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SOME BEST PROXIMITY POINT RESULTS FOR MULTIVALUED MAPPINGS ON PARTIAL METRIC SPACES

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By Doaa Riyadh ABED AL-ZUHAIRI

August 2021

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Doaa Riyadh ABED AL-ZUHAIRI

ABSTRACT

SOME BEST PROXIMITY POINT RESULTS FOR MULTIVALUED MAPPINGS ON PARTIAL METRIC SPACES

Doaa Riyadh ABED AL-ZUHAIRI

Master of Science in Mathematics

Advisor: Asst. Prof. Dr. Mustafa ASLANTAŞ

August 2021

Fixed point theory is an exciting branch of mathematics. It is a mixture of analysis, topology and geometry. Fixed point techniques have been applied in fields such as biology, chemistry, economics, engineering and physics. It has very fruitful applications in control theory, game theory, category theory, functional equations, integral equations, mathematical physics, mathematical chemistry, functional analysis and many other areas. Thus, the study of the fixed point theory has been researched extensively in different metric spaces. On the other hand, by introducing the concept of the best proximity point, taking into account non-essential mappings, the fixed point results are expanded in a different sense. In this thesis, we obtain some best proximity point results for multivalued mappings via partial Hausdorff metric on partial metric spaces. Also, we obtain some new best proximity point results for cyclic multivalued mappings on partial metric spaces by considering Feng-Liu's technique.

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Keywords: Multivalued mapping, partial metric space, fixed point, best proximity point, mixed multivalued mapping, cyclic mapping

ÖZET

KISMİ METRİK UZAYLARDA KÜME DEĞERLİ DÖNÜŞÜMLER İÇİN BAZI EN İYİ YAKINLIK NOKTASI SONUÇLARI

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Sabit nokta teorisi, matematiğin heyecan verici bir dalıdır. Analiz, topoloji ve geometrinin bir karışımıdır. Biyoloji, kimya, ekonomi, mühendislik ve fizik gibi alanlarda sabit nokta teknikleri uygulanmıştır. Kontrol teorisi, oyun teorisi, kategori teorisi, fonksiyonel denklemler, integral denklemler, matematiksel fizik, matematiksel kimya, fonksiyonel analiz ve diğer birçok alanda çok verimli uygulamaları vardır. Bu nedenle, sabit nokta teorisi çalışması, farklı metrik uzaylarda kapsamlı bir şekilde araştırılmıştır. Öte yandan, temel olmayan eşlemeler dikkate alınarak en iyi yakınlık noktası kavramı tanıtılarak, sabit nokta sonuçları farklı bir anlamda genişletilir. Bu tezde, kısmi metrik uzaylar üzerinde kısmi Hausdorff metriği yoluyla çok değerli eşlemeler için en iyi yakınlık noktası sonuçlarını elde ediyoruz. Ayrıca, Feng-Liu'nun tekniğini göz önünde bulundurarak kısmi metrik uzaylarda döngüsel çok değerli eşlemeler için bazı yeni en iyi yakınlık noktası sonuçları elde ettik.

2021, 36 sayfa

Anahtar Kelimeler:Küme değerli dönüşümler,Kısmi metrik uzaylar, Sabit nokta, En iyi yakınlık noktası, Karma küme değerli dönüşümler, döngüsel dönüşümler

PREFACE AND ACKNOWLEDGEMENTS

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Doaa Riyadh ABED AL-ZUHAIRI
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CONTENTS

LIST OF SYMBOLS

$\forall$	For all
$\exists$	There exists
$\ni$	Such that
$\Longrightarrow$	Implies that
$\Leftrightarrow$	If an only if
$\mathbb{N}$	The set of all natural numbers
$\mathbb{Z}$	The set of all integer numbers
$\mathbb{R}$	The set of real numbers
$\bar{A}$	The closure of a set A
$CB(X)$	The set of all closed and bounded subsets of metric space
$CB^p(X)$	The set of all closed and bounded subsets of partial metric space
$\limsup$	Limit superior
$\liminf$	Limit inferior
ρ_ω	The ordinary metric generated by the partial metric ω
$\mathbb{R}^+$	The set of positive real numbers
$\mathbb{R}_0^+$	$\mathbb{R}^+ \cup \{0\}$
τ_ω	The topology generated by the partial metric ω

1. INTRODUCTION

Fixed point theory is an exciting branch of mathematics. It is a mixture of analysis topology and geometry. Fixed point theory is one of the most dynamic research areas in nonlinear analysis. It has a wide range of applications in fields such as economics, computer science and many others. The most important in this direction was given by polish mathematician Banach (1922), popularly known as Banach contraction principle. This principle not only guarantees the existence and uniqueness of the fixed points of self-mappings but also gives a procedure to obtain the fixed points of these mappings (Abdeljawad *et al.* 2012, Agarwal *et al.* 2009, Chi *et al.* 2012, Faraji 2019, Hashim and Singh 2017, Karapınar and Erhan 2011, Karapınar *et al.* 2012, Karapınar and Erhan 2012, Karapınar and Romaguera 2013, Matthews 1995, Oltra and Valero 2004, Pata 2019, Romaguera 2012,1, Wadge 1981) . This technique is used as a powerful tool to solve various problems in different branches of mathematic such as integral equations, ordinary differential equations, partial differential equations, dynamic programming and game theory. Later, the Banach contraction principle is extended to different spaces.

Let W be nonempty set and $\Lambda : W \to W$ be a mapping. A point $\bar{a} \in W$ is called a fixed point of Λ if $\Lambda\bar{a} = \bar{a}$. Let $\Lambda : W \to P(W)$ be a multivalued mapping. A point $\bar{a} \in W$ is a fixed point if $\bar{a} \in \Lambda\bar{a}$.

One of the first studies on fixed point theory in metric spaces was made by Banach (1922) by obtaining a fundamental result known as the Banach contraction principle. Just as the Banach fixed point theorem guarantees the existence of the fixed point of the transformation, it also shows the uniqueness of this fixed point and how it can be found. Later, the Banach contraction principle is extended to different spaces.
It is well known that $\bar{a} = \bar{b}$ if and only if $\rho(\bar{a}, \bar{b}) = 0$ for all $\bar{a}, \bar{b}$ in a metric space (W, ρ). However, motivated by the experience of computer science, there was a tendency to relax this equivalence. In this sense, Matthews (1994) introduced a new concept so called partial metric by weakening the mentioned condition.

Recently, Haghi *et al.* (2012) presented a very interesting paper by associating to each partial metric space (W, ω) with a metric space (W, ρ) by setting $\rho(\bar{a}, \bar{b}) = 0$ if $\bar{a} = \bar{b}$ and

$\rho(\bar{a},\bar{b}) = \omega(\bar{a},\bar{b})$ if $\bar{a} \neq \bar{b}$, and proved that (W,ω) is 0-complete if and only if (W,ρ) is complete. They proved that some fixed point results in partial metric space results are equivalent to the results in the context of a usual metric space. However, some conclusions important for applications of partial metrics in computer sciences cannot be obtained in this way. For example, using the method from Haghi *et al.* (2012) one cannot conclude that $\omega(\bar{a},\bar{a}) = 0 = \omega(\Lambda\bar{a},\Lambda\bar{a})$ when $\bar{a}$ is a fixed point of Λ.

On the other hand, Basha and Veeramani (1977) introduced the concept of best proximity point. Best proximity point theory analyzes the condition under which the optimization problem, namely, $\inf_{\bar{a}\in A} \rho(\bar{a},\Lambda\bar{a})$, has a solution. The point $\bar{a}$ is called the best proximity point of $\Lambda : A \to B$ if $\rho(\bar{a},\Lambda\bar{a}) = \rho(A,B)$, where

$$\rho(A,B) = \inf\left\{\rho\left(\bar{a},\bar{b}\right) : \bar{a} \in A; \bar{b} \in B\right\}$$

and many authors have generalized and extended in different ways (Abkar *et al.* 2016, Al-Thagafi and Shahzad 2009, Aslantas 2021, Basha 2006, Feng and Liu 2006).
In this thesis, we will investigate some best proximity point results for multivalued mappings on partial metric spaces.

2. PRELIMINARIES

2.1 Metric Spaces

Definition 2.1. (Jain and Ahmad 2004) Let W be a nonempty set and $\rho : W \times W \to \mathbb{R}$ be a function. If the next conditions hold: for all $\bar{a}, \bar{b}, \bar{c} \in W$,

p1) $\rho\left(\bar{a}, \bar{b}\right) \geq 0$,

p2) $\rho(\bar{a}, \bar{b}) = 0$ iff $\bar{a} = \bar{b}$,

p3) $\rho(\bar{a}, \bar{b}) = \rho(\bar{b}, \bar{a})$,

p4) $\rho(\bar{a}, \bar{b}) \leq \rho(\bar{a}, \bar{c}) + \rho(\bar{c}, \bar{b})$.

then, ρ is called a metric on W. Also, the pair (W, ρ) is called a metric space.

Geometrically, $\rho(\bar{a}, \bar{b})$ represents distance between two points $\bar{a}$ and $\bar{b}$ on the real line.

Example 2.2. Let $W = \mathbb{R}$ and $\rho : W \times W \to \mathbb{R}$ be a function defined by $\rho\left(\bar{a}, \bar{b}\right) = \left|\bar{a} - \bar{b}\right|$ for all $\bar{a}, \bar{b} \in \mathbb{R}$. Then, (W, ρ) is a metric space. This metric is referred to as the usual metric.

Example 2.3. Let $W = \mathbb{R}^2$ and $\rho : W \times W \to \mathbb{R}$ be a function defined by

$$\rho\left(\left(\bar{a_1}, \bar{b}_1\right), \left(\bar{a_2}, \bar{b}_2\right)\right) = \max\left\{\left|\bar{a_1} - \bar{a_2}\right|, \left|\bar{b}_1 - \bar{b}_2\right|\right\}$$

Then, (W, ρ) is a metric space. This metric is referred to as the supremum or maximum metric.

Example 2.4. Let $W = \mathbb{R}^2$ and $\rho : W \times W \to \mathbb{R}$ be a function defined by $\rho\left(\bar{a}, \bar{b}\right) = \sqrt{\left(\bar{a_1} - \bar{b}_1\right)^2 + \left(\bar{a_2} - \bar{b}_2\right)^2}$, where $\bar{a} = \left(\bar{a_1}, \bar{a_2}\right), \bar{b} = \left(\bar{b}_1, \bar{b}_2\right)$. Then, (W, ρ) is a metric space. This metric is referred to as the Euclidean metric.

Example 2.5. As a set W we take the set of all real-valued continuous functions on closed interval I denoted by $C(I)$. Let $\rho : W \times W \to \mathbb{R}$ be a function defined by $\rho(f, g) = \max_{t \in I} |f(t) - g(t)|$. Then, (W, ρ) is a metric space. This metric is referred to as the uniform metric. The set of all real-valued continuous functions on closed interval I are denoted by $C(I)$.

Definition 2.6. (Maddox 1988) Let (W,ρ) be a metric space and $\bar{a_0} \in W$. The set

$$B(\bar{a_0},r) = \left\{\bar{b} \in W : \rho\left(\bar{a_0},\bar{b}\right) < r\right\}.$$

For each $r \in \mathbb{R}^+$ is called open ball.

Example 2.7. Let $(\mathbb{R},\rho)$ be usual metric. The open ball

$$B(\bar{a_0},r) = (-r+\bar{a_0}, r+\bar{a_0}).$$

Definition 2.8. (Maddox 1988) Let (W,ρ) be a metric space and $\bar{a_0} \in W$. The set

$$\overline{B}(\bar{a_0},r) = \left\{\bar{b} \in W : \rho\left(\bar{a_0},\bar{b}\right) \leq r\right\}.$$

For each $r \in \mathbb{R}^+$ is called closed ball .

