DOKUZ EYLÜL UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

PARAMETER ESTIMATION OF PROBABILITY DISTRIBUTIONS BASED ON RANKED SET SAMPLING

by
Melek ESEMEN

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İZMİR

PARAMETER ESTIMATION OF PROBABILITY DISTRIBUTIONS BASED ON RANKED SET SAMPLING

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We have read the thesis entitled "**PARAMETER ESTIMATION OF PROBABILITY DISTRIBUTIONS BASED ON RANKED SET SAMPLING**" completed by **MELEK ESEMEN** under supervision of **PROF. DR. SELMA GÜRLER** and we certify that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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PARAMETER ESTIMATION OF PROBABILITY DISTRIBUTIONS BASED ON RANKED SET SAMPLING

ABSTRACT

Recently, many researchers focused on the more effective sampling method that is called Ranked Set Sampling (RSS) for estimating the population parameter when the measurement of an observation is costly and/or time consuming. In RSS procedure, we rank the units of the variable of interest without actual measurement and after the ranking process, only selected units are measured exactly. Therefore, we have effectiveness and advantageous as time and cost. Furthermore, it provides more representative sample from the target population by selecting the units almost everywhere from the interested population. RSS method has been studied by the number of researchers under various scopes. While some of them has modified the RSS, the others estimated the parameter of population by using these modifications. In this study, we deal with the estimation of the shape and scale parameters for Generalized Rayleigh (GR) distribution. We propose the maximum likelihood (ML) estimators of unknown parameters of GR distribution based on RSS and its some modifications. As we have no explicit form of estimators of parameters, numerical methods are used for the solutions. For comparison of the performances of estimators, a Monte Carlo simulation study is performed via Mathematica 11.0 with 10,000 repetitions. The biases, mean squared errors and relative efficiencies of estimators are compared in simple random sampling (SRS), RSS, extreme RSS, median RSS and imperfect RSS with different set and cycle sizes. Moreover, the study is supported with a real data example.

Keywords: Ranked set sampling, median RSS, extreme RSS, imperfect RSS, parameter estimation, maximum likelihood estimation, generalized Rayleigh distribution

OLASILIK DAĞILIMLARININ SIRALI KÜME ÖRNEKLEMESİNE DAYALI PARAMETRE KESTİRİMİ

ÖZ

Son zamanlarda, birçok araştırmacı gözlemlerin ölçümünün masraflı ve/veya zaman alıcı olduğu durumlarda kitle parametrelerinin kestirimi için daha etkili bir örnekleme metodu olan sıralı küme örneklemesi (SKÖ) üzerinde yoğunlaşmıştır. SKÖ prosedüründe, ilgilenilen değişkene ait birimler gerçek ölçüm yapılmadan sıralanır ve sıralama süreci sonrasında sadece seçilen birimlere ölçülür.Böylece, zaman ve maliyet açısından etkinlik ve avantaj elde edilir. Ayrıca SKÖ, ilgilenilen kitlenin neredeyse her yerinden birimler seçerek kitleyi daha iyi temsil eden bir örneklem elde etmemizi sağlar. SKÖ yöntemi birçok araştırmacı tarafından çeşitli kapsamlar altında çalışılmıştır. Araştırmacılardan bazıları SKÖ'yü modifiye ederken, diğerleri de bu modifikasyonları kullanarak kitle parametrelerini tahmin etmektedir. Bu çalışmada, Genelleştirilmiş Rayleigh (GR) dağılımının şekil ve ölçek parametrelerinin kestirimlerine değinilmiştir. GR dağılımının bilinmeyen parametrelerinin SKÖ'ye ve onun bazı modifikasyonlarına dayalı en çok olabilirlik kestiricileri sunulmuştur. Parametre kestiricileri kapalı bir formda yazılamadığından, çözümler için nümerik yöntemler kullanılmıştır. Kestiricilerin performanslarının karşılaştırılması amacıyla Mathematica yazılımı kullanılarak 10.000 tekrarlı Monte Carlo benzetim çalışması yapılmıştır. Kestiricilerin basit rasgele örnekleme, SKÖ, uç değer SKÖ, ortanca SKÖ ve kusurlu SKÖ altındaki, yanlılıkları, hata kareler ortalamaları ve etkinlikleri farklı küme ve döngü sayılarıyla karşılaştırılmıştır. Ek olarak, çalışma gerçek veri örneğiyle de desteklenmiştir.

Anahtar kelimeler: Sıralı küme örneklemesi, ortanca SKÖ, uç değer SKÖ, kusurlu SKÖ, parametre kestirimi, en çok olabilirlik kestirimi, genelleştirilmiş Rayleigh dağılımı

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CHAPTER ONE
INTRODUCTION

Ranked Set Sampling (RSS) is a sampling method was first proposed by McIntyre (1952) as an advantegeous alternative to Simple Random Sampling (SRS) when estimating the population mean. An important advantage of this approach is that it improves the efficiency of estimators of the population parameters by providing more representative sample from the target population cost and/or time effectively. However, McIntyre (1952) had not supported his work theoretically. Takahasi & Wakimoto (1968) obtained the first theoretical results about RSS. They proved that the mean estimator in RSS is unbiased with smaller variance compared to the mean estimator in SRS when ranking is perfect. Also, Dell & Clutter (1972) showed the same manner without ranking constraint. The other results can be obtained in the study of Patil et al. (1999), Wolfe (2012) and Al-Omari & Bouza (2014). In RSS, we select m random sets each of size m from the interested population. Each set is ranked by an expert judgment, auxiliary variable or visual inspection without actual measurement. After the ranking, smallest ranked unit is chosen from the first set, then second smallest ranked unit from the second set and continuing until the largest ranked unit is selected in the last set. Only these selected m units are measured for the analysis. If the larger sample size is required, then we repeat this procedure r times as a cycle and we obtain $n = mr$ sample size of ranked set sample.

In the literature, there are great number of studies focused on modifying the RSS scheme. Extreme RSS (ERSS) is the first modification of RSS proposed by Samawi et al. (1996) to estimate the population mean only using maximum or minimum ranked unit from each set. Muttlak (1997) suggested median RSS (MRSS) for the efficient population mean estimation. Al-Saleh & Al-Kadiri (2000) offered Double RSS (DRSS) to estimate the population mean. Then multistage RSS (MSRSS), as a generalization of DRSS proposed by Al-Saleh & Al-Omari (2002) and it is found more effective than SRS when estimating the population mean. Muttlak (2003a) considered quartile RSS (QRSS) in order to estimate the population mean. By using the quartiles of the sets, it provides better results especially for asymmetric distributions. Similar to this method, Muttlak (2003b) used percentiles of the sets instead of quartiles for the selection, that is called percentile RSS (PRSS). On behalf of a robust perspective, Al-Nasser (2007) offered L-RSS which is ground on the idea of L-statistics and an combination of RSS, QRSS and MRSS. These modifications are only some of all studies. For others also one can see Jemain et al. (2008), Al-Nasser & Mustafa (2009), Al-Omari & Al-Saleh (2009), Al-Omari & Raqab (2013), Haq et al. (2013, 2014). Besides, number of researchers studied on estimation of parameters for the distributions by using these modifications of RSS with different estimation methods. Stokes (1995) studied estimation of the parameters of the location-scale family having cumulative distribution function (cdf) of the form $F(\frac{x-\mu}{\sigma})$. They investigated both ML estimators and best linear unbiased estimators (BLUE) of μ and σ using modified RSS. Al-Saleh & Al-Hadhrami (2003) investigated moving extremes ranked set sampling (MERSS) for the location parameter of symmetric distributions. They studied on ML estimators and a modified ML estimators based on RSS. Moreover, they compared the efficiencies of corresponding estimators with respect to SRS for the case of normal distribution under perfect and imperfect ranking. Abu-Dayyeh et al. (2004) proposed different estimators for the location and scale parameters of logistic distribution using SRS and RSS. They provided the estimators in the cases of either one or both parameters are unknown. They also compared the mean square error (MSE)s of method of moment estimators (MOME), ML estimators and BLUE under RSS. Shaibu & Muttlak (2004) compared ML estimators of the parameters of the location-scale parameter family of distributions under the different methods of sampling namely, RSS, median RSS and extreme RSS, and percentile RSS. Helu et al. (2010) provided that ML estimators, MOME and Bayes estimators of the shape and scale parameters of Weibull distribution in SRS, RSS and MRSS. Hassan (2013) studied on estimation of the shape and scale parameters of exponentiated exponential distribution is considered based on SRS and RSS. Hussian (2014) estimated the unknown parameters of the Kumaraswamy distribution using both SRS and RSS techniques. Khamnei & Mayan (2016) introduced the exponentiated Gumbel distribution and studied the ML estimators of the parameters of exponentiated Gumbel distribution based on a SRS. After the description of RSS, they studied the estimation of the parameters based on RSS and finally compared these two sampling methods. Dey et al. (2017) considered the estimation of Rayleigh distribution by using ML method and Bayesian method and also different sampling schemes these are SRS, RSS, median RSS and modified RSS. Also see much more studies about parameter estimation based on RSS; Modarres & Zheng (2004), Abu-Dayyeh et al. (2013), Chen et al. (2013), Biradar & Santosha (2014), Yousef & Al-Subh (2014) and Samuh & Qtait (2015).

