EXACT RENORMALIZATION GROUP ON POINT INTERACTIONS

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ABSTRACT

EXACT RENORMALIZATION GROUP ON POINT INTERACTIONS

The goal of this thesis is to present a non-perturbative renormalization of point interactions in non-relativistic quantum mechanics using Exact Renormalization Group (ERG) techniques. The analyzed model is a particle interacting with a Dirac-Delta potential on various two dimensional Riemannian Manifolds, namely the Euclid Plane $\mathbb{R}^2$, the Hyperbolic Plane $\mathbb{H}^2$, and the Sphere $\mathbb{S}^2$. In all cases the problem contains logarithmic divergences, hence regularization and renormalization are needed. The renormalization is performed using the ERG method, where one gets an expression for the flow equation of the effective coupling constant by integrating out degrees of freedom between the bare scale Λ and the effective scale λ. The main conclusion of this work is to show that the flow equations are identical in all three manifolds under consideration.

ÖZET

NOKTASAL ETKİLEŞİMLERDE TAM RENORMALİZASYON GRUBU

Bu tezin amacı, göreli olmayan kuantum mekaniğindeki noktasal etkileşimlerin, Tam Renormalizasyon Grubu teknikleri kullanılarak yapılan, pertürbasyon dışı bir renormalizasyon yönteminin aktarımıdır. İncelenen model, bir parçacığın çeşitli Riemann katmanlı uzaylarında (Öklid Düzlemi $\mathbb{R}^2$, Hiperbolik Düzlem $\mathbb{H}^2$, Küre $\mathbb{S}^2$) bir Dirac-Delta potansiyeli ile etkileşimidir. Her üç uzayda da, problem logaritmik ıraksaklıklar içerir; dolayısıyla regülarizasyon ve renormalizasyon işlemlerine ihtiyaç vardır. Renormalizasyon, Tam Renormalizasyon Grubu yöntemi ile yapılmıştır ve bunun sonucunda, bir yalın enerji seviyesi Λ ile bir efektif enerji seviyesi λ arasındaki serbestlik dereceleri toplanıp sistemin tanımının dışına alınarak, efektif kuplaj sabiti için bir akış denklemi türetilmiştir. Bu çalışmanın temel sonucu, akış denklemlerinin, incelenen üç katmanlı uzayda da birebir aynı formda olduğudur.

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LIST OF SYMBOLS

B_{2k}	Bernoulli numbers
$C^k(U)$	Space of k-times differentiable functions on U
D	Dimension of the space
dV_g	Volume form on Riemannian manifold with metric g
g_λ	Effective coupling constant
g_Λ	Bare coupling constant
$\mathbb{H}^2$	Hyperbolic plane
$\mathcal{H}$	Hilbert space
$K_{i\tau}(z)$	Modified Bessel functions of the third kind
$L^2(\mathcal{M}, dV_g)$	Space of square integrable functions on $(\mathcal{M}, g)$
$(\mathcal{M}, g)$	Riemannian manifold $\mathcal{M}$ with metric structure g
$[Q]$	Dimensionality of the quantity Q in terms of inverse length
$\mathbb{R}^D$	Euclid space of dimension D
$\mathbb{S}^2$	2-sphere
$\mathrm{vol}\,(\mathcal{M})$	Volume of the Riemannian manifold $\mathcal{M}$
Y_l^m	Spherical harmonics on the 2-sphere
Δ_g	Laplace-Beltrami operator with the metric structure g
$\delta_g(x, x_0)$	Dirac-Delta function on $(\mathcal{M}, g)$ at the position $x_0 \in \mathcal{M}$
Λ	Bare scale
λ	Effective scale
ν^2	Bound state energy
$\sigma(A)$	Spectrum of the operator A

LIST OF ACRONYMS/ABBREVIATIONS

ERG	Exact Renormalization Group
QFT	Quantum Field Theory
RG	Renormalization Group

1. INTRODUCTION

When the de Broglie wavelength of the particle is very large compared to the range of the potential, the interaction can be approximated by a point interaction. The quantum mechanical problem of a particle interacting with a Dirac-Delta potential in one dimension is a standard example in any textbook on elementary quantum mechanics [1]. One of the first studies of point interactions in the literature is the work by Kronig and Penney [2]. In their work, the authors used periodic Dirac-Delta potentials to describe the non-relativistic electrons moving in a one dimensional fixed crystal lattice. Later, this so called *Kronig-Penney Model* became one of the standard models in solid state physics [3].

However when we investigate the point interactions in higher dimensions, we run into difficulties. The problem does possess ultraviolet divergences and as a result of this, a finite value for the bound state energy could not be obtained. This fact was first realized by Thomas in his study of point interactions in three dimensions [4]. Later Bég and Furlong showed that the Dirac-Delta potential in three dimensions is the non-relativistic limit of the $\lambda\phi^4$ theory in $3+1$ space time dimensions and found that choosing a finite constant for the unrenormalized coupling constant gives rise to a trivial S-matrix [5].

Despite all of these problems, Thorn [6] realized that physical results could be obtained in two and three dimensions, after one applies regularization and renormalization which are standard tools of Quantum Field Theory (QFT). Renormalization of point interactions has been studied by many authors; in position space [7–9], and in momentum space [10–14]. The renormalization group equations were derived in [10, 15]. The two dimensional case is additionally interesting due to the fact that the Hamiltonian does not contain any dimensional parameters, so the coupling constant is dimensionless which makes the Hamiltonian scale invariant [16]. Renormalization necessarily breaks this invariance, therefore two dimensional Dirac-Delta potential is one of the simplest examples of quantum mechanical symmetry breaking [16–18].

A mathematically rigorous way to define the two- and three dimensional Dirac-Delta interactions is the method of self-adjoint extensions which was first studied for the three dimensional case by Berezin and Faddeev [19]. In this approach the point interaction Hamiltonian is described as a free Hamiltonian on a space with one point removed, together with a suitable boundary condition specifying the interaction [16]. A detailed study of this view and an extensive list of references can be found in the monograph [20].

In the 70's, Wilson introduced the idea of *Renormalization Group (RG)* [21–25], which dramatically changed the concept of renormalization; promoted it from being a mathematical trick into a novel way of thinking about physics. A pedagogical application of the idea of Wilsonian RG to the point interactions on the Euclid plane, $\mathbb{R}^2$ was given in the lecture notes by Głazek and Maslowski [26]. Basically this thesis is about generalization of these lecture notes to the hyperbolic plane $\mathbb{H}^2$ and 2-sphere $\mathbb{S}^2$.

We shall start by reviewing point interactions on a D-dimensional Euclid space $\mathbb{R}^D$. We will answer why this problem requires renormalization when $D \geq 2$, discuss standard renormalization methods and derive flow equations for the effective coupling constant. We will also comment why these methods do not work when $D \geq 4$ and discuss the scale invariance of the Hamiltonian when $D = 2$. Later, based on Głazek and Maslowski [26], we will mention about the renormalization of point interactions in $\mathbb{R}^2$ using the Wilsonian RG and derive the flow equation. As an addendum to Głazek and Maslowski we also investigate the range of renormalizability using the Banach Contraction Principle. In the next two sections we shall analyze the same problem on $\mathbb{H}^2$ and $\mathbb{S}^2$ and show that the flow equation has the same form on all three manifolds.

2. POINT INTERACTIONS ON EUCLID SPACES

In this section we shall review the standard renormalization of point interactions on Euclid spaces $\mathbb{R}^2$ and $\mathbb{R}^3$. We begin by answering why these problems requires renormalization and we also investigate why the renormalization is not possible when the dimension of the space is higher than 3. Finally we will derive the flow equations for the coupling constant.

2.1. Formulation of the Problem

Consider a particle of mass m interacting with a $D-$dimensional Dirac Delta potential located at some point $\mathbf{a} \in \mathbb{R}^D$. Because of the translational invariance of the problem, without loss of generality we can shift the coordinate system such that the interaction takes place at the origin. The corresponding Schrödinger equation becomes

$$\left(-\frac{\hbar^2}{2m}\Delta_{\mathbb{R}^D} - g\delta^D(\mathbf{x})\right)\psi(\mathbf{x}) = E\psi(\mathbf{x}), \mathbf{x} \in \mathbb{R}^D \tag{2.1}$$

where $\Delta_{\mathbb{R}^D}$ is the Laplacian operator on $D-$dimensional Euclid space and g is a real, positive parameter which determines the strength of the point interaction. In QFT, it is convenient to use *natural units* defined by taking two fundamental constants c and $\hbar$ to be dimensionless and unity, so that all dimensionful quantities can be expressed in terms of a single scale, namely mass or energy. Similarly in one-particle non-relativistic quantum mechanics we can take $\hbar$ and m to be dimensionless. Throughout this thesis we will work with the units defined by

$$\hbar = 1 \quad \text{and} \quad m = \frac{1}{2},$$

so that all dimensionful parameters can be expressed in terms of a single scale, namely momentum or inverse length. In these units the Schrödinger equation for the problem becomes

$$\left(-\Delta_{\mathbb{R}^D} - g\delta^D(\mathbf{x})\right)\psi(\mathbf{x}) = E\psi(\mathbf{x}). \tag{2.2}$$

First we consider the bound state solutions. Let us parametrize the bound state energy by $E = -\nu^2$. Then the equation we need to solve takes the form

$$\left(-\Delta_{\mathbb{R}^D} - g\delta^D(\mathbf{x})\right)\phi(\mathbf{x}) = -\nu^2\phi(\mathbf{x}). \tag{2.3}$$

where $\phi(\mathbf{x})$ is the bound state wavefunction. Now we will write this equation in momentum space. This corresponds to the construction of the spectral representation of $\Delta_{\mathbb{R}^D}$. It is given by a map between position and momentum spaces

$$\mathcal{F} : L^2(\mathbb{R}^D, d^D\mathbf{x}) \to L^2(\mathbb{R}^D, d^D\mathbf{p})$$

which satisfies the following properties [27]:

(i) It diagonalizes the Laplacian, i.e., for all $\tilde{\phi}(\mathbf{p}) \in L^2(\mathbb{R}^D, d^D\mathbf{p})$ there exists a real-valued measurable function f on $\mathbb{R}^D$ such that

$$-(\mathcal{F}\Delta_{\mathbb{R}^D}\mathcal{F}^{-1}\tilde{\phi})(\mathbf{p}) = f(\mathbf{p})\tilde{\phi}(\mathbf{p})$$

(ii) It is an isometric isomorphism, i.e., for all $\phi(\mathbf{x}) \in L^2(\mathbb{R}^D, d^D\mathbf{x})$ we have

$$(\mathcal{F}^{-1}\mathcal{F}\phi)(\mathbf{x}) = \phi(\mathbf{x}) \quad \text{and} \quad \|\phi\|_{L^2(\mathbb{R}^D, d^D\mathbf{x})} = \|\mathcal{F}\phi\|_{L^2(\mathbb{R}^D, d^D\mathbf{p})}.$$

It is easy to check that the Fourier Transform on $\mathbb{R}^D$, which is given by

$$(\mathcal{F}\phi)(\mathbf{p}) \equiv \tilde{\phi}(\mathbf{p}) = \frac{1}{(2\pi)^{D/2}} \int_{\mathbb{R}^D} d^D\mathbf{x}\, e^{-i\mathbf{x}\cdot\mathbf{p}}\phi(\mathbf{x}) \tag{2.4}$$

$$(\mathcal{F}^{-1}\tilde{\phi})(\mathbf{x}) = \frac{1}{(2\pi)^{D/2}} \int_{\mathbb{R}^D} d^D\mathbf{p}\, e^{i\mathbf{x}\cdot\mathbf{p}}\tilde{\phi}(\mathbf{p}) \tag{2.5}$$

satisfies the required conditions. The functions $e^{\pm i\mathbf{x}\cdot\mathbf{p}}$ are the eigenfunctions of $\Delta_{\mathbb{R}^D}$, since

$$-\Delta_{\mathbb{R}^D} e^{\pm i\mathbf{x}\cdot\mathbf{p}} = p^2\, e^{\pm i\mathbf{x}\cdot\mathbf{p}}, \tag{2.6}$$

where we have denoted the norm of the momentum vector by p. Since the eigenvalues depend only on the norm, it is more convenient to work with p instead of $\mathbf{p}$. Therefore we write $\mathbf{p} \in \mathbb{R}^D$ as $(p, \omega) \in [0, \infty) \times \mathbb{S}^{D-1}$, so that

$$\tilde{\phi}(\mathbf{p}) = \tilde{\phi}(p, \omega) \in L^2([0, \infty) \times \mathbb{S}^{D-1}, dp\, d\omega)$$

where $d\omega$ is the volume element on $\mathbb{S}^{D-1}$. Moreover the possibility of expressing any $\omega \in \mathbb{S}^{D-1}$ by a vector $p\hat{\omega}$ where $\hat{\omega}$ is an unit vector on $\mathbb{R}^D$ allows us to write

$$E_0(\mathbf{x}; p, \omega) \equiv e^{-i\mathbf{x}\cdot\mathbf{p}} = e^{-ip\mathbf{x}\cdot\hat{\omega}}. \tag{2.7}$$

This means that the eigenvalues of $\mathbb{R}^D$ are degenerate and for each $\omega \in \mathbb{S}^{D-1}$ there is an eigenfunction $E_0(\mathbf{x}; p, \omega)$ with eigenvalue p^2. Finally using the isomorphism of $\mathcal{F}$ we can write Equation 2.3 in momentum space by

$$-(\mathcal{F}\Delta_{\mathbb{R}^D}\mathcal{F}^{-1}\mathcal{F}\phi)(p, \omega) - g(\mathcal{F}\delta^D(\mathbf{x})\mathcal{F}^{-1}\mathcal{F}\phi)(p, \omega) = -\nu^2(\mathcal{F}\phi)(p, \omega), \tag{2.8}$$

which after a straigtforward calculation gives

$$(p^2 + \nu^2)\tilde{\phi}(p, \omega) = \frac{g}{(2\pi)^D} \int_{\mathbb{S}^{D-1}} d\omega \int_0^\infty dp'\, p'^{D-1}\tilde{\phi}(p', \omega). \tag{2.9}$$

Now let us define

$$\mathcal{N} \equiv \int_{\mathbb{S}^{D-1}} d\omega \int_0^\infty dp'\, p'^{D-1}\tilde{\phi}(p', \omega).$$

Therefore Equation 2.9 implies that

$$\tilde{\phi}(p,\omega) = \frac{1}{(2\pi)^D}\frac{\mathcal{N}g}{p^2+\nu^2}. \tag{2.10}$$

If we put this result into the definition of $\mathcal{N}$ and divide both sides by $\mathcal{N}g$ we get

$$\frac{1}{g} = \frac{1}{(2\pi)^D}\int_{\mathbb{S}^{D-1}} d\omega \int_0^\infty dp'\,\frac{p'^{D-1}}{p'^2+\nu^2} = \frac{\mathrm{vol}\left(\mathbb{S}^{D-1}\right)}{(2\pi)^D}\int_0^\infty dp\,\frac{p'^{D-1}}{p'^2+\nu^2}. \tag{2.11}$$

where $\mathrm{vol}\left(\mathbb{S}^{D-1}\right)$ denotes the volume of the unit sphere in $D-1$ dimensions and given by the formula

$$\mathrm{vol}\left(\mathbb{S}^{D-1}\right) = \frac{2\pi^{d/2}}{\Gamma(d/2)}.$$

For $D=1$, Equation 2.11 implies

$$\frac{1}{g} = \frac{2}{2\pi}\int_0^\infty dp'\frac{1}{p'^2+\nu^2} = \frac{1}{2\nu} \tag{2.12}$$

From this we can conclude that there exist only one bound state and its energy is given by

$$E_B = -\nu^2 = -\frac{g^2}{4}. \tag{2.13}$$

This result is in agreement with the one obtained by using standard wave mechanics [1].

Everything went as expected for $D=1$, but the integral in Equation 2.11 diverges for $D\geq 2$. The divergence is logarithmic for $D=2$, linear for $D=3$ and higher for $D\geq 4$. In all cases the divergence is due to the high values of momenta, therefore it is called *ultra-violet divergence*. To see explicitly what this divergence implies, we can put a large but finite upper bound Λ to the integration variable, find an expression for the bound state in terms of Λ and finally take the $\Lambda\to\infty$ limit. For $D=2$, Equation

2.9 with an upper bound Λ reads

$$\frac{1}{g} = \frac{1}{2\pi} \int_0^{\Lambda} dp' \, \frac{p'}{p'^2 + \nu^2} = \frac{1}{4\pi} \log\left(\frac{\Lambda^2 + \nu^2}{\nu^2}\right). \tag{2.14}$$

Then in the $\Lambda \to \infty$ limit the bound state energy becomes

$$E_b = \lim_{\Lambda \to \infty} -\nu^2 = \lim_{\Lambda \to \infty} -\frac{\Lambda^2}{e^{4\pi/g} - 1} \to -\infty.$$

We see that this formulation of the problem predicts a ground state energy which is not bounded from below. The situation is similar for higher dimensions. Moreover this problem is not specific to the bound state solutions. Consider the Lippmann-Schwinger equation for the T-matrix elements at the energy E

$$\langle \mathbf{p}' | \, T_E \, | \mathbf{p} \rangle = \langle \mathbf{p}' | \, V \, | \mathbf{p} \rangle + \left\langle \mathbf{p}' \right| V G^0(E) T_E \left| \mathbf{p} \right\rangle \tag{2.15}$$

where $G^0(E)$ is the free Green's Function given by [28]

$$G^0(E) = \lim_{\epsilon \to 0^+} (EI - H_0 + i\epsilon)^{-1}. \tag{2.16}$$

Using the completeness relation

$$\int_{\mathbb{R}^D} d^D \, \mathbf{q} \, |\mathbf{q}\rangle \, \langle \mathbf{q}| = 1$$

we get

$$\langle \mathbf{p}' | \, T_E \, | \mathbf{p} \rangle = \langle \mathbf{p}' | \, V \, | \mathbf{p} \rangle + \int_{\mathbb{R}^D} d^D \mathbf{q} \, \langle \mathbf{p}' | \, V \, | \mathbf{q} \rangle \, G^0(q^2; E) \, \langle \mathbf{q} | \, T_E \, | \mathbf{p} \rangle \, . \tag{2.17}$$

where

$$G_0(q^2; E) \equiv \left\langle \mathbf{q}' \right| G^0(E) \left| \mathbf{q} \right\rangle = \lim_{\epsilon \to 0^+} (E - q^2 + i\epsilon)^{-1}. \tag{2.18}$$

We choose the normalization of the eigenkets as $\langle \mathbf{p} | \, \mathbf{p}' \rangle = \delta(\mathbf{p} - \mathbf{p}')$ and $\langle \mathbf{x} | \, \mathbf{x}' \rangle = \delta(\mathbf{x} - \mathbf{x}')$. As a result of this convention the momentum-position brakets are given by

$$\langle \mathbf{p} | \, \mathbf{x} \rangle = \frac{1}{(2\pi)^{D/2}} e^{i\mathbf{p}\cdot\mathbf{x}}. \tag{2.19}$$

Therefore the matrix elements of V in momentum space can be found as

$$\langle \mathbf{p} | \, V \, | \mathbf{p}' \rangle = \int_{\mathbb{R}^{2D}} d^D\mathbf{x} \, d^D\mathbf{x}' \; \langle \mathbf{p} | \, \mathbf{x} \rangle \, \langle \mathbf{x} | \, V \, | \mathbf{x}' \rangle \, \langle \mathbf{x}' | \, \mathbf{p}' \rangle = -\frac{g}{(2\pi)^D}. \tag{2.20}$$

Hence

$$\langle \mathbf{p}' | \, T_E \, | \mathbf{p} \rangle = -\frac{g}{(2\pi)^D} \left(1 + \int_{\mathbb{R}^D} d^D\mathbf{q} \, G^0(q^2; E) \, \langle \mathbf{q} | \, T \, | \mathbf{p} \rangle \right). \tag{2.21}$$

This shows that $\langle \mathbf{p}' | \, T_E \, | \mathbf{p} \rangle$ is actually $\mathbf{p}'$-independent. Thus we can replace $\langle \mathbf{q} | \, T_E \, | \mathbf{p} \rangle$ by $\langle \mathbf{p}' | \, T_E \, | \mathbf{p} \rangle$ and take it outside the integral. After this we find

$$\langle \mathbf{p}' | \, T_E \, | \mathbf{p} \rangle = \frac{1}{(2\pi)^D} \left(I(E) - \frac{1}{g} \right)^{-1} \tag{2.22}$$

where

$$I(E) = \int_{\mathbb{R}^D} \frac{d^D\mathbf{q}}{(2\pi)^D} G^0(q^2; E) = \frac{\mathrm{vol}\left(\mathbb{S}^{D-1}\right)}{(2\pi)^D} \lim_{\epsilon \to 0^+} \int_0^\infty dq \, \frac{q^{D-1}}{q^2 - E - i\varepsilon}. \tag{2.23}$$

It is clear that this integral diverges for $D \geq 2$, therefore a scattering solution could not be obtained. Moreover since the cross section in $D-$dimensions is related to $\langle \mathbf{p}' | \, T_E \, | \mathbf{p} \rangle$ by the formula [13]

$$\sigma = \frac{2^D \pi^{3D/2+1}}{\Gamma(D/2)} E^{\frac{D-3}{2}} \left| \langle \mathbf{p}' | \, T_E \, | \mathbf{p} \rangle \right|^2, \tag{2.24}$$

the divergence of Equation 2.23 implies a zero cross section.

