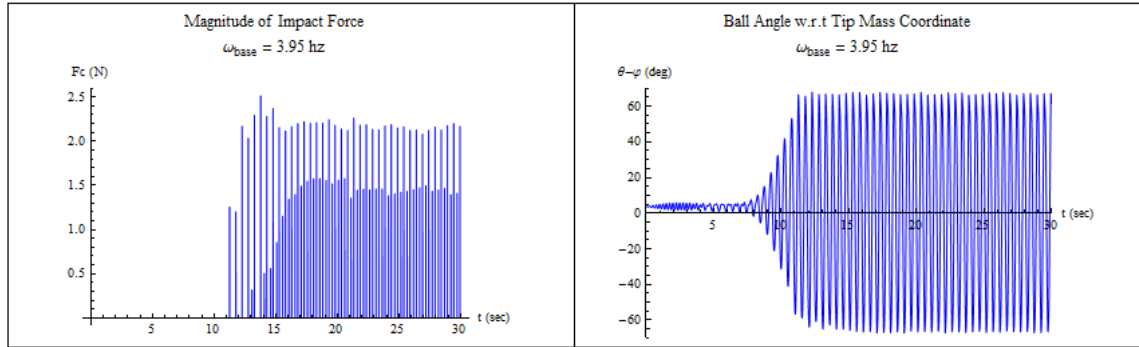


Figure 8-28: Impact details for the forcing frequency of 3.95 Hz

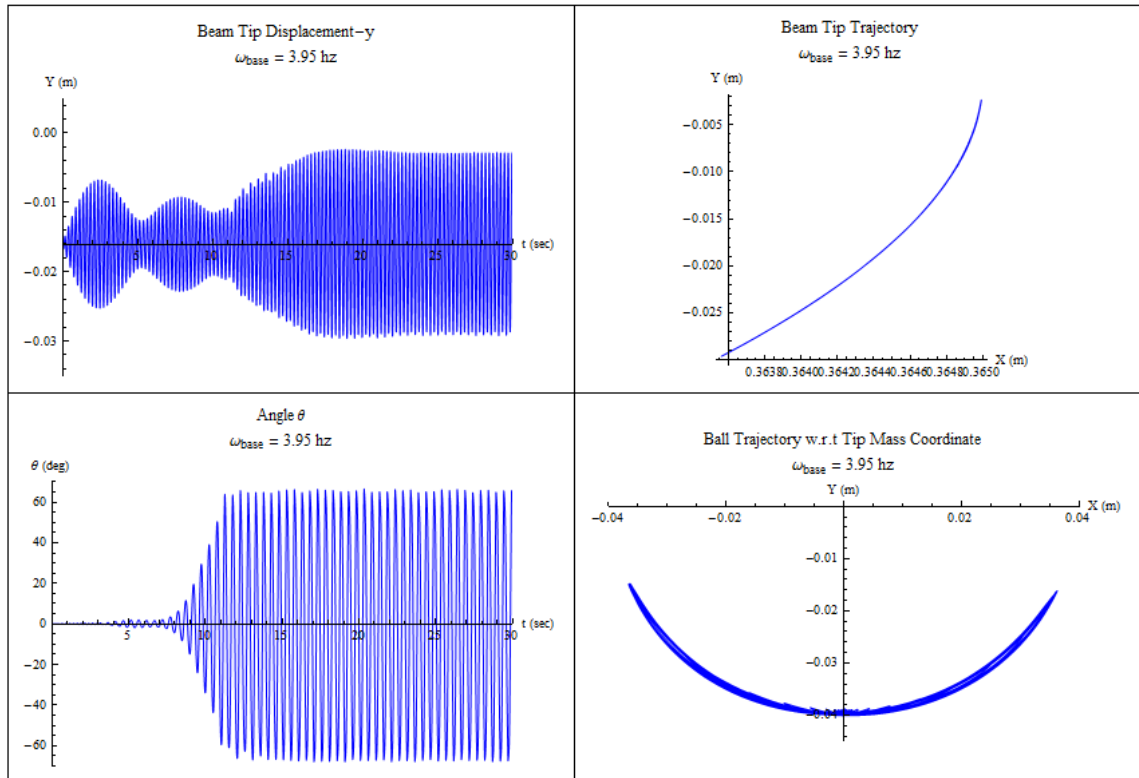


Figure 8-29: System trajectories for the forcing frequency of 3.95 Hz

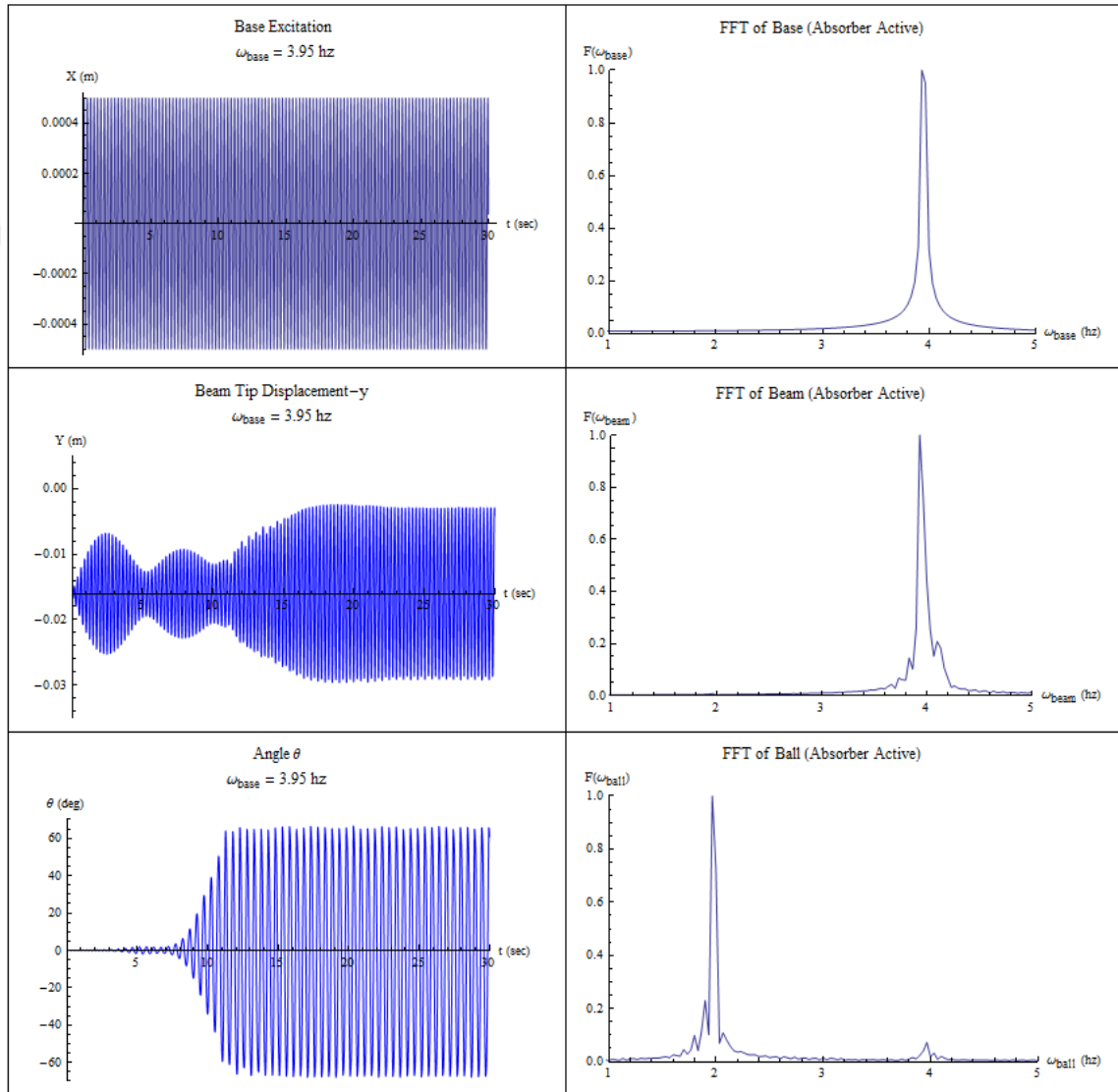


Figure 8-30: FFT of the system for the forcing frequency of 3.95 Hz

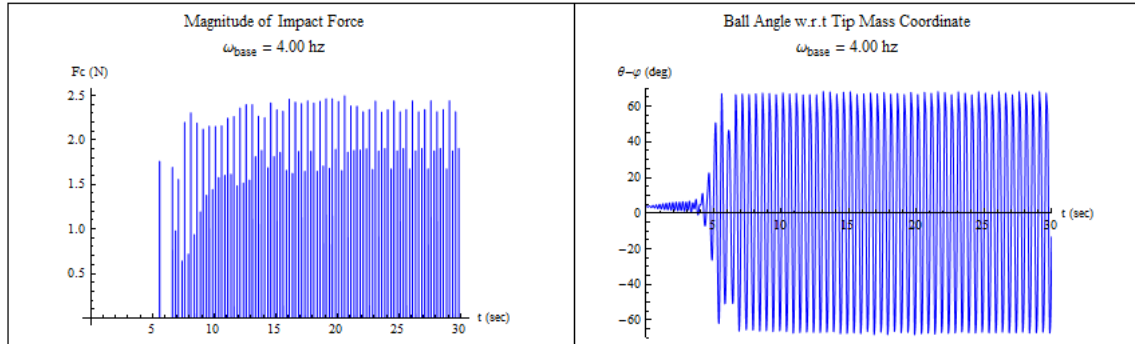


Figure 8-31: Impact details for the forcing frequency of 4.00 Hz

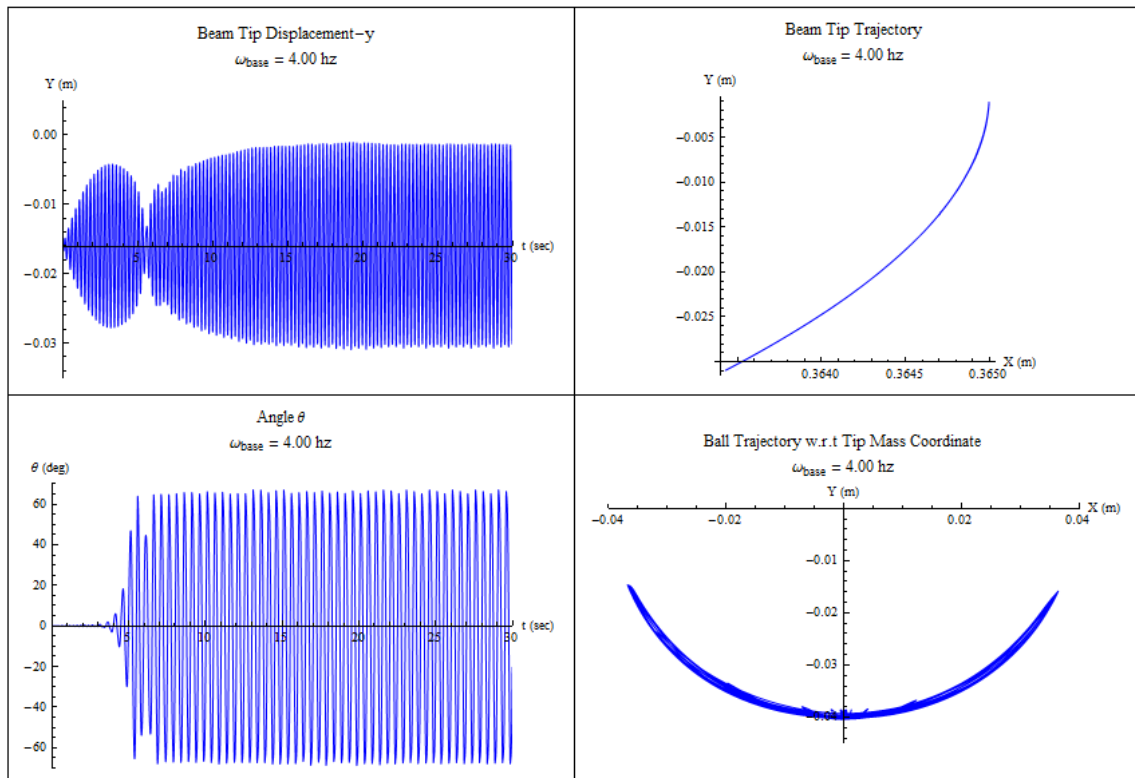


Figure 8-32: System trajectories for the forcing frequency of 4.00 Hz

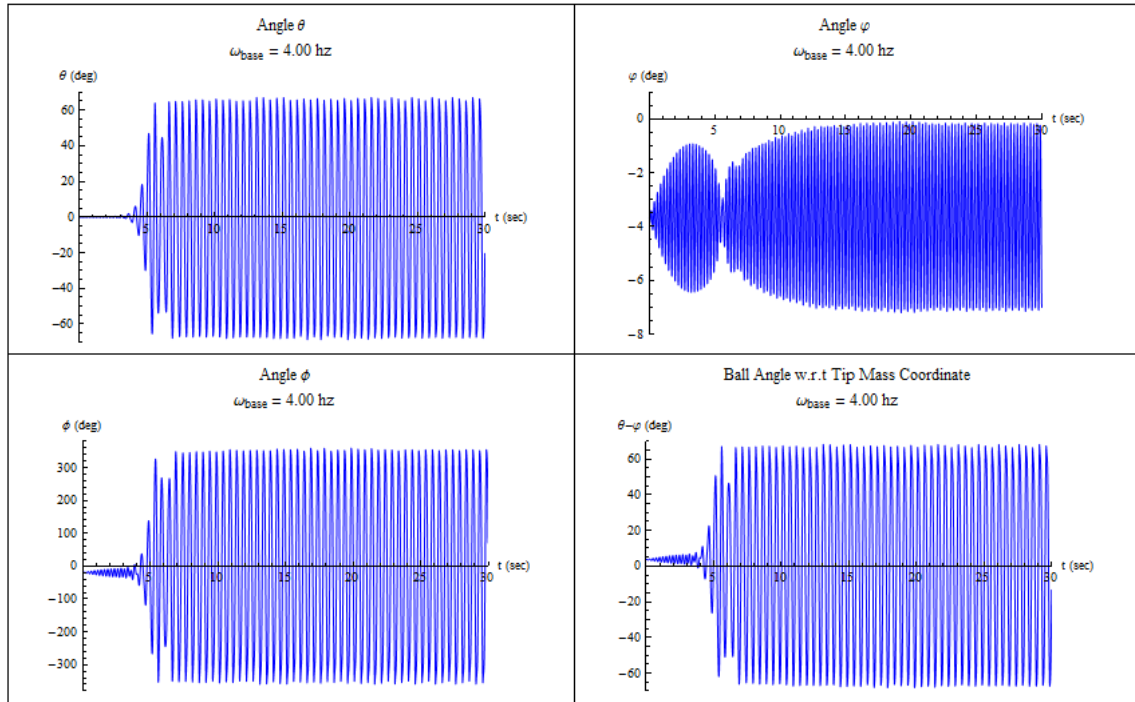


Figure 8-33: Time histories of the system angles for the forcing frequency of 4.00 Hz

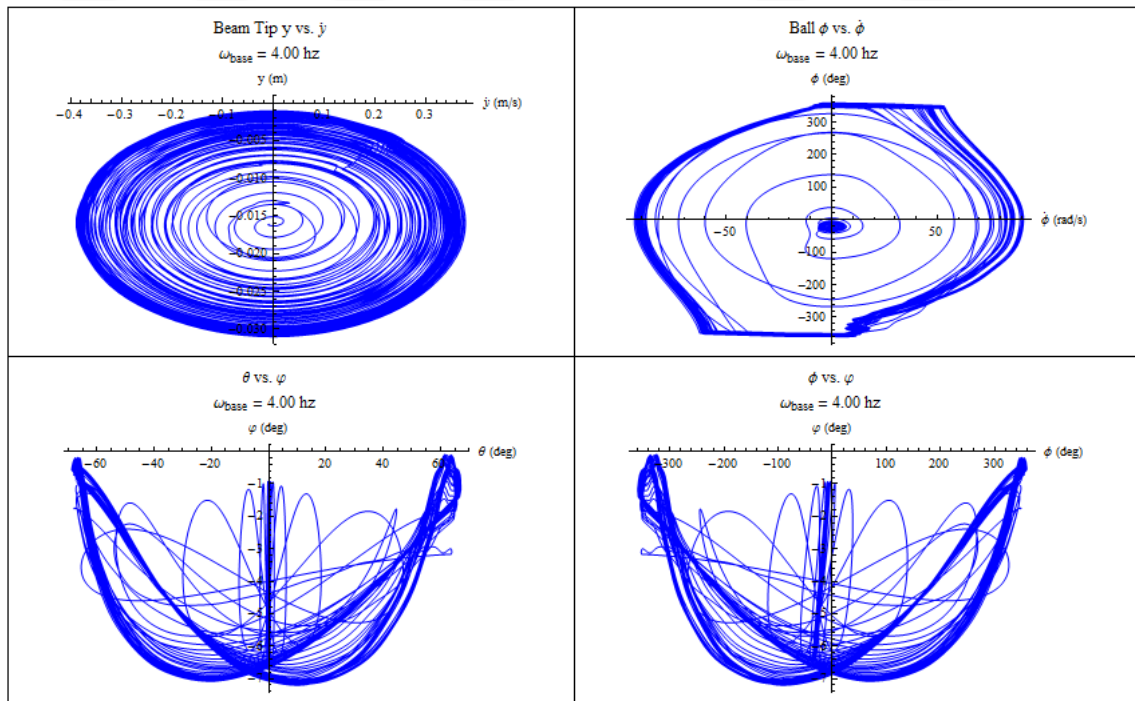


Figure 8-34: Phase plane curves for the forcing frequency of 4.00 Hz

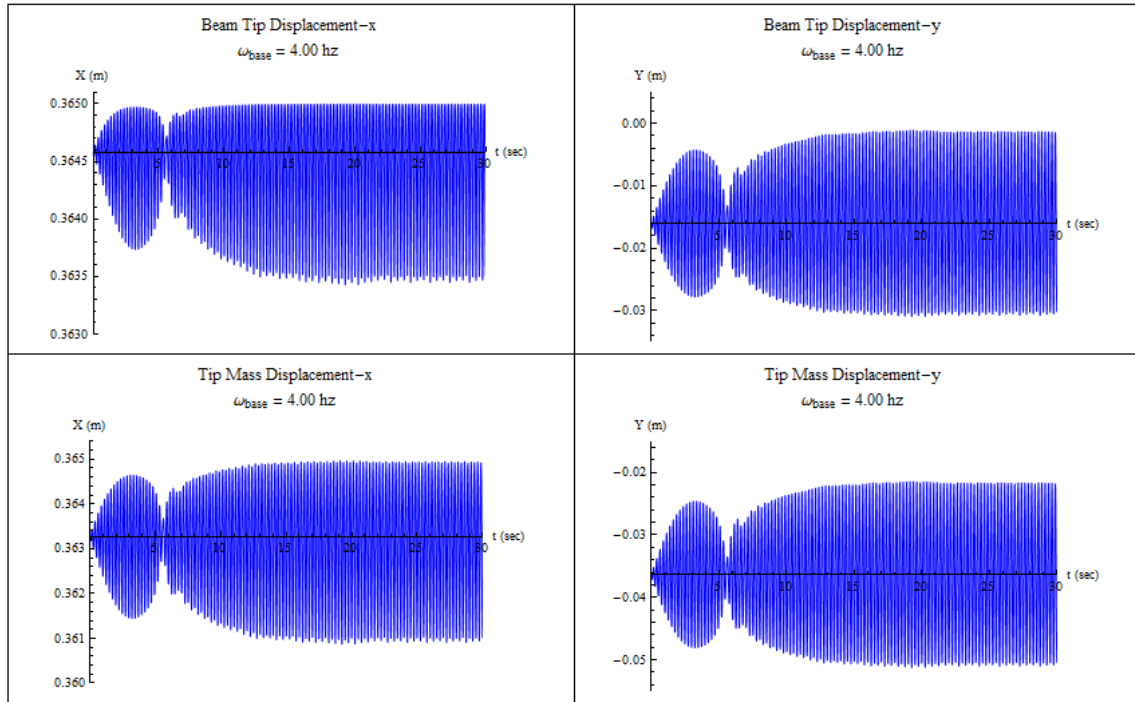


Figure 8-35: Time histories of the system translational displacements for the forcing frequency of 4.00 Hz