1.1 Literature Review of Bivariate Distribution Based on RSS

On literature, there are a few works about bivariate distributions based on RSS while there are a lot of univariate distribution studies depending on RSS. Most of these studies are related to estimation of parameters for the distributions.

Firstly, Stokes (1980) considered the estimation of correlation coefficient for bivariate normal distribution based on RSS with concomitant variable. He investigated the estimators of correlation coefficient for other parameters are known and unknown. After that, Al-Saleh & Samawi (2005) estimated the correlation coefficient for bivariate normal distribution based on bivariate RSS which is a modification of RSS. Then, they considered nonparametric estimation of correlation coefficient when other parameters are known. Also they suggested the modified ML estimator for estimating the correlation coefficient when other parameters are unknown. Al-Saleh & Al-Ananbeh (2007) used MERSS to estimate the two means of bivariate normal distribution with concomitant variable. They also considered ML estimators of means based on MERSS. These estimators are found that unbiased and more effective than obtained by using SRS. Chacko & Thomas (2007) considered the different estimators of a parameter which is associated with study variate Y when (X, Y) is from bivariate Pareto distribution by using RSS. They compared the mean estimator in RSS, BLUE in RSS and BLUE in lower ERSS for the interested parameter. They found that BLUE in RSS of the parameter is more efficient than the

RSS mean estimator and BLUE in lower ERSS of the parameter is more effective than the another estimators. Al-Saleh & Diab (2009) estimated the parameters of Downton's bivariate exponential distribution by using nonparametric and parametric estimation methods based on RSS. They compared the efficiencies of estimators depending on RSS and SRS. Tahmasebi & Jafari (2012) presented that several estimators of a scale parameter of Morgenstern type bivariate uniform distribution; mean estimator based on RSS, a BLUE using RSS and upper RSS. They also used ERSS and MERSS to compare different estimators by a simulation study. Tahmasebi & Jafari (2015) investigated the unbiased estimators for a parameter associated with study variable Y when (X, Y) is from Morgenstern type bivariate gamma distribution based on RSS, ERSS and MERSS. They compared mean estimators depending on RSS, ERSS and MERSS and found that ERSS is more efficient than the another methods. One can see the other works for bivariate distribution on RSS; Al-Saleh & Samawi (2004), Chacko & Thomas (2008), Dieh et al. (2011), Singh & Mehta (2014), Nematollahi & Shahi (2015), Chacko (2016), Hanandeh & Al-Saleh (2016), Chacko (2017), Tahmasebi et al. (2017).

1.2 Outline of the Study

In this thesis, we consider the problem of estimating the parameters of Generalized Rayleigh (GR) distribution based on RSS by using ML estimation method. GR distribution is also named as the two parameters Burr Type X distribution was first introduced by Surles & Padgett (2001). It plays a crucial role in modelling lifetime data analysis. It has a relation with many other distributions such as; Rayleigh distribution, Weibull distribution, gamma distribution, exponentiated Weibull distribution and generalized exponential distribution. We are interested in ML estimations of shape and scale parameters of GR distribution based on SRS, RSS, MRSS and ERSS. In chapter two, we described the procedures of RSS and its some modifications. Third chapter initially introduced the properties of GR distribution and proposed the estimators of parameters based on SRS, RSS, ERSS and MRSS by using the method of ML. Chapter four includes simulation results for GR distribution depend on different sampling schemes under perfect and imperfect ranking for the RSS. Also in this chapter, performances of the estimators are compared with numerical study. Chapter five provides a real data example and the study ends up with the conclusion remarks.

CHAPTER TWO

RANKED SET SAMPLING AND ITS SOME MODIFICATIONS

Recently, RSS is one of the most advantageous sampling method to estimate the parameters of a population, since this method provides efficient results in terms of cost and time. When the actual measurement of an observation is expensive and/or difficult, RSS ensures the convenience for this with visual ranking and selection procedure. Firstly, it was introduced by McIntyre (1952) without any mathematical background and study of Takahasi & Wakimoto (1968) supported it theoretically. It is found that more effective than the SRS when estimating the population mean and provides unbiased results. And then many modifications of RSS have been suggested such as preliminarily; extreme RSS and median RSS. Initially, the procedure of RSS is decribed as follows: Firstly, we select randomly m number of sets of sizes m from the target population. Each set of sizes m is ranked by visually, expert judgment or auxiliary variable without exact measurement. After this ranking process, ith smallest ranked unit is chosen from the ith set for $i = 1, \ldots, m$. We only measure these selected m units and the others are discarded for the analysis. If the larger sample size is required, we can repeat this procedure r times as a cycle and we obtain $n = mr$ sample size of ranked set sample. The procedure for one cycle and case of $m = 3$ can be shown as below:

$$\begin{bmatrix} \mathbf{X_{1(1:3)}} \leq & X_{1(2:3)} \leq & X_{1(3:3)} \\ X_{2(1:3)} \leq & \mathbf{X_{2(2:3)}} \leq & X_{2(3:3)} \\ X_{3(1:3)} \leq & X_{3(2:3)} \leq & \mathbf{X_{3(3:3)}} \end{bmatrix}$$

where $X_{i(h:m)}$ stands for ith set, hth ranked unit and m set size.

If the procedure is repeated r times, we have

$$\begin{bmatrix} \mathbf{X_{1(1:3)1}} \leq & X_{1(2:3)1} \leq & X_{1(3:3)1} \\ X_{2(1:3)1} \leq & \mathbf{X_{2(2:3)1}} \leq & X_{2(3:3)1} \\ X_{3(1:3)1} \leq & X_{3(2:3)1} \leq & \mathbf{X_{3(3:3)1}} \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{X_{1(1:3)2}} \leq & X_{1(2:3)2} \leq & X_{1(3:3)2} \\ X_{2(1:3)2} \leq & \mathbf{X_{2(2:3)2}} \leq & X_{2(3:3)2} \\ X_{3(1:3)2} \leq & X_{3(2:3)2} \leq & \mathbf{X_{3(3:3)2}} \end{bmatrix}$$

$$\vdots$$

$$\begin{bmatrix} \mathbf{X_{1(1:3)r}} \leq & X_{1(2:3)r} \leq & X_{1(3:3)r} \\ X_{2(1:3)r} \leq & \mathbf{X_{2(2:3)r}} \leq & X_{2(3:3)r} \\ X_{3(1:3)r} \leq & X_{3(2:3)r} \leq & \mathbf{X_{3(3:3)r}} \end{bmatrix},$$

where m is the set size, j is the cycle number for ith set and hth ranked unit in $X_{i(h:m)j}$.