We see that solving the $D-$dimensional Dirac Delta potential in $D \geq 2$ dimensions yields unphysical results for both bound states and scattering states. The fact that all these problems arise due to the high values of momenta gives us the hint that we went beyond the domain of the validity of our theory. In turns out that choosing a constant value for the coupling constant which is valid for all energy scales is not possible. So we should modify our theory to obtain physical results.

2.2. Renormalization

The usual renormalization procedure in QFT involves two steps. First step is called *regularization* which means regularizing the theory such that the divergent integrals become finite but dependent on some arbitrary parameters. Most natural regularization scheme is the *cutoff regularization* where a regularizing function is used to suppress high or low degrees of momenta depending on the type of divergence. The regularizing function can be sharp such that it integrates out the unwanted degrees of freedom completely, or it can be smooth in which case the unwanted degrees are just suppressed. There are other regularization schemes such as Pauli-Willars regularization or dimensional regularization [13]. In our problem we will use cutoff regularization, so that regularization corresponds to the following operation

$$\frac{\mathrm{vol}\left(\mathbb{S}^{D-1}\right)}{(2\pi)^D} \int_0^\infty dp' \, \frac{p'^{D-1}}{p'^2+\nu^2} \xrightarrow{Reg.} \frac{\mathrm{vol}\left(\mathbb{S}^{D-1}\right)}{(2\pi)^D} \int_0^\infty dp' \, \rho_\Lambda(p') \frac{p'^{D-1}}{p'^2+\nu^2} \, , \tag{2.25}$$

where $\rho_\Lambda(p')$ is a function which suppress the high degrees of momenta. The forthcoming discussion will be a review of the standard renormalization procedure given in [12–14]. We will consider each dimension separately.

2.2.1. Two Dimensional Case

For $D = 2$, let us choose the step function $\Theta_\Lambda(p)$ defined by

$$\Theta_\Lambda(p) \equiv \begin{cases} 1, & p \leq \Lambda \\ 0, & p > \Lambda \end{cases} \tag{2.26}$$

where $\Lambda \gg 1$, as the regularizing function. After regularization, Equation 2.11 with $D = 2$ becomes

$$\frac{1}{g} = \frac{1}{2\pi} \int_0^\Lambda dp' \, \frac{p'}{p'^2 + \nu^2} = \frac{1}{4\pi} \log \left(\frac{\Lambda^2 + \nu^2}{\nu^2} \right) , \tag{2.27}$$

and we find the bound state energy as

$$E_b(\Lambda) = -\nu^2 = -\frac{\Lambda^2}{e^{4\pi/g} - 1} . \tag{2.28}$$

This expression is finite for a finite Λ, but an expression depending on the cutoff makes no sense since the cutoff is totally arbitrary. Moreover as we have seen previously, if we take the $\Lambda \to \infty$ limit, we find a ground state energy which is not bounded from below. This implies that the regularization alone does not solve the problem. We need another process to obtain an expression for the bound state which is independent of the cutoff. A quick look at the Equation 2.28 tells us that the only way to achieve this is to make the coupling constant g dependent on Λ and divergent in such a way that the bound state energy is finite and independent of Λ. The determination of this coupling constant is the first step in *renormalization*. This is done by redefining the parameter g of the initial Hamiltonian in such a way that it gives the correct answer for an experimentally measured quantity. For instance, let $-\mu^2 < 0$ be the experimentally measured bound state energy of the system. Then the cutoff dependent coupling constant $g(\Lambda)$, called *bare coupling constant*, should be chosen such that

$$-\mu^2 = - \lim_{\Lambda \to \infty} \frac{\Lambda^2}{e^{4\pi/g(\Lambda)} - 1} \tag{2.29}$$

is satisfied. A typical choice would be

$$\frac{1}{g(\Lambda)} = \frac{1}{2\pi}\int_0^\Lambda dp' \, \frac{p'}{p'^2+\mu^2} = \frac{1}{4\pi}\log\left(\frac{\Lambda^2+\mu^2}{\mu^2}\right) \tag{2.30}$$

which clearly satisfies Equation 2.29.

However the renormalization process does not end here. Once we find an expression for the bare coupling constant so that the theory gives the correct answer for an experimentally measured quantity, we need to make sure that with this choice we can get finite and physically acceptable results for other physical observables. If this happens, then the theory is said to be *renormalizable.* We recall that before renormalization, the integral in Equation 2.23 was divergent and as a result we got a zero cross section. Now we check if this choice of $g(\Lambda)$ would cure these problems. Equation 2.22 in regularized form reads

$$\langle \mathbf{p}' | \, T_E^\Lambda \, | \mathbf{p} \rangle = \frac{1}{(2\pi)^2}\left(I^\Lambda(E) - \frac{1}{g(\Lambda)}\right)^{-1} \tag{2.31}$$

where

$$I^\Lambda(E) = \lim_{\epsilon\to 0^+} \frac{1}{2\pi}\int_0^\Lambda dq \, \frac{q}{q^2 - E - i\varepsilon} = \frac{1}{4\pi}\left[\log\left(\frac{\Lambda^2+E}{E}\right) + i\pi\right]. \tag{2.32}$$

If we put this result and the expression for the bare coupling constant as given in Equation 2.30 into Equation 2.31 we find in the $\Lambda \to \infty$ limit

$$\langle \mathbf{p}' | \, T_E \, | \mathbf{p} \rangle = \lim_{\Lambda\to\infty} \langle \mathbf{p}' | \, T_E^\Lambda \, | \mathbf{p} \rangle = \frac{1}{\pi}\frac{1}{\log\left(E/\mu^2\right) + i\pi}, \tag{2.33}$$

which is finite. Then the cross section becomes

$$\sigma = \frac{4\pi^2}{\sqrt{E}}\left[\frac{1}{\log^2(E/\mu^2) + \pi^2}\right], \tag{2.34}$$

which is finite and nonzero. We could also start with an experimentally measured cross section $\sigma_{\rm exp}$, choose a $g(\Lambda)$ such that Equation 2.34 gives $\sigma_{\rm exp}$ and then prove that with this choice of $g(\Lambda)$, the bound state is finite.

Before continuing with the $D = 3$ case we mention an important point which is unique to the $D = 2$ case. Before any kind of renormalization, with our definition of units, $\hbar = 1$ and $m = 1/2$, the non-relativistic Schrödinger equation for the bound state in $D = 2$ is given by

$$\left(-\Delta_{\mathbb{R}^D} - g\delta^2(\mathbf{x})\right)\phi(\mathbf{x}) = -\nu^2\phi(\mathbf{x}). \tag{2.35}$$

Let $[Q]$ denote the dimensionality of the quantity Q in terms of momentum or inverse length, for instance if ν^2 parametrizes the bound state we have $[\nu^2] = 2$. With this notation the dimensionality of the quantities in Equation 2.35 are

$$[\Delta_{\mathbb{R}^D}] = 2 \quad , \quad [\delta^2(\mathbf{x})] = 2 \quad , \quad [\phi(\mathbf{x})] = 1$$

where the dimensionality of $\delta^2(\mathbf{x})$ and $\phi(\mathbf{x})$ are calculated using

$$1 = \int_{\mathbb{R}^2} d^2\mathbf{x}\,\delta^2(\mathbf{x}) \quad \text{and} \quad 1 = \int_{\mathbb{R}^2} d^2\mathbf{x}\;|\phi(\mathbf{x})|^2$$

respectively. After power counting we realize that the coupling constant g has to be dimensionless. This makes the system *scale invariant*, which means that it exhibits no explicit dimensional dynamical parameter at the level of the Lagrangian or Hamiltonian [17]. Because of this, the ground state wavefunction, should have a scaling symmetry given by [18]

$$\phi_E(\mathbf{x}) \to \phi_{\gamma^{-2}E}(\gamma\mathbf{x}).$$

However after renormalization, this symmetry is broken. For a rigorous calculation of the ground state wavefunction, one needs an operator, called *resolvent* which is a very important operator for studying the spectra of linear operators. Below we summarize some of its important properties [29]:

Let A be a linear operator in the Hilbert Space $\mathcal{H}$. The resolvent of A is given by

$$R_A(\zeta) = (A - \zeta)^{-1}, \quad \zeta \in \mathbb{C}$$

and defined at all complex values of ζ at which $(A-\zeta)^{-1}$ exists. The set of all complex numbers ζ at which $R_A(\zeta)$ is a bounded operator defined densely in $\mathcal{H}$ is called the *resolvent set* of A and the complement of the resolvent set is called the *spectrum* of A, denoted by $\sigma(A)$. The spectrum of A is the discrete union of three sets defined as follows:

- *Point Spectrum* $\sigma_p(A)$: the set of all $\zeta \in \sigma(A)$ at which $R_A(\zeta)$ does not exists.
- *Continuous Spectrum* $\sigma_c(A)$: the set of all $\zeta \in \sigma(A)$ at which $R_A(\zeta)$ exists, defined on a dense subset of $\mathcal{H}$, but unbounded.
- *Residual Spectrum* $\sigma_r(A)$: the set of all $\zeta \in \sigma(A)$ at which $R_A(\zeta)$ exists, but not defined on a dense subset of $\mathcal{H}$.

A self-adjoint operator has no residual spectrum [29].

We denote the resolvent of the Hamiltonian by $R(z) = (H - EI)^{-1}$. One can calculate the resolvent for this problem by solving the inhomogeneous equation

$$\left(-\Delta_{\mathbb{R}^2} - g\delta^2(\mathbf{x})\right)\varphi(\mathbf{x}) - E\varphi(\mathbf{x}) = \chi(\mathbf{x}) \tag{2.36}$$

where $\varphi, \chi \in L^2(\mathbb{R}^2, d^2\mathbf{x})$. In momentum space, this equation reads

$$(p^2 - E)\tilde{\varphi}(\mathbf{p}) - \frac{g}{(2\pi)^2}\int_{\mathbb{R}^2} d^2\mathbf{p}'\,\tilde{\varphi}(\mathbf{p}') = \tilde{\chi}(\mathbf{p}). \tag{2.37}$$

After regularization one finds

$$(p^2 - E)\tilde{\varphi}^\Lambda(\mathbf{p}) - \frac{g(\Lambda)}{(2\pi)^2}\Theta_\Lambda(\mathbf{p}) \int_{\mathbb{R}^2} d^2\mathbf{p}'\, \Theta_\Lambda(\mathbf{p}')\tilde{\varphi}^\Lambda(\mathbf{p}') = \tilde{\chi}^\Lambda(\mathbf{p}), \tag{2.38}$$

where $\varphi^\Lambda, \chi^\Lambda \in L^2(\mathbb{R}^2, d^2\mathbf{x})$ and they are defined such that

$$\lim_{\Lambda\to\infty} \varphi^\Lambda = \varphi \quad \text{and} \quad \lim_{\Lambda\to\infty} \chi^\Lambda = \chi.$$

After solving for $\tilde{\varphi}^\Lambda(\mathbf{p})$ we find

$$\tilde{\varphi}^\Lambda(\mathbf{p}) = \frac{\tilde{\chi}^\Lambda(\mathbf{p})}{p^2 - E} + \frac{\Theta_\Lambda(\mathbf{p})}{(2\pi)^2}\frac{N(\Lambda)}{p^2 - E} \tag{2.39}$$

where

$$N(\Lambda) \equiv g(\Lambda) \int_{\mathbb{R}^2} d^2\mathbf{p}'\, \Theta_\Lambda(\mathbf{p}')\tilde{\varphi}^\Lambda(\mathbf{p}'). \tag{2.40}$$

If we put Equation 2.39 into Equation 2.40, we get

$$N(\Lambda) = \frac{1}{(2\pi)^2}\left(\frac{1}{g(\Lambda)} - \int_{\mathbb{R}^2} d^2\mathbf{p}'\, \frac{\Theta_\Lambda(\mathbf{p}')}{p^2 - E}\right)^{-1} \int_{\mathbb{R}^2} d^2\mathbf{p}'\, \Theta_\Lambda(\mathbf{p}')\frac{\tilde{\chi}^\Lambda(\mathbf{p}')}{p'^2 - E} \tag{2.41}$$

Now by plugging this result into Equation 2.39, using Equation 2.30 for $g(\Lambda)$ and finally taking the $\Lambda \to \infty$ limit we find

$$\tilde{\varphi}(\mathbf{p}) = \frac{\chi(\mathbf{p})}{p^2 - E} + \frac{1}{(2\pi)^2}\frac{4\pi}{\log(-E/\mu^2)}\frac{1}{p^2 - E}\int_{\mathbb{R}^2} d^2\mathbf{p}'\, \frac{\tilde{\chi}(\mathbf{p}')}{p'^2 - E}. \tag{2.42}$$

We can directly read the resolvent from this result by using the fact that the resolvent kernel is given by the formula [30]

$$\tilde{\varphi}(\mathbf{p}) = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} d^2\mathbf{p}'\, R(\mathbf{p}, \mathbf{p}'; E)\tilde{\chi}(\mathbf{p}') \tag{2.43}$$

where $R(\mathbf{p},\mathbf{p}';E) \equiv \langle\mathbf{p}|\, R(E)\, |\mathbf{p}'\rangle$. It is given by

$$R(\mathbf{p},\mathbf{p}';E) = \frac{(2\pi)^2\delta^2(\mathbf{p}-\mathbf{p}')}{p^2-E} + \frac{4\pi}{\log(-E/\mu^2)}\frac{1}{p^2-E}\frac{1}{p'^2-E}. \tag{2.44}$$

The bound states correspond to the point spectrum of the resolvent and we see that because of the logarithm, $E=-\mu^2$ is the only point in this spectrum. This implies that the bound state energy is given by $E=-\mu^2$, which should be the case because our definition of $g(\Lambda)$. Since this point is an isolated point of $\sigma(H)$, i.e.

$$\text{there exists a } \varepsilon > 0 \quad \text{such that} \quad \big\{E \mid \big|-\mu^2 - E\big| < \varepsilon\big\} \cap \sigma(H) = \{-\mu^2\},$$

we can find the ground state wavefunction by the following contour integral [31]:

$$\langle\mathbf{p}|\, \mathcal{P}_{-\mu^2}\, |\mathbf{p}'\rangle = \tilde{\phi}^*(\mathbf{p})\tilde{\phi}(\mathbf{p}') = -\frac{1}{2\pi i}\oint_\Gamma dE\, R(\mathbf{p},\mathbf{p}';E) \tag{2.45}$$

where Γ is a small circle with radius $r<\varepsilon$ enclosing the eigenvalue $-\mu^2$ and $\mathcal{P}_{-\mu^2}$ is the projection operator to the subspace corresponding to this eigenvalue. If we expand the logarithm term into a series

$$\frac{1}{\log\left(-\frac{E}{\mu^2}\right)} = \frac{1}{\log\left(1-\frac{E+\mu^2}{\mu^2}\right)} = -\frac{\mu^2}{E+\mu^2} + \frac{1}{2} + \mathcal{O}\left(\frac{\mu^2}{E+\mu^2}\right) \tag{2.46}$$

we see that the only first term has a simple pole at $E=-\mu^2$. Thus Equation 2.45 becomes

$$\begin{aligned}
\langle\mathbf{p}|\, \mathcal{P}_{-\mu^2}\, |\mathbf{p}'\rangle = \tilde{\phi}^*(\mathbf{p})\tilde{\phi}(\mathbf{p}') &= -\mathrm{Res}\left(-\frac{4\pi\mu^2}{E+\mu^2}\frac{1}{p^2-E}\frac{1}{p'^2-E}\right)_{E=-\mu^2} \\
&= -\lim_{E\to-\mu^2}(E+\mu^2)\left(\frac{-4\pi\mu^2}{E+\mu^2}\frac{1}{p^2-E}\frac{1}{p'^2-E}\right) \\
&= 4\pi\mu^2\frac{1}{p^2+\mu^2}\frac{1}{p'^2+\mu^2}
\end{aligned} \tag{2.47}$$

From this result we can directly read the ground state wavefunction as

$$\tilde{\phi}(\mathbf{p}) = \frac{\sqrt{4\pi\mu^2}}{p^2 + \mu^2}. \tag{2.48}$$

We can return to the position space by using the inverse Fourier Transform. The result is

$$\phi(\mathbf{x}) = (\mathcal{F}^{-1}\tilde{\phi})(\mathbf{x}) = \frac{\sqrt{4\pi\mu^2}}{(2\pi)} \int_{\mathbb{R}^2} d^D\mathbf{p}\, \frac{e^{i\mathbf{p}\cdot\mathbf{x}}}{p^2 + \mu^2} = \sqrt{4\pi\mu^2} K_0(\mu\,|\mathbf{x}|), \tag{2.49}$$

where K_0 is the modified Bessel function of the second kind. Although $K_0(\mu\,|\mathbf{x}|)$ is singular at the origin, it is square integrable, i.e. $\phi(\mathbf{x}) \in L^2(\mathbb{R}^2, d^2\mathbf{x})$.

This expression clearly breaks the scaling symmetry of the Hamiltonian. To renormalize the theory, we have used the measured ground state energy as an input, which means that we added a scale to the system and as a result of this, the scale invariance is broken. Because of this fact, the two-dimensional Dirac-Delta potential is one of the simplest examples of quantum mechanical symmetry breaking [18].

2.2.2. Three Dimensional Case

If we perform the dimensional analysis as in the two dimensional case, we find $[g] = -1$. Therefore we say that the problem has a natural scale and the system is not scale invariant. For $D = 3$ Equation 2.11 with the regularization above becomes

$$\frac{1}{g(\Lambda)} = \frac{4\pi}{(2\pi)^3} \int_0^{\Lambda} dp' \, \frac{p'^2}{p'^2 + \nu^2} = \frac{1}{2\pi^2} \left[\Lambda - \nu \tan^{-1}\left(\frac{\Lambda}{\nu}\right) \right]. \tag{2.50}$$

Again if $-\mu^2$ is the measured binding energy we can define the bare coupling constant as

$$\frac{1}{g(\Lambda)} = \frac{1}{2\pi^2} \left(\Lambda - \mu \frac{\pi}{2} \right). \tag{2.51}$$

It is easy to check that by putting this definition into Equation 2.50 one gets the bound state energy in the limit $\Lambda \to \infty$ as $E_B = -\mu^2$. Now we check if this definition of $g(\Lambda)$ yields a finite and nonzero result for the cross section. For $D = 3$, Equation 2.22 in regularized form reads

$$\langle \mathbf{p}' | \, T_E^\Lambda \, |\mathbf{p}\rangle = \frac{1}{(2\pi)^3} \left(I^\Lambda(E) - \frac{1}{g(\Lambda)} \right)^{-1} \tag{2.52}$$

where

$$I^\Lambda(E) = \lim_{\epsilon \to 0^+} \frac{1}{2\pi^2} \int_0^\Lambda dq \, \frac{q^2}{q^2 - E - i\varepsilon} = \frac{1}{2\pi^2} \left(\Lambda + i\sqrt{E}\frac{\pi}{2} \right). \tag{2.53}$$

Using the definition Equation 2.51 we find in the $\Lambda \to \infty$ limit

$$\langle \mathbf{p}' | \, T_E \, |\mathbf{p}\rangle = \lim_{\Lambda \to \infty} \langle \mathbf{p}' | \, T_E^\Lambda \, |\mathbf{p}\rangle = \frac{1}{2\pi^2} \frac{1}{i\sqrt{E} + \mu} \tag{2.54}$$

and the cross section becomes

$$\sigma = \frac{4\pi}{\mu^2 + E} \tag{2.55}$$

which is nonzero and finite. By using the arguments from Equation 2.36 to Equation 2.44 we can find the resolvent for the three dimensional case as

$$R(\mathbf{p}, \mathbf{p}'; E) = \frac{(2\pi)^3 \delta^3(\mathbf{p} - \mathbf{p}')}{p^2 - E} - \frac{4\pi}{\mu + i\sqrt{E}} \frac{1}{p^2 - E} \frac{1}{p'^2 - E} \tag{2.56}$$

which has a simple pole at $E = -\mu^2$ as expected. Since this eigenvalue is also isolated, using Equation 2.45 we can find the ground state wavefunction in momentum space by

$$\tilde{\phi}(\mathbf{p}) = \frac{\sqrt{8\pi\mu}}{p^2 + \mu^2}. \tag{2.57}$$

In position space this becomes

$$\phi(\mathbf{x}) = (\mathcal{F}^{-1}\tilde{\phi})(\mathbf{x}) = \frac{\sqrt{8\pi\mu}}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} d^3\mathbf{p}\, \frac{e^{i\mathbf{x}\cdot\mathbf{p}}}{p^2+\mu^2} = 2\pi\sqrt{\mu}\frac{e^{-\mu\|\mathbf{x}\|}}{\|\mathbf{x}\|}. \tag{2.58}$$

Also this function is singular at the origin, but it is square integrable.

