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ON THE GENERALIZATIONS AND PROPERTIES OF SOLID SEQUENCE SPACES DERIVED FROM CESARO TYPE MATRICES

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By Ahmed Abbas Jebur AL-FURAIJI

August 2021

We certify that we have read this thesis and that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science

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Ahmed Abbas Jebur AL-FURAIJI

ABSTRACT

ON THE GENERALIZATIONS AND PROPERTIES OF SOLID SEQUENCE SPACES DERIVED FROM CESARO TYPE MATRICES

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Master of Science in Mathematics

Advisor: Prof. Dr. Faruk POLAT

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In this thesis, a research on some remarkable pullback solid sequence spaces, denoted by $sld - A^{-1}(E)$, derived from infinite matrices is given. Let E be a set of complex or real sequences and A an infinite matrix with non-negative entries. By definition, $x \in A^{-1}(E)$ if and only if the sequence x falls in the domain of A and $Ax \in E$. Similarly, we define , $x \in sld - A^{-1}(E)$ if and only if the sequence $|x|$ falls in the domain of A and $A|x| \in E$. Thus, if $E = l_p$ $(1 < p < \infty)$ and A is the Cesaro matrix, then $sld - A^{-1}(E)$ becomes the classical Cesaro sequence space $cesr_p$. So solid sequence space, $sld - A^{-1}(E)$ is the generalization of the classical Cesaro sequence space ces_p. In this context, some results and examples on this pullback sequence space obtained by (Johnson and Mohapatra 1985) are considered. For example, some conditions under which the pullback spaces inherit properties of E such as LCC, AK and Hausdorffness properties or E inherits properties of its pullbacks are established, and various counterexamples are given to show that many of the results are not satisfied.

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Keywords: Solid set, Cesaro sequence space, Infinite matrix, $LCC -$property, AK property

ÖZET

CESARO TİPİ MATRİSLERDEN TÜRETİLEN SOLİD DİZİ UZAYLARININ ÖZELLİKLERİ VE GENELLEŞTİRİLMELERİ ÜZERİNE

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Bu tezde, bazı dikkat çekici sonsuz matrislerden türetilen geri çekilme ile elde edilmiş $sld - A^{-1}(E)$ ile gösterilen, solid dizi uzaylarıyla ilgili bir araştırma verilmiştir. E kompleks veya reel dizilerin bir kümesi ve A da negatif olmayan girdilere sahip bir sonsuz matris olsun. Tanım olarak, $x \in A^{-1}(E)$ olması ancak ve ancak x dizisinin A'nın tanım kümesine düşmesi ve $Ax \in E$ olması demektir. Benzer şekilde, $x \in sld - A^{-1}(E)$ olmasınıda ancak ve ancak $|x|$ dizisinin A'nın tanım kümesine düşmesi ve $A|x| \in E$ olacak şekilde tanımlarız. Böylece, $E = l_p$ $(1 < p < \infty)$ ve A da Cesaro matrisi alınırsa, $sld - A^{-1}(E)$ uzayı klasik Cesaro dizi uzayı $cesr_p$ olur. Bu bağlamda, (Johnson and Mohapatra 1985) tarafından bu geri çekme dizi uzayının üzerinde elde edilen bazı sonuçlar ve örnekler göz önüne alınmıştır. Örneğin, LCC, AK ve Hausdorffluk özellikleri gibi geri çekme uzaylarının E'nin özelliklerini devraldığı veya E'nin geri çekmelerinin özelliklerini devraldığı bazı koşullar belirlenmiş ve sonuçların çoğunun karşılanmadığını göstermek için çeşitli ters örnekler verilmiştir.