Example 2.9. Let $(\mathbb{R},\rho)$ be usual metric. The closed ball

$$\overline{B}(\bar{a_0},r) = [-r+\bar{a_0}, r+\bar{a_0}].$$

Definition 2.10. (Maddox 1988) Let (W,ρ) be a metric space and S be a nonempty subset of W. The set S is open if it contains an open ball about each of its points i.e., if

$$\forall \bar{a} \in S : \exists\, \varepsilon > 0 : B(\bar{a},\varepsilon) \subseteq S.$$

Definition 2.11. (Maddox 1988) Let (W,ρ) be a metric space and S be a nonempty subset of W. The set S is closed if it is the complement of an open set.

Example 2.12. Let W be nonempty set and $\rho : W \times W \to \mathbb{R}$ be a function defined by

$$\rho(\bar{a},\bar{b}) = \begin{cases} 0 & , \quad \bar{a} = \bar{b} \\ 1 & , \quad \bar{a} \neq \bar{b} \end{cases}$$

Then, (W,ρ) is a metric space known as discrete metric space. Each point $\{\bar{a}\} = B\left(\bar{a}, \frac{1}{2}\right)$ so is an open set. Hence every set S is open, since for $\bar{a} \in S$ we have $B\left(\bar{a}, \frac{1}{2}\right) \subseteq S$. Then, by taking complements, every set is also closed.

Definition 2.13. (Maddox 1988) Let (W,ρ) be a metric space. A subset $S \subseteq W$ is referred to as bounded if there exists a positive real number $r > 0$ such that $S \subseteq B(\bar{a}, r)$ for some $\bar{a} \in W$. The set S is said to be unbounded if it is not bounded.

Definition 2.14. (Kreyszig 1978) A function whose domain is the set of natural numbers or a subset of the natural numbers is said to be sequence. We usually use the symbol $\{\bar{a}_n\}$ to represent a sequence, where n is a natural number and $\bar{a}_n$ is the value of the function on n.

Definition 2.15. (Maddox 1988) Let (W,ρ) be a metric space and $\{\bar{a_n}\}_{n=1}^{\infty}$ be sequence in W. Then, it is a Cauchy sequence if and only if $\rho\left(\bar{a_n}, \bar{a_m}\right) \to 0$ as $n,m \to \infty$, i.e. for all $\varepsilon > 0$, there exists $N = N(\varepsilon)$ such that $\rho\left(\bar{a_n}, \bar{a_m}\right) < \varepsilon$ for all $n,m > N$.

Example 2.16. Let $(\mathbb{R},\rho)$ be a usual metric space and $\{\bar{a_n}\} = \left\{1 - \frac{1}{2^n}\right\}$ be a sequence in W. Then, the sequence $\{a_n\}$ is a Cauchy sequence on $\mathbb{R}$.

Example 2.17. Let $((0,1],\rho)$ be a usual metric and $\{\bar{a_n}\} = \frac{1}{n}$ be a sequence in W. Then, the sequence $\{\bar{a}_n\}$ is Cauchy sequence.

Definition 2.18. (Maddox 1988) Let (W,ρ) be a metric space and $\{\bar{a}_n\}$ be a sequence in W. It is called convergent (with limit $\bar{a}_0$) if and only if for every $\varepsilon > 0$, there exists $N = N(\delta,\varepsilon)$ such that $\rho\left(\bar{a}_n, \bar{a}_0\right) < \varepsilon$ for all $n \geq N$. We write $\bar{a}_n \to \bar{a}_0$ as $n \to \infty$ or $\lim \bar{a}_n = \bar{a}_0$.

Example 2.19. Let (W,ρ) be a usual metric space and $\{\bar{a}_n\} = \frac{1}{n}$ be a sequence in $\mathbb{R}$. Then, the sequence $\{\bar{a}_n\}$ is convergent.

Theorem 2.20. (Maddox 1988) Let (W,ρ) be a metric space and $\{\bar{a}_n\}$ be a sequence in W. If $\{\bar{a}_n\}$ is convergent, then it is unique.

Proof. Suppose that $\bar{a}_n \to \bar{a} \in W$ and $\bar{a}_n \to \bar{b} \in W$. Let $\varepsilon > 0$ be given. Since $\bar{a}_n \to \bar{a}$, there exists $n_0' \in \mathbb{N}$ such that

$$\rho\left(\bar{a}_n, \bar{a}\right) < \frac{\varepsilon}{2},$$

also, since $\bar{a}_n \to \bar{b}$, there exists $n_0'' \in \mathbb{N}$ such that

$$\rho\left(\bar{a}_n, \bar{b}\right) < \frac{\varepsilon}{2},$$

select $n_0 = \max\left\{n_0', n_0''\right\}$. Then, for all $n > n_0$

$$\begin{aligned} \rho\left(\bar{a}, \bar{b}\right) &\leq \rho\left(\bar{a}, \bar{a}_n\right) + \rho\left(\bar{a}_n, \bar{b}\right) \\ &< \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

Then, $\rho\left(\bar{a}, \bar{b}\right) \leq \lim_{\varepsilon \to 0^+} \varepsilon \Rightarrow \rho\left(\bar{a}, \bar{b}\right) \leq 0 \Rightarrow \rho\left(\bar{a}, \bar{b}\right) = 0 \Rightarrow \bar{a} = \bar{b}$. □

Definition 2.21. (Maddox 1988) Let (W, ρ) be a metric space. (W, ρ) is called complete if each Cauchy sequence in W is convergent to a point of W.

Example 2.22. Let W be a nonempty set and (W, ρ) be a discrete metric space. Then, (W, ρ) is a complete metric space.

Remark 2.23. Every convergent sequence in a metric space is a Cauchy sequence but the converse may not be true. Indeed, let W be set of all rational numbers with the an usual metric space. It is well familiar that the sequence

$$1.4,\ 1.41,\ 1.414, ...$$

converges to $\sqrt{2}$. Therefore, it is a Cauchy. But, it well known not converge to a point of Ω.

Definition 2.24. (Agarwal *et al.* 2009) Let (W, d_1), (Y, d_2) be two metric spaces and $f : W \to Y$ be a function, then f is called continuous at a point $\bar{a}_0$ if and only if for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon, \bar{a}_0) > 0$ such that $d_2\left(f(\bar{a}), f(\bar{a}_0)\right) < \varepsilon$ whenever $d_1(\bar{a}, \bar{a}_0) < \delta$ for all $\bar{a} \in W$.

Example 2.25. Let $(\mathbb{R}, \rho)$ be a usual metric space and $f : (0,1) \to \mathbb{R}$ be a function defined by $f(\bar{a}) = \frac{1}{\bar{a}}$. Then, the function f is continuous on $(0,1)$.

Definition 2.26. (Lafferriere *et al.* 2016) Let $f : W \to \mathbb{R}$ be a function and $\bar{a}_0 \in W$. We

say that f is lower semicontinuous function at $\bar{a_0}$ if every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$f(\bar{a_0}) - \varepsilon < f(\bar{a}) \text{ for all } \bar{a} \in B(\bar{a_0}, \delta) \cap W.$$

Similarly, we say that f is upper semicontinuous function at $\bar{a_0}$ if every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$f(\bar{a}) < f(\bar{a_0}) + \varepsilon \text{ for all } \bar{a} \in B(\bar{a_0}, \delta) \cap W.$$

It is clear that f is continuous at $\bar{a_0}$ if and only if f is lower semicontinuous function and upper semicontinuous function at this point.

Theorem 2.27. (Lafferriere *et al.* 2016) Let $f : W \to \mathbb{R}$ be a function and $\bar{a_0} \in W$ be a limit point of W. Then f is lower semicontinuous function at $\bar{a_0}$ if and only if

$$\lim_{\bar{a} \to \bar{a_0}} \inf f(\bar{a}) \geq f(\bar{a_0}).$$

Similarly, f is upper semicontinuous function at $\bar{a_0}$ if and only if

$$\lim_{\bar{a} \to \bar{a_0}} \inf f(\bar{a}) \leq f(\bar{a_0}).$$

Theorem 2.28. (Lafferriere *et al.* 2016) Let $f : W \to \mathbb{R}$ be a function and $\bar{a_0} \in W$. Then f is lower semicontinuous function at $\bar{a_0}$ if and only if for every sequence $\{\bar{a}_n\}$ in W that converges to $\bar{a}$,

$$\lim_{n \to \infty} \inf f(\bar{a}_n) \geq f(\bar{a_0}).$$

Similarly, f is upper semicontinuous function at $\bar{a_0}$ if and only if for every sequence $\{\bar{a}_n\}$ in X that converges to $\bar{a_0}$,

$$\lim_{n \to \infty} \inf f(\bar{a}_n) \leq f(\bar{a_0}).$$

2.2 Fixed Point and Best Proximity Point Theory

Definition 2.29. (Agarwal *et al.* 2009) Let W be nonempty set and $\Lambda : W \to W$ be a mapping. A point $\bar{a} \in W$ is called a fixed point of Λ if $\Lambda\bar{a} = \bar{a}$. Let $\Lambda : W \to P(W)$ be a multivalued mapping. A point $\bar{a} \in \Lambda\bar{a}$ is called a fixed point of Λ.

Example 2.30. Let $W = [1,\infty)$ and $\Lambda : W \to W$, $F : W \to W$ be two mappings defined by

$$\Lambda\bar{a} = \frac{1+\bar{a}}{2},$$

and

$$F\bar{a} = 1+\bar{a},$$

respectively. Then it can be seen that $\bar{a} = 1$ is a fixed point of Λ, but F has not a fixed point in W. Also, if we take $W = (1,\infty)$, then Λ and F have not a fixed point. Hence, the existence of the fixed point depends on both the mapping and the set in which the mapping is defined.

Furthermore, we also say that the fixed point of the mapping Λ has a solution to the equation $\Lambda\bar{a} = \bar{a}$. Indeed, let's consider the equation

$$\bar{a}^2 - 6\bar{a} + 5 = 0 \tag{2.1}$$

Hence, we can write $\bar{a} = \frac{\bar{a}^2+5}{6}$. If we define $\Lambda : \mathbb{R} \to \mathbb{R}$ as

$$\Lambda\bar{a} = \frac{\bar{a}^2+5}{6}.$$

Then, fixed points of Λ is a solution of the equation (2.1).

Definition 2.31. (Agarwal *et al.* 2009) Let (W,ρ) be a metric space and $\Lambda : W \to W$ be a mapping.

- Λ is called a contraction mapping if it exists k in $[0,1)$ such that

  $$\rho(\Lambda\bar{a},\Lambda\bar{b}) \leq k\rho(\bar{a},\bar{b})$$

  for all $\bar{a},\bar{b} \in W$.
- Λ is called a nonexpansive mapping if the following inequality holds

  $$\rho(\Lambda\bar{a},\Lambda\bar{b}) \leq \rho(\bar{a},\bar{b})$$

  for all $\bar{a},\bar{b} \in W$.

- Λ is called a contractive mapping if the following inequality holds

$$\rho(\Lambda\bar{a}, \Lambda\bar{b}) < \rho(\bar{a}, \bar{b})$$

for all $\bar{a} \neq \bar{b} \in W$.

Note that every contraction mapping is a contractive mapping, and also every contractive mapping is a nonexpansive mapping. However, the opposite may not be true.

Example 2.32. Let $(\mathbb{R}, \rho)$ be a usual metric space.