2.1 Extreme Ranked Set Sampling

ERSS is the first modification of RSS offered by Samawi et al. (1996) to estimate the population mean only using maximum or minimum ranked unit from each set. For estimation based on ERSS, the following procedure can be used. The selected m random sets each of size m units from the population can be ranked within each set with respect to a variable of interest by visual inspection or any cost-free method. According to set size m is even or odd, selection method may be altered. If the set size m is even, the lowest ranked unit of each set is selected from the first $m/2$ sets and the largest ranked unit of each set is chosen from the other $m/2$ sets. If the set size is odd, select the lowest ranked unit from the first $(m-1)/2$ sets, select the largest ranked unit from the other $(m-1)/2$ sets and median is taken from the remaining last set. When the size of ERSS is increased by cycling the procedure r times, then we have $n = mr$ sample size. The procedure for one cycle and case of $m = 4$ and $m = 5$ can be shown as below:

$$\begin{bmatrix} \mathbf{X_{1(1:4)}} \leq & X_{1(2:4)} \leq & X_{1(3:4)} \leq & X_{1(4:4)} \\ \mathbf{X_{2(1:4)}} \leq & X_{2(2:4)} \leq & X_{2(3:4)} \leq & X_{2(4:4)} \\ X_{3(1:4)} \leq & X_{3(2:4)} \leq & X_{3(3:4)} \leq & \mathbf{X_{3(4:4)}} \\ X_{4(1:4)} \leq & X_{4(2:4)} \leq & X_{4(3:4)} \leq & \mathbf{X_{4(4:4)}} \end{bmatrix}$$

and

$$\begin{bmatrix} \mathbf{X_{1(1:5)}} \leq & X_{1(2:5)} \leq & X_{1(3:5)} \leq & X_{1(4:5)} \leq & X_{1(5:5)} \\ \mathbf{X_{2(1:5)}} \leq & X_{2(2:5)} \leq & X_{2(3:5)} \leq & X_{2(4:5)} \leq & X_{2(5:5)} \\ X_{3(1:5)} \leq & X_{3(2:5)} \leq & X_{3(3:5)} \leq & X_{3(4:5)} \leq & \mathbf{X_{3(5:5)}} \\ X_{4(1:5)} \leq & X_{4(2:5)} \leq & X_{4(3:5)} \leq & X_{4(4:5)} \leq & \mathbf{X_{4(5:5)}} \\ X_{5(1:5)} \leq & X_{5(2:5)} \leq & \mathbf{X_{5(3:5)}} \leq & X_{5(4:5)} \leq & X_{5(5:5)} \end{bmatrix}.$$

In literature, ERSS is widely used modification for estimating the population mean effectively. Especially when underlying distribution is asymmetric, it provides unbiased estimators. Also, Samawi & Al-Sagheer (2001) used ERSS and MRSS for estimating the distribution functions. Muttlak (2001) considered two regression estimators based on ERSS and MRSS to estimate the population mean and compared with the regression estimators based on RSS. Samawi et al. (2002) studied the impacts of ERSS on regression and residual analysis.

2.2 Median Ranked Set Sampling

MRSS is the one of well-known modifications of RSS which is suggested by Muttlak (1997) for estimating the population mean effectively. He indicated that the MRSS gives unbiased results when the underlying distribution is symmetric and that estimator was found to be more efficient than both SRS and RSS. The procedure of MRSS is initially like as usual RSS in terms of randomly selection, allocation and ranking within each m sets with respect to variable of interest. After cost-free ranking process, selection is done according to the set size which is even or odd number. For odd set size m, we select median unit of the each m set. If the set size m is even then the $(m/2)$th ranked units are chosen from the first $m/2$ sets and the $((m+2)/2)$ ranked units are selected from the remaining $m/2$ sets. Others are discarded and only selected units are measured. If necessary, procedure can be repeated r times as a cycle and we have $n = mr$ sample of size. The procedure for one cycle and case of $m = 3$ and $m = 4$ can be shown as below:

$$\begin{bmatrix} X_{1(1:3)} \leq & \mathbf{X_{1(2:3)}} \leq & X_{1(3:3)} \\ X_{2(1:3)} \leq & \mathbf{X_{2(2:3)}} \leq & X_{2(3:3)} \\ X_{3(1:3)} \leq & \mathbf{X_{3(2:3)}} \leq & X_{3(3:3)} \end{bmatrix}$$

and

$$\begin{bmatrix} X_{1(1:4)} \leq & \mathbf{X_{1(2:4)}} \leq & X_{1(3:4)} \leq & X_{1(4:4)} \\ X_{2(1:4)} \leq & \mathbf{X_{2(2:4)}} \leq & X_{2(3:4)} \leq & X_{2(4:4)} \\ X_{3(1:4)} \leq & X_{3(2:4)} \leq & \mathbf{X_{3(3:4)}} \leq & X_{3(4:4)} \\ X_{4(1:4)} \leq & X_{4(2:4)} \leq & \mathbf{X_{4(3:4)}} \leq & X_{4(4:4)} \end{bmatrix}.$$

MRSS gives more efficient results than SRS for mean estimation in especially symmetric distributions. Additionally, Muttlak (1998) used MRSS to estimate the population mean when the ranking is depending on a concomitant variable and he showed that MRSS gives more efficient results compared to RSS and regression estimator in most cases. Samawi & Muttlak (2001) sugessted to MRSS for estimating the population ratio. Muttlak & Al-Sabah (2003) offered new control charts by using RSS, MRSS and ERSS for the population mean and they stated that MRSS dominates all other methods in terms of the out-of-control average run length performance. Alodat & Jetschke (2011) studied polynomial regression by using MRSS scheme and they found that least square estimator under MRSS are more efficient than SRS.

CHAPTER THREE
ESTIMATIONS FOR PARAMETERS OF GENERALIZED RAYLEIGH DISTRIBUTION

Burr (1942) proposed cumulative distribution function as 12 different types for modelling lifetime data. The most popular ones of among these are Burr Type X and Burr Type XII. Several researchers consider different aspects of these distributions, one can see for example, Rodriguez (1977), Sartawi & Abu-Salih (1991), Jaheen (1995, 1996), Ahmad et al. (1997), Raqab (1998), Surles & Padgett (1998), Alshunnar et al. (2010) and Pathak & Chaturvedi (2014). Later, Surles & Padgett (2001) (see also Surles & Padgett (2005)) proposed two parameters Burr Type X distribution and also called as the generalized Rayleigh distribution. Note that it is a particular case of exponentiated Weibull distribution suggested by Mudholkar & Srivastava (1993). If X is the random variable from GR distribution with shape parameter $\alpha > 0$ and scale parameter $\lambda > 0$ has the cummulative distibution function

$$F(x;\alpha,\lambda) = (1 - \exp(-(\lambda x)^2))^{\alpha}, \;\; x > 0. \tag{3.1}$$

Then the probability distribution function of GR is

$$f(x;\alpha,\lambda) = 2\alpha\lambda^2 x \exp(-(\lambda x)^2)(1 - \exp(-(\lambda x)^2))^{\alpha-1}, \;\; x > 0, \tag{3.2}$$

and the survival and the hazard functions are given as below

$$S(x;\alpha,\lambda) = 1 - (1 - \exp(-(\lambda x)^2))^{\alpha}, \;\; x > 0, \tag{3.3}$$

$$h(x;\alpha,\lambda) = \frac{2\alpha\lambda^2 x \exp(-(\lambda x)^2)(1 - \exp(-(\lambda x)^2))^{\alpha-1}}{1 - (1 - \exp(-(\lambda x)^2))^{\alpha}}. \tag{3.4}$$

GR distribution will be denoted by $GR(\alpha, \lambda)$ by Raqab & Kundu (2006) and they also indicated that if $\alpha = 1$, GR distribution coincides with the Rayleigh distribution.

If we put another shape parameter instead of 2 in (3.1), it is an exponentiated Weibull distribution. For $\alpha \leq 0.5$, the probability density function (pdf) is a decreasing function and hazard function of $GR(\alpha, \lambda)$ is bathtub type and if $\alpha > 0.5$, it is a right skewed unimodal pdf and increasing hazard function. Figure 3.1 shows that various forms of the density functions of GR distribution. GR distribution is a quite useful when modelling general lifetime data and strength data.

