# ON CLASSIFICATION OF ELEMENTS OF AFFINE WEYL GROUP OF INFINITE DIMENSIONAL LOOP GROUP $LG$ FOR COMPACT LIE GROUP $G = A_3$

by

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I certify that this thesis satisfies all the requirements as a thesis for the degree of Master of Science.

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This is to certify that we have read this thesis and that in our opinion it is fully adequate, in scope and quality as a thesis for the degree of Master of Science.

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## ABSTRACT

ON CLASSIFICATION OF ELEMENTS OF WEYL GROUP OF

THE LOOP GROUP $LG$ FOR COMPACT GROUP $G = A_3$

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In this thesis we describe the general theory of compact connected semi-simple Lie groups and algebras including root system, Weyl group an Dynkin diagram, and we describe the root system and Weyl group of $LG$ as well. We give the group presentation of the affine Weyl group of $G = \hat{A}_3$. We classify all elements of affine Weyl group $\tilde{W}$ and $\hat{W} = \tilde{W}/W$ for $\hat{A}_3$.



Keywords: Lie group, Lie algebra, Loop group, Affine Weyl group, Kač-Moody Lie algebras.

# ÖZET

TIKIZ $G = A_3$ İÇİN DÖNGÜ GRUBU LG'NİN WEYL GRUBUNUN

ELEMANLARININ SINIFLANDIRILMASI

Topraklı, Ali

Master Tezi, Matematik Bölümü

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ŞUBAT 2007, 94 sayfa

Bu tezde döngü grupların genel kuramı ile beraber klasik, tıkız, bağlantılı, yarı-bağlantılı Lie grup ve cebirlerinin karakter sistemi, simetri grupları ve Dynkin çizgelerini içeren genel kuramını tanımlıyoruz. $G = \hat{A}_3$'ın afin Weyl grubunun üreteç ve bağıntılar anlamında grup temsilini veriyoruz. Ayrıca, $\hat{A}_3$ için $\tilde{W}$ ve $\hat{W} = \tilde{W}/W$'nin elemanlarını sınıflandırıyoruz.

Anahtar Kelimeler: Lie grup, Lie cebiri, Döngü grubu, Weyl grup, Kač-Moody Lie cebirleri.

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*To My Family*

## TABLE OF CONTENTS

# CHAPTER 1
# INTRODUCTION

In [23], Kumar described the Schubert classes which are the dual to the closures of the Bruhat cells in the flag varieties of the Kač-Moody groups associated to the infinite dimensional Kač-Moody algebras. These classes are indexed by affine Weyl groups and can be choosen as elements of integral cohomologies of the homogeneous space $\widehat{L}_{\mathrm{pol}}G_{\mathbb{C}}/\widehat{B}$ for any compact simply connected semi-simple Lie group $G$. Later, Kumar & Kostant gave explicit cup product formulas of these classes in the cohomology algebras by using the relation between the invariant-theoratic relative Lie algebra cohomology theory (using the representation module of the nilpotent part) with the purely nil-Hecke rings [22]. These explicit product formulas involve some BGG-type operators $A^i$ and reflections. In the paper [29] of Ozel, using some homotopy equivalences, cohomology ring structures of $LG/T$ have been determined where $LG$ is the smooth loop space on $G$. He has calculated the products and explicit cohomology ring structure of $LG/T$ for $G = \hat{A}_1$ using these ideas. He found that it has a quotient of the divided power algebra. Other details can be found in [29]. Later Ozel & Yılmaz [30] calculated the products and explicit ring structures of integral cohomology of $LG/T$ and $\Omega G$ for $G = \hat{A}_2$ using these ideas. In this thesis, we give explicit presentation of affine Weyl group of the loop group $LSU_4$ and make classify elements of affine Weyl group $LSU(4)$ and determine the minimal elements of each coset $w\tilde{W}$. We note that all multiplications in the Weyl group can be verified in MATHEMATICA.

Some comments about the structure of this thesis are in order. It is written for a reader with a first course in matrix groups and some understanding of the structure of compact semi-simple Lie groups and their representations, plus some mathematical maturity. Some good general references are Bredon [4] for topology and geometry, Pressley-Segal [32] for loop groups and their representations, Kač [18] for Kač-Moody

algebra theory, Hiller [14] for reflection and Coxeter groups, and Humpreys [16] for Lie algebras and representations, [2] for matrix groups.

The organization of this thesis is as follows.

Chapter 2 is about the theory of Lie algebras and their representations on vector spaces. Chapter 2 includes descriptions of manifolds, tangent space, Lie groups, one-parameter groups, adjoint derivative and differentials as well.

In Chapter 3 we describe the general theory of compact connected semi-simple Lie groups and algebras including root system, Weyl group an Dynkin diagram.

In Chapter 4, we describe the root system and Weyl group of $LG$ and we give the group presentation of the affine weyl group of $G = \hat{A}_3$. We classify all elements of affine Weyl group $\tilde{W}$ and $\hat{W} = \tilde{W}/W$ for $\hat{A}_3$.

# CHAPTER 2

# SEMI-SIMPLE LIE GROUPS AND THEIR ALGEBRAS

The general references for this chapter are [16] and [2] respectively.

Lie algebras arise "in nature " as vector spaces of linear transformations endowed with a new operation bracket $[x, y] = xy - yx$ which is in general neither commutative nor associative where the operations on the right side are the usual ones.

**Definition 2.1** *A vector space $L$ over a field $\mathbb{F}$, with an operation $L \times L \to L$, denoted $(x, y) \mapsto [x, y]$ and called* bracket or commutator *of $x$ and $y$, is called a* Lie algebra *over $\mathbb{F}$ if the following axioms are satisfied.*

*(L1) The bracket operation is bilinear.*

*(L2)* $\forall x \in L, [x, x] = 0$.

*(L3)* $\forall x, y, z \in L, [x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ *(Jacobi identity).*

We note that if $(L1)$ and $(L2)$ are applied to $[x + y, x + y]$, then we have anti-commutativity $[x, y] = -[y, x]$. Conversely, if $\operatorname{char} \mathbb{F} \neq 2$, it is clear that the anti-commutativity will imply $(L2)$.

We say that two Lie algebras $L$, $L'$ over $\mathbb{F}$ are *isomorphic* if there exists a vector space isomorphism $\phi : L \to L'$ satisfying $\phi([xy]) = [\phi(x), \phi(y)]$ for all $x, y \in L$ where $\phi$ is called an *isomorphism of Lie algebra*.

A subspace $K$ of $L$ is called a *subalgebra* if $[x, y] \in K$ whenever $x, y \in K$; in particular, $K$ is a Lie algebra in its own right relative to the inherited operations. Note that any nonzero element $x \in L$ defines a one dimensional subalgebra $\mathbb{F}x$, with trivial multiplication, because of $(L2)$.

Let $V$ be a finite dimensional vector space over $\mathbb{F}$, and let $\hom(V)$ be the set of linear transformations $V \to V$. As a vector space over $\mathbb{F}$, $\hom(V)$ has dimension

$n^2$ where $n = \dim V$, and $\hom(V)$ is a ring relative to the usual product operation *composition*. We define a new operation $[x, y] = xy - yx$, called the bracket of $x$ and $y$. With this operation $\hom(V)$ becomes a Lie algebra over $\mathbb{F}$. In order to distinguish this new algebra structure from the old associative one, we write $\mathfrak{g}l$ for $\hom(V)$ viewed as a Lie algebra and call it *general linear algebra* because it is closely associated with the general linear group $GL(V)$ consisting of all invertible endomorphisms of $V$.

**Definition 2.2** *Any subalgebra of a Lie algebra $\mathfrak{g}l(V)$ is called a linear Lie algebra.*

If we fix a basis for $V$, then we are identifying $\mathfrak{gl}(V)$ with the set of all $nxn$ matrices over $\mathbb{F}$, denoted $\mathfrak{gl}(n, \mathbb{F})$. For the standard basis consisting of the matrices $e_{ij} = [E_{\mu\nu}]$ where

$$E_{\mu\nu} = \begin{cases} 1 & \mu = i, \nu = j, \\ 0 & \text{elsewhere,} \end{cases}$$

we can write down the multiplication table for $\mathfrak{g}l(n, \mathbb{F})$. Since $e_{ij}e_{kl} = \delta_{jk}e_{il}$, it follows

$$[e_{ij}, e_{kl}] = \delta_{jk}e_{il} - \delta_{li}e_{kj}$$

where the coefficients are all $\pm 1$ or 0.

They fall into four families $A_l, B_l, C_l, D_l$, $l \geq 1$ and they are called the *classical algebras* because they correspond to certain classical linear Lie groups. In the thesis we will be interested to only classical linear Lie algebra $A_l$, $l \geq 1$.

Let $\dim V = l + 1$ and Let $\mathfrak{sl}(V)$, or $\mathfrak{sl}(l + 1, \mathbb{F})$ be the set of endomorphisms of $V$ having trace zero. Since $\text{trace}(xy) = \text{trace}(yx)$, and $\text{trace}(x + y) = \text{trace}(x) + \text{trace}(y)$, then $\text{sl}(V)$is a subalgebra of $\text{gl}(V)$ which is called *special linear algebra* because of its connection with the special linear group $SL(V)$ of endomorphisms of $\det 1$. On the one hand $\text{sl}(V)$ is a proper subalgebra of $\text{gl}(V)$, hence the dimension is at most $(l+1)^2 - 1$. So, we can exhibit this number of linearly independent matrices of trace zero. Let us take all $e_{ij}$ $(i \neq j)$, along with all $h_i = e_{ii} - e_{i+1,i+1}$ $(1 \leq i \leq l)$, for a total of $l + (l+1)^2 - (l+1) = l(l+2)$ matrices. We shall always view this as the standard basis

for $\mathrm{sl}(l+1,\mathbb{F})$.

Now we introduce other subalgebras of $\mathfrak{gl}(n,\mathbb{F})$. Let $\mathfrak{t}(n,\mathbb{F})$ be the set of upper triangular matrices $[a_{ij}]$ where $a_{ij}=0$ if $i>j$. Let $\mathfrak{n}(n,\mathbb{F})$ be the strictly upper triangular matrices $[a_{ij}]$ where $a_{ij}=0$ if $i\geq j$). Finally, let $\mathfrak{d}(n,\mathbb{F})$ be the set of all diagonal matrices. It is trivial to check that each of these is closed under the bracket. Notice also that $\mathfrak{t}(n,\mathbb{F})=\mathfrak{d}(n,\mathbb{F})\bigoplus\mathfrak{n}(n,\mathbb{F})$ with $[\mathfrak{d}(n,\mathbb{F}),\mathfrak{n}(n,\mathbb{F})]=\mathfrak{n}(n,\mathbb{F})$, hence $[\mathfrak{t}(n,\mathbb{F}),\mathfrak{t}(n,\mathbb{F})]=\mathfrak{n}(n,\mathbb{F})$.

Some Lie algebras of linear transformations arise most naturally as derivations of algebras.

**Definition 2.3 (*Lie algebras of derivations*)** *Let $U$ be an $\mathbb{F}$-algebra. By a derivation of $U$ we mean a linear map $\delta: U\to U$ satisfying the familiar product rule $\delta(ab)=a\delta(b)+\delta(a)b$.*

It is easily checked that the collection $\mathrm{Der}U$ is a subalgebra of $\mathfrak{gl}(U)$.

Since a Lie algebra $L$ is an $\mathbb{F}$-algebra, $\mathrm{Der}L$ can be defined. Certain derivations arise quite naturally, as follows. If $x\in L$, by bilinearity of the bracket operation, the map $y\mapsto[x,y]$ is an endomorphism of $L$ which is denoted by $\mathrm{adj}x$. Using Jacobi identity

$$[x,[y,z]]=[[x,y],z]+[y,[x,z]]=[y,[x,z]+[[x,y],z],$$

hence $\mathrm{adj}x\in DerL$. The map $L\to DerL$ sending $x$ to $\mathrm{adj}x$ is called *adjoint representation* of $L$. Derivations of this form are called inner, all others outer.

We will use $\mathrm{adj}_L x$ or $\mathrm{ad}_K x$ to indicate that $x$ is acting on $L$ and $K$ respectively. For example, if $x$ is a diagonal matrix, then $\mathrm{adj}_{\mathfrak{n}(n,\mathbb{F})}(x)=0$, whereas $\mathrm{adj}_{\mathfrak{gl}(n,\mathbb{F})}(x)$ need not be zero.

## 2.1 Abstract Lie algebras.

So far we have looked at some natural examples of linear Lie algebras. In fact, every finite dimensional Lie algebra is isomorphic to some linear Lie algebra. Later this claim will be proved using matrix representation.

For example, if $L$ is an arbitrary finite dimensional vector space over $\mathbb{F}$, we can view $L$ as a Lie algebra by $[x, y] = 0$ for all $x, y \in L$. Such an algebra having trivial Lie multiplication is called abelian. If $L$ is any Lie algebra with basis $x_1, \ldots, x_n$ it is clear that the entire multiplication table of $L$ can be obtained from the structure constants ${a^k}_{ij}$ which occur in the expressions

$$[x_i, x_j] = \sum_{k=1}^{n} {a^k}_{ij} x_k.$$

By anti-commutativity these constant structures for which $i \geq j$ can be deduced from the others. So, it is possible to define an abstract Lie algebra by specifying a set of structure constants. Of course, just any set of scalars $\{a_{ij}^k\}$ will not do, it is enough to require the following identities which are implied by (L2) and (L3)

$$a_{ii}^k = 0 = a_{ij}^k + a_{ji}^k$$

and

$$\sum_k \left( a_{ij}^k a_{kl}^m + a_{jl}^k a_{ki}^m + a_{li}^k a_{kj}^m \right) = 0.$$

In practice, we will not construct Lie algebras in this artificial way. But, as an application of the abstract point of view, we can determine all Lie algebras of dimension $\leq 2$ up to isomorphism. In dimension 1 there is a single basis vector $x$, with multiplication table $[x, x] = 0$ by $(L2)$. In dimension 2, we start with a basis $x, y$ of $L$. Clearly, all products in $L$ yield scalar multiples of $[x, y]$. If these are all 0, then $L$ is abelian. Otherwise, we can replace $x$ in the basis by a vector spanning one dimensional space of multiplies of the original $[x, y]$, and take $y$ to be any other vector independent of the new $x$. Abstractly, therefore, at most one non-abelian $L$ with $\dim L = 2$ exists.

## 2.2 Ideals and homomorphisms.

**Definition 2.4** *A subspace $I$ of a Lie algebra $L$ is called an* ideal *of $L$ if for $x \in L, y \in I, [x, y] \in I$.*

Since $[x, y] = -[y, x]$, the condition $\forall x \in L, \forall y \in I, [x, y] \in I$ implies $[y, x] \in I$. Ideals play the role in Lie algebra theory which is played by normal subgroups in group theory and by two sided ideals in ring theory. They arise as kernels of homomorphisms.

Obviously $\{0\}$ and $L$ itself are ideals of $L$. A less trivial example is the center

$$Z(L) = \{z \in L \mid [xz] = 0 \, \forall \, x \in L\}.$$

Another important example of ideals is the *derived algebra* of $L$, denoted $[L, L]$, which is analogous to the commutator subgroup of a group. It consists of all linear combinations of commutators $[x, y]$, and is clearly an ideal.

Evidently $L$ is abelian if and only if $[L, L] = 0$. If $I, J$ are two ideals of a lie algebra $L$, $I + J = \{x + y \mid x \in I, y \in J\}$ is also an ideal. Similarly, $[I, J] = \{\sum a_{ij}[x_i, y_j] \mid x_i \in I, y_j \in J\}$ is an ideal; the derived algebra $[L, L]$ is just a special case of this construction.

It is natural to analyze the structure of a Lie algebra by looking at its ideals.

**Definition 2.5** *$L$ is called* simple *If $L$ has no ideals except itself and $\{0\}$.*

Clearly, if $L$ is simple then $Z(L) = 0$ and $L = [L, L]$.

Let $L = \mathfrak{sl}(2, \mathbb{F})$ with $\mathrm{char}\mathbb{F} \neq 2$. Let us consider as standard basis for $L$ the three matrices

$$x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \; y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \; h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The multiplication table is then completely determined by the equations $[x, y] = h, [h, x] = 2x, [h, y] = -2y$. Notice that $x, y, h$ are eigenvectors for $\mathrm{adj}\, h$, corresponding to the eigenvalues $2, -2, 0$ respectively. Let $I \neq 0$ be an ideal of $L$, and let $ax+by+ch$ be an arbitrary non-zero element of $I$. Applying $\mathrm{adj}\, x$ twice, we get $-2ay \in I$. Therefore, if $a$ or $b$ is nonzero, $I$ contains either $x$ or $y$. Then $I = L$ follows. We conclude that $L$ is simple.

**Definition 2.6 (*Homomorphisms and representations*)** *Let $L$ and $L'$ be two Lie algebras over $\mathbb{F}$. A linear transformation $\phi : L \to L'$ is called a* homomorphism *from $L$ to $L'$ if it preserves bracket operations. i.e. $\phi$ is a homomorphism from $L$ to $L'$ if it is a linear map satisfying $\phi[x, y]_L = [\phi(x), \phi(y)]_{L'}$.*

A representation of a Lie algebra $L$ on $V$ *is a homomorphism $\phi : L \to \mathfrak{gl}(V)$ where $V$ is a vector space over $\mathbb{F}$.*

Since a Lie algebra $L$ is a vector space and $DerL \subset \mathfrak{gl}(L)$, the only important example to keep in mind is the adjoint representation $\mathrm{adj} : L \to \mathfrak{gl}(L)$ which sends $x$ to $\mathrm{adj}\, x$, where $\mathrm{adj}\, x(y) = [x, y]$. By bilinearity of the bracket operation $[,]$, adj is a linear transformation. To see that it preserves the bracket, we calculate as follows

$$\begin{aligned}
[\mathrm{adj}\, x, \mathrm{adj}\, y]\,(z) &= \mathrm{adj}\, x\, \mathrm{adj}\, y(z) - \mathrm{adj}\, y\, \mathrm{adj}\, x(z) \\
&= \mathrm{adj}\, x([y, z]) - \mathrm{adj}\, y([x, z]) \\
&= [x, [y, z]] - [y, [x, z]] \\
&= [x, [y, z]] + [[x, z], y] \\
&= [[x, y], z] \\
&= \mathrm{adj}\ [x, y]\,(z).
\end{aligned}$$

What is the kernel of adj? It consists of all $x \in L$ for which $\mathrm{adj}\, x = 0$, for which $[x, y] = 0$, $\forall y \in L$. This has an interesting consequence. If $L$ is simple, then $Z(L) = 0$, so that $\mathrm{adj} : L \to \mathfrak{gl}(L)$ is a monomorphism. It means that *any simple Lie algebra is isomorphic to a linear Lie algebra.*

## 2.3 Lie Groups and their Lie algebras

The general reference for this section is [2].

### *2.3.1 Groups of matrices.*

Let $M_{m,n}(\mathbb{F})$ be the set of $m \times n$ matrices whose entries are in $\mathbb{F}$. We will denote the $(i,j)$ entry of an $m \times n$ matrix A by $A_{ij}$ or $a_{ij}$ and also write

$$\begin{pmatrix} a_{11} & \dots & a_{1n} \\ \dots & \dots & \dots \\ a_{m1} & \dots & a_{mn} \end{pmatrix}.$$

We will use the special notations

$$M_n(\mathbb{F}) = M_{n,n}(\mathbb{F}), \mathbb{F}^n = M_{n,1}(\mathbb{F}).$$

$M_{m,n}(\mathbb{F})$ is a $\mathbb{F}$-vector space with the operations of matrix addition and scalar multiplication. The zero vector is the $m \times n$ zero matrix $O_{m,n}$ which we will often denote O when the size is clear from the context. The matrices $E^{rs}$ with $r = 1, ..., m$,$s = 1, ..., n$ and

$$(E^{rs})_{ij} = \delta_{ir}\delta_{js} = \begin{cases} 1 & \text{if } i = r \text{ and } j = s, \\ 0 & \text{otherwise} \end{cases}$$

form a basis of $M_{m,n}(\mathbb{F})$, hence its dimension as a $\mathbb{F}$-vector space is

$$\dim_{\mathbb{F}} M_{m,n}(\mathbb{F}) = mn.$$

When n=1 we will denote the standard basis vectors of $(\mathbb{F})^n = M_{n,1}(\mathbb{F})$ by

$$e_r = E^{r1}(r = 1, ..., m).$$

As well as being a $\mathbb{F}$-vector space of dimension $n^2$, $M_n(\mathbb{F})$ is also a ring with the usual addition and multiplication of square matrices, with zero $O_n = O_{n,n}$ and the $n \times n$ identity matrix $I_n$ as its unity; $M_n(\mathbb{F})$is not commutative except when n=1.