- If we define $\Lambda : \left[0, \frac{1}{3}\right] \to \mathbb{R}$ as $\Lambda\bar{a} = \bar{a}^2$, then Λ is contraction mapping.
- If we define $\Lambda : \mathbb{R} \to \mathbb{R}$ as $\Lambda\bar{a} = \bar{a}$, then Λ is a nonexpansive mapping. Also, Λ is not a contractive mapping.
- If we define $\Lambda : [1, \infty) \to [1, \infty)$ as $\Lambda\bar{a} = \frac{\bar{a}^2+1}{\bar{a}}$, then Λ is a contractive mapping. Also, Λ is not a contraction mapping.

Banach's Fixed Point Theorem is an existence and uniqueness theorem for fixed points of certain mappings. As we will see from the proof, it also provides us with a constructive procedure for getting better and better approximations of the fixed point. This procedure is called iteration; we start by selecting an arbitrary $\bar{a}_0$ in a given set, and compute recursively a sequence $\bar{a}_1, \bar{a}_2, \bar{a}_3, ...$ by letting

$$\bar{a}_{n+1} = \Lambda\bar{a}_n, \qquad n = 0, 1, 2...$$

Such iteration procedures can be found in nearly every branch of applied mathematics, and Banach's Fixed Point Theorem is frequently used to ensure the scheme's convergence and the uniqueness of the solution.

Theorem 2.33. (Banach 1922) (The Banach contraction principle) Let (W, ρ) be a complete metric space and $\Lambda : W \to W$ be a mapping. If Λ is a contraction mapping, then Λ has a unique fixed point in W.

One of the generalizations of Banach's result is obtained by Nadler (1969) via Hausdorff metric. Firstly, a fixed point result has been obtained for multivalued mappings. Now, we recall some definitions and concepts related to multivalued mappings. Let

$$P(W) = \{A \subseteq W : A \neq \emptyset\},$$

$$C(W) = \{A \subseteq W : A \text{ is nonempty closed subsets of } W.\},$$

$$CB(W) = \{A \subseteq W : A \text{ is nonempty, closed and bounded subsets of } W.\}.$$

Definition 2.34. (Nadler 1969) Let (W,ρ) be a metric space $A,B \in C(W)$, and $H : CB(W) \times CB(W)$

$$H(A,B) = \max\left\{\sup_{\bar{a}\in A}\rho(\bar{a},B), \sup_{\bar{b}\in B}\rho(\bar{b},A)\right\},$$

where $\rho(\bar{a},B) = \inf\left\{\rho(\bar{a},\bar{b}) : \bar{b} \in B\right\}$. Then, H is called Pompeiu-Hausdorff metric on $CB(W)$.

Definition 2.35. (Nadler 1969) Let (W,ρ) be a metric space and $\Lambda : W \to CB(W)$ be a multivalued mapping. Then, Λ is called multivalued contraction mapping if it exists $\lambda \in [0,1)$ such that

$$H(\Lambda\bar{a},\Lambda\bar{b}) \leq \lambda\rho(\bar{a},\bar{b})$$

for all $\bar{a},\bar{b} \in W$.

Nadler (1969) proved that every multivalued contraction mapping on a complete metric space has a fixed point in W. Then, Feng and Liu (2006) obtained the following result which is a generalization of Nadler's result.

Theorem 2.36. Let (W,ρ) be a complete metric space and $\Lambda : W \to C(W)$ be a multivalued mapping. If for all $\bar{a} \in W$ it exists $\bar{b} \in I_\beta^{\bar{a}}$ satisfying

$$\rho(\bar{b},\Lambda\bar{b}) \leq \gamma\rho(\bar{a},\bar{b}), \tag{2.2}$$

where

$$I_\beta^{\bar{a}} = \left\{\bar{b} \in \Lambda\bar{a} : \beta\rho(\bar{a},\bar{b}) \leq \rho(\bar{a},\Lambda\bar{a})\right\}.$$

Then, Λ has a fixed point in W provided that $0 < \gamma < \beta < 1$ and the function $f(\bar{a}) = \rho(\bar{a},\Lambda\bar{a})$ is lower semicontinuous.

On the other hand, Kirk *et al.* (2003) proved that the Banach contraction principle had been further generalized by introducing a new concept of cyclic mapping and they achieved the following nice results.

Theorem 2.37. (Kirk *et al.* 2003) Let (W,ρ) be a complete metric space, $\emptyset \neq A,B \subseteq W$ where A and B are closed and $\Lambda : A\cup B \to A\cup B$ be a mapping. Assume that Λ is a cyclic mapping, that is, $\Lambda(A) \subseteq B$ and $\Lambda(B) \subseteq A$. If it exists k in $[0,1)$ such that

$$\rho(\Lambda\bar{a},\Lambda\bar{b}) \leq k\rho(\bar{a},\bar{b}) \tag{2.3}$$

for all $\bar{a} \in A$ and $\bar{b} \in B$, then Λ has a fixed point in $A\cap B$.

Note that, unlike Banach contraction principle, the mapping Λ is not necessary to be continuous in Theorem 2.37. Because of its applicability, there are many studied on this subject in the literature. In this sense, (Eldred and Veeramani 2006) introduced a concept of cyclic contraction mapping by considering $A\cap B = \emptyset$ and proved a fundamental result.

Definition 2.38. (Eldred and Veeramani 2006) Let (W,ρ) be a metric space and $\emptyset \neq A,B \subseteq W$. A mapping $\Lambda : A\cup B \to A\cup B$ is called a cyclic contraction if it satisfies $\Lambda(A) \subseteq B$ and $\Lambda(B) \subseteq A$ and the following condition:

$$\rho(\Lambda\bar{a},\Lambda\bar{b}) \leq k\rho(\bar{a},\bar{b}) + (1-k)\rho(A,B) \tag{2.4}$$

for all $\bar{a} \in A$ and $\bar{b} \in B$, where $k \in [0,1)$.

Taking into account this situation, the inequality (2.3) was generalized different way from the results in the literature as in inequality (2.4). Consequently, they obtained the existence of best proximity point of Λ as follows:

Theorem 2.39. (Eldred and Veeramani 2006) Let (W,ρ) be a metric space, $\emptyset \neq A,B \subseteq W$ and $\Lambda : A\cup B \to A\cup B$ be a cyclic contraction mapping. Let $\bar{a}_0 \in A$ and define $\bar{a}_{n+1} = \Lambda\bar{a}_n$, for all $n \geq 1$. If $\{\bar{a}_{2n-1}\}$ has a convergent subsequence in A, then it exists $\bar{a} \in A$ such that $\rho(\bar{a},\Lambda\bar{a}) = \rho(A,B)$.

Consequently, they gave a generalization of inequality (2.3). Indeed, if $A\cap B$ is empty, then Λ cannot have a fixed point. In this case, it is sensible to find the existence of solution

of the minimization problem $\min\{\rho(\bar{a},\Lambda\bar{a}) : \bar{a} \in A\}$. Since $\rho(\bar{a},\Lambda\bar{a}) \geq \rho(A,B)$ for all $\bar{a} \in A$, a point $\bar{a}$ satisfying $\rho(\bar{a},\Lambda\bar{a}) = \rho(A,B)$ is an optimal solution of the minimization problem $\min_{\bar{a}\in A} \rho(\bar{a},\Lambda\bar{a})$. This point $\bar{a}$ is called a best proximity point of Λ. Since every best proximity point is a fixed point in special case $A = B = W$, many authors have studied this topic. (Altun *et al.* 2020, Basha 2011, Sahin *et al.* 2020)

Let (W,ρ) be a metric space, A,B be nonempty subsets of W and $\Lambda : A \to B$ be a mapping. We regard the subsets of A and B, respectively:

$$A_0 = \left\{\bar{a} \in A : \rho(\bar{a},\bar{b}) = \rho(A,B) \text{ for some } \bar{b} \in B\right\}$$

and

$$B_0 = \left\{\bar{b} \in B : \rho(\bar{a},\bar{b}) = \rho(A,B) \text{ for some } \bar{a} \in A\right\}$$

where $\rho(A,B) = \inf\left\{\rho(\bar{a},\bar{b}) : \bar{a} \in A \text{ and } \bar{b} \in B\right\}$. Basha (2011) proved a version of the Banach contraction principle for nonself mappings by introducing the concept of proximal contraction mapping.

Definition 2.40. (Basha 2006) Let (W,ρ) be a metric space, A,B be nonempty subsets of W and $\Lambda : A \to B$ be a mapping. Λ is called a proximal contraction mapping if the following implication holds:

$$\left.\begin{array}{rcl} \rho(u_1,\Lambda\bar{a}_1) & = & \rho(A,B) \\ \rho(u_2,\Lambda\bar{a}_2) & = & \rho(A,B) \end{array}\right\} \Longrightarrow \rho(u_1,u_2) \leq k\rho(\bar{a}_1,\bar{a}_2)$$

for all $\bar{a}_1,\bar{a}_2,u_1,u_2 \in A$ and for some $k \in [0,1)$.

Definition 2.41. (Basha 2006) Let (W,ρ) be a metric space and A,B be nonempty subsets of W. B is approximately compact with respect to A if for every sequence $\{\bar{b}_n\}$ satisfying $\rho(\bar{a},\bar{b}_n) \to \rho(\bar{a},B)$ for some $\bar{a} \in A$ has a convergent subsequence in B.

Theorem 2.42. (Basha 2006) Let (W,ρ) be a complete metric space, A,B be nonempty subsets of W where A is closed and B is approximately compact with respect to A and $\Lambda : A \to B$ be a mapping. Assume that $A_0 \neq \emptyset$ and Λ is a proximal contraction mapping satisfying $\Lambda(A_0) \subseteq B_0$, then Λ has a unique best proximity point in A.

Then, Raj (2013) introduced the nice concept of P-Property and obtained a best proximity point.

Definition 2.43. Let (W,ρ) be a metric space and A,B be nonempty subsets of W. The pair (A,B) is said to have the P-Property if and only if

$$\left.\begin{array}{rcl}\rho(u_1,v_1) & = & \rho(A,B)\\ \rho(u_2,v_2) & = & \rho(A,B)\end{array}\right\} \Longrightarrow \rho(u_1,u_2)=\rho(v_1,v_2)$$

for all $u_1,u_2,v_1,v_2 \in A$.

Theorem 2.44. (Raj 2013) Let (W,ρ) be a metric space, A,B be nonempty subsets of W where A,B are closed and $\Lambda : A \to B$ be a mapping. Assume that $A_0 \neq \emptyset$, $\Lambda(A_0) \subseteq B_0$ and the pair (A,B) has the P-Property. If there exists k in $[0,1)$ such that

$$\rho(\Lambda\bar{a},\Lambda\bar{b}) \leq k\rho(\bar{a},\bar{b})$$

for all $\bar{a},\bar{b} \in A$, then Λ has a best proximity point in A.

The next theorem, which is presented by Abkar and Gabeleh (2013), is best proximity point version of Nadler's result:

Theorem 2.45. Let (W,ρ) be a complete metric space, $A,B \subseteq W$ be two nonempty closed subsets of W having P-property and $A_0 \neq \emptyset$. Let $\Lambda : A \to CB(B)$ be a mapping such that $\Lambda(A_0) \subseteq B_0$ and for all $u,v \in A_0$ satisfying

$$H(Tu,Tv) \leq c\rho(u,v)$$

for some $c \in (0,1)$. Then Λ has a best proximity point in A.

Lemma 2.46. (Sahin *et al.* 2020) Let (A,ρ) and (B,ω) be two metric spaces and $\Lambda : A \to P(B)$ be an upper semicontinuous mapping. Then, the function $f : A \times B \to \mathbb{R}$ defined by $f(\bar{a},\bar{b}) = \omega(\bar{b},\Lambda\bar{a})$ is lower semicontinuous.

Considering Feng-Lui approach Sahin *et al.* (2020) obtained the following best proximity point result for the first time.