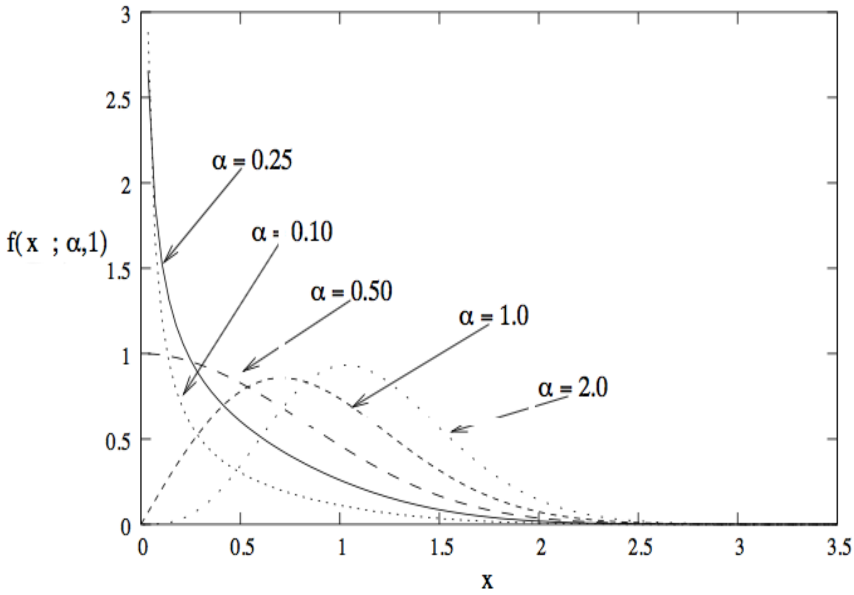

Figure 3.1 The density functions of the GR distribution for different shape parameters (Raqab & Kundu (2006))

In literature, there are many methods to estimate the parameters of target population. In this chapter, we use the one of these estimation methods, namely ML technique. It is the most widely used estimation method in statistical inference to inform about the population.

To give a general definition of maximum likelihood estimates, let $\mathbf{X} = [X_1, X_2, \ldots, X_n]$ is a random sample of size n whose joint distribution is described by a density $f_n(\mathbf{x}; \theta)$ over the n-dimensional Euclidean space $\mathcal{R}^n$ where θ is the vector of unknown parameters in parameter space Ω. For fixed $\mathbf{x}$, define the likelihood function of $\mathbf{x}$ as $L(\theta) = L_x(\theta) = f_n(\mathbf{x}; \theta)$ considered as a function of $\theta \in \Omega$.

Any $\hat{\theta} = \hat{\theta}(\mathbf{x}) \in \Omega$ which maximizes $L(\theta)$ over Ω is called a maximum likelihood estimate of the unknown true parameter θ. We often use $logL(\theta)$ instead of $L(\theta)$ since it provides easiness for computations.

3.1 Estimation for Parameters of GR Distribution in SRS

In this section, we give the ML estimators of the parameters of $GR(\alpha, \lambda)$ when both parameters are unknown. Let $X_1, X_2, \ldots, X_n$ is a random sample of size n from $GR(\alpha, \lambda)$, then likelihood function $L(\alpha, \lambda)$ and log-likelihood function $l(\alpha, \lambda)$ are written by Kundu & Raqab (2005) as

$$\begin{aligned} L(\alpha, \lambda; x) &= \prod_{i=1}^{n} f(x_i; \alpha, \lambda) \\ &= 2^n \alpha^n \lambda^2 n \prod_{i=1}^{n} x_i \exp(-(\lambda x_i)^2)(1 - \exp(-(\lambda x_i)^2)^{\alpha-1}, \end{aligned} \quad (3.5)$$

$$\begin{aligned} l(\alpha, \lambda) &= C + n\log\alpha + 2n\log\lambda + \sum_{i=1}^{n} \log(x_i) - \lambda^2 \sum_{i=1}^{n} x_i^2 + \ldots \\ &\quad +(\alpha - 1)\sum_{i=1}^{n} \log(1 - \exp(-(\lambda x_i)^2), \end{aligned} \quad (3.6)$$

where $C = nlog2$.

When they take the derivatives with respect to parameters, equations become

$$\frac{\partial l}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n} \log(1 - \exp(-(\lambda x_i)^2), \quad (3.7)$$

$$\frac{\partial l}{\partial \lambda} = \frac{2n}{\lambda} - 2\lambda \sum_{i=1}^{n} x_i^2 + 2\lambda(\alpha - 1)\sum_{i=1}^{n} \frac{x_i^2 \exp(-(\lambda x_i)^2)}{1 - \exp(-(\lambda x_i)^2}. \quad (3.8)$$

When (3.7) is equated to zero, ML estimator of α as a function of λ, can be obtained as

$$\hat{\alpha}(\lambda) = -\frac{n}{\sum_{i=1}^{n} \log(1-\exp(-(\lambda x_i)^2))}. \tag{3.9}$$

Substituting $\hat{\alpha}$ in (3.6), loglikelihood of λ can be written as

$$\begin{aligned} l(\hat{\alpha}(\lambda), \lambda) &= C + n\sum_{i=1}^{n} \log(1-\exp(-(\lambda x_i)^2)) + 2n\log\lambda - \lambda^2 \sum_{i=1}^{n} x_i^2 - \dots \\ &\quad - \sum_{i=1}^{n} \log(1-\exp(-(\lambda x_i)^2)). \end{aligned} \tag{3.10}$$

Thus, the ML estimator of λ , say $\hat{\lambda}_{MLE}$, is obtained by maximizing (3.10) with respect to λ. The maximum of (3.10) can be shown as a fixed point solution of the following equation:

$$h(\mu) = \mu, \tag{3.11}$$

where

$$h(\mu) = \left[\frac{\sum_{i=1}^{n} \dfrac{x_i^2 \exp(-\mu x_i^2)}{1-\exp(-\mu x_i^2)}}{\sum_{i=1}^{n} \log(1-\exp(-\mu x_i^2))} + \frac{1}{n}\sum_{i=1}^{n} x_i^2 + \frac{1}{n}\sum_{i=1}^{n} \frac{x_i^2 \exp(-\mu x_i^2)}{1-\exp(-\mu x_i^2)} \right]^{-1}.$$

If $\hat{\mu}$ is a solution of (3.11) , $\hat{\mu}_{MLE} = \sqrt{\hat{\mu}}$. By iteration, $h(\mu^{(j)}) = \mu^{(j+1)}$ where $\mu^{(j)}$ is the j^{th} iterative, $\hat{\lambda}_{MLE}$ is obtained. After that, $\hat{\alpha}_{MLE}$ can be obtained from (3.9). Note that $\hat{\lambda}_{MLE}$ and $\hat{\alpha}_{MLE}$ cannot be written explicit form.

3.2 Estimation for Parameters of GR Distribution in RSS

In this title, ML equations of parameters of GR distribution based on RSS will be attained. For now, we think the perfect ranking mechanism i.e. there is no ranking error in ranking of variable of interest.

Let $X_{i(h:m)j},\ i = h = 1,2,...,m,\ j = 1,2,...,r$ be a RSS drawn from GR distribution with sample size $n = mr$ where m is the set size and r is the cycle number. For simplifying the notations, we denote $Y_{ij} = X_{i(h:m)j}$. Then, Y_{ij} 's are independent and its density equal to the density of the ith order statistic from a sample of size m given by,

$$g_i(y_{ij};\alpha,\lambda) = \frac{m!}{(i-1)!(m-i)!} f(y_{ij};\alpha,\lambda)[F(y_{ij};\alpha,\lambda)]^{i-1}[1-F(y_{ij};\alpha,\lambda)]^{m-i} \tag{3.12}$$

where $f(y_{ij};\alpha,\lambda)$ is pdf and $F(y_{ij};\alpha,\lambda)$ is cdf of X .
Using the RSS algorithm and (3.12), the likelihood function can be written as

$$\begin{aligned} L(\alpha,\lambda;y) &= \prod_{j=1}^{r}\prod_{i=1}^{m} g_i(y_{ij};\alpha,\lambda) \\ &= D\prod_{j=1}^{r}\prod_{i=1}^{m} y_{ij}\exp(-(\lambda y_{ij})^2)(1-\exp(-(\lambda y_{ij})^2))^{\alpha i-1} \\ &\times(1-(1-\exp(-(\lambda y_{ij})^2))^{\alpha})^{m-i}, \end{aligned} \tag{3.13}$$

where $D = \left(\frac{m!}{(m-i)!(i-1)!}\right)^{mr} 2^{mr}\alpha^{mr}\lambda^{2mr}$.
And the natural logarithm of likelihood function is