**Proposition 2.7** *The determinant function* $\det : M_n(\mathbb{F}) \to \mathbb{F}$ *has the following prop-*

*erties.*

*(i) For* $A, B \in M_n(\mathbb{F})$, $\det(AB) = \det A \det B$

*(ii)* $\det I_n = 1$.

*(iii)* $A \in M_n(\mathbb{F})$ *is invertible if and only if* $\det A \neq 0$.

We will use the notation

$$GL_n(\mathbb{F}) = \{A \in M_n(\mathbb{F}) : \det A \neq 0\}$$

for the set of invertible $n \times n$ matrices, and

$$SL_n(\mathbb{F}) = \{A \in M_n(\mathbb{F}) : \det A = 1\} \subseteq GL_n(\mathbb{F})$$

for the set of $n \times n$ *unimodular* matrices.

**Theorem 2.8** *The sets* $GL_n(\mathbb{F})$, $SL_n(\mathbb{F})$ *are groups under matrix multiplication. Furthermore,* $SL_n(\mathbb{F}) \leq GL_n(\mathbb{F})$*,i.e,* $SL_n(\mathbb{F})$ *is a subgroup of* $GL_n(\mathbb{F})$.

Because of these group structures, $GL_n(\mathbb{F})$ is called $n \times n$ *general linear group.* When $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$, we will refer to $GL_n(\mathbb{R}), SL_n(\mathbb{R})$ or $GL_n(\mathbb{C}), SL_n(\mathbb{C})$ as *real or complex general linear or special linear groups.*

### *2.3.2 The matrix exponential and logarithm.*

Throughout this section, we will assume that $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. The power series

$$\mathrm{Exp}\,(X) = \sum_{n \geq 0} \frac{1}{n!} X^n, \; \log(X) = \sum_{n \geq 0} \frac{-1^{n-1}}{n} X^n$$

have radii of convergence (r.o.c) $\infty$ and 1 respectively. If $z \in \mathbb{C}$, the series $\mathrm{Exp}(z), \log(z)$ converge absolutely whenever $|z| <$ (r.o.c).

Let $A \in M_n(\mathbb{F})$. The matrix-valued series

$$\mathrm{Exp}(A) = \sum_{n\geq 0} \frac{1}{n!} A^n = I + A + \frac{1}{2n!} A^2 + \frac{1}{3n!} A^3 + ...,$$

$$\log(A) = \sum_{n\geq 1} \frac{(-1)^{n-1}}{n} A^n = A - \frac{1}{2} A^2 + \frac{1}{3} A^3 - \frac{1}{4} A^4 + \dots,$$

converge whenever $\| A \| <$ (r.o.c). So $\mathrm{Exp}(A)$ makes sense for every $A \in M_n(\mathbb{F})$ while $\log(A)$ only exists if $\| A \| < 1$.

**Proposition 2.9** *Let $A \in M_n(\mathbb{F})$.*

*(i) For $u, v \in \mathbb{C}$, $Exp\,((u+v)A) = Exp\,(uA)Exp\,(vA)$.*

*(ii) $ExpA \in GL_n(\mathbb{F})$ and $Exp\,(A)^{-1} = Exp\,(-A)$.*

**Proof.**

(i)By expanding the first series we obtain

$$\begin{aligned} \mathrm{Exp}((u+v)A) &= \sum_{n\geq 0} \frac{1}{n!} (u+v)^n A^n \\ &= \sum_{n\geq 0} \frac{(u+v)^n}{n!} A^n. \end{aligned}$$

By a sequence of obvious manipulations that are justified since these series are all

absolutely convergent,

$$\begin{aligned}
\mathrm{Exp}(uA)\mathrm{Exp}(vA) &= \left(\sum_{r\geq 0}\frac{u^r}{r!}A^r\right)\left(\sum_{s\geq 0}\frac{v^s}{s!}A^s\right)\\
&= \sum_{r\geq 0, s\geq 0}\frac{u^r v^s}{r!s!}A^{r+s}\\
&= \sum_{n\geq 0}(\sum_{r=0}^{n}\frac{u^r v^{n-r}}{r!(n-r)!A^n}\\
&= \sum_{n\geq 0}\frac{1}{n!}\left(\sum_{r=0}^{n}\binom{n}{r}u^r v^{n-r}\right)A^n\\
&= \sum_{n\geq 0}\frac{(u+v)^n}{n!}A^n\\
&= \mathrm{Exp}((u+v)A).
\end{aligned}$$

(ii) From part (i),

$$I = \mathrm{Exp}(0) = \mathrm{Exp}((1+(-1))A) = \mathrm{Exp}(A)\mathrm{Exp}(-A)$$

so $\mathrm{Exp}(A)$ is invertible with inverse $\mathrm{Exp}(-A)$. □

Using these series we can define the matrix version of the exponential function

$$\exp : M_n(\mathbb{F}) \to GL_n(\mathbb{F}) \text{ by } \exp(A) = \mathrm{Exp}(A).$$

**Proposition 2.10** *If* $A, B \in M_n(\mathbb{F})$ *commute then* $\exp(A+B) = \exp(A)\exp(B)$.

**Proof.** Again we expand the series and perform a sequence of manipulations, all of

which are justified because these series are absolutely convergent. Then we have

$$\begin{aligned}
\exp(A)\exp(B) &= \left(\sum_{r\geq 0}\frac{1}{n!}A^r\right)\left(\sum_{s\geq 0}\frac{1}{s!}B^s\right)\\
&= \sum_{r\geq 0, s\geq 0}\frac{1}{r!s!}A^rB^s\\
&= \sum_{n\geq 0}\left(\sum_{r=0}^{n}\frac{1}{r!(n-r)!}A^rB^{n-r}\right)\\
&= \sum_{n\geq 0}\frac{1}{n!}\left(\sum_{r=0}^{n}\binom{n}{r}A^rB^{n-r}\right)\\
&= \sum_{n\geq 0}\frac{1}{n!}(A+B)^n\\
&= \mathrm{Exp}(A+B)\\
&= \exp(A+B).
\end{aligned}$$

Of course the identity

$$\sum_{r=0}^{n}\binom{n}{r}A^rB^{n-r} = (A+B)^n$$

holds whenever $A$ and $B$ commute. $\square$

We define the logarithm function by

$$\log : N_{M_n(\mathbb{F})}(I;1) \to M_n(\mathbb{F}) \text{ by } \log(A) = \log(A-I).$$

Then for $\| A-I \| < 1$,

$$\log(A) = \sum_{n\geq 1}\frac{(-1)^{n-1}}{n}(A-I)^n.$$

**Proposition 2.11** *The functions* exp *and* log *are inverses each other.*

*(i) If* $\| A-I \| < 1$, *then* $\exp(\log(A)) = A$;

*(ii) If* $\| \exp(B)-I \| < 1$, *then* $\log(\exp(B)) = B$.

**Proof.** These results follow from the formal identities between power series

$$\sum_{m\geq 0}\frac{1}{m!}\left(\sum_{n\geq 1}\frac{(-1)^{n-1}}{n}(A-1)^n\right)^m = A,$$

and

$$\sum_{n\geq 1}\frac{(-1)^{n-1}}{n}\left(\sum_{m\geq 1}\frac{1}{m!}B^m\right)^n = B$$

which are powered by comparing coefficients. □

The functions exp and log are continuous and in fact infinitely differentiable on their domains. By continuity of exp at 0, there is a $\delta_1 > 0$ such that

$$N_{M_n(\mathbb{F})}(0;\delta_1) \subseteq \exp^{-1} N_{GL_n(\mathbb{F})}(I;1)).$$

In fact we can actually take $\delta_1 = \log 2$ since

$$\exp N_{M_n(\mathbb{F})}(0;r) \subseteq N_{M_n(\mathbb{F})}(I;e^r-1))$$

Hence we have the following results.

**Proposition 2.12 (*inverse mapping theorem*)** *The exponential function* exp *is injective when restricted to the open subset* $N_{M_n(\mathbb{F})}(0;\ln 2) \subseteq N_{M_n(\mathbb{F})}$*, hence it is locally a diffeomorphism at* 0 *with local inverse* log*.*

It will sometimes be useful to have a formula for derivative of exp at an arbitrary $A \in M_n(\mathbb{F})$. When $B \in M_n(\mathbb{F})$ commutes with $A$,

$$\begin{aligned}\frac{d}{dt}_{|t=0}\exp(A+tB) &= \lim_{h\to 0}\frac{1}{h}\left(\exp(A+hB)-\exp(A)\right)\\ &= \exp(A)B = B\exp(A).\end{aligned}$$

The general situation is more complicated.

For a variable X consider the series

$$F(X) = \sum_{k\geq 0} \frac{1}{(k+1)!} X^k = \frac{\exp(X) - 1}{X}$$

which has infinite radius of convergence. If we have a linear operator $\Phi$ on $M_n(\mathbb{C})$ we can apply the convergent series of operators

$$F(\Phi) = \sum_{k\geq 0} \frac{1}{(k+1)!} \Phi^k$$

to elements of $M_n(\mathbb{C})$. In particular we can consider the adjoint operator $\mathrm{adj}_A : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ by

$$\mathrm{adj}_A(C) = [A, C] = AC - CA.$$

Then

$$F(\mathrm{adj}_A)(C) = \sum_{k\geq 0} \frac{1}{(k+1)!} (\mathrm{adj}_A)^k(C).$$

**Proposition 2.13** *For $A, B \in M_n(\mathbb{C})$ we have*

$$\frac{d}{dt_{|t=0}} \exp(A + tB) = F(adj_A)(B) \exp(A).$$

*In particular, if $A = 0$ or more generally if $AB = BA$,*

$$\frac{d}{dt_{|t=0}} \exp(A + tB) = B \exp(A).$$

**Proof.** We begin by observing that if $D = \frac{d}{ds}$ and $f(s)$ is a smooth function of the real variable $s$, then

$$F(D)_{|s=0} f(s) = \int_0^1 f(s) ds \tag{2.1}$$

holds since the Taylor expansion of a smooth function $g$ satisfies

$$\sum_{k\geq 1} \frac{1}{k!} D^k g(s) = g(s+1) - g(s).$$

Taking $g(s) = \int f(s) ds$ to be an indefinite integral of $f$ we obtain

$$\sum_{k\geq 0} \frac{1}{(k+1)!} D^k f(s) = g(s+1) + g(s).$$

Evaluating at s=0 gives Equation 2.1.

The matrix-valued function

$$\varphi(s) = \exp(sA) B \exp((1-s)A)$$

satisfies

$$\begin{aligned} \varphi(s) &= \exp(sA) B \exp(A) \exp(-sA) \\ &= \exp(s \operatorname{adj} A)(B \exp(A)) \\ &= \exp(s \operatorname{adj} A)(B) \exp(A), \end{aligned}$$

since for $m, n \geq 1$,

$$(\operatorname{adj} A)^m (BA^n) = (\operatorname{adj} A)^m (B) A^n.$$

So

$$F(D)(\varphi(s)) = \left( \sum_{k\geq 0} \frac{(s+1)^{k+1} - s^{k+1}}{(k+1)!} (\operatorname{adj} A)^k \right) (B) \exp(A)$$

is giving

$$\begin{aligned} F(D)(\varphi(s))_{|s=0} &= \left( \sum_{k\geq 0} \frac{1}{(k+1)!} (\operatorname{adj} A)^k \right) (B) \exp(A) \\ &= F(\operatorname{adj} A)(B) \exp(A), \end{aligned}$$

which is the desired formula. □

### *2.3.3 One-parameter subgroups in matrix groups.*

Let $G \leq GL_n(\mathbb{F})$ be a matrix group. Since $G \subseteq M_n(\mathbb{F})$, in particular the derivative is defined by

$$\gamma'(t) = \lim_{s \to t} \frac{1}{(s-t)} (\gamma(s) - \gamma(t)) \in M_n(\mathbb{F})$$

provided this limit exists.

**Definition 2.14** *A differentiable curve in $G$ is a function $\gamma : (a, b) \to G$ for which the derivative $\gamma'(t)$ exists at each $t \in (a, b)$.*

Since $G \subseteq M_n(\mathbb{F})$, such a curve is also a curve in $M_n(\mathbb{F})$.

**Definition 2.15** *Let $\varepsilon > 0$.* A one parameter semigroup in $G$ *is a continuous function $\gamma : (-\varepsilon, \varepsilon) \to G$ which is differentiable at $0$ and also satisfies*

$$\gamma(s + t) = \gamma(s) \cdot \gamma(t)$$

*whenever $s, t, (s + t) \in (-\varepsilon, \varepsilon)$.*

We will refer to the last condition as the homomorphism property.

If $\varepsilon = \infty$ then $\gamma : \mathbb{R} \to G$ is called *a one-parameter group in $G$ or one-parameter subgroup of $G$.*

Notice that for a one-parameter semigroup in $G$, $\gamma(0) = I_G \in G$.

**Proposition 2.16** *Let $\gamma : (-\varepsilon, \varepsilon) \to G$ be a one-parameter semigroup in $G$. Then for every $t \in (-\varepsilon, \varepsilon)$, $\gamma$ is differentiable at $t$ and*

$$\gamma'(t) = \gamma'(0)\gamma(t) = \gamma(t)\gamma'(0).$$

**Proof.** For a small $h \in \mathbb{R}$,

$$\gamma(h)\gamma(t) = \gamma(h + t) = \gamma(t + h) = \gamma(t)\gamma(h).$$

Hence

$$\begin{aligned}\gamma'(t) &= \lim_{h\to 0}\frac{1}{h}(\gamma(t+h)-\gamma(t)) \\ &= \lim_{h\to 0}\frac{1}{h}(\gamma(h)-I)\gamma(t) \\ &= \gamma'(0)\gamma(t),\end{aligned}$$

and similarly $\gamma'(t) = \gamma(t)\gamma'(0)$.

□

Of course, part of the importance of this result is the implication that $\gamma$ is differentiable curve in $G$ even though we only assumed it was continuous.

**Proposition 2.17** *Let $\gamma : (-\varepsilon, \varepsilon) \to G$ be a one-parameter semigroup in $G$. Then there is a unique extension to a one-parameter group $\tilde{\gamma} : \mathbb{R} \to G$,i.e, a function $\tilde{\gamma}$ for which $\tilde{\gamma}(t) = \gamma(t)$ for all $t \in (-\varepsilon, \varepsilon)$.*

**Proof.** Let $t \in \mathbb{R}$. Then for a large enough natural number $m, t/m \in (-\varepsilon, \varepsilon)$, hence $\gamma(t/m), (\gamma(t/m))^m \in G$. Similarly, for a second such natural number $n$, $\gamma(t/n), (\gamma(t/n))^n \in G$. Since $mn \geq m$ and $mn \geq n$, we also have $t/mn \in (-\varepsilon, \varepsilon)$ therefore

$$\begin{aligned}\gamma(t/n)^n &= \gamma(mt/mn)^n \\ &= \gamma(t/mn)^{mn} \\ &= \gamma(nt/mn)^m \\ &= \gamma(t/m)^m.\end{aligned}$$

Thus $\gamma(t/n)^n = \gamma(t/m)^m$, which shows that we obtain a well-defined element of $G$ for every real number $t$. This defines a function $\tilde{\gamma} : \mathbb{R} \to G$ by $\tilde{\gamma}(t) = \lim_{n\to\infty} \gamma(t/n)^n$. It is easy to see that $\tilde{\gamma}$ is one-parameter group in $G$. □

We can now determine the form of all one-parameter groups in $G$.

**Theorem 2.18** *Let $\gamma : \mathbb{R} \to G$ be a one parameter group in $G$. Then it has the form*

$$\gamma(t) = \exp(tA)$$

*for some $A \in M_n(\mathbb{F})$.*

**Proof.** Let $A = \gamma'(0)$. By Proposition 2.16, $\gamma(t) = \exp(tA)$ satisfies the following differential equation

$$\gamma'(t) = \gamma'(0)\gamma(t),\ \gamma(0) = I.$$

□

We can not yet reverse this process and decide for which $A \in M_n(\mathbb{F})$ the one-parameter group

$$\gamma : \mathbb{R} \to GL_n(\mathbb{F});\ \gamma(t) = \exp(tA)$$

actually takes values in $G$. Notice that we also have the curious phenomenon that although the definition of a one-parameter group only involves first order differentiability, the general form $\exp(tA)$ is always infinitely differentiable and indeed analytic as a function of $t$.

### *2.3.4 Curves, tangent spaces and Lie algebras.*

Let $G \leq GL_n(\mathbb{F})$ be a matrix group.

**Definition 2.19** *: The tangent space to $G$ at $U \in G$ is*

$$T_U G = \{\gamma'(0) \in M_n(\mathbb{F}) : \gamma \text{ is a differentiable curve in} G \text{ with} \gamma(0) = U\}.$$

**Proposition 2.20** *$T_U G$ is a real vector subspace of $M_n(\mathbb{F})$*

**Proof.** Suppose that $\alpha, \beta$ are differentiable curves in $G$ for which $\alpha(0) = \beta(0) = U$. Then

$$\gamma : \mathrm{dom}(\alpha) \cap \mathrm{dom}(\beta) \to G$$

defined by $\gamma(t) = \alpha(t)U^{-1}\beta(t)$ is also differentiable curve in $G$ with $\gamma(0) = U$. The product Rule gives

$$\gamma'(t) = \alpha'(t)U^{-1}\beta(t) + \alpha(t)U^{-1}\beta'(t),$$

hence

$$\gamma'(0) = \alpha'(0)U^{-1}\beta(0) + \alpha(0)U^{-1}\beta'(0) = \alpha'(0) + \beta'(0),$$

which shows that $T_U$ is closed under addition.

Similarly, if $r \in \mathbb{R}$ and $\alpha$ is a differentiable curve in $G$ with $\alpha(0) = U$, then $\eta(t) = \alpha(rt)$ defines another such curve. Since $\eta'(0) = r\alpha'(0)$, we see that $T_U G$ is closed under real scalar multiplication. □

**Definition 2.21** *The dimension of the real matrix group $G$ is* $\dim G = \dim_{\mathbb{R}} T_I G$ *If $G$ is complex then its complex dimension is* $\dim_{\mathbb{C}} G = \dim_{\mathbb{C}} T_I G$.

We will use the notation $\mathfrak{g} = T_I G$ for this real vector subspace of $M_n(\mathbb{F})$. In fact, $\mathfrak{g}$ has the algebraic structure of a real Lie algebra as we now show.

**Theorem 2.22**

*(i) If $G \leq GL_n(\mathbb{R})$ is a matrix subgroup, then $\mathfrak{g}$ is a real Lie subalgebra of $M_n(\mathbb{R})$.*

*(ii) If $G \leq GL_n(\mathbb{C})$ is a matrix subgroup then $\mathfrak{g}$ is a complex Lie subalgebra of $M_n(\mathbb{C})$.*

**Proof.**

(i) We will show that for two differentiable curves $\alpha$ and $\beta$ in $G$ which satisfy $\alpha(0) = \beta(0) = I_n$, there is another such curve $\gamma$ with $\gamma'(0) = [\alpha'(0), \beta'(0)]$.

Consider a function

$$F : \mathrm{dom}(\alpha) \times \mathrm{dom}(\beta) \to G; F(s,t) = \alpha(s)\beta(t)\alpha(s)^{-1}.$$

This is clearly continuous and differentiable with respect to each of the variables $s, t$. For each $s \in \mathrm{dom}\alpha$, the function $F(s,-) : \mathrm{dom}(\beta) \to G$ is a differentiable curve in $G$ with $F(s,0) = I_n$. By differentiating we find

$$\frac{dF(s,t)}{dt}_{|t=o} = \alpha(s)\beta'(0)\alpha(s)^{-1},$$

and so

$$\alpha(0)\beta'(0)\alpha(s)^{-1} \in g.$$

Since $\mathfrak{g}$ is a closed subspace of $M_n(\mathbb{F})$, whenever this limits exists we also have

$$\lim_{s\to 0} \frac{1}{s}(\alpha(0)\beta'(0)\alpha(s)^{-1} - \beta'(0))) \in g.$$

We will use following easily verified matrix version of the usual rule for differentiating inverse:

$$\frac{d}{dt}(\alpha(t)^{-1}) = -\alpha(t)^{-1}\alpha'(t)\alpha(t)^{-1}.$$

We have

$$\begin{aligned}
\lim_{s\to 0} \frac{1}{s}(\alpha(0)\beta'(0)\alpha(s)^{-1} - \beta'(0))) &= \frac{d}{ds_{|s=o}}\alpha(s)\beta'(0)\alpha(s)^{-1} \\
&= \alpha'(0)\beta'(0)\alpha(0) - \alpha(0)\beta'(0)\alpha(0)^{-1}\alpha'(0)\alpha(0)^{-1} \\
&= \alpha'(0)\beta'(0)\alpha(0) - \alpha(0)\beta'(0)\alpha'(0) \\
&= \alpha'(0)\beta'(0) - \beta'(0)\alpha'(0) \\
&= [\alpha'(0), \beta'(0)].
\end{aligned}$$

This shows that $[\alpha'(0), \beta'(0)] \in \mathfrak{g}$, hence it must be of the form $\gamma'(0)$ for some differentiable curve.