2.2.3. Higher Dimensions

Our approach for renormalization was redefining the coupling constant such that it gives finite results for the physical observables. So far we have been successful but as we shall see now this approach cannot be used to renormalize the Dirac-Delta potential for dimensions $D \geq 4$. Instead of solving for bound state, we will show this fact using the expression for the $T-$matrix elements. For $D = 4$, Equation 2.22 in regularized form becomes

$$\langle \mathbf{p}' | \, T^{\Lambda}(E) \, |\mathbf{p}\rangle = \frac{1}{(2\pi)^4}\left(I^{\Lambda}(E) - \frac{1}{g(\Lambda)}\right)^{-1} \tag{2.59}$$

where

$$I^{\Lambda}(E) = \frac{\mathrm{vol}\,(\mathbb{S}^3)}{(2\pi)^4}\lim_{\epsilon\to 0^+}\int_0^{\Lambda} dq\, \frac{q^3}{q^2-E-i\epsilon} = \frac{1}{16\pi^2}\left[\Lambda^2 + E\log\left(\frac{\Lambda^2-E}{E}\right) + i\pi E\right]. \tag{2.60}$$

By a suitable of choice of $g(\Lambda)$ we can cancel the quadratic divergence but the logarithmic divergence is dependent on energy. Therefore it is impossible to find a $g(\Lambda)$ which completely cancels the divergence of $I^{\Lambda}(E)$ for all E. Hence we conclude that $D = 4$ case cannot be renormalized using coupling constant renormalization.

2.3. Flow Equation for the Coupling Constant

We have seen that we cannot choose a constant coupling constant to describe the Dirac-Delta interaction in $D \geq 2$ dimensions. This tells us that the interaction strength should depend on some energy scale and as this scale varies, the strength of the interaction must also vary accordingly. In this section we shall investigate how the coupling constant evolves with respect to the change of the energy scale, in other words we will derive the *flow equations* for the coupling constant. This discussion can also be found in [15].

Recall that in order to get a well-defined theory, we have introduced the bare coupling constant $g(\Lambda)$ which cancels the divergence of the problem and as a result of this we have found finite results for the physical observables. However the bare coupling constant is far from being physical since $g^{-1}(\Lambda)$ diverges in the $\Lambda \to \infty$ limit. Therefore we cannot use it to determine the evolution of the interaction strength. Instead of working with the bare coupling, we define the *renormalized* or *physical* coupling constant at the scale λ by

$$\frac{1}{g_R(\lambda)} \equiv \lim_{\Lambda \to \infty} \left[\frac{1}{g(\Lambda)} - \frac{1}{4\pi} \log\left(\frac{\Lambda^2}{\lambda^2}\right) \right] = \frac{1}{4\pi} \log\left(\frac{\lambda^2}{\mu^2}\right), \quad D = 2 \tag{2.61}$$

and

$$\frac{1}{g_R(\lambda)} \equiv \lim_{\Lambda \to \infty} \left[\frac{1}{g(\Lambda)} - \frac{1}{2\pi^2} \left(\Lambda - \lambda\frac{\pi}{2}\right) \right] = \frac{1}{4\pi}(\lambda - \mu), \quad D = 3. \tag{2.62}$$

Here λ is an arbitrary scale with dimensions of momentum, $[\lambda] = 1$. Unlike μ, λ does not correspond to an experimentally measured quantity and is not a physical observable. We can find the bound state energies in terms of $g_R(\lambda)$ using

$$\frac{1}{g(\Lambda)} = \frac{1}{g_R(\lambda)} + \frac{1}{4\pi} \log\left(\frac{\Lambda^2}{\lambda^2}\right) = \frac{1}{2\pi} \int_0^\Lambda dp' \, \frac{p'}{p'^2 + \nu^2}, \quad D = 2 \tag{2.63}$$

and

$$\frac{1}{g(\Lambda)} = \frac{1}{g_R(\lambda)} + \frac{1}{2\pi^2} \left(\Lambda - \lambda\frac{\pi}{2}\right) = \frac{1}{2\pi^2} \int_0^\Lambda dp' \, \frac{p'^2}{p'^2 + \nu^2}, \quad D = 3. \tag{2.64}$$

The results are

$$E_b = -\lambda^2 \, e^{-4\pi/g_R(\lambda)}, \quad D = 2 \tag{2.65}$$

and

$$E_b = -\left(-\lambda + \frac{4\pi}{g_R(\lambda)}\right)^2, \quad D = 3. \tag{2.66}$$

We note that the bound state energies do not implicitly depend on the scale λ. By using the definitions for the renormalized couplings, Equation 2.61 and Equation 2.62, one can show that in both cases the explicit dependence on λ cancels the implicit dependence on λ through $g_R(\lambda)$ and the bound state energies are given by $E_b = -\mu^2$.

Now we will derive the flow equation for $g_R(\lambda)$. It is determined by the so called $\beta-$*function* defined by

$$\beta(g_R) \equiv \frac{\partial g_R}{\partial \log \lambda} = \lambda \frac{\partial g_R}{\partial \lambda}. \tag{2.67}$$

The derivative is taken with respect to $\log \lambda$ due to the fact that the Renormalization Group is generated by the group of transformations $\lambda \to e^{\gamma} \lambda$. For our problem the $\beta-$functions can be calculated exactly. Using Equation 2.61, we find $\beta(g_R)$ in two dimensions as

$$\beta(g_R) = \lambda \frac{\partial g_R}{\partial \lambda} = -\frac{g_R^2}{2\pi} < 0. \tag{2.68}$$

This result tells us that the physical coupling constant and hence the interaction strength decreases as we increase the energy scale. Hence we say that the Dirac-Delta interaction in two dimensions is *asymptotically free*. If we integrate Equation

2.68 from λ_0 to λ we find

$$g_R(\lambda) = \frac{g_R(\lambda_0)}{1 + \frac{g\lambda_0}{2\pi}\log\left(\frac{\lambda}{\lambda_0}\right)} \tag{2.69}$$

For $D = 3$ it is convenient to work with the dimensionless coupling constant given by

$$\hat{g}_R(\lambda) = \lambda g_R(\lambda). \tag{2.70}$$

The β-function becomes in this case

$$\beta(\hat{g}_R) = \lambda\frac{\partial \hat{g}_R}{\partial \lambda} = \hat{g}_R - \frac{1}{4\pi}\hat{g}_R^2. \tag{2.71}$$

After integrating this equation from λ_0 to λ we find

$$\hat{g}_R(\lambda) = \frac{\hat{g}_R(\lambda_0)}{\frac{1}{4\pi}\hat{g}_R(\lambda_0)\left(1 - \frac{\lambda_0}{\lambda}\right) + \frac{\lambda_0}{\lambda}}. \tag{2.72}$$

Using Equation 2.70 we can return to $g_R(\lambda)$ and this will give us

$$g_R(\lambda) = \frac{g_R(\lambda_0)}{1 + \frac{1}{4\pi}g_R(\lambda_0)(\lambda - \lambda_0)}. \tag{2.73}$$

The equations Equation 2.69 and Equation 2.73 are called *flow equations* for the coupling constant in two- and three dimensions respectively.

3. EXACT RENORMALIZATION GROUP ON THE EUCLID PLANE

In this section, our aim is to arrive Equation 2.69 via *Exact Renormalization Group (ERG)* method. First we will briefly review the concept of ERG. Then we will mention about an application of ERG to renormalization of Hamiltonians and then use it to renormalize the point interaction on the Euclid plane. This part will be mainly based on the lecture notes given by Głazek and Maslowski [26] and form the basis of the rest of this thesis. Finally we will discuss some mathematical concepts regarding the range of renormalizability.

3.1. Introduction to Exact Renormalization Group

The ERG was invented by Ken Wilson in the early 70's to understand the physics behind the renormalization of QFT. Although precise formulation of it is quite difficult, the physical intuition behind it is very simple. It can be formulated by the basic fact that the description of physics changes as we change the scale at which observations are made. For instance, let us consider a gas of molecules. The behavior of the gas at the large scale is governed by the laws of thermodynamics, even though we know that microscopically the interactions between the molecules determine the behavior. In other words, the description of a physical system in terms of degrees of freedom and an action determining how they interact changes with the scale. The ERG is a mathematical formulation of this idea. It allows us to go step by step from the short distance scale to the long distance scale [32].

In order to give a precise mathematical formulation, Wilson introduced the space of theories, which is an infinite dimensional space consisting of all possible interactions [25]. Let us denote this space by $\mathcal{S}$. We start by specifying the system at some high energy scale Λ, called the *bare scale*. The description of this system at that scale is given by the action at that scale and this action corresponds to a point in $\mathcal{S}$. Let $S_\Lambda \in \mathcal{S}$

denote this action. Then we introduce another scale λ, called the *effective scale*, such that $1 \ll \lambda \ll \Lambda$. The ERG procedure consists of integrating out degrees of freedom between these two scales. As a result we obtain the so called *Wilsonian effective action* $S_\lambda \in \mathcal{S}$ which describes the system at the natural scale of the physical problem. The actions at different scales are related by the requirement that the partition function of the system does not change as one changes the scale.

However this integrating out procedure is not performed in a single step. In each step one integrates over an infinitesimal momentum shell, thereby obtains another effective action $S_{\Lambda - d\Lambda} \in \mathcal{S}$. This transformation, which is called a *Renormalization Group (RG) tranformation* creates a trajectory $\Gamma \subset \mathcal{S}$ from S_Λ to S_λ, called *RG trajectory*, which is determined by the condition that the partition function is invariant along this trajectory.

The points which are invariant under RG transformations are called *fixed points*. They act like a source or sink of trajectories depending whether the trajectories are flowing away from it or flowing into it. The fixed points which act like a source are called *ultraviolet (UV) fixed points* and the RG trajectories flowing away from them are called *UV trajectories*. On the other hand, if a fixed point acts like a sink, then it is called an *infrared (IR) fixed point* and the trajectories flowing into it are called *IR trajectories* [33].

After starting from an initial action S_Λ and following a RG trajectory, we arrive to the effective action S_λ which depends on both λ and Λ. If the theory is renormalizable we should be able to define a finite number of renormalized couplings, such that the $\Lambda \to \infty$ limit can safely be taken. This is possible if we could absorb all the Λ-dependent terms into the definitions of renormalized couplings.

3.2. Renormalization of Hamiltonians

Now we will apply the above mentioned ideas of the ERG to our problem. Instead of using effective action, we will use effective Hamiltonians in the space of theories $\mathcal{S}$.

This means that every possible form of the interaction corresponds to a point in $\mathcal{S}$.

In the context of point interactions, the form of the Hamiltonian is completely specified by the form of the coupling constant g. So the space of theories $\mathcal{S}$ can also be interpreted as the space of the coupling constants. Before we begin with the ERG procedure, it is worthwhile to investigate how the renormalized coupling constant at some effective scale is related to the unrenormalized coupling constant. Before any kind of regularization or renormalization, the Schrödinger equation for the bound state can be written as

$$H\,|\phi\rangle = -\nu^2\,|\phi\rangle\,. \tag{3.1}$$

We want to calculate the effective Hamiltonian H^{Λ}_{eff} at some effective scale Λ. This is done by integrating out degrees of freedom above Λ. To this end we introduce the operators $\mathcal{P}$ and $\mathcal{Q}$ which are projections to the subspaces, where the momentum eigenvalue takes the values $0 \leq p \leq \Lambda$ and $p > \Lambda$ respectively. Let us also define $|\phi\rangle_{\mathcal{P}} \equiv \mathcal{P}\,|\phi\rangle$ and $|\phi\rangle_{\mathcal{Q}} \equiv \mathcal{Q}\,|\phi\rangle$. By using $\mathcal{P}+\mathcal{Q}=I$ and $\mathcal{P}\mathcal{Q}=0$ one can split Equation 3.1 as

$$\mathcal{P}H\mathcal{P}\,|\phi\rangle_{\mathcal{P}} + \mathcal{P}H\mathcal{Q}\,|\phi\rangle_{\mathcal{Q}} = -\nu^2\,|\phi\rangle_{\mathcal{P}} \tag{3.2}$$

$$\mathcal{Q}H\mathcal{P}\,|\phi\rangle_{\mathcal{P}} + \mathcal{Q}H\mathcal{Q}\,|\phi\rangle_{\mathcal{Q}} = -\nu^2\,|\phi\rangle_{\mathcal{Q}}\,. \tag{3.3}$$

From Equation 3.3 we find

$$|\phi\rangle_{\mathcal{Q}} = (-\nu^2 - \mathcal{Q}H\mathcal{Q})^{-1}\mathcal{Q}H\mathcal{P}\,|\phi\rangle_{\mathcal{P}}\,. \tag{3.4}$$

If we substitute this result back into Equation 3.2 we get

$$\big(\mathcal{P}H\mathcal{P} + \mathcal{P}H\mathcal{Q}(-\nu^2 - \mathcal{Q}H\mathcal{Q})^{-1}\mathcal{Q}H\mathcal{P}\big)\,|\phi\rangle_{\mathcal{P}} = -\nu^2\,|\phi\rangle_{\mathcal{P}} \tag{3.5}$$

and this implies that the effective Hamiltonian at the scale Λ is given by

$$H_{\text{eff}}^{\Lambda} = \mathcal{P}H\mathcal{P} + \mathcal{P}H\mathcal{Q}(-\nu^2 - \mathcal{Q}H\mathcal{Q})^{-1}\mathcal{Q}H\mathcal{P} \equiv \mathcal{P}H\mathcal{P} + X_\Lambda. \tag{3.6}$$

This result clearly demonstrates why regularization alone does not solve the divergence issue. Although we are working at the effective scale Λ, the effective Hamiltonian contains the X_Λ term and this term, which we call it as *counterterm*, depends on the higher degrees of freedom. And as we shall see now, we will use this counterterm in order to define the effective coupling constant at the effective scale Λ. Let us write the Hamiltonian as $H = H_0 + V$ where H_0 is the free Hamiltonian. We can write Equation 3.5 in momentum space as

$$(p^2+\nu^2)\tilde{\phi}_{\mathcal{P}}(\mathbf{p}) + \int_{\mathbb{R}^D} d^D\mathbf{p}' \langle \mathbf{p}| \mathcal{P}V\mathcal{P} |\mathbf{p}'\rangle \tilde{\phi}_{\mathcal{P}}(\mathbf{p}') + \int_{\mathbb{R}^D} d^D\mathbf{p}' \langle \mathbf{p}| X_\Lambda |\mathbf{p}'\rangle \tilde{\phi}_{\mathcal{P}}(\mathbf{p}') = 0 \tag{3.7}$$

where $\tilde{\phi}_{\mathcal{P}}(\mathbf{p}) = \left\langle \mathbf{p} \middle| \tilde{\phi}_{\mathcal{P}} \right\rangle$. By defining $x_\Lambda(\mathbf{p},\mathbf{p}') \equiv (2\pi)^D \langle \mathbf{p}| X_\Lambda |\mathbf{p}'\rangle$ and using Equation 2.20 we get

$$(p^2+\nu^2)\tilde{\phi}_{\mathcal{P}}(\mathbf{p}) - \frac{1}{(2\pi)^D}\int_{\mathbb{R}^D} d^D\mathbf{p}'\Theta_\Lambda(\mathbf{p})\left(g - x_\Lambda(\mathbf{p},\mathbf{p}')\right)\tilde{\phi}_{\mathcal{P}}(\mathbf{p}') = 0. \tag{3.8}$$

We see that the $g - x_\Lambda(\mathbf{p},\mathbf{p}')$ term plays the role of the effective coupling constant. From now on we denote it by $g_\Lambda(\mathbf{p},\mathbf{p}')$. The counterterm $x_\Lambda(\mathbf{p},\mathbf{p}')$ acts like a correction to the initial theory and by using it we have defined the renormalized coupling constant $g_\Lambda(\mathbf{p},\mathbf{p}')$ at the effective scale Λ. In the next section, we shall say more about the role of the counterterm in renormalizing the theory.

3.3. Applying the ERG Procedure

Finally we are in a position to perform the ERG analysis of our theory. In contrast to the previous case, we choose Λ as the *bare scale* and λ as the *effective scale*. Instead of using $\mathbf{p} \in \mathbb{R}^2$ we will use $(p,\omega) \in [0,\infty) \times \mathbb{S}^1$. Also without loss of generality we will assume that the renormalized coupling constant does not depend on ω so we can safely replace $g_\Lambda(\mathbf{p}, \mathbf{p}')$ by $g_\Lambda(p, p')$. Therefore at the bare scale Λ we can write the following equation:

$$(p^2 + \nu^2)\tilde{\phi}(p, \omega) = \frac{\Theta_\Lambda(p)}{(2\pi)^2} \int_0^\Lambda dp'\, p' g_\Lambda(p, p')\vartheta(p') \tag{3.9}$$

where

$$\vartheta(p) \equiv \int_{\mathbb{S}^1} d\omega\, \tilde{\phi}(p, \omega). \tag{3.10}$$

We remark that we have switched to the unprojected wavefunction $\tilde{\phi}(p, \omega)$ and compensate this change by putting the step function $\Theta_\Lambda(p)$ in front of the integral, which ensures that Equation 3.9 is valid for $p \leq \Lambda$. Following the ERG procedure, we write the analog of Equation 3.9 at the infinitesimally lower scale $\Lambda - d\Lambda$.

$$(p^2 + \nu^2)\tilde{\phi}(p, \omega) = \frac{\Theta_{\Lambda - d\Lambda}(p)}{(2\pi)^2} \int_0^{\Lambda - d\Lambda} dp'\, p' g_{\Lambda - d\Lambda}(p, p')\vartheta(p'). \tag{3.11}$$

We can rewrite Equation 3.9 as

$$(p^2 + \nu^2)\tilde{\phi}(p, \omega) = \frac{\Theta_\Lambda(p)}{(2\pi)^2} \left(\int_0^{\Lambda - d\Lambda} dp'\, p' g_\Lambda(p, p')\vartheta(p') + d\Lambda\, \Lambda\, g_\Lambda(p, \Lambda)\vartheta(\Lambda) \right). \tag{3.12}$$

For $p = \Lambda$ this will give us

$$(\Lambda^2 + \nu^2)\tilde{\phi}(\Lambda, \omega) = \frac{1}{(2\pi)^2} \left(\int_0^{\Lambda - d\Lambda} dp'\, p' g_\Lambda(\Lambda, p')\vartheta(p') + d\Lambda\, \Lambda\, g_\Lambda(\Lambda, \Lambda)\vartheta(\Lambda) \right), \tag{3.13}$$

and from this we can read of $\tilde{\phi}(\Lambda, \omega)$ as

$$\tilde{\phi}(\Lambda, \omega) = \frac{1}{(2\pi)^2(\Lambda^2 + \nu^2)} \int_0^{\Lambda - d\Lambda} dp'\, p' g_\Lambda(\Lambda, p')\vartheta(p') \tag{3.14}$$

where we have ignored the term which is proportional to $d\Lambda$. If we substitute this result into Equation 3.10 and perform the ω integral we find

$$\vartheta(\Lambda) = \frac{1}{(2\pi)(\Lambda^2+\nu^2)} \int_0^{\Lambda - d\Lambda} dp'\, p' g_\Lambda(\Lambda, p')\vartheta(p'). \tag{3.15}$$

Finally we put this result into Equation 3.12 to obtain

$$(p^2+\nu^2)\tilde{\phi}(p,\omega) = \frac{\Theta_\Lambda(p)}{(2\pi)^2} \int_0^{\Lambda - d\Lambda} dp'\, p' \left(g_\Lambda(p,p') + \frac{d\Lambda\,\Lambda}{2\pi(\Lambda^2+\nu^2)} g_\Lambda(p,\Lambda) g_\Lambda(\Lambda,p') \right) \times \vartheta(p'). \tag{3.16}$$

Clearly, we can replace $\Theta_\Lambda(p)$ by $\Theta_{\Lambda - d\Lambda}(p)$ and write

$$(p^2+\nu^2)\tilde{\phi}(p,\omega) = \frac{\Theta_{\Lambda - d\Lambda}(p)}{(2\pi)^2} \int_0^{\Lambda - d\Lambda} dp'\, p' \left(g_\Lambda(p,p') + \frac{d\Lambda\,\Lambda}{2\pi(\Lambda^2+\nu^2)} g_\Lambda(p,\Lambda) g_\Lambda(\Lambda,p') \right) \times \vartheta(p'). \tag{3.17}$$

Now comparing this equation with Equation 3.11 gives us an equation for the coupling constant

$$g_{\Lambda - d\Lambda}(p,p') = g_\Lambda(p,p') + \frac{d\Lambda\,\Lambda}{2\pi(\Lambda^2+\nu^2)} g_\Lambda(p,\Lambda) g_\Lambda(\Lambda,p'), \tag{3.18}$$

which can be put into differential form as

$$-\frac{dg_\Lambda(p,p')}{d\Lambda} = \frac{\Lambda}{2\pi(\Lambda^2+\nu^2)} g_\Lambda(p,\Lambda) g_\Lambda(\Lambda,p'). \tag{3.19}$$

This equation determines the RG trajectory of the coupling constant. To find the effective coupling at the effective scale λ we integrate this from λ to Λ and find

$$g_\lambda(p,p') = g_\Lambda(p,p') + \frac{1}{2\pi} \int_\lambda^\Lambda ds\, \frac{s}{s^2+\nu^2} g_s(p,s) g_s(s,p') \tag{3.20}$$

or

$$g_\lambda(p,p') = g - x_\Lambda(p,p') + \frac{1}{2\pi}\int_\lambda^\Lambda ds\, \frac{s}{s^2+\nu^2} g_s(p,s) g_s(s,p') \tag{3.21}$$

Although this is an ordinary differential equation with three variables and we have one initial condition, there is also the requirement that $g_\lambda(p,p')$ should not depend on Λ when we take the $\Lambda \to \infty$ limit. This can be satisfied by the appropriate choice of the counterterm $x_\Lambda(p,p')$. We try an iteration procedure to obtain a solution.

