HOMOMORPHISMS BETWEEN GROUP ALGEBRAS

by

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A Thesis Submitted to the
Graduate School of Sciences & Engineering
in Partial Fulfillment of the Requirements for
the Degree of

Master of Science

in

Mathematics

Koç University

May, 2008

Koç University

Graduate School of Sciences and Engineering

This is to certify that I have examined this copy of a master's thesis by

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and have found that it is complete and satisfactory in all respects,
and that any and all revisions required by the final
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To my parents and my husband

ABSTRACT

Let G and H be two locally compact abelian groups, equipped with their Haar measures. The corresponding group algebras are the sets of functions which are the measurable functions of whose absolute values have finite integrals. The object of this work is to study the structure of the Banach algebra homomorphisms between these group algebras. The adjoint of such a homomorphism sends the dual group of H into dual group of G union the zero function. The problem comes to find out the mappings between the dual groups that are induced by the adjoint of a homomorphism h between group algebras. To this end, the work will be exploring commutative Banach algebras, dual group of a locally compact group and measure algebra of a locally compact group.

ÖZETÇE

Her lokal kompakt ve kommutatif grup üzerinde tanımlı Haar ölçüsü vardir. G ve H lokal kompakt ve kommutatif iki grup olsun. Bu gruplara ait grup cebirleri, Haar ölçüsüne göre, grup üzerinde tanımlı ölçülebilir ve mutlak değeri sonlu integrale sahip olan fonksiyonların kümesidir. Bu çalışmanın amacı grup cebirleri arasındaki Banach cebiri homomorfizmalarını anlamaktır. Bunun için, sırayla komutatif Banach cebirleri, lokal kompakt grupların dualleri ve ölçü cebirleri incelendi.

ACKNOWLEDGMENTS

I would like to thank Prof. Ali Ülger , my advisor, for his suggestions and patience during this research.

The *TUBITAK Scholarship*, which was awarded to me for the period 2005–2007, was crucial to the successful completion of this project.

Of course, I am grateful to my parents and my husband for their support and *love*. Without them this work would never have come into existence.

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February, 2008

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INTRODUCTION

The structure of homomorphisms between group algebras of locally compact abelian groups has been well established by Paul J. Cohen [3]. Before then, there have been many papers on the subject dealing with certain special cases, especially isomorphisms, such as [6], [12].

When dealing with the homomorphism problem of group algebras, it is inevitable to study the measure algebras, in particular the idempotent measures. It is also natural to establish a background of dual groups and Fourier transform on measure algebras. In that reason, the paper [16] by I.E. Segal, which establishes the concepts on group algebras, and [12] by W. Rudin, which constituted the facts on the measure algebras, were examined in depth and the relevant facts are presented carefully. However, the most crucial papers for the course of the study were [14] by W. Rudin and [2] by P. J. Cohen, and of course [3] by the same author.

Throughout this study, the books by W. Rudin [14] and [13] were indispensable sources of information. Also [5] and [15] were most valuable to establish background information for the study and myself.

Chapter 1

COMMUTATIVE SEMISIMPLE BANACH ALGEBRAS (=CSS-BA)

A complex Banach algebra is an algebra $\mathcal{B}$ over the field of complex numbers with a norm $||\cdot||$ under which it is a Banach space and with the property $||xy|| \leq ||x||\,||y||$. Since every non-unital Banach algebra can be embedded in a unital Banach algebra (see Appendix A.1), throughout this chapter all Banach algebras are unital and commutative if otherwise is not stated.

For a Banach algebra $\mathcal{A}$, intersection of all maximal ideals of $\mathcal{A}$ is called the **radical of** $\mathcal{A}$, denoted as $Rad\mathcal{A}$. And $\mathcal{A}$ is **semisimple** if $Rad\mathcal{A} = \{0\}$.

1.1 Gelfand Spectrum and Gelfand Transform of a CSS-BA

Let $\mathcal{A}$ be a (commutative) Banach algebra. A **multiplicative functional** on $\mathcal{A}$ is a nonzero algebra homomorphism $\chi : \mathcal{A} \to \mathbb{C}$. The **spectrum of** $\mathcal{A}$, denoted as $\sigma(\mathcal{A})$, is the set of all multiplicative functionals on $\mathcal{A}$.

Proposition 1.1.1 *For $\chi \in \sigma(\mathcal{A})$,*

(1) $\chi(e) = 1$.

(2) If x is invertible then $\chi(x) \neq 0$.

(3) And χ is continuous of norm one.

Proof: *(1) As $\chi \in \sigma(\mathcal{A})$, there is $x \in \mathcal{A}$ with $\chi(x) \neq 0$, then*

$\chi(x) = \chi(xe) = \chi(x)\chi(e)$ hence $\chi(e) = 1$.

In particular, the condition $\chi \neq 0$ is equivalent to $\chi(e) = 1$

(2) If x is invertible then $\chi(e) = \chi(x^{-1}x) = \chi(x^{-1})\chi(x) = 1$, so neither $\chi(x^{-1})$ nor $\chi(x)$ is zero.

(3) If $|\lambda| > ||x||$, then $\lambda e - x$ is invertible by the theorem A.1.3, so by (1) and (2) $\chi(\lambda e - x) = \lambda - \chi(x) \neq 0$. Hence $|\chi(x)| \leq ||x||$ for all $x \in \mathcal{A}$. So χ is a bounded linear map, hence continuous. As $\chi(e) = 1$, $||\chi||_\infty = 1$. ■

Let $\mathfrak{M}(\mathcal{A})$ denote the set of maximal ideals of the Banach algebra $\mathcal{A}$. Then there is a correspondence between the sets $\sigma(\mathcal{A})$ and $\mathfrak{M}(\mathcal{A})$ as follows:

Theorem 1.1.2 ***(Gelfand)*** *Let $\mathcal{A}$ be a Banach algebra. Then:*

(i) $\chi \to Ker\chi$ defines a bijection from $\sigma(\mathcal{A})$ onto $\mathfrak{M}(\mathcal{A})$.

(ii) For every $x \in \mathcal{A}$, $\sigma(x) = \{\chi(x) : \chi \in \sigma(\mathcal{A})\}$ (spectrum of x defined in A.1)

Proof: *(i) Let $\chi \in \sigma(\mathcal{A})$, then $Ker\chi$ is a proper ideal of $\mathcal{A}$ as $\chi(e) = 1 \neq 0$. Since for each $\lambda \in \mathbb{C}$, $\lambda e \in \mathcal{A}$ and $\chi(\lambda e) = \lambda = 0$ iff $\lambda = 0$, then $\mathcal{A} = Ker\chi \oplus \mathbb{C}e$. Hence $Ker\chi$ has codimension 1, so that it is a maximal ideal of $\mathcal{A}$.*
Let $Ker\chi = Ker\tau$ for $\chi, \tau \in \sigma(\mathcal{A})$. By above argument, all $x \in \mathcal{A}$ can be written as: $x = \chi(x)e + y$ for some $y \in Ker\chi$, so that $\tau(x) = \chi(x)\tau(e) - \tau(y) = \chi(x)$, i.e. $\tau = \chi$.
Let $\mathcal{I} \in \mathfrak{M}(\mathcal{A})$, then $\mathcal{I}$ is a closed ideal, and $\mathcal{A}/\mathcal{I}$ is a Banach algebra with quotient norm (see Appendix A.1). Moreover, $\mathcal{A}/\mathcal{I}$ has no nontrivial ideals, since if $M \subset \mathcal{A}/\mathcal{I}$ is an ideal, then $\pi^{-1}(M)$ is an ideal of $\mathcal{A}$ where $\pi : \mathcal{A} \to \mathcal{A}/\mathcal{I}$ is the quotient mapping and $\mathcal{I} \subset \pi^{-1}(M)$. But then $M = \{0\}$ or $M = \mathcal{A}/\mathcal{I}$, as $\mathcal{I}$ is maximal . Hence $\mathcal{A}/\mathcal{I}$ has no non-trivial ideals, thus for each element $\tilde{x} \in \mathcal{A}/\mathcal{I}$, $(\mathcal{A}/\mathcal{I})\tilde{x} = \mathcal{A}/\mathcal{I}$, and so $\tilde{x}$ is invertible. Then by Gelfand-Mazur theorem (A.1.6), $\mathcal{A}/\mathcal{I} \equiv \mathbb{C}$, let $\phi : \mathcal{A}/\mathcal{I} \to \mathcal{C}$ be the isomorphism, then $\phi \circ \pi : \mathcal{A} \to \mathbb{C}$ is a multiplicative functional of $\mathcal{A}$ as both ϕ and π are homomorphisms of algebras. Moreover, $Ker\phi \circ \pi = \mathcal{I}$.

(ii) $\lambda \in \sigma(x) \iff \lambda e - x$ is not invertible $\iff \lambda e - x$ is contained in a maximal ideal (since ideal generated by $\lambda e - x$ is proper) $\iff \chi(\lambda e - x) = \lambda - \chi(x) = 0$ for some $\chi \in \sigma(\mathcal{A})$, i.e. $\chi(x) = \lambda$
In particular, an element $x \in \mathcal{A}$ is invertible iff $\chi(x) \neq 0$ for every $\chi \in \sigma(\mathcal{A})$.■

Moreover, $x \in Rad\mathcal{A}$ iff spectral radius of x (defined in A.1) is zero;

$x \in Rad\mathcal{A} \Leftrightarrow x$ is in all maximal ideals of $\mathcal{A} \Leftrightarrow x \in Ker\chi$ for all $\chi \in \sigma(\mathcal{A}) \Leftrightarrow \sigma(x) = \{\chi(x) : \chi \in \sigma(\mathcal{A})\} = \{0\}$ i.e. $r(x) = \sup\{|\chi(x)| : \chi \in \sigma(\mathcal{A})\} = 0$.

Let $\mathcal{A}$ be a Banach algebra. Each $x \in \mathcal{A}$ defines a function $\hat{x}$ on $\sigma(\mathcal{A})$ given by $\hat{x}(\chi) = \chi(x)$ for $\chi \in \sigma(\mathcal{A})$. The weak* topology induced on $\sigma(\mathcal{A})$ by the collection $\{\hat{x} : x \in \mathcal{A}\}$ is called the **Gelfand topology of** $\sigma(\mathcal{A})$. And the map $\Gamma : \mathcal{A} \to C(\sigma(\mathcal{A}))$ defined as $\Gamma(x) = \hat{x}$ is called the **Gelfand transform of** x.
Since there is a one-to-one correspondence between maximal ideals of $\mathcal{A}$ and members of $\sigma(\mathcal{A})$, $\sigma(\mathcal{A})$ with Gelfand topology is called the **maximal ideal space of** $\mathcal{A}$.

Theorem 1.1.3 ***(Gelfand)*** *Let $\mathcal{A}$ be a Banach algebra, and $\sigma(\mathcal{A})$ be its maximal ideal space. Then:*
(i) $\sigma(\mathcal{A})$ is compact.
(ii) Gelfand transform, Γ is a continuous homomorphism from $\mathcal{A}$ onto a subalgebra $\widehat{\mathcal{A}}$ of $C(\sigma(\mathcal{A}))$ such that $Ker\Gamma = Rad\mathcal{A}$. Hence Γ is an injective homomorphism if and only if $\mathcal{A}$ is semisimple.
(iii) For every $x \in \mathcal{A}$, range $\hat{x} = \sigma(x)$ and $||\hat{x}||_\infty = r(x)$, which is the spectral norm of x.

Proof: *(i) By proposition 1.1.1, $\sigma(\mathcal{A}) \subset B^*$ (closed unit ball of $\mathcal{A}^*$). By Banach-Alaoglu theorem, it is enough to show that $\sigma(\mathcal{A})$ is closed in weak* topology.*

Let $f \in \overline{\sigma(\mathcal{A})}^{w*}$ then there is a net $\{\chi_\alpha\}$ in $\sigma(\mathcal{A})$ with $\chi_\alpha \to^{w*} f$, i.e. $\chi_\alpha(x) \to f(x)$ for each $x \in \mathcal{A}$. Then;

$\chi_\alpha(xy) \to f(xy)$ *and* $\chi_\alpha(xy) = \chi_\alpha(x)\chi_\alpha(y) \to f(x)f(y)$, *so* $f(xy) = f(x)f(y)$

$\chi_\alpha(e) = 1 \to f(e)$, *so* $f(e) = 1$

Hence f is a multiplicative functional, i.e $\sigma(\mathcal{A})$ is weak closed, hence compact.*
(ii) For $x, y \in \mathcal{A}$ and $\alpha \in \mathbb{C}$, $\widehat{(\alpha x + y)}(\chi) = \chi(\alpha x + y) = \alpha\chi(x) + \chi(y) = \alpha\hat{x}(\chi) + \hat{y}(\chi)$, so that $\Gamma(\alpha x + y) = \alpha\Gamma(x) + \Gamma(y)$, hence Γ is a linear mapping.

$|\hat{x}(\chi)| = |\chi(x)| \leq ||x||$, *so* $||\Gamma(x)|| = ||\hat{x}|| \leq ||x||$,

hence Γ is continuous.

$\widehat{xy}(\chi) = \chi(xy) = \chi(x)\chi(y) = \hat{x}(\chi)\hat{y}(\chi)$,

hence $\Gamma(xy) = \Gamma(x)\Gamma(y)$, so it is also a homomorphism.

$$\begin{aligned} Ker\Gamma &= \{x \in \mathcal{A} : \Gamma(x) = \hat{x} = 0\} = \{x \in \mathcal{A} : \hat{x}(\chi) = 0, \forall \chi \in \sigma(\mathcal{A})\} \\ &= \{x \in \mathcal{A} : x \in Ker\chi, \forall \chi \in \sigma(\mathcal{A})\} = Rad\mathcal{A} \text{ by theorem 1.1.2.} \end{aligned}$$

(iii) By theorem 1.1.2, range $\hat{x} = \{\hat{x}(\chi) : \chi \in \sigma(\mathcal{A})\} = \{\chi(x) : \chi \in \sigma(\mathcal{A})\} = \sigma(x)$ *and* $||\hat{x}||_\infty = \sup\{|\lambda| : \lambda \in \sigma(x)\} = r(x)$ ■

Example 1.1.4 *In some cases,* $\sigma(\mathcal{A})$ *can be identified explicitly. Let* K *be a compact set and* $\mathcal{A} = C(K)$*. Then* $\sigma(\mathcal{A})$ *can be identified with* K*.*

Let $x \in K$*, define* $\phi_x : C(K) \to \mathbb{C}$ *by* $\phi_x(f) = f(x)$*. Then*

$\phi_x(\alpha f + g) = (\alpha f + g)(x) = \alpha f(x) + g(x) = \alpha\phi_x(f) + \phi_x(g)$

$\phi_x(fg) = (fg)(x) = f(x)g(x) = \phi_x(f)\phi_x(g)$

Hence ϕ_x *is a multiplicative functional on* $C(K)$*.*

Now define $\Phi : K \to \sigma(C(K))$ *by* $\Phi(x) = \phi_x$*. By above argument* Φ *is well-defined.*

Φ *is also one-to-one since for* $x \neq y$ *in* K*, there is a function* $f \in C(K)$ *such that* $f(x) \neq f(y)$*, hence* $\phi_x \neq \phi_y$*.*

If $x_\alpha \to x$ *then* $f(x_\alpha) \to f(x)$ *for each* $f \in C(K)$*, so that* $\Phi(x_\alpha) \to \Phi(x)$ *in the weak* topology, so* Φ *is continuous.*

Claim: Every multiplicative functional on $C(K)$ *is of the form* ϕ_x *for some* $x \in K$*.*

By the theorem 1.1.2, it is enough to show that every maximal ideal of $C(K)$ *is of the form* $Ker\phi_x = \{f \in C(K) : \phi_x(f) = f(x) = 0\}$ *for some* $x \in K$*. So we need to show that every proper ideal* $\mathcal{I} \subset C(K)$ *is contained in some* $Ker\phi_x$*.*

Assume by contradiction that $\mathcal{I}$ *is a proper ideal and for all* $x \in K$ *there exists* $f_x \in \mathcal{I}$ *such that* $f_x(x) \neq 0$*. Then* $\{y \in K : f_x(y) \neq 0\} \neq \emptyset$*, is an open set for each* $x \in K$*, and all such sets cover* K*. As* K *is compact there are* $x_1, \ldots, x_n \in K$ *such that the corresponding sets for these elements cover* K*. So there are* $f_{x_1}, \ldots, f_{x_n} \in \mathcal{I}$ *such that not all of these functions are zero at every point in* K*. Let* $g = \sum_{i=1}^{n} |f_{x_i}|^2$*, then* $g \in \mathcal{I}$ *and* g *is invertible contradicting* $\mathcal{I}$ *being a proper ideal by proposition A.1.11.*

1.2 Shilov Idempotent Theorem

Let $\mathcal{A}$ be a Banach algebra, an element $x \in \mathcal{A}$ is called **idempotent** if $x^2 = x$. Idempotent measures will be important in this study, we will give a few general results on idempotent elements of a Banach algebra in this section.

Proposition 1.2.1 *Let $\mathcal{A}$ be a Banach algebra with radical R.*

(i) The only idempotent element in R is 0.

(ii) If x, y are idempotent elements in $\mathcal{A}$ with $x + R = y + R$ then $x = y$.

Proof: *(i) It is enough to show that for $a \in \mathcal{A}$ and $x \in R$ if $ax = a$ then $a = 0$. Now $\mathcal{A}(x - e)$ is a nonzero ideal of $\mathcal{A}$ as $x \in R$ (so $x \neq e$). If it is proper, it must be contained in a maximal ideal, say M. Then as $x \in R$, x must be in M, so that $ax \in M$, for each $a \in \mathcal{A}$. As $\mathcal{A}(x-e) \subset M$, $ax-a \in M$, for all $a \in \mathcal{A}$. Then $\mathcal{A} \subset M$, i.e. any maximal ideal including $\mathcal{A}(x - e)$ is equal to $\mathcal{A}$, contradicting maximality. Hence $\mathcal{A}(x - e) = \mathcal{A}$. In particular, there exists $y \in \mathcal{A}$ such that $e = y(x - e)$, i.e. $x - e$ is invertible. Then by the assumption above $ax = a$, so $a(e - x) = 0$. Hence $a = 0$ as $(e - x)$ is invertible.*

(ii) Let x, y are idempotent elements in $\mathcal{A}$ with $x + R = y + R$. Then $x - y \in R$. And as R is intersection of ideals, $x(x - y) = x^2 - xy = x - xy \in R$ and also $y(y - x) = y^2 - yx = y - yx \in R$. Moreover,

$(x - xy)^2 = x^2 + xyxy - x^2y - xyx = x + xy - xy - xy = x - xy$ *and*

$(y - yx)^2 = y^2 + yxyx - y^2x - yxy = y + yx - yx - yx = y - yx$,

i.e. $x-xy$ and $y-yx$ are idempotent elements of R, then by (i), $x-xy = 0 = y-yx$, hence $x = xy = yx = y$.

Lemma 1.2.2 *Let $\mathcal{A}$ be a Banach algebra with radical R.If $x \in \mathcal{A}$ with $x^2 - x \in R$ then there is a unique element $y \in R$ with $(x + y)^2 = (x + y)$.*

Proof: *First note that R is a closed ideal as it is intersection of closed ideals. Let $u := x-x^2$. Suppose there are $v, w \in R$ such that $(e-4u)(e-v) = e$ and $2w-w^2 = v$. Then for $y = \frac{1}{2}w - xw$, y is in R (as is ideal) and*

$$\begin{aligned}(x + y) - (x + y)^2 &= x + \frac{1}{2}w - xw - (\frac{1}{4}w^2 + x^2w^2 + x^2 + xw - x^2w - xw^2)\\ &= \frac{1}{4}(2w - w^2) + (x - x^2) - x(2w - w^2) + x^2(2w - w^2)\\ &= \frac{1}{4}v + u - (x - x^2)v = \frac{1}{4}v + u - uv\\ &= \frac{-1}{4}(-v - 4u + 4uv + e - e) = \frac{1}{4}((e - 4u)(e - v) - e) = 0\end{aligned}$$

So if we can find such v, w, then we are done.

As $u \in R$, $4u \in R$ too, so spectral radius of $4u$ which is spectral norm of $4u$: $r(4u) = 0 = ||\hat{4u}||_\infty < 1$, so $e - 4u$ is invertible and $(e-4u)^{-1} = \sum_{n=0}^{\infty}(4u)^n$.

Let $v = -\sum_{n=1}^{\infty}(4u)^n$, so $v \in R$ and, plugging in v in the above sum we have $(e-4u)^{-1} = e - v$, hence $e = (e-4u)(e-v)$.

Now take $T : \{||z|| < \varepsilon < 1\} \subset R \to A$ be defined as $Tz = \frac{1}{2}(v + z^2)$, then

$$\begin{aligned} ||Tz_1 - Tz_2|| &= ||\frac{1}{2}(v+z_1^2) - \frac{1}{2}(v+z_2^2)|| \\ &= \frac{1}{2}||z_1^2 - z_2^2|| = \frac{1}{2}||z_1+z_2||\,||z_1-z_2|| \\ &< \varepsilon||z_1 - z_2|| \end{aligned}$$

hence T is a contraction so it has a fixed point $w \in R$, i.e $Tw = \frac{1}{2}(v+w^2) = w$, hence $2w - w^2 = v$.■

Let D be an open subset of a linear topological space X over $\mathbb{C}$. Fix $n \in \mathbb{N}$ and $x \in X^{n+1}$, product space of X $n+1$ times. Let $\pi_x : \mathbb{C}^n \to X$ be defined by $\pi_x(z) = x_0 + z_1x_1 + \ldots + z_nx_n$. A function $f : D \to \mathbb{C}$ is **analytic** on D if f is locally bounded on D and for each integer n and $x \in X^{n+1}$ the map $f \circ \pi_x$ is holomorphic on $\pi_x^{-1}(D)$.

The following lemma is necessary for the proof of Shilov Idempotent theorem. However, the proof of the lemma (which can be found in [1], §21) will not be included for the sake of completeness and not to be off-topic.

Lemma: Let $\mathcal{A}$ be a Banach algebra and let $\mathcal{A}^*$ be the dual space with the weak* topology. Let D be an open neighborhood of $\sigma(\mathcal{A})$ in $\mathcal{A}^*$ and $f : D \to \mathbb{C}$ be an analytic function. Then there exist $x_0 \in \mathcal{A}$ with $f(\phi) = \phi(x_0)$ for all $\phi \in \sigma(\mathcal{A})$.

Theorem 1.2.3 *Let $\mathcal{A}$ be a Banach algebra and suppose that K is a compact and open subset of $\sigma(\mathcal{A})$. Then there exists unique idempotent element $x \in \mathcal{A}$ with $\hat{x} = \chi_K$.*

Proof: *Suppose $K \subset \sigma(\mathcal{A})$ is compact and open, then K^c is also compact. So K, K^c are disjoint compact subsets of $\mathcal{A}^*$, so we can find disjoint open subsets U_1, U_2 such that $K \subset U_1$ and $K^c \subset U_2$. Let $D = U_1 \cup U_2$. Define $f : D \to \mathbb{C}$ as $f = \chi_{U_1}$ then it is continuous,and moreover it is analytic. Indeed, f is locally bounded on D and for*

$x = (x_0, x_1, \ldots, x_n) \in (\mathcal{A}^*)^{n+1}$, $\pi_x^{-1}(D) = \pi_x^{-1}(U_1) \cup \pi_x^{-1}(U_2)$ *and* $f \circ \pi_x$ *is constant on each set hence is analytic. Hence is* f. *Then by above lemma there exists* $z_0 \in \mathcal{A}$ *with* $f(\phi) = \phi(z_0)$ *for all* $\phi \in \sigma(\mathcal{A})$.

Now $\phi(z_0^2 - z_0) = 0$ *for each* $\phi \in \sigma(\mathcal{A})$ *as* $\phi(z_0)$ *is either* 0 *or* 1. *So* $z_0^2 - z_0 \in Rad\mathcal{A}$. *Then by lemma 1.2.2, there is an element* $z_1 \in Rad\mathcal{A}$ *such that* $(z_0 + z_1)^2 = z_0 + z_1$ *and* $\chi(z_0 + z_1) = \chi(z_0) + \chi(z_1) = \chi(z_0)$ *as* $z_1 \in Rad\mathcal{A}$. *Let* $z = z_0 + z_1$ *then* z *is idempotent and* $\hat{z} = \hat{z_0} = \chi_K$.

The major importance of Shilov idempotent theorem, for this study, is due to the following corollary, since this leads to a partial solution for the duality problem on locally compact abelian groups.