So for each matrix group $G$ there is a Lie algebra $\mathfrak{g} = T_I G$. $\square$

A suitable type of homomorphism $G \to H$ between matrix group gives rice to a linear transformation $g \to h$ which is a homomorphism of Lie algebras.

**Definition 2.23** *: Let $G \leq GL_n(\mathbb{F}), H \leq GL_m(\mathbb{F})$ be matrix group and $\varphi : G \to H$ be a continuous map. Then $\varphi$ is said to be a* differentiable map *if it satisfies the following two conditions:*

- *for every differentiable curve $\gamma : (a, b) \to G$, the curve $\varphi o \gamma : (a, b) \to H$ is differentiable and has derivative*

$$(\varphi o \gamma)'(t) = \frac{d}{dt}\varphi(\gamma(t));$$

- *if two differentiable curves $\alpha, \beta : (a, b) \to G$ satisfies*

$$\alpha(0) = \beta(0), \alpha'(0) = \beta'(0),$$

*then*

$$(\varphi o \alpha)'(0) = (\varphi o \beta)'(0).$$

*A differentiable map which also a group homomorphism is called a* differentiable homomorphism. *A continuous of matrix groups that is also a differentiable map is called a* Lie homomorphism.

**Proposition 2.24** *Let $G, H, K$ be matrix groups with $\varphi : G \to H$ and $\theta : H \to K$ differentiable homomorphisms. Then the following are true.*

*(i) For each $A \in G$ there is an $\mathbb{R}$-linear transformation $d_\varphi : T_A G \to T_{\varphi(A)} H$ given by*

$$d_{\varphi A}(\gamma'(0)) = (\varphi o \gamma)'(0),$$

*for every differentiable curve* $\gamma : (a, b) \to G$ *with* $\gamma(0) = A$

*(ii) We have*

$$d\theta_{\varphi(A)} \; o \; d_{\varphi(A)} = d(\theta o \varphi)_A.$$

*(iii) For the identity map* $dId_G = Id_{T_A}G$.

If $\varphi : G \to H$ is a differentiable homomorphism then since $\varphi(I) = I$, $d_{\varphi I} : T_I G \to T_I H$ is a linear transformation called *derivative of* $\varphi$ which is usually denoted $d_\varphi : \mathfrak{g} \to \mathfrak{h}$.

**Theorem 2.25** *Let* $G, H$ *be matrix groups and* $\varphi : G \to H$ *be a differentiable homomorphism. Then the derivative* $d_\varphi : \mathfrak{g} \to \mathfrak{h}$ *is a homomorphism of Lie algebras.*

**Proof.** Following ideas and notation in the proof of Theorem 2.22, for differentiable curves $\alpha, \beta$ in $G$ with $\alpha(0) = \beta(0) = I$, we can use the composite function $\varphi o F$ given by

$$\varphi o F(s, t) = \varphi(F(s, t)) = \varphi(\alpha(s))\varphi(\beta(t))\varphi(\alpha(s))^{-1},$$

to deduce that

$$d_\varphi([\alpha'(0), \beta'(0)]) = [d_\varphi(\alpha'(0)), d_\varphi(\beta'(0)))].$$

□

As a special case, for each $A \in G$ the *conjugation map*

$$\chi A : G \to G; \; \chi A(U) = AUA^{-1}$$

has *as its derivative at identity* $I$ *the adjoint action of* $A$ *on* $\mathfrak{g}$, namely the linear transformation

$$\mathrm{adj}_A = d_\chi : g \to g;\ \chi_A(X) = AXA^{-1}.$$

### *2.3.5 The Lie algebra of some matrix groups.*

We will start with the real matrix group $GL_n(\mathbb{R}) \subseteq M_n(\mathbb{R})$. For $A \in M_n(\mathbb{R})$ and $\varepsilon > 0$ there is a differentiable curve

$$\alpha : (-\varepsilon, \varepsilon) \to M_n(\mathbb{R});\ \alpha(t) = I + tA.$$

For $t \neq 0$, the roots of the equation $\det(A - tI+) = 0$ are of the form $t = -1/\lambda$ where $\lambda$ is a non-zero eigenvalue of $A$. Hence if we choose $\varepsilon$ as $\varepsilon < min\{\frac{1}{|\lambda|} : \lambda \,\text{a non-zero eigenvalue of}\, A\}$, then $\mathrm{Im}(\alpha) \subseteq GL_n(\mathbb{R})$, so we will view $\alpha$ as a function $\alpha : (-\varepsilon, \varepsilon) \to GL_n(\mathbb{R})$. Calculating the derivative we find that $\alpha'(t) = A$, hence $\alpha'(0) = A$. This shows that $A \in T_I GL_n(\mathbb{R})$. Since $A \in M_n(\mathbb{R})$ was arbitrary, we have

$$\mathfrak{gl}_n(\mathbb{R}) = T_I GL_n(\mathbb{R}) = M_n(\mathbb{R}),$$

$$\dim GL_n(\mathbb{R}) = n^2.$$

Similarly,

$$\mathfrak{gl}_n(\mathbb{C}) = T_I GL_n(\mathbb{C}) = M_n(\mathbb{C}),$$

$$\dim_{\mathbb{C}} GL_n(\mathbb{C}) = n^2.$$

For $SL_n(\mathbb{R}) \leq GL_n(\mathbb{R})$, suppose that $\alpha : (a, b) \to SL_n(\mathbb{R})$ is a curve lying in $SL_n(\mathbb{R})$ and satisfying $\alpha(0) = I$. For $t \in (a, b)$ we have $\det \alpha(t) = 1$, so

$$\frac{d(\det \alpha(t))}{dt} = 0.$$

**Lemma 2.26** *We have*

$$\frac{d\,(\det \alpha(t))}{dt_{|_{t=0}}} = trace\, \alpha'(0).$$

**Proof.** Recall that for $A \in M_n(\mathbb{F})$,

$$\operatorname{trace} A = \sum_{i=1}^{n} A_{ii}.$$

It is easy to verify that the operation $\partial = \frac{d}{dt}_{|t=0}$ on functions has the derivation property

$$\partial(\gamma_1\gamma_2) = (\partial\gamma_1)\gamma_2(0) + \gamma_1(0)\partial\gamma_2.$$

Put $a_{ij} = \alpha(t)_{ij}$ and notice that when $t = 0$, $a_{ij} = \delta_{ij}$. Write $C_{ij}$ for the cofactor matrix obtained from $\alpha(t)$ by deleting the $i$th row and $j$th column. By expanding along the $n$th row we obtain

$$\det \alpha(t) = \sum_{j=1}^{n} (-1)^{n+j} a_{nj} \det C_{nj}.$$

Then we have

$$\begin{aligned}
\partial \det \alpha(t) &= \sum_{j=1}^{n} (-1)^{n+j} \left( (\partial_{a_{nj}}) \det C_{nj} + a_{nj} (\partial \det C_{nj}) \right) \\
&= \sum_{j=1}^{n} (-1)^{n+j} (\partial_{a_{nj}}) \det C_{nj} + \partial \det C_{nn}.
\end{aligned}$$

For $t = 0$, $\det C_{nj} = \delta_{nj}$ since $\alpha(0) = I$, hence

$$\partial \det \alpha(t) = \partial a_{nn} + \partial \det C_{nn}.$$

We can repeat this calculation with the $(n-1) \times (n-1)$ matrix $C_{nn}$ and so on. This yields

$$\begin{aligned}
\partial \det \alpha(t) = \partial a_{nn} + \partial a_{(n-1)(n-1)} + \partial \det C_{(n-1)(n-1)} & \\
&= \partial a_{nn} + \partial a_{(n-1)(n-1)} + \ldots + \partial a_{11} \\
&= \operatorname{trace}(\alpha'(0)),
\end{aligned}$$

giving the result. $\square$

So we have $\text{trace}(\alpha'(0)) = 0$ and hence

$$\mathfrak{sl}_n(\mathbb{R}) = T_I SL_n(\mathbb{R}) \subseteq \ker(\text{trace}) \subseteq M_n(\mathbb{R}).$$

If $A \in \ker(\text{trace}) \subseteq M_n(\mathbb{R})$, the function

$$\alpha : (-\varepsilon, \varepsilon) \to M_n(\mathbb{R}); \alpha(t) = exp(tA) = \sum_{k \geq 0} \frac{t^k}{k!} A^k,$$

is defined for every $\varepsilon > 0$ and satisfy the boundary conditions

$$\alpha(0) = I, \alpha'(0) = A.$$

We will make use of the following lemma.

**Lemma 2.27** *For $A \in M_n(\mathbb{C})$ we have*

$$\det \exp(A) = e^{trace(A)}.$$

**Proof.** Approach using Jordan form. If $S \in GL_n(\mathbb{C})$,

$$\det(\exp(SAS^{-1})) = \det(S \exp(A) S^{-1})$$

$$= \det S \det(\exp(A)) \det S^{-1}$$

$$= \det \exp A,$$

and

$$e^{\text{trace}(SAS^{-1})} = e^{\text{trace}(A)}.$$

So it suffices to prove the identity for $SAS^{-1}$ for a suitably chosen invertible matrix $S$. Using the theory of Jordan forms,there is a suitable choice of such an S for which

$$B = SAS^{-1} = D + N,$$

where $D$ is diagonal, $N$ is strictly upper triangular,i.e. $N_{ij} = 0$ whenever $i \geq j$) and with $D, N$ commuting. Then $N$ is nilpotent, $N^k = 0_n$ for $k$ large.

We have

$$\begin{aligned} exp(B) = \sum_{k\geq 0} \frac{1}{k!}(D+N)^k & \\ &= \sum_{k\geq 0} \frac{1}{k!}(D)^k + \sum_{k\geq 0} \frac{1}{(k+1)!}\left(\sum_{j=0}^{k} \binom{k+1}{j} D^j N^{k+1-j}\right) \\ &= exp(D) + \sum_{k\geq 0} \frac{1}{(k+1)!} N \left(\sum_{j=0}^{k} \binom{k+1}{j} D^j N^{k-j}\right). \end{aligned}$$

For each $k \geq 0$, the matrix

$$N \sum_{j=0}^{k} \binom{k+1}{j} D^j N^{k-j}$$

is strictly upper triangular, hence

$$\exp(B) = \exp(D) + N',$$

where $N'$ is strictly upper triangular. If $D = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$, by calculating the determinant we find that

$$\begin{aligned} \det \exp(A) &= det \exp(B) \\ &= \det \exp(D) = \det(\mathrm{diag}(e^{\lambda_1}, ..., e^{\lambda_n})) \\ &= e^{\lambda_1} \ldots e^{\lambda_n} = e^{(\lambda_1 + \ldots + \lambda_n)}. \end{aligned}$$

Since $\text{trace}(D) = (\lambda_1 + ... + \lambda_n)$, this implies that

$$\det \exp(A) = e^{\text{trace}(D)}.$$

□

Using this lemma and the function $\alpha$, we obtain

$$\mathfrak{sl}_n(\mathbb{R}) = T_I SL_n(\mathbb{R}) = \ker(\text{trace}) \subseteq M_n(\mathbb{R})$$

$$\dim SL_n(\mathbb{R}) = n^2 - 1.$$

Working over $\mathbb{C}$ we also have

$$\mathfrak{sl}_n(\mathbb{C}) = T_I SL_n(\mathbb{C}) = \ker \text{trace} \subseteq M_n(\mathbb{C}).$$

Hence

$$\dim_{\mathbb{C}} SL_n(\mathbb{C}) = n^2 - 1.$$

### *2.3.6 Upper triangular and unipotent groups.*

For $n \geq 1$, recall the upper triangular and unipotent subgroups $UT_n(\mathbb{F})$ and $SUT_n(\mathbb{F})$ of $GL_n(\mathbb{F})$. Let

$$\alpha : (-\varepsilon, \varepsilon) \to UT_n(\mathbb{R})$$

be a differentiable curve with $\alpha(0) = I$. Then $\alpha'(t)$ is upper triangular. Moreover, using the argument for $GL_n(\mathbb{F})$ we see that given any upper triangular matrix $A \in M_n(\mathbb{F})$, there is a curve

$$\alpha : (-\varepsilon, \varepsilon) \to UT_n(\mathbb{F}); \alpha(t) = I + tA,$$

where $\varepsilon > 0$ has to be chosen small and $\alpha'(0) = A$. Then we have

$\mathfrak{t}_n(\mathbb{F})$ = set of upper triangular matrices in $M_n(\mathbb{F})$, and

$$\dim \mathfrak{t}(n, \mathbb{F}) = \binom{n+1}{2}.$$

An upper triangular matrix $A \in M_n(\mathbb{F})$ is strictly upper triangular if all its diagonal entries are 0, i.e, $a_{ii} = 0$. Then

$\mathfrak{n}(n, \mathbb{F})$ = set of strictly upper triangular matrices in $M_n(\mathbb{F})$, and

$$\dim \mathfrak{n}(n, \mathbb{F}) = \binom{n}{2}.$$

### *2.3.7 Unitary and special unitary groups.*

Now consider $n \times n$ unitary group

$$U_n = \{A \in GL_n(\mathbb{C}) : A^*A = I\}.$$

For a curve $\alpha$ in $U_n$ satisfying $\alpha(0) = I$, we obtain

$$\alpha'(0)^* + \alpha'(0) = 0$$

and so $\alpha'(0)^* = -\alpha'(0)$,i.e, $\alpha(0)$ is skew hermitian. So

$$\mathfrak{u}(n) = T_I U(n) \subseteq \text{Skew-Herm}_n(\mathbb{C}),$$

the set of all $n \times n$ skew hermitian matrices.

If $H \in \text{Skew-Herm}_n(\mathbb{C})$ then the curve

$$\eta : (-\varepsilon, \varepsilon) \to GL_n(\mathbb{C});\ \eta(t) = \exp(tH)$$

satisfies

$$\begin{aligned}\eta(t)^*\eta(t) &= \exp(tH)^* \exp(tH) = \exp(tH^*) \exp(tH) \\ &= \exp(-tH) \exp(tH) \\ &= I.\end{aligned}$$

Hence we can view $\eta$ as a curve $\eta : (-\varepsilon, \varepsilon) \to U(n)$. Since $\eta'(0) = H$, this show that

$$\text{Skew-Herm}_n(\mathbb{C}) \subseteq \mathfrak{u}(n) = T_I U(n).$$

Hence since

$$\dim \text{Skew-Herm}_n(\mathbb{C}) = n + 2\binom{n}{2} = n^2,$$

We have

$$\mathfrak{u}(n) = T_I U(n) = \text{Skew-Herm}_n(\mathbb{C}),$$

$$\dim U(n) = n^2.$$

The special unitary group $SU(n)$ can be handled in a similar way. Again we have

$$\mathfrak{su}(n) = T_I SU(n) \subseteq \text{Skew-Herm}_n(\mathbb{C}).$$

But also if $\eta : (a, b) \to SU(n)$ is a curve with $\eta(0) = I$, then as in the analysis for $SL_n(\mathbb{R})$, we have $\eta'(0) = 0$. Writing

$$\text{Skew-Herm}_n^0(\mathbb{C}) = \{H \in \text{Skew-Herm}_n(\mathbb{C}) : \text{trace}(H) = 0\},$$

we see that

$$\dim \text{Skew-Herm}_n{}^0(\mathbb{C}) = \dim \text{Skew-Herm}_n(\mathbb{C}) - 1 = n^2 - 1,$$

and $\mathfrak{su}(n) \subseteq \text{Skew-Herm}_n{}^0(\mathbb{C})$. On the other hand, if $H \in \text{Skew-Herm}_n{}^0(\mathbb{C})$ then the curve

$$\eta : (-\varepsilon, \varepsilon) \to U(n); \eta(t) = \exp(tH),$$

takes values in $SU(n)$ by Lemma 2.27 and satisfies $\eta'(0) = H$. Hence

$$\mathfrak{su}(\mathfrak{n}) = T_I SU(n) \subseteq \text{Skew-Herm}_n{}^0(\mathbb{C}),$$

$$\dim SU(n) = n^2 - 1.$$

### *2.3.8 Some observations on the exponential function of a matrix group.*

The function

$$\exp_G : \mathfrak{g} \to GL_n(\mathbb{R}); \exp_G(X) = \exp(X),$$

has image contained in $G$, $\exp_G g \subseteq G$, so we will usually write $\exp_G : \mathfrak{g} \to G$ for the exponential of $G$ and sometimes even just exp.

If G is compact and connected then $\exp_G(\mathfrak{g}) = G$. y inverse mapping theorem, there is an open disc $N_{\mathfrak{g}}(0; r) \subseteq \mathfrak{g}$ on which exp is injective and gives a homeomorphism

$$\exp : N_{\mathfrak{g}}(0; r) \to \exp(N_g(0; r)),$$

where $\exp(N_{\mathfrak{g}}(0; r)) \subseteq G$ is in fact an open subset.

Here is an example which shows that the exponential map need not be surjective for a connected matrix group.

**Example 2.28** *Consider the exponential* $\exp_{SL_2(\mathbb{R})} : \mathfrak{sl}(2, \mathbb{R}) \to SL_2(\mathbb{R})$. *Then for* $\delta \geq 0$, *the matrix*

$$\begin{pmatrix} -(2+\delta) & 0 \\ 0 & \frac{-1}{(2+\delta)} \end{pmatrix} \in SL_2(\mathbb{R})$$

*is not in the image of* $\exp_{SL_2(\mathbb{R})}$.

**Proof.** Let

$$A \in \mathfrak{sl}_2(\mathbb{R}) = \{A \in M_2(\mathbb{R}) : \text{trace}A = 0\}.$$

By Cayley-Hamilton Theorem, we have

$$A^2 + (\det A)I_2 = 0.$$

If $\det A = 0$ then the only eigenvalue of $A$ is 0 and for any $t \in \mathbb{R}$ we have

$$\exp(tA) = I_2 + tA,$$

from which we obtain

$$\text{trace}\exp(tA) = 2.$$

If $\det A \neq 0$, then A has two distinct non-zero eigenvalues, namely

$$\begin{cases} \pm\sqrt{\det Ai} & \text{if } \det A > 0, \\ \pm\sqrt{-\det A} & \text{if } \det A < 0. \end{cases}$$

□

### *2.3.9 $SO(3)$ and $SU(2)$*

The Lie algebras $\mathfrak{so}(3)$ and $\mathfrak{su}(2)$ are both 3-dimensional real vector spaces, for example having the following bases:

$$\mathfrak{so}(3) : P = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, Q = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, R = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\mathfrak{su}(2) : H = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, E = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, F = \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

The non-trivial Lie brackets amongst these are

$$[P, Q] = R, [Q, R] = P, [R, P] = Q,$$

$$[H, E] = F, [E, F] = H, [F, H] = E.$$

This implies that the $\mathbb{R}$-linear isomorphism

$$\varphi : \mathfrak{su}(2) \to \mathfrak{so}(3); \varphi(xH + yE + zF) = xP + yQ + zR\,(x, y, x \in \mathbb{R}),$$

satisfies

$$\varphi([U, V]) = [\varphi(U), \varphi(V)],$$

and so is an isomorphism of $\mathbb{R}$-Lie algebras. Thus these Lie algebras look the same algebraically. This suggest that there might be a close relationship between the groups themselves.

### *2.3.10 Smooth manifolds.*

**Definition 2.29** *:A continuous map $g : V_1 \to V_2$ where each $V_k \subseteq \mathbb{R}^{m_k}$ is open is* smooth *if it is infinitely differentiable. A smooth bijection $g$ is a* diffeomorphism *if its inverse $g^{-1} : V_2 \to V_1$ is also smooth.*

**Definition 2.30** *: A topological space $X$ is* separable *if it has a countable open cover-*

*ing, an open of the form* $\{U_j\}_{j=1}^{\infty} = \{U_1, U_2, U_3, ...\}$.

Every compact topological space is separable, as is any subspace of $\mathbb{R}^n$ or $\mathbb{C}^n$ for $n \geq 1$, so all matrix groups are separable. A quotient space of a separable space is also separable.

From now on, let $M$ be a separable Hausdorff topological space.

**Definition 2.31** *If* $U \subseteq M$ *and* $V \subseteq \mathbb{R}^n$ *are open subsets, a homomorphism* $f : U \to V$ *is called an* n-chart *for* $U$.

*If* $\mathrm{U} = \{U_\alpha \in A\}$ *is an open covering of* $M$ *and* $\mathrm{F} = \{f_\alpha : U_\alpha \to V_\alpha\}$ *is a collection of charts, then* $F$ *is called an* atlas *for* $M$ *if, whenever* $U_\alpha \cap U_\beta \neq \phi$,

$$f_\beta o f_\alpha^{-1} : f_\alpha(U_\alpha \cap U_\beta) \to f_\beta(U_\alpha \cap U_\beta)$$

*is a diffeomorphism.*

*Sometimes we will denote an atlas by* $(M, U, F)$ *and refer to it as a s*mooth manifold of dimension n *or* smooth n-manifold.