$$\begin{aligned} l(\alpha,\lambda) &= mr\log K + mr\log\alpha + 2mr\log\lambda + \sum_{j=1}^{r}\sum_{i=1}^{m}\log(y_{ij}) \\ &-\lambda^2\sum_{j=1}^{r}\sum_{i=1}^{m} y_{ij}^2 + \sum_{j=1}^{r}\sum_{i=1}^{m}(\alpha i-1)\log(1-\exp(-(\lambda y_{ij})^2)) \\ &+\sum_{j=1}^{r}\sum_{i=1}^{m}(m-i)\log(1-(1-\exp(-(\lambda y_{ij})^2))^{\alpha}), \end{aligned} \tag{3.14}$$

where $K = \frac{m!}{(m-i)!(i-1)!}$. When we differentiate $l(\alpha,\lambda)$ with respect to parameters and equates the zero,

$$\begin{aligned} \frac{\partial l}{\partial\alpha} &= \frac{mr}{\alpha} + \sum_{j=1}^{r}\sum_{i=1}^{m} i\log(1-\exp(-(\lambda y_{ij})^2)) \\ &-\sum_{j=1}^{r}\sum_{i=1}^{m}(m-i)\frac{(1-\exp(-(\lambda y_{ij})^2))^{\alpha}\log(1-\exp(-(\lambda y_{ij})^2))}{1-(1-\exp(-(\lambda y_{ij})^2))^{\alpha}} \end{aligned} \tag{3.15}$$

$$\begin{aligned}\frac{\partial l}{\partial \lambda} &= \frac{2mr}{\lambda} - 2\lambda \sum_{j=1}^{r}\sum_{i=1}^{m} y_{ij}^2 \\ &-\sum_{j=1}^{r}\sum_{i=1}^{m}(m-i)\frac{2\alpha\lambda y_{ij}^2 \exp(-(\lambda y_{ij})^2)(1-\exp(-(\lambda y_{ij})^2))^{\alpha-1}}{1-(1-\exp(-(\lambda y_{ij})^2))^{\alpha}} \\ &+\sum_{j=1}^{r}\sum_{i=1}^{m}(\alpha i-1)\frac{2\lambda y_{ij}^2 \exp(-(\lambda y_{ij})^2)(1-\exp(-(\lambda y_{ij})^2))^{\alpha-1}}{1-\exp(-(\lambda y_{ij})^2)}. \quad (3.16)\end{aligned}$$

Since equations have nonlinear forms, they are solved by using numerical method that is Newton-Raphson method for parameters to obtain ML estimators of α and λ , say $\hat{\alpha}_{RSS}$ and $\hat{\lambda}_{RSS}$.

3.3 Estimation for Parameters of GR Distribution in ERSS

Now, the aim of this section is to estimate the parameters of GR distribution by using ML method based on ERSS.

Let $U = \{X_{i(1:m)j},\ i = 1, ..., \frac{m}{2},\ j = 1, ..., r\} \bigcup \{X_{i(m:m)j},\ i = \frac{m}{2}+1, ..., m,\ j = 1, ..., r\}$. Here, $X_{i(1:m)j}$ is distributed as the first order statistic and $X_{i(m:m)j}$ as the maximum order statistic in a random sample of size m from the original distribution. Then the densities of U_{ij} are given by

$$g_1(u_{ij};\alpha,\lambda) = mf(u_{ij};\alpha,\lambda)[1-F(y_{ij};\alpha,\lambda)]^{m-1} \quad (3.17)$$

and

$$g_m(u_{ij};\alpha,\lambda) = mf(u_{ij};\alpha,\lambda)[F(y_{ij};\alpha,\lambda)]^{m-1}. \quad (3.18)$$

Combining the ERSS procedure and densities, for even set size m, the likelihood function of parameters given $U = u$ is as follows

$$
\begin{aligned}
L_{ERSS_e}(\alpha, \lambda; u) &= \prod_{j=1}^{r}\prod_{i=1}^{m/2} g_1(u_{ij}; \alpha, \lambda) \prod_{j=1}^{r}\prod_{i=\frac{m}{2}+1}^{m} g_m(u_{ij}; \alpha, \lambda) \\
&= E \prod_{j=1}^{r}\prod_{i=1}^{m} u_{ij} \exp(-(\lambda u_{ij})^2)(1 - \exp(-(\lambda u_{ij})^2))^{\alpha-1} \\
&\times \prod_{j=1}^{r}\prod_{i=1}^{m/2}(1 - (1 - \exp(-(\lambda u_{ij})^2))^{\alpha})^{m-1} \\
&\times \prod_{j=1}^{r}\prod_{i=\frac{m}{2}+1}^{m} ((1 - \exp(-(\lambda u_{ij})^2))^{\alpha})^{m-1}, \qquad (3.19)
\end{aligned}
$$

where $E = 2^{mr} m^{mr} \alpha^{mr} \lambda^{2mr}$.

Also let $V = \{X_{i(1:m)j},\ i = 1, ..., \frac{m-1}{2},\ j = 1, ..., r\} \bigcup \{X_{i(m:m)j},\ i = \frac{m+1}{2}, ..., m-1,\ j = 1, ..., r\} \bigcup \{X_{i(\frac{m+1}{2}:m)j},\ i = m,\ j = 1, ..., r\}$. Then the likelihood function of parameters given $V = v$, for odd set size can be written as

$$
\begin{aligned}
L_{ERSS_o}(\alpha, \lambda; v) &= \prod_{j=1}^{r}\prod_{i=1}^{\frac{m-1}{2}} g_1(v_{ij}; \alpha, \lambda) \prod_{j=1}^{r}\prod_{i=\frac{m+1}{2}}^{m-1} g_m(v_{ij}; \alpha, \lambda) \times \dots \\
&\times \prod_{j=1}^{r}\prod_{i=m}^{m} g_{\frac{m+1}{2}}(v_{ij}; \alpha, \lambda) \\
&= M \prod_{j=1}^{r}\prod_{i=1}^{m} v_{ij} \exp(-(\lambda v_{ij})^2)(1 - \exp(-(\lambda v_{ij})^2))^{\alpha-1} \\
&\times \prod_{j=1}^{r}\prod_{i=1}^{\frac{m-1}{2}}(1 - (1 - \exp(-(\lambda v_{ij})^2))^{\alpha})^{m-1} \\
&\times \prod_{j=1}^{r}\prod_{i=\frac{m+1}{2}}^{m-1} ((1 - \exp(-(\lambda v_{ij})^2))^{\alpha})^{m-1} \\
&\times \prod_{j=1}^{r}\prod_{i=m}^{m}((1 - \exp(-(\lambda v_{ij})^2))^{\alpha})^{\frac{m-1}{2}}, \qquad (3.20)
\end{aligned}
$$

where $M = \left(\frac{m!}{((\frac{m-1}{2}!))^2}\right)^{mr} 2^{mr} m^{(m-1)r} \alpha^{mr} \lambda^{2mr}$, $g_i(.;\alpha,\lambda)$, $g_1(.;\alpha,\lambda)$ and $g_m(.;\alpha,\lambda)$ are given by (3.12), (3.17) and (3.18) respectively. As there are no closed form of the ML estimators of the parameters under odd and even sample size m,

$\hat{\alpha}_{ERSS}$ and $\hat{\lambda}_{ERSS}$ are obtained numerically by maximizing the likelihood functions (3.19) and (3.20) respectively.

3.4 Estimation for Parameters of GR Distribution in MRSS

In this section, we will obtain the parameters of GR distribution via ML technique depending on MRSS. Let $W = \{X_{i(\frac{m+1}{2}:m)j},\ i = 1, ..., \frac{m}{2},\ j = 1, ..., r\}$. Then the likelihood function of parameters given $W = w$, for odd set size can be written as

$$\begin{aligned} L_{MRSS_o}(\alpha, \lambda; w) &= \prod_{j=1}^{r}\prod_{i=1}^{m} g_{\frac{m+1}{2}}(w_{ij}; \alpha, \lambda) \\ &= H\prod_{j=1}^{r}\prod_{i=1}^{m} w_{ij}\exp(-(\lambda w_{ij})^2)(1-\exp(-(\lambda w_{ij})^2))^{\frac{m\alpha+\alpha}{2}-1} \\ &\times(1-(1-\exp(-(\lambda w_{ij})^2))^{\alpha})^{\frac{m-1}{2}}, \end{aligned} \tag{3.21}$$

where $H = 2^{mr}\left(\frac{m!}{((\frac{m-1}{2})!)^2}\right)^{mr}\alpha^{mr}\lambda^{2mr}$ and $g_i(.; \alpha, \lambda)$ is given by (3.12).