We start by picking

$$g_\lambda^{(1)}(p,p') = g \quad \text{and} \quad x_\Lambda^{(1)} = 0. \tag{3.22}$$

After substituting these choices to Equation 3.21 we get

$$g_\lambda^{(2)}(p,p') = g - x_\Lambda^{(2)}(p,p') + \frac{g^2}{2\pi}\int_\lambda^\Lambda ds\, \frac{s}{s^2+\nu^2}. \tag{3.23}$$

Since λ was chosen such that $1 \ll \lambda \ll \Lambda$, one can expand the integrand into a series and get

$$g_\lambda^{(2)}(p,p') = g - x_\Lambda^{(2)}(p,p') + \frac{g^2}{2\pi}\int_\lambda^\Lambda ds \left(\frac{1}{s} - \frac{\nu^2}{s^3} + \mathcal{O}\left(\frac{1}{s^5}\right)\right). \tag{3.24}$$

Only first term inside the integral diverges in the $\Lambda \to \infty$ limit, therefore we choose the counterterm as

$$x_\Lambda^{(2)}(p,p') = \frac{g^2}{2\pi}\int_{\lambda_0}^\Lambda \frac{ds}{s} = \frac{g^2}{2\pi}\log\left(\frac{\Lambda}{\lambda_0}\right) \tag{3.25}$$

where λ_0 is an energy scale chosen such that $1 \ll \lambda_0 < \lambda \ll \Lambda$. Then Equation 3.24 becomes

$$g_\lambda^{(2)}(p,p') = g - \frac{g^2}{2\pi}\left(\log\left(\frac{\lambda}{\lambda_0}\right) + c_0^{(2)}(\lambda)\right). \tag{3.26}$$

where

$$c_0^{(2)}(\lambda) = \int_\lambda^\Lambda ds \left(\frac{\nu^2}{s^3} + \mathcal{O}\left(\frac{1}{s^5}\right) \right) \tag{3.27}$$

and represents the finite part of $g_\lambda^{(2)}(p,p')$. Although it depends on Λ, the Λ-dependent terms will go to zero after taking the $\Lambda \to \infty$ limit. We also see that both $x_\Lambda^{(2)}(p,p')$ and $g_\lambda^{(2)}(p,p')$ do not depend on p and p'. Then by induction it is straightforward to see that this condition suggest that both $x_\Lambda^{(n)}(p,p')$ and $g_\lambda^{(n)}(p,p')$ do not depend on p and p' for all n. At the nth iteration we write

$$g_\lambda^{(n+1)} = g - x_\Lambda^{(n+1)} + \frac{1}{2\pi} \int_\lambda^\Lambda ds \, \left(g_s^{(n)}\right)^2 \left(\frac{1}{s} + \mathcal{O}\left(\frac{1}{s^3}\right) \right). \tag{3.28}$$

By successive application of the above iteration procedure it can be shown that $g_s^{(n)}$ is an alternating series of logarithmic terms given by

$$g_s^{(n)} = \sum_{j=0}^{m} (-1)^j \frac{1}{a_j} \frac{g^{j+1}}{(2\pi)^j} \log^j \left(\frac{s}{\lambda_0} \right) + c_0^{(n)}(s). \tag{3.29}$$

Here a_n's are some coefficients and $c_0^{(n)}(s)$ is a power series starting from a term of order ν^2/s^2. Therefore in the $\Lambda \to \infty$ limit, only first term of Equation 3.28 would diverge, so we choose the counterterm as

$$x_\Lambda^{(n+1)} = \frac{1}{2\pi} \int_{\lambda_0}^\Lambda ds \, \left(g_s^{(n)}\right)^2 \frac{1}{s} \tag{3.30}$$

so that

$$g_\lambda^{(n+1)} = g - c_0^{(n+1)}(\lambda) - \frac{1}{2\pi} \int_{\lambda_0}^\lambda ds \, \left(g_s^{(n)}\right)^2 \frac{1}{s}. \tag{3.31}$$

To obtain the solution we need to take the $n \to \infty$ limit. It is not so hard to convince ourselves that a limit of $\{c_0^n(\lambda)\}_{n=1}^\infty$ exists, however one cannot claim directly that $\{g_\lambda^{(n)}\}_{n=1}^\infty$ has a limit. Although it is difficult to speculate about the behavior of a_n's as n gets larger, it is clear that, if $\frac{g}{2\pi}\log(\frac{\lambda}{\lambda_0}) > 1$ then $\{g_\lambda^{(n)}\}_{n=1}^\infty$ may diverge very quickly.

We shall deal with this important issue later in this section and for now we assume that g and λ_0 are chosen such that $\{g_\lambda^{(n)}\}_{n=1}^{\infty}$ has a limit as $n \to \infty$. After taking the limit we find

$$g_\lambda = g - c_0(\lambda) - \frac{1}{2\pi} \int_{\lambda_0}^{\lambda} ds\, \frac{g_s^2}{s}. \tag{3.32}$$

For $\lambda = \lambda_0$ one gets

$$g_{\lambda_0} = g - c_0(\lambda_0). \tag{3.33}$$

Now we write g_λ as

$$g_\lambda = g_{\lambda_0} - \frac{1}{2\pi} \int_{\lambda_0}^{\lambda} ds\, \frac{g_s^2}{s}, \tag{3.34}$$

where we have absorbed the remaining λ-dependent term into the definition of g_λ. Now Equation 3.34 can be put into the following form

$$\int_{\lambda_0}^{\lambda} ds\, \frac{dg_s}{ds} = -\frac{1}{2\pi} \int_{\lambda_0}^{\lambda} ds\, \frac{g_s^2}{s}, \tag{3.35}$$

which implies

$$\frac{dg_s}{g_s^2} = -\frac{1}{2\pi}\frac{ds}{s}. \tag{3.36}$$

After integrating this equation from λ_0 to λ and solving for g_λ, we obtain the final answer.

$$g_\lambda = \frac{g_{\lambda_0}}{1 + \frac{g_{\lambda_0}}{2\pi} \log\left(\frac{\lambda}{\lambda_0}\right)} \tag{3.37}$$

This result is identical with the one obtained in Equation 2.69. There we arrived it by correcting the definition of the coupling constant such that the outcomes of the theory agree with the experimentally measured quantities. Here we obtained it by integrating out degrees of freedom between the bare scale Λ and the effective scale λ and ensuring that the coupling constant at the effective scale does not depend on the bare scale Λ, so $\Lambda \to \infty$ limit can safely be taken.

We can check that with the coupling constant given as in Equation 3.36 we get a finite answer for the bound state energy. For this we plug Equation 3.36 into Equation 3.9 to find

$$(p^2+\nu^2)\tilde{\phi}(p,\omega) = \frac{\Theta_\Lambda(p)}{(2\pi)^2}\frac{g_{\lambda_0}}{1+\frac{g_{\lambda_0}}{2\pi}\log\left(\frac{\Lambda}{\lambda_0}\right)}\int_0^\Lambda dp'\, p' \int_{\mathbb{S}^1} d\omega'\, \tilde{\phi}(p',\omega'). \tag{3.38}$$

By defining

$$\mathcal{N} = \int_0^\Lambda dp'\, p' \int_{\mathbb{S}^1} d\omega'\, \tilde{\phi}(p',\omega'), \tag{3.39}$$

we obtain

$$\tilde{\phi}(p,\omega) = \frac{\Theta_\Lambda(p)}{(2\pi)^2}\frac{g_{\lambda_0}}{1+\frac{g_{\lambda_0}}{2\pi}\log\left(\frac{\Lambda}{\lambda_0}\right)}\frac{\mathcal{N}}{p^2+\nu^2}. \tag{3.40}$$

Substituting this result into Equation 3.39 and diving both sides by $\mathcal{N}$ give us

$$1 = \frac{1}{4\pi}\frac{g_{\lambda_0}}{1+\frac{g_{\lambda_0}}{2\pi}\log\left(\frac{\Lambda}{\lambda_0}\right)}\log\left(\frac{\Lambda^2+\nu^2}{\nu^2}\right). \tag{3.41}$$

From this equation we can solve for ν^2 and in the $\Lambda \to \infty$ limit and find

$$E_b = \lim_{\Lambda\to\infty} -\nu^2 = -\lambda_0^2 e^{-4\pi/g_{\lambda_0}} \tag{3.42}$$

which is finite and in agreement with Equation 2.65.

3.4. Estimating the Range of Renormalizability

Previously we have stated that the renormalization may not be possible for any value of g and λ_0 because of the fact that $\{g_\lambda^{(n)}\}_{n=1}^{\infty}$ may diverge if $\frac{g}{2\pi}\log(\frac{\lambda}{\lambda_0}) > 1$. Moreover determining a closed form expression for the coefficients a_n of Equation 3.29 is not possible, hence we could not obtain a rigorous result for the convergence radius of the series $\{g_\lambda^{(n)}\}_{n=1}^{\infty}$ in this way. An alternative way is to investigate under which circumstances does the integral equation given by

$$g_\lambda = g_{\lambda_0} - \frac{1}{2\pi}\int_{\lambda_0}^{\lambda} ds\, \frac{g_s^2}{s} \tag{3.43}$$

has a unique solution. This can be done by using the tools of ordinary differential equations theory. We begin by defining a compact interval $I = [\lambda_0, \tilde{\lambda}] \subset \mathbb{R}$ where $1 \ll \lambda_0 < \lambda < \tilde{\lambda} \ll \Lambda$. Let $C(I)$ denote the space of continuous functions on I. It becomes a vector space if the vector space operations are defined pointwise. Moreover it is well known that it is a Banach Space if we define a norm on $C(I)$ by

$$\|\mathfrak{g}\| = \sup_{s\in I} |g_s| \,. \tag{3.44}$$

We note that we use $\mathfrak{g}$, $\mathfrak{h}$ as elements of $C(I)$ to avoid confusion with the unrenormalized coupling constant g, that is, we made the definition $\mathfrak{g}(s) \equiv g_s$. Now we introduce a map $T : C(I) \to C(I)$ defined by

$$T(\mathfrak{g})(\lambda) = g_{\lambda_0} - \frac{1}{2\pi}\int_{\lambda_0}^{\lambda} ds\, \frac{g_s^2}{s}. \tag{3.45}$$

Then Equation 3.43 can be expressed as $\mathfrak{g} = T\mathfrak{g}$, in other words the solution of Equation 3.43 is also a *fixed point* of T. The existence and uniqueness of a solution to $\mathfrak{g} = T\mathfrak{g}$ can be proved using *Contraction Principle* which can be stated as follows: Let D be a nonempty closed subset of a Banach Space $\mathcal{B}$. If a map $T : D \to \mathcal{B}$ is a contraction and maps D into itself, i.e. $T(D) \subseteq D$, then T has an exactly one fixed point $\overline{\mathfrak{g}}$ which is in D [34]. T is a contraction means that there exist a positive constant $\theta < 1$ such that

$$\|T(\mathfrak{g}) - T(\mathfrak{h})\| \leq \theta \|\mathfrak{g} - \mathfrak{h}\|, \quad \text{for} \quad \mathfrak{g}, \mathfrak{h} \in D. \tag{3.46}$$

If T is a contraction, then the sequence $\{\mathfrak{g}^{(n)}\}_{n=1}^{\infty}$ defined by

$$\mathfrak{g}^{(n)} = T(\mathfrak{g}^{(n-1)}) \quad \text{with} \quad \mathfrak{g}^{(1)} = T(\mathfrak{g}_0), \tag{3.47}$$

where $\mathfrak{g}_0$ is an arbitrary element of D, converges to the fixed point $\overline{\mathfrak{g}}$, that is

$$\lim_{n\to\infty} \|\mathfrak{g}^{(n)} - \overline{\mathfrak{g}}\| = 0. \tag{3.48}$$

Therefore to estimate the range of renormalizability of our theory, we need to estimate under which cases the map T defined as in Equation 3.45 is a contraction. First of all we need a closed subset of $C(I)$. From Equation 3.43 we can conclude that if $\mathfrak{g}$ is a solution, then it should be monotone decreasing on $I = [\lambda_0, \tilde{\lambda}]$. Thus it is natural to choose our closed subset as

$$D = \{\mathfrak{g} \in C(A) \mid \|\mathfrak{g}\| \leq g_{\lambda_0}\}. \tag{3.49}$$

Then

$$\|T(\mathfrak{g})\| = \sup_{\lambda\in I} |T(\mathfrak{g})(\lambda)| = \sup_{\lambda\in I} \left| g_{\lambda_0} - \frac{1}{2\pi}\int_{\lambda_0}^{\lambda} ds\, \frac{g_s^2}{s} \right| = g_{\lambda_0}, \tag{3.50}$$

thus $T(D) \subseteq D$. So it remains to show that T is a contraction. Let $\mathfrak{g}, \mathfrak{h} \in D$. Then we have the estimate

$$\begin{aligned} |T(\mathfrak{g}) - T(\mathfrak{h})| &= \frac{1}{2\pi}\int_{\lambda_0}^{\lambda} ds\, \frac{1}{s}\left((h_s)^2 - (g_s)^2\right) \\ &\leq \frac{1}{2\pi} \sup_{s\in[\lambda_0,\lambda]} \left|(h_s)^2 - (g_s)^2\right| \log\left(\frac{\lambda}{\lambda_0}\right) \\ &= \frac{1}{2\pi} \sup_{s\in[\lambda_0,\lambda]} |(h_s + g_s)(h_s - g_s)| \log\left(\frac{\lambda}{\lambda_0}\right) \end{aligned} \tag{3.51}$$

By taking the supremum of both sides we find

$$
\begin{aligned}
\|T(\mathfrak{g}) - T(\mathfrak{h})\| &\le \frac{1}{2\pi} \sup_{s\in I} |(h_s + g_s)(h_s - g_s)| \log\left(\frac{\tilde{\lambda}}{\lambda_0}\right) \\
&= \frac{1}{2\pi} \|\mathfrak{g} + \mathfrak{h}\| \|\mathfrak{g} - \mathfrak{h}\| \log\left(\frac{\tilde{\lambda}}{\lambda_0}\right) \\
&\le \frac{1}{2\pi} (2g_{\lambda_0}) \|\mathfrak{g} - \mathfrak{h}\| \log\left(\frac{\tilde{\lambda}}{\lambda_0}\right). \qquad (3.52)
\end{aligned}
$$

This tells us that T is a contraction if

$$
\frac{g_{\lambda_0}}{\pi} \log\left(\frac{\tilde{\lambda}}{\lambda_0}\right) < 1. \qquad (3.53)
$$

We recall that we have found g_{λ_0} in Equation 3.33 as $g_{\lambda_0} = g - c_0(\lambda_0)$ and $c_0(\lambda_0)$ was a power series starting from a term of order ν^2/λ_0^2. Since $\lambda_0 \gg 1$, we can take $g \approx g_{\lambda_0}$, therefore this result is consistent with our previous claim that $\{g_\lambda^{(n)}\}_{n=1}^{\infty}$ may diverge if $\frac{g}{2\pi}\log(\frac{\lambda}{\lambda_0}) > 1$. However Equation 3.53 tells us even more. If we interpret the interval $I = [\lambda_0, \tilde{\lambda}]$ as the range of renormalizability, then from Equation 3.53 we can see that it is directly related to the coupling at the energy scale λ_0. For a small coupling $g_{\lambda_0} \ll 1$, we can shift up $\tilde{\lambda}$ considerably without breaking the contraction property of T, however for couplings $g_{\lambda_0} \sim 1$, the range is quite small or we may not even prove the existence of a solution.

4. EXACT RENORMALIZATION GROUP ON THE HYPERBOLIC PLANE

In this section we start by reviewing some key concepts about the geometry of Riemannian manifolds and define the Laplacian operator on them. Then we shall construct the spectral representation of the Laplacian operator on the hyperbolic plane $\mathbb{H}^2$. By using this construction we shall perform the ERG analysis of a point interaction on the hyperbolic plane.

4.1. Laplacian on Riemannian Manifolds

A topological manifold $\mathcal{M}$ of dimension D is a topological Hausdorff space which has a countable basis of open sets and is locally Euclidean. A topological space is Hausdorff if for any pair of points $p, q \in \mathcal{M}$, there exists open neighborhoods $U_p \subset \mathcal{M}$ and $U_q \subset \mathcal{M}$ such that $U_p \cap U_q = \emptyset$. By locally Euclidean we mean that for any $p \in \mathcal{M}$ there exists an open neighborhood $U_p \subset \mathcal{M}$ which is homeomorphic to an open set of $\mathbb{R}^D$. This homeomorphism $\varphi : U_p \to \mathbb{R}^D$ is called a *chart* on $\mathcal{M}$. The pair (U, φ) is said to be a *coordinate neighborhood*. Now let I be an indexing set and consider a family of coordinate neighborhoods $\mathcal{U} = \{(U_i, \varphi_i)\}_{i \in I}$. This collection is called a *smooth atlas* if the following conditions are met:

(i) $\bigcup_{i \in I} U_i = \mathcal{M}$,

(ii) The sets of the form $\varphi_i(U_i \cap U_j)$ for $i, j \in I$ are all open in $\mathbb{R}^D$,

(iii) Whenever $U_i \cap U_j$ for some $i, j \in I$ is not empty, the so called *overlap maps*

$$\varphi_i \circ \varphi_j^{-1} : \varphi_j(U_i \cap U_j) \to \varphi_i(U_i \cap U_j)$$

are C^∞ diffeomorphisms, in other words they can be differentiated infinitely many times.

Two different smooth atlases $\mathcal{U}_1$ and $\mathcal{U}_2$ on $\mathcal{M}$ are considered to be equivalent if $\mathcal{U}_1 \cup \mathcal{U}_2$ is another smooth atlas on $\mathcal{M}$. An equivalence class of smooth atlases is called a *smooth differentiable structure*. A topological manifold with a smooth differentiable structure is called a *smooth differentiable manifold*. We always assume that the manifolds we are working with are smooth, in other words, whenever we mention a manifold, it should be understood as smooth and differentiable.

Consider a point $p \in \mathcal{M}$ and let $C^\infty(p)$ denote the set of all functions which are infinitely differentiable at p. The *tangent space* $T_p\mathcal{M}$ to $\mathcal{M}$ at p is defined to be the set of all mappings $v_p : C^\infty(p) \to \mathbb{R}$ satisfying for all $\alpha, \beta \in \mathbb{R}$ and $f, h \in C^\infty(p)$, the following two conditions:

(i) Linearity: $v_p(\alpha f + \beta h) = \alpha(v_p f) + \beta(v_p h)$,

(ii) Leibniz Rule: $v_p(fh) = (v_p f)h(p) + f(p)(v_p h)$.