Let $(M, U, F)$ and $(M', U', F')$ be atlases on topological spaces M and M'. A *smooth map* $h : (M, U, F) \to (M', U', F')$ is a continuous map $h : M \to M'$ such that for each pair $\alpha, \alpha'$ with $h(U_\alpha) \cap U'_{\alpha'} \neq \phi$ , the composite

$$f'_{\alpha'} \ o \ h \ o \ f_\alpha^{-1} : f_\alpha(h^{-1}U'_{\alpha'}) \to V'_{\alpha'}$$

is smooth.

### *2.3.11 Tangent spaces and derivatives.*

Let $(M, U, F)$ be a smooth n-manifold and $p \in M$. Let $\gamma : (a, b) \to M$ be a continuous curve with $a < 0 < b$.

**Definition 2.32** $\gamma$ *is* differentiable *at* $t \in (a, b)$ *if for every chart* $f : U \to V$ *with* $\gamma(t) \in U$, *the curve* $f o \gamma : (a, b) \to V$, *is differentiable at* $t \in (a, b), i.e., (f o \gamma)'(t)$ *exists.*

$\gamma$ *is* smooth *at* $t \in (a,b)$ *if all the derivatives of* $(f o \gamma)$ *exists at* $t$.

*The curve* $\gamma$ *is differentiable if it is differentiable at all points in (a, b). Similarly* $\gamma$ *is smooth at all points in (a, b).*

From now on we will often write $fg = fog$ for the composition of functions f and g when no confusion seems likely to result.

**Lemma 2.33** *Let* $f_0 : U_0 \to V_0$ *be a chart with* $\gamma(t) \in U_0$ *and suppose that*

$$f_0 \; o \; \gamma : (a,b) \cap f_0^{-1} V_0 \to V_0$$

*is differentiable at* $t$. *Then for any chart* $f : U \to V$ *with* $\gamma(t) \in U$,

$$f o \gamma : (a,b) \cap f^{-1} V \to V$$

*is differentiable at* $t$.

**Proof.** The composition $f o \gamma$ is defined on a subinterval of $(a,b)$ containing $t$ and is smooth. There is the usual chain rule or function of a function rule for the derivative

$$(f\gamma)'(t) = \mathrm{Jac}_{f f_0^{-1}}(f o \gamma(t))(f o \gamma)'(t) \tag{2.2}$$

Hence, for a differentiable function

$$h : W_1 \to W_2;\; h(x) = \begin{pmatrix} h_1(x) \\ \vdots \\ h_{m_2}(x) \end{pmatrix}$$

where $W_1 \subseteq \mathbb{R}^{m_1}$ and $W_2 \subseteq \mathbb{R}^{m_2}$ are open subsets, $x \in W_1$ and

$$\mathrm{Jac}_h(x) = \left[ \frac{\partial h_i}{\partial x_j}(x) \right] \in M_{m_2, m_1}(\mathbb{R})$$

is the Jacobian matrix of $h$ at $x$.

If $\gamma(0) = p$ and $\gamma$ is differentiable at 0, then for any chart $f_0 : U_0 \to V_0$ with $\gamma(0) \in U_0$, there is a derivative vector $v_0 = (f\gamma)'(0) \in \mathbb{R}^n$. In passing to another chart $f : U \to V$ with $\gamma(0) \in U$ by equation 2.2, we have

$$(f\gamma)'(0) = \mathrm{Jac}_{ff_0^{-1}}(fo\gamma(0))(fo\gamma)'(0)$$

□

In order to define the notion of the tangent space $T_pM$ to the manifold $M$ at $p$, we consider all pairs of the form

$$(f\gamma)'(0), f : U \to V$$

where $\gamma(0) = p \in U$, and then impose an equivalence relation $\sim$ under which

$$((f_1\gamma)'(0), f_1 : U_1 \to V_1) \sim ((f_2\gamma)'(0), f_2 : U_2 \to V_2).$$

Since

$$(f_2\gamma)'(0) = Jac_{f_2f_1^{-1}}((f_1\gamma)(0)(f_1\gamma)'(0),$$

whenever there is a curve $\alpha$ in $M$ for which $\gamma(0) = p$, $(f_1\gamma)'(0) = v$. The set of equivalence classes is $TpM$ and we will sometimes denote the equivalence class of $(v, f : U \to V)$ by $[v, f : U \to V]$.

**Proposition 2.34** *: For $p \in M, TpM$ is an $\mathbb{R}$-vector space of dimension $n$.*

Let $h : (M, U, F) \to (M', U', F')$ be a smooth map between manifolds of dimensions $n, n'$. We will use the notation of Definition 2.31. For $p \in M$, consider a pair of charts with $p \in U_\alpha$ and $h(p) \in U'_{\alpha'}$. Since $h_{\alpha',\alpha} = f'_{\alpha'}ohof_\alpha^{-1}$ is differentiable, the Jacobian matrix $\mathrm{Jac}_{h_{(\alpha',\alpha)}}(f_\alpha(p))$ has an associated $\mathbb{R}$-linear transformation

$$dh_{(\alpha',\alpha)} : \mathbb{R}^n \to \mathbb{R}^{n'}; dh_{(\alpha',\alpha)}(x) = Jac_{h_{(\alpha',\alpha)}}(f_\alpha(p))x.$$

It can be verified that this passes to equivalence classes to give a well-defined $\mathbb{R}$-linear transformation

$$dh_p : TpM \to T_{h(p)}M'.$$

**Proposition 2.35** *: Let $h : (M, U, F) \to (M', U', F')$ and $g : (M', U', F') \to (M'', U'', F'')$ be smooth maps between manifolds $M, M', M''$ of dimensions $n, n', n''$. Then for each $p \in M$ we have*

*(i) $dg_{h_{(p)}} od_{h_p} = d(goh)_p$;*

*(ii) $dId_p = Id_{TpM}$, where $Id : M \to M$ is the identity map.*

**Definition 2.36** *Let $(M, U, F)$ be a manifold of dimension $n$. A subset $N \subseteq M$ is a* submanifold *of dimension $k$ if for every $p \in N$ there is an open neighbourhood $U \subseteq M$ of $p$ and an $n$-chart $f : U \to V$ such that*

$$p \in f^{-1}(V \cap \mathbb{R}^k) = N \cap U.$$

*For such an $N$ we can form $k$-charts of the form*

$$f_0 : N \cap U \to V \cap \mathbb{R}^k; f_0(x) = f(x).$$

*We will denote this manifold by $(N, U_N, F_N)$.*

The following result is immediate.

**Proposition 2.37** *For a submanifold $N \subseteq M$ of dimension $k$, the inclution function $incl : N \to M$ is smooth and for every $p \in N$, $dincl_p : T_pN \to T_pM$ is an injection.*

The next result allows us to recognise submanifolds as inverse image of points under smooth mappings.

**Theorem 2.38 (*Implicit Function Theorem for manifolds*)** *Let $h : (M, U, F) \to (M', U', F')$ be a smooth map between manifolds of dimensions $n, n'$. Suppose that for some $q \in M', dh_p : T_pM \to T_{h(p)}M'$ is surjective for every $p \in N = h^{-1}q$. Then*

*$N \subseteq M$ is submanifold of dimension $n - n'$ and the tangent space at $p \in N$ is given by $T_pN = \ker dh_p$.*

**Theorem 2.39** *: Let $h : (M, U, F) \to (M', U', F')$ be a smooth map between manifolds of dimensions $n, n'$. Suppose that for some $p \in M, dh_p : T_pM \to T_{h(p)}M'$ is an isomorphism.*

Then there is an open neighbourhood $U \subseteq M$ of $p$ and an open neighbourhood $V \subseteq M'$ of $h(p)$ such that $hU = V$ and the restriction of $h$ to the map $h_1 : U \to V$ is a diffeomorphism. In particular, the derivative $dh_p : T_p \to T_{h(p)}$ is an $\mathbb{R}$- linear isomorphism and $n = n'$.

When this occurs we say that $h$ is *locally a diffeomorphism* at $p$.

### *2.3.12 Lie groups.*

**Definition 2.40** *Let $G$ be a smooth manifold which is also a topological group with multiplication map $mult : G \times G \to G$ and inverse map $inv : G \to G$ and view $G \times G$ as a product manifold. Then $G$ is a Lie group if $mult, inv$ are smooth maps.*

**Definition 2.41** *: Let $G$ be a Lie group. A closed subgroup $H \leq G$ that is also a submanifold is called a Lie group of $G$. It is then automatic that the restrictions to $H$ of the multiplication and inverse maps on $G$ are smooth, hence $H$ is also a Lie group.*

For a Lie group $G$, let $g \in G$. The following three functions are of particular importance:

$L_g : G \to G; L_g(x) = gx$, *Left translation*

$R_g : G \to G; R_g(x) = xg$, *Right translation*

$\chi_g : G \to G; \chi_g(x) = gxg^{-1}$ *Conjugation*

**Proposition 2.42** *For each $g \in G$, the maps $L_g, R_g, \chi_g$ are diffeomorphism with inverses*

$$L_g{}^{-1} = L_{g^{-1}}, R_g{}^{-1} = R_{g^{-1}}, \chi_g{}^{-1} = \chi_{g^{-1}}.$$

**Proof.** Charts for $G \times G$ have the form

$$\varphi_1 \times \varphi_2 : U_1 \times U_2 \to V_1 \times V_2,$$

where $\varphi_k : U_k \to V_k$ are charts of $G$. Now suppose that $\mu(U_1 \times U_2) \subseteq W \subseteq G$ where there is a chart $\theta : W \to Z$. By assumption, the composition

$$\theta o \mu o (\varphi_1 \times \varphi_2)^{-1} = \theta o \mu o (\varphi_1^{-1} \times \varphi_2^{-1}) : V_1 \times V_2 \to Z$$

is smooth. Then $L_g(x) = \mu(g, x)$ so if $g \in U_1$ and $x \in U_2$, we have

$$L_g(x) = \theta^{-1} o (\theta o L_g o \varphi_2^{-1}) o \varphi_2(x).$$

But then it is clear that

$$\theta o \varphi_2^{-1} : V_2 \to Z$$

is smooth since it is obtained from $\theta o \mu o (\varphi_1 \times \varphi_2)^{-1}$, but treating the first variable as a constant. A similar argument deals with $R_g$.

For $\chi_g$, notice that

$$\chi_g = L_g o R_g = R_g o L_g,$$

and a composite of smooth maps is smooth. □

The derivatives of these maps at the identity $1 \in G$ are worth studying. Since $L_g$ and $R_g$ are diffeomorphism with inverses $L_{g^{-1}}$ and $R_{g^{-1}}$, the derivatives

$$d(L_g)_1, d(R_g)_1 : \mathfrak{g} = T_1 G \to T_g G$$

are $\mathbb{R}$-linear isomorphisms. We can use this to identify every tangent space of $G$ with $g$. The conjugation map $\chi_g$ fixes 1, so it induces an $\mathbb{R}$- linear isomorphism

$$\mathrm{adj}_g = d(\chi_g)_1 : g \to g.$$

This is the adjoint action of $g \in G$.

**Theorem 2.43** *Let $G, H$ be Lie groups and $\varphi : G \to H$ be a Lie homomorphism. Then the derivative is a homomorphism of Lie algebras. In particular, if $G \leq H$ is a Lie subgroup, the inclusion map $i : G \to H$ induces an injection of Lie algebras $di : \mathfrak{g} \to \mathfrak{h}$.*

### *2.3.13 Some examples of Lie Ggroups.*

**Example 2.44** *When $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}, GL_n(\mathbb{F})$ is a Lie group.*

**Proof.** Since $\det : M_n(\mathbb{F}) \to \mathbb{F}$ is continuous and

$$GL_n(\mathbb{F}) = \det^{-1}(\mathbb{F} - \{0\}),$$

$GL_n(\mathbb{F}) \subseteq M_n(\mathbb{F})$ is an open subset, where as usual we identify $M_n(\mathbb{F})$ with $\mathbb{F}^{n^2}$. For charts we take the open sets $U \subseteq GL_n(\mathbb{F})$ and the identity function $Id : U \to U$ . The tangent space at each point $A \in GL_n(\mathbb{F})$ is just $M_n(\mathbb{F})$. The multiplication and inverse maps are obviously smooth as they are defined by polynomial and rational functions between open subsets of $M_n(\mathbb{F})$. □

**Example 2.45** *For $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}, SL_n(\mathbb{F}) \leq GL_n(\mathbb{F})$ is a Lie subgroup.*

**Proof.** Since $\det : GL_n(\mathbb{F}) \to \mathbb{F}^* = \mathbb{F} - \{0\}$ is a group homomorphism,

$$SL_n(\mathbb{F}) = \ker(\det) \leq GL_n(\mathbb{F}).$$

Since $T_1\mathbb{F}^* = \mathbb{F}$ is a smooth manifold, by the implicit function theorem, $SL_n(\mathbb{F})$ is a submanifold of $GL_n(\mathbb{F})$. Hence $SL_n(\mathbb{F})$ is a Lie subgroup of $GL_n(\mathbb{F})$. □

**Proposition 2.46 (*Left Translation Trick*)** *Let $F : GL_n(\mathbb{R}) \to M$ be a smooth function into a smooth manifold $M$ and suppose that $B \in GL_n(\mathbb{R})$ satisfies $F(BC) = F(C)$ for all $C \in GL_n(\mathbb{R})$. Let $A \in GL_n(\mathbb{R})$ with $dF_A$ is surjective. Then $dF_{BA}$ is surjective.*

**Proof.** Left multiplication by $B \in G, L_B : GL_n(R) \to GL_n(R)$, is a diffeomorphism and its derivative at $A \in GL_n(R)$ is

$$d(L_B) : M_n(R) \to M_n(R); dL_B(X) = BX$$

By assumption $FoL_B = F$ as a function on $GL_n(R)$. Then

$$\begin{aligned} dF_{BA}(X) &= dF_{BA}(B(B^{-1}X)) \\ &= dF_{BA} od(L_B)_A(B^{-1}X) \\ &= d(FoL_B)_A(B^{-1}X) \\ &= dF_A(B^{-1}X). \end{aligned}$$

Since left multiplication by $B^{-1}$ on $M_n(\mathbb{R})$ is surjective, the result follows. □

**Proposition 2.47** *Let $G \leq GL_n(\mathbb{R})$ be a matrix subgroup and let $M$ be a smooth manifold. Let $F : GL_n(\mathbb{R}) \to M$ be a smooth function with $F^{-1}q = G$ for some $q \in M$. Suppose that for every $B \in G, F(BC) = F(C)$ for all $C \in GL_n(\mathbb{R})$. If $dF_I$ is surjective then $dF_A$ is surjective for all $A \in G$ and $\ker dF_A = A\mathfrak{g}$.*

**Theorem 2.48** *Let $G \leq GL_n(\mathbb{R})$ be a matrix group which is also a submanifold, hence a Lie subgroup. Then the tangent space to $G$ at $I$ agrees with the Lie algebra $\mathfrak{g}$ and the dimension of the smooth manifold $G$ is $\dim G$; more generally, $T_AG = A\mathfrak{g}$.*

**Theorem 2.49** *Let $G \leq GL_n(\mathbb{R})$ be a matrix subgroup. Then $G$ is a Lie subgroup of $GL_n(\mathbb{R})$.*

**Theorem 2.50** *Let $H \leq G$ be a closed subgroup of a Lie group $G$. Then $H$ is a Lie subgroup of $G$.*

## 2.4 All matrix groups are Lie groups.

Let $G \leq GL_n(\mathbb{R})$ be a matrix subgroup. Recall that the Lie algebra $\mathfrak{g} = T_1G$ is a Lie subalgebra of $\mathfrak{gl}_n(\mathbb{R}) = M_n(\mathbb{R})$. Let

$$\tilde{\mathfrak{g}} = \{A \in M_n(\mathbb{R}) : \forall t \in \mathbb{R}, \exp(tA) \in G\}.$$

**Theorem 2.51** *$\tilde{\mathfrak{g}}$ is a Lie subalgebra of $M_n(\mathbb{R})$.*

**Proof.** By defining, $\tilde{\mathfrak{g}}$ is closed under multiplication by real scalars. If $U, V \in \tilde{\mathfrak{g}}$ and $r \geq 1$, then following are in $G$:

$$\exp((1/r)U)\exp((1/r)V), \exp((1/r)U)\exp((1/r)V)^r,$$

$$\exp((1/r)U)\exp((1/r)V)\exp(-(1/r)U)\exp(-(1/r)V)^{r^2},$$

$$\exp((1/r)U)\exp((1/r)V)\exp(-(1/r)U)\exp(-(1/r)V)^{r^2}.$$

By Theorem of Trotter, for $t \in \mathbb{R}$, we have

$\exp([tU, tV]) = \lim r \to \infty(\exp((1/r)tU)\exp((1/r)tV))^r$, and $\exp(t[U, V]) = \exp([tU, V]) = \lim r \to \infty(\exp((1/r)tU))\exp((1/r)V)\exp(-(1/r)tU)\exp(-(1/r)V)^{r^2}$.

As these are both limits of elements of the closed subgroup $G \leq GL_n(\mathbb{R})$ they are also in $G$. This shows that $\tilde{\mathfrak{g}}$ is a Lie subalgebra of $\mathfrak{gl}_n(\mathbb{R}) = M_n(\mathbb{R})$. □

**Proposition 2.52** *Let $U \in \tilde{\mathfrak{g}}$. Then the curve*

$$\gamma : \mathbb{R} \to G;\ \gamma(t) = \exp(tU),$$

*has $\gamma(0) = I$ and $\gamma'(0) = U$, hence $U \in \mathfrak{g}$.*

Eventually we will see that $\tilde{\mathfrak{g}} = \mathfrak{g}$.

**Theorem 2.53** *A subgroup of $GL_n(\mathbb{R})$ is a closed Lie group if and only if it is a matrix subgroup, i.e, a closed subgroup.*

Notice that the dimension of a matrix group $G$ as a manifold is $\dim_{\mathbb{R}} \tilde{\mathfrak{g}}$. By Proposition 2.52, $\tilde{\mathfrak{g}} \subseteq \mathfrak{g}$ and $\dim_{\mathbb{R}} \tilde{\mathfrak{g}} \leq \dim_{\mathbb{R}} \mathfrak{g}$. By Theorem 2.51, these dimensions are in fact equal, giving $\tilde{\mathfrak{g}} = \mathfrak{g}$.

**Theorem 2.54** *For a matrix group $G \leq GL_n(\mathbb{R})$, the exponential map* $\exp : g \to M_n(\mathbb{R})$ *has image in $G$, Im*$(\exp) \leq G$*. Moreover,* exp *is a local diffeomorphism at* 0*, mapping some open neighbourhood of* 0 *onto an open neighbourhood of $I$ in $G$.*

# CHAPTER 3

# MAXIMAL TORI IN COMPACT CONNECTED LIE GROUPS

The general reference for this chapter is [2].

**Theorem 3.1** *Let $G$ be a compact Lie group. Then for some $m, n \geq 1$, there are injective Lie homomorphisms $\alpha : G \to O(m)$ and $G \to U(n)$, hence $G$ is a matrix group.*

The familiar circle group

$$\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\} \leq \mathbb{C}^*$$

is a matrix group since $\mathbb{C}^* = GL_1(\mathbb{C})$; its Lie algebra is the 1-dimensional real abelian subalgebra

$$\{ti \in \mathbb{C} : t \in \mathbb{R}\} \leq \mathbb{C}.$$

Notice that $\mathbb{T}$ is a closed and bounded subset of $\mathbb{C}$, hence it is compact; it is also path connected and abelian.

