ESSAYS ON MICROECONOMICS

by
OZAN ALTUĞ ALTUN

Submitted to the Graduate School of Social Sciences
in partial fulfillment of
the requirements for the degree of Master of Arts

Sabancı University
July 2021

ESSAYS ON MICROECONOMICS

Approved by:

Date of Approval: July 13, 2021

ABSTRACT

ESSAYS ON MICROECONOMICS

OZAN ALTUĞ ALTUN

Economics, M.A. Thesis, July 2021

Thesis Supervisor: Assoc. Prof. Mehmet Barlo

Keywords: Contagion, Interbank Network, Bank Failure, Cascade, Nash Bargaining, Proportional Cost Allocation

This thesis analyzes bargaining situations via cost allocation methods pertaining to rescue in an interbank market where there exists contagious financial distress. Our results extend the rescuing structure of Rogers and Veraart (2013) into a bargaining perspective. We consider different cost allocation methods which can be used to save the defaulting banks and to stop the contagious effect of the bank failure. We show that under proportional cost allocation, the solvent banks who have strictly positive income will always save the defaulting bank under mild conditions. On the other hand, we show that rescuing formation might not work under some other cost allocation methods.

ÖZET

MİKROEKONOMİ ÜZERİNE MAKALELER

OZAN ALTUĞ ALTUN

EKONOMİ, YÜKSEK LİSANS TEZİ, TEMMUZ 2021

Tez Danışmanı: Doç. Dr. Mehmet Barlo

Anahtar Kelimeler: Ekonomik Kriz, Bankalar, Ağ Yapıları, Nash Pazarlık Çözümü

Bu tezde, finansal sıkıntıların bankalararası piyasada sebep olabileceği bulaşıcı bir iflas zincirinin, farklı masraf tahsis yöntemleri ile elde edilen pazarlık yapıları ile kurtarılması incelenmektedir. Elde ettiğimiz sonuçlar, Roger and Veraart (2013) tarafından kurtarma yapıları ile ilgili bulguları, pazarlık yapıları ile değerlendirmektedir. Farklı pazarlık yapılarını, batan bir bankayı kurtarma ve yayılan finansal sıkıntıları durdurma açılarından değerlendirdik. Orantılı maliyet tahsisi kullanıldığında, batan bankaları kurtarmaya gücü yeten ve teşviği olan diğer sağlam bankalar grubunun müdahale edeceğini kanıtladık. Ayrıca, başka pazarlık yapıları altında, bu kurtarma grubunun oluşamayacağı durumlar olabileceğini gösterdik.

ACKNOWLEDGEMENTS

I would like to express my sincerest gratitude to my thesis advisor, Mehmet Barlo, for his valuable support, guidance and motivation during my study. I cannot thank him enough for instructing me all the time and inspiring me throughout my M.A.

I would like to present my special thanks to my thesis jury members, Mustafa Oğuz Afacan and Han Özsöylev for their valuable comments and suggestions.

Lastly, I wish to thank my family, Bayram - Fatih - Ayşegül Altun and my beloved girlfriend Ayşe Çağlar for supporting me and believing in me. Without their love, I would not succeed.

To my family and my girlfriend

TABLE OF CONTENTS

1. INTRODUCTION

Financial institutions are connected through financial commitments to each other. The banks "lend to and borrow from each other to smooth idiosyncratic liquidity variations and meet deposit requirements; they collaborate on investment opportunities; and they operate in chains —repackaging and reselling assets to each other" (Jackson and Pernoud (2020)). Therefore, a considerable amount of costs occurs in the case of insolvency of some part of the financial system due to the spread of financial distress through the network.

The resolution of Long-Term Capital Management (LTCM) in 1999 is one of the examples of such insolvencies. LTCM has lost nearly $4.6 billion due to its exposure to Russian and Asian Financial Crises (1997 and 1998, respectively). A financial consortium injected $3.6 billion to save LTCM without government assistance. Another example is the sale of Merrill Lynch to Bank of America in 2008. Merrill Lynch has lost $50 billion because of the subprime mortgage crisis, and it is sold to Bank of America at a price representing a discount of %61 from its September 2007 price. Cypriot Finacial Crisis in 2012 set an example of a bail-in where bondholders in Cyprus banks with more than $100,000$ euros in their accounts were forced to write off a portion of their holdings.

If a bank in the network defaults, then some losses are realized in bank failures due to liquidation. James (1991) finds that the loss on assets is substantial, averaging over 30 percent of the assets of the defaulting bank. Because of the decrease in the value of these assets, the bank cannot meet its obligations to its creditors. Due to the linkages of financial commitments, these losses spread throughout the financial network. Thus, the interconnectedness of banks causes the whole network to be exposed to the risk of a crisis that can stem from a failure of an individual financial institution (a node in the financial network).

The banking crisis in 2008 has shown that regulators and market participants had limited information about the network of obligations between financial institutions, and there was little theoretical understanding of the relationship between interconnectedness and financial stability (Glasserman and Young (2016)). The spread of financial distress through a network of institutions and the risk of interconnectedness across banks have been extensively studied. An incomplete list of papers in that literature are Allen and Gale (2000), Eisenberg and Noe (2001), Furfine (2003), Upper (2011), Elsinger, Lehar, and Summer (2013), Elliott, Golub, and Jackson (2014), Glasserman and Young (2016), Banerjee and Feinstein (2019), Papp and Wattenhofer (2020), and Kanik (2020). The network models in these studies have been instrumental to understand financial systemic risk better.

Some of these models operate with given network structures of financial institutions and analyze the failure of financial systems while investigating policies aimed to prevent the contagion of financial distress.

Eisenberg and Noe (2001) shows the existence of a market clearing vector by employing a fixed point argument. It also obtains the uniqueness of the market clearing vector under a condition called *regularity* that demands "any maximal connected subset of nodes of the financial system has some surplus to transfer among the other nodes". In furtherance, that study characterizes the market clearing vector with the help of an algorithm called the "Fictitious Default Algorithm" (FDA, henceforth).

Another significant contribution on this subject is made by Rogers and Veraart (2013), and it extends the model of Eisenberg and Noe (2001) by including (depreciation) constants that capture the realized fraction of the face values of the assets that a bank owns. The motive behind this is as follows: The defaulting bank would have to sell its loan portfolio probably at a price strictly less than it could sell when the bank is solvent. Rogers and Veraart establish that if a group of banks (rescue merger) can rescue banks in financial distress, then the group also has the incentive to rescue these failing banks.

In this thesis, we utilize both the determination of the market clearing vector (Eisenberg and Noe (2001)) and the idea of loss on assets' values in case of default (Rogers and Veraart (2013)).

On the other hand, other models assume different structures (concerning capitalization levels, degrees of connections between the banks, sizes of financial commitments among the banks, etc.) in conjunction with the financial network and aim to obtain general results for these structures.

Elliott, Golub, and Jackson (2014) approach the literature on bank failure by introducing two key concepts: integration and diversification of the network structure. *Integration* refers to the proportion of the assets held by the investors of the bank versus the proportion of the claims held by the other banks in the network. Meanwhile, *diversification* involves the number of banks that the aforementioned bank has an obligation to. That study concludes that diversification and integration bring about trade-offs that have nonmonotonic effects on the spread of financial distress, and financial networks with an intermediate degree of diversification and integration are the most susceptible to widespread financial cascades.

Nier et al. (2007) approach the effect of the structure of the network on the contagion by varying the level of capitalization of the banks, the degree to which banks are connected, along with the diversification and integration. Their findings on the effect of diversification and integration are parallel to the results of Elliott, Golub, and Jackson. In addition, they conclude that as the aggregate amount of capital in the network (the measure of absorbing shocks) increases, the risk of spread of financial risks decreases. Moreover, they also show that the more concentrated the network is, the more banks fail (where their measure of concentration increases when there are more banks in the network with the same aggregate size of assets).

Notwithstanding, Kanik (2020) extends the model of Elliott, Golub, and Jackson (2014) to the bank rescue context with a game theoretical framework. It concludes that financial contagion risk in banking networks may imply greater stability in particular network structures. Kanik also finds that the networks connected through an intermediate level of liabilities are welfare-maximizing.

Our thesis utilizes the model of Rogers and Veraart (2013) and makes use of the "Fictitious Default Algorithm" of Eisenberg and Noe (2001) to observe and investigate the cascade of bank failures (insolvencies causing "ripple effects" and "domino effects" as put forth by Papp and Wattenhofer (2020) and Upper (2011), respectively).

Even though the FDA is theoretically an algorithm (alternatively, a *tatonnement*), we argue that individual steps of this algorithm are sufficiently rich to model and display the real-world cascades of bank failures. Therefore, the FDA provides real-world applicability not only in terms of the identification of the limit (fixed-point) market clearing vector, but also in terms modeling cascades of bank failures. To our knowledge, this assessment concerning the FDA and other *sequential default algorithms* (Upper (2011) an example of which is the FDA) is shared by each of the following papers: Eisenberg and Noe (2001), Furfine (2003), Upper (2011), Elsinger, Lehar, and Summer (2013), Elliott, Golub, and Jackson (2014), Glasserman and Young (2016), Banerjee and Feinstein (2019), Papp and Wattenhofer (2020), and Kanik (2020).

To exhibit tangible examples displaying the real-world applicability of the sequential default algorithms (including the FDA), we cite Elliott, Golub, and Jackson (2014):

> Globalization brings with it increased financial interdependencies among many kinds of organizations—governments, central banks, investment banks, firms, etc.—that hold each other's shares, debts and other obligations. Such interdependencies can lead to cascading defaults and failures, which are often avoided through massive bailouts of institutions deemed "too big to fail." Recent examples include the U.S. government's interventions in A.I.G., Fannie Mae, Freddie Mac, and General Motors; and the European Commission's interventions in Greece and Spain.

They continue to associate these cascades with sequential default algorithms are follows:

> Some initial failures are enough to cause a second wave of organizations to fail. Once these organizations fail, a third wave of failures may occur, and so on. A variation on a standard algorithm [of Eisenberg and Noe (2001)] then allows us compute the extent of these cascades by using the formula discussed above to propagate the failure costs at each stage and determine which organizations fail in the next wave. Policymakers can use this algorithm in conjunction with the market value formula to run counterfactual scenarios and identify which organizations might be involved in a cascade under various initial scenarios.