$T_p\mathcal{M}$ becomes a vector space if the vector space operations, addition and multiplication with a scalar, are defined pointwise. The elements of $T_p\mathcal{M}$ are called *tangent vectors to* $\mathcal{M}$ *at* p. If (U, φ) is the coordinate neighborhood corresponding to $p \in \mathcal{M}$, then for each $p \in U$, the collection $\{\frac{\partial}{\partial x^i}|_p\}_{i=1}^D$ defined by

$$\frac{\partial}{\partial x^i}\Big|_p f = \partial_i (f \circ \varphi^{-1})|_{\varphi(p)} \tag{4.1}$$

form a basis for $T_p\mathcal{M}$ [35]. In this definition f is C^∞ and ∂_i denotes the partial derivative with respect to ith coordinate of the function $f \circ \varphi^{-1}$. Using this basis, each $v_p \in T_p\mathcal{M}$ can be written as

$$v_p = \sum_{i=1}^{D} \alpha^i \frac{\partial}{\partial x^i}\Big|_p \tag{4.2}$$

Since $T_p\mathcal{M}$ is a vector space, there exists a dual space $T_p^*\mathcal{M}$, which is called the *cotangent space.* $T_p^*\mathcal{M}$ has a dual basis $\{dx^i|_p\}$ defined through

$$dx^i|_p \left(\frac{\partial}{\partial x^i}\bigg|_p \right) = \delta^i_j. \tag{4.3}$$

The *tangent-* and *cotangent bundles* of a manifold $\mathcal{M}$ are the disjoint unions of the tangent- and cotangent spaces respectively. Formally they are given by

$$TM = \bigcup_{p\in\mathcal{M}} T_p\mathcal{M} \quad \text{and} \quad T^*\mathcal{M} = \bigcup_{p\in\mathcal{M}} T_p^*\mathcal{M} \tag{4.4}$$

and are smooth manifolds too. A *smooth vector field* v on $\mathcal{M}$ is a smooth map $v : \mathcal{M} \to T\mathcal{M}$, which assigns to each $p \in \mathcal{M}$ an element of $T_p\mathcal{M}$, i.e. $v(p) \equiv v_p \in T_p\mathcal{M}$ for all $p \in \mathcal{M}$. On a coordinate neighborhood (U, φ), we can define a set of D vector fields $\{\frac{\partial}{\partial x^i}\}_{i=1}^D$ by

$$\frac{\partial}{\partial x^i}(p) \equiv \frac{\partial}{\partial x^i}\bigg|_p, \tag{4.5}$$

and this collection becomes a *coordinate frame field* on U, which means that we can express any $p \in U$ using this basis and write

$$v(p) = v_p = \sum_{i=1}^{D} v^i(p) \frac{\partial}{\partial x^i}\bigg|_p, \tag{4.6}$$

where $v^i(p)$'s are C^∞ functions on U. Therefore on a coordinate neighborhood (U, φ) we can write any vector field v as

$$v|_U = \sum_{i=1}^{D} v^i \frac{\partial}{\partial x^i}. \tag{4.7}$$

Since each point of a manifold will be at least in one coordinate neighborhood, one usually drops the restriction symbol and write

$$v = \sum_{i=1}^{D} v^i \frac{\partial}{\partial x^i}. \tag{4.8}$$

This expression is legitimate as long as we keep in mind that it is valid only on individual coordinate patches.

A *Riemannian metric* g is a symmetric and positive-definite *2-tensor field* on $\mathcal{M}$. A *2-tensor* is a map $g_p : T_p\mathcal{M} \times T_p\mathcal{M} \to \mathbb{R}$ and g, being a 2-tensor field, assigns to each $p \in \mathcal{M}$ a 2-tensor given by $g(p) \equiv g_p$. If $v_p, w_p \in T_p\mathcal{M}$, then symmetry property implies that $g(v_p, w_p) = g(w_p, v_p)$, while positive-definiteness property implies that $g(v_p, v_p) \geq 0$ and is equal to 0 if and only if $v_p = 0$. A Riemannian metric g determines an inner product on each tangent space, therefore one uses $\langle v_p, w_p \rangle$ instead of $g(v_p, w_p)$. A manifold $\mathcal{M}$ with a Riemannian metric g is called a *Riemannian manifold* and is denoted by the pair $(\mathcal{M}, g)$. On a coordinate neighborhood, by using the coordinate frame $\{dx^i\}_{i=1}^D$ dual to $\{\frac{\partial}{\partial x^i}\}_{i=1}^D$, we can write the metric g as

$$g_p = \sum_{i,j=1}^{D} g_{ij}(p) dx^i|_p \otimes dx^j|_p \quad \text{or} \quad g = \sum_{i,j=1}^{D} g_{ij} dx^i \otimes dx^j, \tag{4.9}$$

where $g_{ij}(p) = \langle \frac{\partial}{\partial x^i}|_p, \frac{\partial}{\partial x^j}|_p \rangle$ is C^∞ on U.

If $\gamma : \mathcal{I} \to \mathcal{M}$ is a C^∞ curve parametrized by t, where $\mathcal{I} = (a, b)$ is an open interval of $\mathbb{R}$, then for a given $t_0 \in \mathcal{I}$, $\frac{d}{dt}$ taken at t_0 can be taken as a basis for $T_{t_0}(\mathcal{I})$. Given $p = \gamma(t_0)$, $v_p \in T_p(\mathcal{M})$ and $f \in C^\infty(p)$, the *directional derivative of f at p in the direction v_p* is defined by

$$D_{v_p} f = \frac{d}{dt}(f \circ \gamma)\big|_{t_0}. \tag{4.10}$$

This definition is independent of the curve γ as long as $\gamma(t_0) = p$ and $\frac{d\gamma}{dt} = v_p$. Given a C^∞ function f on $\mathcal{M}$, the *gradient* of f, denoted by grad f, is defined such that

$$\langle \text{grad } f, v_p \rangle = D_{v_p} f \tag{4.11}$$

holds for all $v_p \in T_p\mathcal{M}$. For differentiation of vector fields, one uses the concept of *connection*, which is a rule that assigns to each $p \in \mathcal{M}$, $v_p \in T_p\mathcal{M}$ and vector field w defined on a neighborhood of p; a tangent vector $\nabla_{v_p} w \in T_p\mathcal{M}$ satisfying

(i) $\nabla_{v_p}(w + \tilde{w}) = \nabla_{v_p} w + \nabla_{v_p} \tilde{\omega}$,

(ii) $\nabla_{v_p}(fw) = (D_{v_p} f) w_p + f(p) \nabla_{v_p} w$.

Here $\omega, \tilde{\omega}$ are C^∞ vector fields and f is a C^∞ function, all defined on a neighborhood of p. The vector $\nabla_{v_p} w$ is also called the *covariant derivative of* w *with respect to* v_p.

Given a smooth vector field w on $\mathcal{M}$, the *divergence* of w is a C^∞ function, denoted by div w and defined by

$$(\text{div } w) = \text{Tr}(v_p \mapsto \nabla_{v_p} w) \tag{4.12}$$

where Tr denote the trace. Finally given a C^∞ function f on $\mathcal{M}$, the *Laplacian* of f, Δf, is defined by

$$\Delta f = \text{div}(\text{grad } f). \tag{4.13}$$

In local coordinates, the Laplacian of f can be expressed by the formula [36],

$$\Delta_g f = \frac{1}{\sqrt{\det g}} \sum_{i,j=1}^{D} \frac{\partial}{\partial x^i} \left(g^{ij} \sqrt{\det g} \frac{\partial}{\partial x^j} f \right) \tag{4.14}$$

The operator Δ_g is also called the *Laplace-Beltrami Operator* for $(\mathcal{M}, g)$ and defines a non-negative symmetric operator on the dense set $C_0^\infty(\mathcal{M})$ in the Hilbert space

$L^2(\mathcal{M}, dV_g)$, where dV_g is the *volume form*, which we shall define soon. Here $C_0^\infty(\mathcal{M})$ denotes the linear vector space of all complex-valued smooth functions of compact support on $\mathcal{M}$ [27].

Before we finish our review of Riemannian Geometry, we shall mention some concepts about integration on Riemannian manifolds. An *r-form* Ω on a manifold $\mathcal{M}$ is an alternating covariant tensor field of order r, which assigns to each $p \in \mathcal{M}$ an r-tensor

$$\Omega_p : \underbrace{T_p\mathcal{M} \times \cdots \times T_p\mathcal{M}}_{r \text{ times}} \to \mathbb{R}, \tag{4.15}$$

that changes sign whenever two arguments are interchanged. Let $\bigwedge^r(T_p\mathcal{M})$ denote the space of alternating $r-$tensors on $T_p\mathcal{M}$. On the space $\bigoplus_{r=0}^{\infty} \bigwedge^r(T_p\mathcal{M})$, there is a bilinear and associative operation called *wedge product*, which is a mapping from $\bigwedge^r(T_p\mathcal{M}) \times \bigwedge^s(T_p\mathcal{M}) \to \bigwedge^{r+s}(T_p\mathcal{M})$, defined by

$$(\Omega, \Upsilon) \mapsto \Omega \wedge \Upsilon \equiv \frac{(r+s)!}{r!s!} \mathscr{A}(\Omega \otimes \Upsilon), \tag{4.16}$$

where $\mathscr{A}$ is the alternating mapping.

A manifold $\mathcal{M}$ with $\dim(\mathcal{M}) = D$ is *orientable* if it is possible to define a D-form Ω on $\mathcal{M}$ which is not zero at any point. In that case $\mathcal{M}$ is said to *oriented* by the choice of Ω. If $\mathcal{M}$ is an orientable Riemannian manifold with the Riemannian metric g, then there exist a unique D-form dV_g, which gives the orientation and has the value $+1$ on every oriented orthonormal frame [37]. It is called the *volume element* of the oriented Riemannian manifold and given in local coordinates by

$$dV_g = \sqrt{\det g}\, dx^1 \wedge \cdots \wedge dx^D. \tag{4.17}$$

Using dV_g we can integrate any smooth and compactly supported function f on an oriented Riemannian manifold $\mathcal{M}$ via

$$\int_{\mathcal{M}} dV_g \, f.$$

Then the *volume* of the Riemannian manifold $\mathcal{M}$ is given by

$$\operatorname{vol}(\mathcal{M}) = \int_{\mathcal{M}} dV_g. \tag{4.18}$$

4.2. The Spectra of the Hyperbolic Plane

To proceed with our quantum mechanical problem on the hyperbolic plane $\mathbb{H}^2$, we need to investigate the spectra of the Laplacian on this space, that is, we need to construct the spectral representation of $\Delta_{\mathbb{H}^2}$. We shall do this by using ideas given in [27] and [38]. There are various models for the hyperbolic plane. We will use the *upper half-plane* model, where $\mathbb{H}^2$ is realized as the set

$$\mathbb{H}^2 = \{z = (x, y) \mid x \in \mathbb{R}\,, y \in [0, \infty)\}\,, \tag{4.19}$$

with the Riemannian metric $g_{\mathbb{H}^2}$ given by

$$g_{\mathbb{H}^2} = \frac{R^2}{y^2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{4.20}$$

where $-R^{-2}$ is the constant sectional curvature. From Equation 4.17 we can find the Riemannian volume element as

$$dV_{\mathbb{H}^2} = \sqrt{\det g_{\mathbb{H}^2}}\, dx \wedge dy = \frac{dx\, dy}{y^2/R^2}. \tag{4.21}$$

The Laplacian on $\mathbb{H}^2$ can be found from Equation 4.14 as

$$\Delta_{\mathbb{H}^2} = \frac{y^2}{R^2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right). \tag{4.22}$$

The eigenfunctions can be found by solving the closed eigenvalue problem on $L^2(\mathbb{H}^2, dV_{\mathbb{H}^2})$ expressed by

$$(\Delta_{\mathbb{H}^2} + \lambda) f(z) = 0, \tag{4.23}$$

where $\lambda \in \mathbb{R}$. For notational simplicity, let us define $\tilde{\Delta}_{\mathbb{H}^2} \equiv R^2\Delta_{\mathbb{H}^2}$ and $\tilde{\lambda} \equiv R^2\lambda$. Then Equation 4.23 will be equivalent to

$$(\tilde{\Delta}_{\mathbb{H}^2} + \tilde{\lambda}) f(z) = 0, \tag{4.24}$$

Since $\tilde{\Delta}_{\mathbb{H}^2}$ is separable in (x, y) coordinates we can use separation of variables. So we choose $f(z) = v(x)w(y)$ and put this into Equation 4.24 to obtain

$$\frac{\partial^2 v}{\partial x^2}\frac{1}{v(x)} + \frac{\partial^2 w}{\partial y^2}\frac{1}{w(y)} + \frac{\tilde{\lambda}}{y^2} = 0. \tag{4.25}$$

This implies that there is a constant ξ^2 such that

$$\frac{\partial^2 v}{\partial x^2}\frac{1}{v(x)} = -\xi^2 \quad \text{and} \quad \frac{\partial^2 w}{\partial y^2}\frac{1}{w(y)} + \frac{\tilde{\lambda}}{y^2} = \xi^2. \tag{4.26}$$

The x-part can be solved easily as $v(x) = e^{i\xi x}$. To solve the y-part we introduce a new function by $u(y) \equiv y^{-1/2}w(y)$. After substituting this into the y-part of Equation 4.26 and making some rearrangements we get

$$y^2\frac{\partial^2 u}{\partial y^2} + y\frac{\partial u}{\partial y} - \left(y^2\xi^2 + \frac{1}{4} - \tilde{\lambda}\right) u(y) = 0 \tag{4.27}$$

The eigenvalues of the Laplacian on $\mathbb{H}^2$ starts with $\tilde{\lambda}_0 = \frac{1}{4}$ [39]. Therefore $\frac{1}{4} - \tilde{\lambda} \leq 0$ so we introduce a new variable $\tau \in [0, \infty)$ such that $\frac{1}{4} - \tilde{\lambda} = (i\tau)^2$. Then Equation 4.27

becomes

$$y^2\frac{\partial^2 u}{\partial y^2}+y\frac{\partial u}{\partial y}-\left[(y\xi)^2+(i\tau)^2\right]u(y)=0. \tag{4.28}$$

There are two linearly independent solutions which are the modified Bessel Functions $I_{i\tau}(|y\xi|)$ and $K_{i\tau}(|y\xi|)$. Since $I_{i\tau}(|y\xi|)$ is singular at infinity, we exclude it from our solution space. Moreover $K_{i\tau}(|y\xi|)$ has a singularity at $\xi = 0$ given by [38]

$$K_{i\tau}(|y\xi|)\sim 2^{i\tau-1}\Gamma(i\tau)\,|y\xi|^{-i\tau}+2^{-i\tau-1}\Gamma(-i\tau)\,|y\xi|^{i\tau} \quad \text{as } \xi\to 0^+. \tag{4.29}$$

So we choose $u(y)=|\xi|^{i\tau}K_{i\tau}(|y\xi|)$ as a solution to Equation 4.28 and write the eigenfunctions of $\Delta_{\mathbb{H}^2}$ as

$$E_0(z;\tau,\xi)=\frac{1}{\sqrt{2\pi}}e^{i\xi x}\sqrt{y}\,|\xi|^{i\tau}K_{i\tau}(|y\xi|) \tag{4.30}$$

where we have put an extra $(2\pi)^{-1/2}$ to simplify our construction of the spectral representation. We note that this does not alter the spectrum of the eigenvalues.

To obtain the spectral representation of $\Delta_{\mathbb{H}^2}$ we introduce the following transform,

$$\begin{aligned}(\mathcal{K}\psi)(\tau,\xi)\equiv\tilde{\psi}(\tau,\xi)&=\frac{1}{R^2}\int_{\mathbb{H}^2}dV_{\mathbb{H}^2}\,\psi(z)\overline{E_0(z;\tau,\xi)}\\(\mathcal{K}^{-1}\tilde{\psi})(z)&=\frac{2}{\pi^2}\int_0^\infty d\tau\,\tau\sinh(\pi\tau)\int_{\mathbb{R}}d\xi\,\tilde{\psi}(\tau,\xi)E_0(z;\tau,\xi).\end{aligned} \tag{4.31}$$

which is a combination of the Fourier Transform in (x,ξ) variables and Kontorovich-Lebedev Transform in (y,τ) variables [40]. The range of $\mathcal{K}$ can be formulated by considering $L^2(\mathbb{R},d\xi)$ as the Hilbert Space corresponding to ξ, and then taking a direct integral of it with respect to the measure space $(0,\infty)$. So the map $\mathcal{K}$ can be expressed formally as

$$\mathcal{K}:L^2(\mathbb{H}^2,dV_{\mathbb{H}^2})\to\int_{(0,\infty)}^{\oplus}L^2(\mathbb{R},d\xi). \tag{4.32}$$

To prove that $\mathcal{K}$ provide a spectral representation of $\Delta_{\mathbb{H}^2}$, we need two identities involving modified Bessel Functions which are given by [27]

$$\frac{2}{\pi^2}\int_0^\infty d\tau\,\tau\sinh(\pi\tau)K_{i\tau}(u)K_{i\tau}(v) = v\delta(u-v), \tag{4.33}$$

and

$$\frac{2}{\pi^2}\int_0^\infty \frac{du}{u}K_{i\tau}(u)K_{i\tau'}(u) = \frac{\delta(\tau-\tau')}{\sqrt{\tau\tau'}\sqrt{\sinh(\pi\tau)\sinh(\pi\tau')}}. \tag{4.34}$$

Moreover we shall also use the property $K_{i\tau}(u) = K_{-i\tau}(u)$. Using Equation 4.34 one can show that $\mathcal{K}$ diagonalizes the Laplacian, that is for all $\tilde{\psi} \in \int_{(0,\infty)}^{\oplus} L^2(\mathbb{R}, d\xi)$ we have

$$-(\mathcal{K}\tilde{\Delta}_{\mathbb{H}^2}\mathcal{K}^{-1}\tilde{\psi})(\tau,\xi) = \left(\tau^2+\frac{1}{4}\right)\tilde{\psi}(\tau,\xi), \tag{4.35}$$

and therefore

$$-(\mathcal{K}\Delta_{\mathbb{H}^2}\mathcal{K}^{-1}\tilde{\psi})(\tau,\xi) = \frac{1}{R^2}\left(\tau^2+\frac{1}{4}\right)\tilde{\psi}(\tau,\xi), \tag{4.36}$$

So the spectrum of $\Delta_{\mathbb{H}^2}$ is given by

$$\sigma(\Delta_{\mathbb{H}^2}) = \left[\frac{1}{4R^2},\infty\right). \tag{4.37}$$

By using Equation 4.33 it is straightforward to show that $\mathcal{K}$ is an isomorphism, i.e. for all $\psi \in L^2(\mathbb{H}^2, dV_{\mathbb{H}^2})$ we have

$$(\mathcal{K}^{-1}\mathcal{K}\psi)(z) = \psi(z). \tag{4.38}$$

Therefore, the transformation $\mathcal{K}$ with the eigenfunctions given as in Equation 4.30 provide a complete spectral representation of $\Delta_{\mathbb{H}^2}$.