**Definition 3.2** *For each $r \geq 1$,*

$$\mathbb{T}^r = \{diag(z_1, \ldots, z_r) : |z_1| = \ldots = |z_r| = 1\} \leq GL_r(\mathbb{C})$$

*is the* standard torus of rank $r$ *. More generally, a* torus of rank $r$ *is any Lie group isomorphic to $\mathbb{T}^r$.*

We will often view elements of $\mathbb{T}^r$ as sequences of complex numbers $(z_1, \ldots, z_r)$ with $|z_1| = \ldots = |z_r| = 1$, this corresponds to an identification

$$\mathbb{T}^r \cong \underbrace{\mathbb{T} \times \ldots \times \mathbb{T}}_{\text{r factors}} \leq (\mathbb{C}^*)^r$$

$\mathbb{T}^r$ is an abelian Lie group of dimension $r$ whose Lie algebra is the abelian Lie algebra

$$\{\text{diag}(t_1 i, \ldots, t_r i) : t_1, \ldots, t_r \in \mathbb{R}\}$$

which can also be identified with

$$\{(t_1 i, \ldots, t_r i) : \forall k, t_k \in \mathbb{R}\}.$$

**Proposition 3.3** *Let $T$ be a torus. Then $T$ is a compact, path connected and abelian Lie group.*

**Theorem 3.4** *Let $H$ be a compact Lie group. Then $H$ is a torus if and only if it is connected and abelian.*

**Proposition 3.5** *Let $T$ be a torus of rank $r$. Then the exponential map $\exp : \mathfrak{t} \to T$ is a surjective homomorphism of Lie groups whose kernel is a discrete subgroup isomorphic to $\mathbb{Z}^r$. Hence there is an isomorphism of Lie groups $\mathbb{R}^r/\mathbb{Z}^r \cong T$.*

**Definition 3.6** *Let $G$ be a Lie group. Then an element $g \in G$ is called a* topological generator *or just a* generator *of $G$ if the cyclic subgroup $< g > \leq G$ is dense in $G$, i.e., $\overline{< g >} = G$.*

**Proposition 3.7** *Every torus $T$ has a generator.*

## 3.1 Maximal tori in compact Lie groups.

**Definition 3.8** *Let $G$ be a Lie group and $T \leq G$ be a closed subgroup which is a torus. Then $T$ is* maximal *or a* maximal torus *in $G$ if the only torus $T' \leqq G$ for which $T \leqq T'$ is $T$.*

It is easy to see that every torus $T \leq G$ is contained in a maximal torus $\overline{T} \leq G$ where

$$\dim T \leq \dim \overline{T} \leq \dim G.$$

Later we will also see that if $G$ is connected then every element $g \in G$ is contained in a maximal torus.

We will now describe some important examples of maximal tori in compact connected matrix groups. For $\theta \in [0, 2\pi)$, let

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \in SO(2).$$

More generally, for each $n \geq 1$, and $\theta_i \in [0, 2\pi)$ $(i = 1, \ldots, n)$, let

$$R_{2n}(\theta_1, \ldots, \theta_n) = \begin{pmatrix} R(\theta_1) & 0 & \ldots & \ldots & \ldots & 0 \\ 0 & R(\theta_2) & 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ O & \ldots & \ldots & \ldots & 0 & R(\theta_n) \end{pmatrix}$$

$$R_{2n+1}(\theta_1, \ldots, \theta_n) = \begin{pmatrix} R(\theta_1) & 0 & \ldots & \ldots & \ldots & \ldots & 0 \\ 0 & R(\theta_2) & 0 & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ O & \ddots & \ddots & \ddots & 0 & R(\theta_n) & 0 \\ 0 & \ldots & \ldots & \ldots & \ldots & 0 & 1 \end{pmatrix}$$

where each entry marked $O$ is an appropriately sized block so that these are matrices of sizes $2n \times 2n$ and $(2n+1) \times (2n+1)$ respectively. Then $\mathbb{R}_{2n}(\theta_1, \ldots, \theta_n) \in SO(2n)$ and $\mathbb{R}_{2n+1}(\theta_1, \ldots, \theta_n) \in SO(2n+1)$.

It is obvious that $\mathbb{T} \cong U(1)$ as Lie groups. After first identifying $\mathbb{C}$ with $\mathbb{R}^2$ as real vector spaces using the bases $\{1, i\}$ and $\{e_1, e_2\}$, we also obtain an isomorphism of Lie groups

$$\mathbb{T} \to SO(2); \exp\theta_i \to R_2(\theta).$$

The maximal tori listed in the next result are usually regarded as the *standard maximal tori* for the corresponding groups.

**Proposition 3.9** *Each of the following is a maximal torus in the stated group:*

$$\{\mathbb{R}_2 n(\theta_1, \ldots, \theta_n) : \forall k \theta_k \in [0, 2\pi)\} \leq SO(2n).$$

$$\{R_{(2n+1)}(\theta_1, \ldots, \theta_n) : \forall k \theta_k \in [0, 2\pi)\} \leq SO(2n+1).$$

$$\{diag(z_1, \ldots, z_n) : \forall k,\ |z_k| = 1\} \leq U(n).$$

$$\{diag(z_1, \ldots, z_n) : \forall k, |z_k| = 1, z_1, \ldots, z_n\} \leq SU(n).$$

$$\{diag(z_1, \ldots, z_n) : \forall k, z_k \in \mathbb{C}, |z_k| = 1\} \leq Sp(n).$$

We now begin to study the structure of compact Lie groups in terms of their maximal tori. Let $G$ be a compact connected Lie group and $T \leq G$ be a maximal torus. The next result is fundamental to understanding the relationship between the different maximal tori in $G$.

**Theorem 3.10** *If $g \in G$, there is an $x \in G$ such that $g \in xTx^{-}1$, i.e., $g$ is conjugate to an element of $T$. Equivalently,*

$$G = \bigcup_{x \in G} xTx^{-1}.$$

**Theorem 3.11** *If $T, T' \leq G$ are maximal tori then they are conjugate in $G$,i.e., there is a $y \in G$ such that $T' = yTy^{-1}$.*

**Theorem 3.12 (*Principle Axis Theorem*)** *In each of the following matrix groups*

*every element is conjugate to one of the stated form.*

$$\begin{aligned} SO(2n) &: R_{2n}(\theta_1, \ldots, \theta_n), & \forall k\, \theta_k \in [0, 2\pi); \\ SO(2n+1) &: R_{(2n+1)}(\theta_1, \ldots, \theta_n), & \forall k\, \theta_k \in [0, 2\pi); \\ U(n) &: diag(z_1, \ldots, z_n), & \forall k\, z_k \in \mathbb{C},\ |z_k| = 1; \\ SU(n) &: diag(z_1, \ldots, z_n), & \forall k\, z_k \in \mathbb{C}, |z_k| = 1, z_1 \cdots z_n = 1; \\ Sp(n) &: diag(z_1, \ldots, z_n), & \forall k\, z_k \in \mathbb{C}, |z_k| = 1. \end{aligned}$$

There are related results on the Lie algebra $\mathfrak{g}$ of such a compact, connected matrix group $G$. Recall that for each $g \in G$, there is a linear transformation

$$\mathrm{Ad}_g : G \to \mathfrak{g};\ \mathrm{Ad}_g(t) = gtg^{-1}.$$

**Proposition 3.13** *Suppose that $g \in G$ and $H, K \leq G$ are Lie subgroups with Lie algebras $\mathfrak{h}$, $\mathfrak{k} \leq \mathfrak{g}$. If $gHg^{-1} = K$, then $Ad_g\mathfrak{h} = \mathfrak{k}$.*

We can now give an important result which we have already seen is true for many familiar examples.

**Theorem 3.14** *For a compact connected Lie group $G$, the exponential map $\exp : \mathfrak{g} \to G$ is surjective.*

## 3.2 The normaliser and Weyl group of a maximal torus.

We can continue to develop the general theory for a compact connected Lie group $G$.

**Proposition 3.15** *Let $A \leq G$ be a compact abelian Lie group and suppose that $A_1 \leq A$ is the connected component of the identity element. If $A/A_1$ is cyclic then $A$ has a generator and hence $A$ is contained in a torus in $G$.*

**Proposition 3.16** *Let $A \leq G$ be a connected abelian subgroup and let $g \in G$ commute with all the elements of $A$. Then there is a torus $T \leq G$ containing the subgroup $< A, g > \leq G$ generated by $A$ and $g$.*

**Theorem 3.17** *Let $T \leq G$ be a maximal torus and let $T \leq A \leq G$ where $A$ is abelian. Then $A = T$. Equivalently, every maximal torus is a maximal abelian subgroup.*

**Lemma 3.18** *Let $T \leq G$ be a torus and let $Q \leq N_G(T)$ be a connected subgroup acting on $T$ by conjugation. Then $Q$ acts trivially, i.e., for $g \in Q$ and $x \in T$,*

$$g \cdot x = gxg^{-1} = x.$$

**Theorem 3.19** *Let $T \leq G$ be a maximal torus.*

*i) The Weyl group $W_G(T)$ is isomorphic to the group of components of the normaliser,*

$$W_G(T) = N_G(T)/T = \pi_o N_G(T)$$

*Hence $W_G(T)$ is finite.*

*ii) $W_G(T)$ acts on $T$ by conjugation*

$$gT \cdot x = gxg^{-1}.$$

*This action on $T$ is faithful, i.e., the coset $gT \in N_G(T)/T$ acts trivially on $T$ if and only if $g \in T$.*

**Proposition 3.20** *Let $T \leq G$ be a maximal torus and $x, y \in T$. If $x, y$ are conjugate in $G$ then they are conjugate in $N_G(T)$, hence there is an element $\omega \in W_G(T)$ for which $y = \omega \cdot x$.*

## 3.3 Root systems, Weyl groups and Dynkin diagrams.

## 3.4 Roots systems and their Weyl groups

Let $V$ be an $l$- dimensional real vector space with a positive definite inner product $<,>$; as usual, the length of a vector $\alpha \in V$ is defined to be $|\alpha| = \sqrt{<\alpha, \alpha>}$.

If $u \in V - \{0\}$, there is an associated $\mathbb{R}$-linear transformation

$$s_u(x) = \begin{cases} -x & if x = tu \text{ for some } t \in \mathbb{R}, \\ x & if < u, x >= 0. \end{cases}$$

It is easy to see that for $x \in V$,

$$s_u(x) = x - \frac{2 < u, x >}{< u, u >} u.$$

Geometrically, $s_u$ is a *hyperplane reflection* in the hyperplane

$${H_u}^{\perp} = \{x \in V :< u, x >= 0\} \subseteq V.$$

**Proposition 3.21** *If $u, v \in V - \{0\}$, the following identities are satisfied:*

$${s_u}^2 = s_u \circ s_u = Id_V,$$

$$\begin{cases} s_v = s_u & if v = tu \, for \, some \, t \in \mathbb{R}, \\ s_v \neq s_u & otherwise. \end{cases}$$

**Definition 3.22** *A subset $\Phi \subseteq V$ is a* reduce root system *if it has the following properties:*

- *$\Phi$ spans $V$ and $0 \notin \Phi$;*
- *if $u \in \Phi$ then $-u \in \Phi$ and no other multiplies of $u$ are in $\Phi$;*
- *if $u, v \in \Phi$ then $\frac{2<u,x>}{<u,u>} \in \mathbb{Z}$ and $s_u(v) \in \Phi$.*

An element of $\Phi$ is called a root. The final condition is equivalent to the requirement that for $u, v \in \Phi$ and some $k \in \mathbb{Z}$,

$$s_u(v) - v = ku.$$

Each $s_u$ acts on $\Phi$ as a permutation and it can be viewed as an element of the group

of all permutations of $\Phi, S_\Phi$.

**Definition 3.23** *The Weyl group of* $\Phi$ *is the subgroup* $W(\Phi) \leq S_\Phi$ *generated by the* $s_u$, $u \in \Phi$.

By definition $W(\Phi)$ is finite with order $|W(\Phi)|$ dividing $|\Phi|$.

**Definition 3.24** *A subset* $\prod \subseteq$ *is a set of* fundamental or simple roots *if it satisfies that*

- $\prod$ *is an*$\mathbb{R}$*-basis of* $V$*;*
- *every* $u \in \Phi$ *can be uniquely expressed in the form*

$$\sum_{w \in \prod} c_w w$$

*where* $c_w \in \mathbb{Z}$ *either all satisfy* $c_w \geq 0$ *or all satisfy* $c_w \leq 0$.

The choice of such a fundamental system is not unique, however given $\prod$ there is a partition $\Phi = \Phi^+ \bigcup \Phi^-$ where

$$\Phi^+ = \left\{ \sum_{w \in \prod} c_w w : c_w \geq 0 \right\}, \Phi^- = \left\{ \sum_{w \in \prod} c_w w : c_w \leq 0 \right\}.$$

Notice that

$$-\Phi^+ = \{-u : u \in \Phi^+\} = \Phi^-, -\Phi^- = \{-u : u \in \Phi^-\} = \Phi^+.$$

The elements of $\prod$ are called (positive) *simple roots*, while the elements of $\Phi^+$ or $\Phi^-$ are called *positive or negative roots*. It is easy to see that $W(\Phi)$ is generated by the reflections $s_u, u \in \prod$.

## 3.5 Some examples of root systems.

**The root system of** $U(2)$. Consider $\mathfrak{t}^*$ with orthonormal basis $\varepsilon_1, \varepsilon_2$ dual to the orthonormal basis $\{\widehat{H_1}, \widehat{H_2}\}$ of $\mathfrak{t} \cong \mathfrak{t}^{**}$. Here we take $V = z(\mathfrak{u}(2))^\perp \subseteq \mathfrak{t}^*$. If $\alpha = \varepsilon_1 - \varepsilon_2$

and $\beta = \varepsilon_1 + \varepsilon_2$ then $\alpha \cdot \beta = 0$ and $\mathbb{R}\{\beta\} = z(\mathfrak{u}(2))^*$. So for this root system we have

$$\prod = \{\alpha\}, \Phi = \{-\alpha, \alpha\},$$

with Weyl group $W(\Phi) = S_\Phi \cong S_2$.

In the remaining examples, $\mathbb{R}^l$ and its subspaces have the usual positive definite inner product $< x|y >= x \cdot y$.

**The root system $A_1$.**

Let $V = \mathbb{R}$ and

$$\prod = \{\alpha_1\},\ \Phi = \{-\alpha_1, \alpha_1\}$$

where $\alpha_1 \in \mathbb{R}$ is non-zero. This root system has Weyl group $W(\Phi) = S_\Phi \cong S_2$.

$$-\alpha_1 \longleftarrow \cdot \longrightarrow \alpha_1$$

Figure 3.1: The root system of $A_1$

**The root system $A_2$.**

Let $V = \mathbb{R}^2$ and

$$\prod = \{\alpha_1, \alpha_2\},\ \Phi = \{\alpha_1, -\alpha_1, \alpha_2, -\alpha_2, \alpha_1 + \alpha_2, -\alpha_1 - \alpha_2\}$$

where $\alpha_1, \alpha_2$ are two vectors for which

$$|\alpha_1| = |\alpha_2|,\ \alpha_1 \cdot \alpha_2 = -|\alpha_1| \cdot |\alpha_2|/2.$$

This root system has the hyperplane reflections $s_{\alpha_1}, s_{\alpha_2}, s_{\alpha_1+\alpha_2}$ and Weyl group $S_\Phi = S_3$.

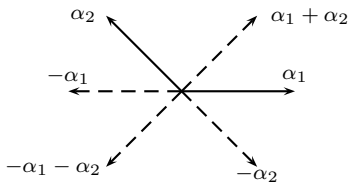



Figure 3.2: The root system of $A_2$

The preceding examples generalize to give the root systems $A_n$.

$$V = \left\{x \in \mathbb{R}^{n+1} : x \cdot (e_1 + \ldots + e_{n+1}) = 0\right\} \subseteq \mathbb{R}^{n+1}$$

and $\alpha = e_k - e_{k+1}$ for $1 \leq k \leq n$. Then $V$ is a vector space of dimension $n$ with basis $\{\alpha_1, \ldots, \alpha_n\}$. Notice that

$$\alpha_i \cdot \alpha_j = \begin{cases} 2 & \text{if } i = j \\ -1 & \text{if } |i-j| = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Taking

$$\begin{aligned} \prod &= \{\alpha_1, \ldots, \alpha_n\}, \\ \Phi &= \{\pm(\alpha_r + \alpha_{r+1} + \ldots + \alpha_s) : 1 \leq r \leq s \leq n\} \\ &= \{\pm(e_r - e_{s+1}) : 1 \leq r \leq s \leq n\}, \end{aligned}$$

we obtain a root system with Weyl group $S_\Phi = S_n$.

### 3.6 The Dynkin diagram of a root system.

Let $\prod$ be a fundamental system of the root system $\Phi \subseteq V$. Choosing an ordering $\alpha_1, \ldots, \alpha_l$ for the elements of $\prod$, there are integers

$$a_{ij} = \frac{2 < \alpha_i, \alpha_j >}{< \alpha_i \alpha_j >}, \ 1 \leq i, j \leq l.$$

The $l \times l$ matrix $[a_{ij}]$ with integer entries is called the *Cartan matrix* of the root system $\Phi$.

We will now show how to associate to a root system $\prod \subseteq \Phi$ with a *Dynkin diagram.* The diagram has one vertex for each simple root labelled by the root, with some bonds joining pairs of them. For distinct simple roots $\alpha_i, \alpha_j \in \prod$, there are $d_{ij}$ bonds between the corresponding vertices, where

$$d_{ij} = \begin{cases} 0 & \text{if } \varphi_{ij} = \frac{\pi}{2}, \\ 1 & \text{if } \varphi_{ij} = \frac{\pi}{3}, \frac{2\pi}{3} \\ 2 & \text{if } \varphi_{ij} = \frac{\pi}{4}, \frac{3\pi}{4} \\ 3 & \text{if } \varphi_{ij} = \frac{\pi}{6}, \frac{5\pi}{6}. \end{cases}$$

In the cases where $d_{ij} = 2, 3$, an inequality sign $>$is added indicating which of the roots is larger. Here are the possible connections between nodes that can occur.

The whole Dynkin diagram is obtained by filling in the bonds between the vertices for all simple roots. Such a diagram is *irreducible or connected* if it has only one path component.

## 3.7 Irreducible Dynkin diagrams.

The connected irreducible Dynkin diagrams are given in Figure 3.3 below. Each of the four classical series $A_n, B_n, C_n, D_n$ consists of infinitely many distinct examples. The classical *Dynkin* diagrams are related to familiar matrix groups, where the entries at the bottom of Figure 3.3 are consequences of certain special isomorphisms.

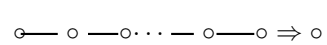
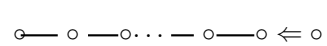
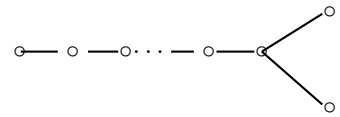
| Root System | *MatrixGroup* | Dynkin Diagrams |
|---|---|---|
| $A_n (n \geq 1)$ | $SU(n)$ |  |
| $B_n (n \geq 2)$ | Spin(2n+1),SO(2n+1) |  |
| $C_n (n \geq 3)$ | $Sp(n)$ |  |
| $D_n (n \geq 4)$ | Spin(2n),SO(2n) |  |
| $A_1$ | $SU(2) \cong Spin(3), SO(3)$ |  |
| $B_2 = C_2$ | $Sp(2) \cong Spin(5), SO(5)$ |  |

Figure 3.3: Classical Dykin diagrams and associated matrix groups

Now assume that $\prod = \{\alpha_1, \ldots, \alpha_l\}$ is a set of simple roots for a root system $\Phi \subseteq \mathfrak{t}^*$, the linear dual of an $\mathbb{R}$-vector space $\mathfrak{t}$ of dimension $l$, equipped with a positive definite inner product $(|)$. Using the duality isomorphism $\nu : \mathfrak{t} \to \mathfrak{t}^*$, we can define elements

$$\breve{a} = \frac{2}{(\alpha|\alpha)} \nu^{-1}(\alpha) \in \mathfrak{t}, \ \alpha \in \Phi$$

which are called *coroots*. We have the formula

$$\begin{aligned} \alpha(\breve{\beta}) &= \frac{2}{(\beta|\beta)}(\nu^{-1}(\alpha)|(\nu^{-1}(\beta))) \\ &= \frac{2}{(\beta|\beta)}(\alpha|\beta). \end{aligned} \tag{3.1}$$

We write

$$\check{\Phi} = \{\check{\alpha} : \alpha \in \Phi\}, \ \check{\prod} = \left\{\check{\alpha} : \alpha \in \prod\right\}$$

for the sets of all coroots and simple coroots respectively.

Then $\check{\Phi} \subseteq \mathfrak{t}$ is a root system with $\check{\prod} \subseteq \check{\Phi}$ which is called t *dual root system* to $\prod \subseteq \Phi$.

# CHAPTER 4
# THE ROOT SYSTEM, WEYL GROUP AND CARTAN MATRIX OF THE LOOP GROUP $LG$

## 4.1 The adjoint decomposition of semi-simple Lie algebras.

We know from compact simply-connected semi-simple Lie theory that the complexified Lie algebra $\mathbf{g}_{\mathbb{C}}$ of the compact Lie group $G$ has a decomposition under the adjoint action of the maximal torus $T$ of $G$. Then, from [16], we have

**Theorem 4.1** *There is a decomposition*

$$\mathbf{g}_{\mathbb{C}} = \mathbf{t}_{\mathbb{C}} \bigoplus_{\alpha} \mathbf{g}_{\alpha},$$

*where* $\mathbf{g}_0 = \mathbf{t}_{\mathbb{C}}$ *is the complexified Lie algebra of* $T$ *and*

$$\mathbf{g}_{\alpha} = \{\xi \in \mathbf{g}_{\mathbb{C}} : t \cdot \xi = \alpha(t)\xi \, \forall t \in T\}.$$

The homomorphisms $\alpha : T \to \mathbb{T}$ for which $\mathbf{g}_{\alpha} \neq 0$ are called the *roots* of $G$. They form a finite subset of the lattice $\breve{T} = \hom(T, \mathbb{T})$.