2021, 35 sayfa

Anahtar Kelimeler: Solid küme, Cesaro dizi uzayı, Sonsuz matris, LCC-özelliği, AK-özelliği

PREFACE AND ACKNOWLEDGEMENTS

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CONTENTS

LIST OF SYMBOLS

$\boldsymbol{L}(\mathbb{K})$	A vector space over the scalar field $\mathbb{K}$
$\mathbb{R}$	The set of real numbers
$\mathbb{N}$	The set of positive integers
$\mathbb{C}$	The set of complex numbers
$\mathbb{F}$	Scalar field
$\boldsymbol{L}^{+}$	The positive cone of the vector lattice L
$\preccurlyeq$	The order relation
sup	Supremum
inf	Infimum
$C[a,b]$	The space of real valued continuous functions on $[a,b]$
A^{d}	The disjoint complement of the set A
l_p	The vector space of absolutely p^{th} power summable sequences
$cesr_p$	Cesàro sequence space
$A^{-1}(\lambda)$	Inverse image of λ under the matrix A
$nor - A^{-1}(\lambda)$	The largest normal subspace of $A^{-1}(\lambda)$

LIST OF ABBREVIATIONS

ζvs	Topological vector space
$neigh.$	Neighborhood
LCC	Locally coordinatewise closed
AK	Abschnitt konvergenz

1. INTRODUCTION

Functional analysis is one of the most important branches in mathematics and studies the variety of spaces of functions. It is a valuable tool in theoretical mathematics. The core of which is formed by the study of vector spaces endowed with some kind of limit-elated structure (e.g. inner product, norm, Hilbert space, etc.) and the linear functions defined on these spaces and respecting these structures in a suitable sense. In modern introductory texts to functional analysis, the subject is seen as the study of vector spaces endowed with a topology, in particular infinite-dimensional spaces. An important part of functional analysis is the extension of the theory of measure, integration, and probability to infinite dimensional spaces, also known as infinite dimensional analysis in the last period. The most of researchers of functional analysis care to find the general solutions of operator equations.

The study of sequence spaces generated by infinite matrices started in 1970 and is an active branch of Functional Analysis. Firstly, Leibowitz (Leibowitz 1971) and Shiue (Shiue 1970) studied solid sequence spaces derived from Cesaro matrix. This space denoted by ces_p and defined as the set $cesr_p = \{x \in \mathbb{R}^{\mathbb{N}}: Cx \in l_p\}$ where $C = [c_{ij}]$ is the Cesaro matrix defined by $c_{ij} = \frac{1}{i}$ if $i \geq j$ and 0 otherwise. Thus, ces_p is a solid vector subspace in the vector lattice $\mathbb{R}^{\boldsymbol{N}}$. For $0 < p \leq 1$, $\mathrm{cesr_p} = \{0\}$ and l_p is a proper subset of ces_p for $1 < p < \infty$. Let C be Cesaro matrix. Then The formula $\left||x|\right|_{cesr_p} = ||\, C|x|\ ||_p$ defines a norm on $cesr_p$ and certainly, it is a solid norm.

Leibowitz (Leibowitz 1971) showed that $cesr_p$ is a separable Banach space with respect to this norm. Note that ces_p is not a subset of l_∞, the space of bounded sequences. But the intersection of ces_p and l_∞ is non-empty since both of these spaces contain l_p.

Let $A = \left[a_{ij}\right]_{i,j\geq 1}$ be an infinite matrix with non-negative entries and no zero columns. The domain of A consists of all scalar (real or complex) sequences $x \in \mathbb{F}^{\mathbb{N}}$ such that the series $\sum_{j=1}^{\infty} a_{ij}x_j$ converges for each $\mathrm{i} \in \mathbb{N}$. The domain of A will be denoted by