Also let $T = \{X_{i(\frac{m}{2}:m)j},\ i = 1, ..., \frac{m}{2},\ j = 1, ..., r\}\bigcup\{X_{i(\frac{m+2}{2}:m)j},\ i = \frac{m+2}{2}, ..., m, j = 1, ..., r\}$. Then the likelihood function of parameters given $T = t$, for even set size can be written as

$$\begin{aligned} L_{MRSS_e}(\alpha, \lambda; t) &= \prod_{j=1}^{r}\prod_{i=1}^{m/2} g_{\frac{m}{2}}(t_{ij}; \alpha, \lambda)\prod_{j=1}^{r}\prod_{i=\frac{m+2}{2}}^{m} g_{\frac{m+2}{2}}(t_{ij}; \alpha, \lambda) \\ &= K\prod_{j=1}^{r}\prod_{i=1}^{m} t_{ij}\exp(-(\lambda t_{ij})^2)(1-\exp(-(\lambda t_{ij})^2))^{\alpha-1} \\ &\times\prod_{j=1}^{r}\prod_{i=1}^{m/2}((1-\exp(-(\lambda t_{ij})^2))^{\alpha})^{\frac{m-2}{2}} \\ &\times(1-(1-\exp(-(\lambda t_{ij})^2))^{\alpha})^{\frac{m}{2}}, \end{aligned} \tag{3.22}$$

where $K = 2^{mr}\left(\frac{m!}{(\frac{m-2}{2})!(\frac{m}{2})!}\right)^{mr}\alpha^{mr}\lambda^{2mr}$. Also, ML estimators of parameters say, $\hat{\alpha}_{MRSS}$ and $\hat{\lambda}_{MRSS}$ are solved by numerical method because of complex forms of (3.21) and (3.22) .

CHAPTER FOUR
SIMULATION STUDY

In this chapter, we analyze the performance of ML estimators of the unknown parameters of GR distribution and a simulation study performed in Mathematica v.11.0 for the analysis. The performance comparisan criteria that are biases, MSEs and relative efficiencies (RE) of estimators rely on SRS, RSS, ERSS, MRSS and imperfect RSS (IRSS). Monte Carlo simulation is conducted with different set sizes, cycle numbers and different values of shape parameter α for the interested population. The following formulas are used for the comparisons:

$$Bias(\hat{\theta}_i, \theta) = \frac{1}{N}\sum_{i=1}^{N}(\hat{\theta}_i - \theta),$$

$$MSE(\hat{\theta}_i, \theta) = \frac{1}{N}\sum_{i=1}^{N}(\hat{\theta}_i - \theta)^2,$$

$$RE(\hat{\theta}_2) = \frac{MSE(\hat{\theta}_1)}{MSE(\hat{\theta}_2)}.$$

where $\hat{\theta}_i$ is the estimated value of ith number of repetitions, θ is the population parameter and $N = 10,000$ is the total repetition number. We denote $\hat{\theta}_1 = \hat{\theta}_{SRS}$ and $\hat{\theta}_2$ is estimated value of the parameter under another sampling method.

4.1 Results under Perfect Ranking in RSS Schemes

In this section, we consider that there is no presence of ranking errors for the performance comparison of estimators. The findings are shown in Table 4.1, Table 4.2 and Figure 4.1. One can conclude from Table 4.1 that estimates of α and λ based on ERSS have smaller biases than the corresponding estimates using SRS, RSS and MRSS. Biases and MSEs of the estimators based on RSS and its modifications decrease when set sizes increase. Figure 4.1 shows us MSE of unknown parameters of GR distribution based on different sampling method for only $\alpha = 0.5$ and $\lambda = 1$

but all α values have the similar results. Also, one can see from Table 4.2 that ML estimators of both parameters derived on ERSS are more efficient than the other methods except the case of $m = 3$. In that condition, RSS and ERSS have the similar results. Actually, the reason of this situation is that case of $m = 3$ in ERSS corresponds to the case of $m = 3$ in usual RSS in terms of the their schemes. Furthermore, we can say that REs based on RSS methods increase as the set sizes increase.

Table 4.1 Biases of the estimators of the parameters of GR distribution

n	$\alpha;\lambda$	$\hat{\alpha}_{SRS}$	$\hat{\lambda}_{SRS}$	m;r	$\hat{\alpha}_{RSS}$	$\hat{\lambda}_{RSS}$	$\hat{\alpha}_{MRSS}$	$\hat{\lambda}_{MRSS}$	$\hat{\alpha}_{ERSS}$	$\hat{\lambda}_{ERSS}$
		0.0898	0.1369	3;4	0.0655	0.1019	0.0854	0.1298	0.0627	0.1001
12	0.4;1.0			4;3	0.0551	0.0852	0.0760	0.1209	0.0429	0.0667
				6;2	0.0443	0.0690	0.0714	0.1122	0.0433	0.0631
		0.1218	0.1280	3;4	0.0820	0.0902	0.1104	0.1198	0.0812	0.0898
	0.5;1.0			4;3	0.0715	0.0779	0.0983	0.1085	0.0530	0.0615
				6;2	0.0703	0.0790	0.1012	0.1070	0.0352	0.0410
		0.1948	0.1013	3;4	0.1427	0.0728	0.1875	0.0868	0.1506	0.0765
	0.7;1.0			4;3	0.1291	0.0659	0.1685	0.0801	0.1013	0.0546
				6;2	0.1037	0.0525	0.1683	0.0755	0.0634	0.0375
		0.3199	0.0995	3;4	0.2139	0.0710	0.3056	0.0874	0.2187	0.0703
	1.0;1.0			4;3	0.1903	0.0647	0.2705	0.0826	0.1466	0.0510
				6;2	0.1433	0.0503	0.2830	0.0842	0.0969	0.0362
		0.0379	0.0632	3;8	0.0253	0.0423	0.0375	0.0624	0.0263	0.0457
24	0.4;1.0			4;6	0.0222	0.0370	0.0320	0.0547	0.0153	0.0273
				6;4	0.0181	0.0302	0.0278	0.0460	0.0077	0.0161
		0.0572	0.0550	3;8	0.0387	0.0443	0.0501	0.0629	0.0390	0.0477
	0.5;1.0			4;6	0.0337	0.0417	0.0467	0.0402	0.0262	0.0328
				6;4	0.0272	0.0358	0.0475	0.0392	0.0187	0.0219
		0.0948	0.0604	3;8	0.0696	0.0468	0.0933	0.0575	0.0706	0.0474
	0.7;1.0			4;6	0.0651	0.0396	0.0799	0.0510	0.0534	0.0334
				6;4	0.0553	0.0357	0.0843	0.0540	0.0400	0.0258
		0.1295	0.0487	3;8	0.0996	0.0364	0.1404	0.0489	0.0980	0.0368
	1.0;1.0			4;6	0.0847	0.0322	0.1281	0.0453	0.0653	0.0267
				6;4	0.0757	0.0297	0.1380	0.0474	0.0356	0.0183
		0.0271	0.0605	3;12	0.0207	0.0479	0.0279	0.0654	0.0209	0.0482
36	0.4;1.0			4;9	0.0189	0.0458	0.0241	0.0573	0.0152	0.0376
				6;6	0.0158	0.0394	0.0255	0.0613	0.0114	0.0296
		0.0268	0.0419	3;12	0.0191	0.0326	0.0222	0.0362	0.0208	0.0358
	0.5;1.0			4;9	0.0176	0.0303	0.0185	0.0299	0.0137	0.0268
				6;6	0.0135	0.0261	0.0191	0.0305	0.0087	0.0220
		0.0630	0.0432	3;12	0.0471	0.0343	0.0613	0.0412	0.0468	0.0336
	0.7;1.0			4;9	0.0424	0.0298	0.0627	0.0427	0.0346	0.0276
				6;6	0.0369	0.0279	0.0570	0.0396	0.0226	0.0202
		0.0760	0.0236	3;12	0.0510	0.0156	0.0668	0.0202	0.0480	0.0143
	1.0;1.0			4;9	0.0701	0.0210	0.0639	0.0193	0.0354	0.0095
				6;6	0.0339	0.0093	0.0567	0.0171	0.0237	0.0058