**Definition 4.2** *Let $G$ be a compact simply connected semi-simple Lie group. Then the set of all smooth functions from the unit circle $\S^1$ to $G$ is called free loop space of $G$ and it is denoted by $LG$. $LG$ has group structure with loop multiplication on $G$.*

By analogy, the complexified Lie algebra $L\mathbf{g}_{\mathbb{C}}$ of the loop group $LG$ has a decomposition

$$L\mathbf{g}_{\mathbb{C}} = \bigoplus_{k \in \mathbb{Z}} \mathbf{g}_{\mathbb{C}} \cdot z^k$$

where $\mathbf{g}_{\mathbb{C}}$ is the complexified Lie algebra of $G$. This is the decomposition into eigenspaces of the rotation action of the circle group $\mathbb{T}$ on the loops. The rotation action commutes with the adjoint action of the constant loops $G$, and from [32], we have

**Theorem 4.3** *There is a decomposition of $L\mathbf{g}_{\mathbb{C}}$ under the action of the maximal torus $T$ of $G$,*

$$L\mathbf{g}_{\mathbb{C}} = \bigoplus_{k\in\mathbb{Z}} \mathbf{g}_0 \cdot z^k \oplus \bigoplus_{(k,\alpha)} \mathbf{g}_\alpha \cdot z^k.$$

The pieces in this decomposition are indexed by homomorphisms

$$(k, \alpha) : \mathbb{T} \times T \to \mathbb{T}.$$

The homomorphisms $(k, \alpha) \in \mathbb{Z} \times \breve{T}$ which occur in the decomposition are called the *roots* of $LG$.

**Definition 4.4** *The set of roots is called the* root system *of $LG$ and denoted by $\widehat{\Delta}$.*

Let $\delta$ be $(0, 1)$. Then

$$\widehat{\Delta} = \bigcup_{k\in\mathbb{Z}} (\Delta \cup \{0\} + k\delta) = \Delta \cup \{0\} + \mathbb{Z}\delta,$$

where $\Delta$ is the root system of $G$. The root system $\widehat{\Delta}$ is the union of real roots and imaginary roots:

$$\widehat{\Delta} = \widehat{\Delta}_{\text{re}} \cup \widehat{\Delta}_{\text{im}},$$

where

$$\begin{aligned} \widehat{\Delta}_{\text{re}} &= \{(\alpha, n) : \alpha \in \Delta,\, n \in \mathbb{Z}\} \\ \widehat{\Delta}_{\text{im}} &= \{(0, r) : r \in \mathbb{Z}\} \end{aligned}$$

**Definition 4.5** *Let the rank of $G$ be $l$. Then, the set of simple roots of $LG$ is*

$$\{(\alpha_i, 0) : \alpha_i \in \Sigma \, for \, 1 \le i \le l\} \cup \{(-\alpha_{l+1}, 1)\},$$

*where $\alpha_{l+1}$ is the highest weight of the adjoint representation of $G$.*

The root system $\widehat{\Delta}$ can be divided into three parts as the positive and the negative and 0:

$$\widehat{\Delta} = \widehat{\Delta}^+ \cup \{0\} \cup \widehat{\Delta}^-$$

where

$$\widehat{\Delta}^+ = \widehat{\Delta}^+_{\text{re}} \cup \widehat{\Delta}^+_{\text{im}},$$
$$\widehat{\Delta}^- = \widehat{\Delta}^-_{\text{re}} \cup \widehat{\Delta}^-_{\text{im}},$$

where

$$\widehat{\Delta}^+_{\text{re}} = \{(\alpha, n) \in \widehat{\Delta}_{\text{re}} : n > 0\} \cup \{(\alpha, 0) : \alpha \in \Delta^+\},$$
$$\widehat{\Delta}^+_{\text{im}} = \{n\delta : n > 0\}$$

and

$$\widehat{\Delta}^-_{\text{re}} = -\widehat{\Delta}^+_{\text{re}},$$
$$\widehat{\Delta}^-_{\text{im}} = -\widehat{\Delta}^+_{\text{im}}.$$

In the case of $LSU_n$, for $n \geq 3$, the root system $\widehat{\Delta}$ of the loop group $LSU_n$ has basis elements $\mathbf{a}_0 = (-\alpha_0, 1)$ and $\mathbf{a}_i = (\alpha_i, 0)$, $1 \leq i \leq n-1$ where $\alpha_i$ is the simple root of $SU_n$ and $\alpha_0 = \sum_{i=1}^{n-1} \alpha_i$. All roots of $LSU_n$ can be written as a sum of the simple roots $\mathbf{a}_i$.

**Theorem 4.6** *(see [18]) The set of roots of $LSU_n$, for $n \geq 3$, is*

$$\widehat{\Delta} = \{k \sum_{r=0}^{i-1} \mathbf{a}_r + l \sum_{r=i}^{j-1} \mathbf{a}_r + k \sum_{r=j}^{n-1} \mathbf{a}_r : |k-l| = 1, k \in \mathbb{Z} \, and \, 0 \leq i \leq j \leq n\}.$$

**Corollary 4.7** *The set of positive roots of $LSU_n$, for $n \geq 3$, is*

$$\widehat{\Delta}^+ = \{k\sum_{r=0}^{i-1}\mathbf{a}_r + l\sum_{r=i}^{j-1}\mathbf{a}_r + k\sum_{r=j}^{n-1}\mathbf{a}_r : |k-l| = 1, k \in \mathbb{Z}^+ \; and \; 0 \leq i \leq j \leq n\}.$$

**Corollary 4.8** *The simple roots of $LSU_4$ are $a_0 = (-\alpha_1 - \alpha_2 - \alpha_3, 1), a_1 = (\alpha_1, 0), a_2 = (\alpha_2, 0), a_3 = (\alpha_3, 0)$ where $\alpha_1$, $\alpha_2$ and $\alpha_3$ are the simple roots of compact Lie group $SU_4$.*

*The set of all positive real roots of $LSU_4$ is $\{(\alpha_1, m), (\alpha_2, m)(\alpha_3, m), (\alpha_1 + \alpha_2 + \alpha_3, m), (-\alpha_1, s), (-\alpha_2, s), (-\alpha_3, s)(-\alpha_1 - \alpha_2 - \alpha_3, s)\; m \geq 0, s > 0\}$.*

## 4.2 Affine Weyl groups.

Now, we will discuss the Weyl group of the loop group $LG$. In order to define this group, we need a larger group structure. We define the semi-direct product $\mathbb{T} \ltimes LG$ of $\mathbb{T}$ and $LG$ in which $\mathbb{T}$ acts on $LG$ by the rotation. From [32], we have

**Theorem 4.9** *$\mathbb{T} \times T$ is a maximal abelian subgroup of $\mathbb{T} \ltimes LG$.*

**Theorem 4.10** *The complexified Lie algebra of $\mathbb{T} \ltimes LG$ has a decomposition*

$$(\mathbb{C} \oplus \mathbf{t}_{\mathbb{C}}) \oplus \left( \bigoplus_{k \neq 0} \mathbf{t}_{\mathbb{C}} \cdot z^k \oplus \bigoplus_{(k,\alpha)} \mathbf{g}_\alpha \cdot z^k, \right)$$

*according to the characters of $\mathbb{T} \times T$.*

We know that the roots of $G$ are permuted by the Weyl group $W$. This is the group of automorphisms of the maximal torus $T$ which arise from conjugation in $G$, i.e. $W = N(T)/T$, where

$$N(T) = \{n \in G : nTn^{-1} = T\}$$

is the normalizer of $T$ in $G$. In exactly same way, the infinite set of roots of $LG$ is permuted by the Weyl group $\widetilde{W} = N(\mathbb{T} \times T)/(\mathbb{T} \times T)$, where $N(\mathbb{T} \times T)$ is the normalizer in $\mathbb{T} \ltimes LG$. The Weyl group $\widetilde{W}$ which was defined above is called the *affine Weyl group.*

**Proposition 4.11** *The affine Weyl group $\widetilde{W}$ is the semidirect product of the coweight lattice $T^\vee = \hom(\mathbb{T}, T)$ by the Weyl group $W$ of $G$.*

We know that the Weyl group $W$ of $G$ acts on the Lie algebra of the maximal torus $T$, it is a finite group of isometries of the Lie algebra $\mathbf{t}$ of the maximal torus $T$. It preserves the coweight lattice $T^\vee$. For each simple root $\alpha$, the Weyl group $W$ contains an element $r_\alpha$ of order two represented by $\exp\left(\frac{\pi}{2}(e_\alpha + e_{-\alpha})\right)$ in $N(T)$. Since the roots $\alpha$ can be considered as the linear functionals on the Lie algebra $\mathbf{t}$ of the maximal torus $T$, the action of $r_\alpha$ on $\mathbf{t}$ is given by

$$r_\alpha(\xi) = \xi - \alpha(\xi)h_\alpha \text{ for } \xi \in \mathbf{t},$$

where $h_\alpha$ is the coroot in $\mathbf{t}$ corresponding to simple root $\alpha$. Also, we can give the action of $r_\alpha$ on the roots by

$$r_\alpha(\beta) = \beta - \alpha(h_\beta)\alpha \text{ for } \alpha, \beta \in \mathbf{t}^*,$$

where $\mathbf{t}^*$ is the dual vector space of $\mathbf{t}$. The element $r_\alpha$ is the reflection in the hyperplane $H_\alpha$ of $\mathbf{t}$ whose equation is $\alpha(\xi) = 0$. These reflections $r_\alpha$ generate the Weyl group $W$. For $G = SU_n$, we have from [17].

**Theorem 4.12** *The Weyl group of $SU_n$ is the symmetric group $S_n$.*

**Proposition 4.13** *The Weyl group $W$ of $SU_4$ is isomorphic to the symmetric group $S_4$ which has the following presentation*

$$\{r_{a_i} : r_i^2 = 1, r_i r_j r_i = r_j r_i r_j, |i-j| \neq 2, \text{ and } i, j = 1, 2, 3, \text{ and } r_1 r_3 = r_3 r_1\}.$$

**Proof.** If we check the action of reflections corresponding to simple roots of $SU_4$ on the root system then we see that there are the relations above. We know that the symmetric group of $S_4$ of an ordered set $\{0, 1, 2, 3\}$ is equal to $S_4 = \{x = (01), y = (0123) \in S_4 | x^2 = y^4 = e\}$ where $x, y$ are cycles. Now if we define a morphism $\phi$ from $S_4$ to the Weyl group of $SU_4$ by $\phi(x) = r_1 r_2 r_3 r_2 r_1$ and $\phi(y) = r_1 r_2 r_3 r_1 r_2$, we have a group isomorphism between $S_4$ and the Weyl group of $SU_4$ which is our aim. $\square$

Now, we want to describe the Weyl group structure of $LG$. By analogy with $\mathbb{R}$ for real form, the roots of the loop group $LG$ can be considered as linear forms on the Lie algebra $\mathbb{R} \times \mathbf{t}$ of the maximal abelian group $\mathbb{T} \times T$. The Weyl group $\widetilde{W}$ acts linearly on $\mathbb{R} \times \mathbf{t}$, the action of $W$ is an obvious reflection in the affine hyperplane $1 \times \mathbf{t}$ and the action of $\lambda \in T^\vee$ is given by

$$\lambda \cdot (x, \xi) = (x, \xi + x\lambda).$$

Thus, the Weyl group $\widetilde{W}$ preserves the hyperplane $1 \times \mathbf{h}$, and $\lambda \in \breve{T}$ acts on it by translation by the vector $\lambda \in T^\vee \subset \mathbf{t}$. If $\alpha \neq 0$, the affine hyperplane $H_{\alpha,k}$ can be defined as follows. For each root $(\alpha, k)$,

$$H_{\alpha,k} = \{\xi \in \mathbf{t} : \alpha(\xi) = -k\}.$$

We know that the Weyl group $W$ of $G$ is generated by the reflections $r_\alpha$ in the hyperplanes $H_\alpha$ for the simple roots $\alpha$. A corresponding statement holds for the affine Weyl group $\widetilde{W}$.

**Proposition 4.14** *Let $G$ be a simply-connected semi-simple compact Lie group. Then the Weyl group $\widetilde{W}$ of the loop group $LG$ is generated by the reflections in the hyperplanes $H_{\alpha,k}$. The affine Weyl group $\widetilde{W}$ acts on the root system $\widehat{\Delta}$ by*

$$r_{(\alpha,k)}(\gamma, m) = (r_\alpha(\gamma), m - \alpha(h_\gamma)k) \text{ for } (\alpha, k), (\gamma, m) \in \widehat{\Delta}.$$

**Proposition 4.15** *The Weyl group of $LSU_n$ is the semi-direct product $S_n \ltimes \mathbb{Z}^{n-1}$ where $S_n$ acts by permutation action on coordinates of $\mathbb{Z}^{n-1}$.*

Actually the symmetric group $S_n$ acts on $\mathbb{Z}^n$ by the permutation action. $\mathbb{Z}^{n-1}$ is the fixed subgroup which corresponds to the eigen-value action.

By 4.15, the Weyl group of $LSU_4$ is $S_4 \ltimes \mathbb{Z}^3$. Moreover, we explicitly give the group presentation of $S_4 \ltimes \mathbb{Z}^3$.

**Proposition 4.16** *The Weyl group $\widetilde{W}$ of $LSU_4$ is isomorphic to the group presentation*

$$\{r_{a_i} : r_i^2 = 1, r_i r_j r_i = r_j r_i r_j, i \neq j, i, j = 0, 1, 2, 3, \text{ and} r_i r_k = r_k r_i, (ijkl) \in S_4\},$$

*where $(ijkl)$ be the cycle of $0, 1, 2, 3$ respectively.*

**Proof.** By 4.14, we know that the Weyl group $\widetilde{W}$ of the loop group $LSU_4$ is generated by the reflections $r_{a_0}$, $r_{a_1}$, $r_{a_2}$, and $r_{a_3}$, in the hyperplanes $H_{a_0}, H_{a_1}, H_{a_2}, H_{a_3}$ respectively.

Now we will find the relations in the presentation by the action of the Weyl group of $LSU_4$ on the real root system

$$\widehat{\Delta}_{\text{re}} = \{(\alpha_i, m) : m \in \mathbb{Z}, \alpha_i \text{ is a root of } SU_4\}.$$

First we show that $r_i^2 = 1, i = 0, 1, 2, 3$. It is enough to check the action of $r_{a_i}$ on the real root $(\alpha_1, m)$ because the Cartan matrix of $SU_4$ is symmetric and the action is bilinear. The action is as follows.

$$r_{a_i}(\alpha_1, m) = \begin{cases} (-\alpha_2 - \alpha_3, m+1) & \text{for } i = 0, \\ (-\alpha_1, m) & \text{for } i = 1, \\ (\alpha_1 + \alpha_2, m) & \text{for } i = 2, \\ (\alpha_1, m) & \text{for } i = 3, \end{cases}$$

and $r_{a_i}^2(\alpha_1, m) = (\alpha_1, m), i = 0, 1, 2, 3$.

So we get $r_{a_i}^2 = 1, i = 0, 1, 2, 3$. Now we are looking for the other equations in the presentation.

For $i = 0$,

$$r_{a_i} r_{a_j} r_{a_i}(\alpha_1, m) = \begin{cases} (\alpha_1 + \alpha_2 + \alpha_3, m-1) & \text{for } j = 1, \\ (-\alpha_2, m+1) & \text{for } j = 3. \end{cases}$$

For $i = 1$,

$$r_{a_i} r_{a_j} r_{a_i}(\alpha_1, m) = \begin{cases} (\alpha_1 + \alpha_2 + \alpha_3, m-1) & \text{for } j = 0, \\ (-\alpha_2, m) & \text{for } j = 2. \end{cases}$$

For $i = 2$,

$$r_{a_i} r_{a_j} r_{a_i}(\alpha_1, m) = \begin{cases} (\alpha_1 + \alpha_2 + \alpha_3, m) & \text{for } j = 3, \\ (-\alpha_2, m) & \text{for } j = 1. \end{cases}$$

For $i = 3$,

$$r_{a_i} r_{a_j} r_{a_i}(\alpha_1, m) = \begin{cases} (-\alpha_2, m+1) & \text{for } j = 0, \\ \alpha_1 + \alpha_2 + \alpha_3, m) & \text{for } j = 2. \end{cases}$$

By these calculations, we see that $r_0 r_1 r_0 = r_1 r_0 r_1$, $r_0 r_3 r_0 = r_3 r_0 r_3$, $r_1 r_2 r_1 = r_2 r_1 r_2$, and $r_2 r_3 r_2 = r_3 r_2 r_3$ hold. Addition to these we also have $r_0 r_2 = r_2 r_0$ and $r_1 r_3 = r_3 r_1$.

Now we will show that the presentation of the Weyl group $\widetilde{W}$ of $LSU_4$ has just the relations in 4.16. To prove that first we classify elements of the Weyl group $\widetilde{W}$ of $LSU_4$ after then we check that there are no more relations using the actions of these classified elements to the root system.

Let $S_4$ be the permutation group on the set $\{0, 1, 2, 3\}$. Let $\sigma$ denote a cycle in $S_4$.

All elements of the Weyl group $\widetilde{W}$ of $LSU_4$ are classified as the following. For every $m, n, p \in \mathbb{N}$,

| i | j | k | l |
|---|---|---|---|
| 0 | 1 | 2 | 3 |
| 1 | 2 | 3 | 0 |
| 2 | 3 | 0 | 1 |
| 3 | 0 | 1 | 2 |

Relations:

$r_i^2 = e$

$r_i r_k = r_k r_i$

$r_i r_j r_i = r_j r_i r_j$

0 Mod 4

1.1 $(r_i r_j r_k r_l)^m$

1.2 $(r_i r_l r_k r_j)^m$

1.3 $(r_i r_j r_l r_k)^m$

1.4 $(r_i r_k r_j r_l)^m$

1.5 $(r_i r_j r_l r_i)(r_k r_j r_l r_i)^m$

2.1.1 $(r_i r_j r_k r_l)^m (r_i r_j r_i)(r_l r_k r_j r_i)^n r_l$

2.1.2 $r_l (r_i r_j r_k r_l)^m (r_i r_j r_i)(r_l r_k r_j r_i)^n$

2.1.3 $r_k r_l (r_i r_j r_k r_l)^m (r_i r_j r_i)(r_l r_k r_j r_i)^n r_l r_k r_j$

2.1.4 $r_j r_k r_l (r_i r_j r_k r_l)^m (r_i r_j r_i)(r_l r_k r_j r_i)^n r_l r_k$

2.2.1 $(r_i r_j r_k r_l)^m (r_i r_j r_i)(r_k r_j r_i r_l)^n r_k$

2.2.2 $r_l (r_i r_j r_k r_l)^m (r_i r_j r_i)(r_k r_j r_i r_l)^n \qquad n \geq 1$

2.2.3 $r_k r_l (r_i r_j r_k r_l)^m (r_i r_j r_i)(r_k r_j r_i r_l)^n r_k r_j r_i$

2.2.4 $r_j r_k r_l (r_i r_j r_k r_l)^m (r_i r_j r_i)(r_k r_j r_i r_l)^n r_k r_j$

2.3.1 $(r_i r_j r_k r_l)^m (r_i r_j r_i)(r_k r_l r_j r_i)^n r_k \qquad n \geq 1$

2.3.2 $r_l (r_i r_j r_k r_l)^m (r_i r_j r_i)(r_k r_l r_j r_i)^n \qquad n \geq 1$

2.3.3 $r_k r_l (r_i r_j r_k r_l)^m (r_i r_j r_i)(r_k r_l r_j r_i)^n r_k r_l r_j$

2.3.4 $r_j r_k r_l (r_i r_j r_k r_l)^m (r_i r_j r_i)(r_k r_l r_j r_i)^n r_k r_l$

2.3.4a $r_j r_k r_l (r_i r_j r_k r_l)^m (r_i r_j r_i)(r_k r_j r_l r_i)^n r_k r_j \qquad n \geq 1$

2.4.1 $(r_i r_j r_l r_k)^m (r_i r_j r_i)(r_l r_k r_j r_i)^n r_l \qquad m \geq 1$

2.4.2 $r_k (r_i r_j r_l r_k)^m (r_i r_j r_i)(r_l r_k r_j r_i)^n$

2.4.3 $r_l r_k (r_i r_j r_l r_k)^m (r_i r_j r_i)(r_l r_k r_j r_i)^n r_l r_k r_j$

2.4.3a $r_j r_k (r_i r_l r_j r_k)^m (r_i r_j r_i)(r_l r_k r_j r_i)^n r_l r_k r_j$