Corollary 1.2.4 *Let* $\mathcal{A}$ *be a CSS-BA. Then* $\mathcal{A}$ *has an identity iff* $\sigma(\mathcal{A})$ *is compact and non-empty.*

Proof: *When* $\mathcal{A}$ *has identity* $\sigma(\mathcal{A})$ *is compact by theorem 1.1.2. So suppose* $\sigma(\mathcal{A}) \neq \emptyset$ *is compact and* $\mathcal{A}$ *does not have an identity. Let* $\mathcal{B} = \mathcal{A} + \mathbb{C}$ *be unitization of* $\mathcal{A}$ *(as in appendix A.1). Let* $\psi : \mathcal{B} \to \mathcal{C}$ *be defined as* $\psi((x, \alpha)) = \alpha$ *for* $x \in \mathcal{A}$ *and* $\alpha \in \mathbb{C}$. *Let define* $H : \sigma(\mathcal{B}) \to \sigma(\mathcal{A})$ *as* $(H\chi)(x) = \chi((x, 0))$, *for* $x \in \mathcal{A}$. *Then* H *is continuous as* $\chi_{|\mathcal{A}}$ *is continuous. Let also* $D_1 = \{\psi\}$ *and* $D_2 = \sigma(\mathcal{B}) \setminus D_1$. *Then,* H *is homeomorphism from* D_2 *onto* $\sigma(\mathcal{A})$ *as for each* $\chi \in \sigma(\mathcal{A})$ *defining* $\hat{\chi}$ *as* $\hat{\chi}((x, \alpha)) = \chi(x)$ *gives the inverse* $H^{-1}\chi = \hat{\chi}$. *So by the assumption,* D_2 *is compact and now* $\sigma(\mathcal{B})$ *is union of two disjoint compact sets, so by Shilov Idempotent Theorem, there is* $(e, \alpha) \in \mathcal{B}$ *with* $\widehat{(e, \alpha)}(\chi) = 1$ *on* D_1 *and* 0 *on* D_2, *so* $\widehat{(e, \alpha)}(\psi) = \psi((e, \alpha)) = \alpha = 0$ *and* $(e, \alpha)^2 = (e, \alpha)$ *so* $e^2 = e \neq 0$. *And* $1 = \widehat{(e, \alpha)}(\chi) = \chi(e)$ *for each* $\chi \in \sigma(\mathcal{A})$. *Hence we have:* $\chi(x - xe) = \chi(x) - \chi(x)\chi(e) = 0$ *for each* $x \in \mathcal{A}$ *and* $\chi \in \sigma(\mathcal{A})$. *Then* $x - xe \in Ker\chi$ *for each* χ, *i.e.* $x - xe \in Rad\mathcal{A} = \{0\}$. *Hence* $x = xe$ *and so* e *is identity for* $\mathcal{A}$.

1.3 Homomorphisms between two CSS-BA

A Banach algebra $\mathcal{A}$ has both algebraic and topological structure, so it is important when a mapping between Banach algebras preserves both algebraic and topological structures. It turns out this preservation is automatic when Banach algebras are semisimple:

Theorem 1.3.1 *Let $\mathcal{A}$ be a Banach algebra, $\mathcal{B}$ be a semisimple Banach algebra and $\mathcal{T} : \mathcal{A} \to \mathcal{B}$ be a homomorphism from $\mathcal{A}$ into $\mathcal{B}$, then $\mathcal{T}$ is continuous.*

Proof: *By closed graph theorem, it is enough to show that for (x_n) a sequence in $\mathcal{A}$, if $x_n \to x$ and $\mathcal{T}(x_n) \to b$ then $b = \mathcal{T}(x)$.*

As $\mathcal{T}(x_n) \to b$ then $\mathcal{T}(x_n - x) \to b - T(x)$. So let $a_n =: x_n - x$ then $a_n \to 0$ and $\mathcal{T}(a_n) \to b - \mathcal{T}(x)$.

For a multiplicative functional $\chi \in \sigma(\mathcal{B})$, $\chi \circ \mathcal{T}$ is a complex homomorphism of $\mathcal{A}$ as both χ and T are nonzero homomorphisms. It may happen that $\chi \circ \mathcal{T}$ is identically zero but in either case it is continuous.

So $\chi \circ \mathcal{T}(a_n) \to \chi \circ \mathcal{T}(0) = 0$ but as $\chi \circ \mathcal{T}(a_n) = \chi(\mathcal{T}(a_n)) \to \chi(b - \mathcal{T}(x))$ then $\chi(b - \mathcal{T}(x)) = 0$ for each $\chi \in \sigma(\mathcal{B})$. This implies that $b - \mathcal{T}(x) \in Ker\chi$, for every $\chi \in \sigma(\mathcal{B})$, then $b - \mathcal{T}(x) \in Rad\mathcal{B} = \{0\}$, hence $b = \mathcal{T}(x)$.■

In particular, every isomorphism between two CSS-BA is a homeomorphism.

Corollary 1.3.2 *On a CSS-BA all Banach algebra norms are equivalent.*

Proof: *Let $\mathcal{A}$ be a CSS-BA and $||\,||_1$, $||\,||_2$ be two Banach Algebra norms on it. Define $I : (\mathcal{A}, ||\,||_1) \to (\mathcal{A}, ||\,||_2)$ by $I(a) = a$. Then I is a homomorphism of algebras. So by above theorem I is continuous, so it is bounded. And $||a||_2 = ||I(a)||_2 \leq ||I||\,||a||_1$, and interchanging norms $||\,||_1$, $||\,||_2$ we get another inequality which is enough for equality of norms.*■

Corollary 1.3.3 *Every involution on a CSS-BA is continuous.*

Proof: *Let $\mathcal{A}$ be a CSS-BA and $f : \mathcal{A} \to \mathcal{A}$ be an involution, i.e. f is an anti-homomorphism of $\mathcal{A}$ such that $f(f(a)) = a$ for each $a \in \mathcal{A}$. As $\mathcal{A}$ is commutative, $f(ab) = f(b)f(a) = f(a)f(b)$ hence f is a homomorphism so it is continuous.*■

Let now $\mathcal{A}, \mathcal{B}$ be Banach algebras and let $\mathcal{T} : \mathcal{A} \to \mathcal{B}$ be a homomophism. Now for a multiplicative functional $\chi \in \sigma(\mathcal{B})$, $\chi \circ \mathcal{T}$ is a complex homomorphism of $\mathcal{A}$ and if it is not identically zero then $\chi \circ \mathcal{T} \in \sigma(\mathcal{A})$. Let denote $\chi \circ \mathcal{T} =: T\chi$, then T maps $\sigma(\mathcal{B})$ into $\sigma(\mathcal{A}) \cup \{\mathbf{0}\}$.

Theorem 1.3.4 *The map $T : \sigma(\mathcal{B}) \to \sigma(\mathcal{A}) \cup \{\mathbf{0}\}$ is continuous with respect to the weak* topology on these sets. If moreover, $\mathcal{T}(\mathcal{A})$ is dense in $\mathcal{B}$, then T is a homeomorphism of $\sigma(\mathcal{B})$ into $\sigma(\mathcal{A})$.*

Proof: *Let $\chi \in \sigma(\mathcal{B})$ and $\{\chi_\alpha\}$ be a net converging to χ in $\sigma(\mathcal{B})$, then for any $y \in \mathcal{B}$, $\chi_\alpha(y) \to \chi(y)$. Then for arbitrary $x \in \sigma(\mathcal{A})$;*

$T\chi_\alpha(x) = \chi_\alpha \circ \mathcal{T}(x) = \chi_\alpha(\mathcal{T}x) \to \chi(\mathcal{T}x) = \chi \circ \mathcal{T}(x) = T\chi(x)$

hence $T\chi_\alpha \to T\chi$ in $\sigma(\mathcal{A})$.

Now let $\mathcal{T}(\mathcal{A})$ be dense and $f = \mathcal{T}(e)$, e identity for $\mathcal{A}$. Then for any $y \in \mathcal{B}$, there is a sequence $\mathcal{T}(x_n)$ converging to y. Hence:

$$\begin{aligned} yf &= (\lim_{n\to\infty} \mathcal{T}(x_n))\mathcal{T}(e) = \lim_{n\to\infty} \mathcal{T}(x_n e) (= \lim_{n\to\infty} \mathcal{T}(x_n) = y) \\ &= \lim_{n\to\infty} \mathcal{T}(ex_n) = \mathcal{T}(e) \lim_{n\to\infty} \mathcal{T}(x_n) = fy \end{aligned}$$

Hence $f = \mathcal{T}(e)$ is identity for $\mathcal{B}$, so for any $\chi \in \sigma(\mathcal{B})$, $T\chi \in \sigma(\mathcal{A})$ as

$$T\chi(e) = \chi \circ \mathcal{T}(e) = \chi(f) = 1.$$

Now if $T\chi_1 = T\chi_2$ then for any $x \in \mathcal{A}$, $\chi_1 \circ \mathcal{T}(x) = \chi_2 \circ \mathcal{T}(x)$. And for $y \in \mathcal{B}$, there is a sequence as above, and since χ_1, χ_2 are continuous:

$\chi_1(y) = \chi_1(\lim_{n\to} \mathcal{T}(x_n)) = \lim_{n\to\infty} \chi_1 \circ \mathcal{T}(x_n) = \lim_{n\to\infty} \chi_2 \circ \mathcal{T}(x_n) = \chi_2(y)$

Hence $\chi_1 = \chi_2$. So T is a continuous one-to-one map from $\sigma(\mathcal{B})$ into $\sigma(\mathcal{A})$. Moreover, it is a homeomorpism as $\sigma(\mathcal{B})$ is compact.

In particular if $\mathcal{A}$ is semisimple, and $\mathcal{T}$ is an automorphism of $\mathcal{A}$, then T is a homeomorphism of $\sigma(\mathcal{A})$ and $\widehat{\mathcal{T}x}(\chi) = \chi(\mathcal{T}x) = T\chi(x) = \hat{x}(T\chi)$. Since we can identify the elements of $\mathcal{A}$ with $\hat{\mathcal{A}} \subset C(\sigma(\mathcal{A}))$ the automorphism $\mathcal{T}$ is defined by the homeomorphism T on the maximal ideal space.

1.4 Identity and Bounded Approximate Identity

Let $\mathcal{A}$ be a Banach algebra. A net $(e_\alpha)_{\alpha\in I}$ in $\mathcal{A}$ is called **bounded approximate identity** if it is bounded and $\lim_\alpha ||e_\alpha a - a|| = 0$ for every $a \in \mathcal{A}$.

Proposition 1.4.1 *If $\mathcal{A}$ contains a bounded set U such that for given $x \in \mathcal{A}$ and $\varepsilon > 0$ there exists $u \in U$ with $||x - ux|| < \varepsilon$. Then $\mathcal{A}$ has a bounded approximate identity.*

Proof: *We need the following lemma:*

Lemma *For $\varepsilon > 0$ and any finite set $F \subset \mathcal{A}$, there is $w \in W = \{u+v-uv : u, v \in U\}$ such that $||x - wx|| < \varepsilon$ for all $x \in F$.*

Suppose the lemma holds. Let $F(\mathcal{A})$ be the set of all finite subsets of $\mathcal{A}$, then it is a directed set with inclusion and $I = \{1/n : n \geq 1\} \times F(\mathcal{A})$ is also a directed set. Now for $\alpha \in I$, $\alpha = (1/n, F)$ with $\varepsilon = 1/n$ and F finite set, there exists $w_\alpha \in W$ chosen with respect to above claim gives an approximate identity. Moreover, as U is bounded, say by M, for any $w \in W$, $||w|| = ||u + v - uv|| \leq 2M + M^2$.

Proof of lemma: *Now let $F = \{x_1, x_2\}$, then by the assumption of proposition,*

(1) there is $u \in U$ such that $||x_1 - ux_1|| < \varepsilon/(1 + M)$ and

(2) there is $v \in U$ such that $||(x_2 - ux_2) - v(x_2 - ux_2)|| < \varepsilon$ (as $x_2 - ux_2 \in \mathcal{A}$).

Letting $w = u + v - uv$, we have $w \in W$ and $||x_2 - wx_2|| < \varepsilon$ by (2) and

$||x_1 - wx_1|| \leq ||x_1 - ux_1|| + ||v||\,||x_1 - ux_1|| < \varepsilon$ *by (1).*

• *Suppose the claim is true for every finite subset with n elements. Let $F \subset \mathcal{A}$ be $\{x_1, \ldots, x_{n+1}\}$ and let $\gamma = \max\{||x_i|| : i = 1, \ldots, n\}$. Then there exists $y \in W$ with $||x_i - yx_i|| < \varepsilon/3M$ for $i = 1, \ldots, n$. Now choose $w \in W$ with $||y - wy|| < \varepsilon/3\gamma$ and $||x_{n+1} - wx_{n+1}|| < \varepsilon$, then for $i = 1, \ldots, n$,*

$$\begin{aligned} ||x_i - wx_i|| &\leq ||x_i - yx_i|| + ||yx_i - wyx_i|| + ||wyx_i - wx_i|| \\ &< ||x_i - yx_i|| + ||y - wy||\,||x_i|| + ||w||\,||yx_i - x_i|| \cdot \\ &< \varepsilon/3 + (\varepsilon/3\gamma)\gamma + M(\varepsilon/3M) < \varepsilon \blacksquare \end{aligned}$$

Note that the converse of this statement is already true by definition, hence we can look for a bounded set with the property in the proposition instead of a net.

Proposition 1.4.2 *Let $\mathcal{I}$, $\mathcal{J}$ be closed ideals of Banach algebra $\mathcal{A}$.*

1. *If $\mathcal{I}, \mathcal{J}$ has bounded approximate identities then so does $\mathcal{I} \cap \mathcal{J}$.*

2. *If $\mathcal{A}$ has bounded approximate identity then so does $\mathcal{A}/\mathcal{I}$.*

3. *If $\mathcal{I}, \mathcal{A}/\mathcal{I}$ have bounded approximate identities then $\mathcal{A}$ has bounded approximate identity.*

Proof: 1. *By above proposition, since $\mathcal{I}, \mathcal{J}$ have bounded approximate identities, there are bounded subsets $W \subset \mathcal{I}$ and $V \subset \mathcal{J}$ with the property:*

for every $x \in \mathcal{I}, y \in \mathcal{J}$, $\varepsilon > 0$ there are $w \in W$ and $v \in V$ with

$||x - wx|| < \varepsilon/||v||$ and $||y - vy|| < \varepsilon$ (*).

Let $U = WV$, then for $x \in \mathcal{I} \cap \mathcal{J}$, there exist $w \in W$, $v \in V$ satisfying (*). Let $u = wv$, then $||x - ux|| \leq ||x - vx|| + ||vx - wvx|| \leq ||x - vx|| + ||v||\,||x - wx|| < 2\varepsilon$, hence by previous proposition $\mathcal{I} \cap \mathcal{J}$ has bounded approximate identity.

2. *Let $(e_\alpha)_{\alpha \in I}$ be bounded approximate identity of $\mathcal{A}$, define $E_\alpha = \mathcal{I} + e_\alpha$, then for $\tilde{B} \in \mathcal{A}/\mathcal{I}$ there is $a \in \mathcal{A}$ with $\tilde{B} = \mathcal{I} + a$, then*

$$\begin{aligned} ||\tilde{B} - E_\alpha \tilde{B}|| &= ||(\mathcal{I} + a) - (\mathcal{I} + e_\alpha)(\mathcal{I} + a)|| = ||\mathcal{I} + (a - e_\alpha a)|| \\ &= \inf_{b \in \mathcal{I}} ||b + (a - e_\alpha a)|| \leq ||a - e_\alpha a|| \to 0 \end{aligned}$$

Hence (E_α) is bounded approximate identity for $\mathcal{A}/\mathcal{I}$.

3. *Let $\varphi : \mathcal{A} \to \mathcal{A}/\mathcal{I}$ be the canonical homomorphism $\varphi x = x + \mathcal{I}$.*

Let $x_0 \in \mathcal{A}$ and $\varepsilon > 0$ be arbitrary. Then $\varphi x_0 = x_0 + \mathcal{I} \in \mathcal{A}/\mathcal{I}$, so by hypothesis and proposition 1.4.1 there is an element $y' \in \mathcal{A}/\mathcal{I}$, with $y' = y + \mathcal{I} = \varphi y$ for some $y \in \mathcal{A}$, such that

$||y'||_{\mathcal{A}/\mathcal{I}} \leq K$ for some $K > 0$, and $||y'\varphi x_0 - \varphi x_0||_{\mathcal{A}/\mathcal{I}} < \varepsilon$ (1)

Now set $z = x_0 - yx_0$ then $||\varphi z||_{\mathcal{A}/\mathcal{I}} = inf_{u \in \mathcal{I}} ||z - u||$, so there is an element $u \in \mathcal{I}$ such that $||z - u|| < ||\varphi z||_{\mathcal{A}/\mathcal{I}} + \varepsilon$ (2).

Again by hypothesis and the proposition there exist $v \in \mathcal{I}$, which is bounded by some $L > 0$, such that $||u - vu|| < \varepsilon$ (3).

Now consider $z - zv = (z - u) + (u - uv) + (uv - vz)$. As $\varphi z = \varphi x_0 - y'\varphi x_0$ (4) then;

$$\begin{aligned} ||z - zv|| &\leq ||z - u|| + ||u - uv|| + ||v||\,||u - z|| \\ &< ||\varphi z||_{\mathcal{A}/\mathcal{I}} + \varepsilon + \varepsilon + L(||\varphi z||_{\mathcal{A}/\mathcal{I}} + \varepsilon) \text{ by (2) and (3)} \\ &< 3\varepsilon + 2L\varepsilon \text{ by (1) and (4)} \end{aligned}$$

Now $||z - zv|| = ||x_0 - yx_0 - vx_0 + vyx_0|| < (3 + 2L)\varepsilon$, *then setting* $w = y + v - yv$ *we have* $||x_0 - wx_0|| < (3 + 2L)\varepsilon$ *and moreover,* $||w|| = ||y + v - yv|| \leq K + L + KL$, *where* K, L *are independent of* ε *and* x_0. *Hence* $\mathcal{A}$ *has a bounded approximate identity.*

■

Chapter 2

DUAL GROUP OF A LOCALLY COMPACT GROUP G

2.1 The Group Algebra $L^1(G)$

Now on m is the Haar measure on a LCA group G and dx, dy will stand for the integral with respect to Haar measure.

Let f, g be Borel functions on LCA group G. Their **convolution** is written as $f * g$ and defined as $(f * g)(x) = \int_G f(x-y)g(y)dy$ if $\int_G |f(x-y)g(y)|dy < \infty$.

Proposition 2.1.1 *(Properties of convolution) Let f, g, h be elements of $L^1(G)$, the set of integrable functions on G, then*

(1) *If $\int_G |f(x-y)g(y)|dy < \infty$ for some $x \in G$ then $(f * g)(x) = (g * f)(x)$*

(2) *If $f, g \in C_c(G)$ with $suppf = A$ and $suppg = B$, then $supp(f * g) \subset A + B$, hence $f * g \in C_c(G)$.*

(3) *$\int_G |f(x-y)g(y)|dy < \infty$ holds for almost all $x \in G$, and $f * g \in L^1(G)$ with $||f * g||_1 \leq ||f||_1 ||g||_1$.*

(4) $(f * g) * h = f * (g * h)$

Proof:(1) $(f * g)(x) = \int_G f(x-y)g(y)dy = \int_G f(-y)g(x+y)dy$ *as Haar measure is translation invariant. Now, as G is abelian, by lemma A.2.7*

$$\int_G f(-y)g(x+y)dy = \int_G f(y)g(x-y)dy = (g * f)(x)$$

(2) *Let $x \in G \setminus (A + B)$, then $x - y \in A$ if $y \notin B$ and $x - y \notin A$ for any $y \in B$. Now $f(x-y) \neq 0$ if $x - y \in A$ which implies $y \notin B$ so that $g(y) = 0$; $g(y) \neq 0$ if $y \in B$ which implies $x - y \notin A$ so that $f(x-y) = 0$, hence $f(x-y)g(y) = 0$ for every $y \in G$, hence $f * g(x) = \int_G f(x-y)g(y)dy = 0$. So $x \notin supp(f * g)$, i.e. $supp(f * g) \subset A + B$. Being a closed subset of a compact set $A + B$, $supp(f * g)$ is compact, hence $f * g \in C_c(G)$.*

(3) $h(x,y) = f(x-y)g(y)$ *is a measurable function as* $f, g \in L^1(G)$ *and* $\phi(x,y) = x-y$ *and* $\psi(x,y) = y$ *are measurable functions from* $G \times G$ *into* G *and* $h = (f \circ \phi)(g \circ \psi)$. *Moreover, by Fubini's theorem and Haar measure being translation invariant,*

$$\begin{aligned}\int_G \int_G |f(x-y)g(y)|dydx &= \int_G \int_G |f(x-y)g(y)|dxdy = \int_G |g(y)| \int_G |f(x-y)|dxdy \\ &= ||f||_1 \int_G |g(y)|dy = ||f||_1||g||_1\end{aligned}$$

Hence $\int_G |f(x-y)g(y)|dy < \infty$ *holds for a.e.* $x \in G$, *so that* $f * g(x)$ *exists for a.e.* $x \in G$, *and*

$$\int_G |f * g(x)|dx \leq \int_G \int_G |f(x-y)g(y)|dydx = ||f||_1||g||_1.$$

Hence $f * g \in L^1(G)$.

(4)

$$\begin{aligned}(f * g) * h(x) &= \int_G (f * g)(x-y)h(y)dy = \int_G \int_G f(x-y-z)g(z)h(y)dzdy \\ &= \int_G \int_G f(x-z)g(z-y)h(y)dzdy \text{ by translation invariance} \\ &= \int_G f(x-z) \int_G g(z-y)h(y)dydz \text{ by Fubini theorem} \\ &= \int_G f(x-z)(g * h)(z)dz = f * (g * h)(x) \blacksquare\end{aligned}$$

We already know that equivalence classes of $L^1(G)$ (with respect to functions being equivalent a.e.) is a Banach space with L^1-norm. More is true with the convolution defined as above:

For any LCA group G, $L^1(G)$ *is a Banach algebra with multiplication as convolution.*

Indeed;

- Multiplication is closed, commutative and associative by the above proposition.
- Distribution laws hold, as

  $f * (g+h)(x) = \int_G f(x-y)(g+h)(y)dy = f * g(x) + f * h(x)$.
- And finally, $||f * g||_1 \leq ||f||_1||g||_1$ by above theorem again.

$L^1(G)$ is called the **group algebra** of the LCA group G.

In first chapter, we assumed every Banach algebra to have an identity. So it is natural to look for an identity of this Banach algebra. However, the existence of an identity is not obvious. Actually,

Theorem 2.1.2 *The group algebra of G has an identity if and only if G is discrete.*

Proof: *Now let G be discrete and $e \in G$ be the identity. Define $f_0 : G \to \mathbb{C}$ as $f_0(e) = 1$ and $f_0(x) = 0$ for $x \neq e$. Then as G is discrete, $f_0 \in L^1(G)$ and for each $f \in L^1(G)$ and for $x \in G$, $f * f_0(x) = \int_G f(x-y)f_0(y)dy = f(x-e) = f(x)$, hence f_0 is identity of the Banach algebra $L^1(G)$.*

For the converse let $f_0 \in L^1(G)$ be identity, then for $g \in L^1(G)$,

$$\int_G g(x-y)f_0(y)dy = g(x) \text{ for a.e. } x \quad (1)$$

If $g \in C_c(G)$, then it is uniformly continuous and so equality in (1) holds for every x. Putting $x = e$ in (1) we have $\int_G g(-y)f_0(y)dy = g(e)$.

Now for a measurable set $E \subset G$ with compact closure, let $\nu(E) = \int_E f_0(y)dy$ and define $\mu(E) = 1$ if $e \in E$, and 0 otherwise. Then for $g \in C_c(G)$,

$$\int_G g(-y)d\nu(y) = \int_G g(-y)f_0(y)dy = g(e) = \int_G g(-y)d\mu(y)$$

and by Riesz representation theorem (Appendix A.2.2), $\nu = \mu$. Since, by definition, ν is absolutely continuous with respect to Haar measure and $\nu(\{e\}) = \mu(\{e\}) = 1$, $\{e\}$ has (Haar) measure 1. So every infinite subset has infinite measure, which in turn implies every compact set is finite (since measure is Radon). Also since G is locally compact, $\{e\}$ is open. ■

For $a \in G$, the translation operator T_a on $L^1(G)$ is defined as: $T_a(f)(x) = f(x-a)$. Translation on the Euclidian spaces is known to be isometry. Similarly, as the Haar measure is translation invariant, it turns out norm of $L^1(G)$ is translation invariant, i.e. $||T_a f||_1 = ||f||_1$ (*). Also by definition of translation: $T_a T_b = T_{a+b}$. Let denote $T_a(f) =: f_a$ for simplicity. Then;

Lemma 2.1.3 *For G a LCA group, and $f \in L^1(G)$ the map $\Phi : G \to L^1(G)$ defined as $\Phi(a) = f_a$ is uniformly continuous.*

Proof: *First note that $C_c(G)$ is dense in $L^1(G)$, so for each $\varepsilon > 0$ there is g in $C_c(G)$ with compact support $K \subset G$ such that $||f - g||_1 < \varepsilon/3$. Since g is uniformly continuous, for the same ε there is $U \in \mathcal{V}(e)$ such that $||g - g_x||_\infty < \varepsilon/(3m(K))$ for each $x \in U$, so that $||g - g_x||_1 < \varepsilon/3$. Then,*

$$\begin{aligned} ||f - f_x||_1 &\leq ||f - g||_1 + ||g - g_x||_1 + ||g_x - f_x||_1 \\ &= ||f - g||_1 + ||g - g_x||_1 + ||(g - f)_x||_1 \\ &= ||f - g||_1 + ||g - g_x||_1 + ||g - f||_1 \text{ by (*)} \\ &< \varepsilon \end{aligned}$$

Now for arbitrary $x, y \in G$ if $x - y \in U$ then

$||\Phi(x) - \Phi(y)||_1 = ||f_x - f_y||_1 = ||(f_{x-y} - f)_y||_1 = ||f - f_{x-y}||_1 < \varepsilon$. ■

Now by previous theorem, $L^1(G)$ has an identity only if G is discrete. So in general, $L^1(G)$ does not have an identity, but it always possess a bounded approximate identity:

Theorem 2.1.4 *For G a LCA group, $L^1(G)$ has a bounded approximate identity.*

Proof: *Let $\mathcal{C} = \{$compact neighborhoods of identity in $G\}$, then it is a directed set with $K_1 \geq K_2$ if $K_1 \subset K_2$. Now for $K \in \mathcal{C}$, let $f_K \in L^1(G)$ be a non-negative function with $||f_K||_1 = 1$ and supp $f_K \subset K$.*

Let $f \in L^1(G)$ and $\varepsilon > 0$ be arbitrary then by the above lemma we can choose $V \in \mathcal{V}(e)$ such that $||f - T_y f|| < \varepsilon$. And we can pick $K \in \mathcal{C}$ with $K \subset V$, and f_K as above. Then;

$(f * f_K)(x) - f(x) = \int_G (f(x - y) - f(x)) f_K(y) dy,$

so that

$||f * f_K - f||_1 \leq \int_G |f_K(y)| \int_G |f(x - y) - f(x)| dx\, dy < \int_G \varepsilon |f_K(y)| dy = \varepsilon$

Hence $(f_K)_{K \in \mathcal{C}}$ is an approximate identity for $L^1(G)$, moreover it is bounded by 1.■

2.2 Dual Group $\widehat{G}$ and Fourier Transform of $L^1(G)$

A **character** on a LCA group G is a continuous homomorphism of G into the multiplicative group of complex numbers of modulus 1, i.e. a continuous complex valued function κ on G with $|\kappa(x)| = 1$ and $\kappa(x + y) = \kappa(x)\kappa(y)$ for all $x, y \in G$. Let $\widehat{G}$ denote the set of all characters on a LCA group G and let define addition on $\widehat{G}$ by;

$(\kappa_1 + \kappa_2)(x) = \kappa_1(x)\kappa_2(x)$ for $\kappa_1, \kappa_2 \in \widehat{G}$ and $x \in G$.