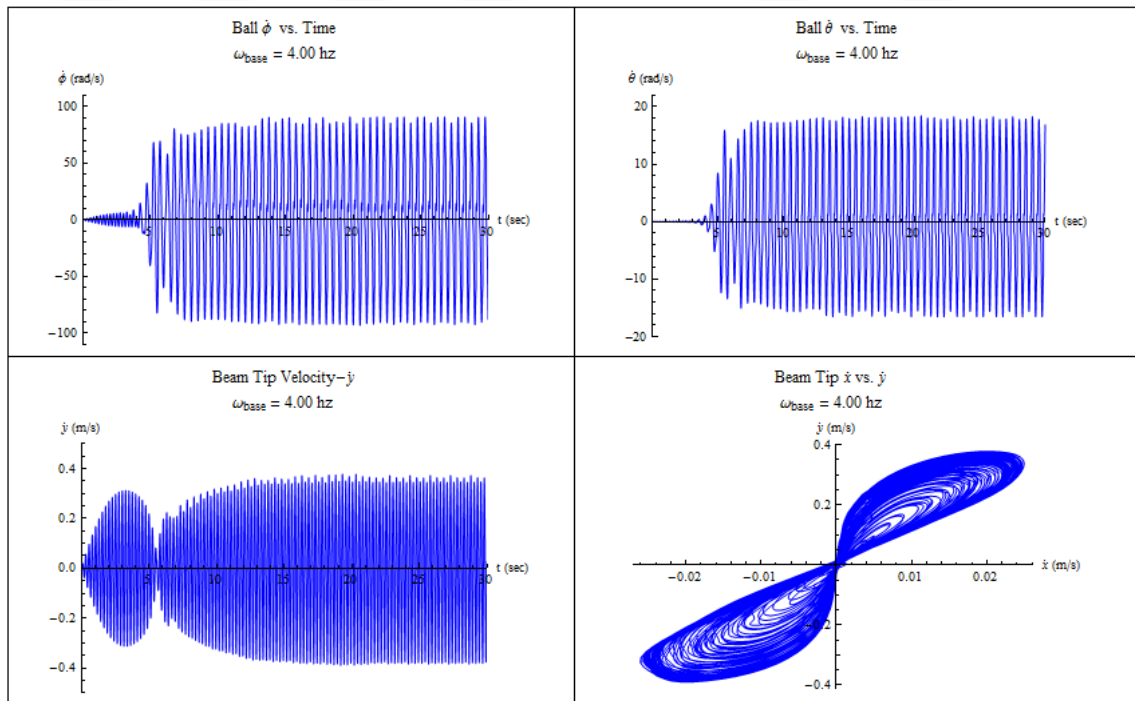


Figure 8-36: System velocity curves for the forcing frequency of 4.00 Hz

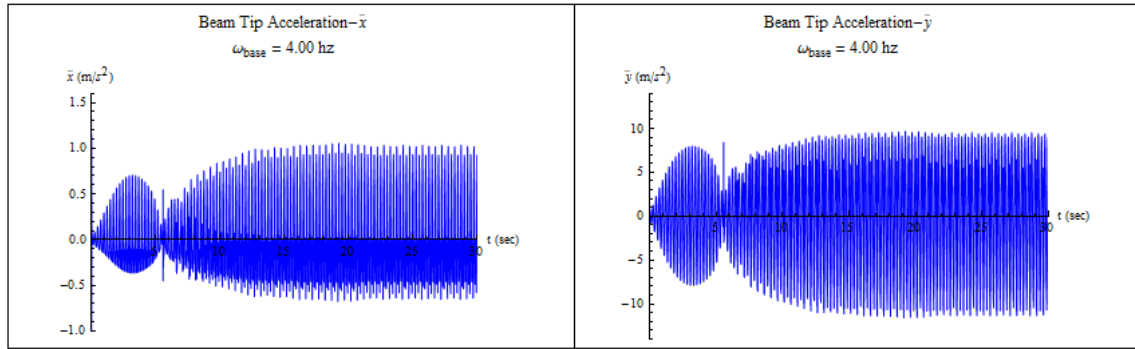


Figure 8-37: Beam tip acceleration for the forcing frequency of 4.00 Hz

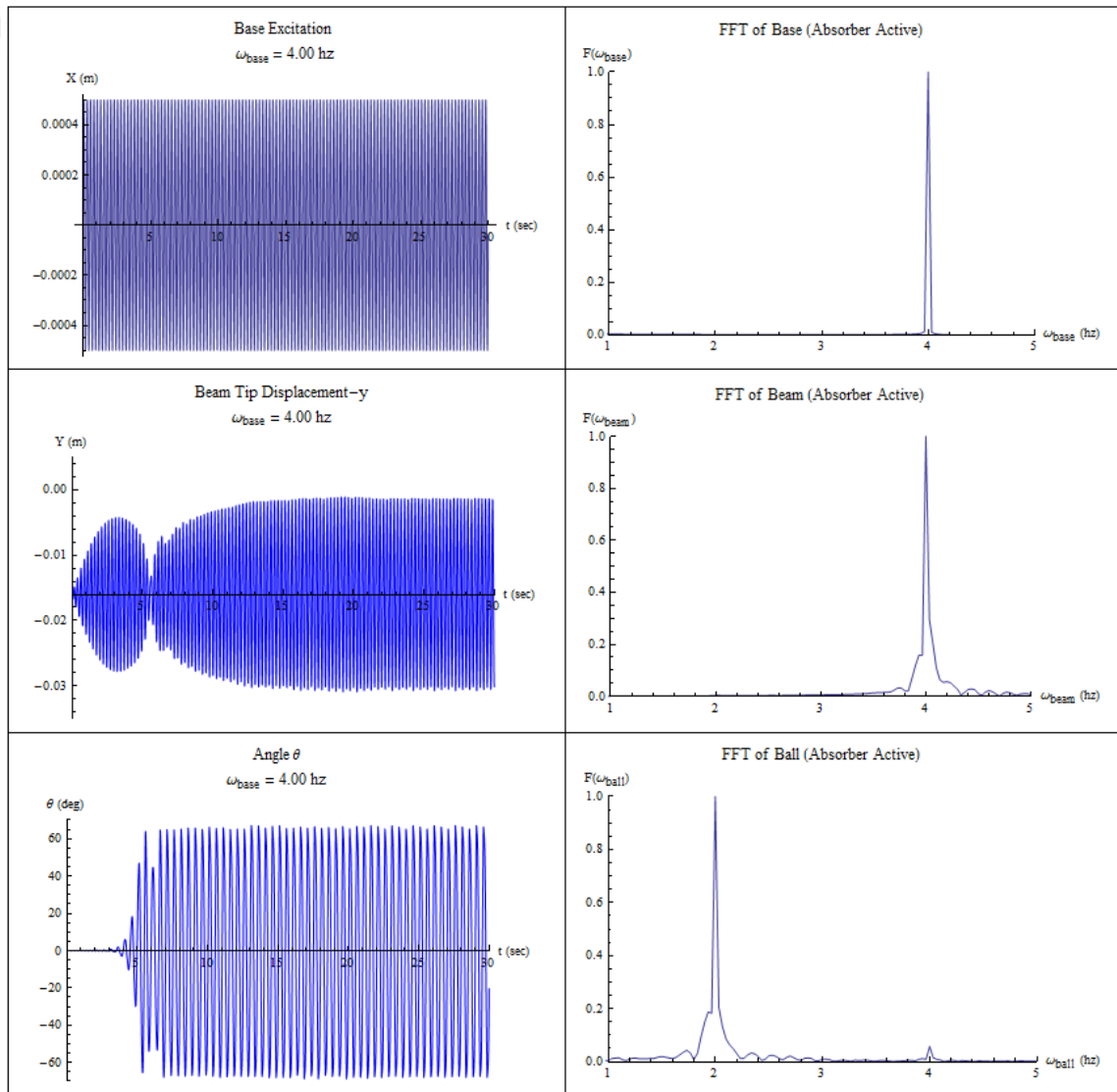


Figure 8-38: FFT of the system for the forcing frequency of 4.00 Hz

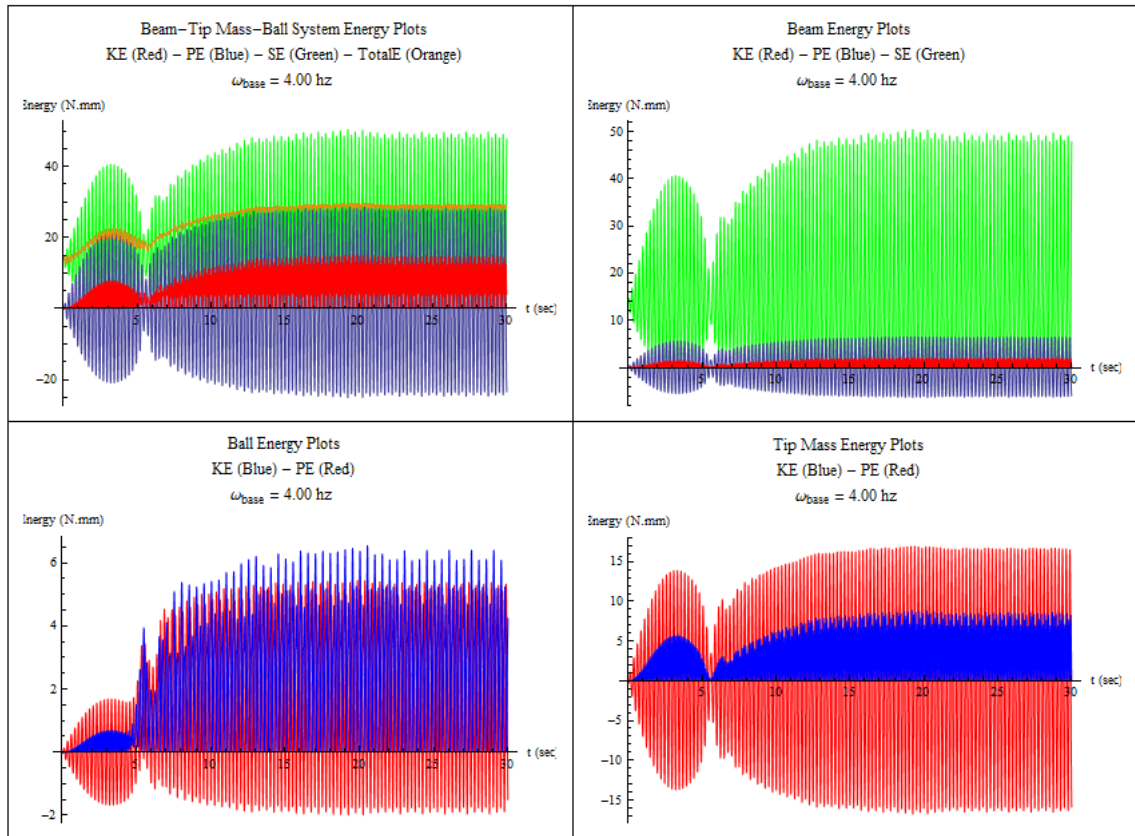


Figure 8-39: System energy curves for the forcing frequency of 4.00 Hz

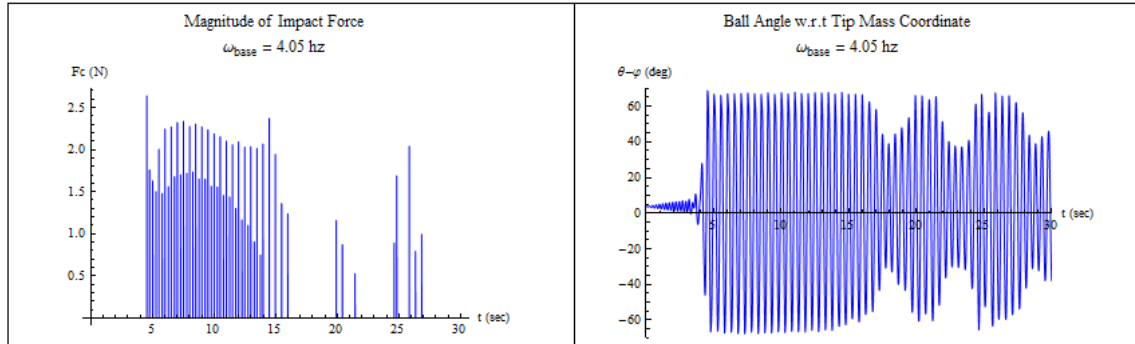


Figure 8-40: Impact details for the forcing frequency of 4.05 Hz

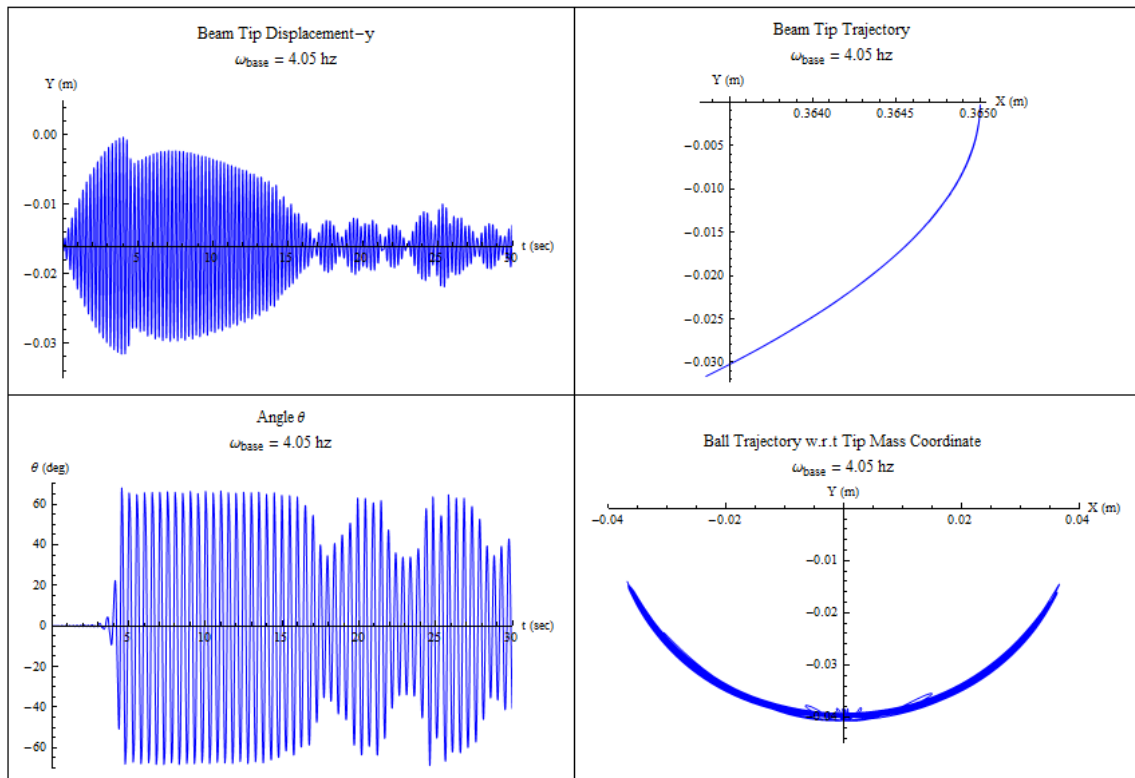


Figure 8-41: System trajectories for the forcing frequency of 4.05 Hz

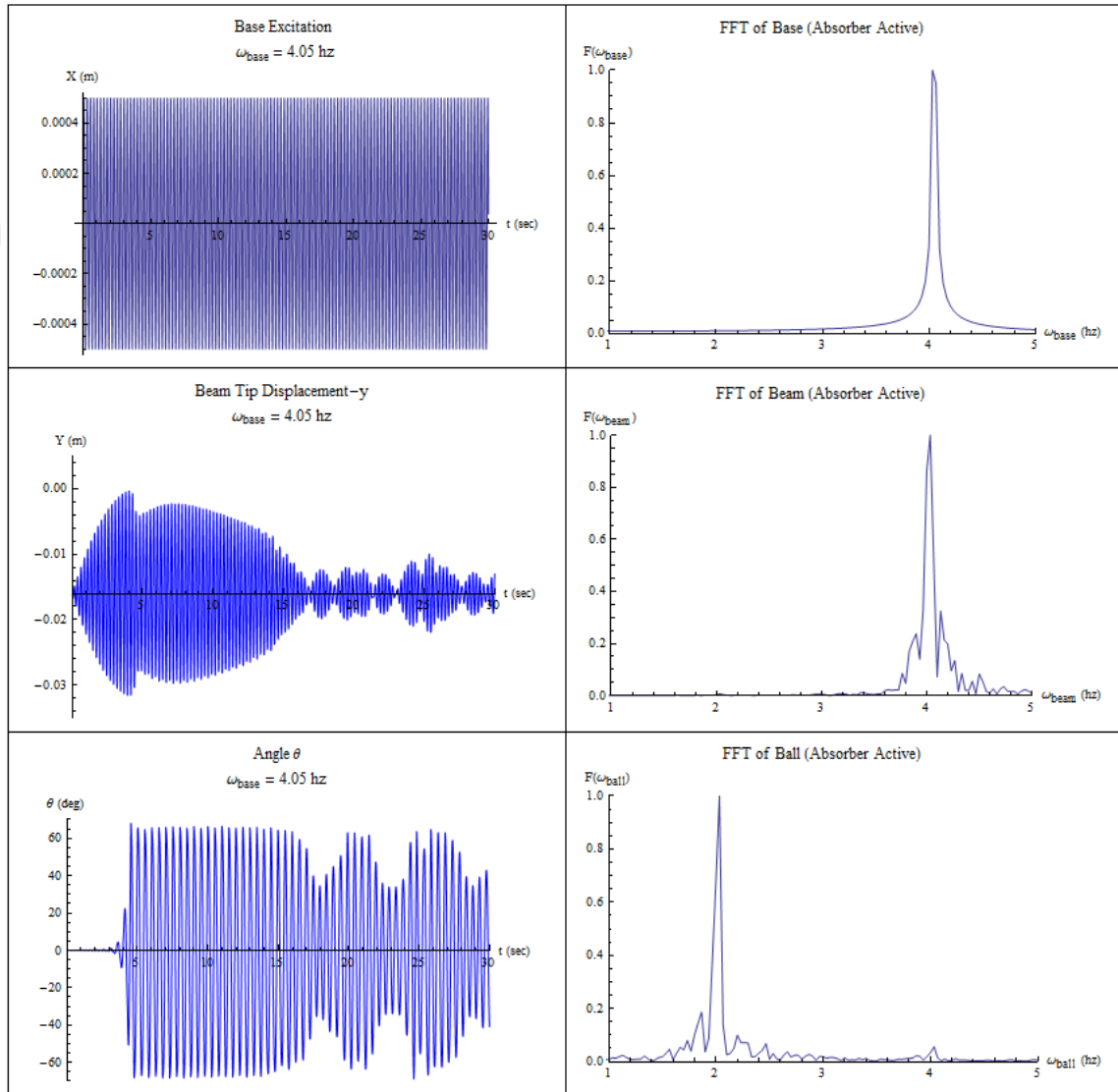


Figure 8-42: FFT of the system for the forcing frequency of 4.05 Hz

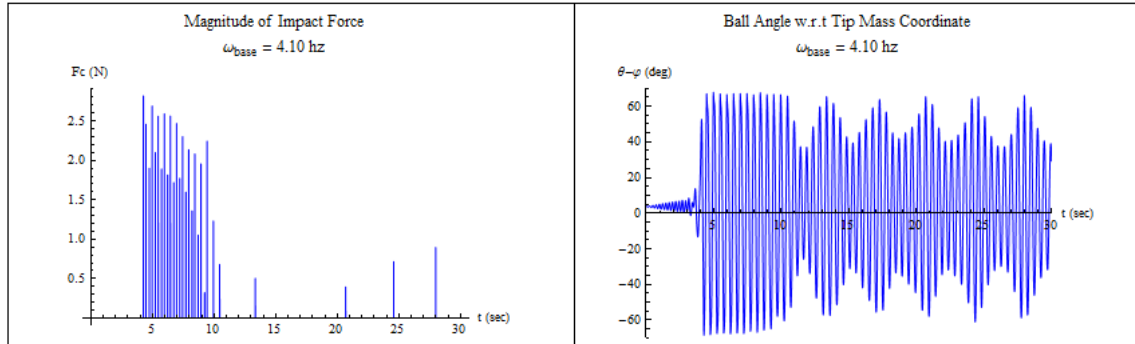


Figure 8-43: Impact details for the forcing frequency of 4.10 Hz

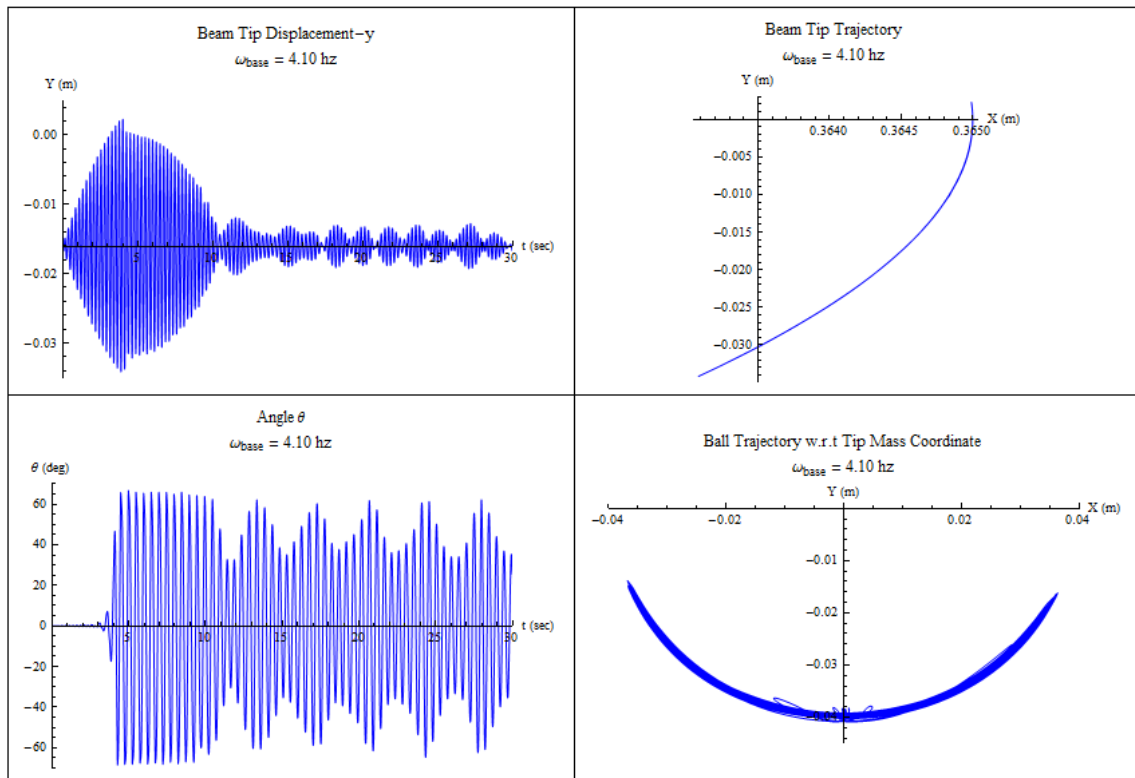


Figure 8-44: System trajectories for the forcing frequency of 4.10 Hz

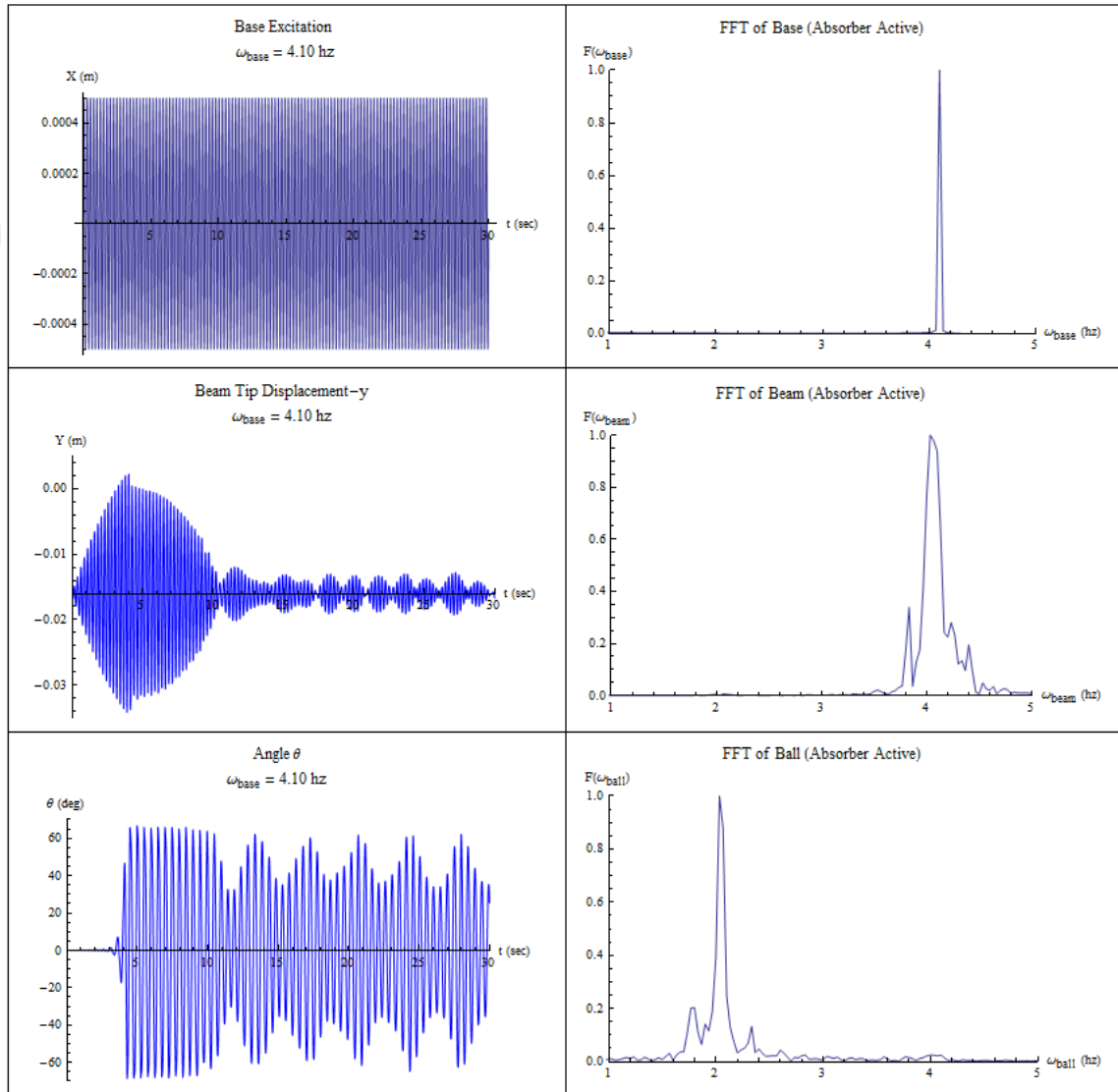


Figure 8-45: FFT of the system for the forcing frequency of 4.10 Hz

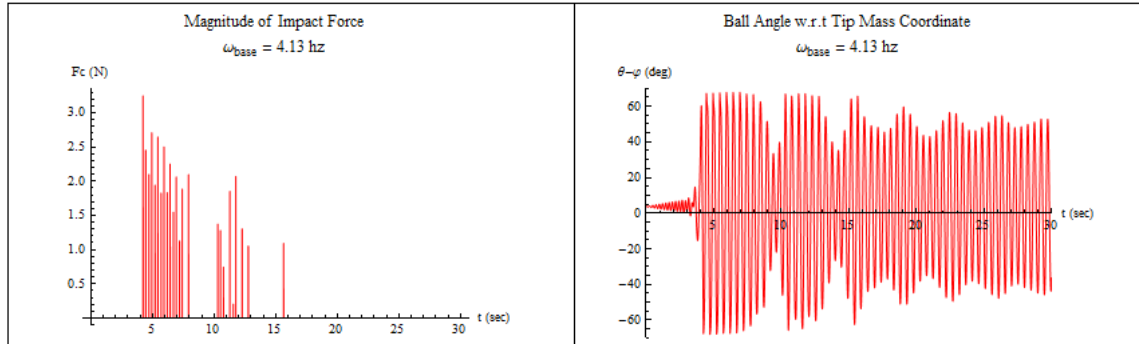


Figure 8-46: Impact details for the forcing frequency of 4.13 Hz

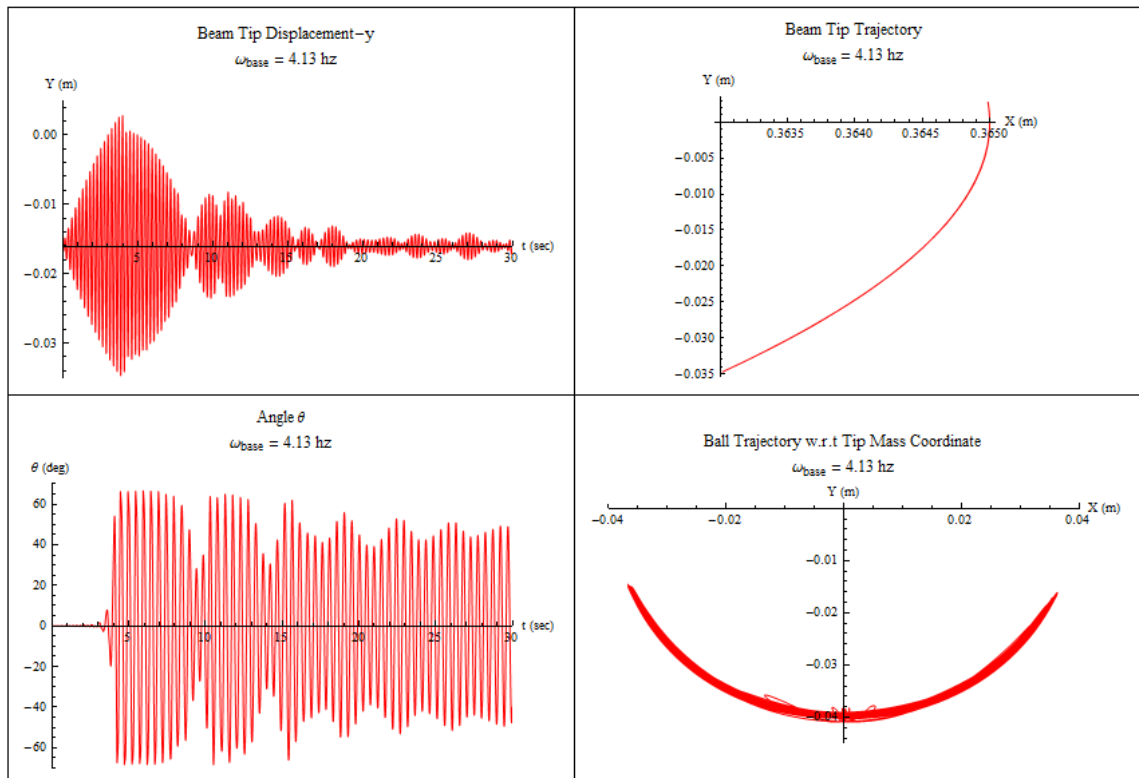


Figure 8-47: System trajectories for the forcing frequency of 4.13 Hz

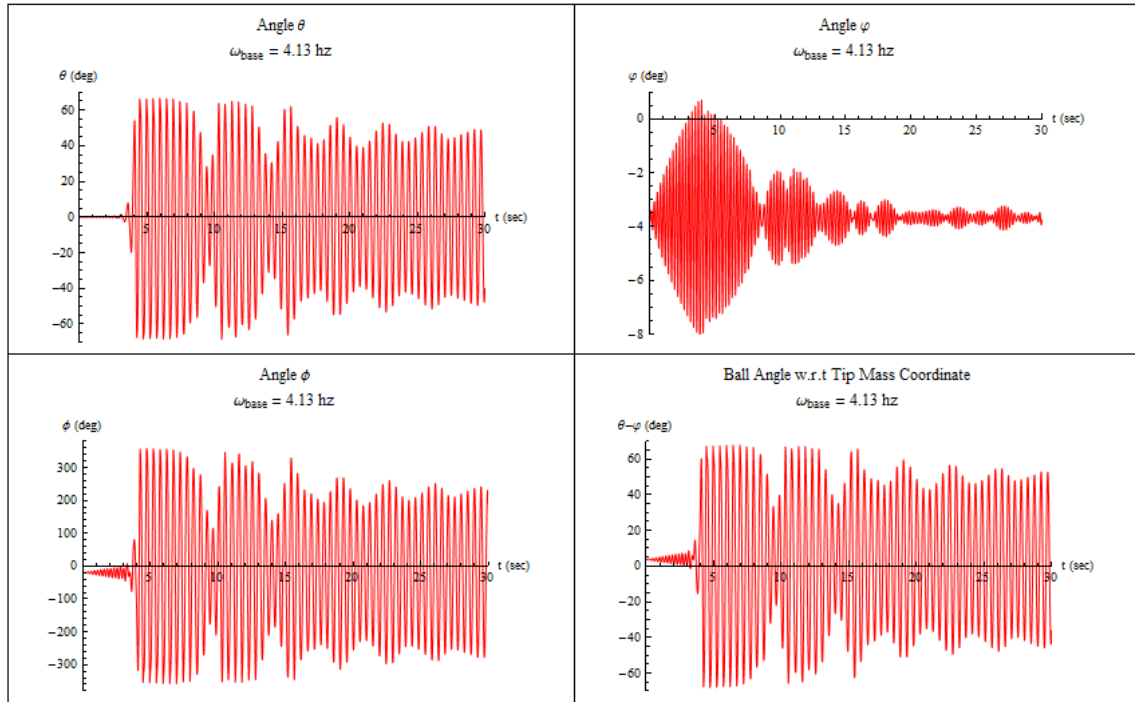


Figure 8-48: Time histories of the system angles for the forcing frequency of 4.13 Hz

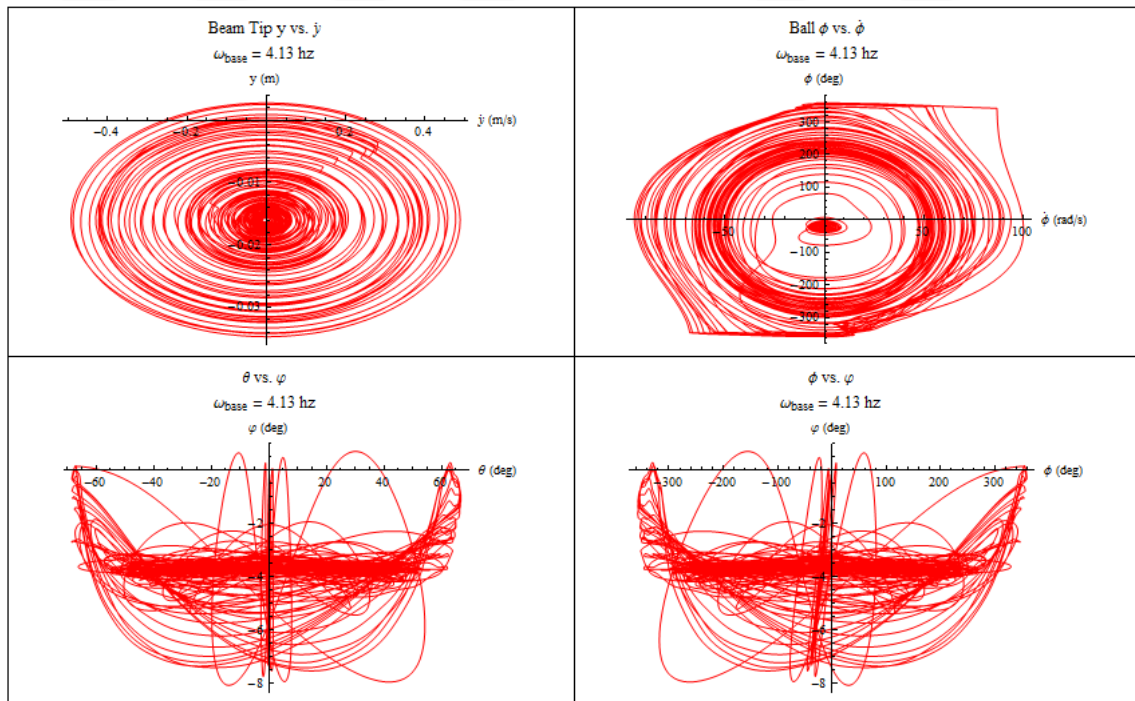


Figure 8-49: Phase plane curves for the forcing frequency of 4.13 Hz

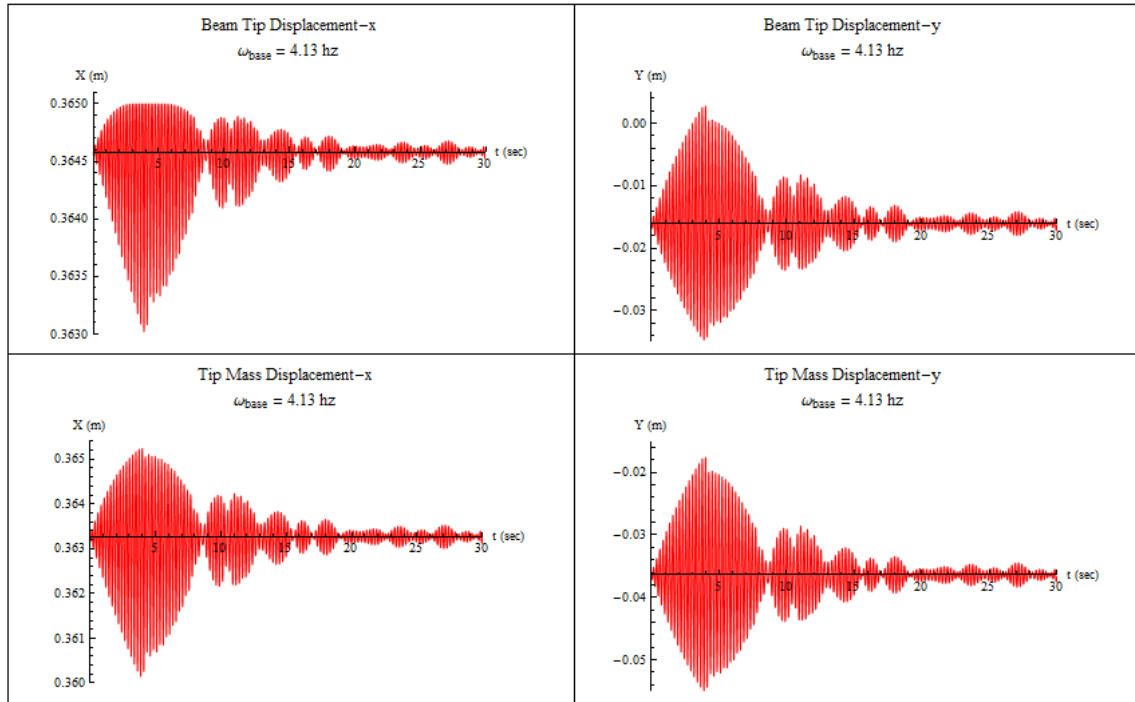


Figure 8-50: Time histories of the system translational displacements for the forcing frequency of 4.13 Hz