2.4.4 $r_j r_l r_k (r_i r_j r_l r_k)^m (r_i r_j r_i)(r_l r_k r_j r_i)^n r_l r_k$

3.1.1 $(r_i r_j r_k r_l)^m (r_i r_j r_i)(r_k r_j r_l r_i)^p (r_k r_l r_k)(r_j r_i r_l r_k)^n r_j r_i$

3.1.2 $r_l(r_ir_jr_kr_l)^m(r_ir_jr_i)(r_kr_jr_lr_i)^p(r_kr_lr_k)(r_jr_ir_lr_k)^nr_j$

3.1.3 $r_kr_l(r_ir_jr_kr_l)^m(r_ir_jr_i)(r_kr_jr_lr_i)^p(r_kr_lr_k)(r_jr_ir_lr_k)^n$

3.1.4 $r_jr_kr_l(r_ir_jr_kr_l)^m(r_ir_jr_i)(r_kr_jr_lr_i)^p(r_kr_lr_k)(r_jr_ir_lr_k)^nr_jr_ir_l$

3.2.1 $(r_ir_jr_kr_l)^m(r_ir_jr_i)(r_kr_jr_lr_i)^p(r_kr_jr_lr_k)(r_jr_ir_lr_k)^nr_j$

3.2.2 $r_l(r_ir_jr_kr_l)^m(r_ir_jr_i)(r_kr_jr_lr_i)^p(r_kr_jr_lr_k)(r_jr_ir_lr_k)^n$

3.2.3 $r_kr_l(r_ir_jr_kr_l)^m(r_ir_jr_i)(r_kr_jr_lr_i)^p(r_kr_jr_lr_k)(r_jr_ir_lr_k)^nr_jr_ir_l$

3.2.4 $r_jr_kr_l(r_ir_jr_kr_l)^m(r_ir_jr_i)(r_kr_jr_lr_i)^p(r_kr_jr_lr_k)(r_jr_ir_lr_k)^nr_jr_i$

3.3.1 $(r_kr_lr_ir_j)^m(r_kr_jr_lr_k)(r_ir_lr_jr_k)^p(r_ir_jr_i)(r_lr_kr_jr_i)^nr_l$

3.3.2 $r_j(r_kr_lr_ir_j)^m(r_kr_jr_lr_k)(r_ir_lr_jr_k)^p(r_ir_jr_i)(r_lr_kr_jr_i)^n$

3.3.3 $r_ir_j(r_kr_lr_ir_j)^m(r_kr_jr_lr_k)(r_ir_lr_jr_k)^p(r_ir_jr_i)(r_lr_kr_jr_i)^nr_lr_kr_j$

3.3.4 $r_lr_ir_j(r_kr_lr_ir_j)^m(r_kr_jr_lr_k)(r_ir_lr_jr_k)^p(r_ir_jr_i)(r_lr_kr_jr_i)^nr_lr_k$

3.4.1 $(r_ir_jr_kr_l)^m(r_ir_jr_i)(r_kr_jr_lr_k)(r_ir_jr_lr_k)^p(r_ir_jr_i)(r_lr_kr_jr_i)^nr_lr_k$

3.4.2 $r_l(r_ir_jr_kr_l)^m(r_ir_jr_i)(r_kr_jr_lr_k)(r_ir_jr_lr_k)^p(r_ir_jr_i)(r_lr_kr_jr_i)^nr_l$

3.4.3 $r_kr_l(r_ir_jr_kr_l)^m(r_ir_jr_i)(r_kr_jr_lr_k)(r_ir_jr_lr_k)^p(r_ir_jr_i)(r_lr_kr_jr_i)^n$

3.4.4 $r_jr_kr_l(r_ir_jr_kr_l)^m(r_ir_jr_i)(r_kr_jr_lr_k)(r_ir_jr_lr_k)^p(r_ir_jr_i)(r_lr_kr_jr_i)^nr_lr_kr_j$

number of elements

1.1 4

1.2 4

1.3 4

1.4 2

1.5 4

2.1.1 $4(\frac{k-4}{4}+1)=k$

2.1.2 $k$

2.1.3 $4(\frac{k-8}{4}+1)=k-4$

2.1.4 $k-4$

2.2.1 $k$

2.2.2 $k-4$

2.2.3 $k-4$

2.2.4 $k-4$

2.3.1 $k-4$

2.3.2 $k-4$

2.3.3 $k-4$

2.3.4 $k-4$

2.3.4$a$ $k-8$

2.4.1 $k-4$

2.4.2 $k$

2.4.3 $k-4$

2.4.3$a$ $k-4$

2.4.4 $k-4$

3.1.1 $4\frac{(k-4)*k}{4*4*2} = \frac{(k-4)k}{8}$

3.1.2 $\frac{(k-4)k}{8}$

3.1.3 $\frac{(k-4)k}{8}$

3.1.4 $4\frac{(k-8)*(k-4)}{4*4*2} = \frac{(k-8)(k-4)}{8}$

3.2.1 $\frac{(k-4)k}{8}$

3.2.2 $\frac{(k-4)k}{8}$

3.2.3 $\frac{(k-8)(k-4)}{8}$

3.2.4 $\frac{(k-8)(k-4)}{8}$

3.3.1 $\frac{(k-4)k}{8}$

3.3.2 $\frac{(k-4)k}{8}$

3.3.3 $\frac{(k-8)(k-4)}{8}$

3.3.4 $\frac{(k-8)(k-4)}{8}$

3.4.1 $\frac{(k-8)(k-4)}{8}$

3.4.2 $\frac{(k-8)(k-4)}{8}$

3.4.3 $\frac{(k-8)(k-4)}{8}$

3.4.4 $\frac{(k-12)(k-8)}{8}$

Type 1= 18

Type 2= $4k + 13(k-4) + (k-8) = 18k - 60$

Type 3= $\frac{7(k-4)k}{8} + \frac{8(k-8)(k-4)}{8} + \frac{(k-12)(k-8)}{8} = 2k^2 - 18k + 44.$

$18 + 18k - 60 + 2k^2 - 18k + 44 = 2k^2 + 2 = 2(k^2 + 1).$

1 Mod 4

1.1 $(r_i r_j r_k r_l)^m r_i$

1.2 $(r_i r_l r_k r_j)^m r_i$

1.3 $(r_i r_j r_l r_k)^m r_i$

1.4 $(r_i r_k r_j r_l)^m r_i$

2.1.1 $(r_i r_j r_k r_l)^m (r_i r_j r_i)(r_l r_k r_j r_i)^n r_l r_k$

2.1.2 $r_l (r_i r_j r_k r_l)^m (r_i r_j r_i)(r_l r_k r_j r_i)^n r_l$

2.1.3 $r_k r_l (r_i r_j r_k r_l)^m (r_i r_j r_i)(r_l r_k r_j r_i)^n$

2.1.4 $r_j r_k r_l (r_i r_j r_k r_l)^m (r_i r_j r_i)(r_l r_k r_j r_i)^n r_l r_k r_j$

2.2.1 $(r_i r_j r_k r_l)^m (r_i r_j r_i)(r_k r_j r_i r_l)^n r_k r_j$

2.2.2 $r_l (r_i r_j r_k r_l)^m (r_i r_j r_i)(r_k r_j r_i r_l)^n r_k$

2.2.3 $r_k r_l (r_i r_j r_k r_l)^m (r_i r_j r_i)(r_k r_j r_i r_l)^n \qquad n \geq 1$

2.2.4 $r_j r_k r_l (r_i r_j r_k r_l)^m (r_i r_j r_i)(r_k r_j r_i r_l)^n r_k r_j r_i$

2.3.1 $(r_i r_j r_k r_l)^m (r_i r_j r_i)(r_k r_l r_j r_i)^n r_k r_l$

2.3.1a $(r_i r_j r_k r_l)^m (r_i r_j r_i)(r_k r_j r_l r_i)^n r_k r_j \qquad n \geq 1$

2.3.2 $r_l (r_i r_j r_k r_l)^m (r_i r_j r_i)(r_k r_l r_j r_i)^n r_k \qquad n \geq 1$

2.3.3 $r_k r_l (r_i r_j r_k r_l)^m (r_i r_j r_i)(r_k r_l r_j r_i)^n \qquad n \geq 1$

2.3.4 $r_j r_k r_l (r_i r_j r_k r_l)^m (r_i r_j r_i)(r_k r_l r_j r_i)^n r_k r_l r_j$

2.4.1 $(r_i r_j r_l r_k)^m (r_i r_j r_i)(r_l r_k r_j r_i)^n r_l r_k \qquad m \geq 1$

2.4.2 $r_k (r_i r_j r_l r_k)^m (r_i r_j r_i)(r_l r_k r_j r_i)^n r_l$

2.4.3 $r_l r_k (r_i r_j r_l r_k)^m (r_i r_j r_i)(r_l r_k r_j r_i)^n$

2.4.3a $r_j r_k (r_i r_l r_j r_k)^m (r_i r_j r_i)(r_l r_k r_j r_i)^n$

2.4.5 $r_j r_l r_k (r_i r_j r_l r_k)^m (r_i r_j r_i)(r_l r_k r_j r_i)^n r_l r_k r_j$

3.1.1 $(r_i r_j r_k r_l)^m (r_i r_j r_i)(r_k r_j r_l r_i)^p (r_k r_l r_k)(r_j r_i r_l r_k)^n r_j r_i r_l$

3.1.2 $r_l (r_i r_j r_k r_l)^m (r_i r_j r_i)(r_k r_j r_l r_i)^p (r_k r_l r_k)(r_j r_i r_l r_k)^n r_j r_i$

3.1.3 $r_k r_l (r_i r_j r_k r_l)^m (r_i r_j r_i)(r_k r_j r_l r_i)^p (r_k r_l r_k)(r_j r_i r_l r_k)^n r_j$

3.1.4 $r_j r_k r_l (r_i r_j r_k r_l)^m (r_i r_j r_i)(r_k r_j r_l r_i)^p (r_k r_l r_k)(r_j r_i r_l r_k)^n$

3.2.1 $(r_i r_j r_k r_l)^m (r_i r_j r_i)(r_k r_j r_l r_i)^p (r_k r_j r_l r_k)(r_j r_i r_l r_k)^n r_j r_i$

3.2.2 $r_l (r_i r_j r_k r_l)^m (r_i r_j r_i)(r_k r_j r_l r_i)^p (r_k r_j r_l r_k)(r_j r_i r_l r_k)^n r_j$

3.2.3 $r_k r_l (r_i r_j r_k r_l)^m (r_i r_j r_i)(r_k r_j r_l r_i)^p (r_k r_j r_l r_k)(r_j r_i r_l r_k)^n$

3.2.4 $r_j r_k r_l (r_i r_j r_k r_l)^m (r_i r_j r_i)(r_k r_j r_l r_i)^p (r_k r_j r_l r_k)(r_j r_i r_l r_k)^n r_j r_i r_l$

3.3.1 $(r_k r_l r_i r_j)^m (r_k r_j r_l r_k)(r_i r_l r_j r_k)^p (r_i r_j r_i)(r_l r_k r_j r_i)^n r_l r_k$

3.3.2 $r_j (r_k r_l r_i r_j)^m (r_k r_j r_l r_k)(r_i r_l r_j r_k)^p (r_i r_j r_i)(r_l r_k r_j r_i)^n r_l$

3.3.3 $r_i r_j (r_k r_l r_i r_j)^m (r_k r_j r_l r_k)(r_i r_l r_j r_k)^p (r_i r_j r_i)(r_l r_k r_j r_i)^n$

3.3.4 $r_l r_i r_j (r_k r_l r_i r_j)^m (r_k r_j r_l r_k)(r_i r_l r_j r_k)^p (r_i r_j r_i)(r_l r_k r_j r_i)^n r_l r_k r_j$

3.4.1 $(r_i r_j r_k r_l)^m (r_i r_j r_i)(r_k r_j r_l r_k)(r_i r_j r_l r_k)^p (r_i r_j r_i)(r_l r_k r_j r_i)^n r_l r_k r_j$

3.4.2 $r_l (r_i r_j r_k r_l)^m (r_i r_j r_i)(r_k r_j r_l r_k)(r_i r_j r_l r_k)^p (r_i r_j r_i)(r_l r_k r_j r_i)^n r_l r_k$

3.4.3 $r_k r_l (r_i r_j r_k r_l)^m (r_i r_j r_i)(r_k r_j r_l r_k)(r_i r_j r_l r_k)^p (r_i r_j r_i)(r_l r_k r_j r_i)^n r_l$

3.4.4 $r_j r_k r_l (r_i r_j r_k r_l)^m (r_i r_j r_i)(r_k r_j r_l r_k)(r_i r_j r_l r_k)^p (r_i r_j r_i)(r_l r_k r_j r_i)^n$

number of elements

1.1 4

1.2 4

1.3 4

1.4 4

2.1.1 $k-1$

2.1.2 $k-1$

2.1.3 $k-1$

2.1.4 $k-5$

2.2.1 $k-1$

2.2.2 $k-1$

2.2.3 $k-5$

2.2.4 $k-5$

2.3.1 $k-1$

2.3.1$a$ $k-5$

2.3.2 $k-5$

2.3.3 $k-5$

2.3.4 $k-5$

2.4.1 $k-5$

2.4.2 $k-1$

2.4.3 $k-1$

2.4.3$a$ $k-1$

2.4.4 $k-5$

3.1.1 $\frac{(k-5)(k-1)}{8}$

3.1.2 $\frac{(k-5)(k-1)}{8}$

3.1.3 $\frac{(k-5)(k-1)}{8}$

3.1.4 $\frac{(k-5)(k-1)}{8}$

3.2.1 $\frac{(k-5)(k-1)}{8}$

3.2.2 $\frac{(k-5)(k-1)}{8}$

3.2.3 $\frac{(k-5)(k-1)}{8}$

3.2.4 $\frac{(k-9)(k-5)}{8}$

3.3.1 $\frac{(k-5)(k-1)}{8}$

3.3.2 $\frac{(k-5)(k-1)}{8}$

3.3.3 $\frac{(k-5)(k-1)}{8}$

3.3.4 $\frac{(k-9)(k-5)}{8}$

3.4.1 $\frac{(k-9)(k-5)}{8}$

3.4.2 $\frac{(k-9)(k-5)}{8}$

3.4.3 $\frac{(k-9)(k-5)}{8}$

3.4.4 $\frac{(k-9)(k-5)}{8}$

Type 1= 16

Type 2= $9(k-1)+9(k-5)=18k-54$

Type 3= $\frac{10(k-5)(k-1)}{8} + \frac{5(k-9)(k-5)}{8} = 2k^2 - 18k + 40.$

$16 + 18k - 54 + 2k^2 - 18k + 40 = 2k^2 + 2 = 2(k^2 + 1).$

2 Mod 4

1.1 $(r_i r_j r_k r_l)^m r_i r_j$

1.2 $(r_i r_l r_k r_j)^m r_i r_l$

1.3 $(r_i r_j r_l r_k)^m r_i r_j$

1.3a $(r_i r_l r_j r_k)^m r_i r_l$

1.4 $(r_i r_k r_j r_l)^m r_i r_k$

2.1.1 $(r_i r_j r_k r_l)^m (r_i r_j r_i)(r_l r_k r_j r_i)^n r_l r_k r_j$

2.1.2 $r_l(r_i r_j r_k r_l)^m (r_i r_j r_i)(r_l r_k r_j r_i)^n r_l r_k$

2.1.3 $r_k r_l(r_i r_j r_k r_l)^m (r_i r_j r_i)(r_l r_k r_j r_i)^n r_l$

2.1.4 $r_j r_k r_l(r_i r_j r_k r_l)^m (r_i r_j r_i)(r_l r_k r_j r_i)^n$

2.2.1 $(r_i r_j r_k r_l)^m (r_i r_j r_i)(r_k r_j r_i r_l)^n r_k r_j r_i$

2.2.2 $r_l(r_i r_j r_k r_l)^m (r_i r_j r_i)(r_k r_j r_i r_l)^n r_k r_j$

2.2.3 $r_k r_l(r_i r_j r_k r_l)^m (r_i r_j r_i)(r_k r_j r_i r_l)^n r_k$

2.2.4 $r_j r_k r_l(r_i r_j r_k r_l)^m (r_i r_j r_i)(r_k r_j r_i r_l)^n \qquad n \geq 1$

2.3.1 $(r_i r_j r_k r_l)^m (r_i r_j r_i)(r_k r_l r_j r_i)^n r_k r_l r_j$

2.3.2 $r_l(r_i r_j r_k r_l)^m (r_i r_j r_i)(r_k r_l r_j r_i)^n r_k r_l$

2.3.2a $r_l(r_i r_j r_k r_l)^m (r_i r_j r_i)(r_k r_j r_l r_i)^n r_k r_j \qquad n \geq 1$

2.3.3 $r_k r_l(r_i r_j r_k r_l)^m (r_i r_j r_i)(r_k r_l r_j r_i)^n r_k \qquad n \geq 1$

2.3.4 $r_j r_k r_l(r_i r_j r_k r_l)^m (r_i r_j r_i)(r_k r_l r_j r_i)^n \qquad n \geq 1$

2.4.1 $(r_i r_j r_l r_k)^m (r_i r_j r_i)(r_l r_k r_j r_i)^n r_l r_k r_j \qquad m \geq 1$

2.4.2 $r_k(r_i r_j r_l r_k)^m (r_i r_j r_i)(r_l r_k r_j r_i)^n r_l r_k$

2.4.3 $r_l r_k(r_i r_j r_l r_k)^m (r_i r_j r_i)(r_l r_k r_j r_i)^n r_l$

2.4.3a $r_j r_k(r_i r_l r_j r_k)^m (r_i r_j r_i)(r_l r_k r_j r_i)^n r_l$

2.4.5 $r_j r_l r_k(r_i r_j r_l r_k)^m (r_i r_j r_i)(r_l r_k r_j r_i)^n$

3.1.1 $(r_i r_j r_k r_l)^m (r_i r_j r_i)(r_k r_j r_l r_i)^p (r_k r_l r_k)(r_j r_i r_l r_k)^n$

3.1.2 $r_l(r_i r_j r_k r_l)^m (r_i r_j r_i)(r_k r_j r_l r_i)^p (r_k r_l r_k)(r_j r_i r_l r_k)^n r_j r_i r_l$

3.1.3 $r_k r_l (r_i r_j r_k r_l)^m (r_i r_j r_i)(r_k r_j r_l r_i)^p (r_k r_l r_k)(r_j r_i r_l r_k)^n r_j r_i$

3.1.4 $r_j r_k r_l (r_i r_j r_k r_l)^m (r_i r_j r_i)(r_k r_j r_l r_i)^p (r_k r_l r_k)(r_j r_i r_l r_k)^n r_j$

3.2.1 $(r_i r_j r_k r_l)^m (r_i r_j r_i)(r_k r_j r_l r_i)^p (r_k r_j r_l r_k)(r_j r_i r_l r_k)^n r_j r_i r_l$

3.2.2 $r_l (r_i r_j r_k r_l)^m (r_i r_j r_i)(r_k r_j r_l r_i)^p (r_k r_j r_l r_k)(r_j r_i r_l r_k)^n r_j r_i$

3.2.3 $r_k r_l (r_i r_j r_k r_l)^m (r_i r_j r_i)(r_k r_j r_l r_i)^p (r_k r_j r_l r_k)(r_j r_i r_l r_k)^n r_j$

3.2.4 $r_j r_k r_l (r_i r_j r_k r_l)^m (r_i r_j r_i)(r_k r_j r_l r_i)^p (r_k r_j r_l r_k)(r_j r_i r_l r_k)^n$

3.3.1 $(r_k r_l r_i r_j)^m (r_k r_j r_l r_k)(r_i r_l r_j r_k)^p (r_i r_j r_i)(r_l r_k r_j r_i)^n r_l r_k r_j$

3.3.2 $r_j (r_k r_l r_i r_j)^m (r_k r_j r_l r_k)(r_i r_l r_j r_k)^p (r_i r_j r_i)(r_l r_k r_j r_i)^n r_l r_k$

3.3.3 $r_i r_j (r_k r_l r_i r_j)^m (r_k r_j r_l r_k)(r_i r_l r_j r_k)^p (r_i r_j r_i)(r_l r_k r_j r_i)^n r_l$

3.3.4 $r_l r_i r_j (r_k r_l r_i r_j)^m (r_k r_j r_l r_k)(r_i r_l r_j r_k)^p (r_i r_j r_i)(r_l r_k r_j r_i)^n$

3.4.1 $(r_i r_j r_k r_l)^m (r_i r_j r_i)(r_k r_j r_l r_k)(r_i r_j r_l r_k)^p (r_i r_j r_i)(r_l r_k r_j r_i)^n$

3.4.2 $r_l (r_i r_j r_k r_l)^m (r_i r_j r_i)(r_k r_j r_l r_k)(r_i r_j r_l r_k)^p (r_i r_j r_i)(r_l r_k r_j r_i)^n r_l r_k r_j$

3.4.3 $r_k r_l (r_i r_j r_k r_l)^m (r_i r_j r_i)(r_k r_j r_l r_k)(r_i r_j r_l r_k)^p (r_i r_j r_i)(r_l r_k r_j r_i)^n r_l r_k$