Similarly, Banerjee and Feinstein (2019) offers the following related observations from the real-world to justify to use the FDA in their analysis:

> The important role that such contingent linkages play is demonstrated by the financial crisis of 2007-2009. As that crisis unfolded, AIG faced bankruptcy after the failure of Lehman Brothers due to the large payouts it was required to make on its CDS contracts referencing Lehman and mortgage backed securities. When the crisis hit, the sudden calls to pay out the CDS contracts put great pressure on AIG, which traditionally had a thin capital base. Consequently AIG had to be rescued by the U.S. Department of Treasury so as to avoid jeopardizing the financial health of firms which bought CDSs from AIG.

On the other hand, we wish to acknowledge that "the unfolding of default cascades and the realization of domino effects of insolvency are rarely observed and there is no reasonable database that would allow a systematic and reliable empirical answer to the question of how big contagion risks actually are." (Elsinger, Lehar, and Summer (2013))

In this thesis, we integrate cost allocation methods into the analysis of how to prevent bank failures and stop contagion cascades in financial networks.

We analyze different cost allocation methods; proportional cost allocation, equal sharing cost allocation, and Talmudic cost allocation methods in the context of cascading failures. Our findings show that the power to stop cascades of failures with equal sharing cost allocation and Talmudic cost allocation methods is limited, while it is possible to prevent the default chain right at the beginning with the proportional cost allocation method. Our results show that the resilient banks can prevent the cascade by relinquishing some portions of their claims, whenever possible. This relinquishment leads to a financial situation where all the banks in the network achieve higher wealth compared to the situation in which at least one bank defaults.

We adopt the model of Eisenberg and Noe (2001) and—as in Rogers and Veraart (2013)—we concentrate on the analysis of the FDA with the distinct feature that defaulting banks cannot extract the whole face value of their assets and have to suffer some depreciation losses to meet their obligations. Indeed, the assets of a financial institution consists of its exogenously given endowments and the financial liabilities owed by other banks to the bank at hand. It needs to be emphasized that the endowments of a bank does not contain shares of ownership of other banks and is rather limited to given monetary holdings. Unlike Rogers and Veraart (2013), we operate with a common depreciation loss with regards to defaulting banks. That is, the depreciation rate for endowments and the depreciation rate for liabilities to be collected from other banks is assumed to equal one another. While restrictive, this condition does not imply important qualitative differences and is rather imposed for reasons of simplicity. Another condition we impose demands that the financial system at hand is one where in the modified FDA in every cascade of bank failures, only one bank is in distress, if any. We confess that this assumption that we keep due to reasons of simplicity, eliminates interesting but complex situations in which the choice of which bank to rescue leads to some bargaining among the resilient banks. The final condition we need for our results is one where each bank is assumed to have a strictly positive income. This condition is related to the "regularity" assumption of Eisenberg and Noe (2001) which is required to obtain uniqueness of the clearing vector of the system.

We analyze the implications of the cost allocation methods on financial contagion and magnitude of bank defaults, and membership composition of a rescue consortia. We show that the spread of financial distress can be prevented at the beginning with the proportional cost allocation method and this prevention (weakly) improves the welfare of every member of the financial system. Additionally, we provide some examples that show the limitations of the other two cost allocation methods, equal cost sharing, and Talmudic cost allocation, in preventing the cascading failure. Therefore, our results establish that the cost allocation method matters for rescuing banks in financial networks.

The thesis proceeds as follows.

In Section 2.1 we describe the preliminaries, the "market clearing vector" used throughout this thesis, and a version of "Fictitious Default Algorithm" introduced by Eisenberg and Noe. The clearing vector is a fixed point of payments or liabilities of the banks to the others and it is characterized by the limit point of the algorithm. A due remark is that our version of the FDA does not suffer from the simultaneity problem of sequential default algorithms elaborated by Upper (2011). Please see Footnote 2 for the details. Using our setup, we are able to divide the cascades of bank defaults into clear steps with the help of the FDA and observe the spread of financial contagion clearly. By examining these steps, we investigate the formation of rescuing banks, their membership composition, and the bargaining procedure in each round. Our results show that behavior of the banks in rescue formation may change from one round to another under different cost allocation methods.

In Section 2.2, we introduce the main assumptions of this thesis. Our first assumption is that in the context of cascading failures, only one bank can go bankrupt at a step. This assumption eliminates complex cases where there exist multiple defaulting banks and a resulting bargaining situation about the determination of which distressed bank(s) to rescue. The second assumption we have is that all banks should have strictly positive income. This assumption is related to the "regularity" assumption of Eisenberg and Noe (2001) which is required to obtain uniqueness of the clearing vector of the system. These two assumptions put some structure on the financial network model we study.

In Section 2.3, we discuss the cost allocation methods that we use in this thesis and present the main result and the required examples of this thesis. An important remark concerning these cost allocation methods is their *time-consistency feature*. We evaluate the rescuing decisions and effort associated with each step of the financial contagion separately. In particular, the banks that have not defaulted yet in a given round will consider their participation in the rescue group under a cost allocation method by evaluating/foreseeing the specifics of the rescue efforts and arrangements in the following rounds with the same cost allocation method. Therefore, whether or not an individual bank wishes to participate and act according to the a rescue effort in a given step of the FDA depends on its value in the current round and its “continuation value” that the bank foresees to achieve in the next round of the rescue effort under the same cost allocation method.

Section 3 concludes the thesis.

2. HOW TO STOP CASCADES: AN ANALYSIS OF COST ALLOCATION METHODS TO FLOAT DEFAULTING BANKS

2.1 Preliminaries and the Fictitious Default Algorithm

Our model parallels that of Rogers and Veraart (2013) and we use the same notations with some small differences.

In our network model, each node represents a bank indexed by $N := \{1,...,n\}$ with $n \geq 2$. Each bank $i \in N$ has a monetary endowment which is denoted by $e_i \geq 0$ that captures the initial net assets of bank i. The profile of endowments is given by $\mathbf{e} := (e_i)_{i \in N}$. We wish to remind that the endowments of a bank does not contain shares of ownership of other banks and is limited to given monetary holdings.

Each of these banks also has nominal liabilities to the other banks in the system, and these liabilities are represented with a matrix that we refer to as the liability matrix.

Definition 2.1. *The* ***liability matrix*** *among the banks indexed by* N *is* $\bar{\mathbf{L}} = [\bar{L}_{ij}]_{ij}$ *where the* $ij^{\boldsymbol{th}}$ *cell,* $\bar{L}_{ij}$*, gives the nominal liability of bank* i *to bank* j*. Total nominal liability of bank* i *to other banks is given by the vector* $\bar{L}_i = \sum_{j \in N} \bar{L}_{ij}$*. We will assume that* $\bar{L}_{ij} \geq 0$ *for all* $i,j \in N$ *with the convention that* $\bar{L}_{ii} = 0$ *for all* $i \in N$*.*

In order to dismiss some of the unnecessary complications, we assume that $\bar{L}_{j,i} > 0$ implies that $\bar{L}_{i,j} = 0$, $i,j \in N$. That is, we consider only the "net" liabilities.

In our model, we operate using a constant rate of depreciation $\delta \in (0,1)$ which acts as a reduction rate of total assets of a defaulting bank i. With this constant, we make sure that the failure of the banks leads to a real cost. We denote $d := 1 - \delta$. As a result, we have $d \in (0,1)$.[1]

[1] When $d = 1$, then the model at hand is that of Eisenberg and Noe (2001). In our analysis, we insist on $d < 1$ in order to force banks to float the other banks whenever the opportunity arises, the main result of Rogers and Veraart (2013).

We define the *financial system* as a triple $(\mathbf{e}, \bar{\mathbf{L}}, d)$ where $\mathbf{e}$ is the endowment vector, $\bar{\mathbf{L}}$ is the liability matrix, and d is the net depreciated amount.

A clearing vector describes the payments of each bank to the others in the network. We assume that the clearing vector is consistent with three criteria proposed by Eisenberg and Noe (2001). These criteria are *limited liability*, which requires that each bank cannot pay more than its available assets, *the priority of debt claims*, which requires that a bank must first pay its outstanding liabilities before stockholders receive their claims, and *proportionality*, which requires that in the case of a default, the defaulting bank pays all creditor banks in proportion to the size of their nominal claims on its assets. We are going to use a similar definition of the clearing vector with Rogers and Veraart (2013).

Definition 2.2. *A* ***market clearing vector*** *for the financial system* $(\mathbf{e}, \bar{\mathbf{L}}, d)$ *is a vector* $\mathbf{L}^* := (L_i^*)_{i \in N}$ *where* $L_i^* \in [0, \bar{L}_i]$ *is such that*

$$\mathbf{L}^* = \phi(\mathbf{L}^*),$$

where $\phi : \mathbb{R}_+^N \to \mathbb{R}_+^N$ *is a function such that for all liability matrices* $\mathbf{L} = [L_{ij}]$ *with* $L_{ii} = 0$*,* $L_{ij} > 0$ *implies* $L_{ji} = 0$*, and* $\sum_j L_{ij} \in [0, \bar{L}_i]$ *for all* $i, j \in N$*,* ϕ *is defined by* $\phi(\mathbf{L}) := (\phi_i(\mathbf{L}))_{i \in N}$ *where we have*

$$\phi_i(\mathbf{L}) := \begin{cases} \bar{L}_i & \text{if } \bar{L}_i \leq e_i + \sum_{j \in N} L_{ji} \\ d\ (e_i + \sum_{j \in N} L_{ji}) & \text{otherwise.} \end{cases}$$

Theorem 1 of Eisenberg and Noe (2001) establishes the existence of a clearing vector for $d = 1$ by using Tarsky's fixed point theorem as well as come up with a condition that guarantees the uniqueness of this vector in general. This condition, which Eisenberg and Noe call *regularity*, says that "any maximal connected subset of nodes of the financial system has some surplus to transfer among the nodes of the system." Moreover, they observe that regularity follows whenever "the financial system is strongly interlinked or each node is endowed with some transferable surplus."

That paper also contains an algorithm and Eisenberg and Noe call it the *Fictitious Default Algorithm* (henceforth, the FDA). Indeed, this is an algorithm that identifies an interbank payment vector among all the financial institutions of a system by iteratively considering consecutive rounds where in each round the insolvent bank(s) at the end of that round is identified.