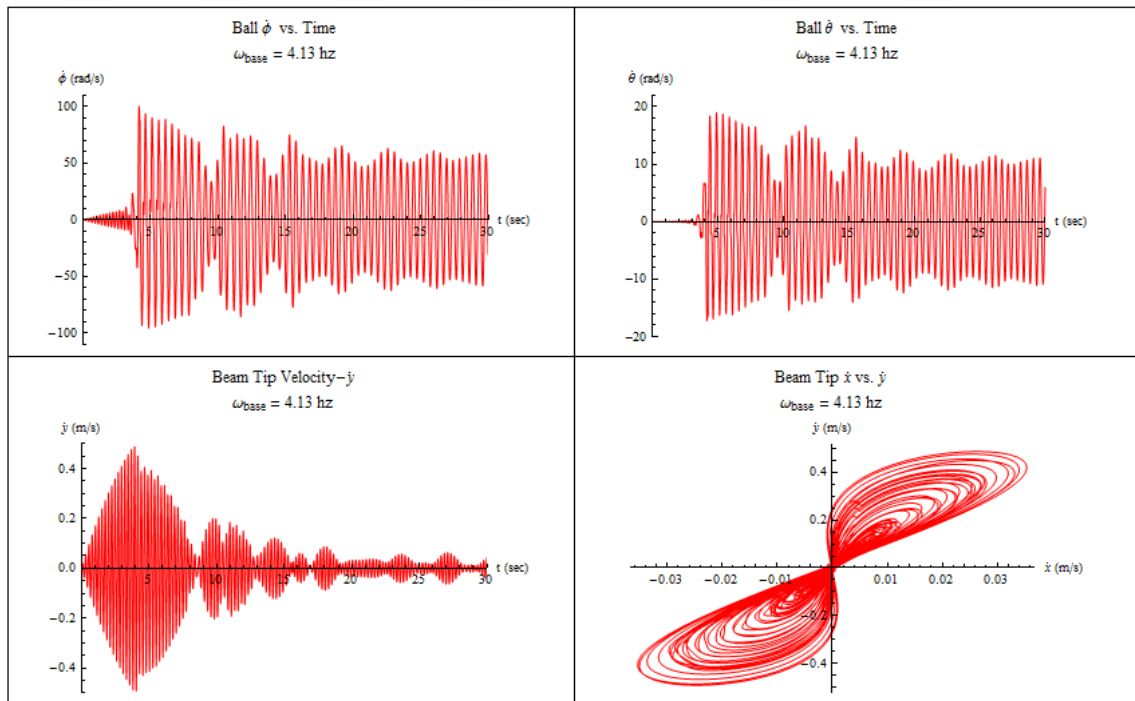


Figure 8-51: System velocity curves for the forcing frequency of 4.13 Hz

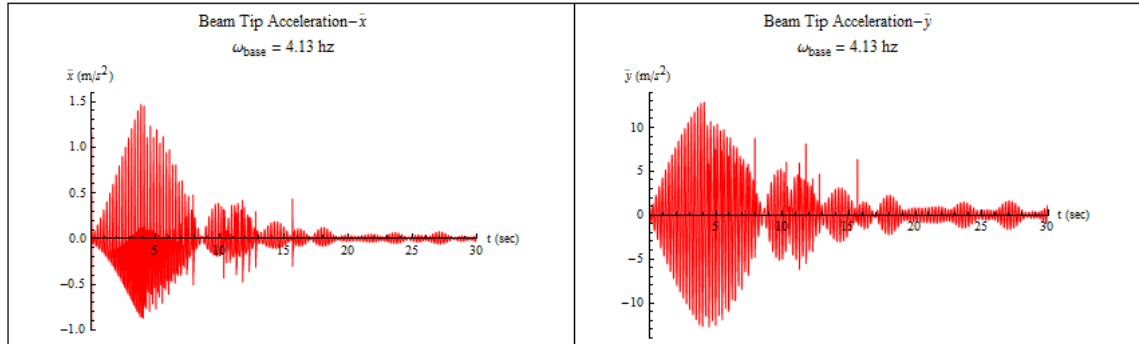


Figure 8-52: Beam tip acceleration for the forcing frequency of 4.13 Hz

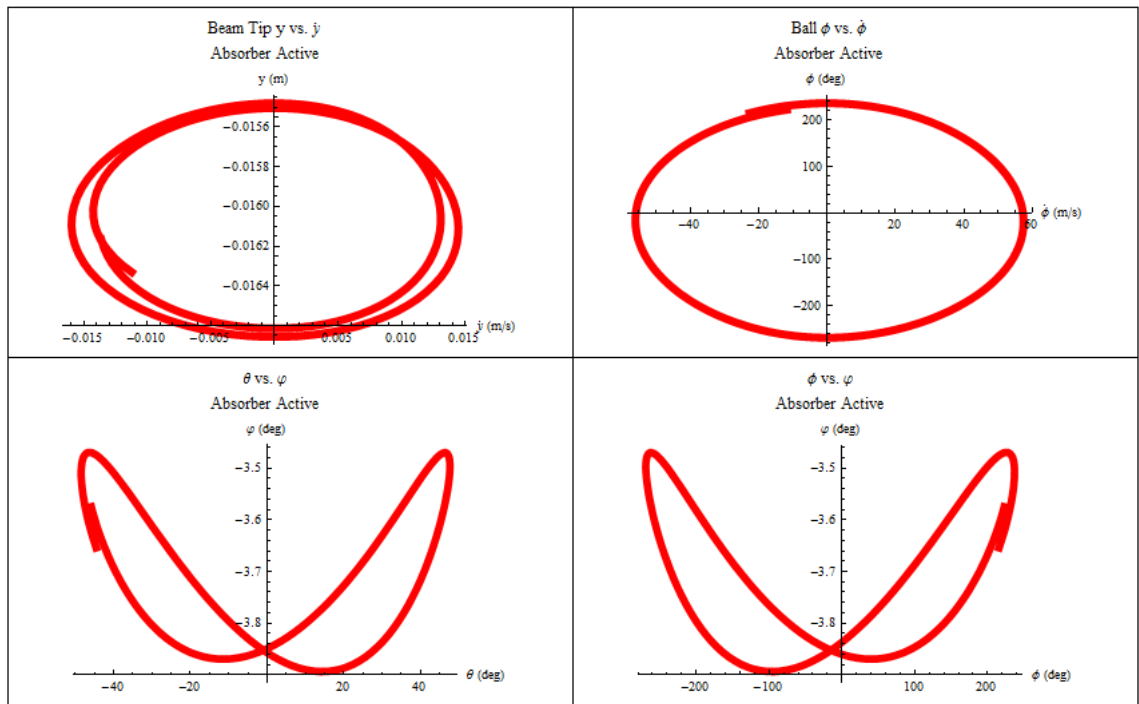


Figure 8-53: Phase plane curves for the forcing frequency of 4.13 Hz at steady-state

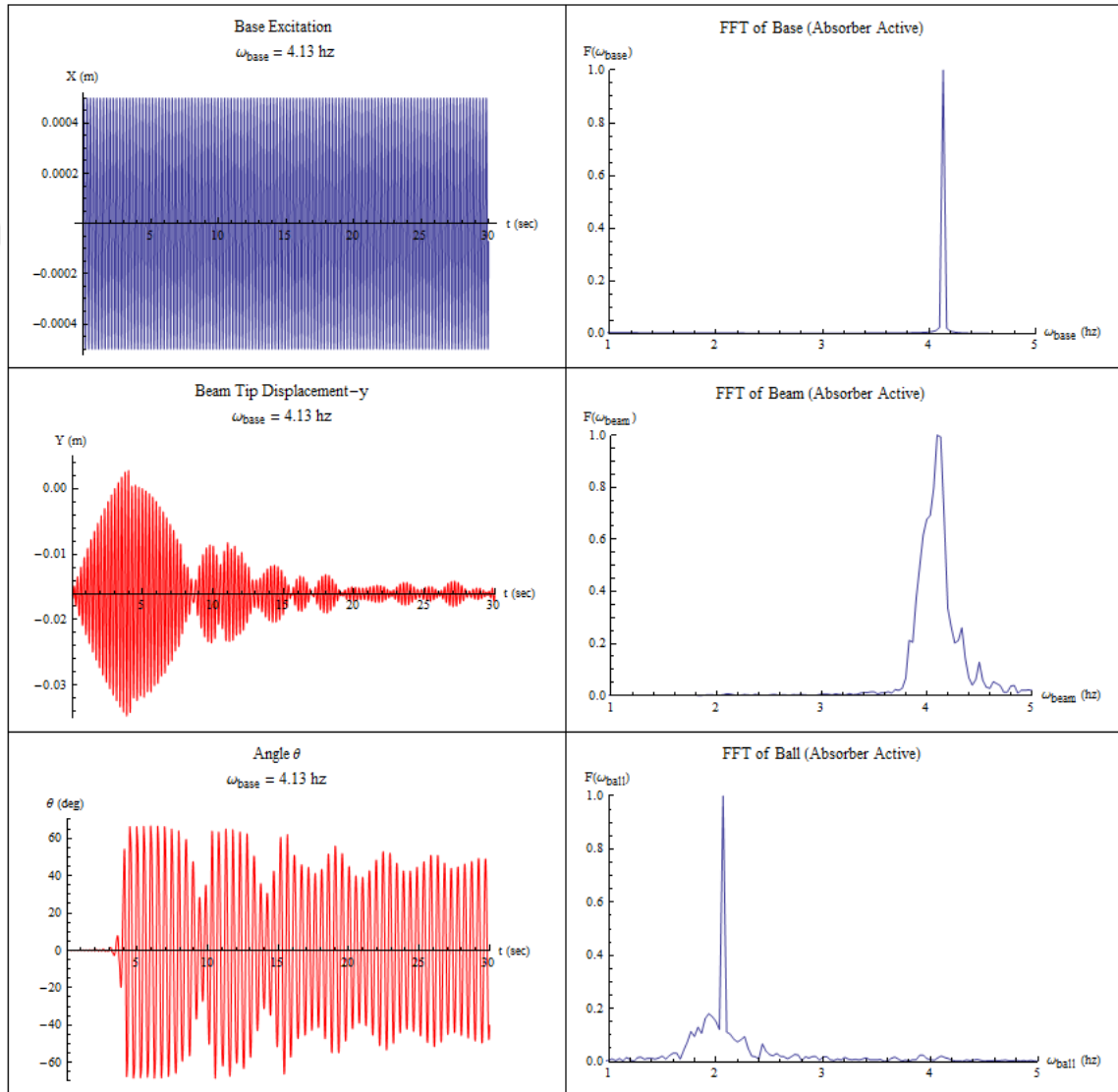


Figure 8-54: FFT of the system for the forcing frequency of 4.13 Hz

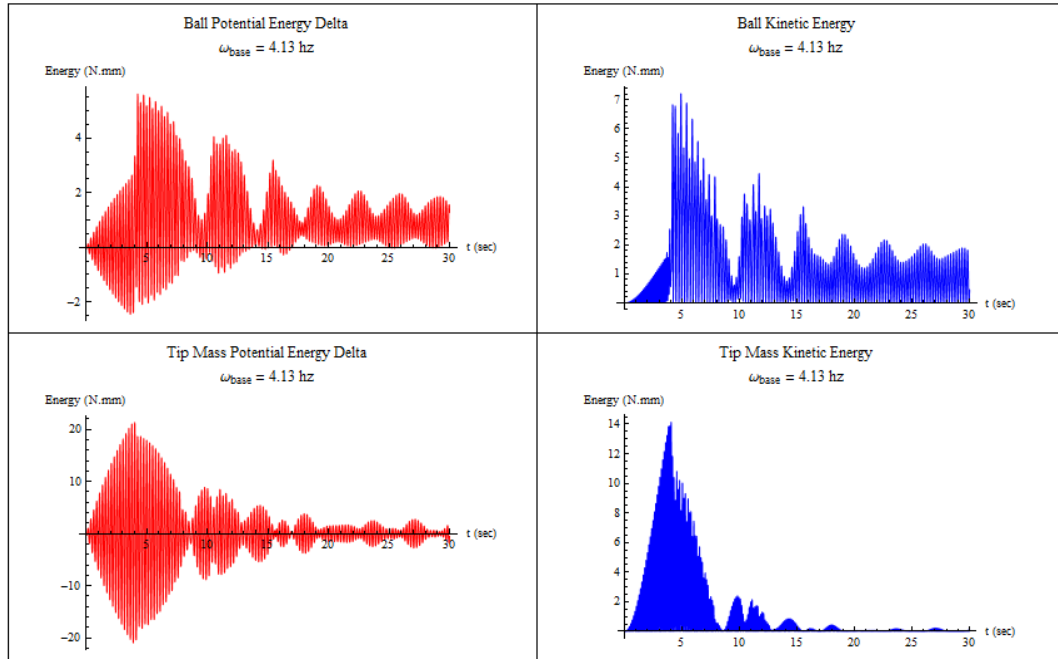


Figure 8-55: Energy curves of the ball and the tip mass for the forcing frequency of 4.13 Hz

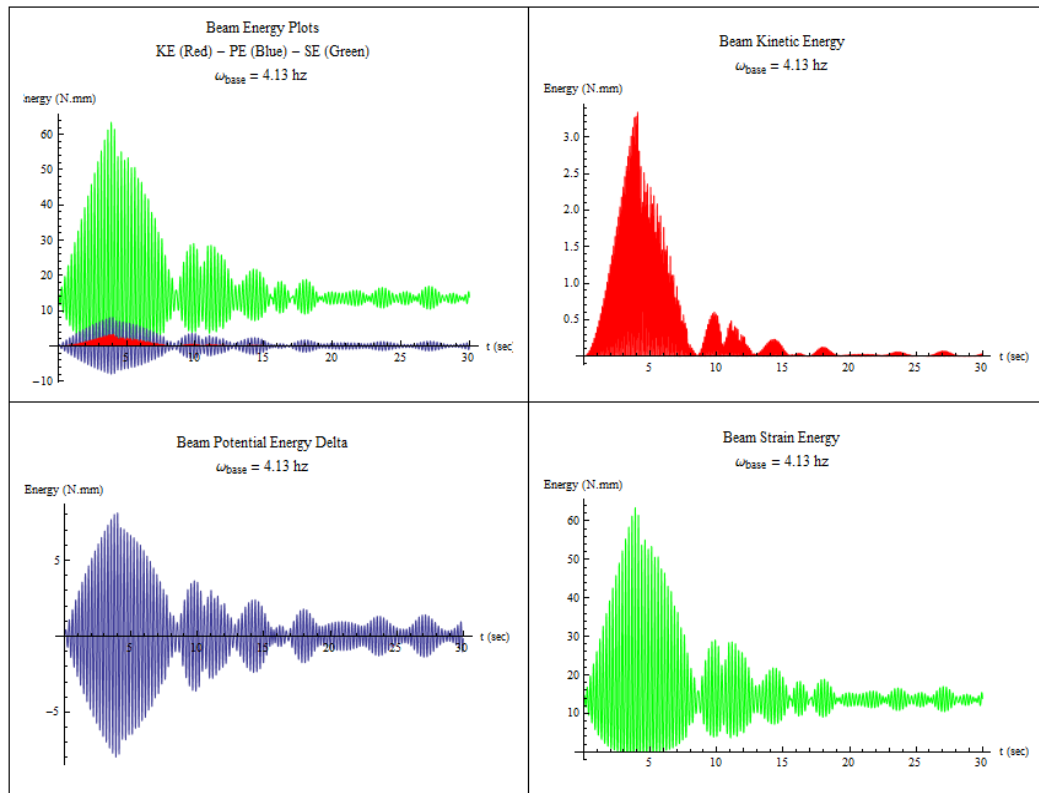


Figure 8-56: Beam energy curves for the forcing frequency of 4.13 Hz

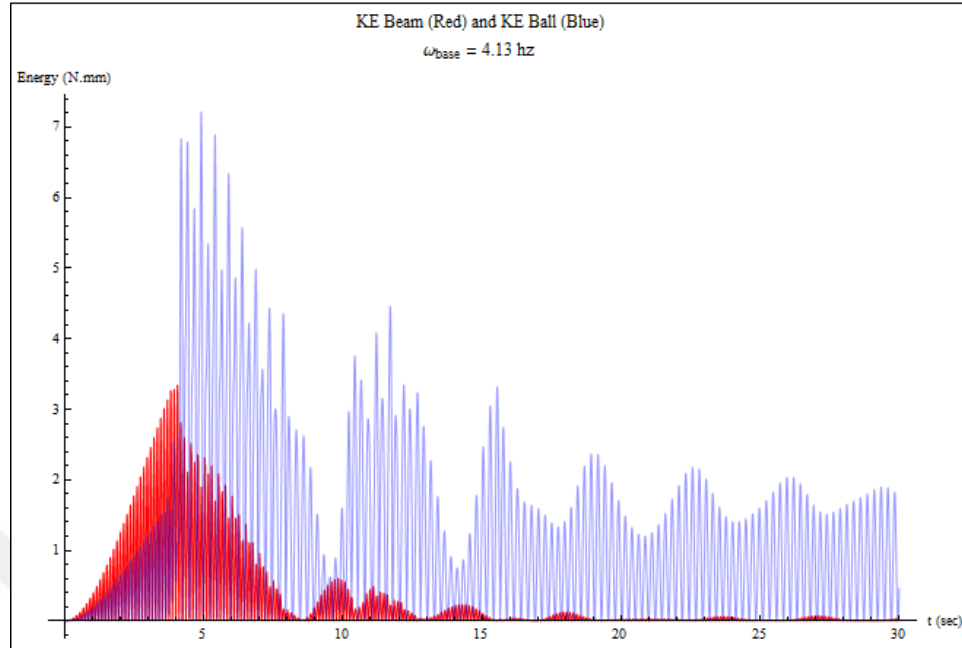


Figure 8-57: Comparison of the beam kinetic energy and the ball kinetic energy for the forcing frequency of 4.13 Hz

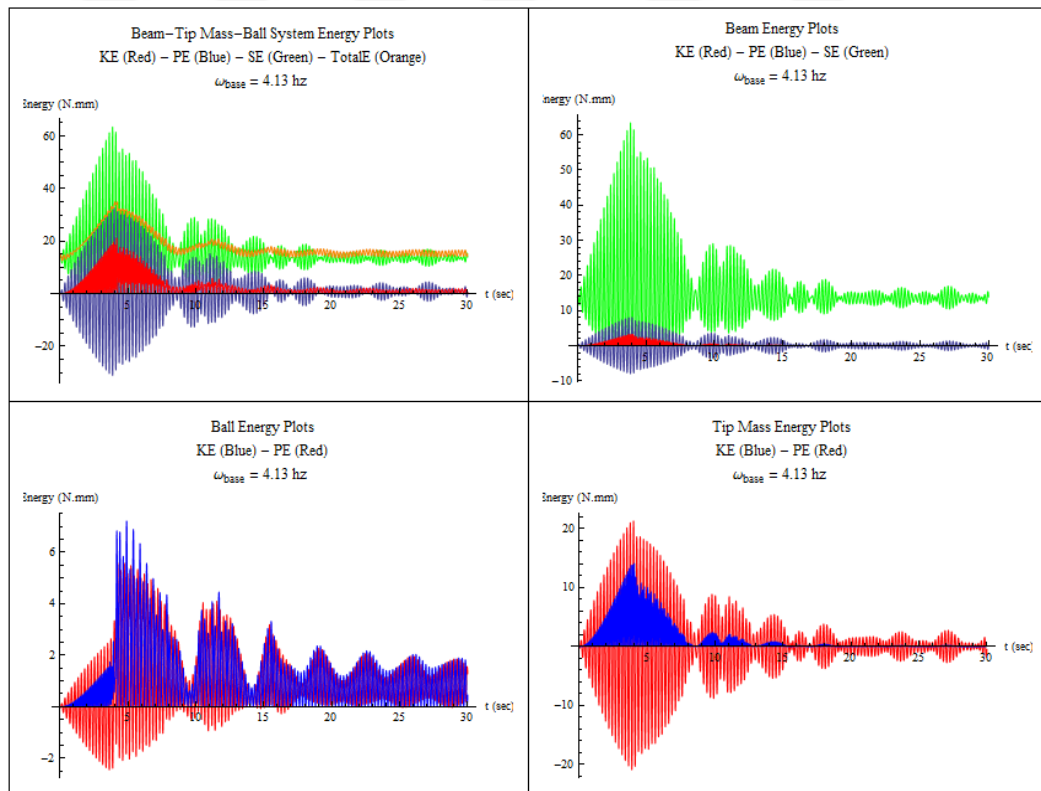


Figure 8-58: System energy curves for the forcing frequency of 4.13 Hz

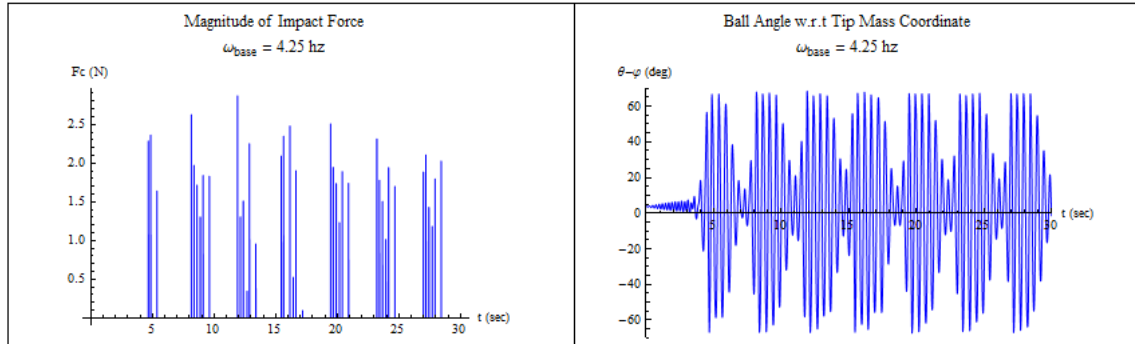


Figure 8-59: Impact details for the forcing frequency of 4.25 Hz

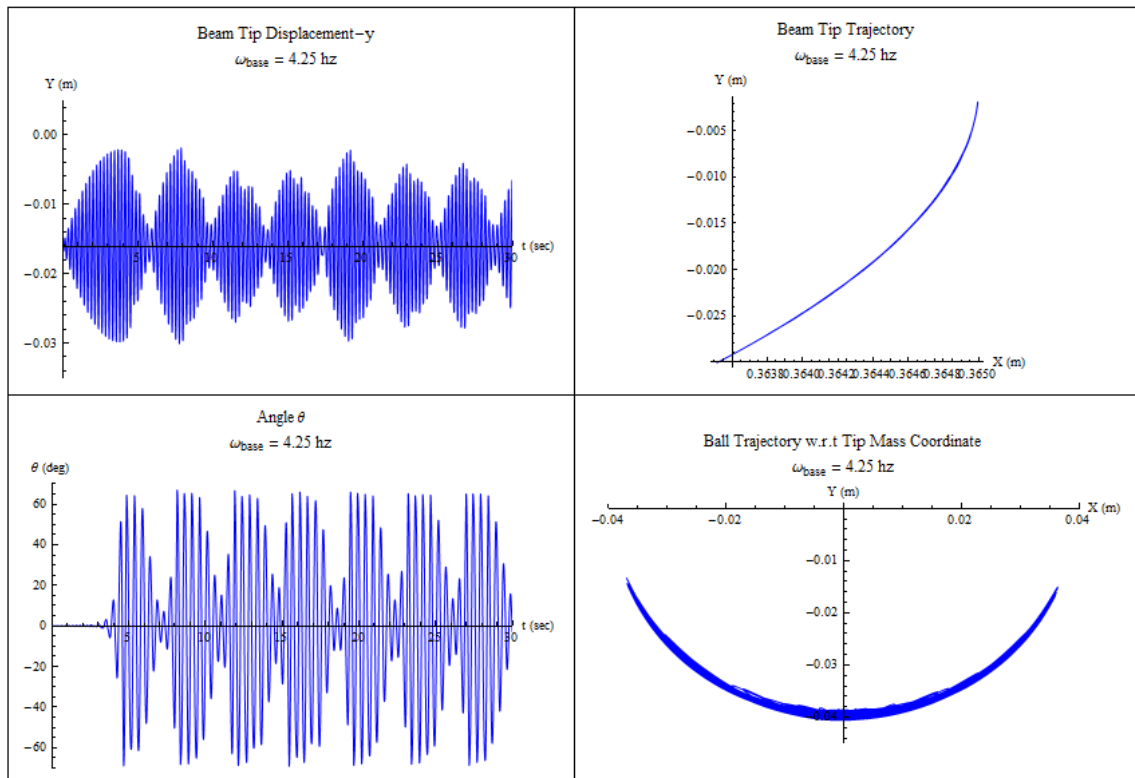


Figure 8-60: System trajectories for the forcing frequency of 4.25 Hz

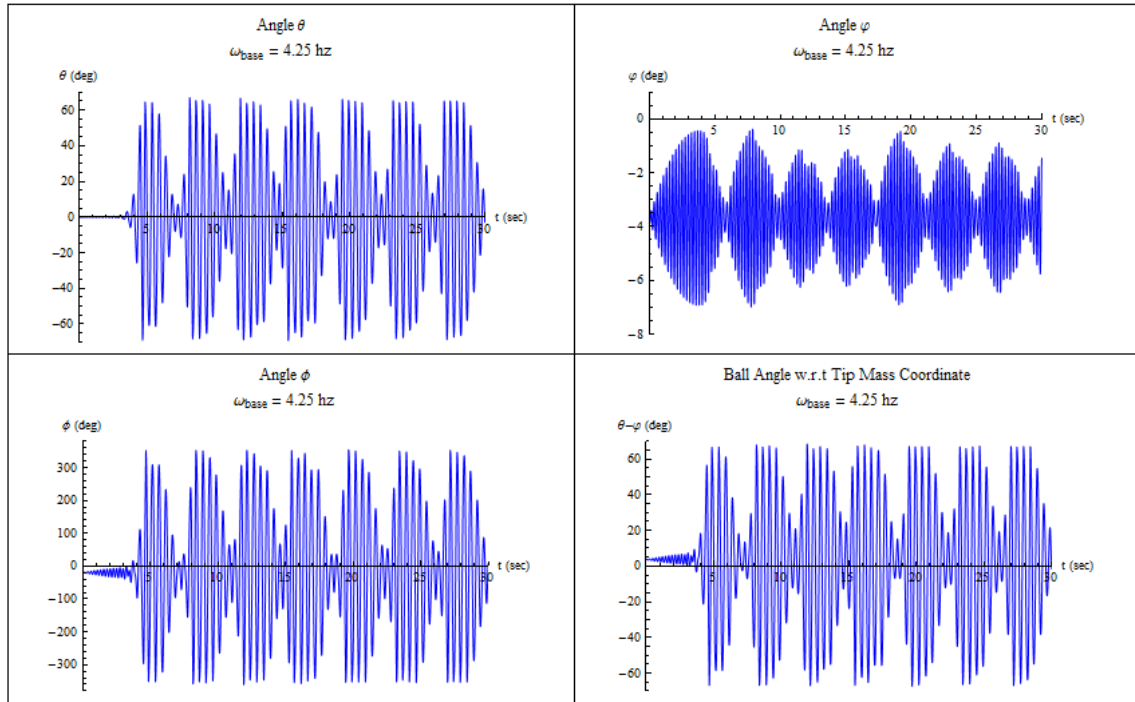


Figure 8-61: Time histories of the system angles for the forcing frequency of 4.25 Hz

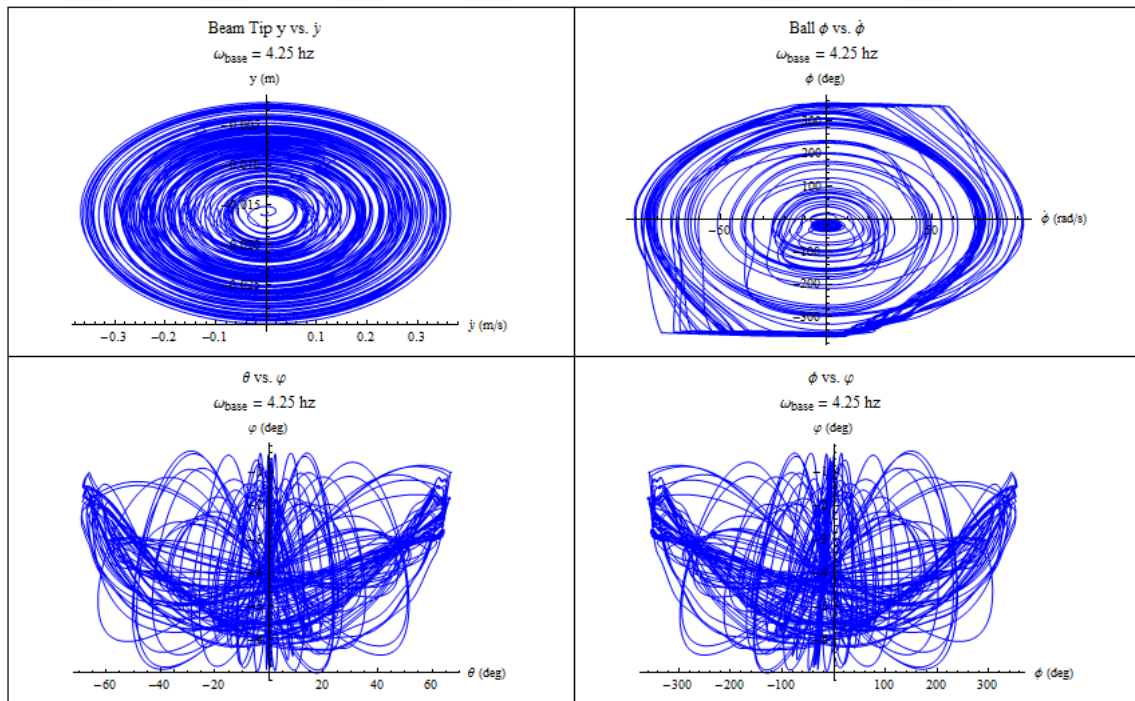


Figure 8-62: Phase plane curves for the forcing frequency of 4.25 Hz

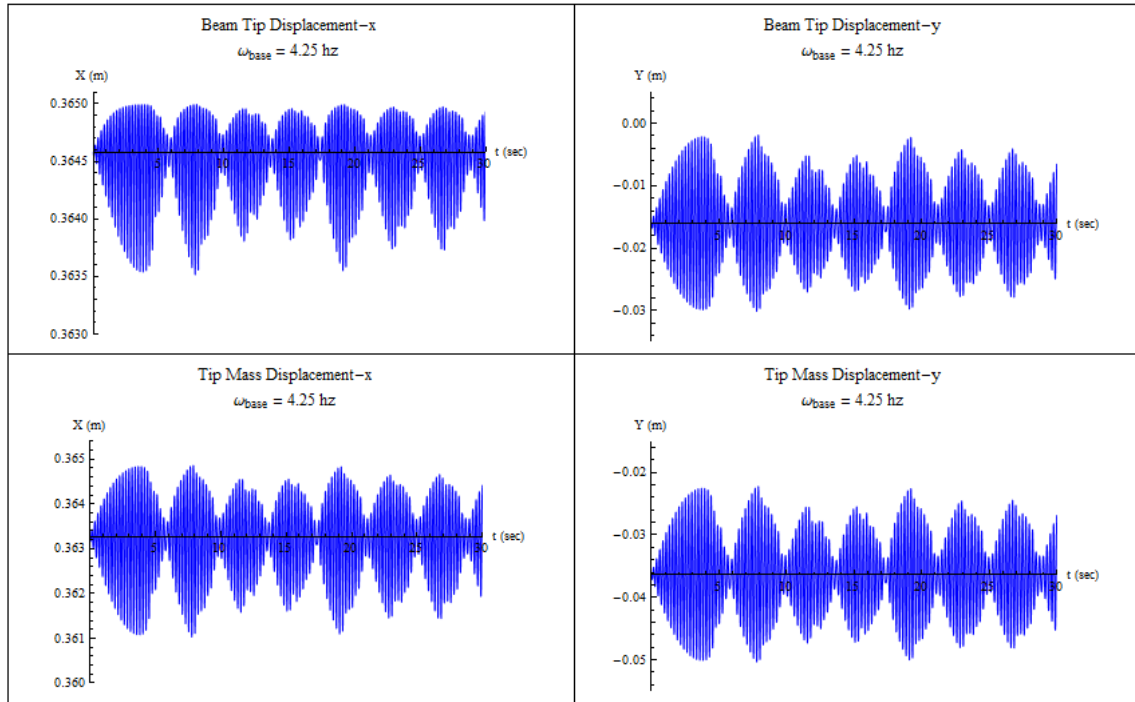


Figure 8-63: Time histories of the system translational displacements for the forcing frequency of 4.25 Hz