3.4.4 $r_j r_k r_l (r_i r_j r_k r_l)^m (r_i r_j r_i)(r_k r_j r_l r_k)(r_i r_j r_l r_k)^p (r_i r_j r_i)(r_l r_k r_j r_i)^n r_l$

number of elements

1.1 4

1.2 4

1.3 4

1.3$a$ 4

1.4 2

2.1.1 $k-2$

2.1.2 $k-2$

2.1.3 $k-2$

2.1.4 $k-2$

2.2.1 $k-2$

2.2.2 $k-2$

2.2.3 $k-2$

2.2.4 $k-6$

2.3.1 $k-2$

2.3.2 $k-2$

2.3.2$a$ $k-6$

2.3.3 $k-6$

2.3.4 $k-6$

2.4.1 $k-6$

2.4.2 $k-2$

2.4.3 $k-2$

2.4.3$a$ $k-2$

2.4.4 $k-2$

3.1.1 $\frac{(k-2)(k+2)}{8}$

3.1.2 $\frac{(k-6)(k-2)}{8}$

3.1.3 $\frac{(k-6)(k-2)}{8}$

3.1.4 $\frac{(k-6)(k-2)}{8}$

3.2.1 $\frac{(k-6)(k-2)}{8}$

3.2.2 $\frac{(k-6)(k-2)}{8}$

3.2.3 $\frac{(k-6)(k-2)}{8}$

3.2.4 $\frac{(k-6)(k-2)}{8}$

3.3.1 $\frac{(k-6)(k-2)}{8}$

3.3.2 $\frac{(k-6)(k-2)}{8}$

3.3.3 $\frac{(k-6)(k-2)}{8}$

3.3.4 $\frac{(k-6)(k-2)}{8}$

3.4.1 $\frac{(k-6)(k-2)}{8}$

3.4.2 $\frac{(k-10)(k-6)}{8}$

3.4.3 $\frac{(k-10)(k-6)}{8}$

3.4.4 $\frac{(k-10)(k-6)}{8}$

Type 1= 18

Type 2= $13(k-2)+5(k-5)=18k-56$

Type 3= $\frac{(k-2)(k+2)}{8}+\frac{12(k-6)(k-2)}{8}+\frac{5(k-10)(k-6)}{8}=2k^2-18k+40.$

$18+18k-56+2k^2-18k+40=2k^2+2=2(k^2+1).$

3 Mod 4

1.1 $(r_i r_j r_k r_l)^m r_i r_j r_k$

1.2 $(r_i r_l r_k r_j)^m r_i r_l r_k$

1.3 $(r_i r_j r_l r_k)^m r_i r_j r_l$

1.4 $(r_i r_k r_j r_l)^m r_i r_k r_j$

2.1.1 $(r_i r_j r_k r_l)^m (r_i r_j r_i)(r_l r_k r_j r_i)^n$

2.1.2 $r_l (r_i r_j r_k r_l)^m (r_i r_j r_i)(r_l r_k r_j r_i)^n r_l r_k r_j$

2.1.3 $r_k r_l (r_i r_j r_k r_l)^m (r_i r_j r_i)(r_l r_k r_j r_i)^n r_l r_k$

2.1.4 $r_j r_k r_l (r_i r_j r_k r_l)^m (r_i r_j r_i)(r_l r_k r_j r_i)^n r_l$

2.2.1 $(r_i r_j r_k r_l)^m (r_i r_j r_i)(r_k r_j r_i r_l)^n \qquad n \geq 1$

2.2.2 $r_l (r_i r_j r_k r_l)^m (r_i r_j r_i)(r_k r_j r_i r_l)^n r_k r_j r_i$

2.2.3 $r_k r_l (r_i r_j r_k r_l)^m (r_i r_j r_i)(r_k r_j r_i r_l)^n r_k r_j$

2.2.4 $r_j r_k r_l (r_i r_j r_k r_l)^m (r_i r_j r_i)(r_k r_j r_i r_l)^n r_k$

2.3.1 $(r_i r_j r_k r_l)^m (r_i r_j r_i)(r_k r_l r_j r_i)^n \qquad n \geq 1$

2.3.2 $r_l (r_i r_j r_k r_l)^m (r_i r_j r_i)(r_k r_l r_j r_i)^n r_k r_l r_j$

2.3.3 $r_k r_l (r_i r_j r_k r_l)^m (r_i r_j r_i)(r_k r_l r_j r_i)^n r_k r_l$

2.3.3a $r_k r_l (r_i r_j r_k r_l)^m (r_i r_j r_i)(r_k r_j r_l r_i)^n r_k r_j \qquad n \geq 1$

2.3.4 $r_j r_k r_l (r_i r_j r_k r_l)^m (r_i r_j r_i)(r_k r_l r_j r_i)^n r_k \qquad n \geq 1$

2.4.1 $(r_i r_j r_l r_k)^m (r_i r_j r_i)(r_l r_k r_j r_i)^n \qquad m \geq 1$

2.4.2 $r_k (r_i r_j r_l r_k)^m (r_i r_j r_i)(r_l r_k r_j r_i)^n r_l r_k r_j$

2.4.3 $r_l r_k (r_i r_j r_l r_k)^m (r_i r_j r_i)(r_l r_k r_j r_i)^n r_l r_k$

2.4.3a $r_j r_k (r_i r_l r_j r_k)^m (r_i r_j r_i)(r_l r_k r_j r_i)^n r_l r_k$

2.4.5 $r_j r_l r_k (r_i r_j r_l r_k)^m (r_i r_j r_i)(r_l r_k r_j r_i)^n r_l$

3.1.1 $(r_i r_j r_k r_l)^m (r_i r_j r_i)(r_k r_j r_l r_i)^p (r_k r_l r_k)(r_j r_i r_l r_k)^n r_j$

3.1.2 $r_l (r_i r_j r_k r_l)^m (r_i r_j r_i)(r_k r_j r_l r_i)^p (r_k r_l r_k)(r_j r_i r_l r_k)^n$

3.1.3 $r_k r_l (r_i r_j r_k r_l)^m (r_i r_j r_i)(r_k r_j r_l r_i)^p (r_k r_l r_k)(r_j r_i r_l r_k)^n r_j r_i r_l$

3.1.4 $r_j r_k r_l (r_i r_j r_k r_l)^m (r_i r_j r_i)(r_k r_j r_l r_i)^p (r_k r_l r_k)(r_j r_i r_l r_k)^n r_j r_i$

3.2.1 $(r_i r_j r_k r_l)^m (r_i r_j r_i)(r_k r_j r_l r_i)^p (r_k r_j r_l r_k)(r_j r_i r_l r_k)^n$

3.2.2 $r_l (r_i r_j r_k r_l)^m (r_i r_j r_i)(r_k r_j r_l r_i)^p (r_k r_j r_l r_k)(r_j r_i r_l r_k)^n r_j r_i r_l$

3.2.3 $r_k r_l (r_i r_j r_k r_l)^m (r_i r_j r_i)(r_k r_j r_l r_i)^p (r_k r_j r_l r_k)(r_j r_i r_l r_k)^n r_j r_i$

3.2.4 $r_j r_k r_l (r_i r_j r_k r_l)^m (r_i r_j r_i)(r_k r_j r_l r_i)^p (r_k r_j r_l r_k)(r_j r_i r_l r_k)^n r_j$

3.3.1 $(r_k r_l r_i r_j)^m (r_k r_j r_l r_k)(r_i r_l r_j r_k)^p (r_i r_j r_i)(r_l r_k r_j r_i)^n$

3.3.2 $r_j (r_k r_l r_i r_j)^m (r_k r_j r_l r_k)(r_i r_l r_j r_k)^p (r_i r_j r_i)(r_l r_k r_j r_i)^n r_l r_k r_j$

3.3.3 $r_i r_j (r_k r_l r_i r_j)^m (r_k r_j r_l r_k)(r_i r_l r_j r_k)^p (r_i r_j r_i)(r_l r_k r_j r_i)^n r_l r_k$

3.3.4 $r_l r_i r_j (r_k r_l r_i r_j)^m (r_k r_j r_l r_k)(r_i r_l r_j r_k)^p (r_i r_j r_i)(r_l r_k r_j r_i)^n r_l$

3.4.1 $(r_i r_j r_k r_l)^m (r_i r_j r_i)(r_k r_j r_l r_k)(r_i r_j r_l r_k)^p (r_i r_j r_i)(r_l r_k r_j r_i)^n r_l$

3.4.2 $r_l (r_i r_j r_k r_l)^m (r_i r_j r_i)(r_k r_j r_l r_k)(r_i r_j r_l r_k)^p (r_i r_j r_i)(r_l r_k r_j r_i)^n$

3.4.3 $r_k r_l (r_i r_j r_k r_l)^m (r_i r_j r_i)(r_k r_j r_l r_k)(r_i r_j r_l r_k)^p (r_i r_j r_i)(r_l r_k r_j r_i)^n r_l r_k r_j$

3.4.4 $r_j r_k r_l (r_i r_j r_k r_l)^m (r_i r_j r_i)(r_k r_j r_l r_k)(r_i r_j r_l r_k)^p (r_i r_j r_i)(r_l r_k r_j r_i)^n r_l r_k$

number of elements

1.1 4

1.2 4

1.3 4

1.4 4

2.1.1 $k+1$

2.1.2 $k-3$

2.1.3 $k-3$

2.1.4 $k-3$

2.2.1 $k-3$

2.2.2 $k-3$

2.2.3 $k-3$

2.2.4 $k-3$

2.3.1 $k-3$

2.3.2 $k-3$

2.3.3 $k-3$

2.3.3$a$ $k-7$

2.3.4 $k-7$

2.4.1 $k-3$

2.4.2 $k-3$

2.4.3 $k-3$

2.4.3$a$ $k-3$

2.4.4 $k-3$

3.1.1 $\frac{(k-3)(k+1)}{8}$

3.1.2 $\frac{(k-3)(k+1)}{8}$

3.1.3 $\frac{(k-7)(k-3)}{8}$

3.1.4 $\frac{(k-7)(k-3)}{8}$

3.2.1 $\frac{(k-3)(k+1)}{8}$

3.2.2 $\frac{(k-7)(k-3)}{8}$

3.2.3 $\frac{(k-7)(k-3)}{8}$

3.2.4 $\frac{(k-7)(k-3)}{8}$

3.3.1 $\frac{(k-3)(k+1)}{8}$

3.3.2 $\frac{(k-7)(k-3)}{8}$

3.3.3 $\frac{(k-7)(k-3)}{8}$

3.3.4 $\frac{(k-7)(k-3)}{8}$

3.4.1 $\frac{(k-7)(k-3)}{8}$

3.4.2 $\frac{(k-7)(k-3)}{8}$

3.4.3 $\frac{(k-11)(k-7)}{8}$

3.4.4 $\frac{(k-11)(k-7)}{8}$

Type 1= 16

Type 2= $(k+1)+15(k-3)+2(k-7)=18k-58$

Type 3= $\frac{4(k-3)(k+1)}{8} + \frac{10(k-7)(k-3)}{8} + \frac{2(k-11)(k-7)}{8} = 2k^2 - 18k + 44.$

$16 + 18k - 58 + 2k^2 - 18k + 44 = 2k^2 + 2 = 2(k^2 + 1).$

If each class in the entries is acted by each reflection $r_{a_i}$ from the left and right sides, by the relations in 4.16, we get new classes which are similar to one of the classes above.

To show that the Weyl group $\widetilde{W}$ of $LSU_4$ has just two relations in 4.16, we will check the action of these classes on the real root system of $LSU_4$.

Since the Cartan matrix of $SU_4$ is symmetric and the action is bilinear on the root system, we have just checked the action of that class on the roots $(\alpha_1, m)$ and $(\alpha_2, m)$. As it is seen in these actions will not give a new relation to us. Hence the proof of 4.16 is done. □

From [14], we have

**Theorem 4.17** *The affine Weyl group $\widetilde{W}$ of LG is a Coxeter group.*

We will give some properties of the affine Weyl group $\widetilde{W}$.

**Definition 4.18** *The* length *of an element $w \in \widetilde{W}$ is the least number of factors in the decomposition relative to the set of the reflections $\{r_{\mathbf{a}_i}\}$, is denoted by $\ell(w)$.*

**Definition 4.19** *Let $w_1, w_2 \in \widetilde{W}, \gamma \in \Delta^+_{\mathrm{re}}$. Then $w_1 \xrightarrow{\gamma} w_2$ indicates the fact that*

$$r_\gamma w_1 = w_2,$$
$$\ell(w_2) = \ell(w_1) + 1.$$

*We put $w \leqslant w'$ if there is a chain*

$$w = w_1 \to w_2 \to \cdots \to w_k = w'.$$

*The relation $\leqslant$ is called the* Bruhat order *on the affine Weyl group $\widetilde{W}$.*

**Proposition 4.20** *Let $w \in \widetilde{W}$ and let $w = r_{\mathbf{a}_1} r_{\mathbf{a}_2} \cdots r_{\mathbf{a}_l}$ be the reduced decomposition of $w$. If $1 \leq i_1 < \ldots < i_k \leq l$ and $w' = r_{\mathbf{a}_{i_1}} r_{\mathbf{a}_{i_2}} \cdots r_{\mathbf{a}_{i_k}}$, then $w' \leqslant w$. If $w' \leqslant w$, then*

*$w'$ can be represented as above for some indexing set $\{i_\xi\}$. If $w' \to w$, then there is a unique index $i$, $1 \leq i \leq l$ such that*

$$w' = r_{\mathbf{a}_1} \cdots r_{\mathbf{a}_{i-1}} r_{\mathbf{a}_{i+1}}.$$

The last proposition gives an alternative definition of the Bruhat ordering on $\widetilde{W}$.

**Proposition 4.21** *In the Weyl group $\widetilde{W}$ of $LSU_4$, the number of elements with length $s$ is $2(s^2+1)$.*

Now we will define the subset $\widehat{W}$ of the affine Weyl group $\widetilde{W}$ which will be used in the text later. We know that the Weyl group $\widetilde{W}$ of the loop group $LG$ is a split extension $T^\vee \to \widetilde{W} \to W$, where $W$ is the Weyl group of the compact group Lie group $G$. Since the Weyl group $W$ is a sub-Coxeter system of the affine Weyl group $\widetilde{W}$, we can define the set of cosets $\widetilde{W}/W$.

**Lemma 4.22** *The subgroup of $\widetilde{W}$ fixing $0$ is the Weyl group $W$.*

**Corollary 4.23** *Let $w, w' \in \widetilde{W}$. Then, $w(0) = w'(0)$ if and only if $wW = w'W$ in $\widetilde{W}/W$.*

By the last corollary, the map $\widetilde{W}/W \to T^\vee$ given by $wW \to w(0)$ is well-defined and has inverse map given by $\chi_i \to r_{\alpha_i} W$, so the coset set $\widetilde{W}/W$ is identified to $T^\vee$ as set. We have from [3],

**Theorem 4.24** *Each coset in $\widetilde{W}/W$ has a unique element of the minimal length.*

We will write $\overline{\ell(w)}$ for the minimal length element occuring in the coset $wW$, for $w \in \widetilde{W}$. We see that each coset $wW, w \in \widetilde{W}$ has two distinguished representatives which are not in the general the same. Let the subset $\widehat{W}$ of the affine Weyl group $\widetilde{W}$ be the set of the minimal representative elements $\overline{\ell(w)}$ in the coset $wW$ for each $w \in \widetilde{W}$. The subset $\widehat{W}$ has the Bruhat order since it identifies the set of the minimal representative elements $\overline{\ell(w)}$. As a example, we calculate the subset $\widehat{W}$ of the Weyl group of $LSU_4$. Our aim

is to find the minimal representative elements $\overline{\ell(w)}$ in the right coset $wW$ for each the element $w \in \widetilde{W}$, where

$$\widetilde{W} = \{r_{a_i} : r_i^2 = 1, r_i r_j r_i = r_j r_i r_j, i \neq j, i, j = 0, 1, 2, 3, \text{ and } r_i r_k = r_k r_i, (ijkl) \in S_4\},$$

and $W = \{r_{a_i} : r_i^2 = 1, r_i r_j r_i = r_j r_i r_j, |i - j| \neq 2, \text{ and } i, j = 1, 2, 3, \text{ and } r_1 r_3 = r_3 r_1\}$. We have the minimal representative elements $\overline{\ell(w)}$ for each coset $wW$, $w \in \widetilde{W}$ as follows.

**Proposition 4.25** *In the Weyl group $\widehat{W} = \widetilde{W}/W$ of $LSU_4$, the number of elements with length $s$ is*

$$\begin{cases} 3k^2 + 3k + 1 & \text{if } s \equiv 0 \text{ Mod } 6, \\ 3k^2 + 4k + 1 & \text{if } s \equiv 1 \text{ Mod } 6, \\ 3k^2 + 5k + 2 & \text{if } s \equiv 2 \text{ Mod } 6, \\ 3k^2 + 6k + 3 & \text{if } s \equiv 3 \text{ Mod } 6, \\ 3k^2 + 7k + 4 & \text{if } s \equiv 4 \text{ Mod } 6, \\ 3k^2 + 8k + 5 & \text{if } s \equiv 5 \text{ Mod } 6, \end{cases}$$

*where $k = \lfloor s/6 \rfloor$ is floor function.*

## 4.3 Kač-Moody Lie algebras

Now we will describe the Lie algebra $L_{\text{pol}}\mathbf{g}_{\mathbb{C}}$ and its universal central extension in terms of generators and relations. For a finite dimensional semi-simple Lie algebra $\mathbf{g}_{\mathbb{C}}$, we can choose a non-zero element $e_\alpha$ in $\mathbf{g}_\alpha$ for each root $\alpha$. From [16], we have

**Theorem 4.26** *$\mathbf{g}_{\mathbb{C}}$ is a Kač-Moody Lie algebra generated by $e_i = e_{\alpha_i}$ and $f_i = e_{-\alpha_i}$ for $i = 1, \ldots, l$ where the $\alpha_i$ are the simple roots and $l$ is the rank of $\mathbf{g}_{\mathbb{C}}$ only if $G$ is semi-simple.*

Let us choose generators $e_j$ and $f_j$ of $L\mathbf{g}_{\mathbb{C}}$ corresponding to simple affine roots. Since $\mathbf{g}_{\mathbb{C}} \subset L\mathbf{g}_{\mathbb{C}}$, we can take

$$e_j = \begin{cases} ze_{-\alpha_0} & \text{for } j = 0, \\ e_i & \text{for } 1 \leq j \leq l \end{cases}$$

and

$$f_j = \begin{cases} z^{-1}e_{\alpha_0} & \text{for } j = 0, \\ f_i & \text{for } 1 \leq j \leq l \end{cases}$$

where $\alpha_0$ is the highest root of the adjoint representation. ¿From [32],

**Theorem 4.27** *Let $\mathbf{g}_{\mathbb{C}}$ be a semi-simple Lie algebra. Then, $L_{\text{pol}}\mathbf{g}_{\mathbb{C}}$ is generated by the elements $e_j$ and $f_j$ corresponding to simple affine roots.*

The Cartan matrix $A_{(l+1)\times(l+1)}$ of $L\mathbf{g}_{\mathbb{C}}$ has the Cartan integers $a_{ij} = \mathbf{a}_j(h_{\mathbf{a}_i})$ as the entries where $\mathbf{a}_0 = -\alpha_0$, and $\mathbf{a}_j = \alpha_j$ if $1 \leq j \leq l$. As an example,

**Proposition 4.28** *Let $G = SU_4$. The Cartan matrix $A_{4\times4}$ of $L\mathbf{g}_{\mathbb{C}}$ is the symmetric matrix*

$$\begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{pmatrix}.$$

Although the relations of the Kač-Moody algebra hold in $L_{\text{pol}}\mathbf{g}_{\mathbb{C}}$, they do not define it. By a theorem of Gabber and Kač in [10], the relations define the universal central extension $\widehat{L}_{\text{pol}}\mathbf{g}_{\mathbb{C}}$ of $L_{\text{pol}}\mathbf{g}_{\mathbb{C}}$ by $\mathbb{C}$ which is described by the cocycle $\omega_K$ given by

$$\omega_K(\xi, \eta) = \frac{1}{2\pi}\int_0^{2\pi} \sigma(\xi(\theta), \eta'(\theta))d\theta.$$

As a vector space $\widehat{L}_{\text{pol}}\mathbf{g}_{\mathbb{C}}$ is $L_{\text{pol}}\mathbf{g}_{\mathbb{C}} \oplus \mathbb{C}$ and the bracket is given by

$$[(\xi, \lambda), (\eta, \mu)] = ([\xi, \eta], \omega_K(\xi, \eta)).$$

**Theorem 4.29** *$\widehat{L}\mathbf{g}_{\mathbb{C}}$ is an affine Kač-Moody algebra.*

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