4.3. Exact Renormalization Group Analysis

4.3.1. Formulation of the Problem

Just like the flat case we consider a particle of mass m interacting with a Dirac-Delta potential on the hyperbolic plane $\mathbb{H}^2$. Let $z_0 = (x_0, y_0), y_0 \neq 0$ denote the location of the Dirac-Delta potential. The corresponding Schrödinger equation for the bound state is written in coordinates $\hbar = 1, m = 1/2$ as

$$(-\Delta_{\mathbb{H}^2} - g\delta_{\mathbb{H}^2}(z, z_0))\,\phi(z) = -\nu^2\phi(z). \tag{4.39}$$

On a Riemannian manifold $(\mathcal{M}, g)$ the Dirac Delta function $\delta_g(z, z_0)$ is defined such that

$$\int_{\mathcal{M}} dV_g(z)\,\delta_g(z, z_0) = 1 \quad \text{for all} \quad z_0 \in \mathcal{M}. \tag{4.40}$$

Thus $\delta_{\mathbb{H}^2}(z, z_0)$ is given by

$$\delta_{\mathbb{H}^2}(z, z_0) = \frac{y^2}{R^2}\delta(x, x_0)\delta(y, y_0). \tag{4.41}$$

Since $\mathcal{K}$ is an isomorphism we can write Equation 4.39 as

$$-\mathcal{K}\Delta_{\mathbb{H}^2}\mathcal{K}^{-1}\tilde{\phi}(\tau, \xi) - g\mathcal{K}\delta_{\mathbb{H}^2}(z, z_0)\mathcal{K}^{-1}\tilde{\phi}(\tau, \xi) = -\nu^2\tilde{\phi}(\tau, \xi). \tag{4.42}$$

First term can be read directly from Equation 4.35, which is

$$-\mathcal{K}\Delta_{\mathbb{H}^2}\mathcal{K}^{-1}\tilde{\phi}(\tau, \xi) = \frac{1}{R^2}\left(\tau^2 + \frac{1}{4}\right)\tilde{\phi}(\tau, \xi). \tag{4.43}$$

Second term can be easily calculated using $\delta_{\mathbb{H}^2}(z, z_0)$. The result is

$$g\mathcal{K}\delta_{\mathbb{H}^2}(z, z_0)\mathcal{K}^{-1}\tilde{\phi}(\tau, \xi) = \frac{2g}{\pi^2 R^2}\int_0^\infty d\tau'\, \tau' \sinh(\pi\tau') \int_{\mathbb{R}} d\xi'\, \overline{E_0(z_0, \tau, \xi)} E_0(z_0, \tau', \xi') \times \tilde{\phi}(\tau', \xi'). \tag{4.44}$$

If we put Equation 4.43 and Equation 4.44 into the Equation 4.39 and rearrange the terms we obtain

$$\left(\tau^2 + \frac{1}{4} + \nu^2 R^2\right)\tilde{\phi}(\tau, \xi) = \frac{2g}{\pi^2}\int_0^\infty d\tau'\, \tau' \sinh(\pi\tau') \int_{\mathbb{R}} d\xi'\, \overline{E_0(z_0, \tau, \xi)} E_0(z_0, \tau', \xi') \times \tilde{\phi}(\tau', \xi'). \tag{4.45}$$

Our next goal is to determine the type and the cause of the divergence. To this end we make an attempt to solve Equation 4.45. We define

$$\mathcal{N} \equiv \int_0^\infty d\tau'\, \tau' \sinh(\pi\tau') \int_{\mathbb{R}} d\xi'\, E_0(z_0, \tau', \xi')\tilde{\phi}(\tau', \xi'). \tag{4.46}$$

Then $\tilde{\phi}(\tau, \xi)$ becomes

$$\tilde{\phi}(\tau, \xi) = \frac{2g}{\pi^2}\mathcal{N}\overline{E_0(z_0, \tau, \xi)}\left(\tau^2 + \frac{1}{4} + \nu^2 R^2\right)^{-1}. \tag{4.47}$$

We put this result back into Equation 4.46 and find

$$\frac{1}{g} = \frac{2}{\pi^2}\int_0^\infty d\tau'\, \tau' \sinh(\pi\tau') \left(\tau'^2 + \frac{1}{4} + \nu^2 R^2\right)^{-1} \int_{\mathbb{R}} d\xi'\, \overline{E_0(z_0, \tau', \xi')} E_0(z_0, \tau', \xi'). \tag{4.48}$$

Let us denote the ξ-integral by $\Upsilon(\tau')$. Using the explicit form of the eigenfunctions as given in Equation 4.30 we can write $\Upsilon(\tau')$ as

$$\begin{aligned}\Upsilon(\tau') &= \frac{y_0}{2\pi}\int_{\mathbb{R}} d\xi'\, K_{i\tau'}(y_0\,|\xi|)K_{-i\tau'}(y_0\,|\xi|)\\ &= \frac{y_0}{\pi}\int_0^\infty d\xi'\, K_{i\tau'}(y_0\xi)K_{i\tau'}(y_0\xi)\end{aligned} \tag{4.49}$$

To evaluate the integral we will use the following integral representation of the modified Bessel functions [41]:

$$K_\nu(z) = \int_0^\infty du\, e^{-z\cosh u}\cosh(\nu u), \quad \operatorname{Re} z > 0 \tag{4.50}$$

Therefore $\Upsilon(\tau')$ becomes

$$\begin{aligned}\Upsilon(\tau') &= \frac{y_0}{\pi}\int_0^\infty d\xi \int_0^\infty du \int_0^\infty dv\, e^{-y_0\xi(\cosh u+\cosh v)}\cosh(i\tau' u)\cosh(i\tau' v)\\ &= \frac{y_0}{\pi}\int_0^\infty du \int_0^\infty dv\cos(\tau' u)\cos(\tau' v)\int_0^\infty d\xi\, e^{-y_0\xi(\cosh u+\cosh v)}\\ &= \frac{1}{\pi}\int_0^\infty du\,\cos(\tau' u)\int_0^\infty dv\frac{\cos(\tau' v)}{\cosh u+\cosh v}.\end{aligned} \tag{4.51}$$

The dv integral can be evaluated using the definite integral [42]

$$\int_0^\infty \frac{\cos(ax)dx}{b\cosh(\beta x)+c} = \frac{\pi\sin\left(\frac{a}{\beta}\cosh^{-1}\left(\frac{c}{b}\right)\right)}{\beta\sqrt{c^2-b^2}\sinh\left(\frac{a\pi}{\beta}\right)}, \qquad \text{for} \quad c > b > 0. \tag{4.52}$$

In our case $a = \tau'$, $b = 1$, $\beta = 1$, $c = \cosh u$ and $\cosh u \geq 1 > 0$, for all $u \in [0,\infty)$ so we can use Equation 4.52. Hence dv integral becomes

$$\int_0^\infty dv\frac{\cos(\tau' v)}{\cosh u+\cosh v} = \frac{\pi\sin\left(\tau'\cosh^{-1}(\cosh u)\right)}{\sqrt{\cosh^2 - 1}\sinh(\tau'\pi)} = \frac{\pi\sin(\tau' u)}{\sinh(u)\sinh(\tau'\pi)}. \tag{4.53}$$

Then we have

$$\Upsilon(\tau') = \frac{1}{\sinh(\tau'\pi)}\int_0^\infty du\,\frac{\cos(\tau' u)\sin(\tau' u)}{\sinh u}. \tag{4.54}$$

Finally we will use [42]

$$\int_0^\infty dx\, \frac{\sin(\alpha x)\cos(\beta x)}{\sinh(\gamma x)} = \frac{\pi \sinh\left(\frac{\pi a}{\gamma}\right)}{2\gamma\left(\cosh\left(\frac{\alpha\pi}{\gamma}\right) + \cosh\left(\frac{\beta\pi}{\gamma}\right)\right)} \tag{4.55}$$

for $\mathrm{Im}\,(\alpha+\beta) < \mathrm{Re}\,\gamma$. In our case $\alpha = \beta = \tau'$, $\gamma = 1$ and $\tau' \in \mathbb{R}$, therefore $\mathrm{Im}\,(\alpha+\beta) = 0 < 1$. Thus

$$\int_0^\infty du\, \frac{\cos(\tau' u)\sin(\tau' u)}{\sinh u} = \frac{\pi \sinh(\pi\tau')}{2\left(\cosh(\pi\tau') + \cosh(\pi\tau')\right)} = \frac{\pi}{4}\tanh(\pi\tau') \tag{4.56}$$

and $\Upsilon(\tau')$ becomes

$$\Upsilon(\tau') = \int_{\mathbb{R}} d\xi'\, \overline{E_0(z_0,\tau',\xi')} E_0(z_0,\tau',\xi') = \frac{\pi}{4}\frac{\tanh(\pi\tau')}{\sinh(\pi\tau')}. \tag{4.57}$$

Finally we put this result into Equation 4.48 and obtain

$$\frac{1}{g} = \frac{1}{2\pi}\int_0^\infty d\tau'\, \tau' \tanh(\pi\tau')\left(\tau'^2 + \frac{1}{4} + \nu^2 R^2\right)^{-1}. \tag{4.58}$$

For large values of τ', $\tanh(\pi\tau') \approx 1$ and the integrand behaves as $1/\tau'$ so just like the flat case we face with a logarithmic divergence. This analysis also shows us that there is no divergence in the ξ term. Therefore we only need to concern with the renormalization of τ.

4.3.2. Applying the ERG Procedure

We start by writing the eigenvalue equation at the bare scale Λ.

$$\left(\tau^2 + \frac{1}{4} + \nu^2 R^2\right)\tilde{\phi}(\tau,\xi) = \frac{2}{\pi^2}\Theta_\Lambda(\tau)\int_0^\Lambda d\tau'\, g_\Lambda(\tau,\tau')\tau' \sinh(\pi\tau')\vartheta(\tau,\tau';\xi), \tag{4.59}$$

where $g_\Lambda(\tau,\tau') = g - x_\Lambda(\tau,\tau')$ and

$$\vartheta(\tau,\tau';\xi) \equiv \int_{\mathbb{R}} d\xi' \, \overline{E_0(z_0,\tau,\xi)} E_0(z_0,\tau',\xi')\tilde{\phi}(\tau',\xi'). \tag{4.60}$$

At an infinitesimally lower scale $\Lambda - d\Lambda$ we write

$$\left(\tau^2 + \frac{1}{4} + \nu^2 R^2\right)\tilde{\phi}(\tau,\xi) = \frac{2}{\pi^2}\Theta_{\Lambda-d\Lambda}(\tau)\int_0^{\Lambda-d\Lambda} d\tau' \, g_{\Lambda-d\Lambda}(\tau,\tau')\tau'\sinh(\pi\tau')\vartheta(\tau,\tau';\xi). \tag{4.61}$$

We can rewrite Equation 4.59 as

$$\begin{aligned}\left(\tau^2 + \frac{1}{4} + \nu^2 R^2\right)\tilde{\phi}(\tau,\xi) = \frac{2}{\pi^2}\Theta_\Lambda(\tau)\Bigg(&\int_0^{\Lambda-d\Lambda} d\tau' \, g_\Lambda(\tau,\tau')\tau'\sinh(\pi\tau')\vartheta(\tau,\tau';\xi) \\ &+ d\Lambda \, g_\Lambda(\tau,\Lambda)\Lambda\sinh(\pi\Lambda)\vartheta(\tau,\Lambda;\xi)\Bigg),\end{aligned} \tag{4.62}$$

and for $\tau = \Lambda$ we obtain

$$\begin{aligned}\left(\Lambda^2 + \frac{1}{4} + \nu^2 R^2\right)\tilde{\phi}(\Lambda,\xi) = \frac{2}{\pi^2}\Bigg(&\int_0^{\Lambda-d\Lambda} d\tau' \, g_\Lambda(\Lambda,\tau')\tau'\sinh(\pi\tau')\vartheta(\Lambda,\tau';\xi) \\ &+ d\Lambda \, g_\Lambda(\Lambda,\Lambda)\Lambda\sinh(\pi\Lambda)\vartheta(\Lambda,\Lambda;\xi)\Bigg).\end{aligned} \tag{4.63}$$

Then $\tilde{\phi}(\Lambda,\xi)$ becomes

$$\tilde{\phi}(\Lambda,\xi) = \frac{2}{\pi^2}\left(\Lambda^2 + \frac{1}{4} + \nu^2 R^2\right)^{-1}\int_0^{\Lambda-d\Lambda} d\tau' \, g_\Lambda(\Lambda,\tau')\tau'\sinh(\pi\tau')\vartheta(\Lambda,\tau';\xi), \tag{4.64}$$

where we have again ignored the term proportional to $d\Lambda$. From this result we can find $\vartheta(\tau,\Lambda;\xi)$ as

$$\begin{aligned}\vartheta(\tau,\Lambda;\xi) =& \frac{2}{\pi^2}\left(\Lambda^2 + \frac{1}{4} + \nu^2 R^2\right)^{-1}\int_0^{\Lambda-d\Lambda} d\tau' \, g_\Lambda(\Lambda,\tau')\tau'\sinh(\pi\tau') \\ &\times \int_{\mathbb{R}} d\xi' \, \overline{E_0(z_0,\tau,\xi)} E_0(z_0,\Lambda,\xi')\vartheta(\Lambda,\tau';\xi')\end{aligned} \tag{4.65}$$

By putting the explicit expression for $\vartheta(\Lambda, \tau'; \xi')$ we get

$$\begin{aligned}
\vartheta(\tau, \Lambda; \xi) =& \frac{2}{\pi^2}\left(\Lambda^2 + \frac{1}{4} + \nu^2 R^2\right)^{-1} \int_0^{\Lambda - d\Lambda} d\tau' \, g_\Lambda(\Lambda, \tau')\tau' \sinh(\pi\tau') \\
&\times \int_{\mathbb{R}} d\xi' \, \overline{E_0(z_0, \tau, \xi)} E_0(z_0, \Lambda, \xi') \int_{\mathbb{R}} d\xi'' \, \overline{E_0(z_0, \Lambda, \xi')} E_0(z_0, \tau', \xi'')\tilde{\phi}(\tau', \xi'') \\
&= \frac{2}{\pi^2}\left(\Lambda^2 + \frac{1}{4} + \nu^2 R^2\right)^{-1} \int_0^{\Lambda - d\Lambda} d\tau' \, g_\Lambda(\Lambda, \tau')\tau' \sinh(\pi\tau') \\
&\times \int_{\mathbb{R}} d\xi' \, \overline{E_0(z_0, \Lambda, \xi')} E_0(z_0, \Lambda, \xi') \int_{\mathbb{R}} d\xi'' \, \overline{E_0(z_0, \tau, \xi)} E_0(z_0, \tau', \xi'')\tilde{\phi}(\tau', \xi'') \\
&= \frac{1}{2\pi}\left(\Lambda^2 + \frac{1}{4} + \nu^2 R^2\right)^{-1} \frac{\tanh(\pi\Lambda)}{\sinh(\pi\Lambda)} \int_0^{\Lambda - d\Lambda} d\tau' \, g_\Lambda(\Lambda, \tau')\tau' \sinh(\pi\tau') \\
&\qquad \times \vartheta(\tau, \tau'; \xi).
\end{aligned} \tag{4.66}$$

If we put this result back into Equation 4.62 we find

$$\begin{aligned}
\left(\tau^2 + \frac{1}{4} + \nu^2 R^2\right)\tilde{\phi}(\tau, \xi) = \frac{2\Theta_\Lambda(\tau)}{\pi^2} & \int_0^{\Lambda - d\Lambda} d\tau'\tau' \sinh(\pi\tau')\vartheta(\tau, \tau'; \xi) \\
& \times \Bigg(g_\Lambda(\tau, \tau') \\
& + \frac{\Lambda \tanh(\pi\Lambda)}{2\pi(\Lambda^2 + \frac{1}{4} + \nu^2 R^2)} g_\Lambda(\Lambda, \tau') g_\Lambda(\tau, \Lambda)\Bigg).
\end{aligned} \tag{4.67}$$

By comparing this with Equation 4.61 we arrive to an equation for the coupling constant

$$g_{\Lambda - d\Lambda}(\tau, \tau') = g_\Lambda(\tau, \tau') + \frac{\Lambda \tanh(\pi\Lambda)}{2\pi(\Lambda^2 + \frac{1}{4} + \nu^2 R^2)} g_\Lambda(\Lambda, \tau') g_\Lambda(\tau, \Lambda), \tag{4.68}$$

which can be put into differential form as

$$-\frac{dg_\Lambda(\tau, \tau')}{d\Lambda} = \frac{\Lambda \tanh(\pi\Lambda)}{2\pi(\Lambda^2 + \frac{1}{4} + \nu^2 R^2)} g_\Lambda(\Lambda, \tau') g_\Lambda(\tau, \Lambda), \tag{4.69}$$

and by integrating from λ to Λ we find

$$g_\lambda(\tau, \tau') = g - x_\Lambda(\tau, \tau') + \frac{1}{2\pi}\int_\lambda^\Lambda ds \, \frac{s}{s^2 + \frac{1}{4} + \nu^2 R^2} \tanh(\pi s) g_s(s, \tau') g_s(\tau, s). \tag{4.70}$$

To obtain a solution we shall use the same procedure we used in the flat case. We begin with

$$g_\lambda^{(1)}(\tau,\tau') = g \quad \text{and} \quad x_\Lambda^{(1)}(\tau,\tau') = 0 \tag{4.71}$$

Then $g_\lambda^{(2)}(\tau,\tau')$ becomes

$$g_\lambda^{(2)}(\tau,\tau') = g - x_\Lambda^{(2)}(\tau,\tau') + \frac{g^2}{2\pi}\int_\lambda^\Lambda ds\, \frac{s}{s^2+\frac{1}{4}+\nu^2R^2}\tanh(\pi s). \tag{4.72}$$

Again using the fact $1 \ll \lambda \ll \Lambda$ we expand the integrand into a series and also write $\tanh(\pi s)$ as

$$\tanh(\pi s) = 1 - \frac{2}{e^{2\pi s}+1}, \tag{4.73}$$

so that Equation 4.72 becomes

$$g_\lambda^{(2)}(\tau,\tau') = g - x_\Lambda^{(2)}(\tau,\tau') + \frac{g^2}{2\pi}\int_\lambda^\Lambda ds\, \left(\frac{1}{s} + \mathcal{O}\left(\frac{1}{s^3}\right)\right)\left(1 - \frac{2}{e^{2\pi s}+1}\right). \tag{4.74}$$

It is clear that the counterterm should be chosen as

$$x_\Lambda^{(2)}(\tau,\tau') = \frac{g^2}{2\pi}\int_{\lambda_0}^\Lambda ds\, \frac{1}{s} = \frac{g^2}{2\pi}\log\left(\frac{\Lambda}{\lambda_0}\right), \tag{4.75}$$

so that $g_\lambda^{(2)}(\tau,\tau')$ becomes

$$g_\lambda^{(2)}(\tau,\tau') = g - \frac{g^2}{2\pi}\left(\log\left(\frac{\lambda}{\lambda_0} + c_0^{(2)}(\lambda)\right)\right) \tag{4.76}$$

where $c_0^{(2)}(\lambda)$ denotes the finite part of Equation 4.74. Again $g_\lambda^{(n)}(\tau,\tau')$ and $x_\Lambda^{(n)}(\tau,\tau')$ do not depend on τ and τ' for all n and $g_s^{(n)}$ is an alternating series of logarithmic terms given as in Equation 3.29. Therefore the counterterm at the nth order is chosen as

$$x_{\Lambda}^{(n+1)} = \frac{1}{2\pi}\int_{\lambda_0}^{\Lambda} ds\,(g_s^{(n)})^2\frac{1}{s} \tag{4.77}$$

so that

$$g_{\lambda}^{(n+1)} = g - c_0^{(n+1)}(\lambda) - \frac{1}{2\pi}\int_{\lambda_0}^{\lambda} ds\,(g_s^{(n)})^2\frac{1}{s}. \tag{4.78}$$

Just like in the flat case, under the assumption $\frac{g_{\lambda_0}}{\pi}\log(\frac{\lambda}{\lambda_0}) < 1$, $\{g_s^{(n)}\}_{n=1}^{\infty}$ has a limit as $n \to \infty$ so we find

$$g_{\lambda} = g - c_0(\lambda) - \frac{1}{2\pi}\int_{\lambda_0}^{\lambda} ds\,\frac{g_s^2}{s}, \tag{4.79}$$

and from this g_λ can be solved as

$$g_{\lambda} = \frac{g_{\lambda_0}}{1 + \frac{g_{\lambda_0}}{2\pi}\log\left(\frac{\lambda}{\lambda_0}\right)}. \tag{4.80}$$

This result is identical with Equation 3.37. Although the geometry of the space has changed, the equation determining the RG flow is the same. The reason for this is that we analyzed the RG flow at the high energy scale, in other words we have chosen $\lambda, \lambda_0 \gg 1$. Since high energy scales correspond to short distance scales and every Riemannian manifold of dimension D is locally an Euclid space of dimension D, it is not a surprise that flow equations at the high energy scales do not depend on the geometry of the space.

Now we check that if Equation 4.80 yields a finite answer for the bound state energy. For this we plug Equation 4.80 into Equation 4.59 to find

$$\begin{aligned}\left(\tau^2+\frac{1}{4}+\nu^2R^2\right)\tilde{\phi}(\tau,\xi)&=\frac{2\Theta_\Lambda(\tau)}{\pi^2}\frac{g_{\lambda_0}}{1+\frac{g_{\lambda_0}}{2\pi}\log\left(\frac{\Lambda}{\lambda_0}\right)}\int_0^\Lambda d\tau'\,\tau'\sinh(\pi\tau')\vartheta(\tau,\tau';\xi)\\&=\frac{2\Theta_\Lambda(\tau)}{\pi^2}\frac{g_{\lambda_0}}{1+\frac{g_{\lambda_0}}{2\pi}\log\left(\frac{\Lambda}{\lambda_0}\right)}\int_0^\Lambda d\tau'\,\tau'\sinh(\pi\tau')\\&\times\int_{\mathbb{R}}d\xi'\,\overline{E_0(z_0,\tau,\xi)}E_0(z_0,\tau',\xi')\tilde{\phi}(\tau',\xi').\end{aligned}\tag{4.81}$$

By defining

$$\mathcal{N}(\Lambda)\equiv\int_0^\Lambda d\tau'\,\tau'\sinh(\pi\tau')\int_{\mathbb{R}}d\xi'\,E_0(z_0,\tau',\xi')\tilde{\phi}(\tau',\xi'),\tag{4.82}$$

we obtain

$$\tilde{\phi}(\tau,\xi)=\mathcal{N}(\Lambda)\frac{2\Theta_\Lambda(\tau)}{\pi^2}\frac{g_{\lambda_0}}{1+\frac{g_{\lambda_0}}{2\pi}\log\left(\frac{\Lambda}{\lambda_0}\right)}\frac{\overline{E_0(z_0,\tau,\xi)}}{\tau^2+\frac{1}{4}+\nu^2R^2}.\tag{4.83}$$

Therefore $\mathcal{N}(\Lambda)$ becomes

$$\begin{aligned}\mathcal{N}(\Lambda)&=\mathcal{N}(\Lambda)\frac{2}{\pi^2}\frac{g_{\lambda_0}}{1+\frac{g_{\lambda_0}}{2\pi}\log\left(\frac{\Lambda}{\lambda_0}\right)}\int_0^\Lambda d\tau'\,\tau'\frac{\sinh(\pi\tau')}{\tau'^2+\frac{1}{4}+\nu^2R^2}\\&\times\int_{\mathbb{R}}d\xi'\,E_0(z_0,\tau',\xi')\overline{E_0(z_0,\tau',\xi')}\\&=\mathcal{N}(\Lambda)\frac{1}{2\pi}\frac{g_{\lambda_0}}{1+\frac{g_{\lambda_0}}{2\pi}\log\left(\frac{\Lambda}{\lambda_0}\right)}\int_0^\Lambda d\tau'\,\tau'\frac{\tanh(\pi\tau')}{\tau'^2+\frac{1}{4}+\nu^2R^2}.\end{aligned}\tag{4.84}$$

Hence we have

$$\begin{aligned}\frac{1}{g_{\lambda_0}}+\frac{1}{2\pi}\log\left(\frac{\Lambda}{\lambda_0}\right)&=\frac{1}{2\pi}\int_0^\Lambda d\tau'\,\tau'\frac{\tanh(\pi\tau')}{\tau'^2+\frac{1}{4}+\nu^2R^2}\\&=\frac{1}{2\pi}\int_0^\Lambda d\tau'\,\frac{\tau'}{\tau'^2+\frac{1}{4}+\nu^2R^2}-\frac{1}{2\pi}\int_0^\Lambda d\tau'\,\frac{2\tau'(e^{2\pi\tau'}+1)^{-1}}{\tau'^2+\frac{1}{4}+\nu^2R^2}\\&=\frac{1}{4\pi}\log\left(\frac{\Lambda^2+\frac{1}{4}+\nu^2R^2}{\frac{1}{4}+\nu^2R^2}\right)-\frac{1}{2\pi}\int_0^\Lambda d\tau'\,\frac{2\tau'(e^{2\pi\tau'}+1)^{-1}}{\tau'^2+\frac{1}{4}+\nu^2R^2}.\end{aligned}\tag{4.85}$$

After we take the $\Lambda \to \infty$ limit we find

$$\frac{1}{g_{\lambda_0}} = \frac{1}{4\pi} \log\left(\frac{\lambda_0^2}{\frac{1}{4} + \nu^2 R^2}\right) - \frac{1}{2\pi} \int_0^\infty d\tau' \, \frac{2\tau'(e^{2\pi\tau'} + 1)^{-1}}{\tau'^2 + \frac{1}{4} + \nu^2 R^2}. \tag{4.86}$$

The integral on the right hand side is convergent. Therefore the effective coupling given as in Equation 4.80 removes all divergences from the problem.