Theorem 2.47. Let (W,ρ) be a complete metric space, $A,B\subseteq W$ be two nonempty closed subsets of W having P-property and $A_0\neq\emptyset$. Let $\Lambda:A\to C(B)$ be a mapping such that $\Lambda(A_0)\subseteq B_0$ and for all $\bar{a}\in A_0$ and $\bar{b}\in\Lambda\bar{a}$ there exists $\bar{c}\in A_0$ satisfying:

$$\rho(\bar{b},\bar{c})=\rho(A,B)$$

and

$$\rho(\bar{b},Tz)\leq c\rho(\bar{a},\bar{c})$$

for some $c\in(0,1)$, then Λ has a best proximity point in A provided that $f(\bar{a},\bar{b})=\rho(\bar{b},\Lambda\bar{a})$ is lower semicontinuous.

2.3 Partial Metric Spaces

To study of denotational semantics of dataflow networks, Matthews (1994) introduced the concept of partial metric as follows:

Definition 2.48. Let W be a nonempty set . A function $\omega:W\times W\to\mathbb{R}_0^+$ is said to be a partial metric on W if for any $\bar{a},\bar{b},\bar{c}\in W$, the next conditions hold:
(p1) $\omega(\bar{a},\bar{a})=\omega(\bar{b},\bar{b})=\omega(\bar{a},\bar{b})$ if and only if $\bar{a}=\bar{b}$;
(p2) $\omega(\bar{a},\bar{a})\leq\omega(\bar{a},\bar{b})$;
(p3) $\omega(\bar{a},\bar{b})=\omega(\bar{b},\bar{a})$;
(p4) $\omega(\bar{a},\bar{c})\leq\omega(\bar{a},\bar{b})+\omega(\bar{b},\bar{c})-\omega(\bar{b},\bar{b})$.

The function ω is called a partial metric and the space (W,ω) is called a partial metric space.

Example 2.49. Let $W=\mathbb{R}$. The function $\omega:\mathbb{R}\times\mathbb{R}\to[0,\infty)$ given as

$$\omega(\bar{a},\bar{b})=\max\{|\bar{a}|,|\bar{b}|\},$$

is a partial metric but not a metric. Note that, for example $\omega(1,1)=\max\{|1|,|1|\}=1\neq 0$.

Example 2.50. Let $W=\mathbb{R}_0^+$. Define a map $\omega:W\times W\to[0,\infty)$ by $\omega(\bar{a},\bar{b})=\max\{\bar{a},\bar{b}\}$ for all $\bar{a},\bar{b}\in\mathbb{R}_0^+$. Then (W,ω) is a partial metric space. We can see that $\omega(\bar{a},\bar{a})\neq 0$ for all $W\in\mathbb{R}_0^+$. Consequently, ω is not a metric on W.

Example 2.51. Let W be a set of all closed finite intervals $[a,b] \subset \mathbb{R}$ where $a \leq b$. Define $\omega : W \times W \to [0,\infty)$ by

$$\omega([a,b],[c,\rho]) = \max\{b,\rho\} \; - \; \max\{a,c\}.$$

Then, (W,ω) is a partial metric space.

Let (W,ω) be a partial metric space. Then, ω generates T_0 topology τ_ω on W which has as a base the family open ω-balls

$$\{B_\omega(\bar{a},\varepsilon) : \bar{a} \in W, \varepsilon > 0\}$$

where

$$B_\omega(\bar{a},\varepsilon) = \left\{\bar{b} \in W : \omega(\bar{a},\bar{b}) < \omega(\bar{a},\bar{a}) + \varepsilon\right\}$$

for all $\bar{a} \in W$ and $\varepsilon > 0$. Let $\{\bar{a}_n\}$ be a sequence in W and $\bar{a} \in W$. It is easy to see that the sequence$\{\bar{a}_n\}$ converges to $\bar{a}$ with respect to τ_ω if and only if

$$\lim_{n \to +\infty} \omega(\bar{a}_n,\bar{a}) = \omega(\bar{a},\bar{a}).$$

Definition 2.52. (Aydi *et al.* 2012) Let (Ω,ω) be a partial metric space.

(*a*) A sequence $\{\bar{a}_n\}$ in Ω is said to be a Cauchy sequence if $\lim_{n,m\to\infty} \omega(\bar{a}_n,\bar{a}_m)$ exists and is finite.

(*b*) (Ω,ω) is said to be complete if every Cauchy sequence $\{\bar{a}_n\}$ in Ω converges with respect to τ_ω to a point $\bar{a} \in \Omega$ such that $\lim_{n\to\infty} \omega(\bar{a}_n,\bar{a}) = \omega(\bar{a},\bar{a})$. In this case, we say that the partial metric ω is complete.

Remark 2.53. It is easy to see that every closed subset of a complete partial metric space is complete.

Remark 2.54. The limit in a partial metric space is not unique.

Example 2.55. if $W = [0,\infty)$ and $\omega\left(\bar{a},\bar{b}\right) = \max\left\{\bar{a},\bar{b}\right\}$ for all $\bar{a},\bar{b} \in W$, then for $\{\bar{a}_n\} = \{1\}, \omega(\bar{a}_n,\bar{a}) = \bar{a} = \omega(\bar{a},\bar{a})$ for each $\bar{a} \geq 1$ and so $\bar{a}_n \to 2, \bar{a}_n \to 3$ as $n \to \infty$.

If (Ω, ω) is a partial metric space, then the function $\rho_\omega : \Omega \times \Omega \to [0, +\infty)$ defined by

$$\rho_\omega(\bar{a}, \bar{b}) = 2\omega(\bar{a}, \bar{b}) - \omega(\bar{a}, \bar{a}) - \omega(\bar{b}, \bar{b})$$

is a metric on Ω.

Lemma 2.56. Let (Ω, ω) be a partial metric space.

(i) $\{\bar{a}_n\}$ is a Cauchy sequence in $(\Omega, \omega) \Leftrightarrow \{\bar{a}_n\}$ is a Cauchy sequence in (Ω, ρ_ω).

(ii) (Ω, ω) is a complete partial metric space $\Leftrightarrow (\Omega, \rho_\omega)$ is a complete metric space.

(iii) Given a sequence $\{\bar{a}_n\}$ in Ω and $\bar{a} \in \Omega$. Then, we have

$$\lim_{n\to+\infty} \rho_\omega(\bar{a}_n, \bar{a}) = 0 \text{ if and only if } \omega(\bar{a}, \bar{a}) = \lim_{n\to+\infty} \omega(\bar{a}_n, \bar{a}) = \lim_{n,m\to+\infty} \omega(\bar{a}_n, \bar{a}_m).$$

It can be seen that

$$\lim_{n\to+\infty} \rho_\omega(\bar{a}_n, \bar{a}) = 0 = \lim_{n\to+\infty} \rho_\omega(\bar{b}_n, \bar{b}) \text{ implies } \lim_{n\to+\infty} \omega(\bar{a}_n, \bar{b}_n) = \omega(\bar{a}, \bar{b}).$$

Definition 2.57. (Romaguera 2009) Let (W, ω) be a partial metric space, then

(i) A sequence $\{\bar{a}_n\}$ in(W, ω) is 0-Cauchy if $\lim_{n,m} \omega(\bar{a}_n, a_m) = 0$.

(ii) A partial metric space (W, ω) is 0-Cauchy complete if every 0-Cauchy sequence in W converges to a point $\bar{a} \in W$ and $\omega(\bar{a}, \bar{a}) = 0$.

We present examples of 0-Cauchy complete partial metric spaces.

Example 2.58. Let $W = [0,1] \cup [2,3]$ and define $\omega : W \times W \to [0, \infty)$ by

$$\omega\left(\bar{a}, \bar{b}\right) = \begin{cases} max\left\{\bar{a}, \bar{b}\right\} & , \quad \{\bar{a}, \bar{b}\} \cap [2,3] \neq \emptyset \\ \left|\bar{a} - \bar{b}\right| & , \quad \{\bar{a}, \bar{b}\} \subset [0,1] \end{cases}$$

for all $\bar{a}, \bar{b} \in W$. Then, (W, ω) is a 0-Cauchy complete partial metric space.

Lemma 2.59. Let (W, ω) be a partial metric space.

i) (Abdeljawad *et al.* 2011) If $\omega(\bar{a}_n, \bar{c}) \to \omega(\bar{c}, \bar{c}) = 0$ as $n \to \infty$, then $\omega(\bar{a}_n, \bar{b}) \to \omega(\bar{c}, \bar{b})$ as $n \to \infty$ for each $\bar{b} \in W$.

ii) (Romaguera 2009) If (W, ω) is complete, then it is 0-complete.

Theorem 2.60. (Romaguera 2009) Let (W,ω) be a partial metric space, then (W,ω) is a metric space if and only if every Cauchy sequence is a 0-Cauchy sequence.

Example 2.61. The space $W=[0,\infty)\cap\mathbb{Q}$ with the partial metric $\omega(\bar{a},\bar{b})=\max\{\bar{a},\bar{b}\}$ is 0-complete, but is not complete. Furthermore, the sequence$\{\bar{a}_n\}$ with $\bar{a}_n=1$ for each $n\in N$ is a Cauchy sequence in (W,ω), but it is not a 0-Cauchy sequence.

Definition 2.62. (Romaguera 2009) Let (W,ω) be a partial metric space. A mapping $f:W\to W$ is called continuous at $\bar{a_0}\in W$ if for every $\varepsilon>0$, there exists $\delta>0$ such that $f(B_\omega(\bar{a_0},\delta))\subset B_\omega(f(\bar{a_0}),\varepsilon)$.

Let (W,ω) be a partial metric space. Let $CB^\omega(W)$ be the family of all nonempty, closed and bounded subsets of the partial metric space (W,ω), induced by the partial metric ω. Note that closedness is take from $(W,\tau_\omega),\tau_\omega$ is the topology induced by ω and boundedness is given as follows: A is a bounded subset in (W,ω) if there exists $\bar{a_0}\in W$ and $M\geq 0$ such that for all $a\in A$, we have $a\in B_\omega(\bar{a_0},M)$, that is, $\omega(\bar{a_0},a)<\omega(a,a)+M$. For $A,B\in CB^\omega(W)$ and $\bar{a}\in W$, define

$$\begin{aligned}\omega(\bar{a},A) &= \inf\{\omega(\bar{a},a),a\in A\}\\ \delta_\omega(A,B) &= \sup\{\omega(a,B):a\in A\} \text{ and } \delta_\omega(B,A)=\sup\{\omega(b,A):b\in B\}\end{aligned}$$

Let (W,ω) be a partial metric space $A,B\in CB^\omega(W)$, and $H_\omega:CB^\omega(W)\times CB^\omega(W)$

$$H_\omega(A,B)=\max\{\delta_\omega(A,B),\delta_\omega(B,A)\}.$$

Then, H_ω is called partial Hausdorff metric on $CB^\omega(W)$.

Remark 2.63. Let (W,ω) be a partial metric space and A any nonempty set in (W,ω), then

$$a\in\bar{A}\ \Leftrightarrow \omega(a,A)=\omega(a,a) \tag{2.5}$$

where A denotes the closure of A with respect to the partial metric ω . Note that A is closed in (W,ω) if and only if $A=\bar{A}$. Now we shall study some properties of mapping $\delta_\omega:CB^\omega(W)\times CB^\omega(W)\to[0,\infty)$.

Lemma 2.64. (Aydi *et al.* 2012) Let (W,ω) be a partial metric space. For all $A,B,C\in$

$CB^{\omega}(W)$, we have the following:

(i) $\delta_{\omega}(A,A) = \sup\{\omega(a,a) : a \in A\}$;

(ii) $\delta_{\omega}(A,A) \leq \delta_{\omega}(A,B)$;

(iii) $\delta_{\omega}(A,B) = 0$ implies that $A \subseteq B$;

(iv) $\delta_{\omega}(A,B) \leq \delta_{\omega}(A,C) + \delta_{\omega}(C,B) - \inf_{c \in C} \omega(c,c)$.