In this thesis, we focus on the following version of the FDA:[2]

1.1 Let each node represent a bank in the network. We denote the set of nodes by $i \in N$.

1.2 In the first round, we assume that all nodes $i \in N$ in the financial system will satisfy their obligations, meaning that they pay their initial liabilities $\bar{L}_i$.

1.3 If all the banks in the system meet their obligations, then the algorithm terminates. That is, if $(e_i + \sum_{j \in N} \bar{L}_{j,i}) \geq \bar{L}_i$, then the algorithm ends.

1.4 If there exists a bank j such that $(e_i + \sum_{j \in N} \bar{L}_{j,i}) < \bar{L}_i$, this bank is in financial distress and if not rescued it will go bankrupt at the end of round 1. Let such financially distressed banks in round $t = 1$ be denoted by $B^1 \subset N$.

1.5 When $B^1 \neq \emptyset$, FDA moves to the second round. The payments become L_i^* by for all $i \in B^1$ while it is $\bar{L}_j$ for all $j \in N \setminus B^1$.

1.6 The FDA procedure ends in round 2 if all the banks $i \in N \setminus B^1$ fulfill their liabilities. That is, we terminate the FDA if $(e_i + \sum_{j \in B^1} L_{j,i}^* + \sum_{j \in N \setminus B^1} \bar{L}_{j,i}) \geq \bar{L}_i$ for all $i \in N \setminus B^1$.

1.7 If there exists a bank $i \notin B^1$ such that $(e_i + \sum_{j \in B^1} L_{j,i}^* + \sum_{j \notin B^1} \bar{L}_{j,i}) < \bar{L}_i$, bank i is in financial distress in round 2 (and if not rescued bank i will go bankrupt at the end of this round). We denote the *union* of the financially distressed banks in this round $t = 2$ and the banks that have started bankrupt in this round by $B^2 \subset N$. So, the financially distressed banks in round 2 are those in $B^2 \setminus B^1$.

1.8 If $B^2 \setminus B^1 \neq \emptyset$, the FDA moves to round 3. The payments become L_i^* for all $i \in B^2$ while they are $\bar{L}_j$ for all $j \in N \setminus B^2$.

1.9 The FDA continues into further rounds by replicating these steps.

This algorithm proposes that in a default cascade, banks that go bankrupt are determined according whether or not they satisfy their obligations in each round. In particular, if all banks satisfy their obligations fully in a round, then the algorithm ends in that round. Otherwise, in the next round, obligations of all the banks and

---

[2] In general, the FDA of Eisenberg and Noe (2001), bears a small difference when compared to ours. In our setting, when a bank defaults, in the next rounds it pays the "fixed point" amount of liabilities. On the other hand, in Eisenberg and Noe's version, the defaulting bank does not necessarily pay the fixed point amount but instead pays all the proceeds it collects in that round from the other banks among which there may be some that will default in future rounds. Thanks to Theorem 1 of Eisenberg and Noe (2001) and our requirement that the defaulting banks pay their fixed point liabilities, both of these versions of the FDA converge to the same unique clearing vector whenever the financial system satisfies the regularity requirement. In furtherance, our formulation is immune to the simultaneity problem of sequential default algorithms as elaborated by Upper (2011).

their payoff amounts are calculated by using the market clearing vector $\mathbf{L}^*$. The algorithm continues until there are no new defaulting banks.

2.2 Floating Insolvent Banks

Eisenberg and Noe (2001) model assumes that defaulting banks realize the face value of their assets fully to meet their liabilities. The model presented by Rogers and Veraart (2013) extends Eisenberg and Noe's model and considers when a bank defaults, it will be realizing some loss given by a proportion (strictly less than one) of its assets to pay its obligations. However, for purposes of simplicity, our model includes the depreciation variable δ (a constant rate of depreciation) in the same fashion as Rogers and Veraart (2013), with the minor modification of using a common depreciation rate (instead of separate rates) for endowments and interbank assets.

Rogers and Veraart (2013) consider a rescue consortium (merger) that is formed among the banks excluding distressed ones. They consider situations where participation into a rescue consortium may be costly. That paper shows that when a rescue consortium is "able" to rescue the failing banks, then the consortium/merger also has an incentive to save the defaulting banks, i.e., the overall rescue costs of the merger are outweighed by the total benefits from rescuing distressed banks.

On the other hand, Rogers and Veraart (2013) is silent on the inner workings of the consortium and does not model how the members of the merger share the cost of rescue. Indeed, the particulars of the cost allocation that emerges when saving defaulting banks, we think, is an essential part of how rescue consortia operate and how they are formed. That is why, in this thesis we aim to model the inner workings of the rescue consortia by using cost allocation methods and thereby analyze the implications on financial contagion as well as understand the membership composition of the mergers.

In our model, we will have a condition similar to the Rescue Incentive condition of Rogers and Veraart (2013) that arises from an assumption for requiring strictly positive income for all banks. Using this condition, we show that when forming consortia is costless and the consortium of rescuing banks share the cost of rescue among themselves proportional to their benefits from saving the insolvent banks, all insolvent banks will be "rescued" meaning that the members of the consortium relinquish their cash inflow from insolvent banks accordingly so that insolvency does not trigger a societal loss due to the depreciation of assets. Moreover, we identify the membership composition of the rescue mergers.

To obtain tangible results and prevent unfruitful technicalities, in this thesis we adopt the following simplification:

Assumption 1. The financial system $(\mathbf{e}, \bar{\mathbf{L}}, d)$ is such that only one bank can go bankrupt at a round. That is, if $B^k \neq \emptyset$ for $k > 1$, then $|B^{k'} \setminus B^{k'-1}| = 1$ for all $k' \leq k$ and $|B^1| = 1$.

When the financial system $(\mathbf{e}, \bar{\mathbf{L}}, d)$ satisfying Assumption 1 is such that there are defaulting banks, then we can calculate the total number of rounds by $\sum_{i=1}^{N} \mathbf{1}_{[L_i^* < \bar{L}_i]} = M$. In such a case, we call the last bank that defaults as the Mth bank by indexing banks relative to the rounds when they become financially distressed. Moreover, if $B^M \neq \emptyset$, then for $t < M$ we define B^t a non-empty subset of N as the set of defaulting banks until the beginning of round $t+1$.

Next, we identify the cost of rescue:

Definition 2.3. *Whenever the financial system* $(\mathbf{e}, \bar{\mathbf{L}}, d)$ *is such that there are defaulting banks (i.e.,* $B^M \neq \emptyset$*), we define* c_k *to be the cost to save the* k*th bank with* $1 \leq k \leq M$ *from default which equals the following:*

$$c_k = \bar{L}_k - \left(e_k + \sum_{i \in B^k} L_{i,k}^* + \sum_{i \notin B^k} \bar{L}_{i,k} \right). \tag{2.1}$$

In words, when there are defaulting banks, the cost to save bank k from default equals banks k's total nominal liabilities subtracted from its endowments and cash inflow from other banks. Bank k's cash inflow, on the other hand, equals the nominal liabilities to bank k from those banks that have not defaulted until the $k-1$th round added to the fixed point payment amounts to bank k from the other banks that have defaulted until the $k-1$th round.

Our second assumption is related to the regularity of Eisenberg and Noe (2001).

Assumption 2. All banks have strictly positive income in the financial system $(\mathbf{e}, \bar{\mathbf{L}}, d)$, i.e., for all $i \in N$ we have that $e_i + \sum_{j \in N} \bar{L}_{j,i} > 0$.

The regularity of Eisenberg and Noe demands that every subset $S \in N$ with banks that are connected within the network via liabilities (this subset excludes the banks that have liabilities to one of the banks in the subset) should have $\sum_{i \in S} e_i > 0$. However, this condition would not be enough for our setting. Consider the example with given liability matrix, $e = [0, 10, 1]$ and the constant rate of depriciation is

$\delta = 0.2$ ($d = 0.8$). The entry $\bar{L}_{ij}$ denotes the liability of bank i to bank j, the row denoted by + shows summation of cash inflow at that round and initial endowments, the column denoted by $-$ shows total liabilities of the banks and the *net* column shows the difference between row + and column $-$ which equals the net amount of assets the banks have.

Round 1:

$$\left(\begin{array}{cccccc|c} & 1 & 2 & 3 & & - & net \\ 1 & 0 & 100 & 0 & & 100 & -100 \\ 2 & 0 & 0 & 100 & & 100 & 10 \\ 3 & 0 & 0 & 0 & & 0 & 101 \\ + & 0 & 110 & 101 & & & \end{array} \right)$$

The financial network in this example satisfies regularity because bank 2 and bank 3 have strictly positive endowments for all connected subsets $S \in N$ (in this case $\{1,2,3\}, \{2,3\}$ and $\{3\}$). However, we wish to exclude the banks like bank 1 in our context since the main focus of this paper is rescue of banks with bargaining rules / cost allocation methods. This type of banks (banks with no endowments and no positive cash inflow) will not be worthy to rescue. We note that Eisenberg and Noe's regularity condition holds whenever our strictly positive income requirement is satisfied.

An important implication of the strictly positive income requirement is that when a bank defaults (in a round up to round M), in the market clearing vector it makes strictly positive payments to the other banks. As a result, we observe that for any defaulting bank, $M^* \in B^M$ that defaults in a round up to round M, we have that $\sum_{j \in N} L^*_{M^*,j} > 0$ (due to the defining property of ϕ in Definition 2.2), while for any other bank $k \notin B^M$, $\sum_{j \in N} L^*_{k,j} = \bar{L}_k > 0$. We use this observation to obtain the following result critical for our thesis:

Proposition 2.1. *Suppose that Assumptions 1 and 2 hold. Then, if M^* is one of the defaulting banks, $M^* \in B^M$, then*

$$\sum_{i \in G^{M^*}} (\bar{L}_{M^*,i} - L^*_{M^*,i}) > c_{M^*} \tag{2.2}$$

where $G^{M^*} := \{i \in N : \bar{L}_{M^*,i} > 0\}$.

Proof. Note that

$$c_{M^*} = \bar{L}_{M^*} - \left(e_{M^*} + \sum_{i\in B^{M^*}} L^*_{i,M^*} + \sum_{i\notin B^{M^*}} \bar{L}_{i,M^*} \right). \tag{2.3}$$

Moreover, due to the definition of the market clearing vector (in particular, ϕ) we have that

$$\sum_{i\in G^{M^*}} L^*_{M^*,i} = d \left(e_{M^*} + \sum_{i\in B^{M^*}} L^*_{i,M^*} + \sum_{i\notin B^{M^*}} \bar{L}_{i,M^*} \right)$$

where $d \in (0,1)$. This follows from the fact that when a bank defaults, it has to pay all of its monetary assets net of some loss given by $\delta = 1 - d$.