5. EXACT RENORMALIZATION GROUP ON THE 2-SPHERE

This section deals with the ERG analysis of point interactions on a compact Riemannian manifold, namely 2-Sphere. Although in the case of compact manifolds obtaining the spectral representation is more simple, the fact that the eigenvalue spectrum is discrete makes the ERG analysis more complicated. However as we shall see in this section, various mathematical techniques allow us to recover the main results of the previous sections.

5.1. The Spectra of Compact Riemannian Manifolds

Intuitively, a *compact* space is a topological space such that each infinite sequence should eventually accumulate at some point which lies in the space. A rigorous definition is as follows: Let X be a topological space and I be an indexing set. The space X is compact if for every arbitrary collection of open sets $\{U_i\}_{i\in I}$ which cover X, that is $X = \bigcup_{i\in I} U_i$, there is a finite subset $J \subset I$ such that $\{U_j\}_{j\in J}$ also cover X. Otherwise the space is called *non-compact*. The D-dimensional sphere $\mathbb{S}^D$ and torus $\mathbb{T}^D$ are examples of compact manifolds, whereas the Euclid Space $\mathbb{R}^D$ and the hyperbolic space $\mathbb{H}^D$ are examples of non-compact manifolds.

In the case of non-compact manifolds, finding the eigenfunctions of the Laplacian is not sufficient. One should also find a mapping which satisfies the conditions we have listed in Section 2.1. However in the case of compact manifolds, the eigenfunctions provide a complete spectral representation of the Laplacian thanks to the *Spectral Theorem* which can be stated as follows [36,43]: If $(\mathcal{M}, g)$ is a compact and connected Riemannian manifold without boundary, then the closed eigenvalue problem given by

$$(\Delta + \lambda)f = 0, \tag{5.1}$$

has a complete orthonormal system of eigenfunctions $\{f_l\}_{l=0}^{\infty}$ in $L^2(\mathcal{M}, dV_g)$ with the corresponding eigenvalues $\{\lambda_l\}_{l=0}^{\infty}$. They satisfy the following properties:

(i) $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots$, with $\lambda_l \to \infty$ as $l \to \infty$.

(ii) For each eigenvalue λ_l, the corresponding eigenspace $\mathcal{E}_{\lambda_l}$ is finite-dimensional.

(iii) The space $L^2(\mathcal{M}, dV_g)$ is the direct sum of these eigenspaces, which means $\{f_l\}_{l=0}^{\infty}$ is a basis for $L^2(\mathcal{M}, dV_g)$.

Here the *eigenspace* $\mathcal{E}_\lambda$ corresponding to λ is the vector space spanned by the solutions of the eigenvalue problem for the given λ. The last property implies that any $\psi \in L^2(\mathcal{M}, dV_g)$ can be written as

$$\psi = \sum_{l=0}^{\infty} (f_l, \psi)_{L^2} f_l \tag{5.2}$$

where $(\cdot, \cdot)_{L^2} : L^2(\mathcal{M}, dV_g) \times L^2(\mathcal{M}, dV_g) \to \mathbb{C}$ is the *inner product* on $L^2(\mathcal{M}, dV_g)$ defined by

$$(\psi, \phi) = \int_{\mathcal{M}} dV_g \, \overline{\psi} \phi. \tag{5.3}$$

The completeness and orthogonality of eigenfunctions imply

$$\int_{\mathcal{M}} dV_g(x) \, \overline{f_k(x)} f_l(x) = \delta_{kl}, \tag{5.4}$$

$$\sum_{l=0}^{\infty} \overline{f_l(x)} f_l(a) = \delta_g(x, a). \tag{5.5}$$

5.2. The Spectra of the 2-Sphere

A *2-sphere* is a compact Riemannian manifold defined by the set

$$\mathbb{S}^2 = \left\{ x \in \mathbb{R}^3 \mid \|x\|_E = R \right\}, \tag{5.6}$$

and by the Riemannian metric

$$g_{\mathbb{S}^2} = \begin{pmatrix} 1 & 0 \\ 0 & R^2 \sin^2\theta \end{pmatrix}, \tag{5.7}$$

where $\|\cdot\|_E$ is the Euclidean norm and we denote a point on $\mathbb{S}^2$ by $\Omega = (\theta, \varphi)$. Also R^2 is the constant sectional curvature. The Laplacian on $\mathbb{S}^2$ is given by

$$\Delta_{\mathbb{S}^2} = \frac{1}{R^2 \sin\theta}\left[\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial}{\partial\theta}\right) + \frac{\partial^2}{\partial\varphi^2}\right]. \tag{5.8}$$

The eigenfunctions are the spherical harmonics $Y_l^m(\Omega)$ expressed by

$$Y_l^m(\Omega) = \sqrt{\frac{(2l+1)}{4\pi}\frac{(l-m)!}{(l+m)!}} P_l^m(\cos\theta) e^{im\varphi} \tag{5.9}$$

where P_l^m's are associated Legendre Polynomials. The corresponding eigenvalues are given by

$$-\Delta_{\mathbb{S}^2} Y_l^m(\Omega) = R^{-2} l(l+1) Y_l^m(\Omega) \tag{5.10}$$

and for each eigenvalue $\lambda_l = R^{-2} l(l+1)$ the corresponding eigenspace is $2l+1$ dimensional. So any function $\psi \in L^2(\mathbb{S}^2, dV_{\mathbb{S}^2})$ can be written as

$$\psi(\Omega) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} C_l^m Y_l^m(\Omega), \tag{5.11}$$

where $C_l^m = (Y_l^m, \psi)_{L^2}$. The volume form on $\mathbb{S}^2$ is given by

$$dV_{\mathbb{S}^2}(\Omega) = R^2 \sin\theta d\theta d\varphi. \tag{5.12}$$

Finally the orthogonality of the Spherical Harmonics implies

$$\frac{1}{R^2}\int_{\mathbb{S}^2} dV_{\mathbb{S}^2}(\Omega)\,\overline{Y_{l'}^{m'}(\Omega)}Y_l^m(\Omega) = \delta_{l,l'}\delta_{m,m'}. \tag{5.13}$$

5.3. Exact Renormalization Group Analysis

5.3.1. Formulation of the Problem

Considering the same problem as in the previous sections we write the eigenvalue equation for the bound state as

$$(-\Delta_{\mathbb{S}^2}+\nu^2)\phi(\Omega) = g\delta_{\mathbb{S}^2}(\Omega,\Omega_0)\phi(\Omega), \tag{5.14}$$

where $\Omega_0 \in \mathbb{S}^2$ is the location of the Dirac-Delta potential. By using the expansion in Equation 5.11 and the eigenvalue relation given as in Equation 5.10 we can write Equation 5.14 as

$$\sum_{l=0}^{\infty}\sum_{m=-l}^{l} C_l^m \left[R^{-2}l(l+1)+\nu^2\right] Y_l^m(\Omega) = g\delta_{\mathbb{S}^2}(\Omega,\Omega_0)\sum_{l=0}^{\infty}\sum_{m=-l}^{l} C_l^m Y_l^m(\Omega). \tag{5.15}$$

Now if we multiply both sides by $\overline{Y_{l'}^{m'}(\Omega)}$ and integrate over $R^{-2}\int_{\mathbb{S}^2} dV_{\mathbb{S}^2}$ we obtain

$$C_l^m \left[l(l+1)+\nu^2R^2\right] = g\sum_{l'=0}^{\infty}\sum_{m'=-l'}^{l'} C_{l'}^{m'}\overline{Y_l^m(\Omega_0)}Y_{l'}^{m'}(\Omega_0), \tag{5.16}$$

where we have also used the orthogonality relation in Equation 5.13. The next step is to determine the type of divergence. For this we define

$$\mathcal{N} = \sum_{l'=0}^{\infty}\sum_{m'=-l'}^{l'} C_{l'}^{m'}Y_{l'}^{m'}(\Omega_0), \tag{5.17}$$

so that C_l^m is given by

$$C_l^m = \mathcal{N} \frac{g\overline{Y_l^m(\Omega_0)}}{l(l+1) + \nu^2 R^2}. \tag{5.18}$$

By plugging this result into Equation 5.17 we find g^{-1} as

$$\frac{1}{g} = \frac{1}{4\pi} \sum_{l'=0}^{\infty} \frac{2l'+1}{l'(l'+1) + \nu^2 R^2}, \tag{5.19}$$

where we have used

$$\sum_{m'=-l'}^{l'} \overline{Y_{l'}^{m'}(\Omega_0)} Y_{l'}^{m'}(\Omega_0) = \frac{2l'+1}{4\pi}. \tag{5.20}$$

By using Maclaurin–Cauchy integral test we can see that the l' sum in Equation 5.19 is logarithmically divergent and the divergence is caused by the large l' values.

5.3.2. Applying the ERG Procedure

We begin by writing the eigenvalue equation at the bare scale Λ.

$$C_l^m \left[l(l+1) + \nu^2 R^2 \right] = \Theta_\Lambda(l) \sum_{l'=0}^{\Lambda} g_\Lambda(l, l') \vartheta(l, l'; m), \tag{5.21}$$

where

$$\vartheta(l, l'; m) \equiv \sum_{m'=-l'}^{l'} C_{l'}^{m'} \overline{Y_l^m(\Omega_0)} Y_{l'}^{m'}(\Omega_0). \tag{5.22}$$

Since the eigenvalue spectrum is discrete, we take the second cutoff as $\Lambda - 1$ instead of $\Lambda - d\Lambda$. We write

$$C_l^m \left[l(l+1) + \nu^2 R^2 \right] = \Theta_{\Lambda-1}(l) \sum_{l'=0}^{\Lambda-1} g_{\Lambda-1}(l, l') \vartheta(l, l'; m). \tag{5.23}$$

We rewrite Equation 5.21 as

$$C_l^m\left[l(l+1)+\nu^2R^2\right]=\Theta_\Lambda(l)\left(\sum_{l'=0}^{\Lambda-1}g_\Lambda(l,l')\vartheta(l,l';m)+g_\Lambda(l,\Lambda)\vartheta(l,\Lambda;m)\right) \tag{5.24}$$

so that by substituting $l=\Lambda$ we get an expression for C_Λ^m:

$$C_\Lambda^m=\frac{1}{\Lambda(\Lambda+1)+\nu^2R^2}\left(\sum_{l'=0}^{\Lambda-1}g_\Lambda(\Lambda,l')\vartheta(\Lambda,l';m)+g_\Lambda(\Lambda,\Lambda)\vartheta(\Lambda,\Lambda;m)\right). \tag{5.25}$$

We note that due to the discrete spectrum we could not ignore the second term. Using Equation 5.22 and Equation 5.25 we can write

$$\begin{aligned}\vartheta(l,\Lambda;m) &= \sum_{m'=-\Lambda}^{\Lambda}C_\Lambda^{m'}Y_\Lambda^{m'}(\Omega_0)\overline{Y_l^m(\Omega_0)}\\ &= \sum_{m'=-\Lambda}^{\Lambda}\frac{Y_\Lambda^{m'}(\Omega_0)\overline{Y_l^m(\Omega_0)}}{\Lambda(\Lambda+1)+\nu^2R^2}\left(\sum_{l'=0}^{\Lambda-1}g_\Lambda(\Lambda,l')\vartheta(\Lambda,l';m)+g_\Lambda(\Lambda,\Lambda)\vartheta(\Lambda,\Lambda;m)\right)\\ &\qquad\qquad + g_\Lambda(\Lambda,\Lambda)\sum_{m'=-\Lambda}^{\Lambda}Y_\Lambda^{m'}(\Omega_0)\overline{Y_l^m(\Omega_0)}\vartheta(\Lambda,\Lambda;m')\Bigg).\end{aligned} \tag{5.26}$$

By putting explicit expressions for $\vartheta(\Lambda,l';m')$ and $\vartheta(\Lambda,\Lambda;m')$ we get

$$\begin{aligned}\vartheta(l,\Lambda;m) &= \frac{1}{\Lambda(\Lambda+1)+\nu^2R^2}\Bigg(\sum_{l'=0}^{\Lambda-1}g_\Lambda(\Lambda,l')\sum_{m'=-\Lambda}^{\Lambda}Y_\Lambda^{m'}(\Omega_0)\overline{Y_\Lambda^{m'}(\Omega_0)}\\ &\qquad\times\sum_{m''=-l'}^{l'}C_{l'}^{m''}Y_{l'}^{m''}(\Omega_0)\overline{Y_l^m(\Omega_0)}\\ &\qquad+g_\Lambda(\Lambda,\Lambda)\sum_{m'=-\Lambda}^{\Lambda}Y_\Lambda^{m'}(\Omega_0)\overline{Y_\Lambda^{m'}(\Omega_0)}\\ &\qquad\times\sum_{m''=-\Lambda}^{\Lambda}C_\Lambda^{m''}Y_\Lambda^{m''}(\Omega_0)\overline{Y_l^m(\Omega_0)}\Bigg)\\ &= \frac{1}{4\pi}\frac{2\Lambda+1}{\Lambda(\Lambda+1)+\nu^2R^2}\left(\sum_{l'=0}^{\Lambda-1}g_\Lambda(\Lambda,l')\vartheta(l,l';m)+g_\Lambda(\Lambda,\Lambda)\vartheta(l,\Lambda;m)\right)\end{aligned} \tag{5.27}$$

From this, $\vartheta(l, \Lambda; m)$ can be solved as

$$\vartheta(l, \Lambda; m) = \left(4\pi \frac{\Lambda(\Lambda+1) + \nu^2 R^2}{2\Lambda+1} - g_\Lambda(\Lambda, \Lambda)\right)^{-1} \sum_{l'=0}^{\Lambda-1} g_\Lambda(\Lambda, l')\vartheta(l, l'; m). \tag{5.28}$$

By putting this result back into Equation 5.24 we get

$$C_l^m[l(l+1) + \nu^2 R^2] = \Theta_\Lambda(l) \sum_{l'=0}^{\Lambda-1} \left[g_\Lambda(l, l') + \left(4\pi \frac{\Lambda(\Lambda+1) - M}{2\Lambda+1} - g_\Lambda(\Lambda, \Lambda)\right)^{-1} \times g_\Lambda(l, \Lambda) g_\Lambda(\Lambda, l') \right] \vartheta(l, l'; m), \tag{5.29}$$

and by comparing this result with Equation 5.23 we obtain a recursion relation for the effective coupling constant.

$$g_{\Lambda-1}(l, l') = g_\Lambda(l, l') + \left(4\pi \frac{\Lambda(\Lambda+1) + \nu^2 R^2}{2\Lambda+1} - g_\Lambda(\Lambda, \Lambda)\right)^{-1} g_\Lambda(l, \Lambda) g_\Lambda(\Lambda, l') \tag{5.30}$$

From this relation we can express the effective coupling at the effective scale λ as

$$g_\lambda(l, l') = g_\Lambda(l, l') + \sum_{s=\lambda+1}^{\Lambda} \left(4\pi \frac{s(s+1) + \nu^2 R^2}{2s+1} - g_s(s, s)\right)^{-1} g_s(l, s) g_s(s, l'), \tag{5.31}$$

or

$$g_\lambda(l, l') = g - x_\Lambda(l, l') + \frac{1}{4\pi} \sum_{s=\lambda+1}^{\Lambda} \left(\frac{s(s+1) + \nu^2 R^2}{2s+1} - \frac{g_s(s, s)}{4\pi}\right)^{-1} g_s(l, s) g_s(s, l'). \tag{5.32}$$

Again we shall use an iteration procedure. We start with

$$g_\lambda^{(1)}(l, l') = g \quad \text{and} \quad x_\lambda^{(1)}(l, l') = 0, \tag{5.33}$$

so that

$$g_\lambda^{(2)}(l,l') = g - x_\Lambda^{(2)}(l,l') + \frac{g^2}{4\pi}\sum_{s=\lambda+1}^{\Lambda}\left(\frac{s(s+1)+\nu^2R^2}{2s+1} - \frac{g}{4\pi}\right)^{-1}. \tag{5.34}$$

Let us define $\beta(s) \equiv \frac{s(s+1)}{2s+1}$. Since λ and Λ are chosen such that $1 \ll \lambda \ll \Lambda$, it is clear that

$$\beta(s) \gg \left|\frac{\nu^2R^2}{2s+1} - \frac{g}{4\pi}\right| \tag{5.35}$$

so we can expand the term inside the sum into a series

$$\left(\beta(s) + \frac{\nu^2R^2}{2s+1} - \frac{g}{4\pi}\right)^{-1} = \frac{1}{\beta(s)} + \frac{1}{(\beta(s))^2}\left(\frac{g}{4\pi} - \frac{\nu^2R^2}{2s+1}\right)^2 + \mathcal{O}\left(\frac{1}{(\beta(s))^3}\right). \tag{5.36}$$

Only the first term of this series will contribute to the divergent part. So we write Equation 5.34 as

$$g_\lambda^{(2)}(l,l') = g - x_\Lambda^{(2)}(l,l') + \frac{g^2}{4\pi}\sum_{s=\lambda+1}^{\Lambda}\frac{2s+1}{s(s+1)} + \text{Regular}, \tag{5.37}$$

where Regular stands for the terms which do not diverge in the $\Lambda \to \infty$ limit. Moreover we can write $\beta(s)^{-1}$ as

$$\frac{2s+1}{s(s+1)} = \frac{2}{s} - \frac{1}{s(s+1)}, \tag{5.38}$$

and since the second term is regular, Equation 5.37 is further reduced to

$$g_\lambda^{(2)}(l,l') = g - x_\Lambda^{(2)}(l,l') + \frac{g^2}{2\pi}\sum_{s=\lambda+1}^{\Lambda}\frac{1}{s} + \text{Regular}. \tag{5.39}$$

The next tool we shall use to reduce Equation 5.39 further is the so called *Euler Summation Formula*, which is stated as follows [44]: For every $k \geq 0$, provided that

$f \in C^{k-1}([0,n])$, we can write

$$\sum_{j=0}^{n} f(j) = \int_0^n dx\, f(x) + \frac{1}{2}\left(f(n) + f(0)\right) + \sum_{j=1}^{k} \frac{B_{2j}}{(2j)!}\left[f^{2j-1}(n) - f^{2j-1}(0)\right] + R_k(n) \tag{5.40}$$

where B_{2j}'s are Bernoulli numbers. R_k is called the *remainder term* and expressed by,

$$R_k(n) = \int_0^n dx\, P_{2k+1}(x)\, f^{2k+1}(x) = -\int_0^n dx\, P_{2k}(x)\, f^{2k}(x) \tag{5.41}$$

where

$$P_{2k}(x) = (-1)^{k-1} \sum_{i=1}^{\infty} \frac{2\cos(2i\pi x)}{(2i\pi)^{2k}}, \tag{5.42}$$

$$P_{2k+1}(x) = (-1)^{k-1} \sum_{i=1}^{\infty} \frac{2\sin(2i\pi x)}{(2i\pi)^{2k+1}}. \tag{5.43}$$

P_k's are bounded from above by [45],

$$|P_{2k}(x)| \leq \frac{|B_{2k}|}{(2k)!}, \quad \text{for} \quad k \in \mathbb{Z}^+. \tag{5.44}$$