Lemma 2.65. (Aydi *et al.* 2012) Let (W,ω) be a partial metric space. For all $A,B,C \in CB^{\omega}(W)$, we have the following:

$(h1)$ $H_{\omega}(A,A) \leq H_{\omega}(A,B)$;

$(h2)$ $H_{\omega}(A,B) = H_{\omega}(B,A)$;

$(h3)$ $H_{\omega}(A,B) \leq H_{\omega}(A,C) + H_{\omega}(C,B) - \inf_{c \in C} \omega(c,c)$.

Corollary 2.66. Let (W,ω) be a partial metric space. For all $A,B \in CB^{\omega}(W)$, the following holds:

$$H_{\omega}(A,B) = 0 \Rightarrow A = B$$

Remark 2.67. The converse of Corollary 2.66 is not true in general as it is clear from the following example.

Example 2.68. Let $W = [0,1]$ be endowed with the partial metric $\omega : W \times W \to \mathbb{R}^{+}$ defined by

$$\omega(\bar{a},\bar{b}) = \max\{\bar{a},\bar{b}\}$$

From (i) of proposition (2.2) we have

$$H_{\omega}(W,W) = \delta_{\omega}(W,W) = \sup\{\bar{a} : 0 \leq \bar{a} \leq 1\} = 1 \neq 0$$

In view of proposition (2.3) and corollary (2.4), we call the mapping $H_{\omega} : CB^{\omega}(W) \times CB^{\omega}(W) \to [0,^{+}\infty)$, a partial Hausdorff metric induced by ω.

We begin to the following very important lemmas to obtain our main results.

Lemma 2.69. (Nadler 1969) Let (W,ω) be a partial metric space and $\emptyset \neq A,B \in CB^{\omega}(W)$, then for every $a \in A$ and $\varepsilon > 0$, there exists $b \in B$ such that

$$\omega(a,b) \leq H_{\omega}(A,B) + \varepsilon \tag{2.6}$$

3. SOME RESULTS FOR MIXED MULTIVALUED MAPPINGS

In this section, we introduce the following new definition. Hence, the first time, the definition of proximal contraction has been given for mixed multivalued mappings.

Definition 3.1. Let (W,ω) be a partial metric space, A,B be subsets of W and $\Lambda : A \to B \cup CB^{\omega}(B)$ be a mixed multivalued mapping. Λ is said to be a proximal Hausdorff contraction mapping if

$$\left.\begin{array}{l} \omega(u_1,\bar{b}_1) = \omega(A,B) \\ \omega(u_2,\bar{b}_2) = \omega(A,B) \end{array}\right\} \Longrightarrow H_{\omega}(Tu_1,Tu_2) \leq k\omega(\Lambda\bar{a}_1,\Lambda\bar{a}_2) \tag{3.1}$$

for all $u_1,u_2,\bar{a}_1,\bar{a}_2 \in A$ and $\bar{b}_1 \in \Lambda\bar{a}_1, \bar{b}_2 \in \Lambda\bar{a}_2$ where k in $[0,1)$.

Theorem 3.2. Let (W,ω) be a complete partial metric space, A,B be nonempty subsets of W where B is closed and A is approximately compact w.r.t B. Assume that $\Lambda : A \to B \cup CB^{\omega}(B)$ is a proximal Hausdorff contraction mapping satisfying $\Lambda(A_0) \subseteq B_0$. If $f : A \times B \to \mathbb{R}$ defined by $f(\bar{a},\bar{b}) = \omega(\bar{b},\Lambda\bar{a})$ is best lower semicontinuos, then Λ has a best proximity point $\bar{a}^*$ in A.

Proof. Let $\bar{a}_0$ be an arbitrary point in A_0, then we consider the following two cases:
Case 1. Let $|\Lambda\bar{a}_0| = 1$. Then, there exists $\bar{b}_0 \in B$ such that $\bar{b}_0 = \Lambda\bar{a}_0$. Since $\bar{b}_0 = \Lambda\bar{a}_0 \in \Lambda(A_0) \subseteq B_0$, there exists $\bar{a}_1 \in A_0$ such that

$$\omega(\bar{a}_1,\bar{b}_0) = \omega(A,B) \tag{3.2}$$

Now, we show that either Λ has a best proximity point or there exist $\bar{b}_1 \in B_0$, $\bar{a}_2 \in A_0$ and $\bar{b}_2 \in \Lambda\bar{a}_2$ such that

$$\omega(\bar{b}_1,\bar{b}_2) \leq \left(\frac{k+1}{2}\right)\omega(\Lambda\bar{a}_0,\Lambda\bar{a}_1) \tag{3.3}$$

If $|\Lambda\bar{a}_1| = 1$, then there exists $\bar{b}_1 \in B$ such that $\bar{b}_1 = \Lambda\bar{a}_1$. Since $\bar{b}_1 = \Lambda\bar{a}_1 \in \Lambda(A_0) \subseteq B_0$, there exists $\bar{a}_2 \in A_0$ such that

$$\omega(\bar{a}_2,\bar{b}_1) = \omega(A,B)$$

Since Λ is a proximal Hausdorff contraction mapping, we have

$$\begin{aligned} H_\omega(\Lambda\bar{a}_1, \Lambda\bar{a}_2) &\leq k\omega(\Lambda\bar{a}_0, \Lambda\bar{a}_1) \\ &\leq \left(\tfrac{k+1}{2}\right)\omega(\Lambda\bar{a}_0, \Lambda\bar{a}_1). \end{aligned}$$

Then, there exists $\bar{b}_2 \in \Lambda\bar{a}_2$ (or $\bar{b}_2 = \Lambda\bar{a}_2$) such that

$$\begin{aligned} \omega(\bar{b}_1, \bar{b}_2) &\leq H_\omega(\Lambda\bar{a}_1, \Lambda\bar{a}_2) \\ &\leq k\omega(\Lambda\bar{a}_0, \Lambda\bar{a}_1) \\ &\leq \left(\tfrac{k+1}{2}\right)\omega(\Lambda\bar{a}_0, \Lambda\bar{a}_1), \end{aligned}$$

and so (3.3) holds. Now, assume that $|\Lambda\bar{a}_1| > 1$. Choose $\bar{b}_1 \in \Lambda\bar{a}_1 \subseteq \Lambda(A_0) \subseteq B_0$. Then, there exists $\bar{a}_2 \in A_0$ such that

$$\omega(\bar{a}_2, \bar{b}_1) = \omega(A, B) \tag{3.4}$$

If $\omega(\Lambda\bar{a}_0, \Lambda\bar{a}_1) = 0$, then since Λ is aproximal Hausdorff contraction mapping, we have $H_\omega(\Lambda\bar{a}_1, \Lambda\bar{a}_2) = 0$. From Lemma 2.69, we get $\Lambda\bar{a}_1 = \Lambda\bar{a}_2$. Hence, $\bar{a}_2$ is a best proximity point of Λ. If $\omega(\Lambda\bar{a}_0, \Lambda\bar{a}_1) > 0$, then, from Lemma 2.69, there exists $\bar{b}_2 \in \Lambda\bar{a}_2$ (or $\bar{b}_2 = \Lambda\bar{a}_2$) such that

$$\begin{aligned} \omega(\bar{b}_1, \bar{b}_2) &\leq H_\omega(\Lambda\bar{a}_1, \Lambda\bar{a}_2) + \tfrac{1-k}{2}\omega(\Lambda\bar{a}_0, \Lambda\bar{a}_1) \\ &\leq k\omega(\Lambda\bar{a}_0, \Lambda\bar{a}_1) + \tfrac{1-k}{2}\omega(\Lambda\bar{a}_0, \Lambda\bar{a}_1) \\ &= \left(\tfrac{k+1}{2}\right)\omega(\Lambda\bar{a}_0, \Lambda\bar{a}_1). \end{aligned}$$

Hence, (3.3) holds.

Case 2. Let $|\Lambda\bar{a}_0| > 1$. Choose $\bar{b}_0 \in \Lambda\bar{a}_0 \subseteq \Lambda(A_0) \subseteq B_0$. Then, there exists $\bar{a}_1 \in A_0$ such that

$$\omega(\bar{a}_1, \bar{b}_0) = \omega(A, B)$$

Now, we show that either Λ has a best proximity point or there exist $\bar{b}_1 \in B_0$, $\bar{a}_2 \in A_0$ and $\bar{b}_2 \in \Lambda\bar{a}_2$ satisfying (3.3). If $|\Lambda\bar{a}_1| = 1$, then there exists $\bar{b}_1 \in B$ such that $\bar{b}_1 = \Lambda\bar{a}_1$. Since $\bar{b}_1 = \Lambda\bar{a}_1 \in \Lambda(A_0) \subseteq B_0$, there exists $\bar{a}_2 \in A_0$ such that

$$\omega(\bar{a}_2, \bar{b}_1) = \omega(A, B)$$

Since Λ is aproximal Hausdorff contraction mapping, we have

$$\begin{aligned} H_\omega(\Lambda\bar{a}_1,\Lambda\bar{a}_2) &\leq k\omega(\Lambda\bar{a}_0,\Lambda\bar{a}_1) \\ &\leq \left(\tfrac{k+1}{2}\right)\omega(\Lambda\bar{a}_0,\Lambda\bar{a}_1), \end{aligned}$$

Then, there exists $\bar{b}_2 \in \Lambda\bar{a}_2$ (or $\bar{b}_2 = \Lambda\bar{a}_2$) such that

$$\begin{aligned} \omega(\bar{b}_1,\bar{b}_2) &\leq H_\omega(\Lambda\bar{a}_1,\Lambda\bar{a}_2) \\ &\leq \left(\tfrac{k+1}{2}\right)\omega(\Lambda\bar{a}_0,\Lambda\bar{a}_1) \end{aligned}$$

and so (3.3) holds. Now, assume that $|\Lambda\bar{a}_1| > 1$. Choose $\bar{b}_1 \in \Lambda\bar{a}_1 \subseteq \Lambda(A_0) \subseteq B_0$. Then, there exists $\bar{a}_2 \in A_0$ such that

$$\omega(\bar{a}_2,\bar{b}_1) = \omega(A,B)$$

If $\omega(\Lambda\bar{a}_0,\Lambda\bar{a}_1) = 0$, then since Λ is a proximal Hausdorff contraction mapping, we have $H_\omega(\Lambda\bar{a}_1,\Lambda\bar{a}_2) = 0$. From Lemma 2.69, we get $\Lambda\bar{a}_1 = \Lambda\bar{a}_2$. Hence, $\bar{a}_2$ is a best proximity point of Λ. If $\omega(\Lambda\bar{a}_0,\Lambda\bar{a}_1) > 0$, then there exists $\bar{b}_2 \in \Lambda\bar{a}_2$ (or $\bar{b}_2 = \Lambda\bar{a}_2$) such that

$$\begin{aligned} \omega(\bar{b}_1,\bar{b}_2) &\leq H_\omega(\Lambda\bar{a}_1,\Lambda\bar{a}_2) + \tfrac{1-k}{2}\omega(\Lambda\bar{a}_0,\Lambda\bar{a}_1) \\ &\leq k\omega(\Lambda\bar{a}_0,\Lambda\bar{a}_1) + \tfrac{1-k}{2}\omega(\Lambda\bar{a}_0,\Lambda\bar{a}_1) \\ &= \left(\tfrac{k+1}{2}\right)\omega(\Lambda\bar{a}_0,\Lambda\bar{a}_1). \end{aligned}$$