As $\bar{L}_{M^*} = \sum_{i\in G^{M^*}} \bar{L}_{M^*,i}$ due to the definition of G^{M^*}, and equation (2.3) we attain that

$$\begin{aligned} c_{M^*} &= \bar{L}_{M^*} - \left(e_{M^*} + \sum_{i\in B^{M^*}} L^*_{i,M^*} + \sum_{i\notin B^{M^*}} \bar{L}_{i,M^*} \right) \\ &= \bar{L}_{M^*} - \frac{1}{d} \sum_{i\in G^{M^*}} L^*_{M^*,i} = \sum_{i\in G^{M^*}} \bar{L}_{M^*,i} - \frac{1}{d} \sum_{i\in G^{M^*}} L^*_{M^*,i}, \\ &< \sum_{i\in G^{M^*}} \bar{L}_{M^*,i} - \sum_{i\in G^{M^*}} L^*_{M^*,i} = \sum_{i\in G^{M^*}} \left(\bar{L}_{M^*,i} - L^*_{M^*,i} \right) \end{aligned}$$

since $\sum_{i\in G^{M^*}} L^*_{M^*,i} > 0$ due to Assumption 2. The last inequality delivers (2.2) and hence finishes the proof of this proposition. □

Intuitively, Proposition 2.1 displays that the cost of saving a financially distressed bank is strictly lower than the forfeited amount of money that this bank would not be able to pay to the other banks in case of its default.

2.3 The Proportional Cost Allocation

The main cost allocation method that has been analyzed in the current thesis is the proportional cost allocation method, the PCA.

This time-consistent cost allocation demands that if there exist a rescue opportunity, the rescuing banks should share the recovery cost *proportional* to their benefits.

Given the financial system $(\mathbf{e}, \bar{\mathbf{L}}, d)$ satisfying Assumptions 1 and 2 with a clearing vector L^* and $B^M \neq \emptyset$ (i.e., there are defaulting banks), the benefit of a bank $i \notin B^M$ to save bank M (in the very last round of the FDA before the fixed point payments are achieved), is given by

$$S_i^M := \left(e_i + \sum_{j\in B^{M-1}} L^*_{j,i} + \sum_{j\notin B^{M-1}} \bar{L}_{j,i} - \bar{L}_i\right) - V_i^*, \tag{2.4}$$

where

$$V_i^* := \left(e_i + \sum_{j\in B^{M}} L^*_{j,i} + \sum_{j\notin B^{M}} \bar{L}_{j,i} - \bar{L}_i\right) \tag{2.5}$$

denotes the monetary surplus that firm i obtains in the fixed point payment scheme. Notwithstanding, for any bank $j \in B^M$ (including M), the benefit of saving bank M is given by 0. Therefore, we obtain the following benefit/surplus representation: In round M of the FDA:

$$S_i^M = \begin{cases} \left(e_i + \sum_{j\in B^{M-1}} L^*_{j,i} + \sum_{j\in N\setminus B^{M-1}} \bar{L}_{j,i} - \bar{L}_i\right) - V_i^* & \text{if } i \notin B^M, \\ 0 & \text{otherwise.} \end{cases} \tag{2.6}$$

Using (2.5) in (2.4), we observe that the benefit of firm $i \notin B^M$ to save bank M is given by (thanks to Assumption 1) S_i^M which equals

$$\begin{aligned} &\left(e_i + \sum_{j\in B^{M-1}} L^*_{j,i} + \sum_{j\in N\setminus B^{M-1}} \bar{L}_{j,i} - \bar{L}_i\right) - \left(e_i + \sum_{j\in B^{M}} L^*_{j,i} + \sum_{j\in N\setminus B^{M}} \bar{L}_{j,i} - \bar{L}_i\right) \\ &= \left(\sum_{j\in B^{M-1}} L^*_{j,i} - \sum_{j\in B^{M}} L^*_{j,i}\right) + \left(\sum_{j\in N\setminus B^{M-1}} \bar{L}_{j,i} - \sum_{j\in N\setminus B^{M}} \bar{L}_{j,i}\right) = \bar{L}_{M,i} - L^*_{M,i}. \end{aligned}$$

Then, the proportional payments of banks in round M in the FDA to save bank M are determined as follows: For any $i \in N$,

$$b_{i,M} := \frac{S_i^M}{\sum_{j\in N} S_j^M}\; c_M. \tag{2.7}$$

Then, by using the above we obtain (2.7) is equivalent to

$$b_{i,M} = \frac{\bar{L}_{M,i} - L^*_{M,i}}{\sum_{j\in N\setminus B^M} \bar{L}_{M,j} - L^*_{M,j}}\; c_M. \tag{2.8}$$

In round $M-1$, the benefit of bank M to rescue bank $M-1$ equals

$$S_M^{M-1} := \left(e_M + \sum_{j\in B^{M-2}} L^*_{j,M} + \sum_{j\notin B^{M-2}} \bar{L}_{j,M} - \bar{L}_M\right)$$

as M will be needing rescue in the next round and hence obtain a value of $V_M^* = 0$. Moreover, it is straightforward to see that for any bank $j \in B^{M-2}$, the benefit of rescuing bank $M-1$, $S_j^{M-1} = 0$. On the other hand, in round $M-1$, the benefit of a bank $i \notin B^M$ to rescue bank $M-1$ equals

$$S_i^{M-1} := \left(e_i + \sum_{j \in B^{M-2}} L^*_{j,i} + \sum_{j \notin B^{M-2}} \bar{L}_{j,i} - \bar{L}_i \right) - V_i^M,$$

where

$$V_i^M := \max \left\{ \left(e_i + \sum_{j \in B^{M-1}} L^*_{j,i} + \sum_{j \notin B^{M-1}} \bar{L}_{j,i} - \bar{L}_i - b_{i,M} \right),\ V_i^* \right\}.$$

For obvious notational reasons, we let $V_i^{M+1} := V_i^*$.

Then, the proportional payments of banks in round $M-1$ of the FDA to save bank $M-1$ are determined as follows: For any $i \in N$,

$$b_{i,M-1} := \frac{S_i^{M-1}}{\sum_{j \in N} S_j^{M-1}}\, c_{M-1}. \tag{2.9}$$

Continuing in this fashion, we obtain the PCA defined formally.

We include the example below to provide a better understanding for the propotional payment $b_{i,k}$. The constant rate of depriciation, δ, is 0.2 ($d = 0.8$) and $e = [200, 10, 0]$ for this example.

Round 1:

$$\left(\begin{array}{cccccc|c} & 1 & 2 & 3 & - & & net \\ 1 & 0 & 1000 & 0 & 1000 & & -800 \\ 2 & 0 & 0 & 1000 & 1000 & & 10 \\ 3 & 0 & 0 & 0 & 0 & & 1000 \\ + & 200 & 1010 & 1000 & & & \end{array} \right)$$

Round 2:

$$\left(\begin{array}{cccccc|c} & 1 & 2 & 3 & - & & net \\ 1 & 0 & 160 & 0 & 160 & & \\ 2 & 0 & 0 & 1000 & 1000 & & -830 \\ 3 & 0 & 0 & 0 & 0 & & 1000 \\ + & 200 & 170 & 1000 & & & \end{array} \right)$$

Round 3:

$$\left(\begin{array}{cc|c} & \begin{matrix} 1 & 2 & 3 \end{matrix} \quad - & net \\ \begin{matrix} 1 \\ 2 \\ 3 \\ + \end{matrix} & \begin{pmatrix} 0 & 160 & 0 \\ 0 & 0 & 136 \\ 0 & 0 & 0 \\ 200 & 170 & 136 \end{pmatrix} \begin{matrix} 160 \\ 0 \\ 0 \\ \\ \end{matrix} & \begin{matrix} \\ \\ 136 \\ \\ \end{matrix} \end{array}\right)$$

At the last round, we observe the clearing vector payments for all banks, which means $L^*_{1,2} = 160$, $L^*_{2,3} = 136$ and bank 3 does not have a payment to make. We reach this step if banks do not make any rescue initiative. At the last round, the only solvent bank is the third bank and only this bank can offer a rescue. Under any cost allocation rule, the third bank will have to pay the rescue amount of the second bank, $c_2 = 830$, meaning that $b_{3,2} = 830$. However, as we can observe from the third matrix, bank 3 will have only \$136 in net terms to cover those costs.

The logic behind the payment share $b_{3,2}$ is as follows: We consider this payment share as the amount of money that bank 3 is willing to relinquish from the net amount it has at the second round. That means, by giving up its claims of \$830 (bank 3's net claims were \$1000) and accepting a total payment of \$170, bank 3 can rescue bank 2 in the second round. That will make bank 3 better of by $170 - 136 = \$44$ compared to the net amount it would have in round 3. As a result, by forfeiting its claims of \$830, bank 3 ensures that \$44 is not lost due to depreciation of bank 2's assets as a consequence of its default.

In round 1, we observe the total amount of money that has to be relinquished is \$800 which is the payment that bank 1 cannot fulfill. By using the equation (2.9), we calculate $b_{2,1} = 9.52$ and $b_{3,1} = 790.48$ under the PCA. That means, the amount of money that bank 3 and bank 2 have to relinquish is \$790.48 and \$9.52 respectively. Since bank 2 has only \$10 in net terms before the bankruptcy of the first bank, by giving up \$9.52, it will have \$0.48 when bank 1 is rescued. With the same logic, Bank 3 will have $1000 - 790.48 = \$209.52$. Both of these values are greater than the amount that bank 3 and bank 2 would have when the first bank defaults. We want to stress that, when we calculate the net amount they have when the first bank is saved, we observe that bank 2 transfers some amount of its endowments to bank 3 (bank 2 had an initial endowment of \$10 at the beginning and the net amount after rescuing bank 1 is \$0.48). That means $b_{j,k}$ cannot be interpreted as the amount of money that bank j relinquishes just from the payment it would get from bank k. The relinquishment related to the amount of money that bank j gives up from the net amount it would have if the default at issue would have not occurred.