Using this bound and the mean-value theorem we can get a bound for $R_k(n)$.

$$|R_k| \leq \frac{|B_{2k}|}{(2k)!} \int_0^n dx\, \left|f^{2k}(x)\right|. \tag{5.45}$$

To convert the sum in Equation 5.39 into an integral we take $f(s) = 1/s$ and use Equation 5.40. We obtain

$$\begin{aligned}\sum_{s=\lambda+1}^{\Lambda} \frac{1}{s} &= \sum_{s=0}^{\Lambda} \frac{1}{s} - \sum_{s=0}^{\lambda} \frac{1}{s} \\ &= \int_\lambda^\Lambda \frac{ds}{s} + \frac{1}{2}\left(\frac{1}{\Lambda} - \frac{1}{\lambda}\right) + \sum_{j=1}^{k} \frac{B_{2j}}{(2j)!}\left[\frac{d^{2j-1}}{ds^{2j-1}}\left(\frac{1}{s}\right)\Big|_{s=\Lambda} - \frac{d^{2j-1}}{ds^{2j-1}}\left(\frac{1}{s}\right)\Big|_{s=\lambda}\right] \\ &+ \left(R_k(\Lambda) - R_k(\lambda)\right),\end{aligned} \tag{5.46}$$

where for the remainder terms we have the following bounds:

$$|R_k(\Lambda)| \leq \frac{|B_{2k}|}{(2k)!} \int_0^\Lambda ds \left| \frac{d^{2k}}{ds^{2k}} \left(\frac{1}{s} \right) \right| , \tag{5.47}$$

$$|R_k(\lambda)| \leq \frac{|B_{2k}|}{(2k)!} \int_0^\lambda ds \left| \frac{d^{2k}}{ds^{2k}} \left(\frac{1}{s} \right) \right| . \tag{5.48}$$

These results tell us that for any $k \geq 1$ only first term of Equation 5.46 would contribute to the divergent part. So we choose the counterterm as

$$x_\Lambda^{(2)}(l, l') = \frac{g^2}{2\pi} \int_{\lambda_0}^\Lambda \frac{ds}{s} = \frac{g^2}{2\pi} \log \left(\frac{\Lambda}{\lambda_0} \right) , \tag{5.49}$$

where $1 \ll \lambda_0 < \lambda \ll \Lambda$, and find

$$g_\lambda^{(2)}(l, l') = g - \frac{g^2}{2\pi} \left(\log \left(\frac{\lambda}{\lambda_0} \right) + c_0^{(2)}(\lambda) \right) \tag{5.50}$$

where again $c_0^{(2)}(\lambda)$ denotes the regular terms. Starting with the unrenormalized coupling g ensures that g_λ is independent of l and l' in all orders. We now prove the following claim: If $g_s^{(n)}$ at some order n is given by the an alternating series of logarithmic terms

$$g_s^{(n)} = \sum_{j=0}^{M} (-1)^n \frac{1}{a_j} \frac{g^{j+1}}{(2\pi)^j} \log^j \left(\frac{s}{\lambda_0} \right) + c_0^{(n)}(s), \tag{5.51}$$

then the counterterm chosen as

$$x_\Lambda^{(n+1)} = \frac{1}{2\pi} \int_{\lambda_0}^\Lambda ds \, \frac{(g_s^{(n)})^2}{s} \tag{5.52}$$

removes all divergences of $g_\lambda^{(n+1)}$, hence $g_\lambda^{(n+1)}$ also becomes an alternating series of logarithmic terms given by

$$g_\lambda^{(n+1)} = \sum_{j=0}^{N} (-1)^n \frac{1}{a_j} \frac{g^{j+1}}{(2\pi)^j} \log^j \left(\frac{\lambda}{\lambda_0} \right) + c_0^{(n+1)}(\lambda), \tag{5.53}$$

with $N > M$. To prove this claim we start with

$$g_\lambda^{(n+1)} = g - x_\Lambda^{(n+1)} + \frac{1}{4\pi}\sum_{s=\lambda+1}^{\Lambda}\left(\frac{s(s+1)+\nu^2R^2}{2s+1} - \frac{g_s^{(n)}}{4\pi}\right)^{-1}(g_s^{(n)})^2, \tag{5.54}$$

where we assume that $g_s^{(n)}$ has the form as given in Equation 5.51. By using the definition $\beta(s) \equiv \frac{s(s+1)}{2s+1}$ and the expansion Equation 5.36 we can write

$$g_\lambda^{(n+1)} = g - x_\Lambda^{(n+1)} + \frac{1}{4\pi}\sum_{s=\lambda+1}^{\Lambda}\left[\frac{1}{\beta(s)} + \frac{1}{(\beta(s)^2)}\left(\frac{g_s^{(n)}}{4\pi} - \frac{\nu^2R^2}{2s+1}\right)^2 + \mathcal{O}\left(\frac{1}{(\beta(s)^3)}\right)\right] \times (g_s^{(n)})^2. \tag{5.55}$$

By the comparison test, the sum

$$\sum_{s=\lambda+1}^{\infty}\frac{\log^n(s)}{s^p}$$

converges for all $n \in \mathbb{N}$ and $p \geq 2$, so Equation 5.55 is reduced to

$$g_\lambda^{(n+1)} = g - x_\Lambda^{(n+1)} + \frac{1}{2\pi}\sum_{s=\lambda+1}^{\Lambda}\frac{(g_s^{(n)})^2}{s} + \text{Regular}. \tag{5.56}$$

We shall again use the Euler summation formula, but first we need to make an important remark. In the previous two cases, $\mathbb{R}^2$ and $\mathbb{H}^2$, g_s was a continuous differentiable function on the interval $[\lambda_0, \Lambda]$. But in this case g_s is only defined for $s \in \sigma(\Delta_{\mathbb{S}^2})$ and since the eigenvalues are discrete, one cannot treat g_s as a continuous function. To use the Euler summation formula, we assume the existence of a function $\tilde{g}$, which is at least C^2 on $[\lambda_0, \Lambda]$ and defined such that

$$\tilde{g}_s = g_s, \quad \text{for all } s \in \sigma(\Delta_{\mathbb{S}^2}).$$

To avoid confusion, we will parametrize $\tilde{g}$ by s'. Then using the Euler summation formula, the sum in Equation 5.56 can be written as

$$\sum_{s=\lambda+1}^{\Lambda} \frac{(g_s^{(n)})^2}{s} = \int_{\lambda}^{\Lambda} ds' \frac{(\tilde{g}_{s'}^{(n)})^2}{s'} + \frac{1}{2}\left(\frac{(g_\Lambda^{(n)})^2}{\Lambda} - \frac{(g_\lambda^{(n)})^2}{\lambda}\right)$$
$$+ \sum_{j=1}^{k} \frac{B_{2j}}{(2j)!}\left[\frac{d^{2j-1}}{ds'^{2j-1}}\left(\frac{(\tilde{g}_{s'}^{(n)})^2}{s'}\right)\Big|_{s'=\Lambda} - \frac{d^{2j-1}}{ds'^{2j-1}}\left(\frac{(\tilde{g}_{s'}^{(n)})^2}{s'}\right)\Big|_{s'=\lambda}\right]$$
$$+ (R_k(\Lambda) - R_k(\lambda)). \tag{5.57}$$

Let us choose $k = 1$ and write

$$\sum_{s=\lambda+1}^{\Lambda} \frac{(g_s^{(n)})^2}{s} = \int_{\lambda}^{\Lambda} ds' \frac{(\tilde{g}_{s'}^{(n)})^2}{s'} + \frac{1}{2}\left(\frac{(g_\Lambda^{(n)})^2}{\Lambda} - \frac{(g_\lambda^{(n)})^2}{\lambda}\right)$$
$$+ \frac{B_2}{2}\left[\frac{d}{ds'}\left(\frac{(\tilde{g}_{s'}^{(n)})^2}{s'}\right)\Big|_{s'=\Lambda} - \frac{d}{ds'}\left(\frac{(\tilde{g}_{s'}^{(n)})^2}{s'}\right)\Big|_{s'=\lambda}\right]$$
$$+ (R_1(\Lambda) - R_1(\lambda)). \tag{5.58}$$

We need to investigate three limits, which are

$$\lim_{\Lambda\to\infty} \frac{(g_\Lambda^{(n)})^2}{\Lambda} \quad , \quad \lim_{\Lambda\to\infty} \frac{d}{ds'}\left(\frac{(\tilde{g}_{s'}^{(n)})^2}{s'}\right)\Big|_{s'=\Lambda} \quad , \quad \lim_{\Lambda\to\infty} R_1(\Lambda). \tag{5.59}$$

The first limit is

$$\lim_{\Lambda\to\infty} \frac{(g_\Lambda^{(n)})^2}{\Lambda} \sim \lim_{\Lambda\to\infty} \frac{\log^{2M}\left(\frac{\Lambda}{\lambda_0}\right)}{\Lambda} = 0. \tag{5.60}$$

The second limit is

$$\lim_{\Lambda\to\infty} \frac{d}{ds'}\left(\frac{(\tilde{g}_{s'}^{(n)})^2}{s'}\right)\Big|_{s'=\Lambda} = \lim_{\Lambda\to\infty} \frac{2g_\Lambda^{(n)}}{\Lambda}\frac{d\tilde{g}_{s'}^{(n)}}{ds'}\Big|_{s'=\Lambda} - \frac{(g_\Lambda^{(n)})^2}{\Lambda^2}$$
$$\sim \lim_{\Lambda\to\infty} \frac{\log^{M(M-1)}\left(\frac{\Lambda}{\lambda_0}\right)}{\Lambda^2} - \frac{\log^{2M}\left(\frac{\Lambda}{\lambda_0}\right)}{\Lambda^2} = 0, \tag{5.61}$$

where we have used

$$\frac{d}{dx}\left(\log^p(x)\right) = \frac{p\log^{p-1}(x)}{x}. \tag{5.62}$$

And for the final term we have the bound

$$\begin{aligned}
\lim_{\Lambda\to\infty}|R_1(\Lambda)| &\leq \frac{|B_2|}{2}\int_0^\Lambda ds' \left|\frac{d^2}{ds'^2}\left(\frac{(\tilde{g}_{s'}^{(n)})^2}{s'}\right)\right| \\
&= \frac{|B_2|}{2}\int_0^\infty ds' \left|\frac{2(\tilde{g}_{s'}^{(n)})^2}{s'^3} - \frac{4\tilde{g}_{s'}^{(n)}}{s'^2}\frac{d\tilde{g}_{s'}^{(n)}}{ds'} + \frac{2}{s'}\left(\frac{d\tilde{g}_{s'}^{(n)}}{ds'}\right)^2 + \frac{2\tilde{g}_{s'}^{(n)}}{s'}\frac{d^2\tilde{g}_{s'}}{ds'^2}\right| \\
&\sim \frac{|B_2|}{2}\int_0^\infty \frac{ds'}{s'^3}\log^M\left(\frac{s'}{\lambda_0}\right).
\end{aligned} \tag{5.63}$$

Since the integral is convergent $\lim_{\Lambda\to\infty} R_1(\Lambda)$ is finite. Therefore only the integral term of Equation 5.58 contribute to the divergent part, so we choose the counterterm as

$$x_\Lambda^{(n+1)} = \frac{1}{2\pi}\int_{\lambda_0}^\Lambda ds' \, \frac{(\tilde{g}_{s'}^{(n)})^2}{s'}, \tag{5.64}$$

so that $g_\lambda^{(n+1)}$ becomes

$$g_\lambda^{(n+1)} = g - \frac{1}{2\pi}\int_{\lambda_0}^\Lambda ds' \, \frac{(\tilde{g}_{s'}^{(n)})^2}{s'} - c_0^{n+1}(\lambda). \tag{5.65}$$

We remark that although the eigenvalue equation for the ERG problem on $\mathbb{S}^2$ looks much different and more complicated than $\mathbb{R}^2$ and $\mathbb{H}^2$, at the end the effective coupling is still described by an alternating series of logarithmic terms. The conditions at which the sequence $\{g_\lambda^{(n)}\}_{n=1}^\infty$ does have a limit is the same as in the previous cases, so we won't repeat it here. By assuming that a limit exists, Equation 5.65 becomes

$$g_\lambda = g - \frac{1}{2\pi}\int_{\lambda_0}^\Lambda ds' \, \frac{\tilde{g}_{s'}^2}{s'} - c_0(\lambda), \tag{5.66}$$

and by picking $g_{\lambda_0} = g - c_0(\lambda_0)$ and absorbing the remaining λ-dependent part into the definition of g_λ as in the flat case, we find the RG flow equation for the effective coupling as

$$g_\lambda = \frac{g_{\lambda_0}}{1 + \frac{g_{\lambda_0}}{2\pi} \log\left(\frac{\lambda}{\lambda_0}\right)}. \tag{5.67}$$

This result is identical with the RG flows in $\mathbb{R}^2$ and $\mathbb{H}^2$, i.e. with the Equations 3.37 and 4.80 respectively. As we have mentioned in Section 4.3.2, this fact is related to the locally Euclidean nature of the Riemannian manifolds. We should also note that g_λ can take only the values which are in the spectrum of $\Delta_{\mathbb{S}^2}$.

Finally we check if the effective coupling given as in Equation 5.67 yields a finite value for the bound state energy. For this we put Equation 5.67 into Equation 5.21 to obtain

$$C_l^m \left[l(l+1) + \nu^2 R^2\right] = \Theta_\Lambda(l) \frac{g_{\lambda_0}}{1 + \frac{g_{\lambda_0}}{2\pi} \log\left(\frac{\Lambda}{\lambda_0}\right)} \sum_{l'=0}^{\Lambda} \sum_{m'=-l'}^{l'} C_{l'}^{m'} \overline{Y_l^m(\Omega_0)} Y_{l'}^{m'}(\Omega_0). \tag{5.68}$$

By defining

$$\mathcal{N}(\Lambda) \equiv \sum_{l'=0}^{\Lambda} \sum_{m'=-l'}^{l'} C_{l'}^{m'} Y_{l'}^{m'}(\Omega_0), \tag{5.69}$$

we obtain

$$C_l^m = \mathcal{N} \Theta_\Lambda(l) \frac{g_{\lambda_0}}{1 + \frac{g_{\lambda_0}}{2\pi} \log\left(\frac{\Lambda}{\lambda_0}\right)} \frac{\overline{Y_l^m(\Omega_0)}}{l(l+1) + \nu^2 R^2}. \tag{5.70}$$

Therefore $\mathcal{N}$ becomes

$$\begin{aligned}\mathcal{N}(\Lambda) &= \mathcal{N}\frac{g_{\lambda_0}}{1+\frac{g_{\lambda_0}}{2\pi}\log\left(\frac{\Lambda}{\lambda_0}\right)}\sum_{l'=0}^{\Lambda}\frac{1}{l'(l'+1)+\nu^2R^2}\sum_{m'=-l'}^{l'}\overline{Y_{l'}^{m'}(\Omega_0)}Y_{l'}^{m'}(\Omega_0) \\ &= \frac{\mathcal{N}}{4\pi}\frac{g_{\lambda_0}}{1+\frac{g_{\lambda_0}}{2\pi}\log\left(\frac{\Lambda}{\lambda_0}\right)}\sum_{l'=0}^{\Lambda}\frac{2l'+1}{l'(l'+1)+\nu^2R^2}. \end{aligned} \tag{5.71}$$

Hence we have

$$\frac{1}{g_{\lambda_0}}+\frac{1}{2\pi}\log\left(\frac{\Lambda}{\lambda_0}\right)=\frac{1}{4\pi}\sum_{l'=0}^{\Lambda}\frac{2l'+1}{l'(l'+1)+\nu^2R^2}. \tag{5.72}$$

This equation can be out into the following form:

$$\frac{1}{g_{\lambda_0}}+\frac{1}{4\pi}\log\left(\frac{\Lambda^2+\Lambda+\lambda_0^2}{\lambda_0^2}\right)-\frac{1}{4\pi}\log\left(\frac{\Lambda^2+\Lambda+\lambda_0^2}{\Lambda^2}\right)=\frac{1}{4\pi}\sum_{l'=0}^{\Lambda}\frac{2l'+1}{l'(l'+1)+\nu^2R^2}, \tag{5.73}$$

and from this we can write

$$\frac{1}{g_{\lambda_0}}+\frac{1}{4\pi}\int_0^{\Lambda}ds\frac{2s+1}{s(s+1)+\lambda_0^2}-\frac{1}{4\pi}\log\left(\frac{\Lambda^2+\Lambda+\lambda_0^2}{\Lambda^2}\right)=\frac{1}{4\pi}\sum_{l'=0}^{\Lambda}\frac{2l'+1}{l'(l'+1)+\nu^2R^2}. \tag{5.74}$$

We know apply the Euler Summation Formula to the integral on the left hand side, but with an alternative form given as [44]

$$\sum_{j=0}^{n}=\int_0^n dx\, f(x)+\frac{1}{2}(f(0)+f(n))+\int_0^n dx\, f'(x)\left(x-[x]-\frac{1}{2}\right) \tag{5.75}$$

where $[\cdot]$ is the nearest integer function. After some rearrangements we find

$$\begin{aligned}\frac{1}{g_{\lambda_0}} &= \frac{1}{4\pi}\sum_{l'=0}^{\Lambda}\left[\frac{2l'+1}{l'(l'+1)+\nu^2R^2} - \frac{2l'+1}{l'(l'+1)+\lambda_0^2}\right] \\ &+ \frac{1}{4\pi}\log\left(\frac{\Lambda^2+\Lambda+\lambda_0^2}{\Lambda^2}\right) + \frac{1}{8\pi}\left(\frac{2\Lambda+1}{\Lambda(\Lambda+1)+\lambda_0^2} + \frac{1}{\lambda_0^2}\right) \\ &+ \frac{1}{4\pi}\int_0^{\Lambda} ds\left(s-[s]-\frac{1}{2}\right)\frac{d}{ds}\left(\frac{2s+1}{s(s+1)+\lambda_0^2}\right)\end{aligned} \tag{5.76}$$

In the $\Lambda \to \infty$ limit we obtain

$$\begin{aligned}\frac{1}{g_{\lambda_0}} &= \frac{1}{4\pi}\sum_{l'=0}^{\infty}\left[\frac{2l'+1}{l'(l'+1)+\nu^2R^2} - \frac{2l'+1}{l'(l'+1)+\lambda_0^2}\right] \\ &+ \frac{1}{12\pi\lambda_0^2} + \frac{1}{4\pi}\int_0^{\infty} ds\left(s-[s]-\frac{1}{2}\right)\frac{d}{ds}\left(\frac{2s+1}{s(s+1)+\lambda_0^2}\right).\end{aligned} \tag{5.77}$$

Since the sum and integral in Equation 5.77 are convergent, we have proved that the choice of the effective coupling as in Equation 5.67 removes all the divergences from the problem.

6. CONCLUSION

In this thesis we investigated a non-perturbative renormalization of point interactions on various two dimensional Riemannian manifolds, using the ERG method.

At the beginning we reviewed point interactions on Euclid spaces, discussed why this problem requires renormalization for $D \geq 2$, explicitly performed the coupling constant renormalization in two and three dimensions and answered why this type of renormalization would not work in higher dimensions. We mentioned that the two dimensional case is an example of quantum mechanical symmetry breaking by calculating the ground state wavefunction after renormalization explicitly. We also derived the flow equations for the effective coupling constant and found that the theory is asymptotically free in two dimensions.

In the next chapter, after a short review of the ERG concept, based on Głazek and Maslowski [26], we talked about how the idea of ERG can be used to renormalize point interactions on the Euclid plane $\mathbb{R}^2$ and derived the RG flow equation. We found that this flow equation is identical to the one derived in the previous section by the standard renormalization methods. Additionally we performed a mathematical analysis of the range on renormalizability using the contraction principle of Banach [34].

In the final two chapters, we generalize the ideas of Głazek to the cases, where the underlying space is the Hyperbolic plane $\mathbb{H}^2$ and 2-sphere $\mathbb{S}^2$. We remarked that although discrete spectrum of $\mathbb{S}^2$ makes it difficult to obtain an expression for the RG flow, we could derive one by using the Euler summation formula. The main conclusion of this thesis is the fact that the RG flow equations are identical in all three manifolds we considered. A possible reason for this is that by definition, any Riemannian manifold is locally a Euclid space of the same dimension.

A rigorous proof of the fact that the flow equations derived by ERG methods are identical in all Riemannian manifolds is not trivial because of our lack of information about the behavior of eigenfunctions on a general Riemannian manifold. However in their quite recent papers, Erman and Turgut presented a non-perturbative renormalization of point interactions on a general two dimensional Riemannian manifold using heat kernel techniques and derived an expression for the flow equation of the coupling constant [46,47]. The form of this equation is identical to the one we obtained. Therefore it is not unreasonable to hope that the ERG method, once properly applied, would yield an identical RG flow equation for all Riemannian manifolds. This would be an interesting challenge to take.

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