Hence (3.3) holds. Continuing this process, we can construct two sequences $\{\bar{a}_n\}$ and $\{\bar{b}_n\}$ such that $\bar{b}_n \in \Lambda\bar{a}_n$ (or $\bar{b}_n = \Lambda\bar{a}_n$) (we can suppose $\omega(\Lambda\bar{a}_n,\Lambda\bar{a}_n+1) > 0$, otherwise the proof is complete.

$$\omega(\bar{a}_n+1,\bar{b}_n) = \omega(A,B) \tag{3.5}$$

and

$$\omega(\bar{b}_n,\bar{b}_n+1) \leq \left(\frac{k+1}{2}\right)\omega(\Lambda\bar{a}_n-1,\Lambda\bar{a}_n)$$

for all $n \in \mathbb{N}$. Then, we have

$$\begin{aligned}
\omega(\bar{b}_n, \bar{b}_n + 1) &\leq \left(\tfrac{k+1}{2}\right) \omega(\Lambda\bar{a}_n - 1, \Lambda\bar{a}_n) \\
&\leq \left(\tfrac{k+1}{2}\right) \omega(y_{n-1}, \bar{b}_n) \\
&\vdots \\
&\leq \left(\tfrac{k+1}{2}\right)^n \omega(\bar{b}_0, \bar{b}_1)
\end{aligned}$$

Then, for all $m > n$, we get

$$\begin{aligned}
\omega(\bar{b}_n, y_m) &\leq \omega(\bar{b}_n, \bar{b}_n + 1) + \omega(\bar{b}_n + 1, \bar{b}_n + 2) + \cdots + \omega(y_{m-1}, y_m) \\
&\leq \left(\tfrac{k+1}{2}\right)^n \omega(\bar{b}_0, \bar{b}_1) + \left(\tfrac{k+1}{2}\right)^{n+1} \omega(\bar{b}_0, \bar{b}_1) \\
&\quad + \cdots + \left(\tfrac{k+1}{2}\right)^{m-1} \omega(\bar{b}_0, \bar{b}_1) \\
&= \left(\tfrac{k+1}{2}\right)^n \omega(\bar{b}_0, \bar{b}_1) \left(1 + \left(\tfrac{k+1}{2}\right) + \cdots + \left(\tfrac{k+1}{2}\right)^{m-n-1}\right) \\
&\leq \tfrac{1-\left(\frac{k+1}{2}\right)^{m-n}}{1-\left(\frac{k+1}{2}\right)} \left(\tfrac{k+1}{2}\right)^n \omega(\bar{b}_0, \bar{b}_1) \\
&\leq \tfrac{1}{1-\left(\frac{k+1}{2}\right)} \left(\tfrac{k+1}{2}\right)^n \omega(\bar{b}_0, \bar{b}_1).
\end{aligned}$$

Hence, we have $\lim_{n,m\to\infty} \omega(\bar{b}_n, y_m) = 0$. Then, $\{\bar{b}_n\}$ is a 0-Cauchy sequence in B. Since (W, ω) is a complete partial metric space and B is a closed subset of W, there exists $\bar{b}^* \in B$ such that

$$\lim_{n,m\to\infty} \omega(\bar{b}_n, y_m) = \lim_{n\to\infty} \omega(\bar{b}_n, \bar{b}^*) = \omega(\bar{b}^*, \bar{b}^*) = 0. \tag{3.6}$$

Also, we get

$$\begin{aligned}
\omega(\bar{b}^*, A) &\leq \omega(\bar{b}^*, \bar{a}_n + 1) \\
&\leq \omega(\bar{b}^*, \bar{b}_n) + \omega(\bar{b}_n, \bar{a}_n + 1) \\
&= \omega(\bar{b}^*, \bar{b}_n) + \omega(A, B) \\
&\leq \omega(\bar{b}^*, \bar{b}_n) + \omega(\bar{b}^*, A)
\end{aligned}$$

Using (3.6), we have $\omega(\bar{b}^*, \bar{a}_n + 1) \to \omega(\bar{b}^*, A)$ as $n \to \infty$. Since A is an approximately compact with respect to B, there exists a subsequence $\{x_{n_k}\}$ of $\{\bar{a}_n\}$ such that

$$\lim_{k,l\to\infty} \omega(x_{n_k}, x_{n_l}) = \lim_{k\to\infty} \omega(x_{n_k}, \bar{a}^*) = \omega(\bar{a}^*, \bar{a}^*). \tag{3.7}$$

From (3.5) and Lemma 2.69, we have

$$\omega(\bar{a}^*, \bar{b}^*) = \omega(A, B).$$

Since $f(\bar{a}, \bar{b}) = \omega(\bar{b}, \Lambda\bar{a})$ is best lower semicontinuos, from (3.6) and (3.7), we have

$$\begin{aligned} \omega(\bar{b}^*, \Lambda\bar{a}^*) &= f(\bar{a}^*, \bar{b}^*) \\ &\leq \lim_{k\to\infty} \inf f(x_{n_k}, y_{n_k}) \\ &= \lim_{k\to\infty} \inf \omega(y_{n_k}, Tx_{n_k}) \\ &= 0 \end{aligned}$$

Hence, we have $\bar{b}^* \in \Lambda\bar{a}^*$, and so $\bar{a}^*$ is a best proximity point of Λ in A. □

Example 3.3. Let $W = [0,\infty) \times [0,\infty)$ and $\omega : W \times W \to [0,\infty)$ be a function defined by

$$\omega(\bar{a}, \bar{b}) = \max\{\bar{a}_1, \bar{b}_1\} + \left|\bar{a}_2 - \bar{b}_2\right| \tag{3.8}$$

for all $\bar{a} = (\bar{a}_1, \bar{a}_2), \bar{b} = (\bar{b}_1, \bar{b}_2) \in W$, then (W, ω) is a partial metric space. Let $A = \{(\bar{a}, 0) : \bar{a} \in [0,\infty)\}$ and $B = \{(\bar{a}, 1) : \bar{a} \in [0,\infty)\}$ be subset of W. Hence, we have $\omega(A, B) = 1$, $A_0 = \{(0,0)\}$ and $B_0 = \{(0,1)\}$. It can be easily seen that A is approximately compact with respect to B and B is closed. Define a mapping $\Lambda : A \to B \cup CB^{\omega}(B)$ as

$$\Lambda\bar{a} = \left[0, \frac{\bar{a}_1}{2}\right] \times \{1\}$$

for all $\bar{a} = (\bar{a}_1, \bar{a}_2) \in A$. It is clear that Λ is a proximal Hausdorff contraction mapping for all k in $[0,1)$. Also, we have $\Lambda(A_0) \subseteq B_0$. Further, we get

$$\begin{aligned} f(\bar{a}, \bar{b}) &= \omega(\bar{b}, \Lambda\bar{a}) \\ &= \bar{b}_1 \end{aligned}$$

for all $\bar{a} = (\bar{a}_1, \bar{a}_2) \in A$ and $\bar{b} = (\bar{b}_1, \bar{b}_2) \in B$. Then, it can be seen that f is lower semicontinuous. Hence, all hypothesis of Theorem 3.2 hold, and so Λ has a best proximity point in A.

Since Theorem 3.2 is a generalization for single valued mappings and multivalued

mappings, we present the following result.

Corollary 3.4. Let (W,ω) be a complete partial metric space, A,B be nonempty subsets of W where B is closed and A is approximately compact w.r.t B. Assume that $\Lambda : A \to CB^{\omega}(B)$ is a proximal Hausdorff contraction mapping satisfying $\Lambda(A_0) \subseteq B_0$. If $f : A \times B \to \mathbb{R}$ defined by $f(\bar{a},\bar{b}) = \omega(\bar{b},\Lambda\bar{a})$ is best lower semicontinuos, then Λ has a best proximity point $\bar{a}^*$ in A.

Proof. Let (W,ω) be a complete partial metric space, A,B be nonempty subsets of W where B is closed and A is approximately compact w.r.t B. Assume that $\Lambda : A \to B$ is a proximal Hausdorff contraction mapping satisfying $\Lambda(A_0) \subseteq B_0$. If $f : A \times B \to \mathbb{R}$ defined by $f(\bar{a},\bar{b}) = \omega(\bar{b},\Lambda\bar{a})$ is best lower semicontinuos, then Λ has a best proximity point $\bar{a}^*$ in A. □

4. SOME RESULTS FOR MULTIVALUED MAPPINGS

4.1 Feng-Liu type multivalued contraction mapping

In this subsection, we introduce the following new definitions. Then, we prove a best proximity point result.

Definition 4.1. Let (W,ω) be a partial metric space and $\emptyset \neq A,B \subseteq W$ with $A_0 \neq \emptyset$. If

$$g(\bar{a_0},\bar{b}_0) \leq \lim_{n\to\infty} \inf g(\bar{a_n},\bar{b}_n).$$

For every sequences $\{\bar{a}_n\}$ in A_0 and $\{\bar{b}_n\}$ in B_0 satisfying

$$\bar{a}_n \to \bar{a_0} \in A \text{ and } \bar{b}_n \to \bar{b}_0 \in B \quad \text{as} \quad n \to \infty$$

Then, a function $g : A \times B \to \mathbb{R}$ is called best lower semicontinuous at a point $(\bar{a_0},\bar{b}_0) \in A \times B$.

Definition 4.2. Let (W,ω) be a partial metric space, $\emptyset \neq A,B \subseteq W$ and $\Lambda : A \to C(B)$ be a multivalued mapping. The mapping Λ is said to be Feng-Liu type multivalued contraction mapping if for all $\bar{a} \in A_0$ and $\bar{b} \in \Lambda\bar{a}$ there exists $\vartheta \in A_0$ satisfying

$$\omega(\bar{b},\vartheta) = \omega(A,B) \tag{4.1}$$

and

$$\omega(\bar{b},\Lambda\vartheta) \leq \gamma\omega(\bar{a},\vartheta) \tag{4.2}$$

for some $\gamma \in [0,1)$.

Theorem 4.3. Let (W,ω) be a complete partial metric space, $\emptyset \neq A,B \subseteq W$ where A,B are closed and $\Lambda : A \to C(B)$ is a Feng-Liu type multivalued contraction mapping. Assume that the following conditions are satisfied:

- $A_0 \neq \emptyset$ and $\Lambda(A_0) \subseteq B_0$
- The pair (A,B) has the P-Property
- A function $g(\bar{a},\bar{b}) = \omega(\bar{b},\Lambda\bar{a})$ is best lower semicontinuous on $A \times B$.

Then, Λ has a best proximity point $\bar{a}^*$ in A. Moreover, $\omega(\bar{a}^*,\bar{a}^*)=0$.