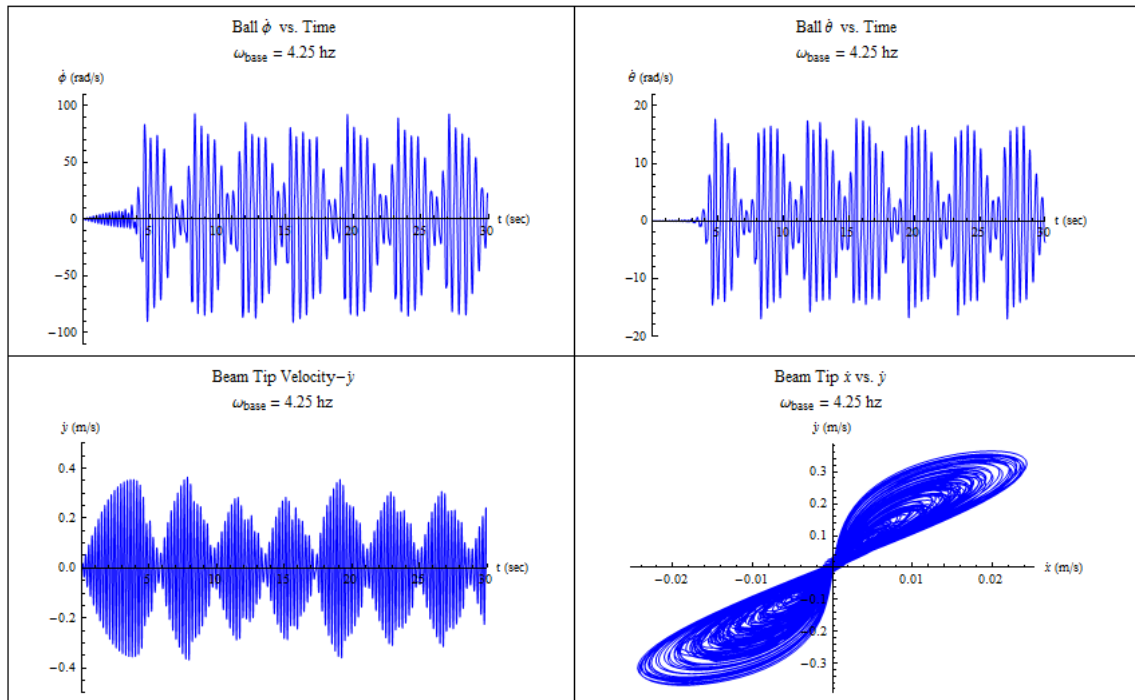


Figure 8-64: System velocity curves for the forcing frequency of 4.25 Hz

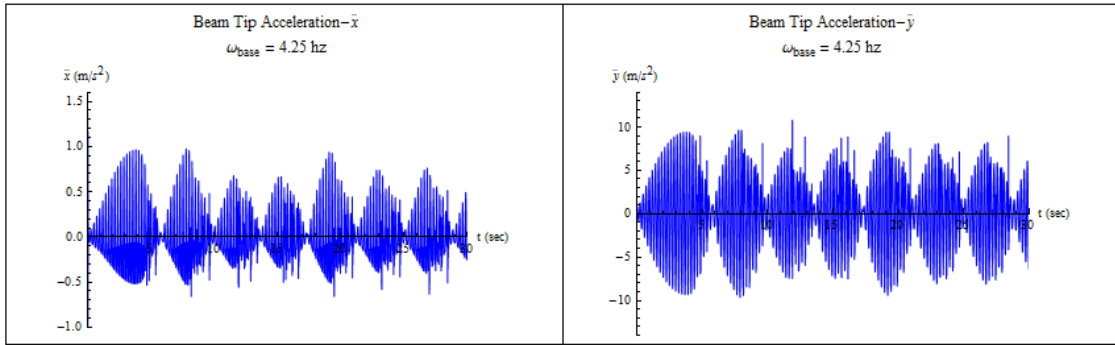


Figure 8-65: Beam tip acceleration for the forcing frequency of 4.25 Hz

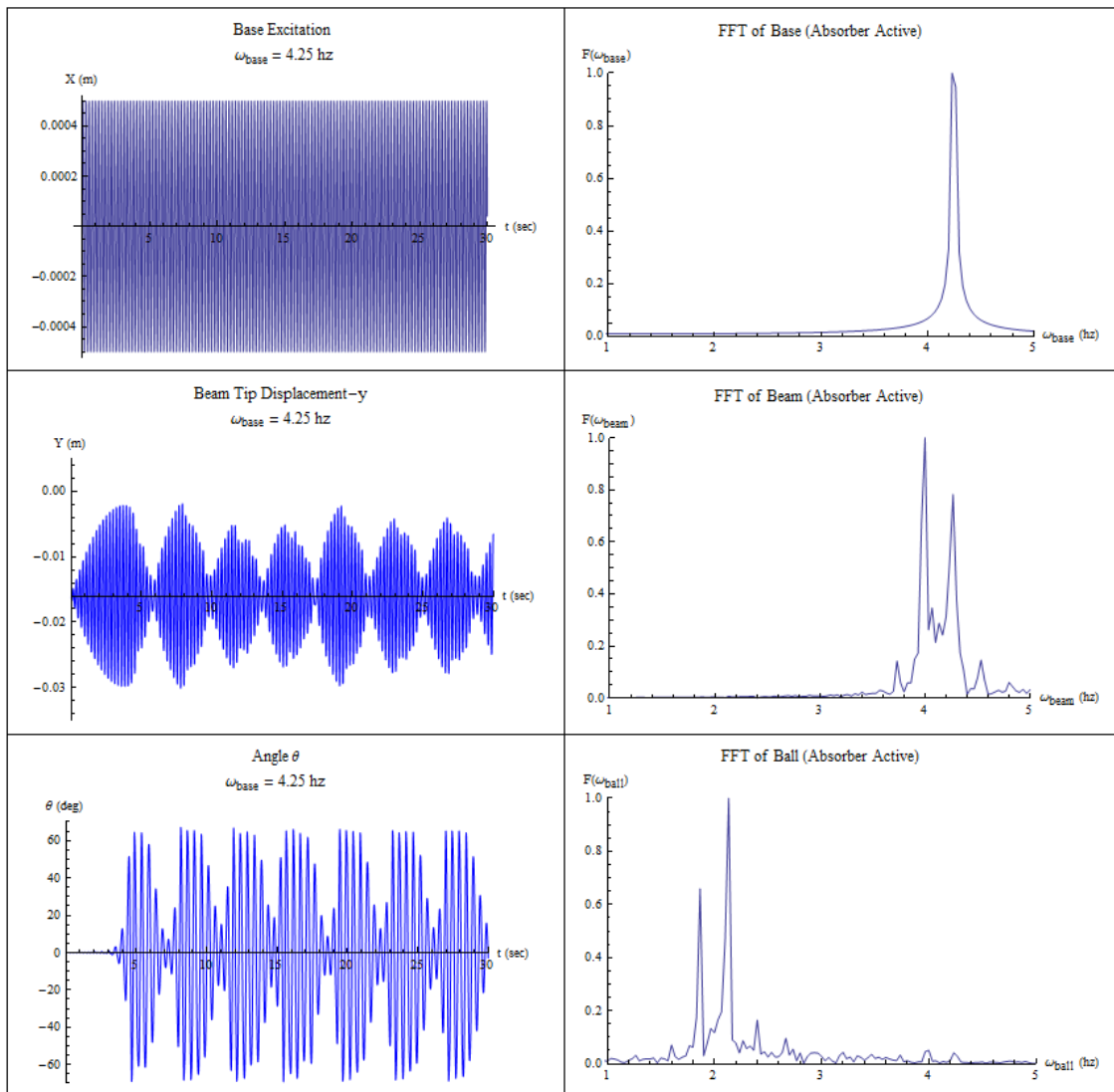


Figure 8-66: FFT of the system for the forcing frequency of 4.25 Hz

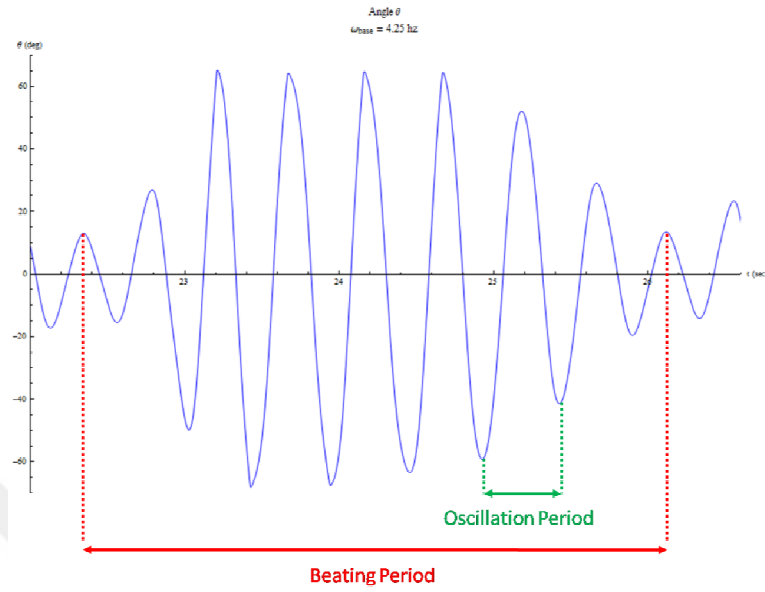


Figure 8-67: Detail of the beating in the ball response

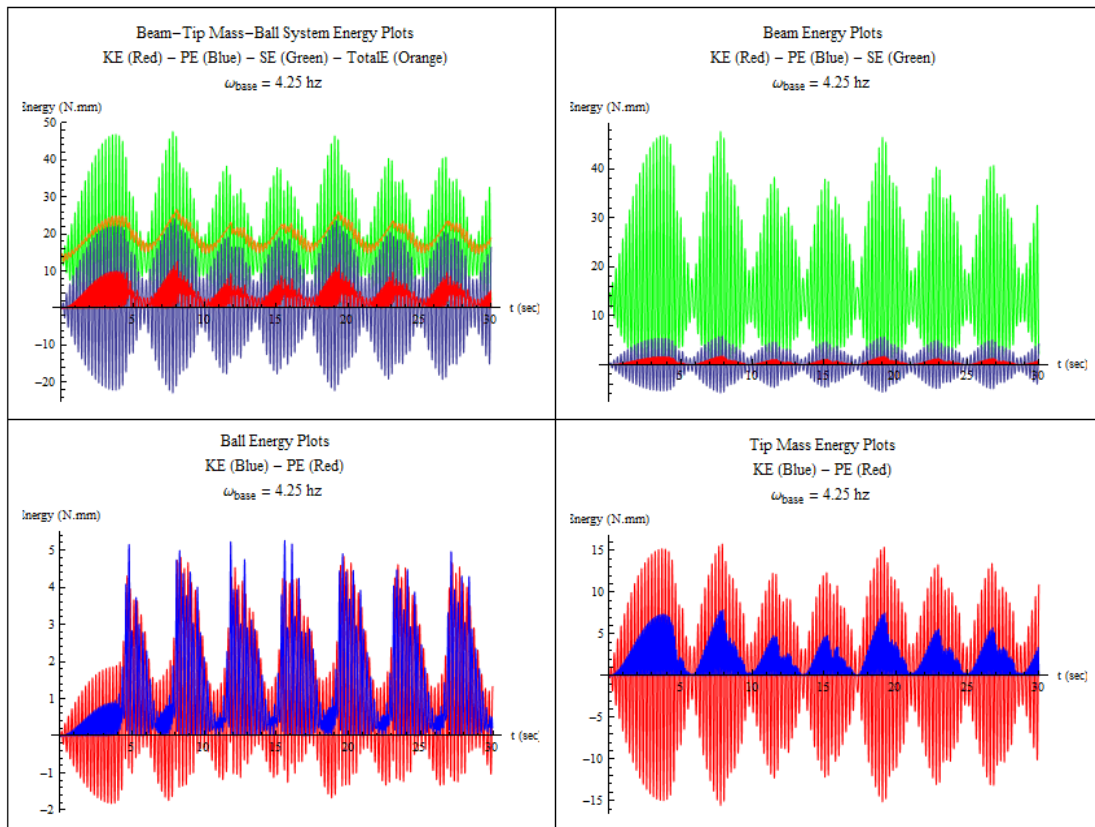


Figure 8-68: System energy curves for the forcing frequency of 4.25 Hz

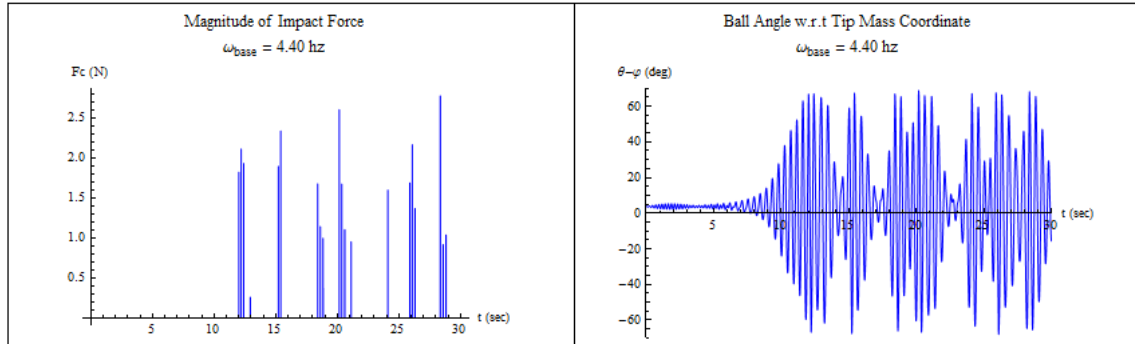


Figure 8-69: Impact details for the forcing frequency of 4.40 Hz

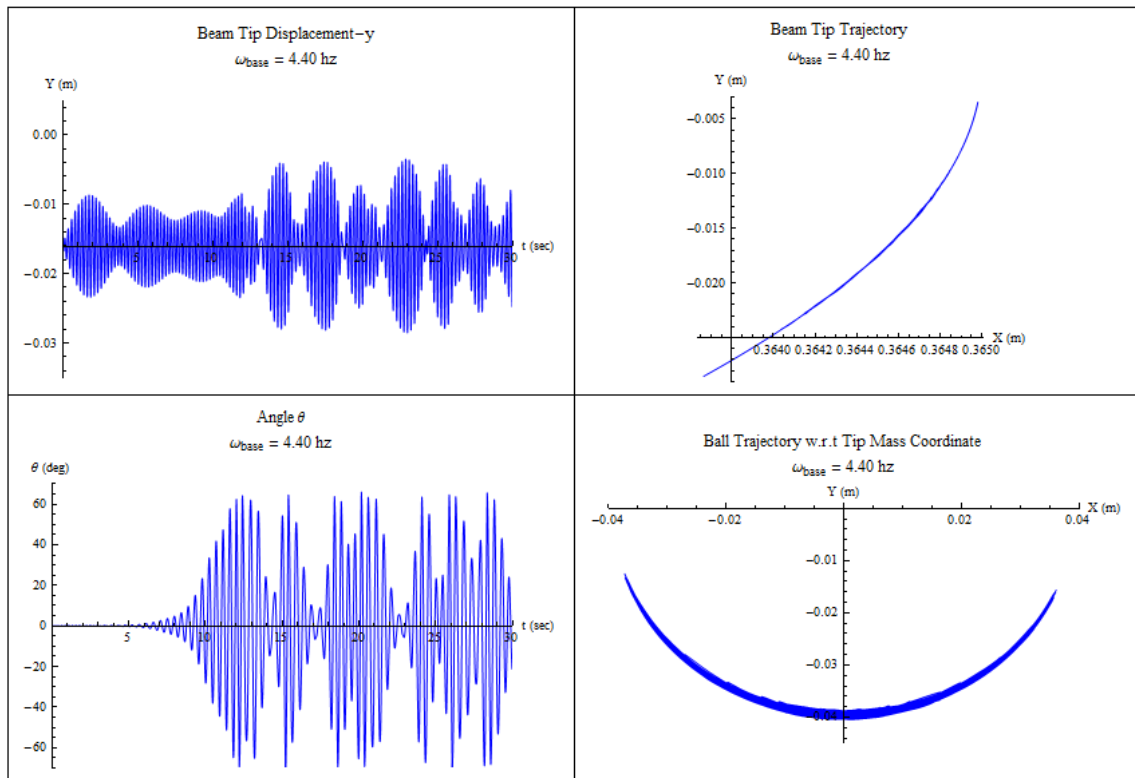


Figure 8-70: System trajectories for the forcing frequency of 4.40 Hz

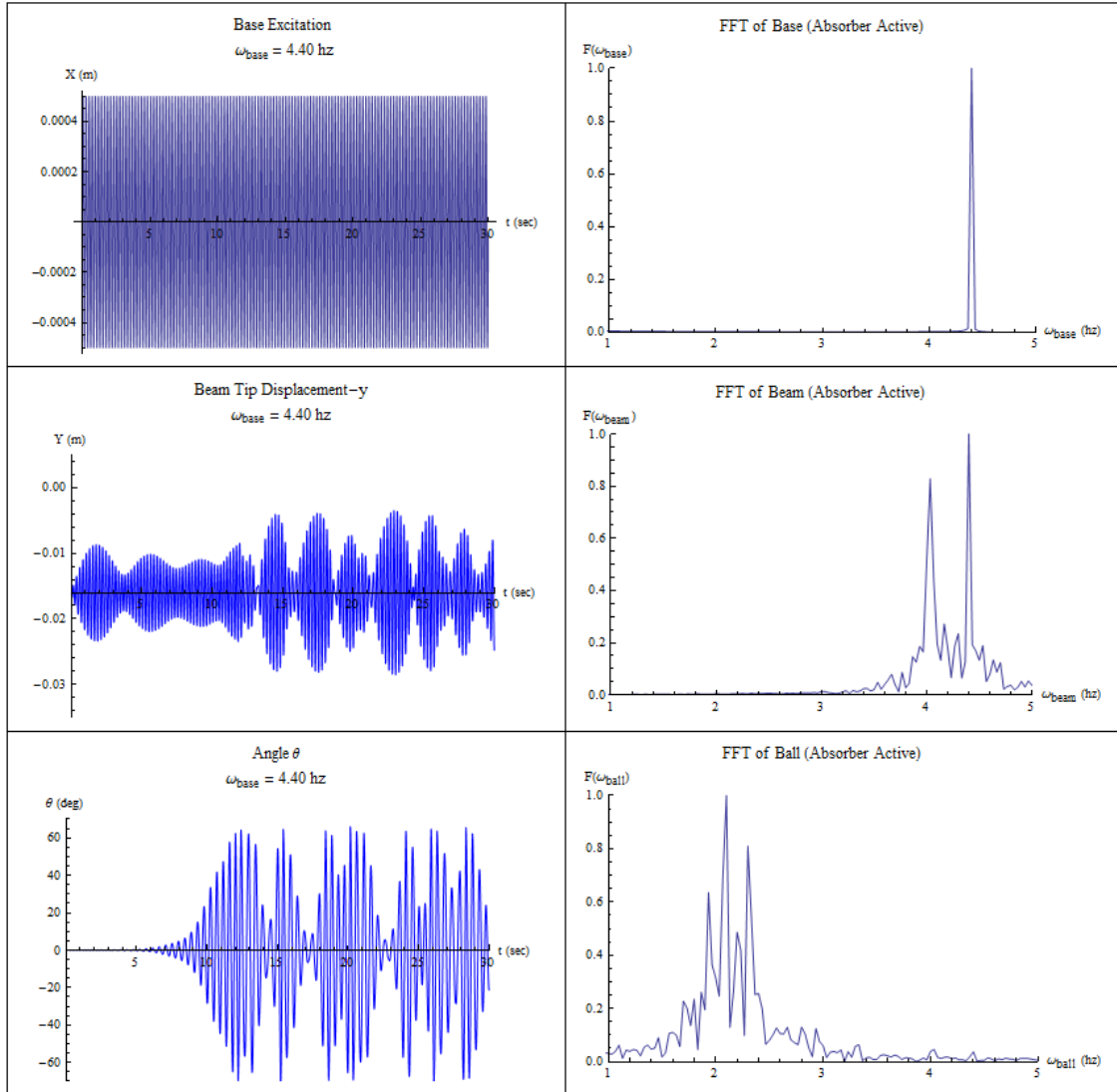


Figure 8-71: FFT of the system for the forcing frequency of 4.40 Hz

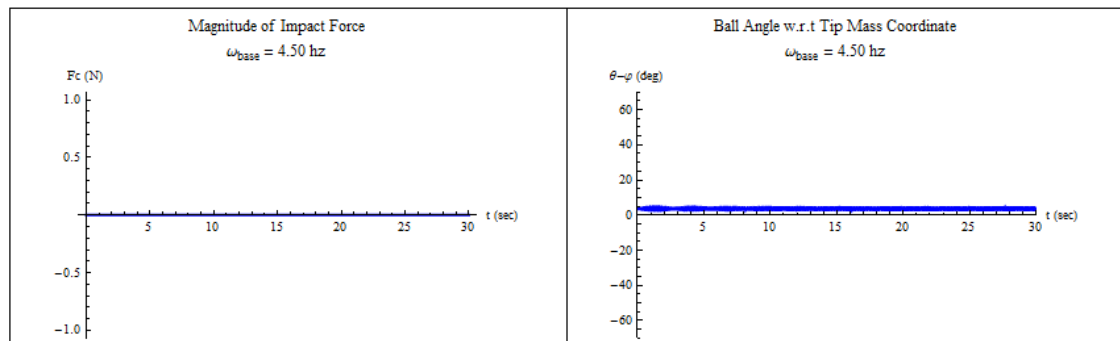


Figure 8-72: Impact details for the forcing frequency of 4.50 Hz

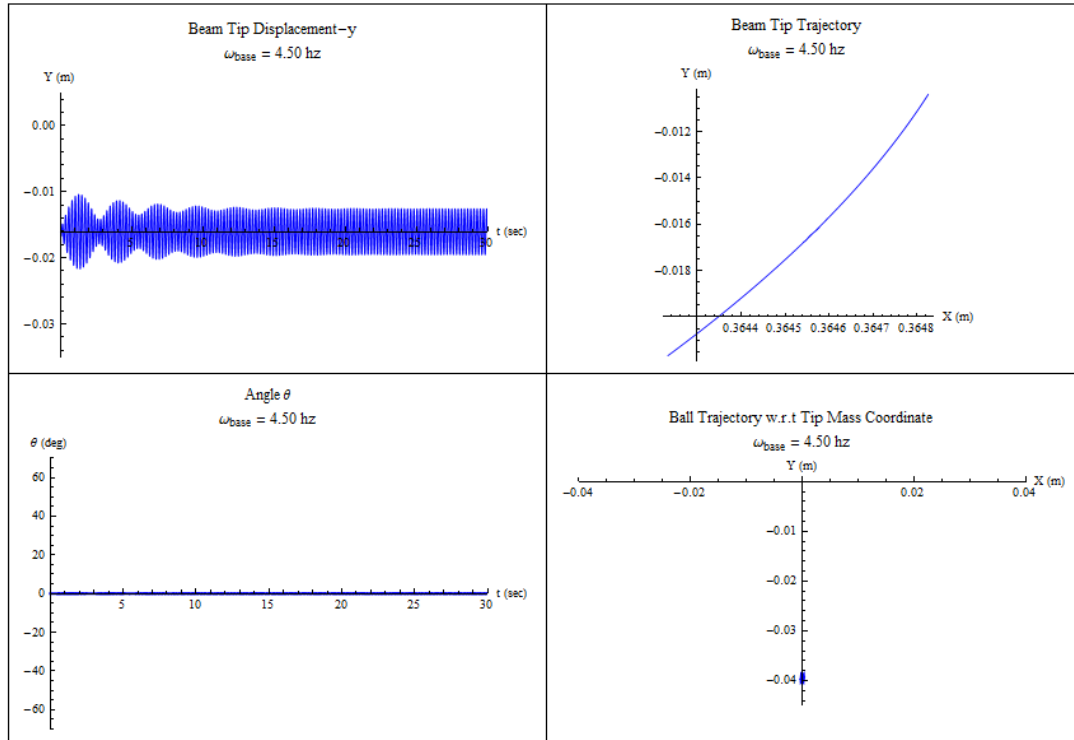


Figure 8-73: System trajectories for the forcing frequency of 4.50 Hz

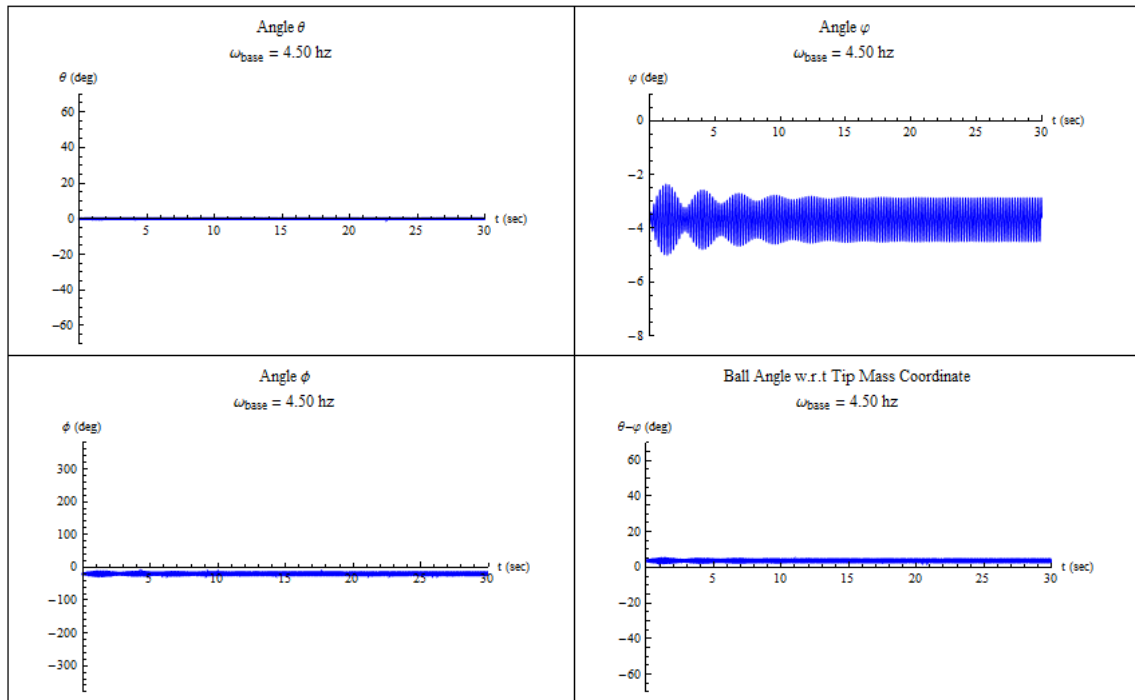


Figure 8-74: Time histories of the system angles for the forcing frequency of 4.50 Hz

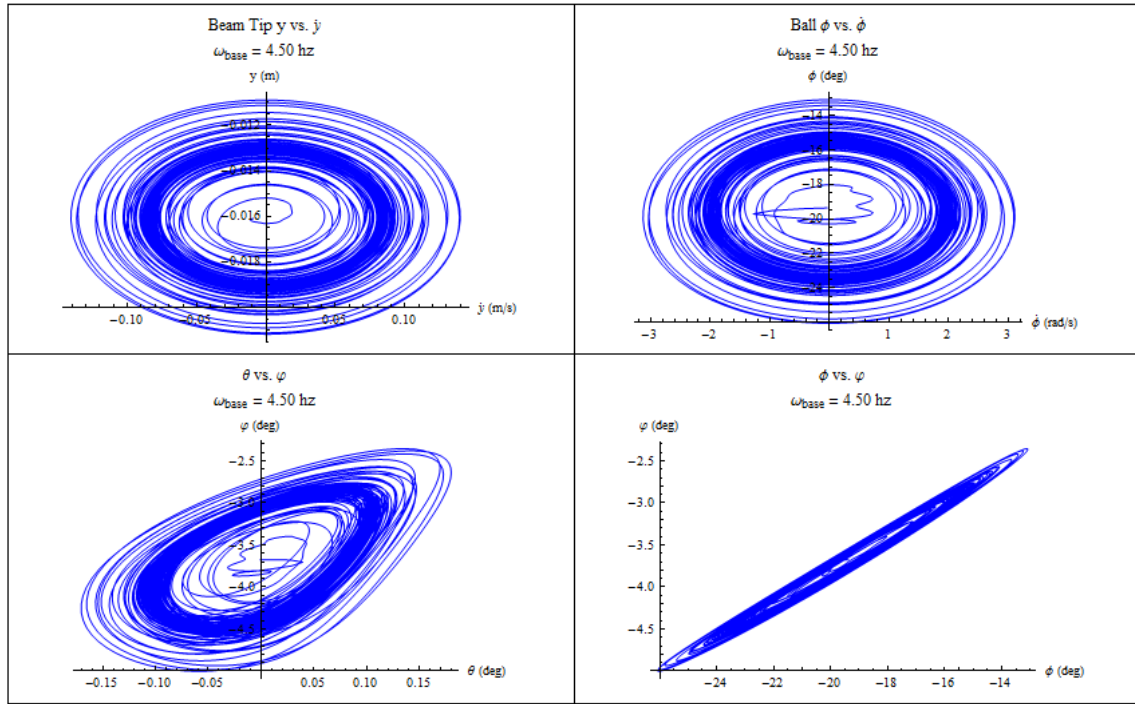


Figure 8-75: Phase plane curves for the forcing frequency of 4.50 Hz

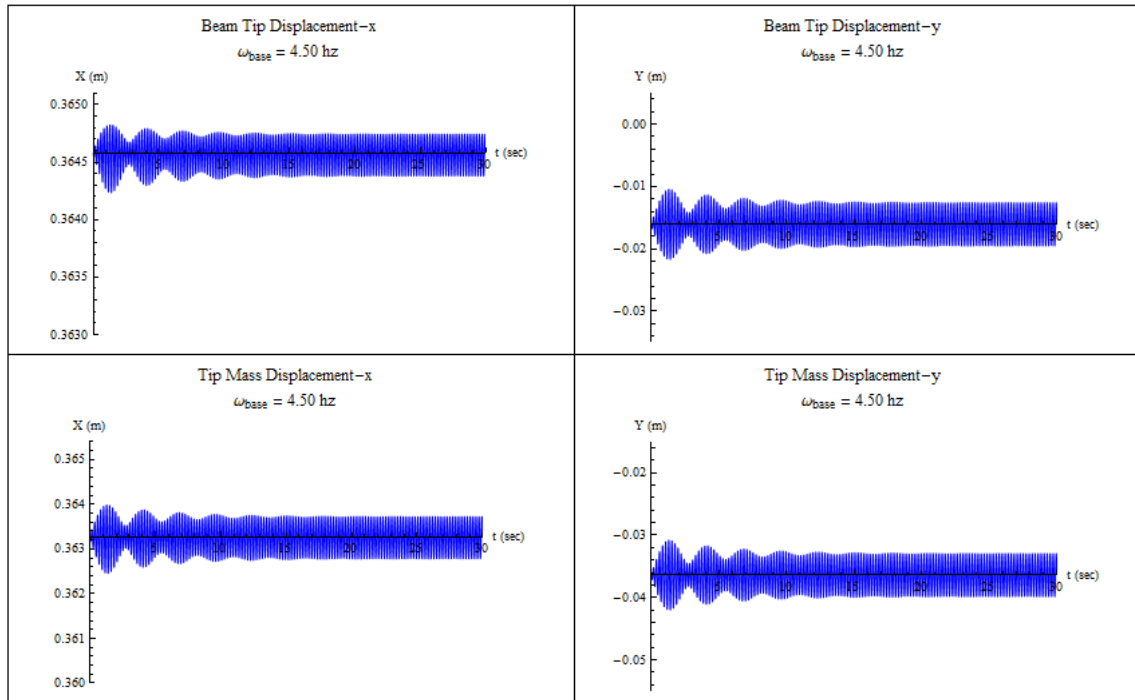


Figure 8-76: Time histories of the system translational displacements for the forcing frequency of 4.50 Hz