The following result, establishes that if a bank i is an active contributor of the rescuing consortium for bank $1 < k \leq M$, then bank i must be an active contributor of also the rescuing consortium for all banks $k' < k$.

Proposition 2.2. *Suppose that Assumptions 1 and 2 hold. Then if a defaulting bank $k \in B^M$ is saved and we have that $b_{i,k} > 0$ for some $i \in N \setminus B^k$, then $b_{i,k-1} > 0$ even if $\bar{L}_{k-1,i} = 0$.*

Proof. First, we wish to remind that

$$V_i^{k-1} = \max\left\{e_i + \sum_{j \in B^{k-2}} L_{j,i}^* + \sum_{j \in N \setminus B^{k-2}} \bar{L}_{j,i} - \bar{L}_i - b_{i,k-1}, V_i^k\right\} \tag{2.10}$$

and as $k \in B^k$ (i.e., bank k defaults in round k)

$$V_i^k = e_i + \sum_{j \in B^{k-1}} L_{j,i}^* + \sum_{j \in N \setminus B^{k-1}} \bar{L}_{j,i} - \bar{L}_i - b_{i,k},$$

where the last equality holds by hypothesis that bank i will save bank k by hypothesis as $b_{i,k} > 0$ by assumption, thus $V_i^k \geq V_i^{k+1}$.

As a result, the benefit of bank i to save bank $k-1$ (provided that bank i will save bank k by hypothesis as $b_{i,k} > 0$ by assumption) is

$$\left(e_i + \sum_{j \in B^{k-2}} L_{j,i}^* + \sum_{j \in N \setminus B^{k-2}} \bar{L}_{j,i} - \bar{L}_i\right) - V_i^k = \bar{L}_{k-1,i} - L_{k-1,i}^* + b_{i,k}.$$

We note that the group of banks saving bank $k-1$, consortium at $k-1$ denoted by G^{k-1}, obtain a total surplus of

$$\sum_{j \in G^{k-1} \setminus \{k\}} \left(\bar{L}_{k-1,j} - L_{k-1,j}^* + b_{j,k}\right) + \left(e_k + \sum_{j \in B^{k-2}} L_{j,k}^* + \sum_{j \in N \setminus B^{k-2}} \bar{L}_{j,k} - \bar{L}_k\right) \tag{2.11}$$

since $V_k^k = 0$ and hence equation (2.10) evaluated for $i = k$ shows that the benefit of bank k by saving bank $k-1$ (given that bank k will default in the next round) equals

$$e_k + \sum_{j \in B^{k-2}} L_{j,k}^* + \sum_{j \in N \setminus B^{k-2}} \bar{L}_{j,k} - \bar{L}_k. \tag{2.12}$$

By using the definition of the cost to save bank k

$$c_k = \bar{L}_k - \left(e_k + \sum_{j \in B^k} L^*_{j,k} + \sum_{j \in N \setminus B^k} \bar{L}_{j,k} \right)$$

in equation (2.12) we obtain (as $L^*_{k,k} = \bar{L}_{k,k} = 0$)

$$\begin{aligned} & e_k + \sum_{j \in B^{k-2}} L^*_{j,k} + \sum_{j \in N \setminus B^{k-2}} \bar{L}_{j,k} - \left[c_k + e_k + \sum_{j \in B^k} L^*_{j,k} + \sum_{j \in N \setminus B^k} \bar{L}_{j,k} \right] \\ & e_k + \sum_{j \in B^{k-2}} L^*_{j,k} + \sum_{j \in N \setminus B^{k-2}} \bar{L}_{j,k} - \left[c_k + e_k + \sum_{j \in B^{k-1}} L^*_{j,k} + \sum_{j \in N \setminus B^{k-1}} \bar{L}_{j,k} \right] \\ & = \bar{L}_{k-1,k} - L^*_{k-1,k} - c_k. \end{aligned}$$

Ergo, substituting this finding into the total surplus of the consortium at $k-1$ shows that equation (2.11) becomes

$$\begin{aligned} & \sum_{j \in G^{k-1} \setminus \{k\}} \left(\bar{L}_{k-1,j} - L^*_{k-1,j} + b_{j,k} \right) + \left(\bar{L}_{k-1,k} - L^*_{k-1,k} - c_k \right) \\ & \quad = \sum_{j \in G^{k-1}} \left(\bar{L}_{k-1,j} - L^*_{k-1,j} \right) + \sum_{j \in G^{k-1} \setminus \{k\}} b_{j,k} - c_k \\ & \quad = \sum_{j \in G^{k-1}} \left(\bar{L}_{k-1,j} - L^*_{k-1,j} \right) \end{aligned}$$

as $\sum_{j \in G^{k-1} \setminus \{k\}} b_{j,k} = c_k$.

Therefore, the proportional payment share of bank i is to be calculated via the following equation:

$$b_{i,k-1} = \frac{\left(\bar{L}_{k-1,i} - L^*_{k-1,i} + b_{i,k} \right)}{\sum_{j \in G^{k-1}} (\bar{L}_{k-1,j} - L^*_{k-1,j})} \, c_{k-1}. \tag{2.13}$$

By Proposition (2.1), we know that $\sum_{i \in G^{k-1}} (\bar{L}_{k-1,i} - L^*_{k-1,i}) > c_{k-1}$. Hence, $b_{i,k-1} < \bar{L}_{k-1,i} - L^*_{k-1,i} + b_{i,k}$, meaning that the benefit of bank i to save bank $k-1$ is strictly greater than the payment i is to make to save $k-1$ under the PCA rule.

Finally, even if $\bar{L}_{k-1,i} = 0$, then it is clear that the gain for bank i to save bank $k-1$ is $b_{i,k}$ and by Proposition 2.1, we observe that $b_{i,k} > b_{i,k-1}$. This means that even if there is no positive inflow from bank $k-1$ to bank i, bank i will be willing to participate the consortium to save bank $k-1$. □

The following is our main result that says any bank will be saved under the proportional cost allocation.

Theorem 2.1. *Suppose that Assumptions 1 and 2 hold. Then, any bank can be rescued with the proportional cost allocation rule.*

Proof. Consider the last defaulting bank M. By Proposition 2.1 we have obtained equation (2.2), i.e., $\sum_{i\in G^M}(\bar{L}_{M,i}-L^*_{M,i})>c_M$ where $G^M\subset N\setminus B^M$. Ergo, $M\notin G^M$. We can calculate the net gains for all banks $i\in G^M$ using their value functions which are given by

$$V_i^M=\max\left\{e_i+\sum_{j\in B^{M-1}}L^*_{j,i}+\sum_{i\in N\setminus B^{M-1}}\bar{L}_{j,i}-\bar{L}_i-b_{i,M},V_i^{M+1}\right\} \tag{2.14}$$

where we remind that $V_i^{M+1}=V_i^*=e_i+\sum_{j\in B^M}L^*_{j,i}+\sum_{i\in N\setminus B^M}\bar{L}_{j,i}-\bar{L}_i$. Then, the benefit of bank i to rescue bank M is given by

$$e_i+\sum_{j\in B^{M-1}}L^*_{j,i}+\sum_{i\in N\setminus B^{M-1}}\bar{L}_{j,i}-V_i^*=\bar{L}_{M,i}-L^*_{M,i}. \tag{2.15}$$

Therefore, the payment share of all healthy banks $i\in B^M$ under the PCA can be written as

$$b_{i,M}=\frac{\bar{L}_{M,i}-L^*_{M,i}}{\sum_{i\in G^M}(\bar{L}_{M,i}-L^*_{M,i})}\,c_M$$

By equation (2.2), we conclude that the total net gains for all banks exceeds the payment share or individual rescue costs, $\bar{L}_{M,i}-L^*_{M,i}>b_{i,M}$, so bank M will be saved.

For induction purposes, suppose bank $N^*+1\in B^M$ is rescued and we need to show that bank N^* will be saved. Consider $i\in G^{N^*}$ with

$$V_i^{N^*}=\max\left\{e_i+\sum_{j\in B^{N^*-1}}L^*_{j,i}+\sum_{j\in N\setminus B^{N^*-1}}\bar{L}_{j,i}-\bar{L}_i-b_{i,N^*},\ V_i^{N^*+1}\right\}.$$

For this bank i, there can be only three cases: $b_{i,N^*+1}>0$ meaning that bank i would contribute to rescuing operation in round N^*+1 and we can categorize this type of banks under the set $T1$. Second case is $i=N^*+1$, i.e., bank i could be the bank that will face financial distress next round. Our last case will be $b_{i,N^*+1}=0$ with $i\neq N^*+1$, meaning that bank i would not attend any rescuing operation until this round and we label these type of banks under the set $T3$.

If $b_{i,N^*+1}>0$ (alternatively, $i\in T1$), then $V_i^{N^*+1}=e_i+\sum_{j\in B^{N^*}}L^*_{j,i}+\sum_{j\in N\setminus B^{N^*}}\bar{L}_{j,i}-b_{i,N^*+1}>0$. So the benefit of this type of bank can be written as

$$e_i + \sum_{j\in B^{N^*-1}} L^*_{j,i} + \sum_{j\in N\setminus B^{N^*-1}} \bar{L}_{j,i} - \bar{L}_i - V_i^{N^*+1} = \bar{L}_{N^*,i} - L^*_{N^*,i} + b_{i,N^*+1}. \tag{2.16}$$

If $i = N^*+1$, then $V_{N^*+1}^{N^*} = \max\{e_{N^*+1} + \sum_{j\in B^{N^*-1}} L^*_{j,N^*+1} + \sum_{j\in N\setminus B^{N^*-1}} \bar{L}_{j,N^*+1} - \bar{L}_{N^*+1} - b_{i,N^*}, 0\}$. In this case, the benefit of this bank is

$$e_{N^*+1} + \sum_{j\in B^{N^*-1}} L^*_{j,N^*+1} + \sum_{j\in N\setminus B^{N^*-1}} \bar{L}_{j,N^*+1} - \bar{L}_{N^*+1} = \bar{L}_{N^*,N^*+1} - L^*_{N^*,N^*+1} - c_{N^*+1}. \tag{2.17}$$