Proof. Let $\bar{a}_0 \in A_0$ be an arbitrary point. Choose $\bar{b}_0 \in \Lambda\bar{a}_0$. Then, since Λ is a Feng-Liu type multivalued contraction mapping, there exists $\bar{a}_1 \in A_0$ such that

$$\omega(\bar{b}_0,\bar{a}_1)=\rho(A,B).$$

and

$$\omega(\bar{b}_0,\Lambda\bar{a}_1)\leq\gamma\omega(\bar{a}_0,\bar{a}_1).$$

Let $\beta\in(\gamma,1)$ be a constant. Then, from definition of infimum, there exists $\bar{b}_1\in\Lambda\bar{a}_1$ such that

$$\beta\,\omega(\bar{b}_0,\bar{b}_1)\leq\omega(\bar{b}_0,\Lambda\bar{a}_1).$$

Again, since Λ is a Feng-Liu type multivalued contraction mapping, there exists $\bar{a}_2\in A_0$ such that

$$\omega(\bar{b}_1,\bar{a}_2)=\rho(A,B).$$

and

$$\omega(\bar{b}_1,\Lambda\bar{a}_2)\leq\gamma\omega(\bar{a}_1,\bar{a}_2).$$

Also there exists $\bar{b}_2\in\Lambda\bar{a}_2$ such that

$$\beta\,\omega(\bar{b}_1,\bar{b}_2)\leq\omega(\bar{b}_1,\Lambda\bar{a}_2).$$

Repeating this process we can construct sequences $\{\bar{a}_n\}$ and $\{\bar{b}_n\}$ in A and B with $\bar{b}_n\in\Lambda\bar{a}_n$ and

$$\begin{aligned}
\omega(\bar{b}_n,\bar{a}_n+1) &= \omega(A,B), &(4.3)\\
\omega(\bar{b}_n,\Lambda\bar{a}_n+1) &\leq \gamma\omega(\bar{a}_n,\bar{a}_n+1), &(4.4)\\
\beta\,\omega(\bar{b}_n,\bar{b}_n+1) &\leq \omega(\bar{b}_n,\Lambda\bar{a}_n+1) &(4.5)
\end{aligned}$$

for all $n\geq 1$. On the other hand, since the pair (A,B) has the P-property, then we have

$$\omega(\bar{a}_n,\bar{a}_n+1)=\omega(y_{n-1},\bar{b}_n) \tag{4.6}$$

for all $n \geq 1$. Therefore we get

$$\begin{aligned} \omega(\bar{a}_n, \bar{a}_n + 1) &= \omega(y_{n-1}, \bar{b}_n) \\ &\leq \tfrac{1}{\beta}\omega(y_{n-1}, \Lambda\bar{a}_n) \\ &\leq \tfrac{\gamma}{\beta}\omega(\bar{a}_n - 1, \bar{a}_n), \end{aligned}$$

for all $n \geq 1$. Hence, we obtain

$$\begin{aligned} \omega(\bar{a}_n, \bar{a}_n + 1) &\leq \tfrac{\gamma}{\beta}\omega(\bar{a}_n - 1, \bar{a}_n) \\ &\leq \left(\tfrac{\gamma}{\beta}\right)^2 \omega(\bar{a}_n - 2, \bar{a}_n - 1) \\ &\vdots \\ &\leq \left(\tfrac{\gamma}{\beta}\right)^n \omega(\bar{a}_0, \bar{a}_1) \end{aligned}$$

and so

$$\lim_{n\to\infty} \omega(\bar{a}_n, \bar{a}_n + 1) = 0 \tag{4.7}$$

From (4.4) and (4.7), we also have

$$\lim_{n\to\infty} \omega(\bar{b}_n, \Lambda\bar{a}_n + 1) = 0 \tag{4.8}$$

Then, for all $m > n > n_0$, we have

$$\begin{aligned} \omega(\bar{a}_n, x_m) &\leq \omega(\bar{a}_n, \bar{a}_n + 1) + \omega(\bar{a}_n + 1, \bar{a}_n + 2) + \cdots + \omega(x_{m-1}, \bar{a}_m) \\ &\leq \omega(\bar{a}_n, \bar{a}_{n+1}) + \omega(\bar{a}_{n+1}, \bar{a}_{n+2}) + \cdots + \omega(\bar{a}_{m-1}, \bar{a}_m) \\ &\leq \left(\tfrac{\gamma}{\beta}\right)^n \omega(\bar{a}_0, \bar{a}_1) + \left(\tfrac{\gamma}{\beta}\right)^{n+1} \omega(\bar{a}_0, \bar{a}_1) + \cdots + \left(\tfrac{\gamma}{\beta}\right)^{m-1} \omega(\bar{a}_0, \bar{a}_1) \\ &= \left(\tfrac{\gamma}{\beta}\right)^n \omega(\bar{a}_0, \bar{a}_1) \left(1 + \tfrac{\gamma}{\beta} + \cdots + \left(\tfrac{\gamma}{\beta}\right)^{m-n-1}\right) \\ &= \left(\tfrac{\gamma}{\beta}\right)^n \omega(\bar{a}_0, \bar{a}_1) \tfrac{1-\left(\frac{\gamma}{\beta}\right)^{m-n}}{1-\frac{\gamma}{\beta}} \\ &\leq \tfrac{\left(\frac{\gamma}{\beta}\right)^n}{1-\frac{\gamma}{\beta}} \omega(\bar{a}_0, \bar{a}_1). \end{aligned}$$

Hence, $\lim_{n,m\to\infty} \omega(\bar{a}_n, x_m) = 0$, and so $\{\bar{a}_n\}$ is a Cauchy sequence in A. Since A is closed subset of the complete partial metric space (W, ω), there exists $\bar{a}^* \in A$ such that

$$\lim_{n,m\to\infty} \omega(\bar{a}_n, x_m) = \lim_{n\to\infty} \omega(\bar{a}_n, \bar{a}^*) = \omega(\bar{a}^*, \bar{a}^*) = 0 \tag{4.9}$$

from (4.6), we also get $\{\bar{b}_n\}$ is a Cauchy sequence in B. Similarly, there exists $\bar{b}^* \in B$ such that

$$\lim_{n,m\to\infty} \omega(\bar{b}_n, y_m) = \lim_{n\to\infty} \omega(\bar{b}_n, \bar{b}^*) = \omega(\bar{b}^*, \bar{b}^*) = 0$$

Letting $n \to \infty$ in (4.3), we have

$$\omega(\bar{a}^*, \bar{b}^*) = \rho(A,B). \tag{4.10}$$

Finally, since $g(\bar{a},\bar{b}) = \omega(\bar{b}, \Lambda\bar{a})$ is best lower semicontinuous on $A \times B$ and for the sequences $\{\bar{a}_n + 1\}$ in A_0 and $\{\bar{b}_n\}$ in B_0, we have $\bar{a}_n + 1 \to \bar{a}^*$ and $\bar{b}_n \to \bar{b}^*$ as $n \to \infty$, from (4.8) we get

$$\omega(\bar{b}^*, \Lambda\bar{a}^*) = g(\bar{a}^*, \bar{b}^*) \leq \liminf g(\bar{a}_n + 1, \bar{b}_n) = \liminf \omega(\bar{b}_n, \Lambda\bar{a}_n + 1) = 0$$

Hence $\bar{b}^* \in \overline{\Lambda\bar{a}^*} = \Lambda\bar{a}^*$. Therefore, from (4.10) we have

$$\omega(A,B) \leq \omega(\bar{a}^*, \Lambda\bar{a}^*) \leq \omega(\bar{a}^*, \bar{b}^*) = \omega(A,B).$$

That is,

$$\omega(\bar{a}^*, \Lambda\bar{a}^*) = \rho(A,B).$$

Hence, Λ has a best proximity point $\bar{a}^*$ in A. Moreover, from (4.9) we have $\omega(\bar{a}^*, \bar{a}^*) = 0$. □

Example 4.4. Let $W = [0,\infty) \times [0,\infty)$ and $\omega : W \times W \to [0,\infty)$ be a function defined by:

$$\omega(\bar{a},\bar{b}) = \max\{\bar{a}_1, \bar{b}_1\} + \left|\bar{a}_2 - \bar{b}_2\right|$$

for all $\bar{a} = (\bar{a}_1, \bar{a}_2), \bar{b} = (\bar{b}_1, \bar{b}_2) \in W$. Then, (W, ω) is a complete partial metric space. Let $A = \{(u,1) : u \in [0,\infty)\}$ and $B = \{(v,0) : v \in [0,\infty)\}$ be subsets of W. It can be easily seen that $\omega(A,B) = 1$ and A,B are closed subsets. Also, we have $A_0 = \{(0,1)\}$, $B_0 = \{(0,0)\}$ and the pair (A,B) has the P-Property. Now, we define a mapping $\Lambda : A \to C(B)$ as

$$\Lambda\bar{a} = \begin{cases} \{(0,0)\} & , \quad \bar{a} = (0,1) \\ \{(v,0) : v \geq u\} & , \quad \bar{a} = (u,1) \end{cases}$$

for all $\bar{a} = (u,0) \in A$. Hence, we have $\Lambda(A_0) \subseteq B_0$. It can be easily seen that Λ is a Feng-Liu type multivalued contraction mapping for all $\gamma \in [0,1)$ and the function $g(\bar{a},\bar{b}) = \omega(\bar{b},\Lambda\bar{a})$ is best lower semicontinuous on $A \times B$. Then, all hypotheses of Theorem 4.3 are hold. Therefore, Λ has a best proximity point $\bar{a}^*$ in A. Moreover, $\omega(\bar{a}^*,\bar{a}^*) = \omega((0,1),(0,1)) = 0$.

If we take $A = B = W$ and $\omega(A,B) = 0$ in Theorem 4.3, we can present the following result.

Corollary 4.5. Let (W,ω) be a complete partial metric space and $\Lambda : W \to C(W)$ be a multivalued mapping. If there exists $\gamma \in (0,1)$ such that

$$\omega(\bar{b},\Lambda\bar{b}) \leq \gamma\omega(\bar{a},\bar{b}).$$

For all $\bar{a} \in W$ and $\bar{b} \in \Lambda\bar{a}$ and $g(\bar{a},\bar{b}) = \omega(\bar{b},\Lambda\bar{a})$ is best lower semicontinuous on $W \times W$, then Λ has a fixed point in W.

4.2 Cyclic Feng-Liu type multivalued contraction mapping

In this subsection, we introduce the following new definitions. Then, we prove a best proximity point result for cyclic multivalued mappings

Definition 4.6. Let (W,ω) be a partial metric space, $\emptyset \neq A,B \subseteq W$ and $\Lambda : A \cup B \to P(A \cup B)$ be a multivalued mapping, then Λ is said to be cyclic Feng-Liu type multivalued contraction mapping if there exist $\beta,\gamma \in (0,1)$ with $\gamma < \beta$ satisfying for all $\bar{a} \in A$ with $\omega(\bar{a},\Lambda\bar{a}) > \omega(A,B)$, there exists $\bar{b} \in \Lambda_\beta^{\bar{a}} = \{\bar{b} \in \Lambda\bar{a} : \beta\omega(\bar{a},\bar{b}) \leq \omega(\bar{a},\Lambda\bar{a})\}$ such that

$$\omega(\bar{b},\Lambda\bar{b}) \leq \gamma\omega(\bar{a},\bar{b}) + (\beta-\gamma)\omega(A,B).$$

Theorem 4.7. Let (W,ω) be a partial metric space and $\emptyset \neq A,B \subseteq W$. Let $\Lambda : A \cup B \to P(A \cup B)$ be a cyclic Feng-Liu type multivalued contraction mapping, $\bar{a}_0 \in A$ and define a sequence $\{\bar{a}_n\}$ by $\bar{a}_n + 1 = \Lambda\bar{a}_n$ for all $n \geq 1$. If $\{\bar{a}_{2n}\}$ has a convergent subsequence in A and $g(\bar{a}) = \rho(\bar{a},\Lambda\bar{a})$ is lower semicontinuous on A, then Λ has a best proximity point in A.