4.2 Results under Imperfect Ranking in RSS Schemes

Now, we consider that the performances of the estimators under imperfect RSS (IRSS) where IRSS scheme is regarded as existence of ranking errors. This means that there is a mismatching between the ranked order of the items and their actual numerical orders. There are some works in the literature about impacts of imperfect ranking on efficiencies of estimators. One can see; Park & Lim (2012), Frey (2014) and references therein. In literature, the concomitant variable is mostly used one to observe effects of imperfect ranking, but here we prefer to use a mathematical model that is called fraction of random ranking mentioned in Vock & Balakrishnan (2011). The model for the judgement ordering is as follows:

$$F_{[i]} = (1-\beta)F_{(i)} + \beta F, \;\; i = 1, \ldots, m, \;\; \beta \in [0, 1], \tag{4.1}$$

where the mixture distribution $F_{[i]}$ of the ith judgement order statistic, $F_{(i)}$ is the distribution of true ith order statistic, F is the original cdf of GR distribution and β is the mixing parameter. We used mixing parameter β=0.1, 0.4 and 0.9 in model 4.1 and also different schemes have been generated from the pdf of GR distribution when $\alpha = 0.5, \; \lambda = 1$. Figure 4.2 illustrates that RE of estimators of parameters under various schemes of sampling including IRSS when $n = 12(n = mr)$ with different mixing parameters for IRSS. As we have expected, REs of estimators decreases while ranking error i.e. mixing parameter is rising. ERSS is still the most efficient sampling method for this analysis. However, IRSS is more efficient than SRS and MRSS even under high ranking error.

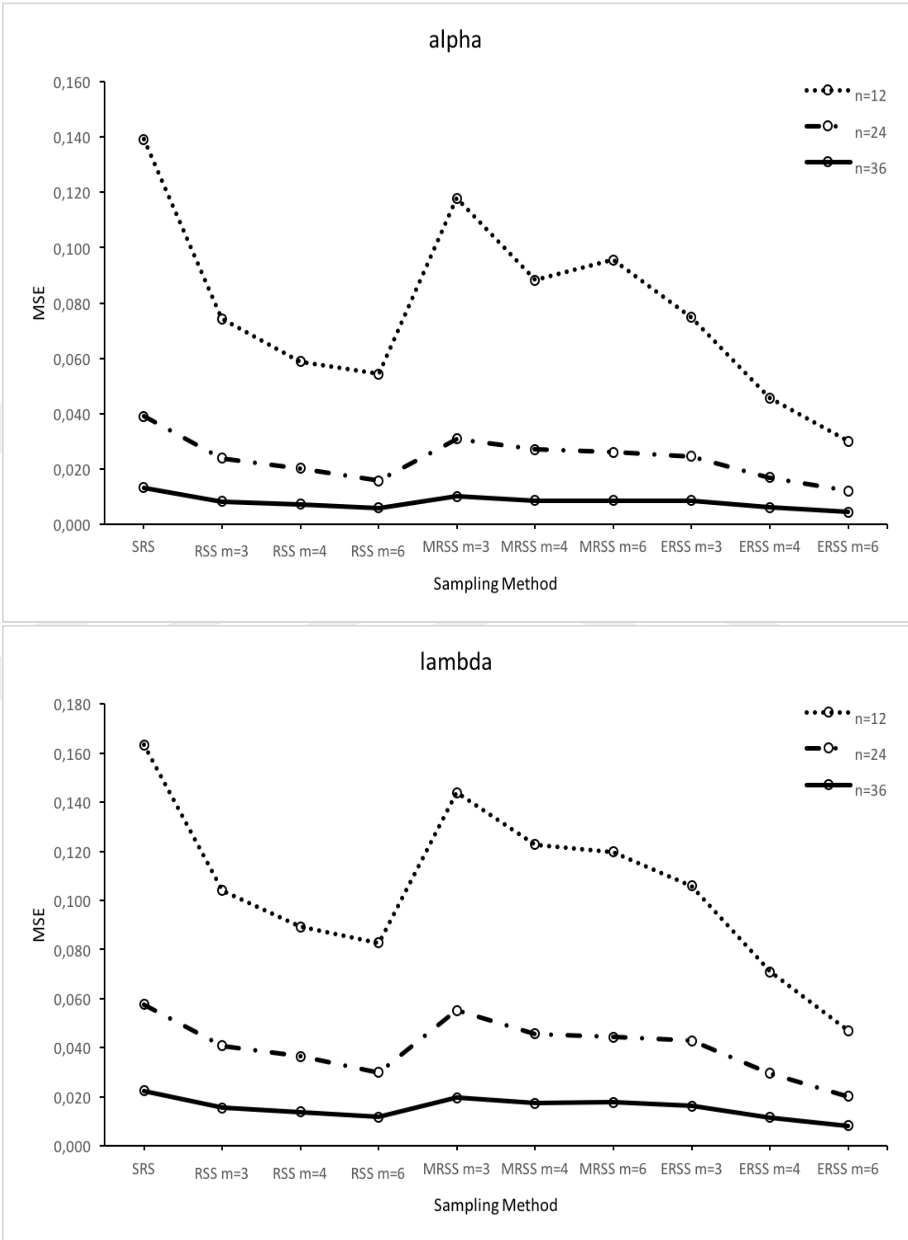

Figure 4.1 MSEs of the estimators versus different sampling methods for $\alpha = 0.5$, $\lambda = 1.0$ and $n = mr$

Table 4.2 RE values of the estimators of the parameters of GR distribution

			RE					
n	$\alpha;\lambda$	m;r	$\hat{\alpha}_{RSS}$	$\hat{\lambda}_{RSS}$	$\hat{\alpha}_{MRSS}$	$\hat{\lambda}_{MRSS}$	$\hat{\alpha}_{ERSS}$	$\hat{\lambda}_{ERSS}$
		3;4	1.9840	1.5915	1.3562	1.1521	2.0565	1.5700
12	0.4;1.0	4;3	2.5238	1.8783	1.6243	1.2779	3.2773	2.4733
		6;2	3.6414	2.4641	1.7383	1.3601	3.2651	2.5234
		3;4	1.9843	1.6717	1.1487	1.1912	1.9872	1.6773
	0.5;1.0	4;3	2.5854	2.0002	1.6300	1.3737	3.4737	2.5526
		6;2	2.5864	1.9995	1.4411	1.4024	5.5651	3.9471
		3;4	1.8032	1.5762	1.1136	1.2268	1.6435	1.5106
	0.7.1.0	4;3	2.1695	1.7713	1.4072	1.4330	3.0278	2.2628
		6;2	3.2265	2.3649	1.4055	1.4350	5.5332	3.3819
		3;4	1.9069	1.5641	0.9353	1.2106	1.8045	1.5687
	1.0;1.0	4;3	2.3969	1.8464	1.2018	1.3757	3.4117	2.2930
		6;2	3.5283	2.3659	1.0594	1.3717	6.2216	3.3959
		3;8	1.6939	1.4864	1.2262	1.0897	1.6571	1.4342
24	0.4;1.0	4;6	2.0155	1.7104	1.5031	1.2848	2.3991	2.1345
		6;4	2.5695	2.1124	1.7467	1.3678	3.5168	3.0011
		3;8	1.6436	1.3946	1.2508	0.9942	1.6182	1.3246
	0.5;1.0	4;6	1.9873	1.5576	1.4124	1.2386	2.4290	1.9651
		6;4	2.6584	1.9304	1.4744	1.3278	3.4496	2.9443
		3;8	1.7016	1.4208	1.1620	1.0817	1.6958	1.4321
	0.7;1.0	4;6	1.8864	1.6135	1.4199	1.2705	2.4654	1.9866
		6;4	2.5650	2.0376	1.3795	1.1972	3.6788	2.9606
		3;8	1.6074	1.4600	1.1062	1.1376	1.5829	1.4378
	1.0;1.0	4;6	1.9846	1.6851	1.2299	1.2496	2.4473	2.0050
		6;4	2.2846	1.9039	1.2404	1.2825	3.9869	2.8635
		3;12	1.6148	1.3903	1.2424	0.9764	1.5834	1.3569
36	0.4;1.0	4;9	1.8301	1.5388	1.4439	1.1391	2.1366	1.9818
		6;6	2.3704	1.9389	1.4430	1.0826	3.0766	2.9329
		3;12	1.6098	1.4423	1.2983	1.1383	1.5490	1.3799
	0.5;1.0	4;9	1.8375	1.6169	1.5278	1.2961	2.1376	1.9348
		6;6	2.2241	1.9175	1.5427	1.2585	2.9195	2.7145
		3;12	1.5348	1.4059	1.1789	1.1888	1.5241	1.4126
	0.7;1.0	4;9	1.7428	1.6137	1.2335	1.2149	2.1516	1.9156
		6;6	2.2172	2.0189	1.3702	1.2760	3.2244	2.7151
		3;12	1.5225	1.3799	1.1399	1.0843	1.5280	1.4102
	1.0;1.0	4;9	1.0189	1.0040	1.2701	1.2307	2.2081	1.8981
		6;6	2.2607	1.9485	1.3187	1.2273	3.2514	2.6422