Then;

Lemma 2.2.1 *$\widehat{G}$ is a commutative group, with addition defined as above.*

Proof: *$(\kappa_1 + \kappa_2) \in \widehat{G}$ as*

$$\begin{aligned}(\kappa_1+\kappa_2)(x+y) &= \kappa_1(x+y)\kappa_2(x+y) = \kappa_1(x)\kappa_1(y)\kappa_2(x)\kappa_2(y)\\ &= \kappa_1(x)\kappa_2(x)\kappa_2(y)\kappa_2(y)\\ &= (\kappa_1+\kappa_2)(x)(\kappa_1+\kappa_2)(y)\end{aligned}$$

κ_0 is the identity for $\widehat{G}$ which is defined as $\kappa_0(x) = 1$ for all $x \in G$. So let us denote the identity element κ_0 as $\mathbf{1}$.

Associativity and commutativity follows from properties of complex numbers.

For $\kappa \in \widehat{G}$, inverse is denoted as $-\kappa$ (since it is a commutative group) and defined as $-\kappa(x) = \overline{\kappa(x)}$. ■

For $\kappa \in \widehat{G}$ and $x \in G$, let $< x, \kappa >:= \kappa(x)$, then

$< x+y, \kappa >=< x, \kappa >< y, \kappa >$ and $< x, \kappa_1 + \kappa_2 >=< x, \kappa_1 >< x, \kappa_2 >$.

With these properties it follows that $< 0, \kappa >=< x, \mathbf{1} >= 1$ for all $\kappa \in \widehat{G}$ and $x \in G$. Moreover, for all $x \in G$ and $\kappa \in \widehat{G}$, $< x, \kappa >< -x, \kappa >=< 0, \kappa >= 1$ hence $< -x, \kappa >= \overline{< x, \kappa >} =< x, -\kappa >$.

For $f \in L^1(G)$, define a function on $\widehat{G}$ by $\hat{f}(\kappa) = \int_G f(x) < -x, \kappa > dx$, where $\kappa \in \widehat{G}$. This function $\hat{f}$ is called the **Fourier transform of** f.

Theorem 2.2.2 *For fixed $\kappa \in \widehat{G}$ the function $\Psi : L^1(G) \to \mathbb{C}$ with $\Psi(f) = \hat{f}(\kappa)$ is an algebra homomorphism and $\Psi \neq 0$.*

Conversely, every multiplicative functional of $L^1(G)$ is of this form and distinct characters induce distinct homomorphisms.

Proof:

$$\begin{aligned}\Psi(f+\lambda g) &= \widehat{(f+\lambda g)}(\kappa) = \int_G (f+\lambda g)(x) < -x, \kappa > dx\\ &= \int_G f(x) < -x, \kappa > dx + \lambda \int_G g(x) < -x, \kappa > dx = \hat{f}(\kappa) + \lambda \hat{g}(\kappa)\\ &= \Psi(f) + \lambda\Psi(g)\end{aligned}$$

$$\begin{aligned}
\Psi(f*g) &= \widehat{(f*g)}(\kappa) = \int_G (f*g)(x) < -x, \kappa > dx = \int_G \int_G f(x-y)g(y) < -x, \kappa > dydx \\
&= \int_G \int_G f(x-y)g(y) < -x, \kappa > dxdy \text{ (by Fubini)} \\
&= \int_G \int_G f(x)g(y) < -x-y, \kappa > dxdy \text{ (by translation invariance)} \\
&= \int_G f(x) < -x, \kappa > dx \int_G g(y) < -y, \kappa > dy = \hat{f}(\kappa)\hat{g}\kappa \\
&= \Psi(f)\Psi(g) \\
\Psi = 0 &\Leftrightarrow \Psi(f) = \hat{f}(\kappa) = \int_G f(x) < -x, \kappa > dx = 0, \forall f \in L^1(G) \\
&\Rightarrow f(x) < -x, \kappa >= 0 \text{ a.e. } x \in G, \forall f \in L^1(G) \\
&\Rightarrow < -x, \kappa >= 0 \text{ a.e. } x \in G
\end{aligned}$$

But $| < x, \kappa > | = 1$ *for every* $x \in G$ *and* $\kappa \in \widehat{G}$*, so we have a contradiction, hence* $\Psi \neq 0$.

For the converse, let $\chi : L^1(G) \to \mathbb{C}$ *be a multiplicative functional, then* χ *is a linear functional of norm* 1*, i.e.* $\chi \in (L^1(G))^*$*. Then there is* $\psi \in L^\infty(G)$ *such that* $\chi(f) = \int_G f(x)\psi(x)dx$ *for* $f \in L^1(G)$*, and* $||\psi||_\infty = ||\chi||_\infty = 1$.

For $\kappa_1, \kappa_2 \in \widehat{G}$ *let* $\hat{f}(\kappa_1) = \hat{f}(\kappa_2)$ *for any* $f \in L^1(G)$*, then*

$$\forall f \in L^1(G), \quad \int_G f(x) < -x, \kappa_1 > dx = \int_G f(x) < -x, \kappa_2 > dx$$

$$\text{i.e.} \qquad \int_G f(x)(< -x, \kappa_1 > - < -x, \kappa_2 >)dx = 0$$

Hence $< -x, \kappa_1 >=< -x, \kappa_2 >$ *for a.e.* $x \in G$*, as* κ_1, κ_2 *are continuous, if they differ at a point, they must have different values on an open subset of* G*, but open sets do not have measure zero, hence* $\kappa_1 = \kappa_2$. ■

Now as a result, we can identify the maximal ideal space, $\sigma(L^1(G))$, of $L^1(G)$ with the group $\widehat{G}$. Then, $\hat{f}$ is the Gelfand transform of f by this identification.
Moreover, $\widehat{G}$ is a locally compact (as $L^1(G)$ does not necessarily have a unit) space with weak topology induced by the set $\mathcal{F}(\widehat{G}) =: \{\hat{f} : f \in L^1(G)\}$, which is Gelfand topology.
In addition, by the weak topology on $\widehat{G}$, Fourier transform $\hat{f}$ of any $f \in L^1(G)$ is continuous and by definition $\hat{f}$ is bounded. Also since Gelfand transform is continuous, $||\hat{f}||_\infty \leq ||f||_1$, for every $f \in L^1(G)$.

Lemma 2.2.3 *$< x, \kappa >$ is a continuous function on $G \times \widehat{G}$, ($\widehat{G}$ with above topology.)*
Proof: *Let define $\hat{f}_x(\kappa) = \hat{f}(\kappa) < x, \kappa >$ for $x \in G$ and $\kappa \in \widehat{G}$. Then if $\hat{f}_x(\kappa)$ is continuous on $G \times \widehat{G}$, as $\hat{f}$ is already continuous on $\widehat{G}$(by weak topology), we have $< x, \kappa >$ to be continuous.*

Now fix $x_0 \in G$ and $\kappa_0 \in \widehat{G}$ and let $\varepsilon > 0$. As $\hat{f}_{x_0}$ is continuous on $\widehat{G}$, let W be a neighborhood of κ_0 such that $|\hat{f}_{x_0}(\kappa) - \hat{f}_{x_0}(\kappa_0)| < \varepsilon$ for each $\kappa \in W$.

By the lemma 2.1.3, the map $x \to f_x$ is uniformly continuous, so that there is a neighborhood V of x_0 such that $||f_x - f_{x_0}|| < \varepsilon$ for every $x \in V$. Now as Gelfand transform is continuous, $|\hat{f}_x(\kappa) - \hat{f}_{x_0}(\kappa)| \leq ||f_x - f_{x_0}||$ for every $x \in V$.

Then $|\hat{f}_x(\kappa) - \hat{f}_{x_0}(\kappa_0)| \leq |\hat{f}_x(\kappa) - \hat{f}_{x_0}(\kappa)| + |\hat{f}_{x_0}(\kappa) - \hat{f}_{x_0}(\kappa_0)| < 2\varepsilon$ for every $\kappa \in W$ and $x \in V$, hence $< x, \kappa >$ is continuous on $G \times \widehat{G}$. ■

Now $\widehat{G}$ is a topological space and has a group structure, it is natural to inquire for compatibility of these structures. For this pupose, first we will investigate the topology on $\widehat{G}$: for compact $F \subset G$ and $\varepsilon > 0$, let define

$B(F, \varepsilon) = \{\kappa \in \widehat{G} : |\kappa(x) - \mathbf{1}| < \varepsilon, \forall x \in F\}$.

The family $\{B(F, \varepsilon) : F \subset G$ compact, $\varepsilon > 0\}$ forms a local basis at the identity. Furthermore,

Theorem 2.2.4 *The topology defined by the translations of the above local base is the same as the Gelfand topology of $\widehat{G}$ as the maximal ideal space.*
Proof: *Sufficient to show that $B(F, \varepsilon)$ is open for the locally compact space $\widehat{G}$ and every neighborhood of identity includes some $B(F, \varepsilon)$.*

Let $\kappa_0 \in B(F, \varepsilon)$. As $< x, \kappa >$ is continuous in both components, for each $y \in F$ there is an open neighborhood V_y of y and W_y of κ_0 such that $| < x, \kappa > -1| < \varepsilon$ for $x \in V_y$ and $\kappa \in W_y$. Since F is compact we can choose $y_1, \ldots, y_n \in F$ such that $F \subset \bigcup_i V_{y_i}$. Let $W = \bigcap_i W_{y_i}$, then W is an open neighborhood of κ_0 and $W \subset B(F, \varepsilon)$.

Let $W \in \mathcal{V}(\mathbf{1})$ in $\widehat{G}$, then by definition of Gelfand topology there are functions $f_1, \ldots, f_n \in L^1(G)$ and $\delta > 0$ such that $\{\kappa \in \widehat{G} : |\hat{f}_i(\kappa) - \hat{f}_i(\mathbf{1})| < \delta, \forall i = 1, \ldots, n\}$ is subset of W. Since G is locally compact, $\overline{C_c(G)} = L^1(G)$ and we may assume that

for each $i = 1, \ldots, n$, $suppf_i$ is subset a compact set F. Now let $\varepsilon < \delta / \max_i ||f_i||_1$, then for $\kappa \in B(F, \varepsilon)$,

$$\begin{aligned} |\hat{f}_i(\kappa) - \hat{f}_i(\mathbf{1})| &\leq \int_G | < -x, \kappa > -1||f_i(x)|dx = \int_K | < -x, \kappa > -1||f_i(x)|dx \\ &\leq \varepsilon ||f_i||_1 < \delta \end{aligned}$$

Hence $B(F, \varepsilon) \subset W$.■

Group operations of $\widehat{G}$ are continuous with respect to this topology. Hence $\widehat{G}$ is a topological group, which is LCA. And $\widehat{G}$ is called the **dual group** of G.

Indeed, let $\kappa_\alpha \to \kappa$ and $\chi_\alpha \to \chi$ in the topology defined on $\widehat{G}$. So that for any compact subset K, $\sup_{x \in K} |\kappa_\alpha(x) - \kappa(x)| \to 0$ and $\sup_{x \in K} |\chi_\alpha(x) - \chi| \to 0$. Then as

$$|(\kappa_\alpha + \chi_\alpha)(x) - (\kappa + \chi)(x)| \leq |\kappa_\alpha(x)||\chi_\alpha(x) - \chi(x)| + |\kappa_\alpha(x) - \kappa(x)||\chi(x)|$$

for all $x \in G$, we have $\kappa_\alpha + \chi_\alpha \to \kappa + \chi$.

Also as $\kappa_\alpha^{-1}(x) = \overline{\kappa_\alpha(x)}$ and $|\overline{\kappa_\alpha(x)} - \overline{\kappa(x)}| = |\overline{\kappa_\alpha(x) - \kappa(x)}| = |\kappa_\alpha(x) - \kappa(x)|$, $\sup_{x \in K} |\kappa_\alpha^{-1}(x) - \kappa^{-1}(x)| = \sup_{x \in K} |\kappa_\alpha(x) - \kappa(x)|$, hence inverse operation is also continuous.

Now, as $\widehat{G}$ is LCA group, there are Radon measures defined on this group. In particular, there is a Haar measure of $\widehat{G}$ as well.

Lemma 2.2.5 *Let μ be a complex Radon measure on $\widehat{G}$ and $\int_{\widehat{G}} < x, \kappa > d\mu(\kappa) = 0$ for every $x \in G$, then $\mu = 0$.*

Proof: *Let $f \in L^1(G)$, then*

$$\begin{aligned} \int_{\widehat{G}} \hat{f}(\kappa) d\mu(\kappa) &= \int_{\widehat{G}} \int_G f(x) < -x, \kappa > dx d\mu(\kappa) \\ &= \int_G f(x) \int_{\widehat{G}} < -x, \kappa > d\mu(\kappa) dx = 0 \end{aligned}$$

So for each $f \in L^1(G)$, $\int_{\widehat{G}} \hat{f} d\mu = 0$. As $\mathcal{F}(\widehat{G})$ is a separating subalgebra of $C_0(\widehat{G})$ (by theorem 2.2.2), it is dense in $C_0(\widehat{G})$ by Stone-Weierstrass theorem (A.2.5). Hence $\int_{\widehat{G}} \phi d\mu = 0$ for every $\phi \in C_0(\widehat{G})$, but then $\mu = 0$. ■

For any LCA group G, we have proved that $L^1(G)$ is a commutative Banach algebra with multiplication defined as convolution, in fact more is true:

Theorem 2.2.6 *For any LCA group G, the group algebra $L^1(G)$ is semisimple.*

Proof: *$L^1(G)$ is semisimple iff the Gelfand transform Γ is an injective homomorphism by theorem 1.1.3. Let $f_1, f_2 \in L^1(G)$ and $\Gamma(f_1) = \hat{f}_1 = \hat{f}_2 = \Gamma(f_2)$. Then;*

$$\hat{f}_1(\kappa) = \int_G f_1(x) < -x, \kappa > dx = \hat{f}_2(\kappa) = \int_G f_2(x) < -x, \kappa > dx \text{ for each } \kappa \in \widehat{G}$$

So $f_1(x) = f_2(x)$ for a.e. $x \in G$, being elements of $L^1(G)$, $f_1 = f_2$. ■

Seeing the relation between the group algebra $L^1(G)$ and the dual group $\widehat{G}$, we can now prove one particular result about the duality of LCA groups:

Theorem 2.2.7 *The dual group of any compact group is discrete and the dual group of any discrete group is compact.*

Proof: • *Let G be compact, then we can normalize the Haar measure, so $m(G) = 1$.*

Claim: *$\{\mathbf{1}\}$ is open. Let $f(x) = 1$ for all $x \in G$, as G is compact $f \in L^1(G)$ and*

$$\hat{f}(\mathbf{1}) = \int_G < -x, \mathbf{1} > dx = \int_G dx = 1.$$

For $\kappa \in \widehat{G} \setminus \{\mathbf{1}\}$, there exists $x_0 \in G$ with $< x_0, \kappa > \neq 1$ then $\hat{f}(\kappa) = 0$ as

$$\hat{f}(\kappa) = \int_G < -x, \kappa > dx = \int_G < x_0, \kappa >< -x - x_0, \kappa > dx =< x_0, \kappa > \hat{f}(\kappa)$$

Hence $\hat{f}(\kappa) = 0$ for all $\kappa \neq \mathbf{1}$. Now as $\hat{f}$ is continuous on $\widehat{G}$, $\{\mathbf{1}\}$ is open (and closed), hence $\widehat{G}$ is discrete.

• *Let G be discrete, then $L^1(G)$ is CSS-BA with unit. By the corollary 1.2.4, $\sigma(L^1(G)) \neq \emptyset$ and compact. And by the theorem 2.2.2, $\sigma(L^1(G)) \simeq \widehat{G}$.* ■

Let H be a subset of the LCA group G, the **annihilator** of H, denoted $N(H)$, is defined as $N(H) = \{\kappa \in \widehat{G} :< x, \kappa >= 1, \forall x \in H\}$. $N(H)$ is a subgroup of $\widehat{G}$ as for $\chi_1, \chi_2 \in N(H)$, $< x, \chi_1 - \chi_2 >=< x, \chi_1 >< -x, \chi_2 >= 1$ for each $x \in H$.

For H closed subgroup of G, if $\chi \in N(H)$ then χ is constant on cosets of H, indeed, for $a \in G$, if $x \in \bar{a}$, then $x = a + h$ for some $h \in H$ and $\chi(x) = \chi(a + h) = \chi(a)$ as $\chi(h) = 1$, for each $h \in H$.

Now let $\chi^* : G/H \to \mathbb{C}$ be defined as $\chi^*(\bar{x}) = \chi(x)$ where $x \in \bar{x}$. Then;

Theorem 2.2.8 *The function χ^* defined above is a character of G/H and $N(H)$ is (isomorphic to) the dual group of G/H.*

Proof: *Let $\bar{x}, \bar{y} \in G/H$, then for some $x \in \bar{x}$ and $y \in \bar{y}$, $x + y \in \overline{x+y}$, so $\chi^*(\bar{x}+\bar{y}) = \chi(x+y) = \chi(x)\chi(y) = \chi^*(\bar{x})\chi^*(\bar{y})$. Moreover, $|\chi^*(\bar{x})| = |\chi(x)| = 1$.*
If $f : G \to G/H$ is the natural projection, then $\chi^* \circ f = \chi$. For $W \in \mathcal{V}^\circ(1)$ in $\mathbb{T}$, there is $U \in \mathcal{V}^\circ(e)$ in G, such that $\chi(U) \subset W$ as χ is continuous. Then $\chi^*(f(U)) = \chi(U) \subset W$. As f is an open mapping $f(U) \in \mathcal{V}^\circ(\bar{e})$ in G/H, hence χ^* is continuous.

Now let X^* be the dual group of G/H and $\phi : N(H) \to X^*$ be the map $\phi(\chi) = \chi^*$.
-If $\phi(\chi_1) = \phi(\chi_2)$, then $\chi_1^*(\bar{x}) = \chi_2^*(\bar{x})$ for all $\bar{x} \in G/H$. Then $\chi_1(x) = \chi_2(x')$ for some $x, x' \in \bar{x}$. As χ_1, χ_2 are constant on cosets $\bar{x}$, $\chi_1(x) = \chi_2(x)$ for every $x \in G$.
-For $\chi_1, \chi_2 \in N(H)$, for any $\bar{x} \in G/H$, $(\chi_1+\chi_2)^*(\bar{x}) = (\chi_1+\chi_2)(x)$ for some $x \in \bar{x}$. $(\chi_1+\chi_2)(x) = \chi_1(x)\chi_2(x) = \chi_1^*(\bar{x})\chi^*(\bar{x})$. Hence $\phi(\chi_1+\chi_2) = \phi(\chi_1)\phi(\chi_2)$.
-Let $\kappa \in X^*$ and define $h = \kappa \circ f$, then $h : G \to \mathbb{T}$ is a homomorphism as is composition of homomorphisms. Let $W \in \mathcal{V}^\circ(1)$ in $\mathbb{T}$ then there is some $\tilde{U} \in \mathcal{V}^\circ(\bar{e})$ in G/H with $\kappa(\tilde{U}) \subset W$. By definition of topology of G/H there exists $U \in \mathcal{V}^\circ(e)$ in G with $f(U) = \tilde{U}$, then $h(U) = \kappa(f(U)) \subset \tilde{U}$, hence $h \in \widehat{G}$. For $x \in H$, $h(x) = \kappa(f(x)) = \kappa(\bar{e}) = 1$ so $h \in N(H)$.

Hence ϕ is a group isomorphism.
-Let $F^* \subset G/H$ be compact and $\varepsilon > 0$, if there is some open set $W \in \mathcal{V}(\mathbf{1}) \cap N(H)$ with $\phi^{-1}(B(F^*, \varepsilon)) \subset W$, then ϕ is continuous. Let $U \in \mathcal{V}^\circ(e)$ in G such that $\overline{U}$ is compact. As $f(x+U) \subset G/H$ is open for each $x \in G$ and $F^* \subset G/H$ is compact, there are $x_1, \ldots, x_n \in G$ such that $F^* \subset \bigcup_i^n f(x_i+U)$. Let $F := \bigcup_i^n (x_i+U)$, then $F^* \subset f(F)$. For $\chi \in N(H) \cap B(F, \varepsilon)$, there is $\chi^* \in X^*$ such that $\chi = \chi^* \circ f$, then $\chi^*(F^*) \subset \chi^*(f(F)) = \chi(F)$. And for every $\bar{x} \in F^*$, $|\chi^*(\bar{x}) - \mathbf{1}| = |\chi(x) - \mathbf{1}| < \varepsilon$ for some $x \in \bar{x} \cap F$, so $\chi^* \in B(F^*, \varepsilon)$. Hence $\phi^{-1}(B(F^*, \varepsilon)) \subset B(F, \varepsilon) \cap N(H)$.■

Formerly, we have shown that the topology on the dual group $\widehat{G}$ can be defined by the compact subsets of the group G. Similarly the following sets are open in G for any compact $\tilde{C} \subset \widehat{G}$ and, $\varepsilon > 0$: $\xi(\tilde{C}, \varepsilon) = \{x \in G : | < x, \kappa > -1| < \varepsilon, \forall \kappa \in \tilde{C}\}$.

Indeed let $x_0 \in \xi(\tilde{C}, \varepsilon)$. As $< x, \kappa >$ is continuous in both components, for each

$\kappa \in \tilde{C}$ there is $W_\kappa \in \mathcal{V}(\kappa)$ and $V_\kappa \in \mathcal{V}(x_0)$ such that $| < x, \gamma > -1| < \varepsilon$ for each $x \in V_\kappa$ and $\gamma \in W_\kappa$. Since $\tilde{C}$ is compact there are $\kappa_1, \ldots, \kappa_n \in \tilde{C}$ such that $\tilde{C} \subset \bigcup_i W_{\kappa_i}$. Now let $V = \bigcap_i^n V_{\kappa_i}$, then $V \in \mathcal{V}(x_0)$ and $V \subset \xi(\tilde{C}, \varepsilon)$, hence it is open.