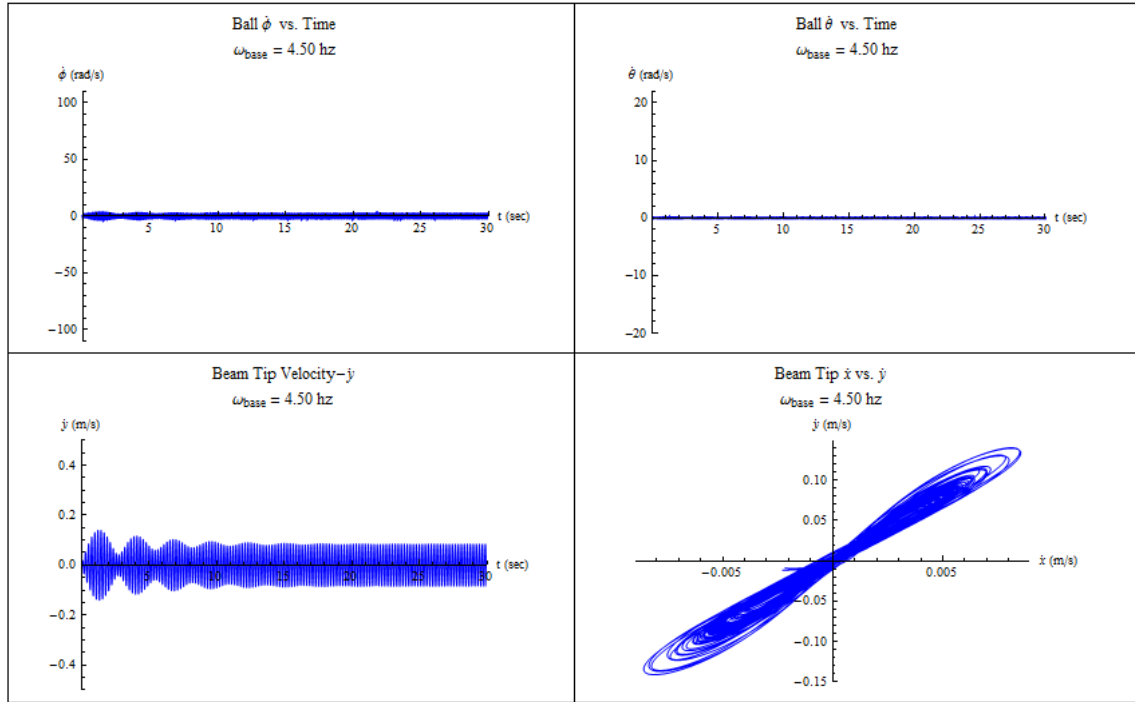


Figure 8-77: System velocity curves for the forcing frequency of 4.50 Hz

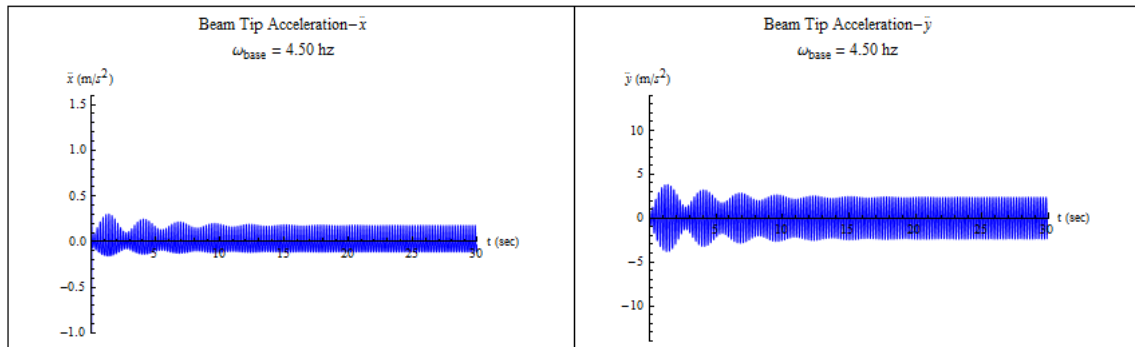


Figure 8-78: Beam tip acceleration for the forcing frequency of 4.50 Hz

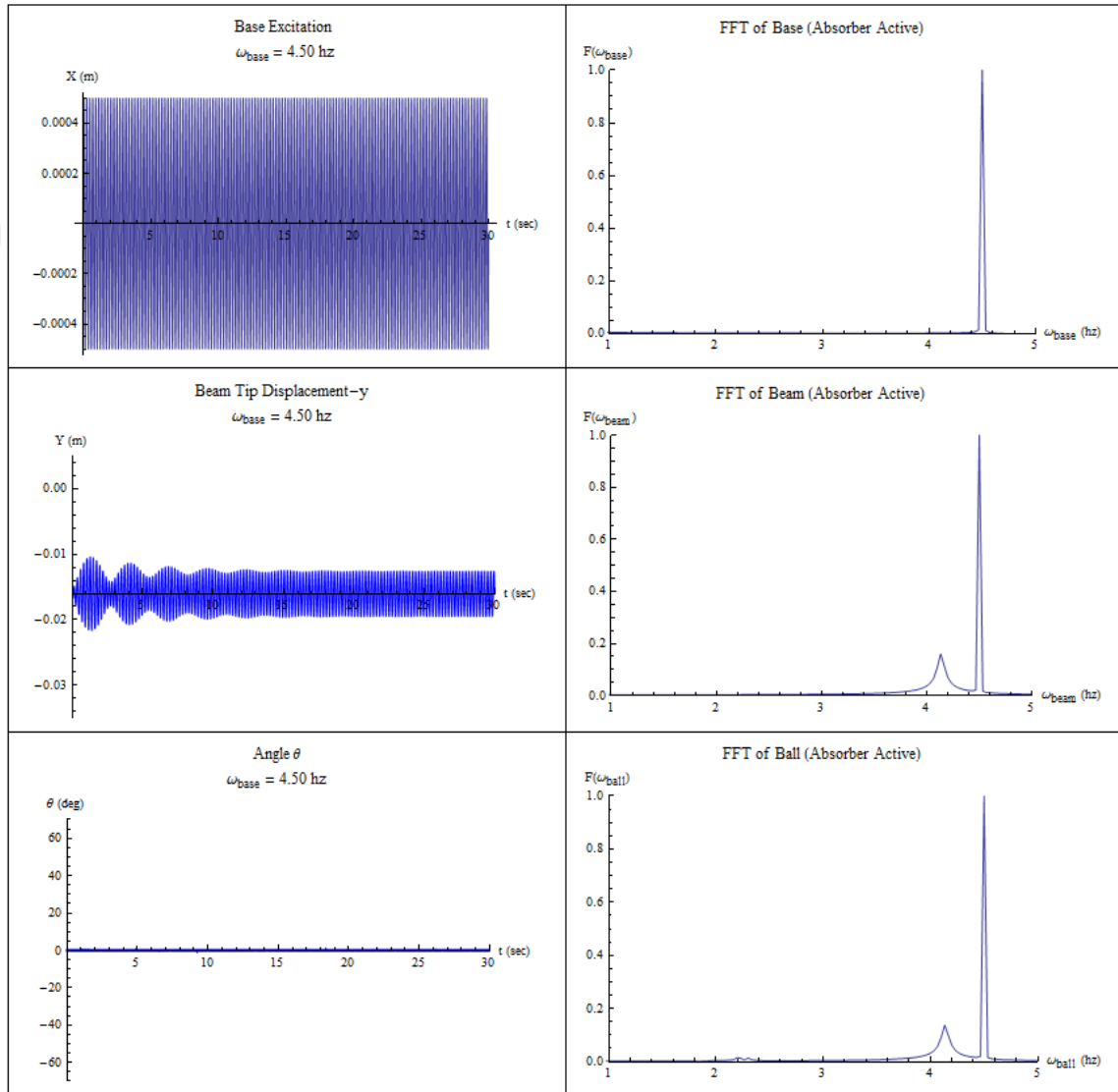


Figure 8-79: FFT of the system for the forcing frequency of 4.50 Hz

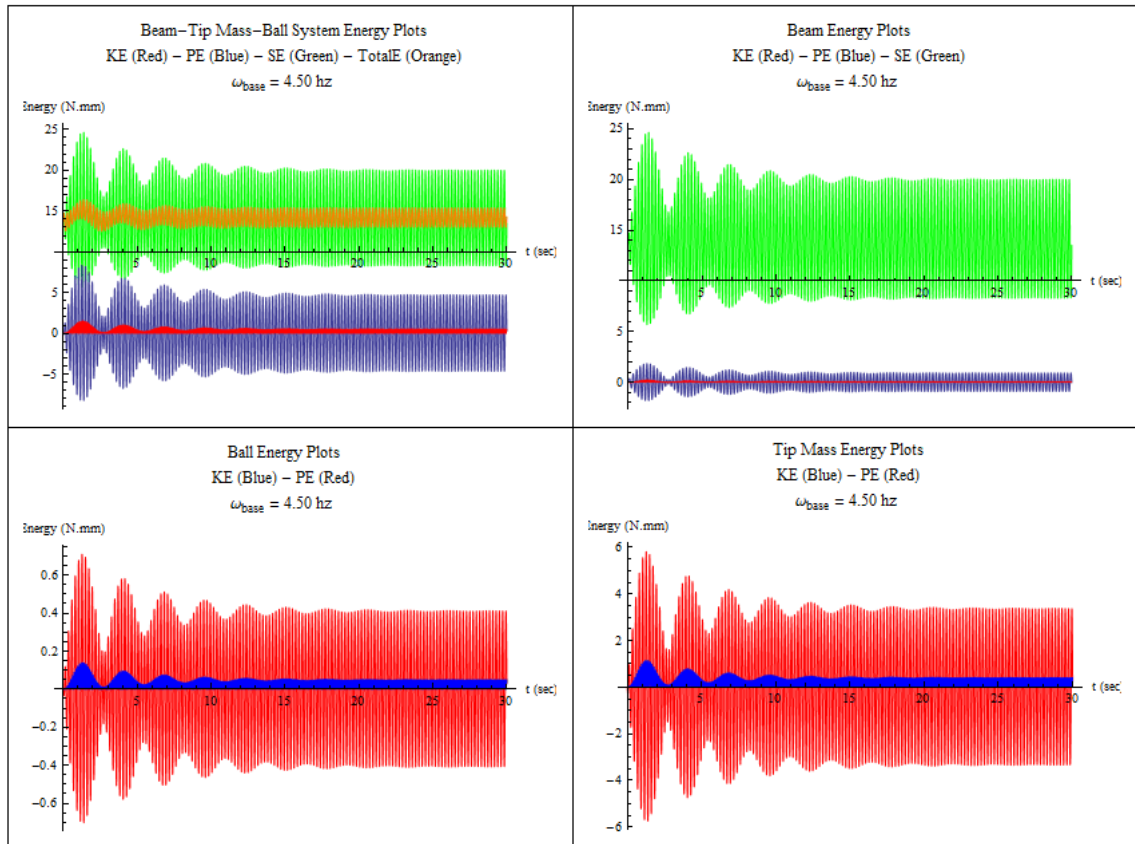


Figure 8-80: System energy curves for the forcing frequency of 4.50 Hz

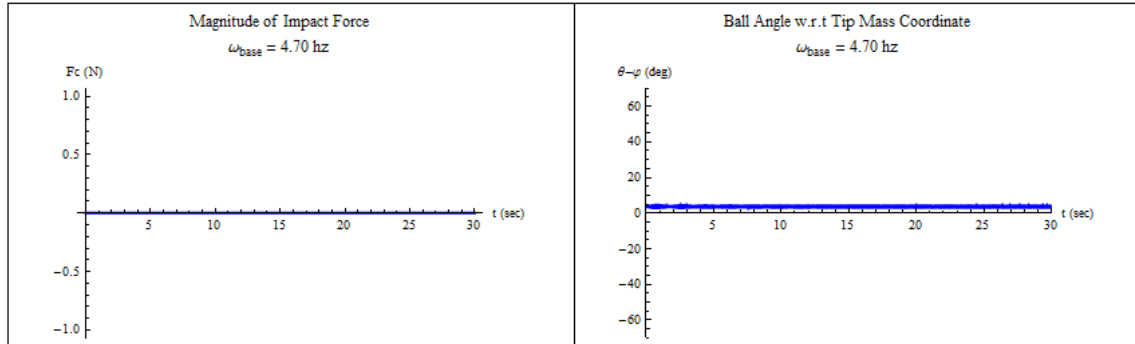


Figure 8-81: Impact details for the forcing frequency of 4.70 Hz

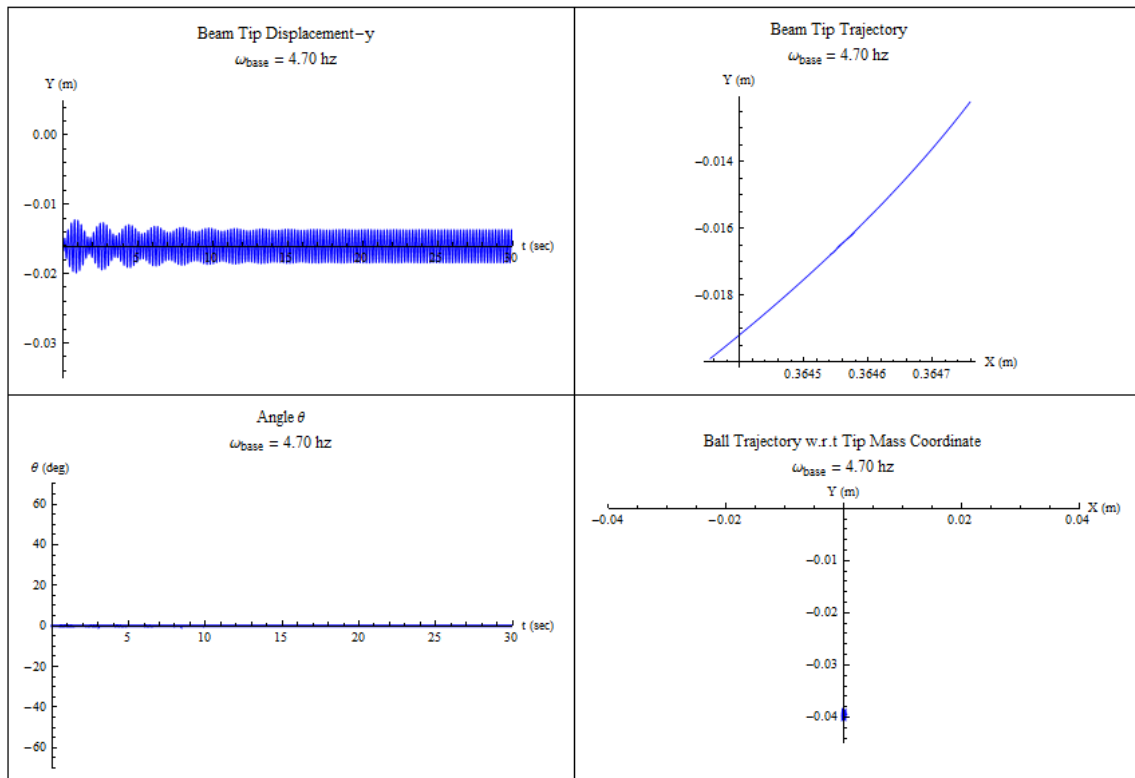


Figure 8-82: System trajectories for the forcing frequency of 4.70 Hz

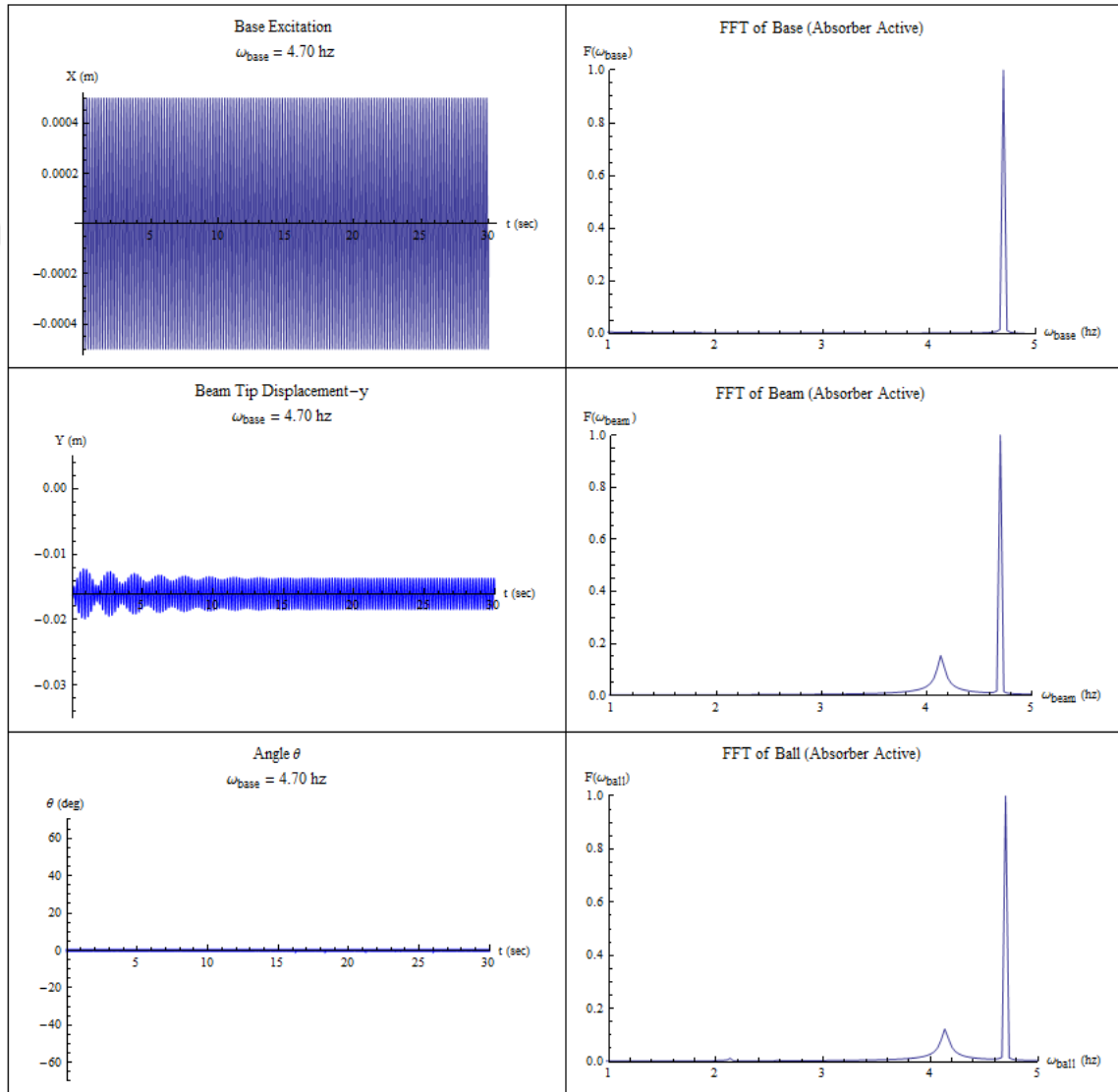


Figure 8-83: FFT of the system for the forcing frequency of 4.70 Hz

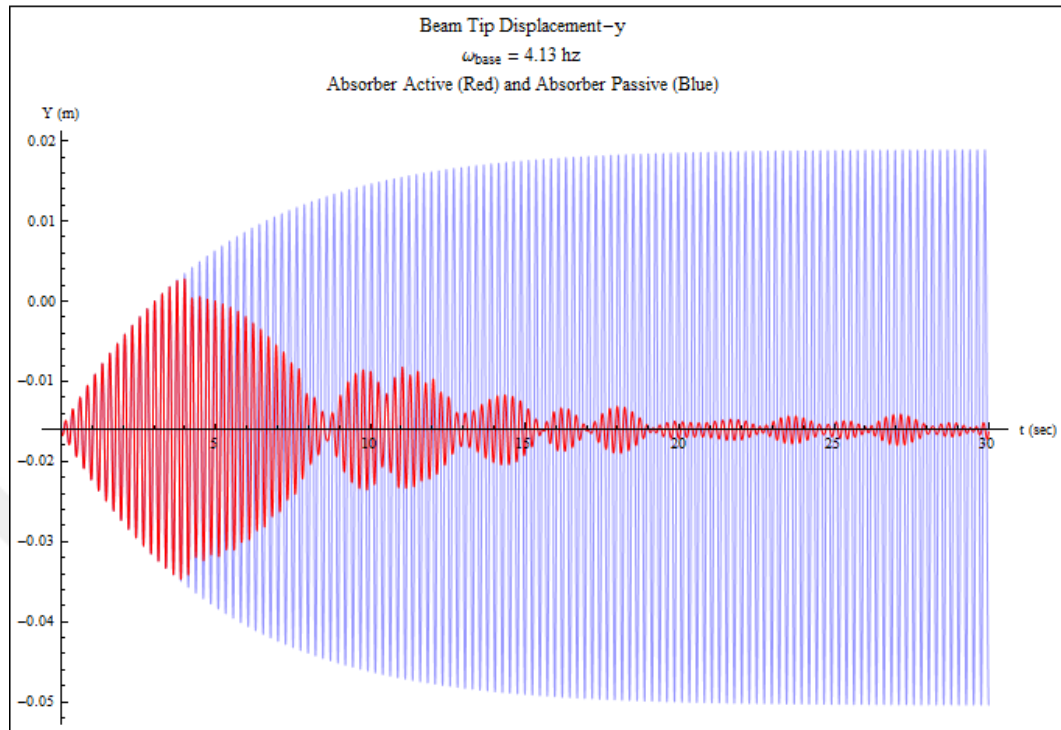


Figure 8-84: Comparison of the beam tip displacement-y in active and passive absorber cases

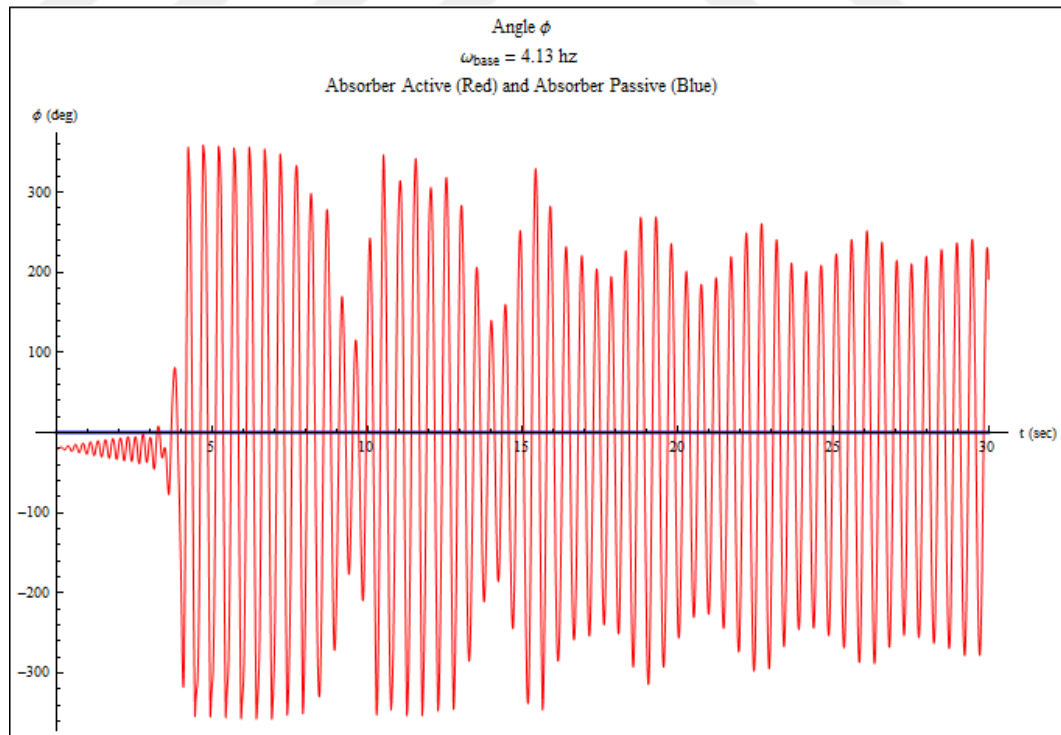


Figure 8-85: Comparison of ball rotation angle in active and passive absorber cases

8.2 Adams Results

Due to the complexity of the tip mass geometry and the existence of the flexible beam, the solution of the beam-tip mass-ball system with ADAMS requires additional programs. Therefore, SolidWorks is used to create the tip mass geometry and ANSYS is used to produce the flexible body description file (MNF) for the beam.

One should note that the SolidWorks model not only provides the geometrical properties of the tip mass but also the center of mass, mass moment of inertia, and area moment of inertia information.

The “Beam 188” element is used in ANSYS while modeling the flexible beam. This element is suitable for analyzing slender to moderately stubby/thick beam structures. Beam 188 is based on the Timoshenko beam theory and has six degrees of freedom at each node. These nodes include translations in the x, y, and z directions and rotations about the x, y, and z directions.

The beam model in ANSYS has 60 elements, 61 nodes (two of them being the attachment nodes), and 366 DOF's. The MNF file that will be used in ADAMS has the following information

- Beam properties: mass, moments of inertia, center of mass
- Reduced stiffness and mass matrices in terms of the interface points
- 18 normal modes
- 12 constraint modes due to the attachment nodes

Therefore, one can import beam, tip mass, and ball data into ADAMS/View, then create joints between the bodies, apply boundary conditions, and perform the numerical analysis.

Since the system has complete energy transfer at the forcing frequency of 4.13 Hz, ADAMS simulation is performed at this frequency, and the results are given from Figure 8-86 to Figure 8-100.

Comparing the transverse displacement curves of the beam shown in Figure 8-50 and Figure 8-88, one can see that both curves start from the static equilibrium position - 0.016m and have their maximum value of 0.003m before 5 seconds. After reaching the maximum value, they decrease gradually, and toward the end of the simulation have small fluctuations around the static equilibrium position.

Similar observations can be written for the beam displacements in the x direction given in Figure 8-50 and Figure 8-87 and tip mass displacements in x and y directions given in Figure 8-50, Figure 8-89, and Figure 8-90, respectively.

Ball position information obtained using ADAMS gives translational displacements as shown in Figure 8-91 and Figure 8-92. However, numerical results in Figure 8-48 give angular displacement. Therefore, it is not possible to have a direct numerical comparison, but one can note the similarities in the displacement patterns. Until 4 seconds, both curves do not show any oscillations, and after that time the ball starts to have significant displacement.

Further validation can be done by comparing the energy curves. Figure 8-93 shows beam energy curves in a single plot and Figure 8-94, Figure 8-95, and Figure 8-96 show details of the beam kinetic, potential, and strain energies, respectively. Comparing Figure 8-96 with the beam strain energy curve given in Figure 8-56, maximum strain energy is observed before 5 seconds and its value is around 60 N.mm. Similarly, for potential energy and kinetic energy curves, maximum values are approximately 6 N.mm and 7.5 N.mm, respectively. In addition to the numerical matches, similarities can be seen between the decreasing and increasing trends of the strain energy, kinetic energy, and potential energy curves of the two models.

Similar observations can be seen between the kinetic energy and the potential energy curves of the ball and the tip mass.

In conclusion, the results obtained from ADAMS and ANCF are observed to be in excellent quantitative and qualitative agreement even though two methods used different solution approaches.

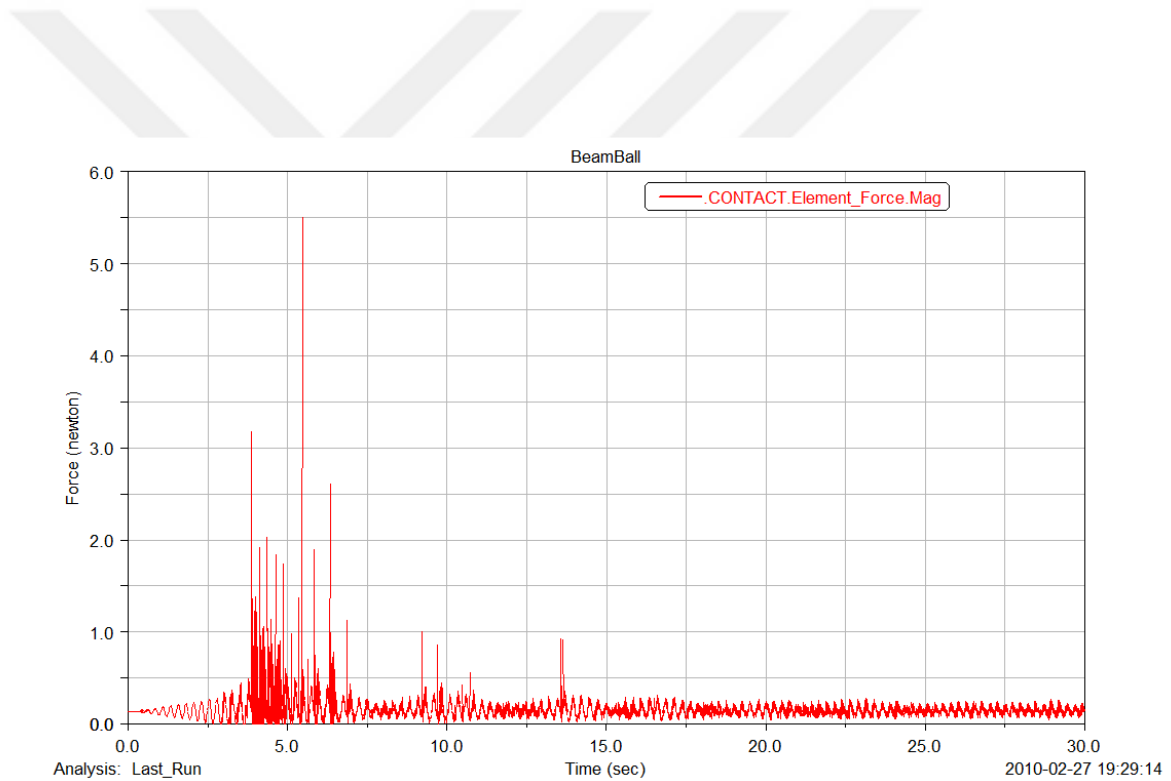


Figure 8-86: Adams result / Contact force for the forcing frequency of 4.13 Hz

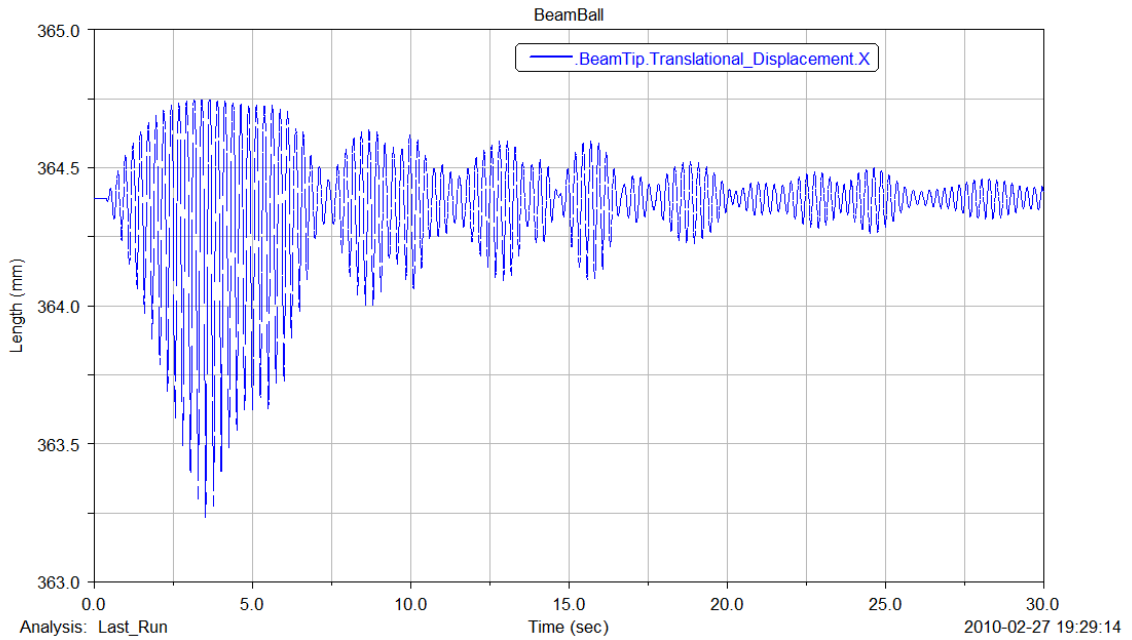


Figure 8-87: Adams result / Beam tip displacement-x for the forcing frequency of 4.13 Hz