Lastly, if $b_{i,N^*+1} = 0$ with $i \neq N^*+1$ (or $i \in T3$), then $V_i^{N^*} = \max\{e_i + \sum_{j\in B^{N^*-1}} L^*_{j,i} + \sum_{j\in N\setminus B^{N^*-1}} \bar{L}_{j,i} - \bar{L}_i - b_{i,N^*}, V_i^{N^*+1}\}$. By Proposition 2.2, $b_{i,N^*+1} = 0 = b_{i,N^*+2} = \ldots = b_{i,M+1}$ and that means $V_i^{N^*+1} = V_i^{N^*+2} = \ldots = V_i^{M+1} = e_i + \sum_{j\in B^{N^*}} L^*_{j,i} + \sum_{j\in N\setminus B^{N^*}} \bar{L}_{j,i} - \bar{L}_i$ the benefit of this bank is

$$e_i + \sum_{j\in B^{N^*-1}} L^*_{j,i} + \sum_{j\in N\setminus B^{N^*-1}} \bar{L}_{j,i} - \bar{L}_i - V_i^{N^*+1} = \bar{L}_{N^*,i} - L^*_{N^*,i}. \tag{2.18}$$

If we sum the benefits of these three type of banks, $\sum_{i\in T1} \bar{L}_{N^*,i} - L^*_{N^*,i} + b_{i,N^*+1} + \bar{L}_{N^*,N^*+1} - L^*_{N^*,N^*+1} - c_{N^*+1} + \sum_{i\in T3} \bar{L}_{N^*,i} - L^*_{N^*,i}$, we would get $\sum_{i\in N\setminus B^{N^*-1}} (\bar{L}_{N^*,i} - L^*_{N^*,i}) = \sum_{i\in G^{(N^*)}} (\bar{L}_{N^*,i} - L^*_{N^*,i})$.

By Proposition (2.1), we have the equation (2.2) and this implies for all banks $i \in N \setminus B^{N^*-1}$, $V_i^{N^*} - b_{i,N^*} > V_i^{N^*+1}$. From this inequivalence, we observe that these banks increase their values by saving the financially distressed bank N^* and hence this bank will be rescued. Since $N^* \in B^M$ is arbitrary, by induction we can conclude that any bank can be rescued with the PCA rule. □

This theorem shows that under the PCA rule, the contagion of financial distress will be stopped by other banks in the network before it starts. Every bank in the financial network will be achieving a higher value compared to the situation when defaulting cascade happens.

2.4 Other Cost Allocation Methods

In this section, we discuss two other time-consistent cost allocation methods that can be used in the case of rescue opportunities. Then, we compare these methods to the PCA and identify examples where the PCA stops the failure cascade at the beginning while the other methods involve cascades of bank failures.

Suppose that the given financial system $(\mathbf{e}, \bar{\mathbf{L}}, d)$ satisfies Assumptions 1 and 2.

Equal Cost Sharing Method: This time-consistent cost allocation method resembles the PCA with the critical difference demanding that the solvent banks share the rescue payment for bank k equally, i.e., for all $i, j \notin B^k$ with $S_i^k > 0$ and $S_j^k > 0$, it should be the case that $b_{i,k} = b_{j,k}$.

In the example [3] below, we compare proportional cost allocation and equal cost sharing method. The constant rate of depriciation, δ, for the example below is 0.2 ($d = 0.8$) and $e = [200, 700, 500, 250, 200]$.

Round 1:

	1	2	3	4	5	−	*net*
1	0	700	400	200	100	1400	−1200
2	0	0	500	400	300	1200	200
3	0	0	0	600	400	1000	400
4	0	0	0	0	1000	1000	450
5	0	0	0	0	0	0	2000
+	200	1400	1400	1450	2000		

Round 2:

	1	2	3	4	5	−	*net*
1	0	80	45.71	22.86	11.43	160	
2	0	0	500	400	300	1200	−420
3	0	0	0	600	400	1000	45.71
4	0	0	0	0	1000	1000	272.86
5	0	0	0	0	0	0	1911.43
+	200	780	1045.71	1272.86	1911.43		

[3]This example is a modified and corrected version of the one that appeared in ECON 399 Independent Study Report of U. Mergen (2018) supervised by Mehmet Barlo.

Round 3:

	1	2	3	4	5	$-$	*net*
1	0	80	45.71	22.86	11.43	160	
2	0	0	260	208	156	624	
3	0	0	0	600	400	1000	-194.29
4	0	0	0	0	1000	1000	80.86
5	0	0	0	0	0	0	1767.43
+	200	780	805.71	1080.86	1767.43		

Round 4:

	1	2	3	4	5	$-$	*net*
1	0	80	45.71	22.86	11.43	160	
2	0	0	260	208	156	624	
3	0	0	0	386.74	257.83	644.57	
4	0	0	0	0	1000	1000	-132.4
5	0	0	0	0	0	0	1625.26
+	200	780	805.71	867.6	1425.26		

Round 5:

	1	2	3	4	5	$-$	*net*
1	0	80	45.71	22.86	11.43	160	
2	0	0	260	208	156	624	
3	0	0	0	386.74	257.83	644.57	
4	0	0	0	0	694.08	694.08	
5	0	0	0	0	0	0	1319.34
+	200	780	805.71	867.6	1319.34		

At the last round Bank 5 has to pay 132.4 to save bank 4, $c_4 = 132.4$. The value of bank 5 at round 5 (in the case where the system achieves the fixed point payments) is $V_5^5 = 1319.34$ and bank 5 will compare this value with the value it would have when bank 4 is rescued, $V_5^4 = \max\{1625.26 - b_{5,4}, V_{5,5}\}$. Since the only healthy bank is bank 5, bank 4 can be rescued by only bank 5. As $c_4 = b_{5,4}$, $V_5^4 = 1625.26 - 132.4 = 1492.88$ because $1492.88 > 1319.34$. Notice that since bank 5 is the only resilient bank, cost in PCA and cost in Equal Cost Sharing will be equal.

Now, consider rescuing of bank 3. The cost of saving bank 3 is $c_3 = 194.29$. With the same fashion, bank 5 will compare its maximum value until this round with what it would achieve when bank 3 is rescued, $V_5^3 = \max\{1767.43 - b_{5,3}, V_5^4 = 1492.88\}$. Also bank 4's maximum value until this round is 0 since it would default next round. Therefore, we have $V_4^3 = \max\{80.96 - b_{4,3}, V_4^4 = 0\}$. Now, we can examine the cost allocations for PCA and Equal Sharing.

Proportional Cost: Bank 5's Gain: $1767.43 - 1492.88 = 274.55$. Bank 4's Gain: $80.96 - 0 = 80.96$. $b_{5,3} = \frac{274.55}{355.51} \times 194.29 = 150.044$ and $b_{4,3} = \frac{80.96}{355.51} \times 194.29 = 57.29$. $V_5^3 = 1767.43 - 150.044 = 1617.39$ and $V_4^3 = 80.96 - 57.29 = 23.67$.

Equal Cost Sharing: Since $c_3 = 194.29$, $b_{5,3} = b_{4,3} = 97.145$. $V_5^3 = \max\{1767.43 - 97.145, 1492.88\} = 1670.285$. However, $V_4^3 = \max\{80.96 - 97.145, 0\} = 0$, bank 4 wouldn't contribute to rescuing bank 3.

The cost of rescuing bank 2 is $c_2 = 420$ in round 2. Now, bank 5 and bank 4 will be comparing their maxiumum value until this round with what they would get when bank 2 is rescued, $V_5^2 = \max\{1911.43 - b_{5,2}, V_5^3 = 1617.39\}$, $V_4^2 = \max\{272.86 - b_{4,2}, V_4^3 = 23.67\}$ respectively. Bank 3 will be comparing the value it will have when bank 2 is rescued with 0 because it would default next round if bank 2 is not rescued, $V_3^2 = \max\{45.71 - b_{3,2}, V_3^3 = 0\}$. We can now compare the two cost allocation methods with the same logic above.

Proportional Cost: Bank 5's Gain: $1911.43 - 1617.39 = 294.04$. Bank 4's Gain: $272.86 - 23.67 = 249.19$. Bank 3's Gain: $45.71 - 0 = 45.71$. $b_{5,2} = \frac{294.04}{588.94} \times 420 = 209.69$, $b_{4,2} = \frac{249{,}19}{588.94} \times 420 = 177.71$ and $b_{3,2} = \frac{45.71}{588.94} \times 420 = 32.6$. $V_5^2 = 1911.43 - 209.69 = 1701.74$, $V_4^2 = 272.86 - 177.71 = 95.15$ and $V_3^2 = 45.71 - 32.6 = 10.11$.

Equal Cost Sharing: Since $c_2 = 420$, $b_{5,2} = b_{4,2} = b_{3,2} = 140$. $V_5^2 = \max\{1911.43 - 140, 1492.88\} = 1771.43$ and $V_4^2 = \max\{272.86 - 140, 0\} = 132.86$. However, $V_3^2 = \max\{45.71 - 140, 0\} = 0$, bank 3 wouldn't contribute to rescuing bank 2.

We can now consider the first round where cost of rescuing bank 1 is $c_1 = 1200$. Again, all the banks will be comparing the value what they would achieve when bank 1 is saved with the maximum value they achieve until this round, meaning that $V_5^1 = \max\{2000 - b_{5,1}, V_5^2 = 1701.74\}$, $V_4^1 = \max\{450 - b_{4,1}, V_4^2 = 95.15\}$, $V_3^1 = \max\{400 - b_{3,1}, V_3^2 = 10.11\}$ and $V_2^1 = \max\{200 - b_{2,1}, 0\}$ respectively. Below, we compute the cost allocations for each bank with PCA and Equal Sharing methods.

Proportional Cost: Bank 5's Gain: $2000 - 1701.74 = 298.26$. Bank 4's Gain: $450 - 95.15 = 354.85$. Bank 3's Gain: $400 - 10.11 = 389.89$. Bank 2's Gain: $200 - 0 = 200$. $b_{5,1} = \frac{298.26}{1243} \times 1200 = 287.94$, $b_{4,1} = \frac{354.85}{1243} \times 1200 = 342.57$,

$b_{3,1} = \frac{389.89}{1243} \times 1200 = 372.54$, $b_{2,1} = \frac{200}{1243} \times 1200 = 193.08$. $V_5^1 = 2000 - 287.94 = 1712.06$, $V_4^1 = 450 - 342.57 = 108.43$, $V_3^1 = 400 - 372.54 = 27.46$, $V_2^1 = 200 - 193.08 = 6.92$ and $V_1^1 = 0$.
Equal Cost Sharing: Since $c_1 = 1200$, $b_{5,1} = b_{4,1} = b_{3,1} = b_{2,1} = 300$. $V_5^1 = \max\{2000 - 300, 1492.88\} = 1700$ and $V_4^1 = \max\{450 - 300, 0\} = 150$ and $V_3^1 = \max\{400 - 300, 0\} = 100$. However, $V_2^1 = \max\{200 - 300, 0\} = 0$, bank 2 wouldn't contribute to rescuing bank 1.