Lemma 2.2.9 *If $\tilde{U}$ is a non-empty open subset of $\widehat{G}$ then there is $\hat{f} \in \mathcal{F}(\widehat{G})$ such that $\hat{f} \neq 0$ and $\hat{f} = 0$ on $\widehat{G} \setminus \tilde{U}$.*

Proof: *Let $\tilde{K} \subset \tilde{U}$ be compact with $m(\tilde{K}) > 0$ and $\tilde{C}$ be a compact neighborhood of* ***1*** *such that $\tilde{K} + \tilde{C} \subset \tilde{U}$. Let $g = \chi_{\tilde{K}}$ and $h = \chi_{\tilde{C}}$, be characteristic functions and set $\hat{f} = g * h$. Then $\hat{f}(\kappa) = (g * h)(\kappa) = \int_{\widehat{G}} g(\kappa - \gamma)h(\gamma)dm(\gamma) = 0$ if $\kappa \notin \tilde{K} + \tilde{C}$, so that supp $\hat{f} \subset \tilde{K} + \tilde{C}$, which is compact hence $\hat{f} \in \mathcal{F}(\widehat{G})$. In particular, $\hat{f} = 0$ outside $\tilde{U}$. And $\hat{f} \neq 0$ as:*

$$\begin{aligned}\int_{\widehat{G}} \hat{f}(\kappa)dm(\kappa) &= \int_{\widehat{G}}\int_{\widehat{G}} g(\kappa - \gamma)h(\gamma)dm(\gamma)dm(\kappa) \\ &= \int_{\widehat{G}} h(\gamma)\int_{\widehat{G}} g(\kappa - \gamma)dm(\kappa)dm(\gamma) = m(\tilde{K})m(\tilde{C}) > 0 \blacksquare\end{aligned}$$

Now, for $\kappa \in \widehat{G}$ and $x \in G$, the mapping $\hat{x} : \kappa \to < x, \kappa >$ defines a character on $\widehat{G}$;

-$\hat{x}(\kappa_1 + \kappa_2) =< x, \kappa_1 + \kappa_2 >=< x, \kappa_1 >< x, \kappa_2 >= \hat{x}(\kappa_1)\hat{x}(\kappa_2)$

-$|\hat{x}(\kappa)| = | < x, \kappa > | = 1$

-And $\hat{x}$ is continuous: (enough to show the continuity at identity as $\widehat{G}$ is a topological group.) Indeed, for $\varepsilon > 0$ and $F \subset G$ compact, $B(F, \varepsilon)$ is open and for $\kappa \in B(x, \varepsilon)$, $|\hat{x}(\kappa) - \mathbf{1}| = | < x, \kappa > -\mathbf{1}| < \varepsilon$, hence $\hat{x}$ is continuous at $\mathbf{1}$.

Moreover, for identity $e \in G$, $\hat{e}(\kappa) =< e, \kappa >= 1$ for any element in $\widehat{G}$.

Question: Are there any other characters for $\widehat{G}$?

Theorem 2.2.10 ***(Pontryagin Duality)*** *If $\widehat{G}$ is the dual group of G then G is (isomorphic to) the dual group of $\widehat{G}$.*

Proof: *Let $\digamma : G \to \widehat{\widehat{G}}$ be the function $\digamma(x) = \hat{x}$, with $\hat{x}$ as defined above. We want to show that $\digamma$ is an isomorphism of topological groups G and $\widehat{\widehat{G}}$.*

-$\digamma(x + y)(\kappa) =< x + y, \kappa >=< x, \kappa >< y, \kappa >= \digamma(x)(\kappa)\digamma(y)(\kappa)$, hence $\digamma$ is a group homomorphism.

-By Gelfand-Raikov theorem (A.2.8), there are enough characters to separate G, hence for $x_1 \neq x_2$ in G, there exists $\kappa \in \widehat{G}$ with $< x_1 - x_2, \kappa > \neq 1$, so $\digamma$ is one-to-one.

-To prove $\digamma$ is continuous let (x_α) be a net in G such that $x_\alpha \to x$ in G. Then for every $\kappa \in \widehat{G}$, $< x_\alpha, \kappa >= \kappa(x_\alpha) \to \kappa(x) =< x, \kappa >$ as κ is continuous. So $\digamma(x_\alpha)(\kappa) \to \digamma(x)(\kappa)$ for any $\kappa \in \widehat{G}$. Moreover, for $\tilde{K} \subset \widehat{G}$ compact, $\digamma(x_\alpha) \to \digamma(x)$ uniformly on $\tilde{K}$. (We can choose $\kappa_1, \ldots, \kappa_n \in \tilde{K}$ whose ε-neighborhoods cover $\tilde{K}$)

-To prove $\digamma^{-1}$ is continuous it is enough to show that it is continuous at the identity. Let $\tilde{K}$ be compact subset of $\widehat{G}$ and $\varepsilon > 0$, then $B(\tilde{K}, \varepsilon)$ is a neighborhood of $\hat{e}$. And $B(\tilde{K}, \varepsilon) = \{\gamma \in \widehat{\widehat{G}} : |\gamma(\kappa) - 1| < \varepsilon, \forall \kappa \in \tilde{K}\}$. Hence,

$$\begin{aligned} \digamma^{-1}(B(\tilde{K}, \varepsilon)) &= \{x \in G : \hat{x} = \gamma \text{ for some } \gamma \in B(\tilde{K}, \varepsilon)\} \\ &= \{x \in G : |\hat{x}(\kappa) - 1| < \varepsilon, \forall \kappa \in \tilde{K}\} = \xi(\tilde{K}, \varepsilon) \end{aligned}$$

As $\xi(\tilde{K}, \varepsilon)$ is open in G, $\digamma$ is an isomorphism and homeomorphism of G into $\widehat{\widehat{G}}$.

-$\digamma$ is onto: enough to show $\digamma(G)$ is closed and dense in $\widehat{G}$.

Since $\digamma(G)$ is locally compact it is open. But open subgroups are closed, hence $\digamma(G)$ is closed.

If $\digamma(G)$ is not dense in $\widehat{\widehat{G}}$, then there is an element $\gamma_0^* \in \widehat{\widehat{G}}$ and a neighborhood V of that element such that $V \cap \digamma(G) = \emptyset$. Then by above lemma, there is a non-zero $f \in L^1(\widehat{G})$ such that $\hat{f}(\digamma(x)) = 0$ for every $x \in G$, i.e.

$$0 = \int_{\widehat{G}} f(\kappa) < -\kappa, \digamma(x) > dm(\kappa) = \int_{\widehat{G}} f(\kappa) < -x, \kappa > dm(\kappa),$$

then by lemma 2.2.5, $f = 0$, contradicting the previous lemma. ■

Corollary 2.2.11 *Let H be a closed subgroup of G. If $N(H)$ is annihilator of H then H is the annihilator of $N(H)$.*

Proof: *Let Υ be the annihilator of $N(H)$, then by Pontryagin duality $\Upsilon \subset G$ so $\Upsilon = \{x \in G :< x, \kappa >= 1, \forall \kappa \in N(H)\}$. Then by definition of $N(H)$, $H \subset \Upsilon$.*

If $x_0 \in \Upsilon \setminus H$, then G/H is a non-trivial group, hence there is a character ϕ on G/H which is not identically $\mathbf{1}$. Then the map $\chi : x \to \phi(x + H)$ is a character on G, which is not identically $\mathbf{1}$ and $< x, \chi >= 1$ iff $x \in H$. So that $\chi \in N(H)$ but $< x_0, \chi > \neq 1$, contradicting x_0 being in the annihilator Υ.

Corollary 2.2.12 *For H a closed subgroup of G, the annihilator $N(H)$ is a closed subgroup of $\widehat{G}$ and $\widehat{G}/N(H)$ is the dual group of H.*

Proof: *Let $\{\kappa_\alpha\}$ be a net in $N(H)$ with $\kappa_\alpha \to \kappa \in \widehat{G}$, then $\kappa_\alpha \to \kappa$ uniformly on every compact subset of G, so $< x, \kappa_\alpha > \to < x, \kappa >$ for each $x \in G$. Then for $x \in H$, $(< x, \kappa_\alpha >= 1 \to < x, \kappa >) \Rightarrow < x, \kappa >= 1$, hence $\kappa \in N(H)$.*

Now by the theorem 2.2.8, previous corollary and the Pontryagin duality theorem the result follows. ■

Theorem 2.2.13 *If H is a closed subgroup of G, then every character on H can be extended to a character on G.*

Proof: *Let ϕ be a character on H and $N(H)$ be the annihilator of H, then by previous corollaries, H is annihilator of $N(H)$ and $\widehat{G}/N(H) \cong$ the dual group of H. Let $\gamma \in \widehat{G}/N(H)$ and $\phi : \widehat{G} \to \widehat{G}/N(H)$ be the natural onto homomorphism. Then there is $\alpha \in \widehat{G}$ such that $\phi(\alpha) = \alpha + N(H) = \gamma$, so that*

$$< x, \gamma >=< x, \alpha + \beta >=< x, \alpha >< x, \beta >=< x, \alpha >$$

for any $\beta \in N(H)$, and for $x \in H$, hence α is the extension of γ. ■

2.3 Examples

1. *The additive group* $\mathbb{R}$: is the group of real numbers with addition and the usual topology.

For $\alpha \in \mathbb{R}$, let $\chi_\alpha : \mathbb{R} \to \mathbb{T}$ be defined as $\chi_\alpha(x) = e^{i\alpha x}$, then $\chi_\alpha \in \widehat{\mathbb{R}}$ as;

-$\chi_\alpha(x + y) = e^{i\alpha(x+y)} = e^{i\alpha x} e^{i\alpha y} = \chi_\alpha(x)\chi_\alpha(y)$ and

-For $x_n \to x$ in $\mathbb{R}$, $\chi_\alpha(x_n) = e^{i\alpha x_n} \to e^{i\alpha x} = \chi_\alpha(x)$

So identifying each $\alpha \in \mathbb{R}$ with χ_α, $\mathbb{R} \subset \widehat{\mathbb{R}}$.

Now let $\chi \in \widehat{\mathbb{R}}$, and let $N = Ker\chi$, so N is a closed subgroup of $\mathbb{R}$. Then $N = \mathbb{R}$ or $N = \{0\}$ or a proper closed subgroup.

-If $N = \mathbb{R}$, then $\chi = \chi_0$.

-If $N = \{0\}$: Then as $\chi([0, 1])$ is a compact connected subset of $\mathbb{T}$ including 1, so it is either an arc including 1 or the set $\{1\}$. But the latter can not hold because for any $x \in \mathbb{R}$ there is some $n \in \mathbb{Z}$ with $x \in [n, n+1]$ (WLOG all these numbers are ≥ 0), and

as $0 \leq \dfrac{x}{n+1} \leq 1$, $\chi(x/n+1) = 1$, so that $\chi(x) = \chi(x/n+1)^{n+1} = 1$, contradicting $N = \{0\}$. Hence $\chi([0,1])$ is an arc including 1. So that there is $x_0 \in]0,1]$ with $\chi(x_0) = e^{2\pi i/m}$ for some $m \in \mathbb{Z}^*$, thus $\chi(mx_0) = (e^{2\pi i/m})^m = 1$, again contradicting $N = \{0\}$. So if $\chi \neq \chi_0$ then $Ker\chi$ is non-trivial.

-If $N \neq \mathbb{R}$, then by above arguments the set $\{x \in N : x > 0\}$ is not empty and so $t = \min\{x \in N : x > 0\}$ exists. If not, for every $\varepsilon > 0$ there is $x \in N \cap [-\varepsilon, \varepsilon]$ with $x \neq 0$, and every integer multiple of x is in N; thus N is dense and closed in $\mathbb{R}$ so $N = \mathbb{R}$. Hence such t exists and $t\mathbb{Z} \subset N$. If $x \in N \setminus t\mathbb{Z}$, we may assume $x > 0$, then $nt < x < (n+1)t$ for some $n \in \mathbb{N}$, then $x - nt \in N$ and $0 < x - nt < t$, contradicting minimality of t, so $N = t\mathbb{Z}$.

Now, $t > 0$ and $1 = \chi(t) = \chi(t/2)^2$ as $0 < t/2 < t$, $\chi(t/2) = -1$. And similarly $\chi(t/4)^2 = \chi(t/2) = -1$, so that $\chi(t/4) = e^{\pi i/2}$ or $e^{-\pi i/2}$. WLOG say the former holds. Suppose $\chi(t/2^k) = (e^{2\pi i})^{1/2^k}$, hence $\chi(t/2^{k+1}) = e^{2\pi i/2^{k+1}}$ or $\chi(t/2^{k+1}) = e^{2\pi i/2^{k+1}+\pi i}$. Suppose the latter occurs, then $\chi([t/2^k, t/2^{k+1}])$ is a compact connected subset of $\mathbb{T}$ and must contain one of $-1, 1$. Thus for some $y_0 \in]t/2^k, t/2^{k+1}[$, $\chi(y_0) = 1$ or -1 and in either case $\chi(2y_0) = 1$ and $0 < 2y_0 < 2t/2^{k+1} < t$, contradicting minimality of t. Hence $\chi(t/2^k) = e^{2\pi i/2^k}$ for every $k \in \mathbb{N}$, i.e. for any dyadic rational number r, $\chi(tr) = e^{2\pi ir}$, by continuity of χ and the fact $\chi(-x) = \chi(x)^{-1}$, $\chi(tx) = e^{2\pi ix}$ for any $x \in \mathbb{R}$. Hence $\chi(x) = e^{2\pi ix/t} = \chi_\alpha(x)$ with $\alpha = 2\pi/t$.

Now the identification $\alpha \to \chi_\alpha$ is a bijection between $\mathbb{R}$ and $\widehat{\mathbb{R}}$. And it is an isomorphism as $\chi_{\alpha+\beta}(x) = e^{(\alpha+\beta)ix} = e^{\alpha ix}e^{\beta ix} = \chi_\alpha\chi_\beta(x)$. Moreover, the topologies agree:

By theorem 2.2.4, the sets $B(n,r) = \{\chi_\alpha \in \widehat{\mathbb{R}} : |\chi_\alpha(x) - 1| < r, \forall |x| \leq n\}$ form a neighborhood base at $\chi_0 = \mathbf{1}$ with respect to Gelfand topology. And y identified with χ_y is in $B(n,r)$ iff $|e^{yxi} - 1| < r$, for every $|x| \leq n$, and this holds if and only if $|y| < (2/n)\sin^{-1}(r/2)$, which is an open subset of $\mathbb{R}$.

Now the Fourier transform of a function $f \in L^1(\mathbb{R})$ is
$\hat{f}(x) = \displaystyle\int_{\mathbb{R}} f(y)\chi_x(y)^{-1}dy = \int_{\mathbb{R}} f(y)e^{-ixy}dy$ for $x \in \mathbb{R}$.

2. *The circle group* $\mathbb{T}$: is the multiplicative group of all complex numbers of absolute value 1. Also $\mathbb{T} = \{e^{i\theta} : \theta \in [0, 2\pi]\}$. $\mathbb{T}$ has the natural subspace topology, and since the multiplication and inversion are continuous on $\mathbb{C} \setminus \{0\}$, the circle group has the structure of a topological group. Moreover it is compact.

• For $n \in \mathbb{Z}$ define $\alpha_n : \mathbb{T} \to \mathbb{T}$ as $\alpha_n(e^{i\theta}) = e^{in\theta}$, then $\alpha_n \in \widehat{\mathbb{T}}$ as:

-$\alpha_n(e^{i\theta}e^{i\gamma}) = \alpha_n(e^{i(\theta+\gamma)}) = e^{in(\theta+\gamma)} = e^{in\theta}e^{in\gamma} = \alpha_n(e^{i\theta})\alpha_n(e^{i\gamma})$ and

-For (x_m) a sequence in $\mathbb{T}$, $x_m = e^{i\theta_m}$ for some $\theta_m \in [0, 2\pi]$. If $x_m \to x$, then $x = e^{i\theta}$ for some $\theta \in [0, 2\pi]$ and $\theta_m \to \theta$. Then;

$\alpha_n(e^{i\theta_m}) = e^{in\theta_m} = (e^{i\theta_m})^n \to (e^{i\theta})^n = e^{in\theta} = \alpha_n(e^{i\theta})$, i.e. $\alpha_n(x_m) \to \alpha_n(x)$.

So identifying each $n \in \mathbb{Z}$ with α_n, $\mathbb{Z} \subset \widehat{\mathbb{T}}$.

• Now let $\alpha \in \widehat{\mathbb{T}}$ and $N = Ker\alpha$.

-If N is infinite, then it contains a limit point (as $\mathbb{T}$ is compact). Then there are arbitrarily close elements $e^{i\theta}, e^{i\gamma} \in N$, so $e^{i(\theta-\gamma)} \in N$ is arbitrarily close to $1 \in \mathbb{T}$. As $e^{in(\theta-\gamma)} \in N$ for each n, N is dense in $\mathbb{T}$, but N is closed so $N = \mathbb{T}$, i.e. $\alpha = \alpha_0$.

-Now if $\alpha \neq \alpha_0$, then $N = Ker\alpha$ is finite, say order n. Then for $e^{i\theta} \in N$, $e^{in\theta} = 1$, so $e^{i\theta}$ is an n^{th}-root of unity in $\mathbb{C}$. Since there are exactly n n^{th}-roots of unity, $N = \{e^{i\theta} : \theta = 2k\pi/n \text{ for } k = 1, \ldots, n\}$. Then by the first isomorphism theorem, there is an isomorphism ϕ from $\mathbb{T}/N$ into $\mathbb{T}$. As $\phi(\mathbb{T}/N)$ is infinite subgroup of $\mathbb{T}$, from above arguments, ϕ is onto. Now, $f : \mathbb{T}/N \to \mathbb{T}$ with $f(e^{i\theta}) = e^{in\theta}$ is an isomorphism and $\phi \circ f^{-1}$ is an automorphism of $\mathbb{T}$, so $\phi \circ f^{-1}(e^{i\theta}) = \phi(e^{i\theta/n}) = e^{i\theta}$ or $e^{-i\theta}$, so $\phi(e^{i\theta}) = e^{ir\theta}$ where $r = \mp n$, i.e. $\alpha = \alpha_n$ or α_{-n}, hence $\widehat{\mathbb{T}} = \mathbb{Z}$ (with the identification above). And the two topologies coincide as in the case of $\mathbb{R}$.

Now the Fourier transform of a function $f \in L^1(\mathbb{T})$ is

$$\hat{f}(n) = \int_{\mathbb{T}} f(y)\alpha_n(y)^{-1}dy = \frac{1}{m([-\pi,\pi])}\int_{[-\pi,\pi]} f(e^{i\theta})e^{-in\theta}d(\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\theta})e^{-in\theta}d(\theta)$$

for $n \in \mathbb{Z}$.

3. *The additive group* $\mathbb{Z}$: is the additive group of integers with discrete topology.

Let $\chi \in \widehat{\mathbb{Z}}$. Then as $\mathbb{Z}$ is cyclic with generator 1 and $\chi(n) = \chi(1)^n$ for any $n \in \mathbb{Z}$, χ is determined by the value of $\chi(1)$. And $\chi(1)$ can take any value $\alpha \in \mathbb{T}$ since $\{\alpha^n : n \in \mathbb{Z}\}$ is a cyclic subgroup of $\mathbb{T}$ for any $\alpha \in \mathbb{T}$. Let denote the character χ

with χ_α if $\chi(1) = \alpha$. Then the map $\alpha \to \chi_\alpha$ is bijective and an isomorphism from $\mathbb{T}$ onto $\widehat{\mathbb{Z}}$ as $\chi_{\alpha\beta}(1) = \alpha\beta = \chi_\alpha(1)\chi_\beta(1)$. Moreover, it is continuous:
Indeed, for finite $F \subset \mathbb{Z}$ and $r > 0$, identifying y with χ_y,

$y \in B(F, r) = \{\chi_\alpha \in \widehat{\mathbb{Z}} : |\chi_\alpha(n) - 1| < r, \forall n \in F\}$ iff $|y^n - 1| < r$ for each $n \in F$ which holds if $|y - 1|$ is small enough.

Hence the map is homeomorphism, and the dual group of $\mathbb{Z}$ is $\mathbb{T}$.

Now the Fourier transform of a function $f \in L^1(\mathbb{Z})$ is of the form

$$\hat{f}(x) = \int_{\mathbb{Z}} f(n)\chi_x(n)^{-1} dn = \sum_{n\in\mathbb{Z}} f(n)x^{-n}$$

for $x \in \mathbb{T}$.

Chapter 3

MEASURE ALGEBRA $M(G)$ OF A LOCALLY COMPACT GROUP G

3.1 Definition and Properties of $M(G)$

For a LCA group G, let $M(G)$ be the set of all complex-valued Radon measures μ on G. On $M(G)$ we can define a norm by the total variation of the measure, i.e. $||\mu|| = |\mu|(G) = \sup \sum |\mu(E_i)|$ where the supremum taken over all Borel measurable partitions of G. And by the Hahn-Jordan decomposition(A.2.4), $||\mu|| < \infty$ for all $\mu \in M(G)$ ([14],6.5).

In particular, the Haar measure m on G is in $M(G)$ iff G is compact.

$M(G)$ is a linear space with the following algebraic operations:

$(\mu_1 + \mu_2)(E) = \mu_1(E) + \mu_2(E)$ and $(\alpha\mu)(E) = \alpha\mu(E)$

for every Borel set E, and $\alpha \in \mathbb{C}$.

Moreover, it is Banach space with the norm above:

Indeed, if $(\mu_n)_n$ is a Cauchy sequence in $M(G)$, then $||\mu_n - \mu_m|| < \varepsilon$ for sufficiently large m, n. As $|\,||\mu_n|| - ||\mu_m||\,| \leq ||\mu_n - \mu_m||$, the sequence $||\mu_n||$ is Cauchy, hence convergent in $\mathbb{R}$, to some α.

Now for $f \in C_0(G)$, define $\Phi f = \lim\limits_{n\to\infty} \int_G f d\mu_n$, then Φ is a linear. And

$$||\Phi f|| \leq \lim_{n\to\infty} \int_G |f| d|\mu_n| \leq \lim_{n\to\infty} \int_G ||f||_\infty d|\mu_n| = \lim_{n\to\infty} ||f||_\infty ||\mu_n|| = \alpha ||f||_\infty$$

Hence Φ is a bounded linear functional on $C_0(G)$, and $||\Phi|| \leq \alpha$. Now by the Riesz theorem (A.2.2), there is a unique measure $\mu \in M(G)$ such that $\Phi f = \int_G f d\mu$ and $||\Phi|| = ||\mu|| < \infty$. Hence for every measurable E, $\Phi\chi_E = \mu(E) = \lim_{n\to\infty} \mu_n(E)$, i.e. μ_n converges in $M(G)$.

Let $\mu, \lambda \in M(G)$. Let $\mu \times \lambda$ be the product measure on the product space $G \times G$. Associate to each Borel set E in G the set $E_2 = \{(x, y) \in G \times G : x + y \in E\}$, then E_2 is a Borel set in product space and we define the **convolution of μ and λ** by $(\mu * \lambda)(E) = (\mu \times \lambda)(E_2)$.

We can write the convolution in the following way (by Fubini's theorem):

$$\begin{aligned}(\mu * \nu)(E) &= (\mu \times \nu)(E_2) = \int_{G\times G} \chi_{E_2} d(\mu \times \nu) \\ &= \int_G \int_G \chi_{E_2}(x, y) d\mu(x) d\nu(y) = \int_G \mu(E - y) d\nu(y) \\ &= \int_G \int_G \chi_{E_2}(x, y) d\nu(y) d\mu(x) = \int_G \nu(E - x) d\mu(x)\end{aligned}$$

And also for a Borel set E in G,

$$(\mu * \nu)(E) = \textstyle\int_G \chi_E d(\mu * \nu) = \displaystyle\int_G \int_G \chi_E(x + y) d\mu(x) d\nu(y).$$

Let now f be a simple function, being a finite linear combination of some characteristic functions of Borel sets

$$\int_G f d(\mu * \nu) = \int_G \int_G f(x + y) d\mu(x) d\nu(y)$$

But then this formula is true for any bounded Borel function f by Dominated Convergence theorem.

Theorem 3.1.1 *$M(G)$ is a commutative Banach algebra with unit if algebraic operations are defined as above.*

Proof: *$M(G)$ is already a Banach space with the algebraic operations defined above.*
1) Let $\mu_1, \mu_2 \in M(G)$ and $E \subset G$ be a Borel set with the disjoint union $E = \cup E_i$ then

$$\begin{aligned}(\mu_1 * \mu_2)(E) &= \int_G \int_G \chi_E(x + y) d\mu_1(x) d\mu_2(y) = \int_G \int_G \sum_i \chi_{E_i}(x + y) d\mu_1(x) d\mu_2(y) \\ &= \sum_i \int_G \int_G \chi_{E_i}(x + y) d\mu_1(x) d\mu_2(y) = \sum_i (\mu_1 * \mu_2)(E_i)\end{aligned}$$

As $\mu_1, \mu_2 \in M(G)$ are regular Borel measures, $\mu_1 \times \mu_2$ is a regular Borel measure. So for a Borel set E and $\varepsilon > 0$, there is a compact subset $K \subset G \times G$ such that $|\mu_1 \times \mu_2(E_2) - \mu_1 \times \mu_2(K)| < \varepsilon$. Now as the group operation is continuous, the set $C = \{x + y : (x, y) \in K\}$ is a compact subset of E and (as $C_2 \supset K$)

$$\begin{aligned}|(\mu_1 * \mu_2)(E) - (\mu_1 * \mu_2)(C)| &= |\mu_1 \times \mu_2(E_2) - \mu_1 \times \mu_2(C_2)| \\ &\leq |\mu_1 \times \mu_2(E_2) - \mu_1 \times \mu_2(K)| < \varepsilon\end{aligned}$$

Now it is enough to show that $\mu_1 * \mu_2$ is finite on compact sets to show it is regular, so let $K \subset G$ be a compact set, then

$$(\mu_1 * \mu_2)(K) = \int_G \mu_1(K-y)d\mu_2(y) = \mu_1(K)\int_G d\mu_2(y) \leq \mu_1(K)||\mu_2|| < \infty.$$

Hence $\mu_1 * \mu_2 \in M(G)$.