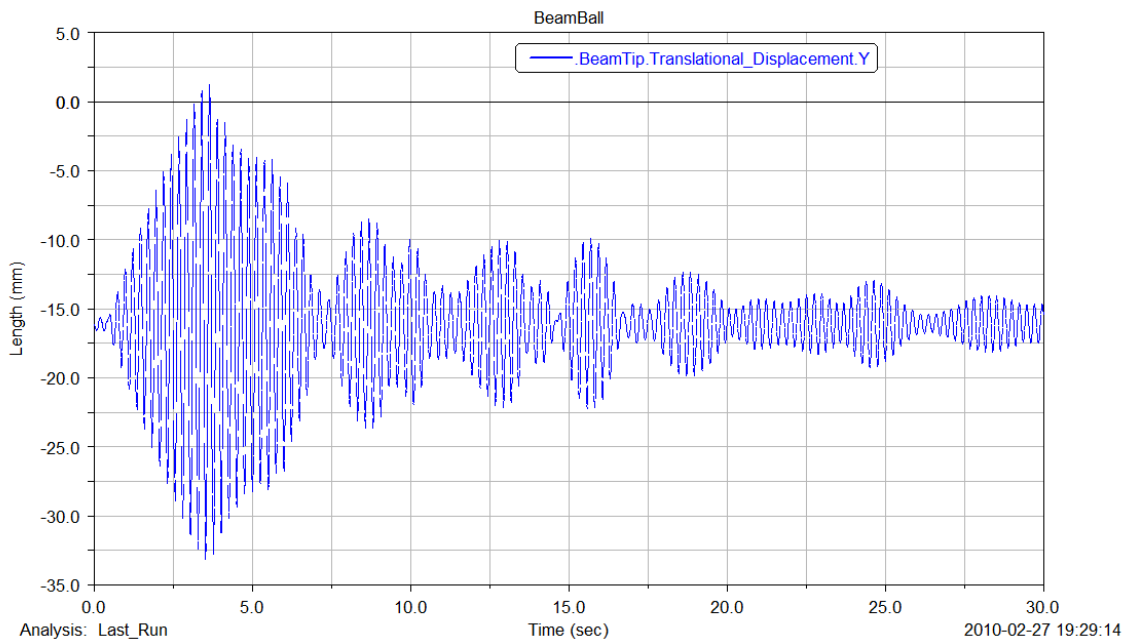


Figure 8-88: Adams result / Beam tip displacement-y for the forcing frequency of 4.13 Hz

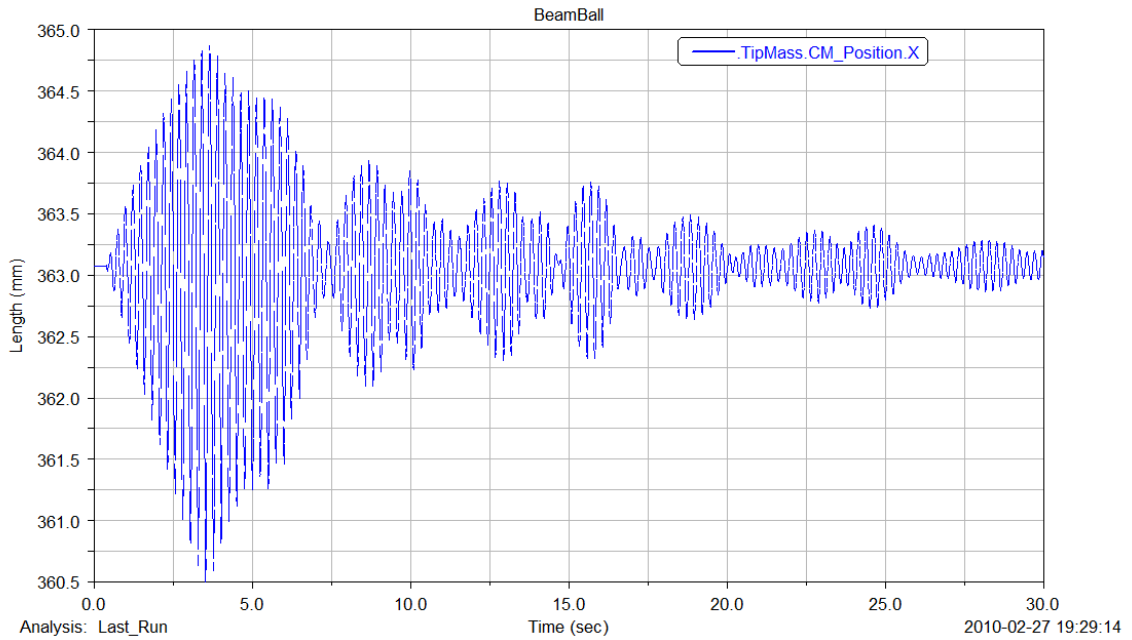


Figure 8-89: Adams result / Tip mass displacement-x for the forcing frequency of 4.13 Hz

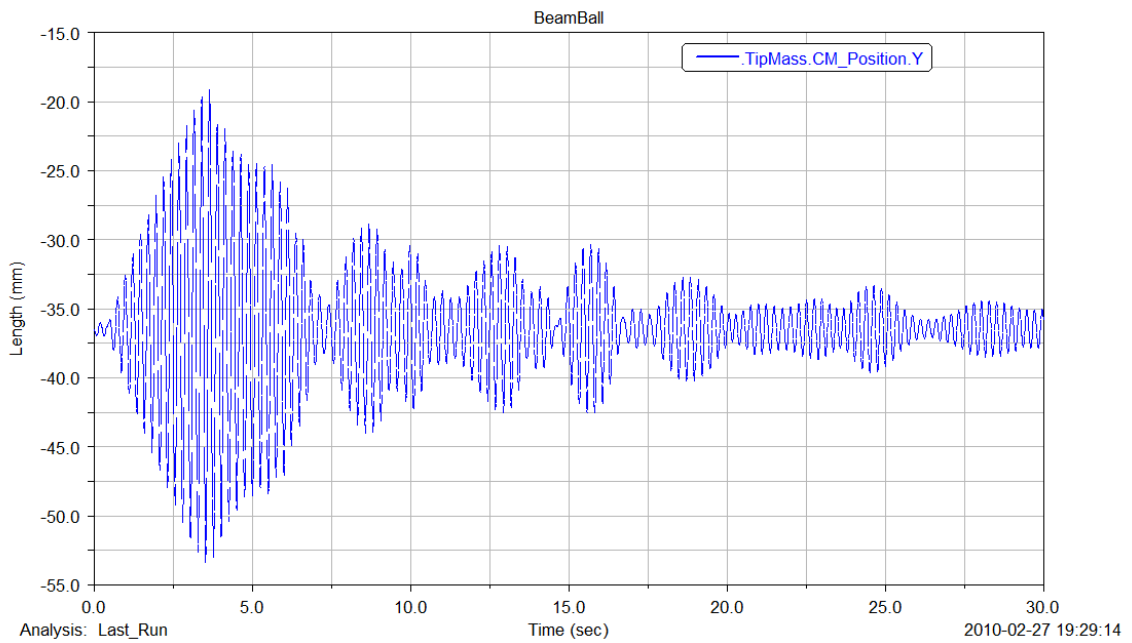


Figure 8-90: Adams result / Tip mass displacement-y for the forcing frequency of 4.13 Hz

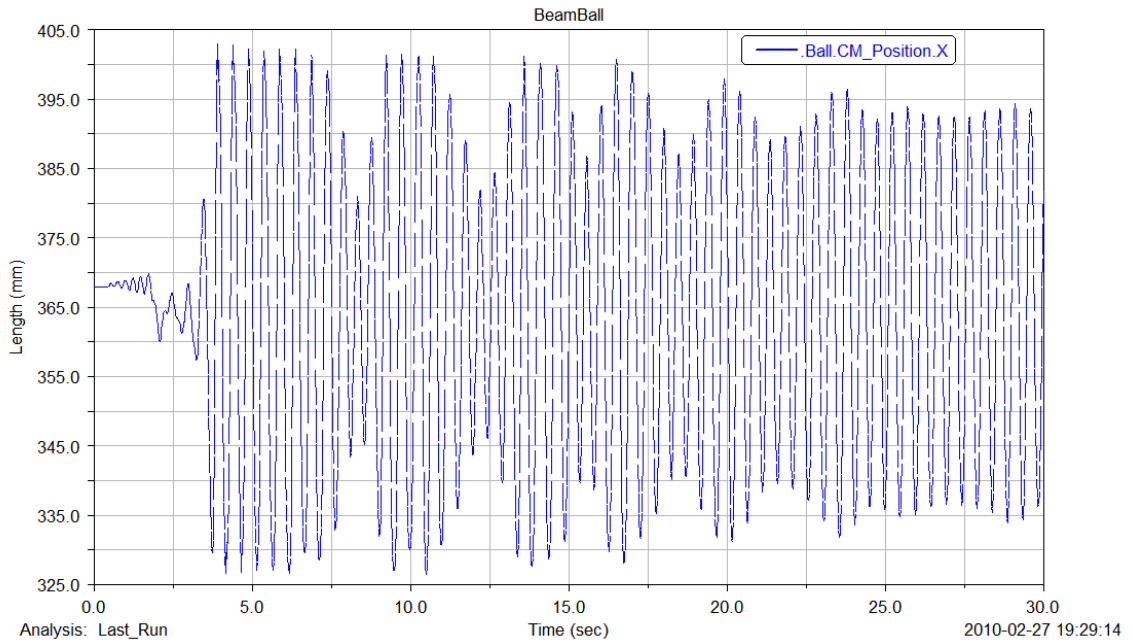


Figure 8-91: Adams result / Ball displacement-x for the forcing frequency of 4.13 Hz

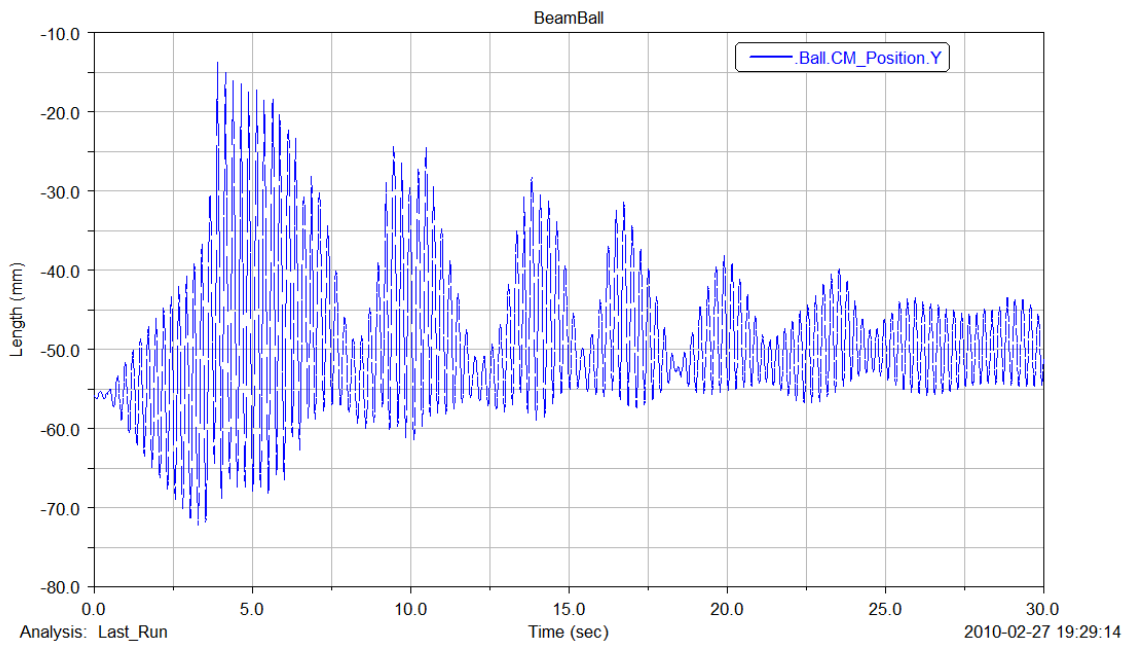


Figure 8-92: Adams result / Ball displacement-y for the forcing frequency of 4.13 Hz

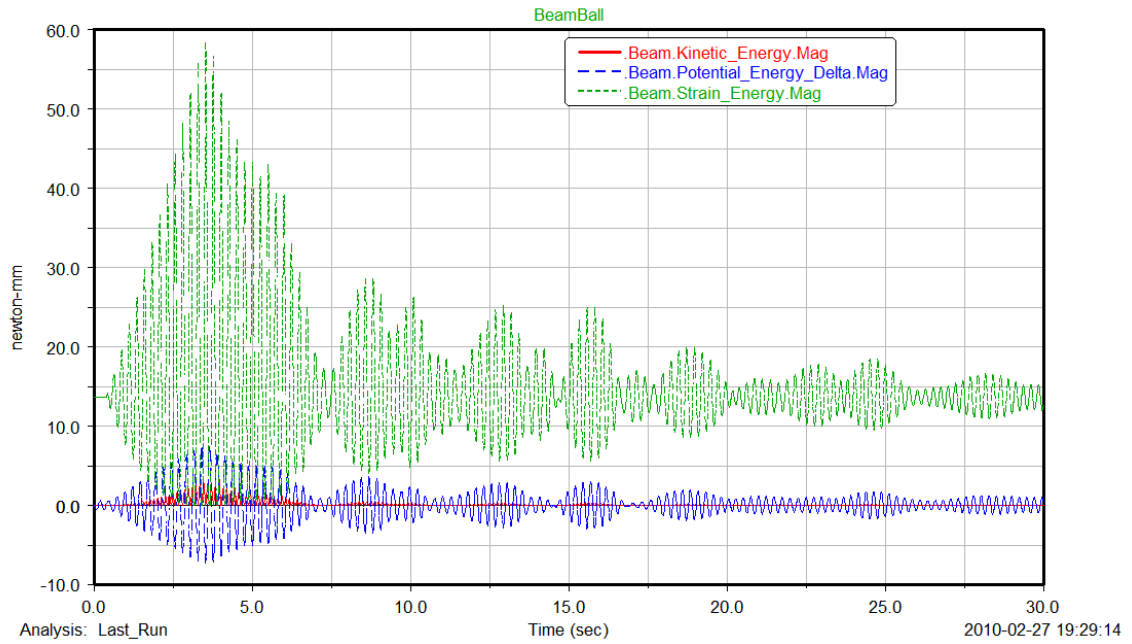


Figure 8-93: Adams result / Beam energy curves for the forcing frequency of 4.13 Hz

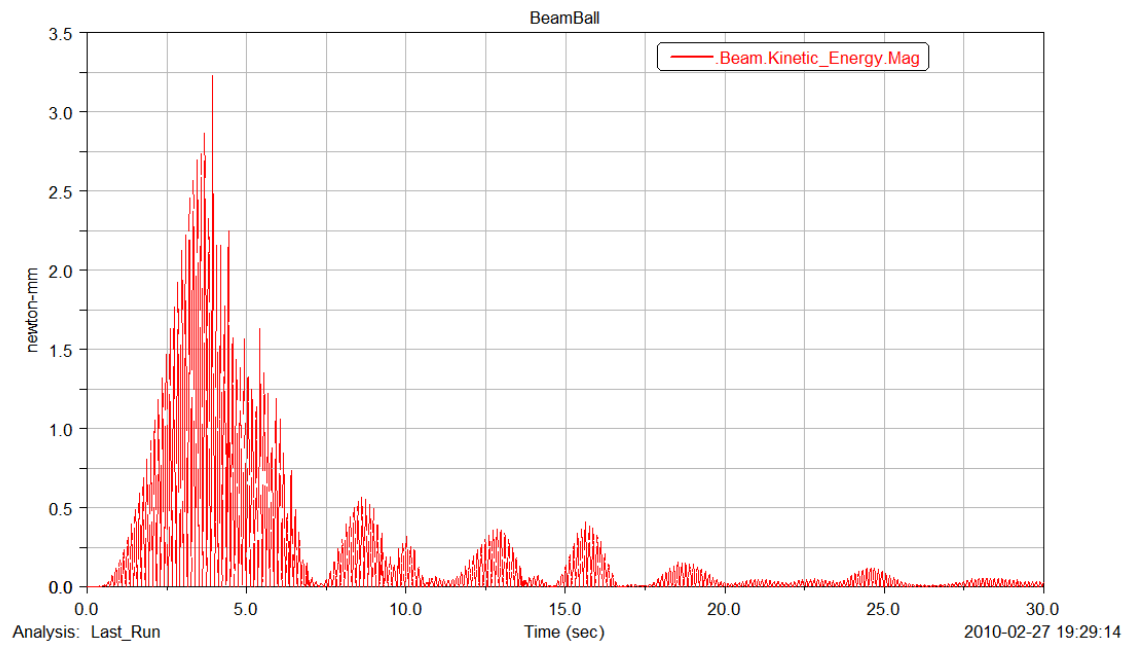


Figure 8-94: Adams result / Beam kinetic energy curve for the forcing frequency of 4.13 Hz

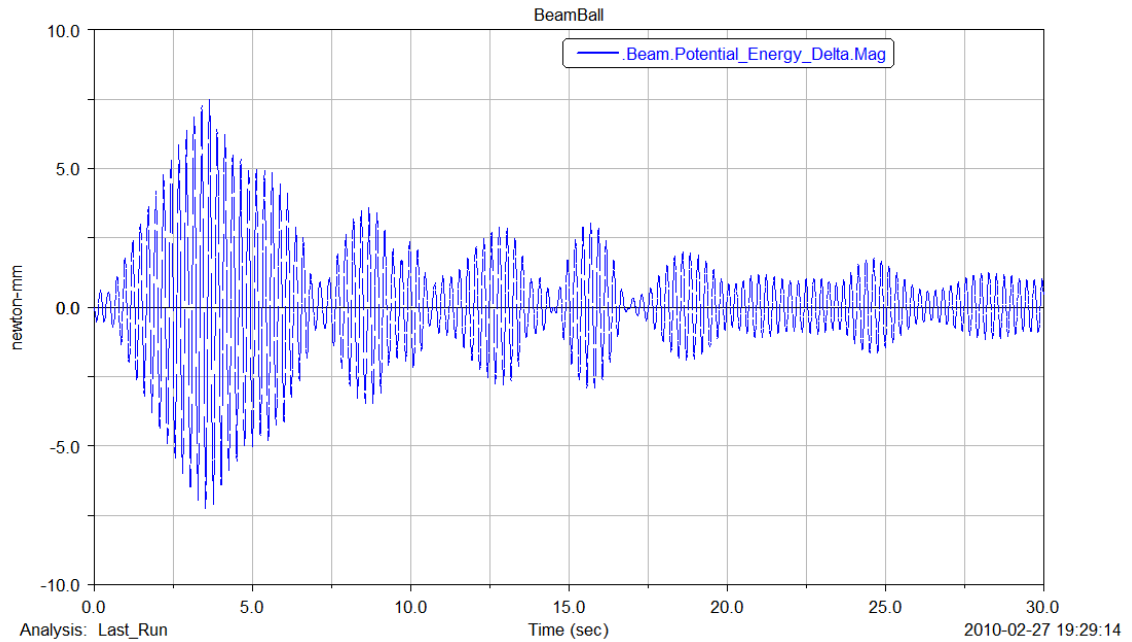


Figure 8-95: Adams result / Beam potential energy delta curve for the forcing frequency of 4.13 Hz

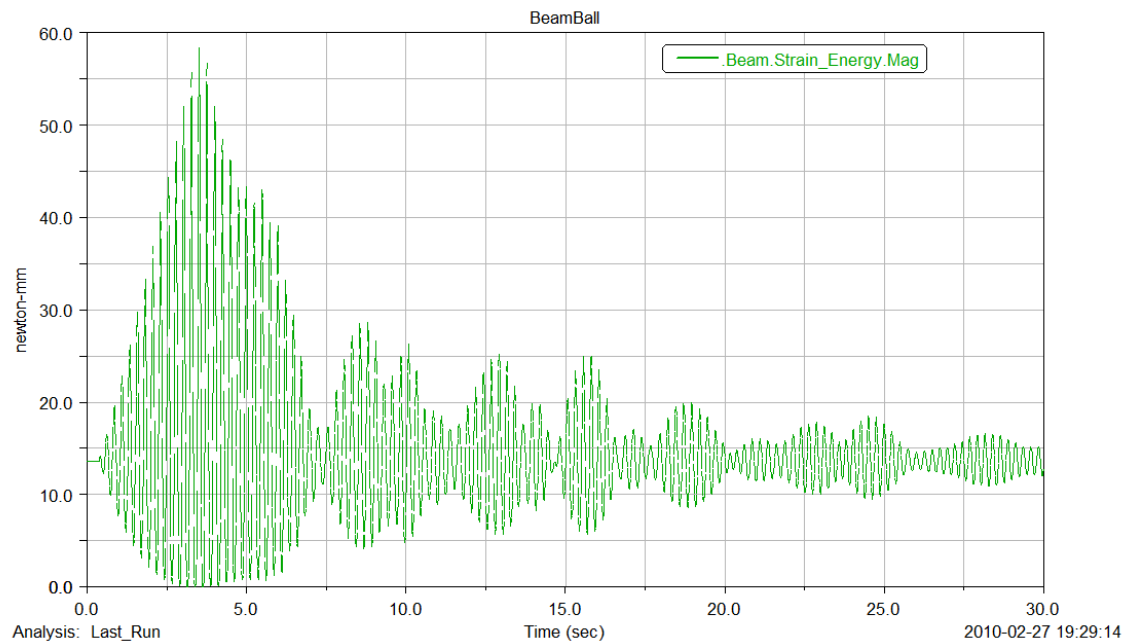


Figure 8-96: Adams result / Beam strain energy curve for the forcing frequency of 4.13 Hz

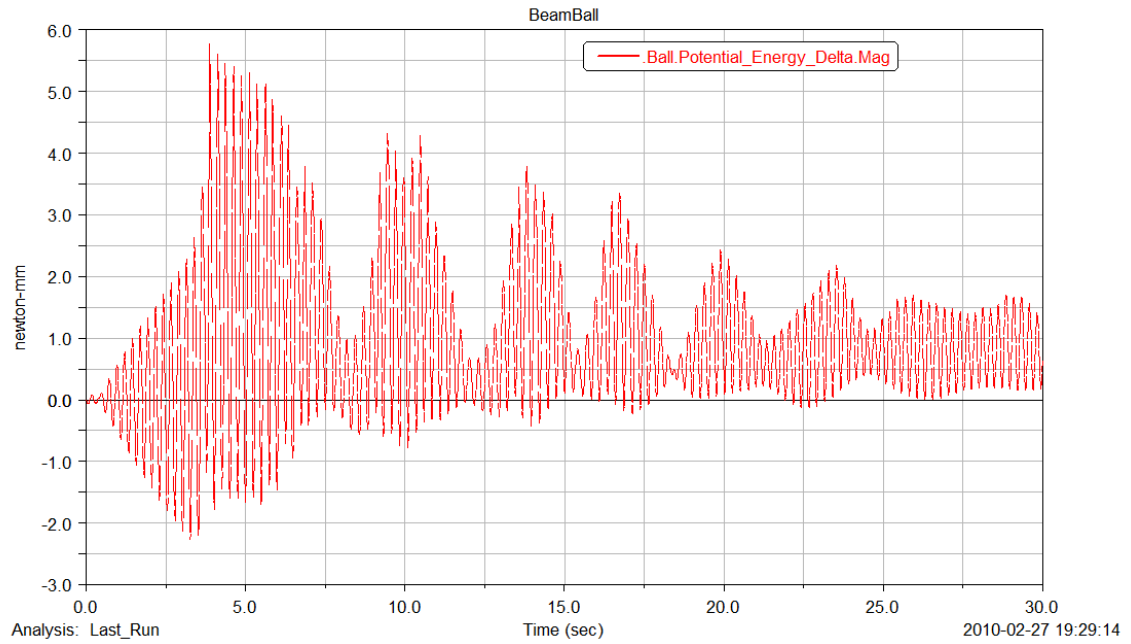


Figure 8-97: Adams result / Ball potential energy delta curve for the forcing frequency of 4.13 Hz

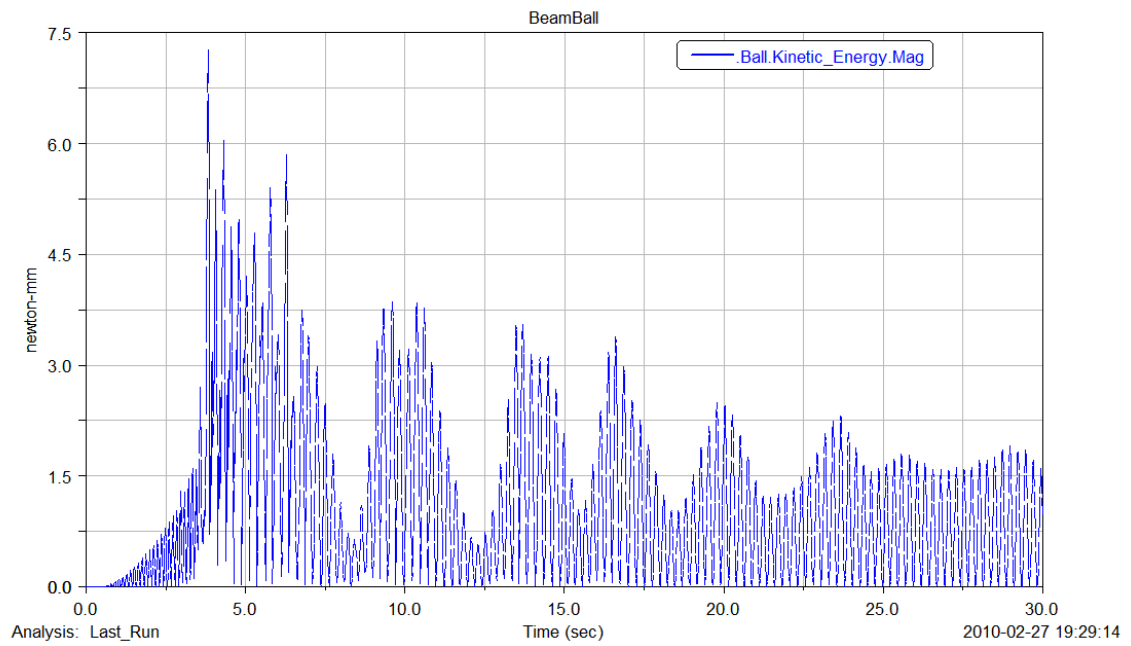


Figure 8-98: Adams result / Ball kinetic energy curve for the forcing frequency of 4.13 Hz

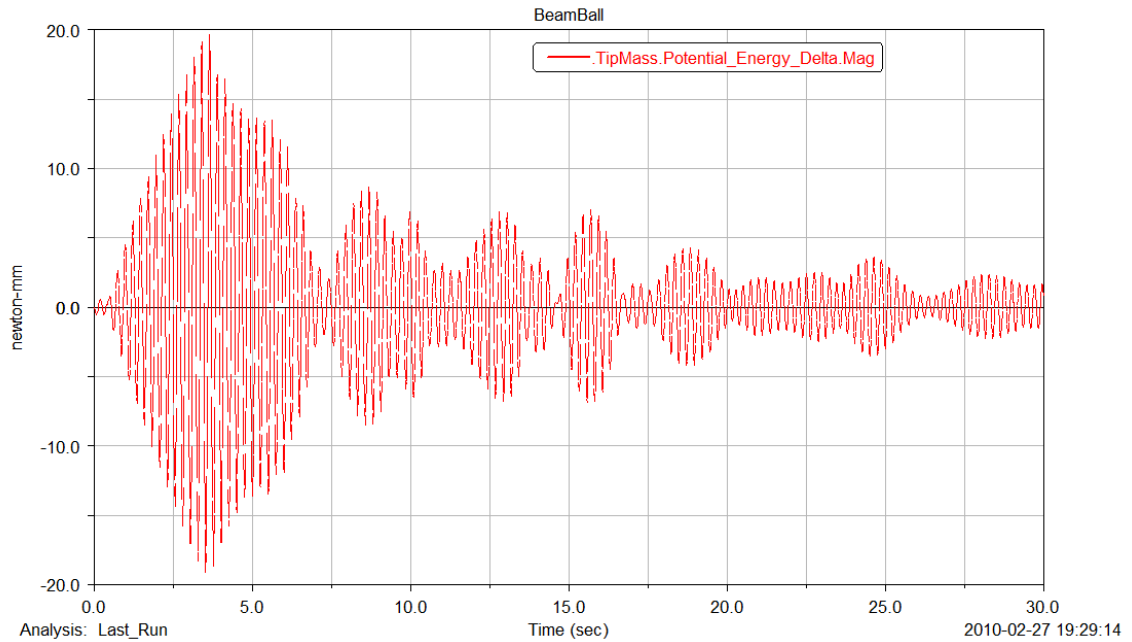


Figure 8-99: Adams result / Tip mass potential energy delta curve for the forcing frequency of 4.13 Hz

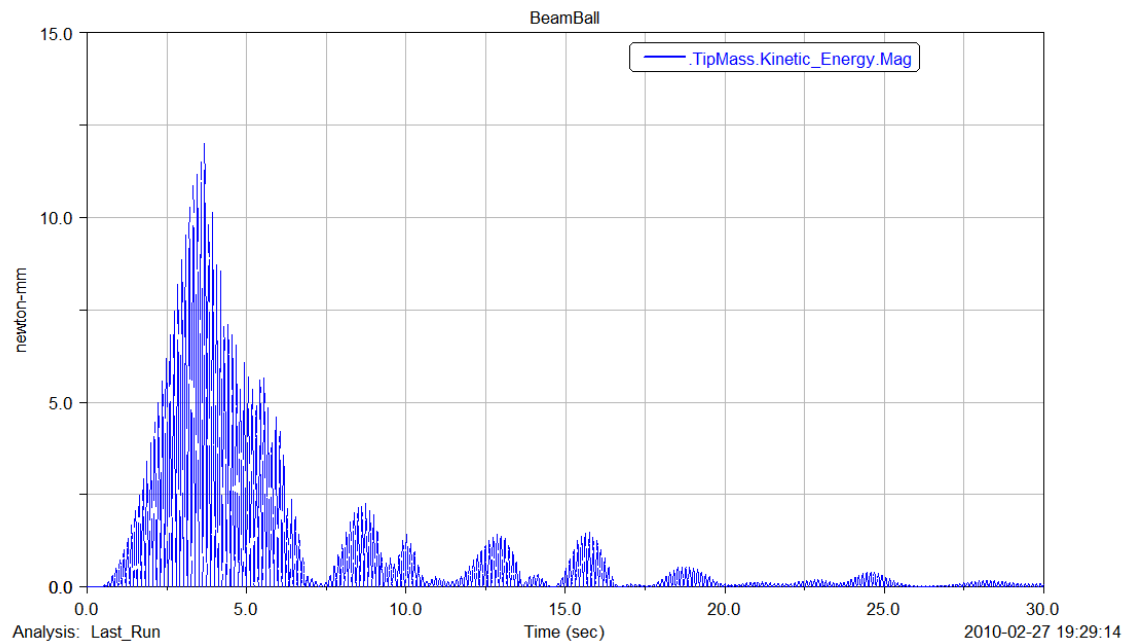


Figure 8-100: Adams result / Tip mass kinetic energy curve for the forcing frequency of 4.13 Hz

8.3 Two-Ball Case

The aim of this section is to observe the dynamics of the two-ball arrangement and compare the results with the results of the one-ball case. In order to keep the total mass of the system the same and the system natural frequency at 4.13 Hz, weights of the ball and the tip mass are rearranged, resulting in changes to the mass moment of inertia terms. Therefore, new input parameters for the numerical solution are given in Table 5.