So, this example shows that it is possible to stop the cascade of failiures at the start (round 1) under PCA, as we proved in Theorem 2.1 while the rescue operation would be limited under Equal Cost Sharing method.

Talmudic Cost Allocation Method: The Talmudic division is first stated in the Babylonian Talmud. In this text, this division method is used for the inheritance divisions between the right holders, However, in Aumann and Maschler (1985), this method is used to analysis of Bankruptcy problem which we will also apply it in our cost allocation framework. Consider the example

	100	200	300
100	$33\frac{1}{3}$	$33\frac{1}{3}$	$33\frac{1}{3}$
200	50	75	75
300	50	100	150

According to traditional Talmud, suppose that there are three creditors in the above system and their debts are 100,200 and 300 respectively and also corresponding estates are 100, 200 and 300 as well. When $E = 100$ the estate equals to smallest debt, therefore equal division will be applied. However, the figures for 200 and 300 are not suitable to fit any obvious extension of either equal or proportional division. In Aumann and Maschler (1985) by looking at the above divisions that are taken directly from the Talmud, a cost allocation formula is constructed, therefore in our context we call this Talmudic cost allocation method.

Let us say there is two-creditor bankruptcy problem with estate E and claims d_1, d_2. The amount that each claimant i concedes to the other claimant j is $(E - d_i)_+$ where;

$$\theta_+ = \max(\theta, 0) \tag{2.19}$$

The amount at issue is therefore;

$$E-(E-d_1)_+-(E-d_2)_+ \tag{2.20}$$

It will be shared equally between the two claimants and in addition to this each claimant will receive the amount conceded to her by the other one. Thus the total amount awarded to i is

$$x_i=\frac{E-(E-d_1)_+-(E-d_2)_+}{2}+(E-d_j)_+ \tag{2.21}$$

To use the above equation in our framework we will modify it a little to fit in the context. In our context, the cost allocation process is done between the saver banks and cash flow will happen from them to the insolvent bank if all the saver banks will be better than their initial situation. Therefore, in the above formula, we will change to claimants as savers and we will change their claims as payments that they should allocate the recovery cost between the saver banks.

The example below compares proportional cost allocation and Talmudic cost sharing method in the case of rescue opportunities. The constant rate of depriciation, δ, for the example below is 0.2 ($d=0.8$) and $e=[200,100,160,0,0]$.

Round 1:

	1	2	3	4	5	−	*net*
1	0	200	200	200	200	800	−600
2	0	0	100	100	0	200	100
3	0	0	0	300	0	300	160
4	0	0	0	0	0	0	600
5	0	0	0	0	0	0	200
+	200	300	460	600	200		

Round 2:

	1	2	3	4	5	−	*net*
1	0	40	40	40	40	160	
2	0	0	100	100	0	200	−60
3	0	0	0	300	0	300	0
4	0	0	0	0	0	0	440
5	0	0	0	0	0	0	40
+	200	140	300	440	40		

Round 3:

	1	2	3	4	5	−	*net*
1	0	40	40	40	40	160	
2	0	0	56	56	0	112	
3	0	0	0	300	0	300	−44
4	0	0	0	0	0	0	396
5	0	0	0	0	0	0	40
+	200	140	256	396	40		

Round 4:

	1	2	3	4	5	−	*net*
1	0	40	40	40	40	160	
2	0	0	56	56	0	112	
3	0	0	0	204.8	0	204.8	
4	0	0	0	0	0	0	300.8
5	0	0	0	0	0	0	40
+	200	140	256	300.8	40		

Consider the last round where bank 3 defaults. Bank 5 would not contribute to rescue amount because Bank 3 is the last defaulting bank and bank 5 is not connected to bank 3 ($L_{3,5} = 0$, $V_5^3 = \max\{40, 40 - b_{5,3}\} = 40$ implies $b_{5,3} = 0$.). Thus, only bank 4 would be willing to contribute to the rescue and it will compare the value what it would achieve when bank 3 is saved with the maximum value it achieves until this round which is the fixed point value, $V_4^3 = \max\{396 - b_{4,3}, 300.8\}$. Therefore, all rescue amount will be paid by Bank 4, $b_{4,3} = 44$ and $V_4^3 = \max\{396 - 44, 300.8\}$ means that Bank 3 will be rescued by Bank 4.

Now, consider the next round where bank 2 defaults. Again, bank 5 and bank 3 would not contribute the rescue amount because of the same reason above (we can also observe this by value comparisons for bank 5 and bank 3, $V_5^2 = \max\{40 - b_{5,2}, 40\}$ and $V_3^2 = \max\{0 - b_{3,2}, 0\}$ implies $b_{5,2} = 0$ and $b_{3,2} = 0$). Again bank 4 will compare the value what it would achieve when bank 2 is rescued with the maximum value it achieves until this round, $V_4^2 = \max\{440 - b_{4,2}, 352\}$. Therefore, all rescue amount should be met by Bank 4 if possible, so $V_4^2 = \max\{440 - b_{4,2}, 352\}$ and $b_{4,2} = 60$ together implies that Bank 2 will also be rescued by Bank 4.

We can now examine the round where bank 1 defaults. This time all the banks are getting some amount of payments from bank 1, and that's why they all would contribute to the rescue amonut if possible. The cost of rescuing the first bank is $c_1 = 600$. With the same logic, all banks will compare the value they would achieve when bank 1 is saved with the maximum value they would get until this round, $V_5^1 = \max\{200 - b_{5,1}, 40\}$, $V_4^1 = \max\{600 - b_{4,1}, 380\}$, $V_3^1 = \max\{160 - b_{3,1}, 0\}$, $V_2^1 = \max\{100 - b_{2,1}, 0\}$ respectively. Until this round, the banks were rescued by only bank 4, and therefore PCA cost and Talmudic cost were the same. In this case, we have to compute them seperately.

Proportional Cost: Bank 5's Gain: $200 - 40 = 160$. Bank 4's Gain: $600 - 380 = 220$. Bank 3's Gain: $160 - 0 = 160$. Bank 2's Gain: $100 - 0 = 100$. $b_{5,1} = \frac{160}{640} \times 600 = 150$, $b_{4,1} = \frac{220}{640} \times 600 = 206.25$, $b_{3,1} = \frac{160}{640} \times 600 = 125$, $b_{2,1} = \frac{100}{640} \times 600 = 93.75$
Talmudic Cost: Since the lowest gain is 100 by Bank 2, all banks will share the cost of this proportion. $b_{2,1} = \frac{100-0}{4} \times k$, $b_{3,1} = b5,1 = \frac{100}{4} \times k + \frac{160-100}{3} \times k$ and $b_{4,1} = \frac{100}{4} \times k + \frac{160-100}{3} \times k + (220 - 160) * k$. To meet the cost, sum of all this payments should add up to $c_1 = 600$. That means $k = \frac{600}{220} = 2.73$. That means $b_{4,1} = 25 * 2.73 + 20 * 2.73 + 60 * 2.73 = 286.65$. From the value function of Bank 4, $V_4^1 = \max\{600 - 286.65, 380\}$, we can conclude that Bank 4 would not contribute to rescue operation for Bank 1 and Bank 1 cannot be saved under Talmudic Cost Allocation rule.

Thus, this example shows that it is possible to stop the cascade of failiures at the start (round 1) under PCA, as we proved in Theorem 2.1 while the rescue operation would be limited under Talmudic Cost Sharing method.

2.5 The Dismissal of Time-Consistency

In this section, we discuss rescue and cost allocation without considering the various steps of the bank failure cascades.[4] In this alternative approach, time-consistency is dismissed, as rounds of the FDA is not considered.

[4]We thank our jury member Han Özsöylev for his input leading to the findings in this section.

Given a financial system $(\mathbf{e}, \bar{\mathbf{L}}, d)$, suppose that Assumptions 1 and 2 hold. The rescuing consortium decides whether to rescue (the insolvent) bank i^* such that $\{i^*\} = B^1$. If i^* is saved at a cost

$$c_{i^*} = \sum_{k \in N} \bar{L}_{i^*,k} - \left(e_{i^*} + \sum_{k \in N} \bar{L}_{k,i^*} \right), \tag{2.22}$$

then each of the banks $j \in N$ with $j \neq i^*$ get to enjoy a level of wealth of

$$\bar{V}_j := e_j + \sum_{k \in N} \left(\bar{L}_{k,j} - \bar{L}_{j,k} \right), \tag{2.23}$$

as the cascade of bank failures will not be triggered in the first place. Notice that c_{i^*} is strictly positive and $\bar{V}_{i^*} = 0$ since $\{i^*\} = B^1$.