2) Convolution is commutative as $(\mu * \nu)(E) = \int_G \mu(E-y)d\nu(y) = \int_G \nu(E-x)d\mu(x)$ *and associative as:*

$$\begin{aligned}
[(\mu_1 * \mu_2) * \mu_3](E) &= \int_G (\mu_1 * \mu_2)(E-y)d\mu_3(y) \\
&= \int_G \int_G \mu_1(E-y-x)d\mu_1(x)d\mu_3(y) \text{ by commutativity} \\
&= \int_G \int_G \mu_1(E-y-x)d\mu_3(y)d\mu_1(x) \text{ by Fubini} \\
&= \int_G (\mu_2 * \mu_3)(E-x)d\mu_1(x) = [\mu_1 * (\mu_2 * \mu_3)](E)
\end{aligned}$$

3) Let f be a Borel function bounded by 1 then

$$\begin{aligned}
\left|\int_G f d(\mu_1 * \mu_2)\right| &\leq \int_G \left|\int_G f(x+y)d\mu_1(x)\right| d|\mu_2|(y) \\
&\leq \int_G \int_G |f(x+y)|d|\mu_1|(x)\, d|\mu_2|(y) \leq \int_G ||\mu_1||\, d|\mu_2|(y) \\
&\leq ||\mu_1||\,||\mu_2||
\end{aligned}$$

Hence $||\mu_1 * \mu_2|| \leq ||\mu_1||\,||\mu_2||$

4) Let for E a Borel set, $\upsilon_0(E) = 1$ if $e \in E$ and 0 if $e \notin E$. Then $\upsilon_0 \in M(G)$. And $(\mu * \upsilon_0)(E) = (\mu \times \upsilon_0)(E_2) = \sum_{x \in G} \mu(E_x)\upsilon_0(\{x\})$ where $E_x = \{y \in G : x + y \in E\}$, as $\upsilon_0(\{x\}) \neq 0$ iff $x = e$, and $E_e = E$, so $(\mu * \upsilon_0)(E) = \mu(E)$. Hence υ_0 is identity of $M(G)$. ■

The group algebra $L^1(G)$ can be identified with a subset of $M(G)$ with the identification as follows:
For $f \in L^1(G)$, define $\mu_f(E) = \int_E f dm$, then $\mu_f \in M(G)$ is an absolutely continuous measure with respect to Haar measure and by Radon-Nikodym theorem (A.2.3) this identification is bijective between $L^1(G)$ and absolutely continuous measures with respect to m on G. Moreover, $||f|| = ||\mu_f|| = \int_G |f| dm$.

For $\mu_f, \mu_g \in L^1(G)$, $\mu_f + \alpha\mu_g \in L^1(G)$ as $L^1(G)$ is a linear space itself. $L^1(G)$ is also closed in the Banach space $M(G)$, as it is already a Banach algebra itself.

Moreover;

Theorem 3.1.2 *$L^1(G)$ is a closed ideal of $M(G)$.*

Proof: *$L^1(G)$ is already a closed subspace of $M(G)$. So let $\mu, \nu \in M(G)$, with μ absolutely continuous (so there is $f \in L^1(G)$ identified with μ). Let E be a Borel set with $m(E) = 0$. Then $(\mu * \nu)(E) = \int_G \mu(E - x)d\nu(x)$ as being translation invariant $m(E - x) = 0$, so $\mu(E - x) = 0$ for each $x \in G$, hence $(\mu * \nu)(E) = 0$, i.e. $\mu * \nu$ is absolutely continuous with respect to m.* ■

3.2 Fourier-Stieltjes Transform of $M(G)$

For $\mu \in M(G)$, the **Fourier-Stieltjes Transform** of μ, denoted as $\hat{\mu}$, is the function defined on $\widehat{G}$ by $\hat{\mu}(\kappa) = \int_G < -x, \kappa > d\mu(x)$ for $\kappa \in \hat{G}$.

Then for $\nu, \mu \in M(G)$,

$$\begin{aligned}(\widehat{\mu + \alpha\nu})(\kappa) &= \int_G < -x, \kappa > d(\mu + \alpha\nu)(x) \\ &= \int_G < -x, \kappa > d\mu(x) + \int_G < -x, \kappa > \alpha d\nu(x) = \hat{\mu}(\kappa) + \alpha\hat{\nu}(\kappa)\end{aligned}$$

Theorem 3.2.1 *Let $\mathcal{B}(\widehat{G}) = \{\hat{\mu} : \mu \in M(G)\}$.*

1) Each $\hat{\mu} \in \mathcal{B}(\widehat{G})$ is bounded and uniformly continuous.

2) If $\sigma = \mu * \lambda$ for $\mu, \lambda \in M(G)$ then $\hat{\sigma} = \hat{\mu}\hat{\lambda}$.

3) $\mathcal{B}(\widehat{G})$ is invariant under translation, under multiplication by $< x, \kappa >$ for all $x \in G$ and complex conjugation.

Proof: *1) $|\hat{\mu}(\kappa)| = \left| \int_G < -x, \kappa > d\mu(x) \right| \leq \int_G | < -x, \kappa > |d|\mu|(x) = ||\mu||$, hence $\hat{\mu}$ is bounded.*

For $\varepsilon > 0$, as G is LCA group, there is a compact subset K such that $\mu(G \backslash K) < \varepsilon$. Then $B(K, \varepsilon) = \{\chi \in \widehat{G} : | < x, \kappa - \chi > | < \varepsilon\}$ is a neighborhood for κ and then for $\chi \in B(K, \varepsilon)$,

$$\begin{aligned}|\hat{\mu}(\kappa) - \hat{\mu}(\chi)| &\leq \int_G | < -x, \kappa - \chi > |d|\mu|(x) \\ &= \int_K | < -x, \kappa - \chi > |d|\mu|(x) + \int_{G\backslash K} | < -x, \kappa - \chi > |d|\mu|(x) \\ &< \varepsilon|\mu|(K) + \varepsilon \leq \varepsilon(||\mu|| + 1)\end{aligned}$$

hence $\hat{\mu}$ is uniformly continuous.

$$\begin{aligned} 2)\,\hat{\sigma}(\kappa) &= \int_G < -x, \kappa > d\sigma(x) = \int_G < -x, \kappa > d(\mu * \lambda)(x) \\ &= \int_G < -x, \kappa > \int_G d\mu(x-y)d\lambda(y) \\ &= \int_G < -(x-y), \kappa > d\mu(x-y) \int_G < -y, \kappa > d\lambda(y) = \hat{\mu}(\kappa)\hat{\lambda}(\kappa) \end{aligned}$$

3) For $\kappa_0 \in \widehat{G}$ fixed,

$$\begin{aligned} \hat{\mu}(\kappa - \kappa_0) &= \int_G < -x, \kappa - \kappa_0 > d\mu(x) \text{ letting } d\nu(x) =< x, \kappa_0 > d\mu(x) \\ &= \int_G < -x, \kappa >< x, \kappa_0 > d\mu(x) = \int_G < -x, \kappa > d\nu(x) = \hat{\nu}(\kappa) \end{aligned}$$

As then $\nu \in M(G)$, $\mathcal{B}(\widehat{G})$ is invariant under translations.

For $y \in G$ fixed let $\nu(E) = \mu(E - y)$ then $\nu \in M(G)$, $d\nu(x) = d\mu(x - y)$ and

$$\begin{aligned} \hat{\nu}(\kappa) &= \int_G < -x, \kappa > d\nu(x) = \int_G < -x, \kappa > d\mu(x-y) \\ &= \int_G < y, \kappa >< -(x-y), \kappa > d\mu(x-y) =< y, \kappa > \hat{\mu}(\kappa) \end{aligned}$$

Hence $\mathcal{B}(\widehat{G})$ is invariant under multiplication by $< y, \kappa >$ for any $y \in G$.

Now let $\nu(E) = \overline{\mu(-E)}$, then $\nu \in M(G$, $d\nu(x) = d\overline{\mu}(-x)$ and

$$\begin{aligned} \hat{\nu}(\kappa) &= \int_G < -x, \kappa > d\nu(x) = \int_G < -x, \kappa > d\overline{\mu}(-x) = \int_G < x, \kappa > d\overline{\mu}(x) \\ &= \int_G \overline{< -x, \kappa >} d\overline{\mu}(x) = \int_G \overline{< -x, \kappa > d\mu(x)} = \overline{\hat{\mu}(\kappa)} \end{aligned}$$

Hence the conjugate of $\hat{\mu}$ is in $\mathcal{B}(\widehat{G})$. ■

Hence $\mathcal{B}(\widehat{G})$ is an algebra of continuous functions on $\widehat{G}$. And the mapping $\mu \to \hat{\mu}$ is an isomorphism of $M(G)$ onto $\mathcal{B}(\widehat{G})$. Furthermore, a measure on G is determined by its Fourier-Stieltjes transform:

Lemma 3.2.2 *For $\mu \in M(G)$, if $\hat{\mu}(\kappa) = 0$ for every $\kappa \in \widehat{G}$, then $\mu = 0$.*

Proof: *By Pontryagin Duality theorem G is the dual of the LCA group $\widehat{G}$ and*

$\hat{\mu}(\kappa) = \int_G < -x, \kappa > d\mu(x) = 0$ *for each $\kappa \in \widehat{G}$.*

Now let m be the Haar measure on $\widehat{G}$ and $\phi \in L^1(\widehat{G})$, then the Fourier transform $\hat{\phi}$ is a function on the dual G and

$$\begin{aligned} \int_G \hat{\phi}(x)d\mu(x) &= \int_G \int_{\widehat{G}} \phi(\kappa) < -x, \kappa > dm(\kappa)d\mu(x) \\ &= \int_{\widehat{G}} \phi(\kappa) \left[\int_G < -x, \kappa > d\mu(x) \right] dm(\kappa) = 0 \text{ (by Fubini)} \end{aligned}$$

Now let $\mathcal{F}(G) = \{\hat{\phi} : \phi \in L^1(\widehat{G})\}$, by theorem 2.2.2, $\mathcal{F}(G)$ is separating, so by Stone-Weierstrass theorem it is dense in $\mathcal{C}_0(G)$, hence $\int_G \phi d\mu = 0$ for every $\phi \in \mathcal{C}_0(\widehat{G})$, and thus $\mu = 0$ by Riesz Representation(A.2.2). ■

Now $M(G)$ has a compact maximal ideal space, $\sigma(M(G))$, as it is a Banach algebra with unit. The Gelfand transform on $M(G)$, $\Gamma : M(G) \to C(\sigma(M(G)))$, is defined by $\Gamma(\mu)(h) = h(\mu)$, for h a non-zero complex homomorphism of $M(G)$.

For a fixed $\kappa \in \widehat{G}$, the mapping $h_\kappa : M(G) \to \mathbb{C}$, $h_\kappa(\mu) = \hat{\mu}(\kappa)$ is a complex homomorphism of $M(G)$ as;

$h_\kappa(\mu + \alpha\nu) = \widehat{(\mu + \alpha\nu)}(\kappa) = \hat{\mu}(\kappa) + \alpha\hat{\nu}(\kappa) = h_\kappa(\mu) + \alpha h_\kappa(\nu)$

$h_\kappa(\mu * \nu) = \widehat{(\mu * \nu)}(\kappa) = \hat{\mu}(\kappa)\hat{\nu}(\kappa) = h_\kappa(\mu)h_\kappa(\nu)$ and

$h_\kappa(\upsilon_0) = \hat{\upsilon_0}(\kappa) = \int_G < -x, \kappa > d\upsilon_0(x) =< -e, \kappa >= 1.$

Moreover, the map $\kappa \to h_\kappa$ is a homeomorphism of $\widehat{G}$ into $\sigma(M(G))$ as $\kappa \in \widehat{G}$ are continuous and $\hat{\mu} \in \mathcal{B}(\widehat{G})$ are uniformly continuous. Thus we can identify $\widehat{G}$ with a subspace of $\sigma(M(G))$.

In particular, when G is discrete, the homeomorphism above is onto; as any $\mu \in M(G)$ is absolutely continuous on a discrete space, so $M(G) = L^1(G)$, and the maximal ideal space of $L^1(G)$ is $\widehat{G}$ by theorem 2.2.2.

Now restricting the Gelfand transform on $M(G)$ to $\widehat{G}$, $\Gamma(\mu)(h_\kappa) = h_\kappa(\mu) = \hat{\mu}(\kappa)$, we get the Fourier- Stieltjes transform on $\widehat{G}$. Hence without confusion, we can assume $\Gamma(\mu) = \hat{\mu}$ for any $\mu \in M(G)$. As $L^1(G)$ is an ideal of $M(G)$, $\Gamma(L^1(G)) = \mathcal{F}(\widehat{G})$ is an ideal of $\mathcal{B}(\widehat{G})$.

Theorem 3.2.3 *$M(G)$ is a CSS-BA (with unit).*

Proof: *$M(G)$ is already a commutative Banach algebra. Let $\Gamma(\mu) = \hat{\mu} = 0$ for some $\mu \in M(G)$, then by the lemma 3.2.2, $\mu = 0$, hence by the theorem 1.1.3, $M(G)$ is semisimple.* ■

3.3 The Coset Ring $\Re(\widehat{G})$ of $\widehat{G}$

A measure μ on G is **concentrated on** a subset H of G if for every Borel set E with $E \cap H = \emptyset$, $\mu(E) = 0$. For $\mu \in M(G)$, the **support group of** μ is the smallest closed subgroup of G on which μ is concentrated.

Let $\mu \in M(G)$ and H be a closed subgroup which contains the support group of μ, then one can regard $\mu \in M(H)$. And if $\nu \in M(H)$ letting $\nu(E) = 0$ for every Borel set $E \subset G$ with $E \cap H = \emptyset$, we have $\nu \in M(G)$. Moreover, if H contains the support of two measures μ, ν, then H also contains the support of $\mu * \nu$: indeed, as ν is concentrated on H, $(\mu * \nu)(E) = \int_G \mu(E - x)d\nu(x) = \int_H \mu(E - x)d\nu(x)$.
And if $H \cap E = \emptyset$, then $H \cap (E - x) = \emptyset$ for any $x \in H$ (as H is subgroup), hence $(\mu * \nu)(E) = 0$, i.e. support of $\mu * \nu$ is in H.

Theorem 3.3.1 *Let H be a closed subgroup of G and $\mu \in M(G)$. μ is concentrated on H iff $\hat{\mu}$ is constant on each coset of $N(H)$, the annihilator of H.*

Proof: *($\Rightarrow$) Suppose μ is concentrated on H and let $\kappa_0 \in N(H)$, then $< x, \kappa_0 >= 1$ for $x \in H$ and $< -x, \kappa_0 > d\mu(x) = d\mu(x)$. So for $\kappa \in \widehat{G}$ and for $\kappa_0 \in N(H)$,*

$$\begin{aligned}\hat{\mu}(\kappa + \kappa_0) &= \int_G < -x, \kappa + \kappa_0 > d\mu(x) = \int_G < -x, \kappa >< -x, \kappa_0 > d\mu(x) \\ &= \int_G < -x, \kappa > d\mu(x) = \hat{\mu}(\kappa)\end{aligned}$$

($\Leftarrow$) Suppose $\hat{\mu}(\kappa + \kappa_0) = \hat{\mu}(\kappa)$ for any $\kappa \in \widehat{G}$ and any $\kappa_0 \in N(H)$. Then

$$\int_G < -x, \kappa + \kappa_0 > d\mu(x) = \int_G < -x, \kappa > d\mu(x)$$

for any $\kappa \in \widehat{G}$ and any $\kappa_0 \in N(H)$.

So for any $\kappa_0 \in N(H)$, $< -x, \kappa_0 >= 1$ for a.e. $x \in G$. As $N(H)$ is annihilator of H and $\kappa_0 \in N(H)$, if $< x, \kappa_0 > \neq 1$ then $x \notin H$, so $\mu(G \setminus H) = 0$, i.e. μ is concentrated on H. ■

The **coset ring of** $\widehat{G}$ is the smallest ring of subsets of $\widehat{G}$ which contains all open cosets of $\widehat{G}$, denoted by $\Re(\widehat{G})$. The coset ring of $\widehat{G}$ is an important tool in characterization of idempotent measures and homomorphisms between group algebras. And the sets in the coset ring may be in some complicated finite union and intersection form, however there are some rather simpler form of sets in $\Re(\widehat{G})$:

Lemma 3.3.2 *Every open compact subset of $\widehat{G}$ belongs to $\Re(\widehat{G})$.*

Proof: *Claim: If $\Omega \subset \widehat{G}$ is open and compact then it is a finite union of cosets of open subgroups of $\widehat{G}$.*

Now, as $\widehat{G}$ is a topological group, for each $\chi \in \Omega$, there exists a symmetric neighborhood $V_\chi \in \mathcal{V}^\circ(\mathbf{1})$ such that $\chi + V_\chi + V_\chi \subset \Omega$. As Ω is compact and $\Omega = \bigcup_{\chi\in\Omega}(\chi + V_\chi)$ then $\Omega = \bigcup_{i=1}^n(\chi_i + V_{\chi_i})$ for some χ_i. Now let $V = \bigcap_{i=1}^n V_{\chi_i}$ then $V \in \mathcal{V}^\circ(\mathbf{1})$ is symmetric. And if H is the subgroup of $\widehat{G}$ generated by V, then H is open as $\kappa + V \subset H$, for each $\kappa \in H$.

For any $\kappa \in \Omega$, $\kappa \in \chi_i + V_{\chi_i}$ for some $i = 1, \ldots, n$ and for any $\omega \in V$,

$$\kappa + \omega \in \chi_i + V_{\chi_i} + V \subset \chi_i + V_{\chi_i} + V_{\chi_i} \subset \Omega, \text{ so } \Omega + V = \Omega.$$

Then for each $\kappa \in \Omega$, $\kappa + H$, which is a coset of an open subgroup, is a subset of Ω. And as Ω is compact there are finitely many elements κ_i, $i = 1, \ldots, m$ such that $\Omega = \bigcup_{i=1}^m(\kappa_i + H)$. ■

Lemma 3.3.3 *E is a coset in $\widehat{G}$ iff $E + E - E \subset E$.*

Proof: *Let E be coset of a subgroup Θ in $\widehat{G}$, then $E = \gamma + \Theta$ for some $\gamma \in \widehat{G}$, so $E - E = \Theta$. Hence $E = E + \Theta = E + E - E$.*

Suppose conversely, $E+E-E \subset E$. Let $\Theta = E-E$ and fix $\kappa_0 \in E$, then $\kappa-\kappa_0 \in \Theta$ for all $\kappa \in E$, so $E \subset \kappa_0 + \Theta = \kappa_0 + E - E \subset E$, the last inclusion following from the assumption. If $\gamma_1, \gamma_2 \in \Theta$, then $\gamma_1 - \gamma_2 = (\gamma_1 + \kappa_0) - (\kappa_0 + \gamma_2) \in E - E = \Theta$, hence Θ is a subgroup.

3.4 The Idempotent Elements of $M(G)$

A measure $\mu \in M(G)$ is idempotent if $\mu * \mu = \mu$, let $\mathcal{J}(G)$ denote the set of all idempotent elements of $M(G)$. For $\mu \in \mathcal{J}(G)$ non-zero, $||\mu|| = ||\mu*\mu|| \leq ||\mu||^2$, hence $||\mu|| \geq 1$.

If $\mu, \nu \in \mathcal{J}(G)$, then so are $\mu * \nu$, $\mu \vee \nu = \mu + \nu - \mu * \nu$, and $\upsilon_0 - \mu$ as:

$$(\mu * \nu)^2 = (\mu * \nu) * (\mu * \nu) = (\mu * \mu) * (\nu * \nu) = \mu * \nu$$

$$\begin{aligned}(\mu + \nu - \mu * \nu)^2 &= \mu + \mu * \nu - \mu * \nu + \nu + \mu * \nu - \mu * \nu - \mu * \nu - \mu * \nu + \mu * \nu \\ &= \mu + \nu - \mu * \nu\end{aligned}$$

$$(\upsilon_0 - \mu)^2 = \upsilon_0^2 - \upsilon_0 * \mu - \upsilon_0 * \mu + \mu^2 = \upsilon_0 - \mu - \mu + \mu = \upsilon_0 - \mu$$

For $\mu \in \mathcal{J}(G)$ as $\hat{\mu^2} = \hat{\mu}$, $\hat{\mu}(\kappa) = 0$ or 1, for each $\kappa \in \widehat{G}$.
Define $S(\mu) = \{\kappa \in \widehat{G} : \hat{\mu}(\kappa) = 1\}$, then $\hat{\mu} = \chi_{S(\mu)}$, the characteristic function of $S(\mu)$. $S(\mu)$ is closed and open subset of $\widehat{G}$ for each $\mu \in \mathcal{J}(G)$ as: for every $\mu \in M(G)$, $\hat{\mu}$ is uniformly continuous; in particular $\hat{\mu} : \widehat{G} \to D$ is continuous for $\mu \in \mathcal{J}(G)$, $D = \{0, 1\}$ discrete and $S(\mu) = \hat{\mu}^{-1}(\{1\})$, hence it is both open and closed.

Moreover, if $\widehat{G}$ is connected then $S(\mu) = \widehat{G}$ or $\emptyset$, i.e. $\mathcal{J}(G) = \{\upsilon_0, \mathbf{1}\}$.

Lemma 3.4.1 *For $\mu \in \mathcal{J}(G)$, the support group of μ is compact.*

Proof: *Let H be the smallest closed subgroup of G which contains the support group of μ. As then μ is concentrated on H we can consider $H = G$. And so μ is not concentrated on any proper closed subgroup of G, hence by theorem 3.3.1, $\hat{\mu}$ is not constant on any non- trivial closed subgroup of $\widehat{G}$. If for $\kappa \in \widehat{G}$, μ_κ is defined by $d\mu_\kappa(x) =< x, \kappa > d\mu(x)$ then $\mu_\kappa \neq \mu$ for $\kappa \neq \mathbf{1}$. So that*

$$\begin{aligned}\hat{\mu_\kappa}(\chi) &= \int_G < -x, \chi > d\mu_\kappa(x) = \int_G < -x, \chi >< x, \kappa > d\mu(x) \\ &= \int_G < -x, \chi - \kappa > d\mu(x) = \hat{\mu}(\chi - \kappa) = 0 \text{ or } 1\end{aligned}$$

Hence $||\mu_\kappa - \mu|| \geq ||\hat{\mu_\kappa} - \hat{\mu}||_\infty \geq 1$ for $\kappa \neq \mathbf{1}$ (as every $\hat{\mu}$ is bounded above by $||\mu||$).

As G is locally compact there is a compact subset $C \subset G$ with $|\mu|(G \setminus C) < 1/4$. Let $V = \{\gamma \in \widehat{G} : |1- < x, \gamma > | < 1/(3||\mu||), \forall x \in C\}$, then V is an open neighborhood of $\mathbf{1}$ in $\widehat{G}$. For $\gamma \in V$,

$$\begin{aligned}||\mu - \mu_\gamma|| &\leq \int_G |1- < x, \gamma > | \, d|\mu|(x) = \int_C + \int_{G \setminus C} \\ &\leq \int_C \frac{1}{3||\mu||} d|\mu|(x) + \int_{C^c} 2d|\mu|(x) = \frac{1}{3} + \frac{1}{2} < 1\end{aligned}$$

As $||\mu_\kappa - \mu|| \geq 1$ for $\kappa \neq \mathbf{1}$, then $V = \{\mathbf{1}\}$, hence $\widehat{G}$ is discrete, so G is compact by Pontryagin duality and theorem 2.2.7.■

Hence to study idempotent elements in $M(G)$, we can study idempotent elements on a compact group G. Now on let G be compact and so $\widehat{G}$ is discrete. Then, we can link the elements of the coset ring of dual group with the idempotent elements as follows:

Theorem 3.4.2 *For every $\tilde{H} \in \Re(\widehat{G})$ there is some $\mu \in \mathcal{J}(G)$ such that $\tilde{H} = S(\mu)$.*

Proof: *Suppose $\tilde{H} \in \Re(\widehat{G})$, and let $H \subset G$ be the annihilator of $\tilde{H}$, then H is a closed subgroup in G, so it is compact. Let m_H be the Haar measure on H (which is normalized), then $m_H \in M(G)$, as it can be extended to all G still concentrated on H. And, $m_H \in \mathcal{J}(G)$:*

$$\begin{aligned}(m_H * m_H)(E) &= \int_G m_H(E-x)dm_H(x) = \int_H m_H(E-x)dm_H(x) \\ &= \int_H m_H(E)dm_H(x) = m_H(E) \text{ as } m_H(H) = 1\end{aligned}$$

Moreover, $S(m_H) = \tilde{H}$. Indeed, for $\kappa \in \tilde{H}$, $< -x, \kappa >= 1$ for all $x \in H$, and

$$\hat{m}_H(\kappa) = \int_G < -x, \kappa > dm_H(x) = \int_H < -x, \kappa > dm_H(x) = \int_H dm_H = 1$$

For $\kappa \notin \tilde{H}$, there is an $x_0 \in H$ such that $< x_0, \kappa > \neq 1$, so that

$$\begin{aligned}\int_G < -x, \kappa > dm_H(x) &=< x_0, \kappa > \int_G < -x - x_0, \kappa > dm_H(x) \\ &=< x_0, \kappa > \int_G < -x, \kappa > dm_H(x) \text{ (translation invariant)}\end{aligned}$$

As $< x_0, \kappa > \neq 1$, $\int_G < -x, \kappa > dm_H(x) = \hat{m}_H(\kappa) = 0$. ■

For $\tilde{H} = S(\mu)$ as in above theorem, $\tilde{H}$ is an open and closed subset of $\widehat{G}$. And if for some $\kappa \in \widehat{G}$, we define a measure by $\nu = \kappa\mu$ then $d\nu(x) =< x, \kappa > d\mu(x)$ and,

$$\hat{\nu}(\gamma) = \int_G < -x, \gamma > d\nu(x) = \int_G < -x, \gamma - \kappa > d\mu(x) = \begin{cases} 1 & \text{if } \gamma - \kappa \in \tilde{H}, \\ 0 & \text{otherwise.} \end{cases}$$

So, the characteristic function of $\tilde{H} + \kappa \subset \widehat{G}$ is a Fourier-Stieltjes transform of a measure ν, hence $\nu \in \mathcal{J}(G)$. And every coset of $\tilde{H}$ is $S(\nu)$ for some $\nu \in \mathcal{J}(G)$.