Table 5: Rigid body data / Two ball case

Tip Mass
$J_t = 1.644 \times 10^{-5} \text{ kg.m}^2$
$m_t = 0.1014 \text{ kg}$
$u_{t2} = 0.0200 \text{ m}$
Ball
$J_{btotal} = 6.3 \times 10^{-7} \text{ kg.m}^2$
$m_{btotal} = 0.028 \text{ kg}$
$r_b = 0.0075 \text{ m}$
$R = 0.0395 \text{ m}$

Detailed dynamics of the system for the two-ball arrangement are shown from Figure 8-101 to Figure 8-111.

The reason behind the numerical investigation of the two-ball case was to answer the following question: Is it possible to further decrease the beam response by increasing the number of balls? In order to have an answer to this question, a comparison of the transverse displacement of the beam tip as shown in Figure 8-110 and a comparison of the ball angular displacement as shown in Figure 8-111 are investigated. As seen from these figures, the two-ball system has a smaller angle of swing than the one-ball system, and the beam has a higher amplitude of oscillations than the one-ball case. At first look, these results can be confusing because one may expect an increase in the kinetic energy

of the ball due to the increase in its mass and therefore expect the two-ball system to absorb more energy, which would result in a decrease of the beam response. However, one should note that the kinetic energy depends mass linearly, but velocity quadratic. A decrease in the rotational velocity (even with double the mass) can reduce the total kinetic energy of the ball, which means less energy will go to the ball from the beam, causing higher transverse displacement of the beam. This result can be proven by inspecting the energy curves of the one-ball and the two-ball systems shown in Figure 8-55 and Figure 8-109, respectively. As seen from these figures, for the one-ball case, the minimum kinetic energy of the ball is 1.8 N.mm at the steady state. However, in the two-ball case, the minimum ball kinetic energy is 1N.mm, which explains the reason of having higher beam oscillations for the two-ball arrangement.

In the experimental study performed by Landis (2009), the decrease in the angle of swing of the ball was also observed. Moreover, Cuvalci et al. (1996) performed experiments on a similar system (beam-tip mass-pendulum system) and found that the smaller pendulum absorbed more energy since it has less inertia.

Investigating the FFT of the system shown in Figure 8-108, one can observe that the frequency of the ball is one half of the frequency of the beam. However, comparing Figure 8-108 with the FFT of the one-ball case, one can claim that the frequency information of the two-ball case has more minor peaks, which may be due to the beating. This result can be verified by looking to the time history curves of the two-ball system.

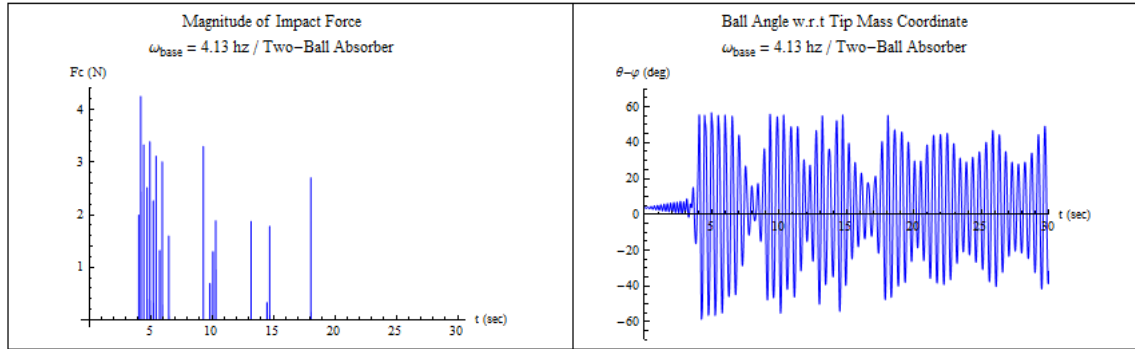


Figure 8-101: Two ball case / Impact details for the forcing frequency of 4.13 Hz

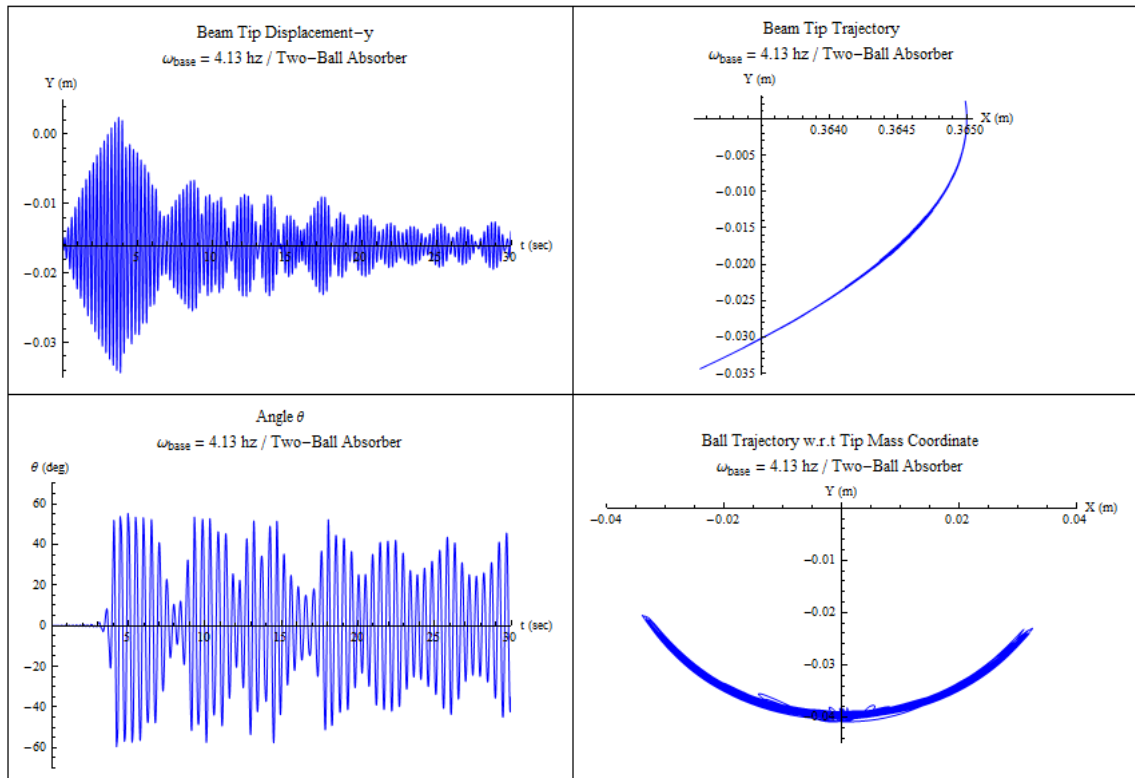


Figure 8-102: Two ball case / System trajectories for the forcing frequency of 4.13 Hz

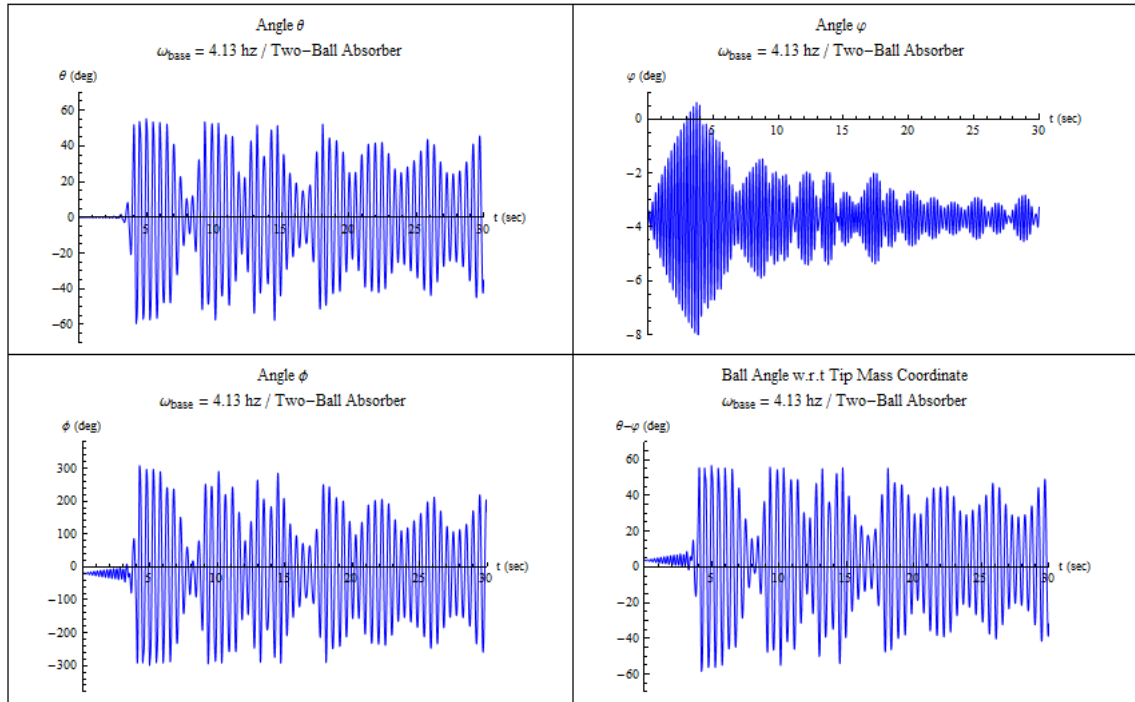


Figure 8-103: Two ball case / Time histories of the system angles for the forcing frequency of 4.13 Hz

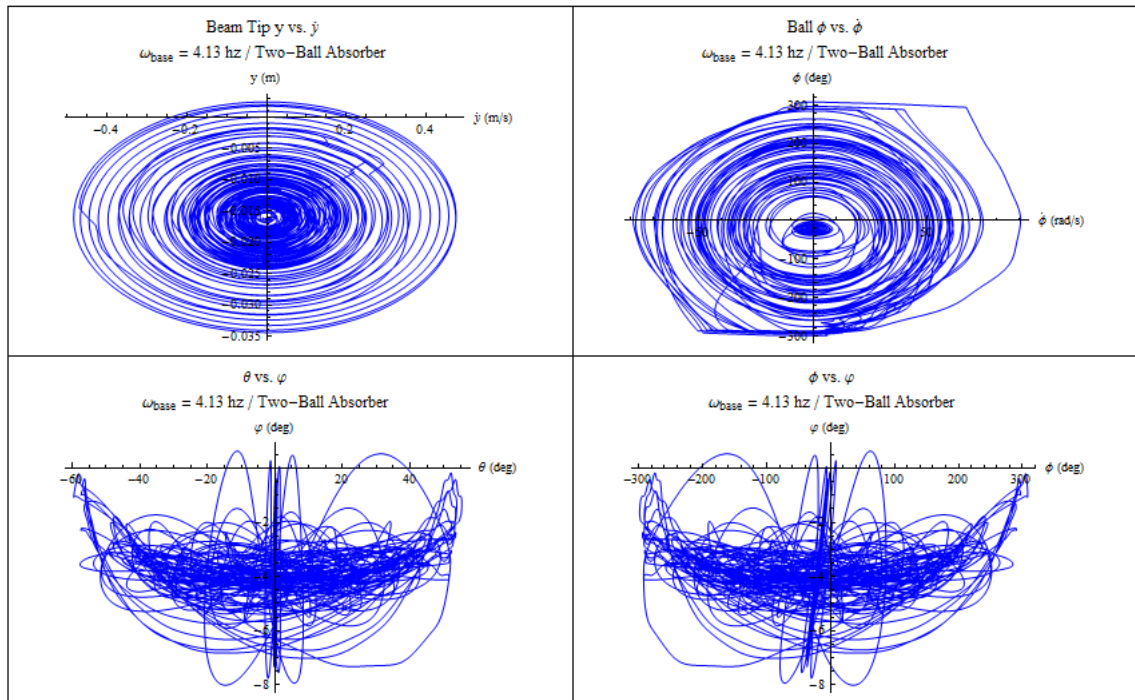


Figure 8-104: Two ball case / Phase plane curves for the forcing frequency of 4.13 Hz

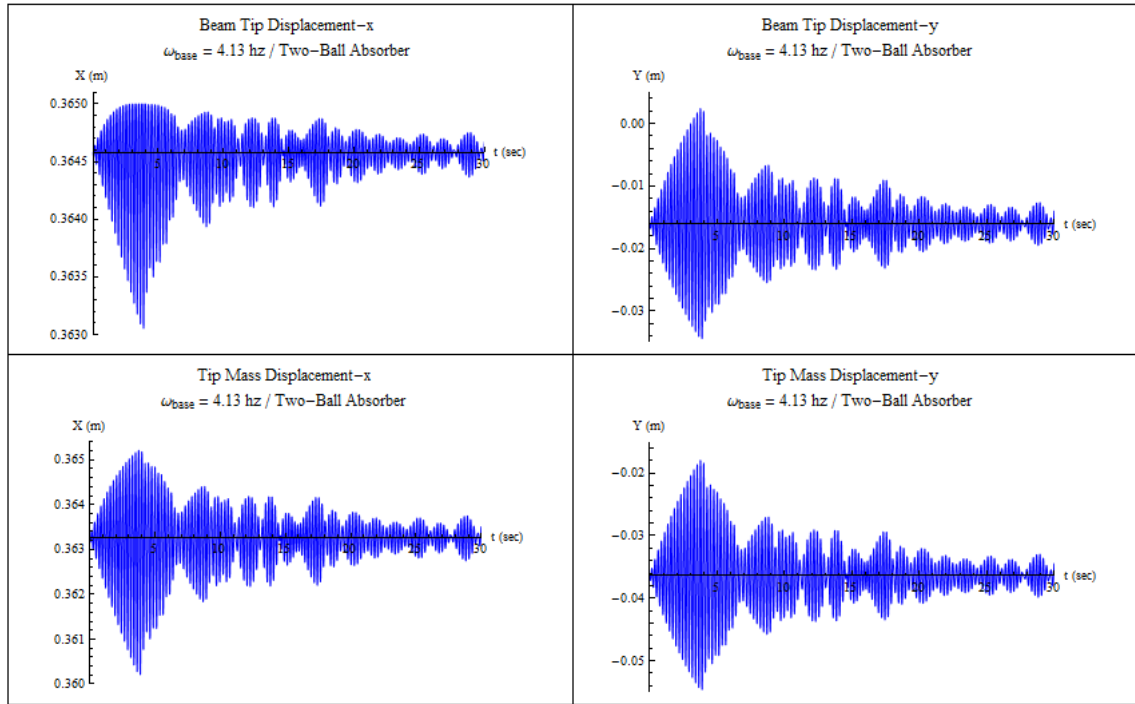


Figure 8-105: Two ball case / Time histories of the system translational displacements for the forcing frequency of 4.13 Hz

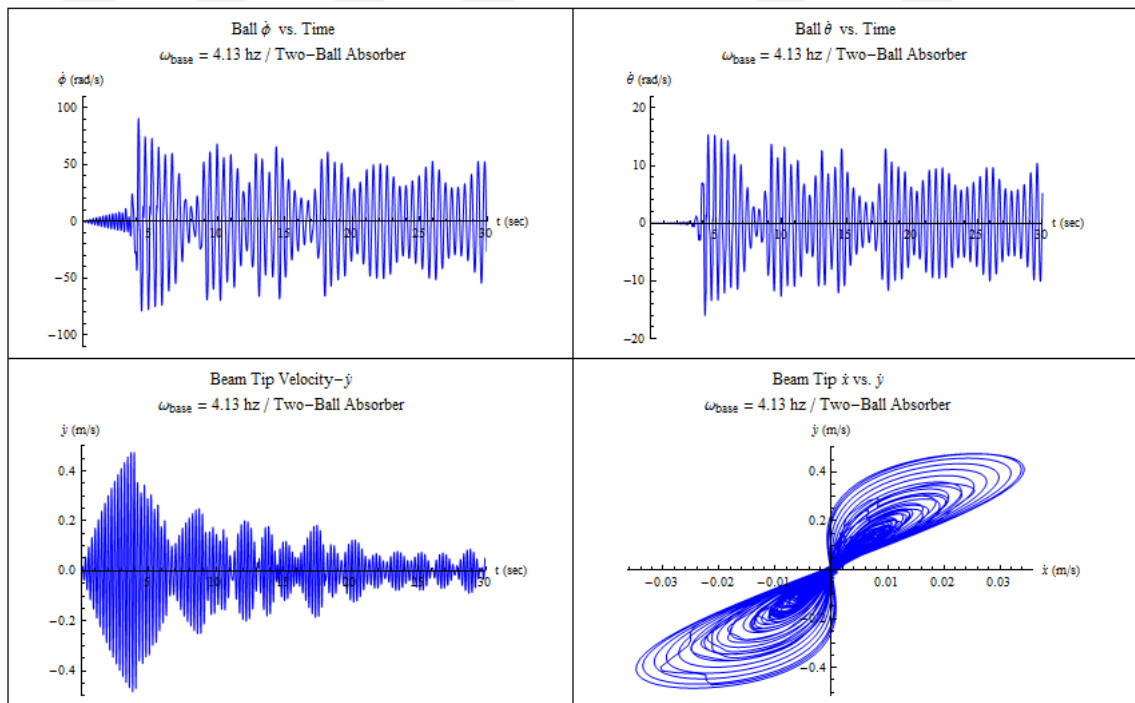


Figure 8-106: Two ball case / System velocity curves for the forcing frequency of 4.13 Hz

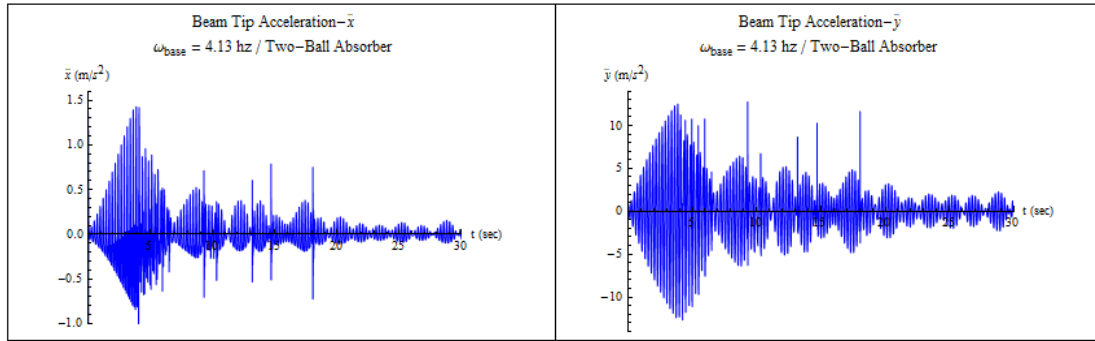


Figure 8-107: Two ball case / Beam tip acceleration for the forcing frequency of 4.13 Hz

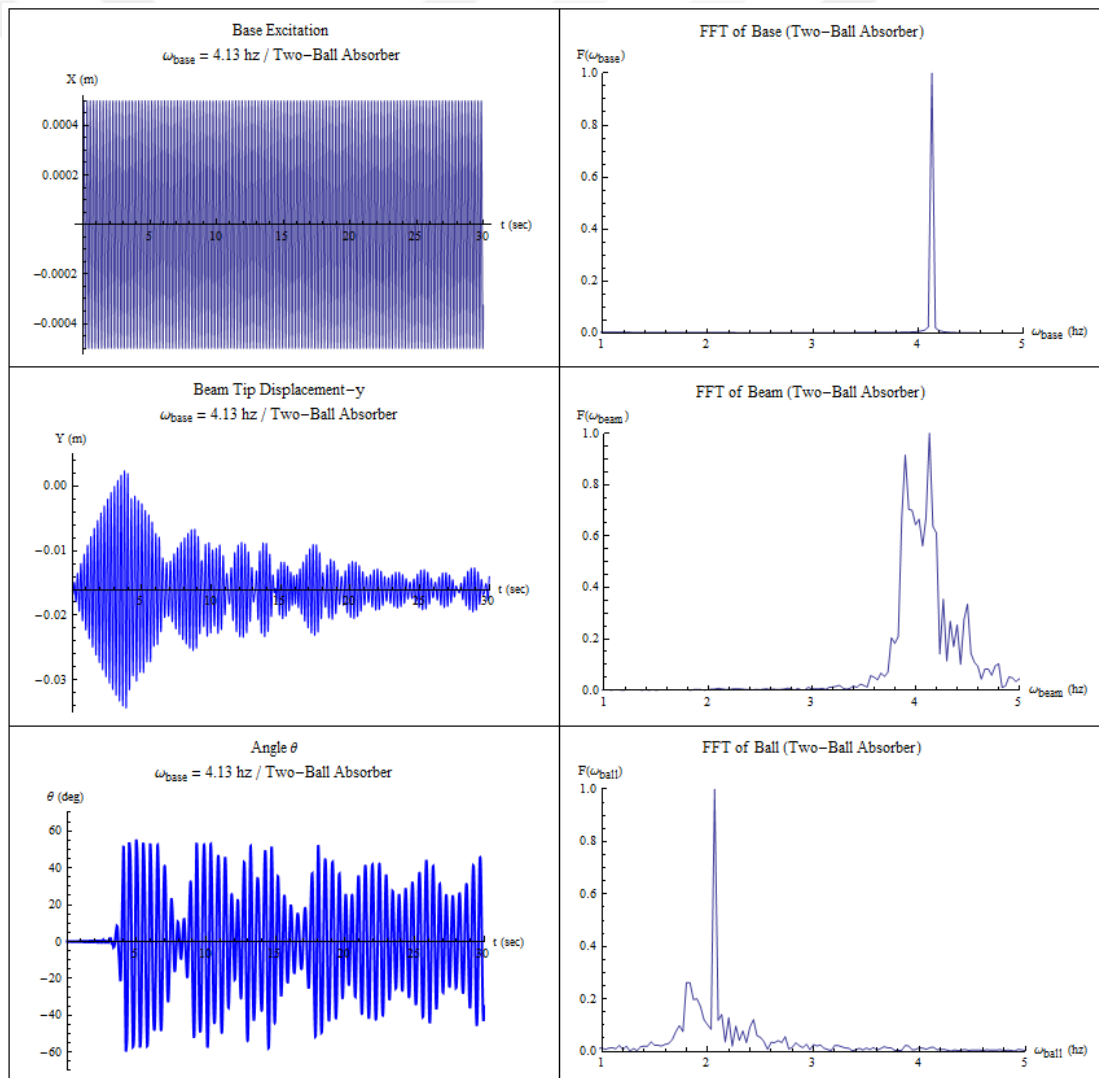


Figure 8-108: Two ball case / FFT of the system for the forcing frequency of 4.13 Hz

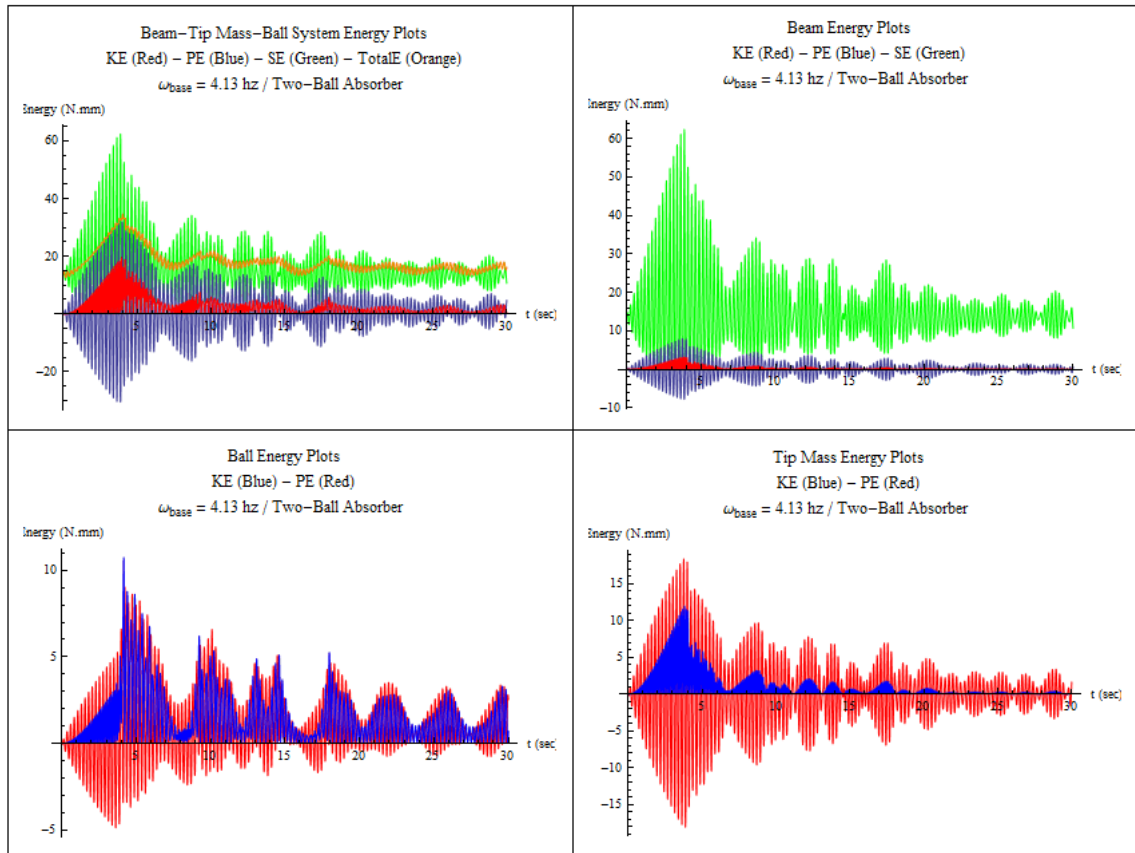


Figure 8-109: Two ball case / System energy curves for the forcing frequency of 4.13 Hz

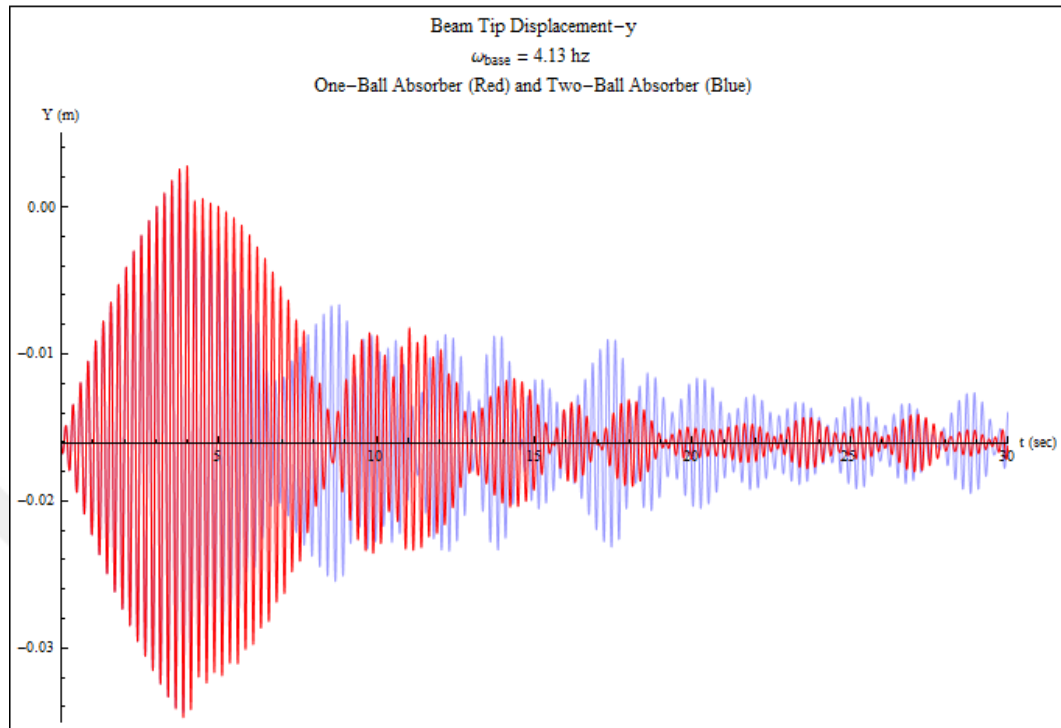


Figure 8-110: Comparison of the beam tip displacement-y in one-ball and two-ball cases



Figure 8-111: Comparison of the ball angle in one-ball and two-ball cases

8.4 Impact Case

In this section, the aim is to investigate the efficiency of the impact dampers in vibration absorption. Therefore, beam-tip mass –one-ball system is excited at a higher excitation amplitude, namely, 2mm peak-to-peak, to observe the impact at all times of the simulation. The forcing frequency is set to 4.13 Hz since the complete energy transfer is observed in the system at that frequency.

Referring to Figure 7-6, the impact condition for the system can be written as $|\theta - \psi| \geq 65^\circ$. As shown in Figure 8-112, at 3 seconds, the swing angle of the ball reaches 65degrees. Therefore, the system starts to have impact and continues to have it until the end of the simulation.

Figure 8-112 to Figure 8-122 give the detailed dynamics of the one-ball impact absorber system.

The FFT curves are given in Figure 8-119. From this figure, it is obvious that the system has an autoparametric interaction and the ball frequency is one half of the frequency of the beam. Moreover, it is interesting to see the frequency information of the beam inside the ball FFT. This coupling effect between the modes of the system can also be seen by looking to the θ vs ψ and ϕ vs ψ curves shown in Figure 8-115.

Investigating y vs $\dot{y}$ and ϕ vs $\dot{\phi}$ curves in Figure 8-115, one can state that the responses of the beam and the ball are periodic. It is important to note that the ϕ vs $\dot{\phi}$ curve also gives information about the impact. For the non-impact cases discussed in previous sections, the ball velocity changes periodically. Therefore, the phase plane curve ϕ vs $\dot{\phi}$ gets an elliptical shape. However, for the impact case, since the ball velocity has sudden direction and magnitude changes at the time of impact, the phase plane curve ϕ vs $\dot{\phi}$ in Figure 8-115 does not have an elliptical shape; instead it has horizontal lines whenever there exists impact between the ball and the tip mass.

A comparison of the beam tip transverse displacement curves of the one-ball system with and without impact as shown in Figure 8-121 reveals that the impact of the ball does not reduce the oscillation of the beam; instead it increases the response amplitude considerably. Similarly, comparing the ball angle curves as shown in Figure 8-122, one can see that the swing angle of the ball is higher for the impact case. Similar observations were obtained in a previous experimental study performed by Landis (2009).