Proof. Let $\bar{a}_0 \in A$ be an arbitrary point. If $\omega(\bar{a}_0, \Lambda\bar{a}_0) = \omega(A,B)$, then the proof is finished. Then, we assume $\omega(\bar{a}_0, \Lambda\bar{a}_0) > \omega(A,B)$. Since Λ is cyclic Feng-Liu type multivalued contraction mapping, there exists $\bar{a}_1 \in \Lambda_\beta^{\bar{a}_0}$ such that

$$\omega(\bar{a}_1, \Lambda\bar{a}_1) \leq \gamma\omega(\bar{a}_0, \bar{a}_1) + (\beta - \gamma)\omega(A,B).$$

If $\omega(\bar{a}_1, \Lambda\bar{a}_1) = \omega(A,B)$, then the proof is finished. Hence, we assume $\omega(\bar{a}_1, S\bar{a}_1) > \omega(A,B)$. Since Λ is cyclic Feng-Liu type multivalued contraction mapping, there exists $\bar{a}_2 \in \Lambda_\beta^{\bar{a}_1}$ such that

$$\omega(\bar{a}_2, \Lambda\bar{a}_2) \leq \gamma\omega(\bar{a}_1, \bar{a}_2) + (\beta - \gamma)\omega(A,B).$$

Repeating this process, we can construct a sequence $\{\bar{a}_n\}$ in $A \cup B$ (we can suppose $\omega(\bar{a}_{2n+1}, \Lambda\bar{a}_{2n+1}) > \omega(A,B)$ for all $n \geq 1$, otherwise the proof is finished.

$$\omega(\bar{a}_{2n+1}, \Lambda\bar{a}_{2n+1}) \leq \gamma\omega(\bar{a}_{2n}, \bar{a}_{2n+1}) + (\beta - \gamma)\omega(A,B) \tag{4.11}$$

for all $n \geq 1$. Since $\bar{a}_{2n+1} \in \Lambda_\beta^{\bar{a}_{2n}}$, we have

$$\beta\,\omega(\bar{a}_{2n}, \bar{a}_{2n+1}) \leq \omega(\bar{a}_{2n}, \Lambda\bar{a}_{2n}) \tag{4.12}$$

for all $n \geq 1$. Hence, from (4.11) and (4.12) we get

$$\beta\,\omega(\bar{a}_{2n}, \bar{a}_{2n+1}) \leq \gamma\omega(\bar{a}_{2n-1}, \bar{a}_{2n}) + (\beta - \gamma)\omega(A,B) \tag{4.13}$$

for all $n \geq 1$. Therefore, we have

$$\begin{aligned}
\omega(A,B) &\leq \omega(\bar{a}_{2n},\bar{a}_{2n+1}) \\
&\leq \tfrac{\gamma}{\beta}\omega(\bar{a}_{2n-1},\bar{a}_{2n}) + \left(1-\tfrac{\gamma}{\beta}\right)\omega(A,B) \\
&\leq \tfrac{\gamma}{\beta}\left\{\tfrac{\gamma}{\beta}\omega(x_{2n-2},\bar{a}_{2n-1}) + \left(1-\tfrac{\gamma}{\beta}\right)\omega(A,B)\right\} + \left(1-\tfrac{\gamma}{\beta}\right)\omega(A,B) \\
&= \left(\tfrac{\gamma}{\beta}\right)^2\omega(x_{2n-2},\bar{a}_{2n-1}) + \left(1+\tfrac{\gamma}{\beta}\right)\left(1-\tfrac{\gamma}{\beta}\right)\omega(A,B) \\
&\leq \left(\tfrac{\gamma}{\beta}\right)^2\left\{\tfrac{\gamma}{\beta}\omega(\bar{a}_{2n}-3,x_{2n-2}) + \left(1-\tfrac{\gamma}{\beta}\right)\omega(A,B)\right\} \\
&\quad + \left(1+\tfrac{\gamma}{\beta}\right)\left(1-\tfrac{\gamma}{\beta}\right)\omega(A,B) \\
&= \left(\tfrac{\gamma}{\beta}\right)^3\omega(\bar{a}_{2n}-3,x_{2n-2}) + \left(1+\tfrac{\gamma}{\beta}+\left(\tfrac{\gamma}{\beta}\right)^2\right)\left(1-\tfrac{\gamma}{\beta}\right)\omega(A,B) \\
&\quad \vdots \\
&\leq \left(\tfrac{\gamma}{\beta}\right)^{2n}\omega(\bar{a}_0,\bar{a}_1) + \left(1+\tfrac{\gamma}{\beta}+\cdots+\left(\tfrac{\gamma}{\beta}\right)^{2n-1}\right)\left(1-\tfrac{\gamma}{\beta}\right)\omega(A,B) \\
&= \left(\tfrac{\gamma}{\beta}\right)^{2n}\omega(\bar{a}_0,\bar{a}_1) + \left(1-\left(\tfrac{\gamma}{\beta}\right)^{2n}\right)\omega(A,B),
\end{aligned}$$

for all $n \geq 1$. Since $\frac{\gamma}{\beta} \in (0,1)$, we have

$$\lim_{n\to\infty}\omega(\bar{a}_{2n},\bar{a}_{2n+1}) = \omega(A,B). \tag{4.14}$$

Now, since $\{\bar{a}_{2n}\}$ has a convergent subsequence, then there exists a subsequence $\{\bar{a}_{2n_k}\}$ of $\{\bar{a}_{2n}\}$ such that $\bar{a}_{2n_k} \to \bar{a}^* \in A$. Because of the fact that g is a lower semicontinuous function, from (4.14), we have

$$\begin{aligned}
\omega(A,B) &\leq \omega(\bar{a}^*,\Lambda\bar{a}^*) \\
&= g(\bar{a}^*) \\
&\leq \lim_{k\to\infty}\inf g(\bar{a}_{2n_k}) \\
&= \lim_{k\to\infty}\inf \omega(\bar{a}_{2n_k},\Lambda\bar{a}_{2n_k}) \\
&\leq \lim_{k\to\infty}\inf \omega(\bar{a}_{2n_k},\bar{a}_{2n_k+1}) \\
&= \omega(A,B).
\end{aligned}$$

Therefore, Λ has a best proximity point $\bar{a}^*$ in A. □

Remark 4.8. Note that, if $g(\bar{a}) = \rho(\bar{a},\Lambda\bar{a})$ is lower semicontinuous on B and $\{\bar{a}_{2n+1}\}$ has a convergent subsequence in Theorem 4.7, then Λ has a best proximity point in B.

Example 4.9. Let $W = [0,\infty)$ and $\omega : W \times W \to [0,\infty)$ be a function defined by

$$\omega(\bar{a},\bar{b}) = \max\{\bar{a},\bar{b}\}$$

for all $\bar{a},\bar{b} \in W$. Then, it can be seen that (W,ω) is a partial metric space. Let $A = \left\{\frac{1}{3^{2n}} : n \geq 1\right\} \cup \{0\}$ and $B = \left\{\frac{1}{3^{2n+1}} : n \geq 1\right\}$ be subsets of W. Hence, we have $\omega(A,B) = 0$. Now, we define a mapping $\Lambda : A \cup B \to P(A \cup B)$ as

$$\Lambda\bar{a} = \begin{cases} \left\{\frac{1}{3^{2n+1}}, \frac{1}{3^{2n+3}}, \cdots\right\} & , \quad \bar{a} = \frac{1}{3^{2n}} \\ B & , \quad \bar{a} = 0 \\ \left\{\frac{1}{3^{2n+2}}, \frac{1}{3^{2n+4}}, \cdots\right\} & , \quad \bar{a} = \frac{1}{3^{2n+1}} \end{cases}$$

for all $n \geq 1$. We shall show that Λ is a cyclic Feng-Liu type multivalued contraction mapping with $\gamma = \frac{1}{3}$ and $\beta \in (\gamma, 1)$. Indeed, let $\bar{a} \in A \cup B$ with $\omega(\bar{a},\Lambda\bar{a}) > \omega(A,B)$. We consider the following cases:

Case 1: Let $\bar{a} = \frac{1}{3^{2n}}$ for all $n \geq 1$. Then, for $\bar{b} = \frac{1}{3^{2n+1}}$, we have

$$\beta\omega(\bar{a},\bar{b}) = \frac{\beta}{3^{2n}} \leq \frac{1}{3^{2n}} = \max\left\{\frac{1}{3^{2n}}, \Lambda\frac{1}{3^{2n}}\right\} = \omega(\bar{a},\Lambda\bar{a})$$

and

$$\begin{aligned} \omega\left(\tfrac{1}{3^{2n+1}}, \Lambda\tfrac{1}{3^{2n+1}}\right) &= \tfrac{1}{3^{2n+1}} \\ &\leq \tfrac{1}{3}\max\left\{\tfrac{1}{3^{2n}}, \tfrac{1}{3^{2n+1}}\right\} \\ &= \gamma\omega(\bar{a},\bar{b}) + (\beta-\gamma)\omega(A,B). \end{aligned}$$

Case 2: Let $\bar{a} = \frac{1}{3^{2n+1}}$ for all $n \geq 1$. Then, for $\bar{b} = \frac{1}{3^{2n+2}}$, we have

$$\beta\omega(\bar{a},\bar{b}) = \frac{\beta}{3^{2n+1}} \leq \frac{1}{3^{2n+1}} = \max\left\{\frac{1}{3^{2n+1}}, \Lambda\frac{1}{3^{2n+1}}\right\} = \omega(\bar{a},\Lambda\bar{a})$$

and

$$\begin{aligned} \omega\left(\tfrac{1}{3^{2n+2}}, \Lambda\tfrac{1}{3^{2n+2}}\right) &= \tfrac{1}{3^{2n+2}} \\ &\leq \tfrac{1}{3}\max\left\{\tfrac{1}{3^{2n+1}}, \tfrac{1}{3^{2n+2}}\right\} \\ &= \gamma\omega(\bar{a},\bar{b}) + (\beta-\gamma)\omega(A,B). \end{aligned}$$

Hence, Λ is a cyclic Feng-Liu type multivalued contraction mapping with $\gamma = \frac{1}{3}$ and $\beta \in (\gamma, 1)$. Also, every sequence in A has a convergent subsequence. Further, it is clear that

$$g(\bar{a}) = \begin{cases} \frac{1}{3^{2n}} & , \quad \bar{a} = \frac{1}{3^{2n}} \\ 0 & , \quad \bar{a} = 0 \end{cases}$$

is a lower semicontinuous function on A. Therefore, all condition of Theorem 4.7 are hold, and so Λ best proximity point $\bar{a}^*$ in A.

If we take $A = B = W$ and $\omega(A, B) = 0$ in Theorem 4.7, we obtain the following result.

Corollary 4.10. Let (W, ω) be a partial metric space, $\Lambda : W \to P(W)$ be a multivalued mapping and define a sequence $\{\bar{a}_n\}$ by $\bar{a}_n + 1 = \Lambda \bar{a}_n$ for all $n \geq 1$ with any initial point $\bar{a}_0 \in W$. Assume that $\{\bar{a}_{2n}\}$ has a convergent subsequence in W and $g(\bar{a}) = \rho(\bar{a}, \Lambda \bar{a})$ is lower semicontinuous on W. If for all $\bar{a} \in W$ there exists $\bar{b} \in I_{\beta}^{\bar{a}}$ satisfying

$$\omega(\bar{b}, \Lambda \bar{b}) \leq \gamma \omega(\bar{a}, \bar{b})$$

where $\beta, \gamma \in (0, 1)$ with $\gamma < \beta$, then Λ has a fixed point in W.

5. CONCLUSIONS AND RECOMMENDATION

In this thesis, some original best proximity point results for multivalued mappings are obtained by using the information in the literature. First, the definition of proximal Hausdorff contraction mapping definition is given and some best proximity point results are obtained for these mapping. Secondly, the definitions of Feng-Liu type multivalued and cyclic Feng-Liu type multivalued contraction mapping are given and some best proximity point results are obtained for these mappings. In addition, our results are supported by examples.

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SCIENTIFIC ACTIVITIES

a) Publications

1) Aslantas, M., and Abed, A. Z. D. R. 2021. Some best proximity point results for multivalued mappings on partial metric spaces. Mathematica Moravica, 25(1):99-111.

b) Full Paper

1) Aslantas, M., and Abed, A. Z. D. R. Best Proximity Point Result For Mixed Multivalued Mappingses. ICRAPAS(2021)(Accepted).