Now the converse of this statement also holds, and thus the correspondence between coset ring of the dual group and the idempotent measures will be established. For, this we need the following lemmas (due to Ito and Amemiya, [8]);

The setting for the lemmas:

Fix $\mu \in \mathcal{J}(G)$ and $A = \{\kappa\mu : \kappa \in \widehat{G}, \hat{\mu}(\kappa) \neq 0\}$. Let ν be an accumulation point of A in the weak* topology of $M(G)$ (as the dual of $C(G)$).

Lemma 3.4.3 *For every compact $K \subset G$, either the restriction $\nu|_K = \kappa\mu$ for some $\kappa\mu \in A$ or else $\nu|_K$ and m_K are singular, where m_K is the Haar measure of K.*

Proof: *Since ν is an accumulation point, $\nu|_K$ either coincides with a restriction of $\kappa\mu$ in A to K or it is also an accumulation point of the elements of A restricted to K, in the weak* topology of $M(K)$. In the second case, by a lemma of Rudin ([15], p.66), $\nu|_K$ is singular with respect to the Haar measure of K.* ■

Lemma 3.4.4 *With the above setting $||\nu|| < ||\mu||$.*

Proof: *We want to show $||\nu||/||\mu|| < 1$, so let $0 < \delta < ||\nu||/||\mu||$, then by Radon-Nikodym theorem, there is some $f \in C(G)$ such that $||f||_\infty \leq 1$ and $\int_G f d\nu > \delta||\mu||$. Now the set $\left\{\sigma \in M(G) : Re\left(\int_G f d\sigma\right) > \delta||\mu||\right\}$ is an open subset of $M(G)$.*

As ν is accumulation point of A, there are $\kappa_1, \kappa_2 \in \widehat{G}$ such that $\kappa_1\mu \neq \kappa_2\mu$ and

$$Re\left(\int_G f\kappa_j d\mu\right) > \delta||\mu|| \text{ for both } j = 1, 2.$$

Take θ such that $|\theta| = 1$ and $d\mu = \theta d|\mu|$. Setting $f\kappa_j\theta = g_j + ih_j$ for $j = 1, 2$ then

$$Re\left(\int_G f\kappa_j d\mu\right) = Re\left(\int_G f\kappa_j\theta d|\mu|\right) = \int_G g_j d|\mu| > \delta||\mu||.$$

And since $||\mu|| \geq \left|\int_G (g_j + i|h_j|)d\mu\right| = \left|\int_G g_j d\mu + i\int_G |h_j|d\mu\right|$, hence

$$\int_G |h_j|d\mu \leq \left(||\mu||^2 - \left(\int_G g_j d\mu\right)^2\right)^{1/2} \leq (||\mu||^2 - (\delta||\mu||)^2)^{1/2}.$$

As $|f\kappa_j\theta| \leq 1$, $g_j \leq 1$ and so;

$$\int_G |1 - f\kappa_j\theta|d|\mu| \leq \int_G (1 - g_j)d|\mu| + \int_G |h_j|d|\mu| \leq (1 - \delta)||\mu|| + (1 - \delta^2)^{1/2}||\mu||.$$

Also as $|\kappa_j| = 1$;

$$\int_G |\kappa_j - f\kappa_1\kappa_2\theta|d|\mu| \leq (1 - \delta + (1 - \delta^2)^{1/2})||\mu||.$$

Hence, $||\kappa_1\mu - \kappa_2\mu|| = \int_G |\kappa_1 - \kappa_2|d|\mu| \leq 2(1 - \delta + (1 - \delta^2)^{1/2})||\mu||$.

Also, as $\kappa_1 \neq \kappa_2$, $\hat{\nu_1} \neq \hat{\nu_2}$ for $\nu_j = \kappa_j\mu$ and $\hat{\nu_1} - \hat{\nu_2}$ takes values of 0 or 1, then $||\kappa_1\mu - \kappa_2\mu|| \geq 1$. Thus we have $1 \leq 2(1 - \delta + (1 - \delta^2)^{1/2})||\mu||$, which implies $\delta < 1$.

Hence for any $\delta < ||\nu||/||\mu||$ we have $\delta < 1$, so $||\nu|| < ||\mu||$. ■

Now we can complete the correspondence:

Theorem 3.4.5 *$S(\mu) \in \Re(\widehat{G})$ for every $\mu \in \mathcal{J}(G)$.*

Proof: *Let $\mu \in \mathcal{J}(G)$ and A as in the above setting. Then the closure of A in the weak* topology is compact (by Banach-Alaoglu as $(1/||\mu||)A \subset$ unit ball of $M(G)$) and does not contain 0, as $||\kappa\mu|| \geq 1$ for each $\kappa\mu \in A$. As norm is a continuous convex function in usual topology of $M(G)$, it is lower semi-continuous in the weak* topology, so the norms of the elements of $\overline{A}$ attain the minimum value, say at $\nu \in \overline{A}$.*
If $\int_G \kappa d\nu \neq 0$ for some κ, then $\kappa\nu \in \overline{A}$. So the set $B = \{\kappa\nu : \hat{\nu}(\kappa) \neq 0\} \subset \overline{A}$ and can not include an accumulation point. Since otherwise, this accumulation point is in $\overline{A}$ and by lemma 3.4.4 must have norm less than $||\nu||$, contradicting minimality. Thus B is finite, and so is $S(\nu) = \{\kappa \in \widehat{G} : \hat{\nu}(\kappa) = 1\}$. Hence $S(\nu)$ is compact open, and thus in $\Re(\widehat{G})$ by lemma 3.3.2.

(1) If ν is not an accumulation point of A then $\nu = \gamma\mu$ for some γ, so $S(\mu) = S(\nu) + \gamma$ hence $S(\mu)$ is in the coset ring too.

(2) If ν is an accumulation point of A, then by lemma 3.4.3, as ν and m_K (Haar measure of K) are not singular for any compact subset K, $\nu|_K = (\gamma\mu)|_K$ for some γ. Let $\mu_1 = \mu|_K$, then $\mu = \mu_1 + (\mu - \mu_1)$ and $S(\mu_1) \in \Re(\widehat{K})$. As $||\mu_1|| \geq 1$ and $||\mu|| = ||\mu_1|| + ||\mu - \mu_1||$, $||\mu - \mu_1|| \leq ||\mu|| - 1$ and $S(\mu) = S(\mu_1) \cup S(\mu - \mu_1)$.
Now taking $\mu' = \mu - \mu_1$, μ' is idempotent and so we can apply the above arguments for μ'. If (1) holds then we are done as finite union of elements of $\Re(\widehat{G})$ is in the coset ring. Otherwise, (2) holds and $\mu' = \mu_2 + (\mu' - \mu_2) = \mu_2 + \mu''$ with $||\mu''|| \leq ||\mu|| - 2$. Since $||\mu||$ is finite and every idempotent measure has measure ≥ 1, we can have at most finitely many steps and at last we have $S(\mu)$ is a union of finitely many elements from the coset ring. Hence $S(\mu) \in \Re(\widehat{G})$. ■

Chapter 4

HOMOMORPHISMS BETWEEN GROUP ALGEBRAS

Let G, H be LCA groups, $L^1(G), L^1(H)$ their group algebras, and $M(G), M(H)$ measure algebras. Let $T : L^1(G) \to L^1(H)$ be a mapping such that:

$T(f+g) = Tf + Tg$, $T(f*g) = Tf * Tg$ and $T(\lambda f) = \lambda Tf$

for any $f, g \in L^1(G)$ and $\lambda \in \mathbb{C}$, then T is a **homomorphism** of $L^1(G)$ into $L^1(H)$.

Since $L^1(H)$ is a closed ideal in $M(H)$, we will look at the homomorphisms from $L^1(G)$ into $M(H)$ and characterize the ones between group algebras.

Let $\psi : L^1(G) \to M(H)$ be a homomorphism, then as $M(H)$ is semisimple, ψ is continuous. Hence for $\alpha \in \widehat{H}$, the map $f \to \widehat{\psi f}(\alpha)$ from $L^1(G)$ into $\mathbb{C}$ is either a multiplicative functional or identically zero. Let Y be the set of all $\alpha \in \widehat{H}$, for which this map is non-zero. Then for $\alpha \in Y$, there is a character $\psi_*(\alpha) \in \widehat{G}$ such that $\widehat{\psi f}(\alpha) = \hat{f}(\psi_*(\alpha))$, by theorem 2.2.2.

Hence, the homomorphism ψ induces a map $\psi_* : \widehat{H} \to \widehat{G}$ defined as

$$\widehat{\psi f}(\alpha) = \begin{cases} \hat{f}(\psi_*(\alpha)), & \text{if } \alpha \in Y, \\ 0, & \text{otherwise.} \end{cases} \qquad (f \in L^1(G) \text{ and } \alpha \in \widehat{H})$$

Let E be a coset in $\widehat{H}$ and $\phi : E \to \widehat{G}$ be continuous, then ϕ is said to be **affine** if $\phi(\kappa_1 + \kappa_2 - \kappa_3) = \phi(\kappa_1) + \phi(\kappa_2) - \phi(\kappa_3)$.

This affine map is uniformly continuous: Indeed let fix $\kappa_0 \in E$ and let $\kappa_1, \kappa_2 \in E$, then $\phi(\kappa_1) - \phi(\kappa_2) = \phi(\kappa_0 + \kappa_1 - \kappa_2) - \phi(\kappa_0)$, so by continuity of ϕ at κ_0, for $V \in \mathcal{V}(\mathbf{1})$ in $\widehat{G}$, there is a neighborhood U of $\mathbf{1}$ in $\widehat{G}$, such that for $\kappa_1 - \kappa_2 \in U$ we have $\phi(\kappa_1) - \phi(\kappa_2) \in V$.

Let now $S_1, \ldots, S_n$ be pairwise disjoint elements of $\Re(\widehat{H})$ with each $S_i \subset H_i$, where H_i is open coset in $\widehat{H}$. And let $\phi_i : H_i \to \widehat{G}$ be affine map for each i. Then the map $\phi : Y = \bigcup_i^n S_i \to \widehat{G}$ is said to be a **piecewise affine** if ϕ coincides with ϕ_i on S_i.

Main Result: *If $\psi : L^1(G) \to M(H)$ is a homomorphism, then the induced map ψ_* is a piecewise affine map of Y into $\widehat{G}$ and $Y \in \Re(\widehat{H})$.*

Conversely, if $Y \in \Re(\widehat{H})$ and $\phi : Y \to \widehat{G}$ is a piecewise affine map then there is a homomorphism ψ of $L^1(G)$ into $M(H)$ which induces it.

Moreover, ψ maps $L^1(G)$ into $L^1(H)$ iff $\psi_*^{-1}(K)$ is compact for every K compact subset of $\widehat{G}$.

4.1

First, we will prove a special case of the second part of the main result:

Lemma 4.1.1 *Let Y be an open coset in $\widehat{H}$ and $\psi_* : Y \to \widehat{G}$ be an affine map. Then there is a homomorphism $\psi : L^1(G) \to M(H)$ such that $\widehat{\psi f}(\alpha) = \hat{f}(\psi_*(\alpha))$.*

Proof: *Fix $\kappa_0 \in Y$ such that $Y - \kappa_0$ is an open subgroup of $\widehat{H}$. Now define on this subgroup $\phi(\kappa) = \psi_*(\kappa + \kappa_0) - \psi_*(\kappa_0)$. Then for $\kappa_1, \kappa_2 \in Y - \kappa_0$,*

$$\begin{aligned}\phi(\kappa_1 - \kappa_2) &= \psi_*(\kappa_1 - \kappa_2 + \kappa_0) - \psi_*(\kappa_0) \\ &= \psi_*(\kappa_1 + \kappa_0) - \psi_*(\kappa_2 + \kappa_0) + \psi_*(\kappa_0) - \psi_*(\kappa_0) \\ &= \phi(\kappa_1) - \phi(\kappa_2)\end{aligned}$$

Hence ϕ is a homomorphism, and also it is continuous as ψ_* is affine (which is continuous).

Now for $\mu \in M(G)$, let $\widehat{\psi(\mu)} = \widehat{\mu}(\phi)$ on $Y - \kappa_0$. As ϕ is a homomorphism of topological groups and the map $\mu \to \hat{\mu}$ is an isomorphism (theorem 3.2.1), then ψ is a homomorphism between measure algebras. Then, $\gamma_0 = \psi_*(\kappa_0)$ is a fixed element in $\widehat{G}$ and $\widehat{\psi\mu}(\kappa) = \widehat{\mu}(\phi(\kappa)) = \widehat{\mu}(\psi_*(\kappa + \kappa_0) - \gamma_0)$.

Since $\mathcal{B}(\widehat{G})$ is invariant under translations there is a measure $\nu \in M(G)$ such that $\hat{\mu}(\gamma - \gamma_0) = \hat{\nu}(\gamma)$ so $\widehat{\mu}(\psi_*(\kappa + \kappa_0) - \gamma_0) = \hat{\nu}(\psi_*(\kappa + \kappa_0))$.

Since $\mathcal{B}(\widehat{H})$ is invariant under translations there is a measure $\epsilon \in M(H)$ such that $\widehat{\psi\mu}(\kappa) = \hat{\epsilon}(\kappa + \kappa_0)$. Hence $\hat{\epsilon}(\kappa + \kappa_0) = \widehat{\psi\mu}(\kappa) = \widehat{\mu}(\psi_*(\kappa + \kappa_0) - \gamma_0) = \hat{\nu}(\psi_*(\kappa + \kappa_0))$.

Now with these translations there is a similar homomorphism of $M(G)$ into $M(H)$ inducing ψ_*. Since $L^1(G)$ is a subspace of $M(G)$, it is also a homomorphism from $L^1(G)$ into $M(H)$. ■

Now we can prove the second part of the main result;

Theorem 4.1.2 *If $Y \in \Re(\widehat{H})$ and $\psi_* : Y \to \widehat{G}$ is a piecewise affine map then there is a homomorphism ψ of $L^1(G)$ into $M(H)$ which induces it.*

Proof: *Now let S_i, H_i, ϕ_i be as in the definition of a piecewise affine map, where ψ_* coincides with ϕ_i on S_i. By theorem 3.4.2, for each S_i there is an idempotent measure $\mu_i \in \mathcal{J}(H)$ such that $\widehat{\mu_i}$ is the characteristic function of S_i. Now since ϕ_i is affine on H_i, by the previous lemma, there is a homomorphism ψ_i of $L^1(G)$ into $M(H)$ such that $\widehat{\psi_i f} = \hat{f}(\phi_i)$ on H_i. Now the map $\psi(f) = \sum_i \mu_i * \psi_i(f)$ is a homomorphism from $L^1(G)$ into $M(H)$ as is a finite sum of homomorphisms. And ψ induces ψ_* as $\widehat{\psi(f)}(\gamma) = \sum_i \widehat{\mu_i}(\gamma)\widehat{\psi_i(f)}(\gamma) = \hat{f}(\phi_i(\gamma)) = \hat{f}(\psi_*(\gamma))$ for $\gamma \in S_i$.* ■

4.2

In this section, we will prove the first part of the main result.
Note that, for any locally compact group H the set $\{g \in L^1(H) : \hat{g}$ has compact support$\}$ is dense in $L^1(H)$. Also $\{\hat{g} : g \in L^1(H), \hat{g}$ has compact support$\}$ is dense in $C_0(\widehat{H})$.

Let ψ, ψ_*, Y be as in the setting of main result:

Lemma 4.2.1 *$\psi : L^1(G) \to M(H)$ can be extended to a homomorphism of $M(G)$ into $M(H)$, in the form $\widehat{\psi(\mu)} = \hat{\mu}(\psi_*)$, where the extension has the same norm as ψ. Moreover, this extension is unique iff $Y = \widehat{H}$ or G is discrete.*

Proof: *(i) Let fix a measure $\mu \in M(G)$ and define $\lambda(\kappa) = \hat{\mu}(\psi_*(\kappa))$ for $\kappa \in Y$. If we can show that $\lambda \in \mathcal{B}(\widehat{H})$ then there is a measure $\psi\mu \in M(H)$ with $\widehat{\psi\mu} = \lambda$, defining ψ on $M(G)$ by $\widehat{\psi(\mu)} = \hat{\mu}(\psi_*)$; hence ψ becomes a homomorphism on $M(G)$ and it coincides with the given homomorphism on $L^1(G)$.*

To this end, let define a map by $Tg = \int_{\widehat{H}} \hat{g}\lambda dm_{\widehat{H}}$ for $g \in C_0(H)$ with $\hat{g}$ has a compact support say K. Then T is a functional and T is also bounded:
Let C be a bound for ψ, then $||\psi f|| \leq C||f||$ for each $f \in L^1(G)$. For $\varepsilon > 0$, let f_ε be in $L^1(G)$ with $||f_\varepsilon|| = 1$ and $|\hat{f}_\varepsilon - 1| < \varepsilon$ on $\psi_*(Y \cap K)$. Then for λ and g as above, $\widehat{\psi(f_\varepsilon)}\lambda = \hat{f}_\varepsilon(\psi_*)\hat{\mu}(\psi_*) = (\widehat{f_\varepsilon * \mu})(\psi_*) = \psi\widehat{(f_\varepsilon * \mu)}$, and so

$$\begin{aligned}\left|\int_{\widehat{H}} \hat{g}\widehat{\psi(f_\varepsilon)}\lambda dm_{\widehat{H}}\right| &= \left|\int_{\widehat{H}} \hat{g}\widehat{\psi(f_\varepsilon * \mu)}dm_{\widehat{H}}\right| = \left|\int_H g\psi(f_\varepsilon * \mu)dm_H\right| \\ &\leq ||g||_\infty||\psi(f_\varepsilon * \mu)|| \\ &\leq C||g||_\infty||\mu|| \text{ as } ||f_\varepsilon|| = 1\end{aligned}$$

Now as ε tends to zero $\widehat{\psi(f_\varepsilon)} = \hat{f}_\varepsilon(\psi_*)$ tends to 1 and the left side of above inequality tends to $|Tg|$. Since right side is independent of ε, $|Tg| \leq C||g||_\infty||\mu||$.

Since the set of all such g are dense in $C_0(H)$, then T can be extended to be a bounded functional on $C_0(H)$. And then by Riesz representation theorem, there is a Radon measure $\nu \in M(H)$ such that $\int_{\widehat{H}} \hat{g}\lambda dm_{\widehat{H}} = Tg = \int_H g d\nu$ and as $\int_H g d\nu = \int_{\widehat{H}} \hat{g}\hat{\nu}dm_{\widehat{H}}$ we have $\int_{\widehat{H}} \hat{g}\lambda dm_{\widehat{H}} = \int_{\widehat{H}} \hat{g}\hat{\nu}dm_{\widehat{H}}$ for every $\hat{g} \in C_c(\widehat{H})$, which is dense in $C_0(\widehat{H})$, hence $\hat{\nu} = \lambda$. Hence $\psi\mu = \nu$ gives the extension of ψ to $M(G)$.

(ii) If G is discrete $M(G) = L^1(G)$, hence extension is the function itself. So suppose G is not discrete and ψ is extended to $M(G)$.

Assume $Y = \widehat{H}$. Fix $f \in L^1(G)$, $\mu \in M(G)$ and set $\sigma = \psi f$ and $\nu = \psi\mu$, then $f * \mu \in L^1(G)$ and $\widehat{(f * \mu)}(\psi_*(\kappa)) = \hat{f}(\psi_*(\kappa))\hat{\mu}(\psi_*(\kappa))$ for $\kappa \in Y(= \widehat{H})$.

Since $\psi(f * \mu) = \psi f * \psi\mu = \sigma * \nu$, and $\widehat{\psi(f * \mu)} = \widehat{(f * \mu)}(\psi_*)$, then

$\hat{f}(\psi_*)\hat{\mu}(\psi_*) = \widehat{\psi(f)}\,\widehat{\psi(\mu)} = \hat{\sigma}\hat{\nu}$ on $\widehat{H}$.

Since these equalities hold for any $f \in L^1(G)$, then $\hat{\mu}(\psi_*(\kappa)) = \hat{\nu}(\kappa)$ for any $\kappa \in \widehat{H}$. So that $\hat{\nu}$ is uniquely determined on $\widehat{H}$, which determines ν uniquely by lemma 3.2.2.

For the converse, suppose $Y \neq \widehat{H}$. Let h be a complex homomorphism of $M(G)$ such that $h = 0$ on $L^1(G)$ which is not identically zero (such an element exist since group algebra is a closed ideal in mesure algebra). For a fixed measure $\mu \in M(G)$, the map $\lambda(\gamma) = \begin{cases} \hat{\mu}(\psi_*(\gamma)), & \text{if } \gamma \in Y, \\ h(\mu), & \text{if } \gamma \notin Y. \end{cases}$

gives a measure $\psi\mu \in M(H)$ such that $\widehat{\psi\mu} = \lambda$ as proved in the first part.

Also replacing that h above with $h = 0$ on $M(G)$, gives another map λ_0 similar to above, which is different from λ, hence the measures corresponding to μ in the two cases are different. ■

Let now G, H be compact and, ψ, ψ_*, Y be as in the setting of main result. A subset $E \subset \widehat{G} \times \widehat{H}$ is a **graph** if for every $\kappa \in \widehat{H}$ there is at most one $\gamma \in \widehat{G}$ such that $(\gamma, \kappa) \in E$. The graph of ψ_* is the set $E = \{(\psi_*(\alpha), \alpha) \in \widehat{G} \times \widehat{H} : \alpha \in Y\}$.

For an abelian group G, let E_1, E_2 be cosets of the subgroups A_1, A_2 of G. The **index of** E_i is the index of A_i in G. And the **index of** E_1 **in** E_2 is the index of $A_1 \cap A_2$ in A_2. Note that an abelian group is not a finite union of cosets of infinite index.

For a finite collection of subgroups of $\widehat{G} \times \widehat{H}$, call $\mathcal{S}$, let $R(\mathcal{S})$ be the ring generated by the cosets of elements in $\mathcal{S}$.

Now if E is in the coset ring of $\widehat{G} \times \widehat{H}$, then $E \in R(\mathcal{S})$ for some finite $\mathcal{S}$. We may assume $\widehat{G} \times \widehat{H} \in \mathcal{S}$. We can also enlarge $\mathcal{S}$ so that the intersection of any two members of $\mathcal{S}$ is in $\mathcal{S}$, without changing the ring $R(\mathcal{S})$. An element $A \in \mathcal{S}$ is **minimal** if there is no proper subgroup of A of finite index contained in $\mathcal{S}$. Let $\mathcal{S}'$ be the set of all minimal elements in $\mathcal{S}$. Then for any $A_1, A_2 \in \mathcal{S}'$ the index of A_1 in A_2 is either 1 or infinite. Since $\mathcal{S}$ is finite and every non-minimal element has a proper subgroup of finite index, $\mathcal{S} \subset R(\mathcal{S}')$. Hence $E \in R(\mathcal{S}')$. Thus E is a finite union of sets of the form: $(\bigcap_{j=1}^{r} K_{i_j}) \cap (\bigcap_{j=r+1}^{n} K_{i_j}^c)$ where K_{i_j}'s are cosets of elements of $\mathcal{S}'$. Let $L = \bigcap_{j=1}^{r} K_{i_j}$ and $M_j = L \cap K_{i_j}$, then above set can be written as $L \cap (\bigcap_{j=r+1}^{n} M_j^c)$. Let L be coset of L' and M_j be coset of M_j', since finite intersections of minimal elements is minimal, then L', M_j' are in $\mathcal{S}'$. Hence the index of M_j in L is either 1 or infinite. If it is 1 then either $M_j = L$, hence $L \cap (\bigcap_{j=r+1}^{n} M_j^c) = \emptyset$ or $M_j \cap L = \emptyset$ hence the intersection will not change if we drop this M_j. Thus re-numbering if necessary, the index of each M_j in L is infinite.