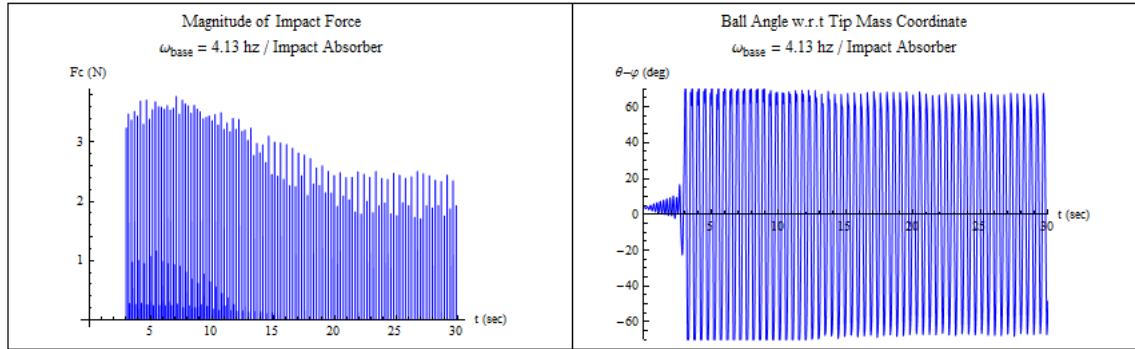


Figure 8-112: Impact case / Impact details for the forcing frequency of 4.13 Hz

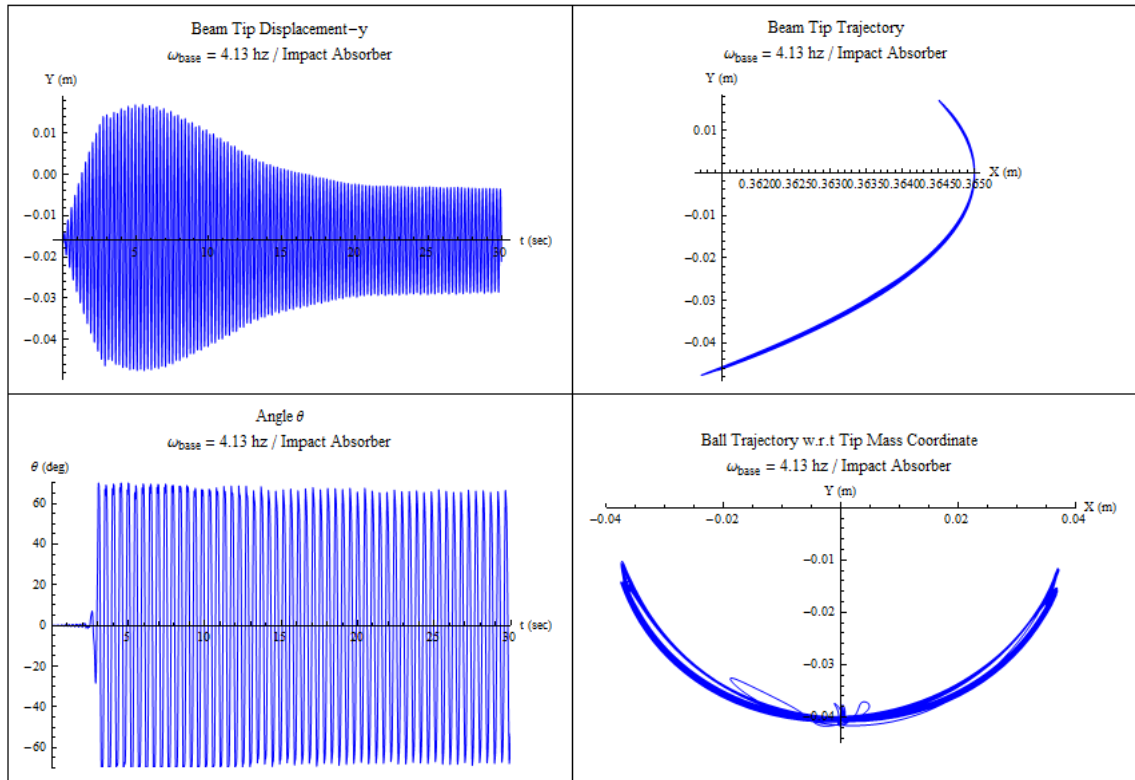


Figure 8-113: Impact case / System trajectories for the forcing frequency of 4.13 Hz

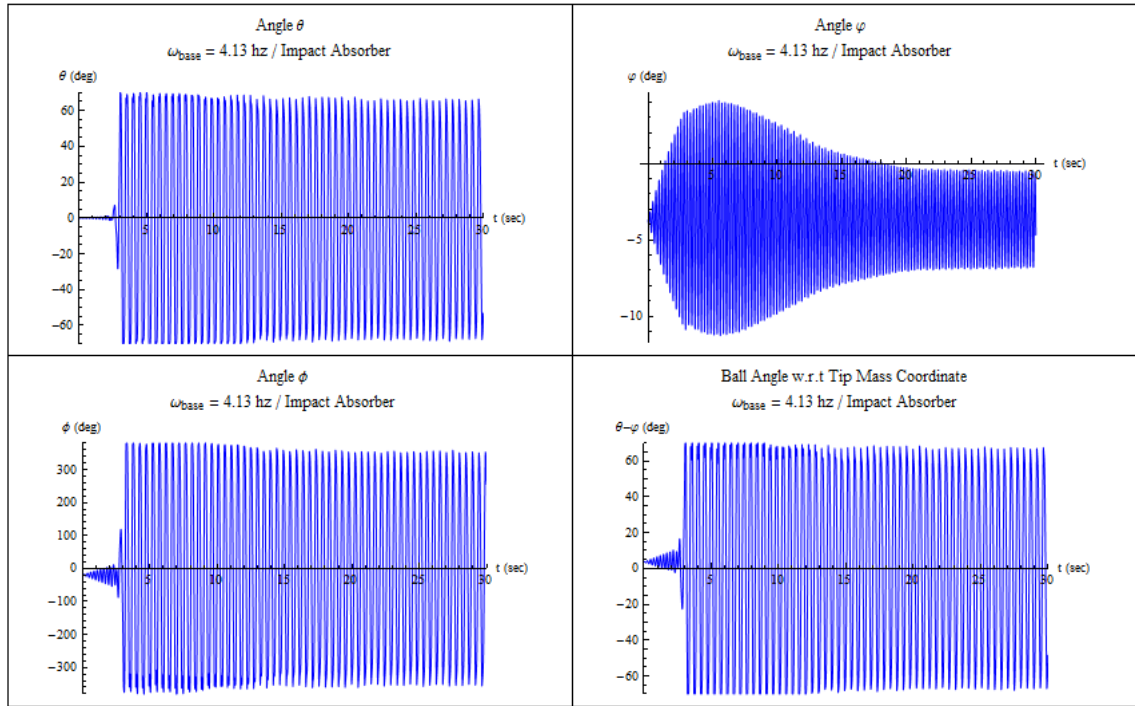


Figure 8-114: Impact case / Time histories of the system angles for the forcing frequency of 4.13 Hz

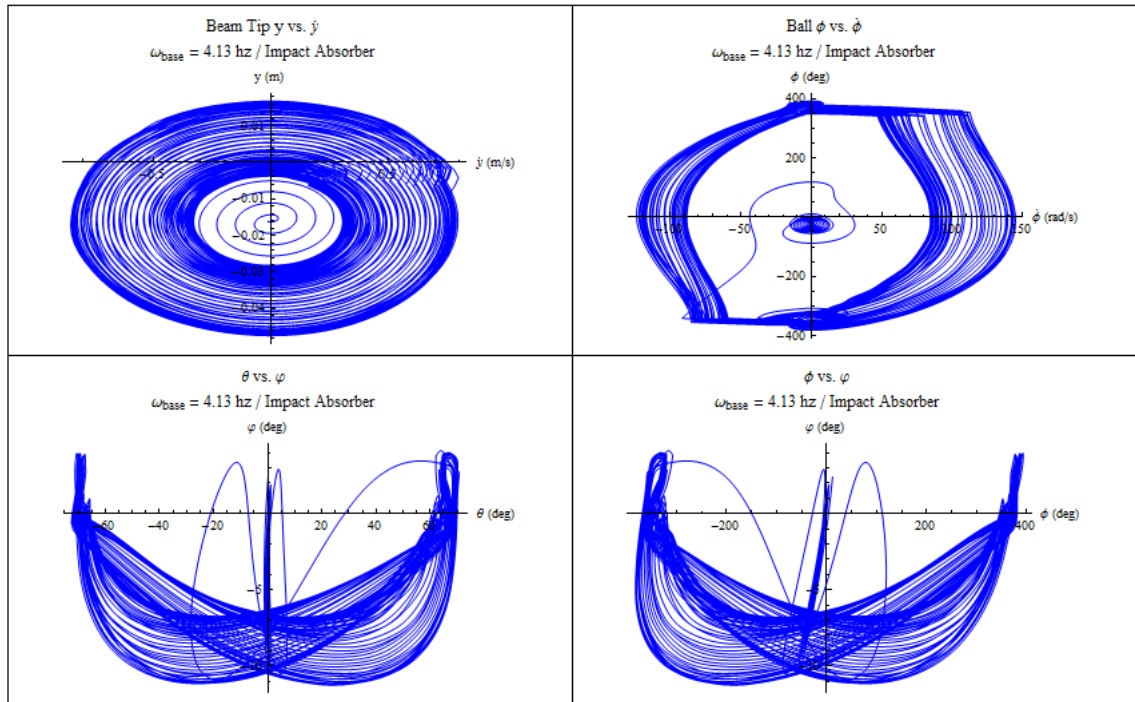


Figure 8-115: Impact case / Phase plane curves for the forcing frequency of 4.13 Hz

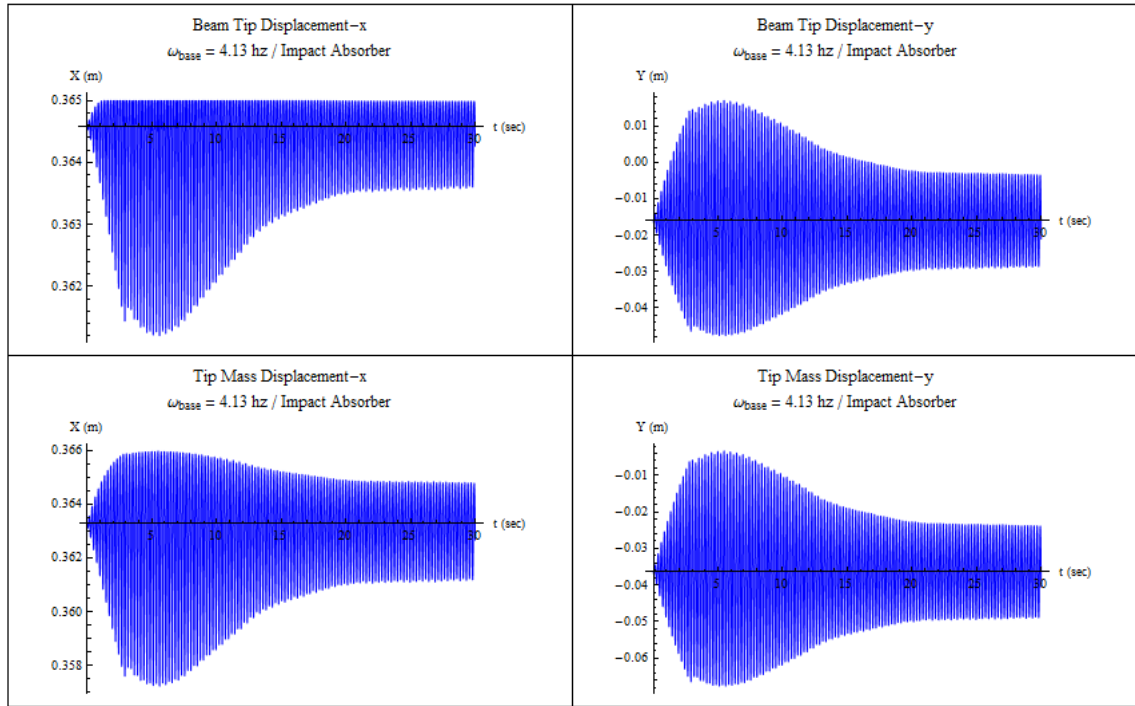


Figure 8-116: Impact case / Time histories of the system translational displacements for the forcing frequency of 4.13 Hz

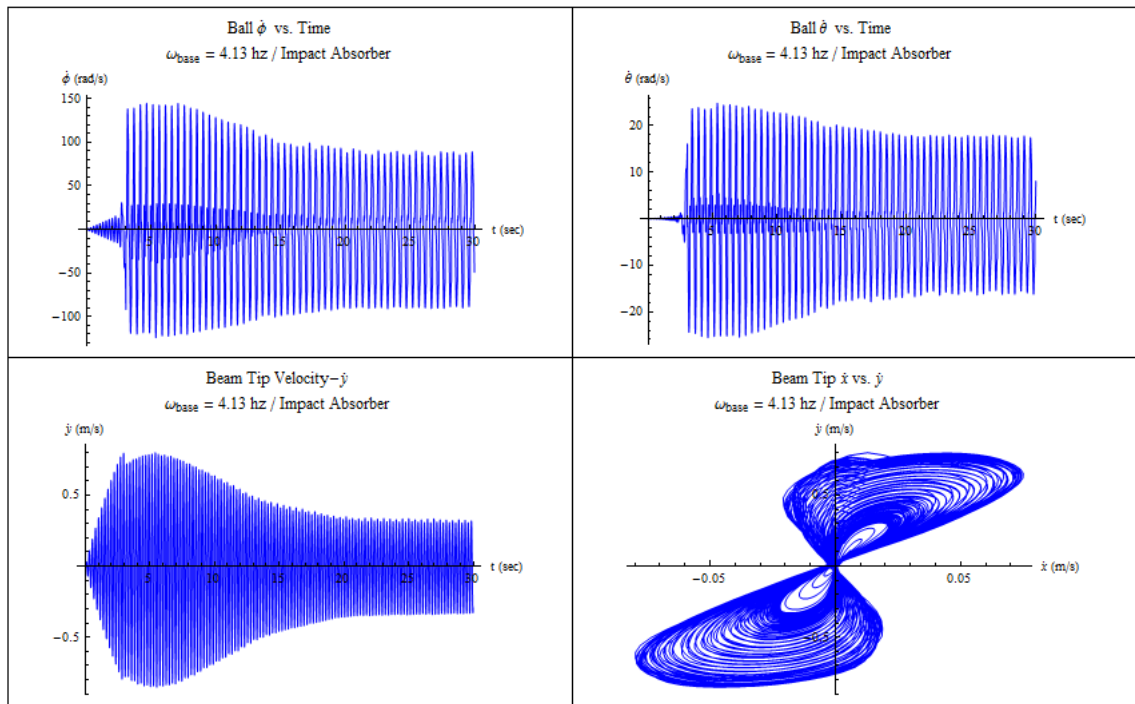


Figure 8-117: Impact case / System velocity curves for the forcing frequency of 4.13 Hz

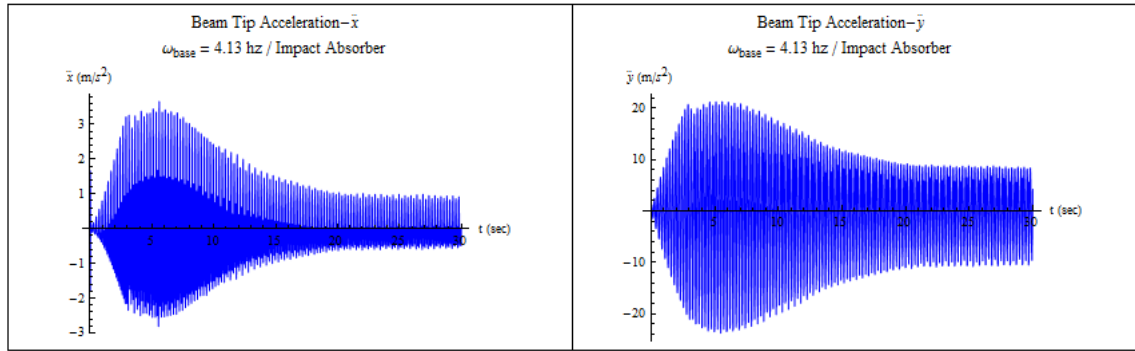


Figure 8-118: Impact case / Beam tip acceleration for the forcing frequency of 4.13 Hz

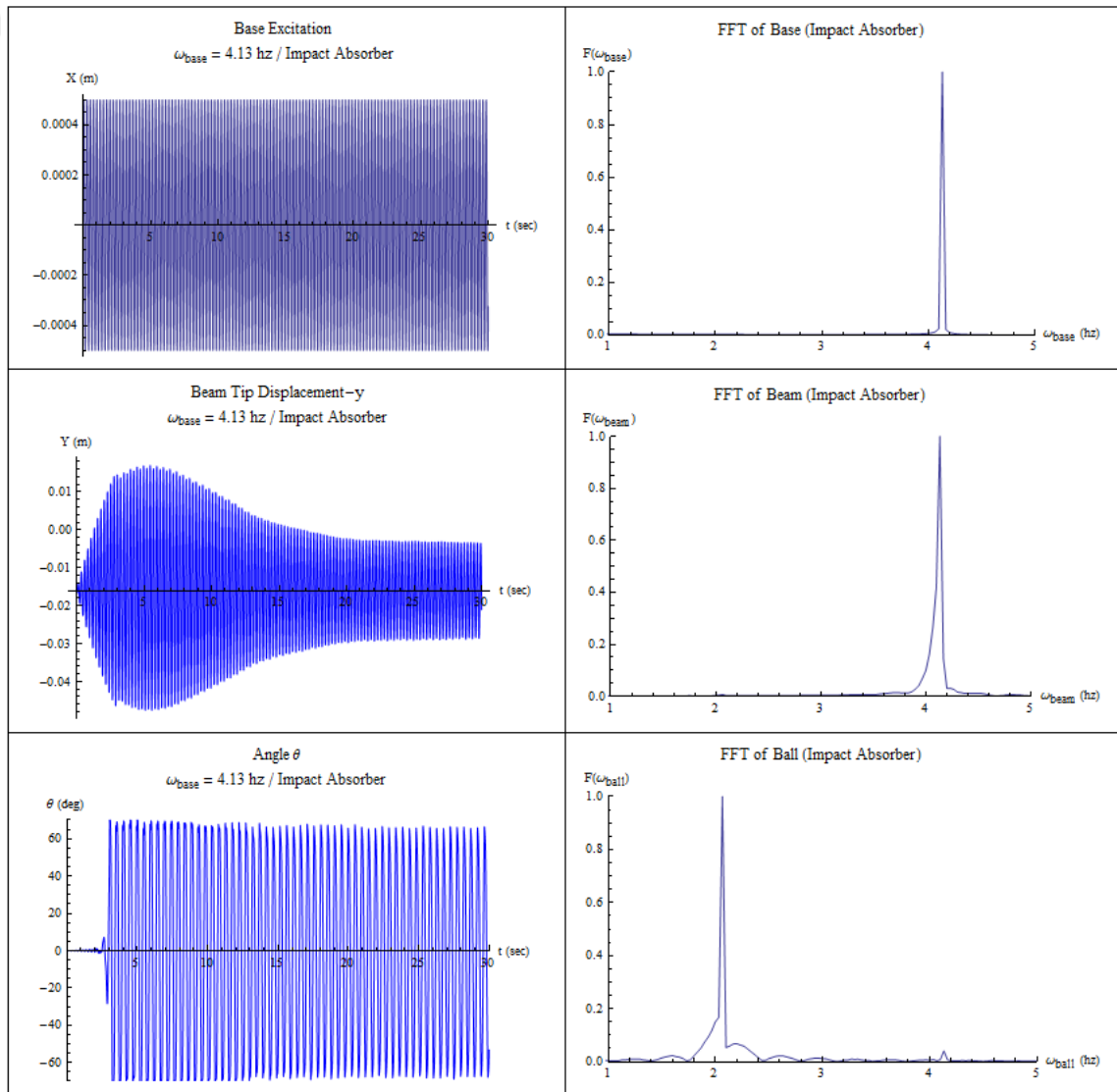


Figure 8-119: Impact case / FFT of the system for the forcing frequency of 4.13 Hz

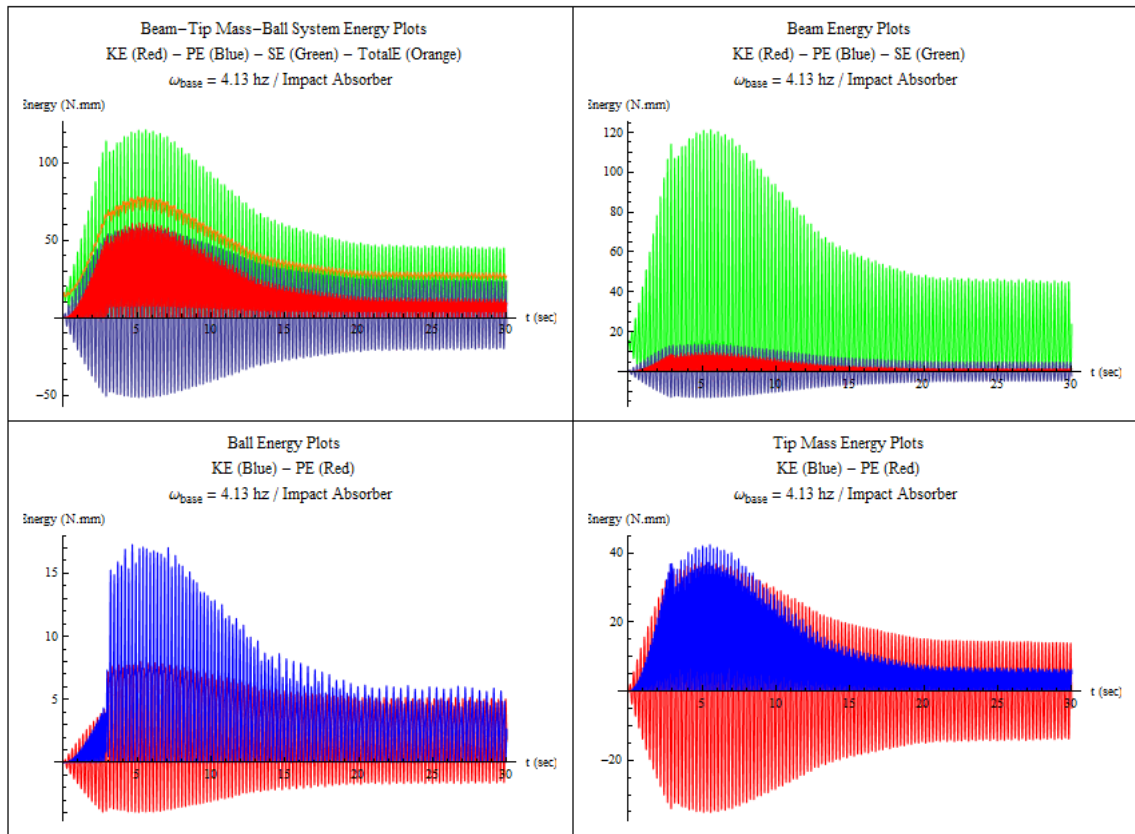


Figure 8-120: Impact case / System energy curves for the forcing frequency of 4.13 Hz

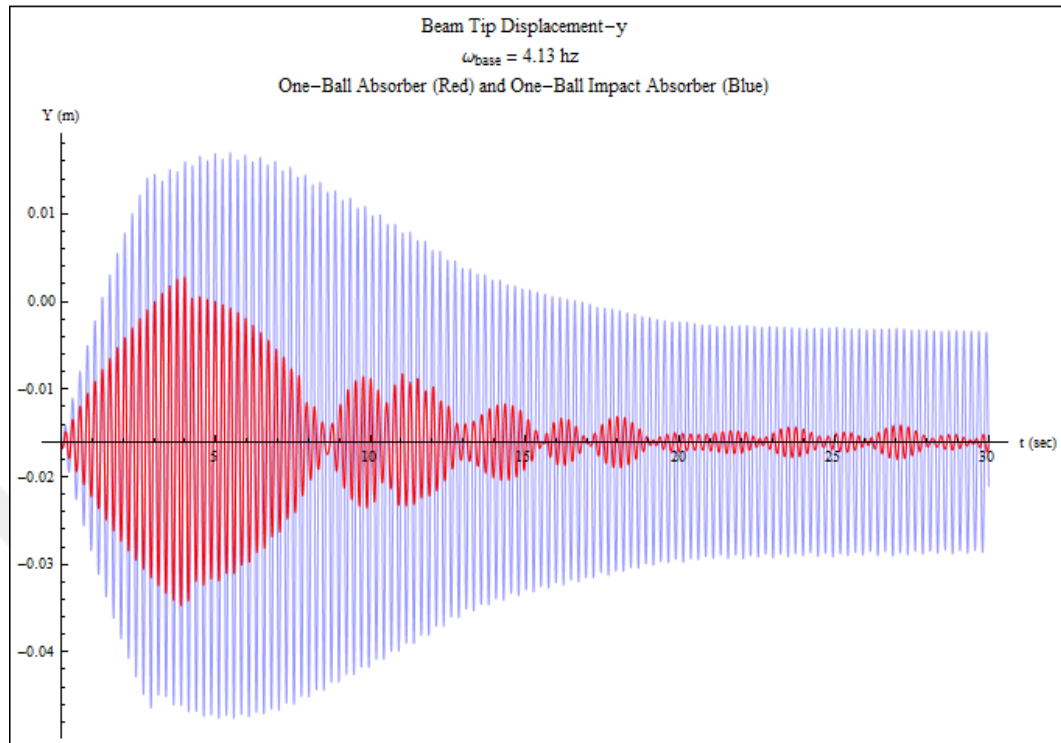


Figure 8-121: Comparison of the beam tip displacement-y in one-ball and impact cases

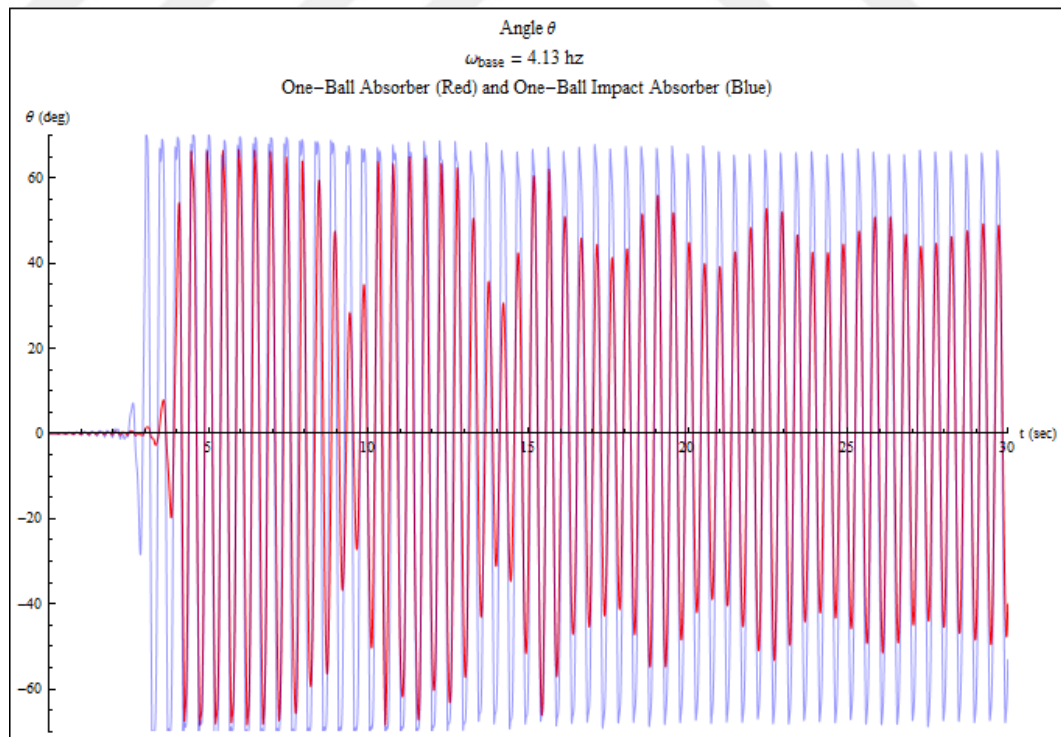


Figure 8-122: Comparison of the ball angle in one-ball and impact cases

CHAPTER 9

CONCLUSION and CONTRIBUTIONS

9.1 Conclusion

This dissertation is concerned with the dynamics of a flexible beam with a tip mass-ball arrangement. The system is treated as a flexible multi-body system interconnected with joints. The tip mass and ball are assumed to be rigid, and the beam is treated as a flexible body. Connection between the tip mass and the free end of the beam is modeled using a fixed joint, and the contact between the ball and the tip mass is modeled using the geometry of the bodies.

The Absolute Nodal Coordinate Formulation (ANCF) is used to determine the mass matrix, stiffness matrix, and generalized forces of the system. Generalized elastic forces for the flexible beam are found using the continuum mechanics approach. Non-linear equations of motion of the system are found using the Lagrangian Formulation in which constraints are treated explicitly as extra equations by using Lagrange Multipliers. The resulting differential algebraic equations are solved using a two-loop sparse matrix numerical integration method in which the kinematic constraint equations are satisfied at the position, velocity, and acceleration levels.

For the first part of the numerical investigation, beam–tip mass-one-ball arrangement is considered and the detailed system dynamics including frequency response curves, time history curves, FFT curves, phase plane curves, and energy curves are plotted for various base excitation frequencies. Numerical results are compared with the results of previously studied similar systems and a good qualitative agreement is observed. Moreover, the same system with the same parameters is modeled using the mechanical analysis software ADAMS, and the results are observed to be in excellent quantitative agreement although the two methods use different formulations. Therefore, in view of the numerical results, it is found that the ball can be considered a suitable autoparametric vibration absorber

under periodic excitation, and it is possible to see the complete energy transfer between two modes occurring when the beam frequency is twice the ball frequency.

In the second part of the numerical investigation, the beam–tip mass-two-ball arrangement is considered in detail for the forcing frequency equal to the natural frequency of the beam. Time history curves, FFT curves, phase plane curves and energy curves are plotted for the system. The autoparametric interaction effect between the two modes of the system is also observed as it is in the one-ball case. However, the beam displacement is larger and the ball displacement is smaller as compared to the one-ball case. A similar observation was obtained in a previous study of a beam-tip mass-pendulum system whenever they increased the pendulum mass. Moreover, numerical results are in good agreement with the previously done experimental results.

Lastly, the excitation amplitude of the beam-tip mass-one-ball arrangement is increased in order to observe impact absorption. As it was observed in a previous experimental study, numerical investigation shows that the beam and the ball responses increase whenever impact exists in the system.

9.2 Contribution

Contributions of this study to the dynamics and vibration literature can be stated as follows:

- Beam-tip mass-ball arrangement as an autoparametric vibration absorber is investigated numerically for the first time;
- ANCF is applied for the numerical solution of an autoparametric system for the first time;
- Detailed time histories, energy plots, velocity, and acceleration curves are found numerically for an autoparametric vibration absorber system for the first time;
- Beating phenomenon in autoparametric vibration absorber systems is shown numerically for the first time; and

- To the author's knowledge, the two-matrix sparse integration method used for the solution of DAE's is applied for the first time.



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