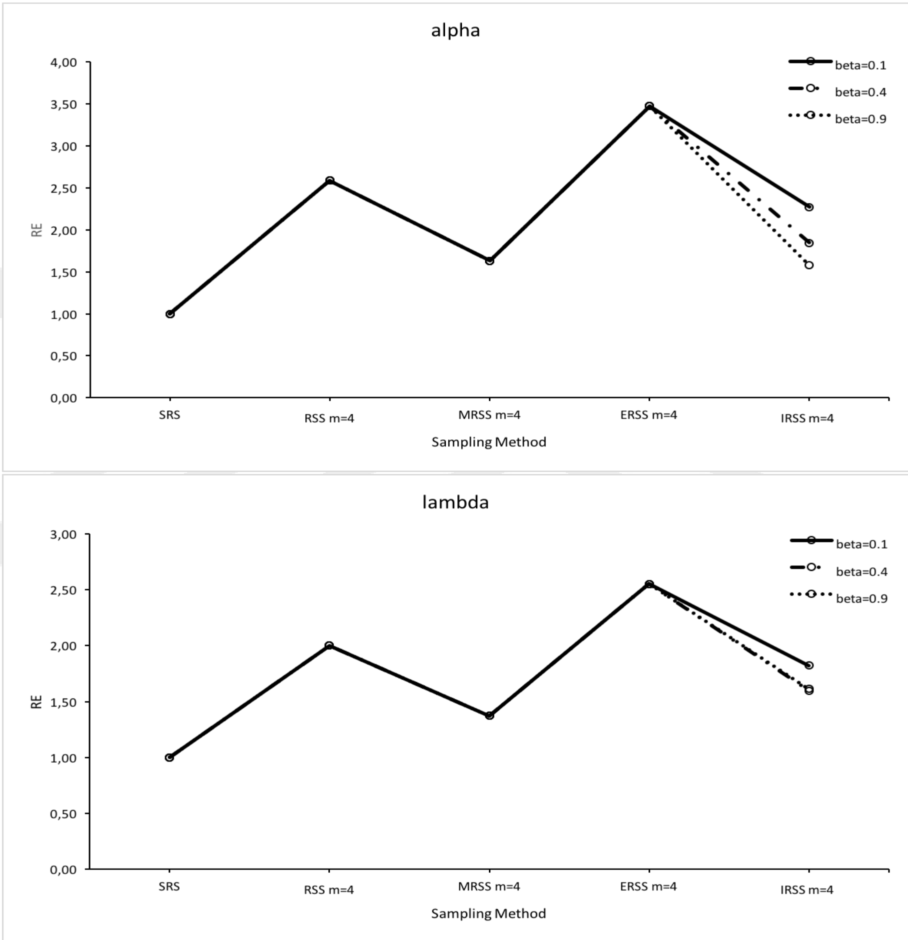

Figure 4.2 REs of the estimators versus different sampling methods for $\alpha = 0.5$, $\lambda = 1.0$, $\beta = 0.1, 0.4, 0.9$ and $n = 12$

CHAPTER FIVE
REAL DATA EXAMPLE

In this chapter, we give a data analysis with a real data was originated by Bader & Priest (1982). Data set in Table 5.1 contains the 69 strength measured in GPA (giga-Pascals), for single carbon fibers and impregnated 1000-carbon fiber tows. When we applied the Kolmogorov-Smirnov test in R software, we see that data set fitted with GR distribution very well in Figure 5.1. The ML estimators of the parameter α, λ and the p-value of the Kolmogorov-Smirnov test are 3.24615, 0.77510 and 0.9033 respectively. Therefore, we conclude that data follow the GR distribution since p-value $>\alpha = 0.05$

Table 5.1 The Strength Data

0.562	0.564	0.729	0.802	0.950	1.053	1.111	1.115	1.194	1.208
1.216	1.247	1.256	1.271	1.277	1.305	1.313	1.348	1.390	1.429
1.474	1.490	1.503	1.520	1.522	1.524	1.551	1.551	1.609	1.632
1.632	1.676	1.684	1.685	1.728	1.740	1.761	1.764	1.785	1.804
1.816	1.824	1.836	1.879	1.883	1.892	1.898	1.934	1.947	1.976
2.020	2.023	2.050	2.059	2.068	2.071	2.098	2.130	2.204	2.262
2.317	2.334	2.340	2.346	2.378	2.483	2.683	2.835	2.835	

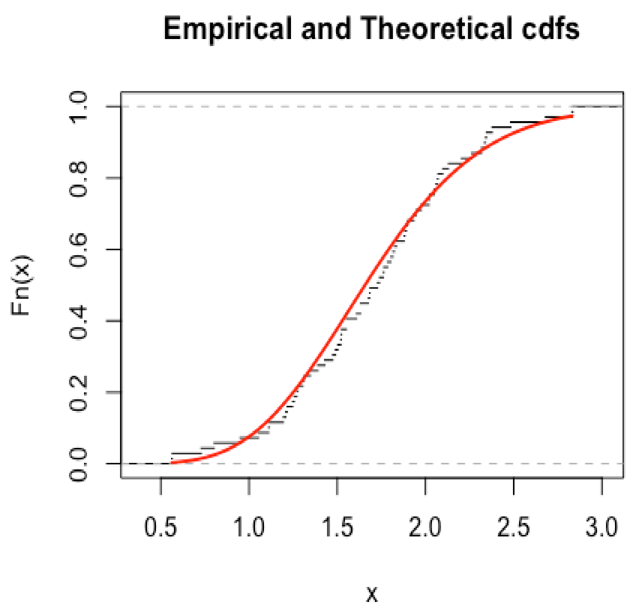

Figure 5.1 The cdf and emprical cdf of GR distribution for real data

For the analysis, a random sample of size 12 is drawn without replacement with different sampling scheme; SRS, RSS, ERSS and MRSS. In RSS and its modifications, we select $m = 6$ and $r = 2$. The result of the analysis presented in Table 5.2. We infer from the result, estimates based on ERSS are closer to the given values for both parameters α and λ. Also, we can say that our findings in this chapter sort together simulation results.

Table 5.2 The ML estimators of parameters for GR distribution

$\hat{\alpha}$				$\hat{\lambda}$			
SRS	RSS	MRSS	ERSS	SRS	RSS	MRSS	ERSS
3.51296	3.10667	3.17405	3.45956	0.866591	0.729072	0.722887	0.754149

CHAPTER SIX
CONCLUSION

In this thesis, we make an inference for unknown parameters of GR distribution based on RSS, ERSS and MRSS by using ML method. These estimators have been derived theoretically but we have no closed form solutions for the estimators. Therefore the performance comparisons have been done by simulation study and it has been supported with a real data example. Biases, MSEs and REs are calculated as comparison criteria under both perfect and imperfect ranking process. We can interpret from the study, biases and MSEs of the estimators for true parameters under ERSS are smaller than the corresponding estimators calculated under SRS and other RSS methods. This shows that ERSS is more effective than the other mentioned approaches. Furthermore, even under the presence of ranking error, we see that it works still better than SRS and MRSS. Finally, when we investigated the real data, we have sight an analogous with a simulation result. Therefore, ERSS has a closer values to the given values.

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