Lemma 4.2.2 *If the graph of ψ_*, E is in the coset ring of $\widehat{G} \times \widehat{H}$ then Y is in the coset ring of $\widehat{H}$ and ψ_* is piecewise affine.*

Proof: *Suppose E is in the coset ring of $\widehat{G} \times \widehat{H}$, then by above arguments we can assume E is finite union of sets of the form $E_i = L_i \cap (\bigcap_j M_{ij}^c)$, $1 \leq i \leq n$, where L_i, M_{ij} are cosets in $\widehat{G} \times \widehat{H}$ and there are finitely many M_{ij} for each i, each of which has infinite index in L_i.*

Now since E is a graph, each E_i is also a graph. Each L_i is also a graph: Indeed, since L_i is a coset of a subgroup L_i', it is a graph iff the L_i' is a graph. If by contradiction, L_i' is not a graph then $(\gamma_0, \mathbf{1}) \in L_i'$ for some $\gamma_0 \neq \mathbf{1}$. If $(\gamma, \kappa) \in E_i$, then $(\gamma_0, \mathbf{1}) + (\gamma, \kappa) = (\gamma_0 + \gamma, \kappa) \in \bigcup_j M_{ij}$, since otherwise $(\gamma_0 + \gamma, \kappa) \in E_i$ contradicting being a graph. So $E_i \subset \bigcup_j M_{ij} - (\gamma_0, \mathbf{1})$, that is $L_i \cap (\bigcap_j M_{ij}^c) \subset \bigcup_j M_{ij} - (\gamma_0, \mathbf{1})$. And so $L_i \subset \bigcup_j [M_{ij} - (\gamma_0, \mathbf{1})] \cup (\bigcup_j M_{ij})$, where each M_{ij} has infinite index in L_i. Then since L_i is coset of L_i', L_i' is subset of a finite union of cosets of infinite index, which is not possible since L_i' is an abelian group.

Now let $\pi : \widehat{G} \times \widehat{H} \to \widehat{H}$ be the natural homomorphism $\pi(\gamma, \kappa) = \kappa$ which is onto. Set $H_i = \pi(L_i)$ and $S_i = \pi(E_i)$. Since L_i is a graph for every $\kappa \in H_i$ there is a unique element in $\widehat{G}$, call $\phi_i(\kappa)$ such that $(\phi_i(\kappa), \kappa) \in L_i$, hence $\phi_i : H_i \to \widehat{G}$ is a well-defined map. Moreover, ϕ_i is affine: Indeed, ϕ_i is continuous since defined on discrete spaces. And for $\kappa_1, \kappa_2, \kappa_3 \in H_i$ there are corresponding unique elements $\phi_i(\kappa_j) \in \widehat{G}$. Since L_i is a coset, it is of the form $G_i + (\gamma_i, \kappa_i)$ where G_i is a subgroup and (γ_i, κ_i) fixed. Hence, H_i is a coset in $\widehat{H}$ and
$(\phi_i(\kappa_1), \kappa_1) + (\phi_i(\kappa_2), \kappa_2) - (\phi_i(\kappa_3), \kappa_3) = (\phi_i(\kappa_1) + \phi_i(\kappa_2) - \phi_i(\kappa_3), \kappa_1 + \kappa_2 - \kappa_3)$ is in L_i, so that $\phi_i(\kappa_1 + \kappa_2 - \kappa_3) = \phi_i(\kappa_1) + \phi_i(\kappa_2) - \phi_i(\kappa_3)$.

Now $Y = \pi(E) = \bigcup_i \pi(E_i) = \bigcup_i S_i$. Moreover, each S_i is in the coset ring of $\widehat{H}$. Indeed, $S_i = \pi(E_i) = \pi(L_i \cap (\bigcap_j (L_i \cap M_{ij})^c)) = H_i \cap (\bigcap_j \pi(L_i \cap M_{ij})^c)$. As both L_i and M_{ij} are cosets, $\pi(L_i \cap M_{ij})$ is in the coset ring of $\widehat{H}$, hence so is S_i. And for $\kappa \in S_i$, $(\phi_i(\kappa), \kappa) \in E_i$, hence is in E, but also there is $(\psi_*(\kappa), \kappa) \in E$, hence $\psi_* = \phi_i$ on S_i (as E is graph). To make ψ_* piecewise affine, set $S_i' = \{\kappa \in S_i : \kappa \notin S_1 \cup \ldots \cup S_{i-1}\}$ for $1 \leq i \leq n$, then S_i' is in the coset ring of $\widehat{H}$ and $Y = \bigcup S_i'$ and ϕ_i on S_i' is affine and agrees with ψ_*. ■

Now to show that first part of the main result holds in the compact case, it is enough to show the following (by theorem 3.4.5):

Lemma 4.2.3 *Let E be the graph of ψ_*, as above. Then the characteristic function of E is the Fourier transform of an idempotent measure on $G \times H$.*

Proof: *Let λ be the characteristic function of E. Let $f \in L^1(H)$ be such that $||f|| = 1$ and support of $\hat{f}$ is compact, so $\hat{f}$ is non-zero at finitely many points.*
Define $\hat{f}$ as a function on $\widehat{G} \times \widehat{H}$ by $\hat{f}(\gamma, \kappa) = \hat{f}(\kappa)$.
And for $(x, y) \in G \times H$, define ϕ as $\phi(x, y) = \sum_{\kappa \in \widehat{H}} \hat{f}(\kappa) < x, \psi_*(\kappa) >< y, \kappa >$, then ϕ is a map of $G \times H$ into $\mathbb{C}$. Moreover,

$$\begin{aligned}\hat{\phi}(\gamma, \alpha) &= \int_{G \times H} < -(x, y), (\gamma, \alpha) > \sum_{\kappa \in \widehat{H}} \hat{f}(\kappa) < x, \psi_*(\kappa) >< y, \kappa > d(x, y) \\ &= \sum_{\kappa \in \widehat{H}} \hat{f}(\kappa) \int_H < y, \kappa - \alpha > dy \int_G < x, \psi_*(\kappa) - \gamma > dx\end{aligned}$$

Then $\hat{\phi}(\gamma, \alpha) \neq 0$ only if $\alpha = \kappa$ and $\gamma = \psi_*(\kappa)$ for some $\kappa \in Y$, and in that case $\hat{\phi}(\gamma, \alpha) = \hat{f}(\kappa)$. Hence $\hat{\phi} = \lambda \hat{f}$.

Let ψ be extended to all $M(G)$ as in lemma 4.2.1, for δ_x (unit mass concentrated at $x \in G$) set $\mu_x = \psi \delta_x$. Then for a fixed $x \in G$,

$< x, \psi_*(\kappa) >= \int_G < -t, \psi_*(\kappa) > d\delta_{-x}(t) = \hat{\delta}_{-x}(\psi_*(\kappa)) = \hat{\psi}(\delta_{-x})(\kappa) = \hat{\mu}_{-x}(\kappa)$.

And also inverse Fourier transformation holds for f, so that

$$\begin{aligned}\phi(x, y) &= \sum_{\kappa \in \widehat{H}} \hat{f}(\kappa) < y, \kappa > \int_H < -t, \kappa > d\mu_{-x}(t) \\ &= \int_H \sum_{\kappa \in \widehat{H}} \hat{f}(\kappa) < y - t, \kappa > d\mu_{-x}(t) = \int_H f(y - t) d\mu_{-x}(t)\end{aligned}$$

Then, as $||\delta_{-x}|| = 1 = ||f||$,

$$\int_H |\phi(x, y)| dy \leq \int_H \int_H |f(y - t)| d|\mu_{-x}|(t) dy \leq \int_H ||f|| d|\mu_{-x}| = ||f||\, ||\mu_{-x}|| \leq ||\psi||$$

Hence, $||\phi|| = \int_G \int_H |\phi(x, y)| dy dx \leq ||\psi||$.

Now as in lemma 4.2.1, for $g \in C(\widehat{G} \times \widehat{H})$ define $Tg = \int_{\widehat{G} \times \widehat{H}} \hat{g} \lambda dm_{\widehat{G} \times \widehat{H}}$. And for every such choice of f, we have

$$\int_{\widehat{G} \times \widehat{H}} \hat{g} \lambda \hat{f} dm_{\widehat{G} \times \widehat{H}} = \int_{\widehat{G} \times \widehat{H}} \hat{g} \hat{\phi} dm_{\widehat{G} \times \widehat{H}} = \int_{G \times H} g d\phi$$

so that $\left| \int_{\widehat{G} \times \widehat{H}} \hat{g} \lambda \hat{f} dm_{\widehat{G} \times \widehat{H}} \right| \leq ||g||_\infty ||\psi||$, hence this inequality holds for f_ε with the additional property $|\hat{f}_\varepsilon - 1| < \varepsilon$ on support of $\hat{f}_\varepsilon$. And so as $\varepsilon \to 0$ (the left side of inequality becomes $|Tg|$), so that $|Tg| \leq ||g||_\infty ||\psi||$, hence there is $\upsilon \in M(G \times H)$ such that $\hat{\upsilon} = \lambda$. Thus υ is idempotent and $E = S(\upsilon)$. ■

Let now G, H be LCA groups and $\widehat{G}, \widehat{H}$ be the dual groups. Suppose $\widehat{G}_d, \widehat{H}_d$ denote the groups $\widehat{G}, \widehat{H}$ with the discrete topology, and $\tilde{G}, \tilde{H}$ the dual groups of them. Hence $\tilde{G}, \tilde{H}$ are compact abelian groups. Moreover, there are continuous isomorphisms β_G of G onto a dense subgroup of $\tilde{G}$ and β_H of H onto a dense subgroup of $\tilde{H}$. So $\tilde{G}, \tilde{H}$ are compactifications (called Bohr compactification) and we can consider G and H as subsets of $\tilde{G}$ and $\tilde{H}$. Suppose Y_d is the set Y with discrete topology.

Lemma 4.2.4 *ψ_* as a map from Y_d into $\widehat{G}_d$ is induced from a homomorphism ψ' from $L^1(\tilde{G})$ into $M(\tilde{H})$.*

Proof: *As there is an embedding of H as a subset of $\tilde{H}$, there is an induced mapping π of $M(H)$ into $M(\tilde{H})$ which is a homomorphism as induced from an isomorphism. Then the composition $\pi \circ \psi$ is a homomorphism of $L^1(G)$ into $M(\tilde{H})$. And the induced map $(\pi \circ \psi)_* = \psi_*$, as $\widehat{(\pi \circ \psi)}(f)(\alpha) \neq 0$ iff $\widehat{\psi(f)}(\alpha) \neq 0$ for α in dual of $\tilde{H}$ which is $\widehat{H}_d$. Also by lemma 4.2.1, there is an extension of $\pi \circ \psi$ to a homomorphism of $M(G)$ into $M(\tilde{H})$ by $\pi \widehat{\circ \psi(\mu_1)} = \hat{\mu_1}(\psi_*)$.*

Let $\mu \in M(\tilde{G})$, $\varepsilon > 0$ and $\kappa_1, \ldots, \kappa_n \in \widehat{H}$. Then there is a measure $\mu_1 \in M(G)$ such that $||\mu_1|| \leq ||\mu||$ and $|\hat{\mu}(\psi_*(\kappa_i)) - \hat{\mu_1}(\psi_*(\kappa_i))| < \varepsilon$ as G is dense in $\tilde{G}$. By lemma 3.2.2, as a measure is determined by its Fourier-Stieltjes transform, there is a measure $\psi'(\mu) \in M(\tilde{H})$ such that $\widehat{\psi'(\mu)} = \hat{\mu}(\psi_*)$ for $\mu \in M(\tilde{G})$. Hence ψ' is a homomorphism from $M(\tilde{G})$ into $M(\tilde{H})$ inducing ψ_*.

Then by lemmas 4.2.3 and 4.2.2, Y_d is in the coset ring of $\widehat{H}_d$ and ψ_* is a piecewise affine map. Now the following steps will lead to the general case:

1. *Y is open in $\widehat{H}$*: Fix an element $\kappa_0 \in Y$, need to find an open set $V \subset \widehat{H}$ containing κ_0 and subset of Y. Let $U \subset \widehat{G}$ be an open subset containing $\psi_*(\kappa_0)$. There is an f in $L^1(G)$, with *supp* $\hat{f}$ in U such that $\hat{f}(\psi_*(\kappa_0)) \neq 0$. Let $W = \{\kappa \in \widehat{H} : \hat{f}(\psi_*(\kappa)) \neq 0\}$. As $\psi f \in M(H)$ and $\hat{f}(\psi_*) = \widehat{\psi f}$ is (uniformly) continuous on $\widehat{H}$, W is an open subset of $\widehat{H}$. Moreover, $\kappa_0 \in W$ and $W \subset Y$.

Also, $\psi_*(W) = \{\psi_*(\kappa) : \hat{f}(\psi_*(\kappa)) \neq 0\} \subset$ *supp* $f \subset U$, hence ψ_* is continuous as a function from Y into $\widehat{G}$.

Now we have the following situation:

$Y = \bigcup_i^n S_i$ is open in $\widehat{H}$, where $S_i \in \Re(\widehat{H}_d)$ are of the form $S_i = H_i \cap (\bigcap_j^{r_i} N_{ij}^c)$ with H_i, N_{ij} cosets in $\widehat{H}_d$ and N_{ij} has infinite index in H_i. On each H_i there is an affine map $\phi_i : H_i \to \widehat{G}_d$ and $\psi_* = \phi_i$ on S_i. Since the groups $\widehat{H}_d$ and $\widehat{H}$ are the same groups with different topologies, H_i, N_{ij} cosets in $\widehat{H}$ and N_{ij} has infinite index in H_i as subsets of $\widehat{H}$. The map $\phi_i : H_i \to \widehat{G}$ has the property $\phi_i(\kappa_1 + \kappa_2 - \kappa_3) = \phi_i(\kappa_1) + \phi_i(\kappa_2) - \phi_i(\kappa_3)$ and is continuous on S_i, since coincides with ψ_*, hence is affine on S_i.

2. *ϕ_i is affine from H_i into $\widehat{G}$*: As H_i is not contained in the union of a finite number of translates of the N_{ij}'s, there are elements $\alpha_1, \ldots, \alpha_{r_i}, \alpha_{r_i+1}$ in S_i such that $\alpha_n - \alpha_k$ for $k \neq n$ is not in $\cup_j N'_{ij}$, where N_{ij} is coset of N'_{ij}. Then for $x, y \in H_i$, there is some $n \leq r_i + 1$ such that $\alpha_n + x - y \in S_i$: Indeed, otherwise $\alpha_k + x - y \notin S_i$ for each k, and so there are at least two of them, say $\alpha_k + x - y$ and $\alpha_l + x - y$, in the same N_{ij}. Thus $\alpha_k - \alpha_l$ is in N'_{ij}, which is not the case. So for $x, y \in H_i$, $\phi_i(x) - \phi_i(y) = \phi_i(\alpha_n + x - y) - \phi_i(\alpha_n)$.

Now since ψ_* is uniformly continuous on S_i, for a neighborhood U of $\mathbf{1}$ in $\widehat{G}$, there are neighborhoods V_k of α_k such that if $z_1, z_2 \in V_k \cap S_i$, then $\psi_*(z_1) - \psi_*(z_2) \in U$. For some neighborhood V of identity of $\widehat{H}$, $\alpha_k + V \subset V_k$, hence for every $x, y \in H_i$, if $x - y \in V$, then $\phi_i(x) - \phi_i(y) = \psi_*(\alpha_n + x - y) - \psi_*(\alpha_n) \in U$. Hence ϕ_i is uniformly continuous on H_i, so is affine on H_i.

Lemma 4.2.5 *Each ϕ_i can be extended to an affine map of $\overline{H_i}$ (closure).*

Proof: *By uniform continuity of ϕ_i, if W is a compact neighborhood of $\mathbf{1}$ in $\widehat{G}$, there is an open neighborhood V of $\mathbf{1}$ in $\widehat{H}$ such that $\phi_i((\kappa + V) \cap H_i) \subset \phi_i(\kappa) + W$ for each $\kappa \in H_i$. Let A be a subset of H_i with compact closure. Since $\overline{A} \subset A + V$, there are $\alpha_1, \ldots, \alpha_n \in A$ such that $A \subset \bigcup_j (\alpha_j + V)$. Hence $\phi_i(A) \subset \bigcup_j \phi_i(\alpha_j) + W$. Thus $\phi_i(A)$ has compact closure.*

Now fix $\kappa_0 \in \overline{H_i}$ then for every compact neighborhood U of κ_0, the set $F_U = \overline{\phi_i(H_i \cap U)}$ is compact, so that $\bigcap F_U$ has a unique element. Calling it $\phi_i(\kappa_0)$, we have extended ϕ_i to closure of H_i, which is well-defined as intersection has unique element. And also continuous as for any neighborhood W of $\phi_i(\kappa_0)$, $W \cap F_U \neq \emptyset$ for some U, and by

uniform continuity of ϕ_i there is an open subset U' of U such that

$\phi_i(H_i \cap U') \subset W \cap F_U$.

Now since group operations and ϕ_i is continuous on $\overline{H_i}$, ϕ_i is affine on the closure.

3. *Y is closed in $\widehat{H}$*: Suppose now ϕ_i are extended affine maps. Fix $\kappa_0 \in \overline{Y}$. Then $\kappa_0 \in \overline{S_i}$ for some i and $\phi_i(\kappa_0) \in \widehat{G}$. Choose $f \in L^1(G)$ such that $\hat{f}(\phi_i(\kappa_0)) \neq 0$. Since $\hat{f}(\phi_i)$ is continuous, there are $\kappa \in S_i$ arbitrarily close enough to κ_0 such that $\hat{f}(\psi_*(\kappa)) = \hat{f}(\phi_i(\kappa)) \neq 0$. Since $\hat{f}(\psi_*)$ is continuous, $\hat{f}(\psi_*(\kappa_0)) \neq 0$, hence $\kappa_0 \in Y$.

4. Suffices to show *$\overline{H_i}$ is open coset and S_i is in the coset ring* since then ψ_* is piecewise affine and Y is in the coset ring.

Let F be the set of i such that $\overline{S_i}$ has non-empty interior. Since Y is open, F is non-empty finite and $Y \subset \bigcup_{i\in F} \overline{S_i}$. But Y is also closed so $Y = \bigcup_{i\in F} \overline{S_i}$.
Now for $i \in F$, $\overline{H_i}$ and $\overline{\bigcap_j N_{ij}^c}$ has nonempty interiors. As H_i is a coset, then $\overline{H_i}$ is a coset and hence it is open.
For $i \in F$, if N_{ij} has non-empty interior then it is open, so that the complement is closed and N_{ij} does not intersect $\overline{S_i}$. If N_{ij} has empty interior, then the complement is dense in $\widehat{H}$, so the closure of intersection without this term will be the same. Hence for $i \in F$, $\overline{S_i} = \overline{H_i} \cap (\bigcap_j N_{ij}^c)$ where the intersection is over the open N_{ij}'s. (If none of them is open, then $\overline{S_i} = \overline{H_i}$). Hence $\overline{S_i}$ is in the coset ring of $\widehat{H}$ for $i \in F$.

Now to satisfy piecewise affine definition, we need $\overline{S_i}$'s to be disjoint. Let enumerate the sets $\overline{S_i}$ with $i \in F$ so that $F = \{1, 2, \ldots, n\}$. Then the sets

$E_i = \{\kappa \in \overline{S_i} : \kappa \notin \overline{S_1} \cup \ldots \cup \overline{S_{i-1}}\}$ for $1 \leq i \leq n$ are in the coset ring of $\widehat{H}$ and $Y = \bigcup E_i$ and ϕ_i on E_i is affine and agrees with ψ_*.

4.3

So far, we have characterized the homomorphisms from a group algebra into a measure algebra. And as group algebra is a closed ideal of measure algebra, the set of homomorphisms characterized above include the homomorphisms between group algebras. Now in this final step, we will characterize the ones between group algebras.

Theorem 4.3.1 *The map ψ defined in main theorem carries $L^1(G)$ into $L^1(H)$ iff $\psi_*^{-1}(K)$ is compact for every compact $K \subset \widehat{G}$.*

Proof: *If there is a compact subset $K \subset \widehat{G}$ such that $\psi_*^{-1}(K)$ is not compact, then choose $f \in L^1(G)$ such that $\hat{f} = 1$ on K. Then the set $\{\gamma \in Y : \hat{f}(\psi_*(\gamma)) = 1\}$ contains $\psi_*^{-1}(K)$, which is closed and non-compact, so the set is not compact either. Then $\psi(f)$ is not in $L^1(H)$*

Conversely, suppose $\psi_*^{-1}(K)$ is compact for every compact $K \subset \widehat{G}$ and let f be in $L^1(G)$. Then there is a sequence f_n in $L^1(G)$ such that $f_n \to f$ in $L^1(G)$ and $\hat{f}_n$ has compact support. So $\hat{f}_n(\psi_*)$ also has compact support for each n and so $\psi(f_n) \in L^1(H)$. As ψ is continuous $\psi(f_n) \to \psi(f)$; since $L^1(H)$ is closed $\psi(f) \in L^1(H)$.

Appendix A

A.1 Banach Algebras

Proposition A.1.1 *Every Banach algebra $\mathcal{A}_0$ can be embedded in a Banach algebra $\mathcal{A}$ with unit.*

Proof: *Let $\mathcal{A} = \mathcal{A}_0 \times \mathbb{C}$. Then define addition and multiplication in $\mathcal{A}$ as follows:*

$(x, \alpha) + (y, \beta) = (x + y, \alpha + \beta)$

$(x, \alpha)(y, \beta) = (xy + \alpha y + \beta x, \alpha\beta)$

Since both $\mathcal{A}_0$ and $\mathbb{C}$ are vector spaces over complex numbers, $\mathcal{A}$ is a vector space. Clearly, associativity and bilinearity of multiplication holds. Hence it is an associative algebra.

Define a norm on $\mathcal{A}$ by: $||(x, \alpha)|| = ||x|| + |\alpha|$. Then as both $\mathcal{A}_0$ and $\mathbb{C}$ are complete, $\mathcal{A}$ is a Banach algebra.

Moreover, $(0, 1) \in \mathcal{A}$ is unit for the algebra as: $(x, \alpha)(0, 1) = (0, 1)(x, \alpha) = (x, \alpha)$.

And we can embed $\mathcal{A}_0$ into $\mathcal{A}$ by the mapping $x \to (x, 0)$.

Hence now on all Banach algebras are assumed to be unitary.

For the identity element $e \in \mathcal{A}$, $||e|| = ||ee|| \leq ||e||^2$, so that $||e|| \geq 1$. It is always possible to re-norm $\mathcal{A}$ so that $||e|| = 1$. (let $|||y||| = \sup_{x \in \mathcal{A}} \frac{||yx||}{||x||}$ then $|||e||| = 1$)

Let $\mathcal{A}$ be a Banach algebra and e be unit of the algebra.

Lemma A.1.2 *For $x \in \mathcal{A}$ if $||x|| < 1$ then $e - x$ is invertible and $(e-x)^{-1} = \sum_{n=0}^{\infty} x^n$.*

Proof: *As $||\sum_{n=0}^{\infty} x^n|| \leq \sum_{n=0}^{\infty} ||x||^n$ and $||x|| < 1$, the series converges absolutely, so converges.*

For $k > 0$, $(e - x)\sum_{n=0}^{k} x^n = (e + x + \ldots + x^k) - (x + \ldots + x^k + x^{k+1}) = e - x^{k+1}$, so $(e - x)\sum_{n=0}^{\infty} x^n = e - \lim_{n\to\infty} x^n = e$ as $\lim_{n\to\infty} ||x||^n = 0$.

Theorem A.1.3 *$x, y \in \mathcal{A}$, $\lambda_0, \lambda \in \mathbb{C}$,*

(a) If $|\lambda| > ||x||$ then $\lambda e - x$ is invertible and its inverse is $\sum_{n=0}^{\infty} \lambda^{-n-1} x^n$.

(b) If $\lambda_0 e - x$ is invertible and $|\lambda - \lambda_0| < ||(\lambda_0 e - x)^{-1}||^{-1}$, then $\lambda e - x$ is invertible and $(\lambda e - x)^{-1} = \sum_{n=0}^{\infty} (\lambda_0 - \lambda)^n ((\lambda_0 e - x)^{-1})^{n+1}$

(c)If x is invertible and $||y|| < ||x^{-1}||^{-1}$ then $x - y$ is invertible and its inverse is $x^{-1} \sum_{n=0}^{\infty} (x^{-1}y)^n$.

(d) If x is invertible and $||y|| \leq \frac{1}{2}||x^{-1}||^{-1}$ then $||(x-y)^{-1} - x^{-1}|| \leq 2||x^{-1}||^2||y||$.

(e) The set of invertible elements of $\mathcal{A}$ is open and the map $x \to x^{-1}$ is continuous on this set.

Proof: *(a) If $|\lambda| > ||x||$ then $||\lambda^{-1}x|| = |\lambda^{-1}|||x|| < |\lambda^{-1}||\lambda| = 1$, so by lemma A.1.2 $e - \lambda^{-1}x$ is invertible and $(e - \lambda^{-1}x)^{-1} = \sum_{n=0}^{\infty} (\lambda^{-1}x)^n$. But $\lambda e - x = \lambda(e - \lambda^{-1}x)$, so $(\lambda e - x)^{-1} = \lambda^{-1}(e - \lambda^{-1}x)^{-1} = \lambda^{-1} \sum_{n=0}^{\infty} (\lambda^{-1}x)^n$*

(b) $$|| \sum_{n=0}^{\infty} (\lambda_0 - \lambda)^n ((\lambda_0 e - x)^{-1})^{n+1} || \leq \sum_{n=0}^{\infty} |\lambda_0 - \lambda|^n ||(\lambda_0 e - x)^{-1}||^{n+1}$$

The term on the right-hand-side is $\leq ||(\lambda_0 e - x)^{-1}|| \sum_{n=0}^{\infty} (\lambda_0 - \lambda)^n ||(\lambda_0 e - x)^{-1}||^n$, hence by the assumption the series on the left-hand-side is convergent. And,

$$\begin{aligned}(\lambda e - x) \sum_{n=0}^{k} (\lambda_0 - \lambda)^n ((\lambda_0 e - x)^{-1})^{n+1} &= [(\lambda_0 - \lambda) - (\lambda_0 - \lambda)] \sum_{n=0}^{k} (\lambda_0 - \lambda)^n ((\lambda_0 e - x)^{-1})^{n+1} \\ &= \sum_{n=0}^{k} (\lambda_0 - \lambda)^n ((\lambda_0 e - x)^{-1})^n - \sum_{n=0}^{k} (\lambda_0 - \lambda)^{n+1} ((\lambda_0 e - x)^{-1})^{n+1} \\ &= e - (\lambda_0 - \lambda)^{k+1} ((\lambda_0 - x)^{-1})^{k+1} \longrightarrow e \text{ as } k \to \infty\end{aligned}$$

Hence the inverse of $\lambda e - x$ is the given series.