On the other hand, if i^* is not rescued, unlike our time-consistent cost allocation rules, this time we assume that the system goes to the limit (fixed-point) market clearing vector $\mathbf{L}^*$. We let the set of banks defaulting B given by B^M. Then, the wealth of banks $\ell \in B$ (including i^*) are all 0 (i.e., $V^*_\ell = 0$ for all $\ell \in B$), while the wealth of bank $j \notin B$ is given by (rewritten version of equation (2.5))

$$V^*_j := e_j + \sum_{k \notin B} \bar{L}_{k,j} + \sum_{k \in B} L^*_{k,j} - \sum_{k \in N} \bar{L}_{j,k}. \tag{2.24}$$

As a result, the net gains of the banks emerge from using equations (2.23) and (2.24) as follows: For any bank $j \notin B$, the net gains from the rescue of bank i^* (and hence the prevention of cascade of bank defaults) equals

$$\sum_{k \in B} \left[\bar{L}_{k,j} - L^*_{k,j} \right], \tag{2.25}$$

as

$$\left(e_j + \sum_{k \in N} \left(\bar{L}_{k,j} - \bar{L}_{j,k} \right) \right) - \left(e_j + \sum_{k \notin B} \bar{L}_{k,j} + \sum_{k \in B} L^*_{k,j} - \sum_{k \in N} \bar{L}_{j,k} \right) = \sum_{k \in B} \left[\bar{L}_{k,j} - L^*_{k,j} \right].$$

These deliver the following net gains (of wealth) from the rescue of bank i^*:

$$W_j = \begin{cases} \sum_{k \in B} \left[\bar{L}_{k,j} - L^*_{k,j} \right] & \text{if } j \notin B, \\ e_j + \sum_{k \in N} \left(\bar{L}_{k,j} - \bar{L}_{j,k} \right) & \text{if } j \in B \setminus \{i^*\}, \\ 0 & \text{if } j = i^*. \end{cases} \tag{2.26}$$

Therefore, for any given a financial system $(\mathbf{e}, \bar{\mathbf{L}}, d)$ satisfying Assumptions 1 and 2, using the net gains from rescue (given by equation (2.26)) and costs of rescue (given by equation (2.22)), we can define the *proportional cost allocation without time-consistency* as follows: Bank $j \in N \setminus \{i^*\}$ pays

$$b_j := \frac{W_j}{\sum_{\ell \in N} W_\ell} \, c_{i^*}. \tag{2.27}$$

It is clear that for any bank $j \in B$ with $W_j = 0$ (which includes i^*), we have that $b_j = 0$. Similarly, we can formalize the *equal cost sharing allocation without time-consistency* as follows: Bank $j \in N \setminus \{i^*\}$ pays

$$b_j^E := \begin{cases} \frac{1}{\#\{\ell \in N | W_\ell > 0\}} \, c_{i^*} & \text{if } W_j > 0, \\ \\ 0 & \text{otherwise.} \end{cases} \tag{2.28}$$

Meanwhile, we skip the definition of the Talmudic cost allocation without time-consistency as it can be defined accordingly using the net gains from rescue (given by equation (2.26)) and costs of rescue (given by equation (2.22)).

Comparing the PCA (involving time-consistency) with the proportional cost allocation without time-consistency, we observe that in any given a financial system $(\mathbf{e}, \bar{\mathbf{L}}, d)$ satisfying Assumptions 1 and 2 (thanks to Theorem 2.1) under the PCA, the cascade of bank failures will be stopped before they start. Thus, the cost of rescue both under the PCA and under the proportional cost allocation without time-consistency are the same. However, the benefits are different.

Notice that under the PCA we know that thanks to Theorem 2.1 the cascade of bank failures will be stopped at every round of the FDA. Thus, the benefit of bank $j \in N$ at the first round of the FDA equals

$$\bar{V}_j - \left[e_j + \tilde{L}_{i*,j} + \sum_{k \neq i^*} \bar{L}_{k,j} - \sum_{k \in N} \bar{L}_{j,k} \right] = \bar{L}_{i*,j} - \tilde{L}_{i*,j} \geq 0,$$

where $\tilde{L}_{i^*,j}$ is such that

$$\tilde{L}_{i^*,j} := (1-\delta) \left(\frac{\bar{L}_{i^*,j}}{\sum_{k \in N} \bar{L}_{i^*,k}} \right) \left[e_{i^*} + \sum_{k \in N} \bar{L}_{k,i^*} \right].$$

When $B = B^1$, then $\tilde{L}_{i^*,j} = L^*_{i^*,j}$. However, in general, as $\mathbf{L}^*$ is the fixed-point market clearing vector, $\tilde{L}_{i^*,j} \geq L^*_{i^*,j}$ and this inequality may become strict when B^m is non-empty for $m > 1$. Therefore, the benefit of bank $j \in N$ at the first round of the FDA is greater or equal to $\bar{L}_{i*,j} - L^*_{i*,j}$.

As a result the benefit of a resilient bank $j \notin B$ under the PCA (from rescuing i^*) equals $\bar{L}_{i*,j} - \tilde{L}_{i*,j} \leq \bar{L}_{i*,j} - L^*_{i*,j}$. On the other hand, under the proportional cost allocation without time-consistency, the benefit of $j \notin B$ equals $\sum_{k \in B} \bar{L}_{k,j} - L^*_{k,j}$. Therefore, when $B = \{i^*\}$, the benefit of $j \notin B$ under the PCA and the proportional cost allocation without time-consistency equal one another. However, when B contains other financial institutions (that default due to the so-called domino effects), it is clear that

$$\sum_{k \in B} \bar{L}_{k,j} - L^*_{i*,j} = (\bar{L}_{i*,j} - L^*_{i*,j}) + (\sum_{k \neq i^*} \bar{L}_{k,j} - L^*_{k,j}) \geq \bar{L}_{i*,j} - L^*_{i*,j} \geq \bar{L}_{i*,j} - \tilde{L}_{i*,j},$$

establishing that the benefit of a resilient bank $j \notin B$ under the proportional cost allocation without time-consistency is bloated when compared with the benefit of j under the PCA. On the other hand, for $j \in B \setminus \{i^*\}$, we see that the benefit of j under the proportional cost allocation without time-consistency equals

$$e_j + \sum_{k \in N} \left(\bar{L}_{k,j} - \bar{L}_{j,k} \right), \tag{2.29}$$

while j's benefit under the PCA equals the expression in the previous equation (2.29) when $j \in B^2$, but when $j \in B^m$ for $m > 2$ we observe that j's benefit cannot exceed (and at the time it may be strictly lower than) the expression given in equation (2.29). As a result, for non-resilient banks $j \notin B$ too, the benefit of $j \notin B$ under the proportional cost allocation without time-consistency is bloated when compared with the benefit of j under the PCA.

Even though the benefits of all banks (apart from i^*) are bloated under the proportional cost allocation without time-consistency when compared with benefits under the PCA, we cannot conclude an implication pertaining to the rescue operations and emergence of cascades of bank failures. This is because, the costs are the same and the banks' payments depend on total benefits. For example, for a bank $k \in B$ but $k \notin B^1 \cup B^2$, the benefit under the proportional cost allocation without time-consistency could be strictly more than the benefit under the PCA, while this also holds for the total benefits. Hence, we cannot predict which way the "ratio" will go, and hence, cannot produce implications.

Notwithstanding, these suggest that the proportional cost allocation without time-consistency is not qualitatively different than the PCA, thanks to Theorem 2.1. The differences are merely quantitative. The same techniques can be used to analyze equal sharing with and without time consistency. However, whether or not the differences are merely quantitative (not qualitative) in such analysis is an open question.

3. CONCLUDING REMARKS

This thesis utilizes the model presented by Rogers and Veraart (2013) and the FDA of Eisenberg and Noe (2001) with cost allocation methods to analyze in the context of preventing cascading failures in banking networks. By doing so, we are able to evaluate the financial contagion in steps and we can observe what the financial network look like without any intervention to stop the cascade.

In this thesis, we show that the PCA method can be utilized to stop cascading bank failures and prevent the spread of financial distress before it starts. Solvent banks in the system have an incentive to save the defaulting banks with the PCA method. The outcome reached with PCA shows that all the banks have higher wealth when compared to the situation in which at least a bank defaults. We have discussed two other cost allocation methods (equal cost sharing and Talmudic cost allocation) and show some examples where they might fail to put a stop to these failures.

One can extend this model by relaxing the assumption of single failure and allowing multiple bank failures. This relaxation will make the model more complex since new bargaining situations emerge. For example, there can be some cases in which solvent banks cannot rescue both of the financially distressed banks and have to bargain over which banks should be rescued. In that case, the bargaining will be on two dimensions, the amount of money to rescue and the choice of banks to save. Another future direction of this thesis would be examining some other cost allocation methods and finding general features that a cost allocation method should have to guarantee a rescue before the cascade of defaults starts as it is the case for PCA. With this way, one can group the cost allocation methods that have common features and gives the same rescuing outcome for rescue in financial networks. Moreover, another avenue of future research involves the study of the effect of the network structure on the bargaining among the banks and vary some key parameters in the system such as the capitalization level of the banks, diversification, and integration of the network, and the concentration of the banks.

BIBLIOGRAPHY

Allen, Franklin, and Douglas Gale. 2000. "Financial contagion." *Journal of political economy* 108(1): 1–33.

Aumann, Robert J, and Michael Maschler. 1985. "Game theoretic analysis of a bankruptcy problem from the Talmud." *Journal of economic theory* 36(2): 195–213.

Banerjee, Tathagata, and Zachary Feinstein. 2019. "Impact of contingent payments on systemic risk in financial networks." *Mathematics and Financial Economics* 13(4): 617–636.

Eisenberg, Larry, and Thomas H. Noe. 2001. "Systemic Risk in Financial Systems." *Management Science* 47(2): 236–249.

Elliott, Matthew, Benjamin Golub, and Matthew O Jackson. 2014. "Financial networks and contagion." *American Economic Review* 104(10): 3115–53.

Elsinger, Helmut, Alfred Lehar, and Martin Summer. 2013. "Network models and systemic risk assessment." *Handbook on Systemic Risk* 1: 287–305.

Furfine, Craig H. 2003. "Interbank exposures: Quantifying the risk of contagion." *Journal of money, credit and banking* pp. 111–128.

Glasserman, Paul, and H Peyton Young. 2016. "Contagion in financial networks." *Journal of Economic Literature* 54(3): 779–831.

Jackson, Matthew O, and Agathe Pernoud. 2020. "Systemic risk in financial networks: A survey." *Annual Review of Economics* 13.

James, Christopher. 1991. "The losses realized in bank failures." *The Journal of Finance* 46(4): 1223–1242.

Kanik, Zafer. 2020. "From Lombard Street to Wall Street: systemic risk, rescues, and stability in financial networks." *Rescues, and Stability in Financial Networks (April 30, 2020) .NET Institute Working Paper* (17-17).

Nier, Erlend, Jing Yang, Tanju Yorulmazer, and Amadeo Alentorn. 2007. "Network models and financial stability." *Journal of Economic Dynamics and Control* 31(6): 2033–2060.

Papp, Pál András, and Roger Wattenhofer. 2020. “Sequential defaulting in financial networks.” *arXiv preprint arXiv:2011.10485* .

Rogers, Leonard CG, and Luitgard AM Veraart. 2013. “Failure and rescue in an interbank network.” *Management Science* 59(4): 882–898.

Upper, Christian. 2011. “Simulation methods to assess the danger of contagion in interbank markets.” *Journal of Financial Stability* 7(3): 111–125.