(c) $||x^{-1}y|| \leq ||x^{-1}||\,||y|| < 1$ by the hypothesis, so $e - x^{-1}y$ is invertible by lemma A.1.2, and $(e - x^{-1}y)^{-1} = \sum_{n=0}^{\infty} (x^{-1}y)^n$.

As $x - y = x(e - x^{-1}y)$ and both x and $e - x^{-1}y$ are invertible then their multiplication is invertible and inverse of the multiplication is multiplication of inverses.

(d) Using (c) and hypothesis of d, we have;

$$\begin{aligned}||(x-y)^{-1}-x^{-1}|| &= ||x^{-1}\sum_{n=0}^{\infty}(x^{-1}y)^n - x^{-1}|| \leq ||x^{-1}||\,||\sum_{n=1}^{\infty}(x^{-1}y)^n|| \\ &\leq ||x^{-1}||\sum_{n=1}^{\infty}(||x^{-1}||\,||y||)^n = ||x^{-1}||\sum_{n=0}^{\infty}(||x^{-1}||\,||y||)^{n+1} \\ &= ||x^{-1}||^2||y||\sum_{n=0}^{\infty}(||x^{-1}||\,||y||) \\ &\leq ||x^{-1}||^2||y||\sum_{n=0}^{\infty}(1/2)^n = 2||x^{-1}||^2||y||.\end{aligned}$$

(e) Let $G(\mathcal{A})$ be the set of all invertible elements of $\mathcal{A}$.

Then for $x \in \mathfrak{U}$,

$||xx^{-1}|| = ||e|| = 1 \leq ||x||\,||x^{-1}||$ *which implies that* $||x^{-1}||^{-1} \leq ||x||$

So let $\varepsilon = ||x^{-1}||^{-1}$ then for each $y \in B_\varepsilon(x)$ in $\mathcal{A}$, $||x-y|| < \varepsilon = ||x^{-1}||^{-1}$

Then by (c) $x-(x-y) = y$ is invertible, i.e. $y \in \mathfrak{U}$, hence $B_\varepsilon(x) \subset \mathfrak{U}$, so $\mathfrak{U}$ is open.

Now let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathfrak{U}$ with $x_n \to x \in \mathcal{A}$.

Let $y = x - x_n$ and x be itself in the setting of (d), then as $||x-x_n|| \to 0$, there exist $N \in \mathbb{N}$: for each $n \geq N$, $||x-x_n|| \leq \frac{1}{2}||x^{-1}||^{-1}$, $||x_n^{-1}-x^{-1}|| \leq 2||x^{-1}||^2||x-x_n||$ for all $n \geq N$; so that $x_n^{-1} \to x^{-1}$, hence the mapping $x \to x^{-1}$ is continuous. ■

For $x \in \mathcal{A}$ the **spectrum** of x is $\sigma(x) = \{\lambda \in \mathbb{C} : \lambda e - x$ is not invertible$\}$.

Proposition A.1.4 *The spectrum, $\sigma(x)$ is a closed subset of $\{\lambda \in \mathbb{C} : |\lambda| \leq ||x||\}$.*

Proof: *By the theorem A.1.3 (a), $\sigma(x) \subset \{\lambda \in \mathbb{C} : |\lambda| \leq ||x||\}$*

Now $\mathbb{C} \setminus \sigma(x) = \{\lambda \in \mathbb{C} : \lambda e - x$ is invertible$\}$.

For $\lambda_0 \in \mathbb{C} \setminus \sigma(x)$ let $\varepsilon = ||(\lambda_0 e - x)^{-1}||^{-1}$ then by theorem A.1.3 (b),

$B_\varepsilon(\lambda_0) \subset \mathbb{C} \setminus \sigma(x)$. Hence $\mathbb{C} \setminus \sigma(x)$ is an open set, so $\sigma(x)$ is closed.

The function $R_x : \mathbb{C} \setminus \sigma(x) \to \mathcal{A}$ defined as $R_x(\lambda) = (\lambda e - x)^{-1}$ is called the **resolvent operator of** x.

Theorem A.1.5 *For all $x \in \mathcal{A}$, $\sigma(x) \neq \emptyset$.*

Proof: *For $x \in \mathcal{A}$, consider the resolvent operator of x, $R_x(\mu) = (\mu e - x)^{-1}$.*

For $\mu \in \mathbb{C} \setminus \sigma(x)$ let $\varepsilon = ||(\mu e - x)^{-1}||^{-1}$ then by theorem A.1.3 (b), $B_\varepsilon(\mu) \subset \mathbb{C} \setminus \sigma(x)$.

And for any $\lambda \in B_\varepsilon(\mu)$, $R_x(\lambda) = (\lambda e - x)^{-1} = \sum_{n=0}^{\infty}(\mu - \lambda)^n((\mu e - x)^{-1})^{n+1}$. Hence R_x is an analytic function on $\mathbb{C} \setminus \sigma(x)$.

If for some x, $\sigma(x) = \emptyset$ then $f \circ R_x$ is an entire function for each $f \in \mathcal{A}^*$. As it is also bounded, by Liouville's theorem, $f \circ R_x$ is constant for each f.

As $\mu \to \infty$, $||R_x(\mu)|| = |\mu|^{-1}||(e - \mu^{-1}x)^{-1}|| \to 0$ so by Hahn-Banach theorem $R_x \equiv 0$, but 0 cannot be an inverse, hence result follows.

Theorem A.1.6 ***(Gelfand-Mazur)*** *If $\mathcal{A}$ is a Banach algebra in which every nonzero element is invertible, then $\mathcal{A}$ is isomorphic to $\mathbb{C}$.*

Proof: *$\mathbb{C}e \subset \mathcal{A}$ where e is the identity of the algebra.*

If $x \notin \mathbb{C}e$, then $\lambda e - x \neq 0$ for all $\lambda \in \mathbb{C}$, then by hypothesis $\lambda e - x$ is invertible for all $\lambda \in \mathbb{C}$, but this implies that $\sigma(x) = \emptyset$, contradicting above theorem. Hence $\mathcal{A} = \mathbb{C}e$, and $\lambda \to \lambda e$ gives the isomorphism of $\mathbb{C}$ onto $\mathcal{A}$.

Let $\mathcal{A}$ be a Banach algebra and $x \in \mathcal{A}$. The **spectral radius of** x is the number $r(x) = \sup\{|\lambda| : \lambda \in \sigma(x)\}$.

Theorem A.1.7 *For $x \in \mathcal{A}$, $r(x) = \lim_{n\to\infty} ||x^n||^{1/n}$.*

Proof: *As $\lambda^n e - x^n = (\lambda e - x)(\lambda^{n-1}e + \lambda^{n-2}x + \ldots + \lambda x^{n-2} + x^{n-1})$, $\lambda \in \sigma(x)$ implies that $\lambda^n \in \sigma(x^n)$. So by theorem A.1.3 (a), $|\lambda|^n \leq ||x^n||$.*

Then $r(x) = \liminf |\lambda| \leq \liminf ||x^n||^{1/n}$

Now assume that $x \neq 0$ and choose $\lambda \in \mathbb{C}$ such that $|\lambda| < 1/r(x)$ (for $r(x) = 0$ take any λ). <u>Claim:</u> $\{(\lambda x)^n : n \in \mathbb{Z}^+\}$ is bounded.

By Banach-Steinhaus theorem it is enough to show that $\{f((\lambda x)^n) : n \in \mathbb{Z}^+\}$ is bounded for all $f \in \mathbb{B}(\mathcal{A})$, we know that such an f is differentiable.

Consider the function ϕ defined on $\{z \in \mathbb{C} : |z| < 1/r(x)\}$ by $\phi(z) = f((1-zx)^{-1})$. ϕ is analytic since: for $|z| < 1/||x||$, $(1-zx)^{-1} = \sum_{n\in\mathbb{Z}^+}(zx)^n$ and since f is a bounded linear functional, $\phi(z) = \sum_{n\in\mathbb{Z}^+} f(x^n)z^n$. And for the region $\{z : 0 < z < 1/r(x)\}$, $(1 - zx)^{-1} = (1/z)(z^{-1} - x)^{-1}$ so $\phi(z) = (1/z)f((z^{-1} - x)^{-1})$

ϕ is analytic on $R = \{z : z < 1/r(x)\}$ and $\phi(z) = \sum_{n\in\mathbb{Z}^+} f(x^n)z^n$ on R. So the sequence $(f(x^n)z^n)$ is bounded for all $z \in R$, and $|f(x^n)z^n| = |f((zx)^n)|$ so $\{(\lambda x)^n : n \in \mathbb{Z}^+\}$ is bounded. So for each λ with $0 < \lambda < 1/r(x)$, there is an $M < \infty$ such that $|\lambda^n|\,||x^n|| = ||(\lambda x)^n|| \leq M$ for all $n \in \mathbb{Z}^+$. Then

$\limsup_{n\to\infty} ||x^n||^{1/n} \leq \limsup_{n\to\infty} \frac{M^{1/n}}{|\lambda|} = \frac{1}{|\lambda|}$ *for each λ with $0 < \lambda < 1/r(x)$. So it must be true for $\inf\{\frac{1}{\lambda} : 0 < \lambda < 1/r(x)\} = r(x)$, i.e. $\limsup_{n\to\infty} ||x^n||^{1/n} \leq r(x)$.*

Lemma A.1.8 *Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $G(\mathcal{A})$, set of invertible elements in $\mathcal{A}$ and x be a boundary point of $G(\mathcal{A})$ where $x_n \to x$. Then $||x_n^{-1}|| \to \infty$.*

Proof: *Suppose $||x_n^{-1}|| \nrightarrow \infty$, then there is $M < \infty$ such that $||x_n^{-1}|| \leq M$ for infinitely many n. As $x_n \to x$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $||x_n - x|| \leq 1/M$. So there is $n_0 \geq N$ such that $||x_{n_0}^{-1}|| \leq M$.*

And $||e - x_{n_o}^{-1}x|| = ||x_{n_0}^{-1}(x_{n_0} - x)|| \leq ||x_{n_0}^{-1}||\,||x_{n_0} - x|| \leq 1$, so $e-(e-x_{n_o}^{-1}x) = x_{n_o}^{-1}x$ is invertible by the lemma A.1.2. Hence $x \in G(\mathcal{A})$. But $G(\mathcal{A})$ is open contradicting x being in the boundary. Hence the result follows.

Theorem A.1.9 *If there exists $M < \infty$ such that $||x||\,||y|| \leq M||xy||$ for each $x, y \in \mathcal{A}$ then $\mathcal{A} \cong \mathbb{C}$.*

Proof: *Let x and x_n be as in the above lemma, then for each $n \in \mathbb{N}$ we have:*

$||x_n^{-1}||\,||x_n|| \leq M||e||$ by the hypothesis.

As $||x_n^{-1}|| \to \infty$ by the lemma, $||x_n|| \to 0$, then $||x|| = 0 \Rightarrow x = 0$, so 0 is the only boundary point of $G(\mathcal{A})$ which implies that 0 is the only non-invertible element of $\mathcal{A}$, hence by Gelfand-Mazur theorem (A.1.6) the result follows.

If $\mathcal{A}$ is any algebra, a **left(right) ideal** of $\mathcal{A}$ is a subalgebra $\mathcal{I}$ such that $xy \in \mathcal{I}$, for every $x \in \mathcal{A}$, $y \in \mathcal{I}(x \in \mathcal{I}$, $y \in \mathcal{A})$

Proposition A.1.10 *If $\mathcal{I}$ is a closed ideal in a Banach algebra $\mathcal{A}$, then $\mathcal{A}/\mathcal{I}$ is a Banach algebra with the quotient norm: $||\bar{a}|| = ||a + \mathcal{I}|| = \inf\{||a + b|| : b \in \mathcal{I}\}$.*

Proof: *As $\mathcal{I}$ is a closed subspace, we know that $\mathcal{A}/\mathcal{I}$ is a Banach space with the following operations:*

$(x+\mathcal{I})+(y+\mathcal{I})=(x+y)+\mathcal{I}$

$\alpha(x+\mathcal{I})=\alpha x+\mathcal{I}$

$||x+\mathcal{I}||=\inf_{y\in\mathcal{I}}||x+y||$

Now define multiplication as: $(x+\mathcal{I})(y+\mathcal{I})=xy+\mathcal{I}$

This multiplication is well-defined as: if $a-a'\in\mathcal{I}$ *and* $b-b'\in\mathcal{I}$ *then* $a'b'-ab=(a'-a)b'+a(b'-b)\in\mathcal{I}$ *(since* $\mathcal{I}$ *is an ideal), so* $a'b'+\mathcal{I}=ab+\mathcal{I}$.

Moreover,

$$\begin{aligned}||ab+\mathcal{I}|| &= \inf_{x\in\mathcal{I}}||ab+x||=\inf_{x_1,x_2\in\mathcal{I}}||ab+ax_2+bx_1+x_1x_2||\\ &= \inf_{x_1,x_2\in\mathcal{I}}||(a+x_1)(b+x_2)||\leq\inf_{x_1,x_2\in\mathcal{I}}||a+x_1||\,||b+x_2||\end{aligned}$$

Hence $||ab+\mathcal{I}||\leq||a+\mathcal{I}||\,||b+\mathcal{I}||$, *and* $\mathcal{A}/\mathcal{I}$ *is a Banach algebra.*

Proposition A.1.11 *Let* $\mathcal{A}$ *be a commutative Banach algebra and* $\mathcal{I}$ *be a proper ideal. Then:*

1. $\mathcal{I}$ *does not contain invertible elements.*
2. $\overline{\mathcal{I}}$ *is a proper ideal.*
3. $\mathcal{I}$ *is contained in a maximal ideal.*
4. *If* $\mathcal{I}$ *is maximal then it is closed.*

Proof: *1. Suppose* $x\in\mathcal{I}$ *is an invertible element. Then* $x^{-1}\in\mathcal{A}$, *and* $x^{-1}x=e\in\mathcal{I}$ *which implies that* $x=xe\in\mathcal{I}$ *for each* $x\in\mathcal{A}$ *contradicting to* $\mathcal{I}$ *being a proper ideal.*
2. As $\mathcal{I}$ *is proper,* $\mathcal{I}\subset\mathcal{A}\backslash G(\mathcal{A})$, *which is a closed as* $G(\mathcal{A})$ *is open, then* $\overline{\mathcal{I}}\subset\mathcal{A}\backslash G(\mathcal{A})$, *and* $e\notin\mathcal{A}\setminus G(\mathcal{A})$, *hence* $\overline{\mathcal{I}}$ *is proper.*
3. Consider the set $\mathfrak{F}=\{\mathcal{J}\subset\mathcal{A}:\mathcal{I}\subset\mathcal{J}$ *and* $\mathcal{J}$ *is a proper ideal of* $\mathcal{A}\}$. *Order* $\mathfrak{F}$ *by inclusion, then it is inductive (take a chain* $(\mathcal{J}_\alpha)$, *then let* $\mathcal{J}=\cup_\alpha\mathcal{J}_\alpha$ *so that* $\mathcal{J}$ *is an upper bound for this chain which is in the set* $\mathfrak{F}$*). Hence by Zorn's lemma,* $\mathfrak{F}$ *has a maximal element, say* $\mathcal{J}_0$. *This* $\mathcal{J}_0$ *is a maximal ideal, since otherwise there is a proper ideal* $\mathcal{J}_1$ *such that* $\mathcal{J}_1\supset\mathcal{J}_0\supset\mathcal{I}$, *then* $\mathcal{J}_1$ *must be in* $\mathfrak{F}$, *contradicting maximality of* $\mathcal{J}_0$.
4. $\overline{\mathcal{I}}$ *is proper ideal by 2, and* $\mathcal{I}\subset\overline{\mathcal{I}}$, *but* $\mathcal{I}$ *is maximal ideal, so* $\mathcal{I}=\overline{\mathcal{I}}$

For a commutative Banach algebra $\mathcal{A}$, $\mathfrak{M}(\mathcal{A})$ denotes the set of all maximal ideals of $\mathcal{A}$.

Theorem A.1.12 *Let $\mathcal{A}$ be an algebra with unit $\mathbf{1}$. Then the intersection of all maximal left ideals of $\mathcal{A}$ and the intersection of all maximal right ideals of $\mathcal{A}$ are the same sets.*

Let $\mathcal{A}$ be a Banach algebra, the intersection of all maximal left(or right) ideals of $\mathcal{A}$ is called **radical** of $\mathcal{A}$; denoted as $Rad\mathcal{A}$. And $\mathcal{A}$ is **semisimple** if $Rad\mathcal{A} = \{0\}$.

Example A.1.13 *Let X be a Banach space, then $\mathcal{L}(X)$ is semisimple.*
Let $\xi \neq 0$ be fixed in X and let $I_\xi = \{T : T \in \mathcal{L}(X), T\xi = 0\}$, then it is a left ideal of $\mathcal{L}(X)$. Claim: It is a maximal left ideal.
Suppose J is another left ideal containing it, with $I_\xi \neq J$, then $J\xi = \{T\xi : T \in J\}$ is a linear subspace of X different from $\{0\}$ and it is invariant under all $S \in \mathcal{L}(X)$ as J is a left ideal. If $J\xi \neq X$, then there is $x_1 \notin J\xi$ in X and there is $x_2 \neq 0$ in $J\xi$. Then there is $S \in \mathcal{L}(X)$ such that $Sx_2 = x_1$, contradicting the invariance of $J\xi$. So $J\xi = X$ and there exists $W \in \mathcal{L}(X)$ such that $W\xi = \xi$.
Then for each $T \in \mathcal{L}(X)$, $T - TW \in I_\xi$, so $T \in J + I_{xi} \subset J$, hence $J = \mathcal{L}(X)$.
Then for any nonzero element $\xi \in X$, I_ξ is a maximal left ideal, hence

$$Rad\mathcal{L}(X) \subset \cap_{\xi \neq 0} I_\xi = \{0\}$$

A.2 Measures on Locally Compact Spaces/Groups

A **Radon measure** on a topological space X is a regular Borel measure which is finite on compact sets.

Theorem A.2.1 ***(Riesz-Markov)*** *Let X be a locally compact (Hausdorff) space. To each bounded linear functional T on $C_0(X)$, there corresponds a unique Radon measure m on X such that $Tf = \int_X f dm$, for $f \in C_0(X)$.*

Theorem A.2.2 ***(Riesz Representation)*** *(1) Let X be a compact space and Λ in $\mathcal{L}(C(X), \mathbb{C})$ be positive then there is a unique regular Borel measure μ on X such that $\Lambda f = \int_G f d\mu$ for every $f \in C(X)$.*

(2) Let X be a locally compact space and $\Lambda : C_c(X) \to \mathbb{C}$ is positive linear then there is a unique Radon measure μ on X such that $\Lambda f = \int_X f d\mu$ for every $f \in C_c(X)$.

A measure μ is **absolutely continuous** with respect to a measure m if for every Borel set E $m(E) = 0$ implies that $\mu(E) = 0$.

Theorem A.2.3 ***(Radon-Nikodym)*** *If $\mu \in M(X)$ where X is a locally compact (Hausdorff) space and if m is a non-negative measure on X and μ is absolutely continuous with respect to m then there is $f \in L^1(m)$ such that $\mu(E) = \int_E f dm$ for each Borel set E. Moreover, $||\mu|| = \int_X |f| dm = ||f||_1$.*

Theorem A.2.4 ***(Hahn-Jordan Decomposition)*** *If $(X, \mathcal{M})$ is a measurable space and μ is a signed measure on $\mathcal{M}$ then there exist two sets $P, N \in \mathcal{M}$ such that $P \cup N = X$, $P \cap N = \emptyset$ and P is positive, N is negative for μ. Moreover, there are positive measures μ^+, μ^- such that $\mu^+(E) = \mu(E \cap P)$ and $\mu^-(E) = \mu(E \cap N)$ and $\mu = \mu^+ - \mu^-$*

Theorem A.2.5 ***(Stone-Weierstrass)*** *Let X be a compact Hausdorff space. If $A \subset C(X)$ is a separating subalgebra containing $\mathbb{C}$ then $\overline{A} = C(X)$.*

If X is a locally compact (non-compact) Hausdorff space and $\mathcal{A}$ is a separating subalgebra of $C_0(X)$ then either $\mathcal{A}$ is dense in $C_0(X)$ or $\mathcal{A} = \{f \in C_0(X) : f(x_0) = 0\}$ for some $x_0 \in X$.

Theorem A.2.6 ***(Fubini)*** *Let X, Y be a locally compact (Hausdorff) spaces and μ, λ be regular measures on them respectively. If $\mu \geq 0$, $\lambda \geq 0$ and f is a Borel function on $X \times Y$ such that $f \geq 0$, then*

$\int_{X \times Y} f d(\mu \times \lambda) = \int_X \int_Y f(x, y) d\lambda(y) d\mu(x) = \int_Y \int_X f(x, y) d\mu(x) d\lambda(y)$.

If $\mu \in M(X)$, $\lambda \in M(Y)$, f is a Borel function on $X \times Y$ and

$\int_X \int_Y |f(x, y)| d|\lambda|(y) d|\mu|(x) < \infty$ *then above also holds for this setting.*

For any locally compact group G there is a non-zero Radon measure m on G that is invariant under left translations, i.e. $m(xE) = m(E)$ for every Borel set $E \subset G$ and $x \in G$. If ν is another such measure then there is a constant $c > 0$ such that $\nu = cm$.

The proof of this statement can be found in various sources. In the case of LCA groups, the proof is simpler and given by Alexander J. Izzo for existence. Also uniqueness is easy to prove:

Uniqueness Fix $g \in C_c(G)$ so that $\int_G g dm = 1$. Define $c = \int_G g(-x)d\nu$. Then for any $f \in C_c(G)$ we have

$$\begin{aligned}\int_G f d\nu &= \int_G g(x)dm(x) \int_G f(y)d\nu(y) = \int_G g(x)dm(x) \int_G f(x+y)d\nu(y) \\ &= \int_G \int_G g(x)f(x+y)dm(x)d\nu(y) = \int_G \int_G g(x-y)f(x)dm(x)d\nu(y) \\ &= \int_G f(x)dm(x) \int_G g(x-y)d\nu(y) = c \int_G f dm\end{aligned}$$

Hence $\nu = cm$.

The measure defined above which is unique up to a constant multiple is called the **(left) Haar measure** on G.

Proposition A.2.7 *For G a LCA group and $f \in L^1(G)$, $\int_G f(x)dm = \int_G f(-x)dm$.*

Proof: *Let $m'(E) = m(-E)$ for every Borel set $E \subset G$, then as $-E$ is Borel set for every Borel set E, m' is a non-zero Radon measure on G. Moreover, for E Borel subset, $x \in G$, $m'(x+E) = m(-x-E) = m(-E) = m'(E)$, hence m' is a Haar measure on G. Then by uniqueness of Haar measure, there is a constant $c > 0$ such that $m' = cm$. Now for E Borel set, $A = -E \cap E$ is also a Borel set and $A = -A$. $m'(A) = m(-A) = m(A)$ and $m'(A) = cm(A)$, hence $c = 1$.*
Now $\int_G f(-x)dm(x) = \int_G f(x)dm'(x) = \int_G f(x)cdm(x) = \int_G f(x)dm(x)$

Theorem A.2.8 ***(Gelfand-Raikov)*** *If G is locally compact group, the irreducible representations of G separate points on G.*

Theorem A.2.9 *(Plancherel) The Fourier transform restricted to $L^1(G) \cap L^2(G)$ is an isometry onto a dense linear subspace of $L^2(G)$. Hence it can be extended to a unique isometry of $L^2(G)$.*

Complementary Notes:

- The set $\{f \in L^1(G) : \hat{f} \text{ has compact support}\}$ is dense in $L^1(G)$.
- For $\mu \in M(G)$, $\int_G f d\mu = \int_{\widehat{G}} \hat{f}\hat{\mu} dm_{\widehat{G}}$
- A subgroup is open iff it has non-empty interior. (by translation invariance, same is true for cosets)
- If M closed convex subset of a Banach space and $f : M \to \mathbb{R}$ is continuous convex function then f is lower semicontinuous in the weak topology.
- *Inverse Fourier transform*(See [13], 1.5):

  Let $B(G) = \{f : G \to \mathbb{C} : f \text{ is in the form } f(x) = \int_{\widehat{G}} < x, \kappa > d\mu(\kappa),\ x \in G\}$. For fixed Haar measure on G, the Haar measure on $\widehat{G}$ can be normalized so that, for each $f \in L^1(G) \cap B(G)$, the following inversion formula holds:

  $f(x) = \int_{\widehat{G}} \hat{f}(\kappa) < x, \kappa > d\kappa$, for each $x \in G$

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