$domain(A)$. If $A = [a_{ij}]_{i,j\geq 1}$ is a lower triangular infinite matrix with non-negative entries and positive entries ($a_{ii} > 0$) on the main diagonal, then $domain(A)$= $\mathbb{F}^{\mathbb{N}}$. The assumption of non-zero entires on the main diagonal assures the existence of matricial inverse of A. This inverse A^{-1} is lower triangular. If A is not completely diagonal matrix, then we can not guarentee that all the entries of A^{-1} is positive. The bounded operators defined by infinite matrices provide us examples of invariant subspaces. For example, from Hardy's inequality ($||Cx||_p \leq \; ||C|x|\,||_p \;\; \leq \frac{p}{p-1}||x||_p$) it follows that Cesaro matrix transforms l_p into l_p. Moreover, it is easy to see that Cesaro matrix transforms c, the space of convergent sequences into itself. For $x \in domain(A)$, Ax , the transformation under A, is given by the formula $(Ax)_i$=$\sum_{j=1}^{\infty} a_{ij}x_j$ for each i ∈ ℕ. If $\lambda \in \mathbb{F}^{\mathbb{N}}$ is a sequence space and A is an infinite matrix with non-negative entries, then Johnson and Mohapatra (Johnson and Mohapatra 1985) defined $A^{-1}(\lambda) = \{x \in domain(A): Ax \in \lambda\}$ and $sld - A^{-1}(\lambda) = \{x \in domain(A): A|x| \in \lambda\}$. When $\lambda \in \mathbb{F}^{\mathbb{N}}$ is solid, $sld - A^{-1}(\lambda)$ is the largest solid subspace of $A^{-1}(\lambda)$. Actually, $sld - A^{-1}(\lambda)$ is the generalization of $cesr_p$ by interchanging Cesaro matrix with an arbitrary infinite matrix A with non-negative entries and l_p with a solid sequence space λ .

The first chapter of this study outlines the thesis as an introduction.

The second chapter of this study contains some basic definitions, theorem, examples and results about vector spaces, ordered vector spaces, Riesz spaces (vector lattices) and topological vector spaces. Throughout this chapter and thesis, reader can consult with the book (Luxemburg 2000) for unexplained terminology.

The third chapter focuses on an article of Johnson and Mohaptra (Johnson and Mohapatra 1985). In this chapter, some results and examples on this pullback sequence space $sld - A^{-1}(\lambda)$ obtained by (Johnson and Mohapatra 1985) are considered. For example, some conditions under which the pullback spaces $sol - A^{-1}(\lambda)$ inherit properties of λ such as LCC, AK and Hausdorffness properties or λ inherits properties of its pullbacks $sld - A^{-1}(\lambda)$ are established, and various counterexamples are constructed to show that many of the results can not be satisfied.

2. PRELIMINARIES

In this chapter, we will see some basic definitions, theorem, examples and results about vector spaces, ordered vector spaces, Riesz spaces (vector lattices) and topological vector spaces. Throughout this chapter and thesis, reader can consult with the books (Luxemburg 2000) and (Schaefer 1966) for unexplained terminologies on these topics.

2.1 Vector Spaces

In this section, we recall the definition of a vector space over the field $\mathbb{K}$ where $\mathbb{K}$ here denotes the field of real number numbers $\mathbb{R}$ or field of complex numbers $\mathbb{C}$, and give some standard examples of vector spaces. Let us recall the following definition.

Definition 2.1.1: Let $(\mathbb{K}, +, .)$ be a field whose elements are called scalars. Let L be a non-empty set whose elements are called vectors. Then L is a linear space (or a vector space) over the field $\mathbb{K}$ if it satisfies the following:

(1) $(\mathrm{L}, +)$ is a commutative group;

(2) For all $\alpha \in \mathbb{K}, x \in L$, then $\alpha . x \in L$;

(3) The scalar multiplication and addition satisfy;

(i) For all $\alpha \in \mathbb{K}, x, y \in L$, then $\alpha .(x + y) = \alpha . x + \alpha . y$,

(ii) For all $\alpha, \beta \in \mathbb{K}, x \in L$, then $(\alpha + \beta). x = \alpha . x + \beta . x$,

(iii) For all $\alpha, \beta \in \mathbb{K}, x \in L$, then $(\alpha . \beta). x = \alpha .(\beta . x)$,

(iv) For all $x \in L$ and 1 is the identity of $\mathbb{K}$, then $1. x = x$.

Then the triple $(L, +, .)$ is a vector space (linear space) over the field $\mathbb{K}$. Sometimes, the notation $L(\mathbb{K})$ denotes a vector space (linear space) or only L denotes a vector space (linear space) over the field $\mathbb{K}$.

Remark 2.1.2: If $\mathbb{K} = \mathbb{R}$, then the vector space $L(\mathbb{K})$ is the real vector space and when

$\mathbb{K} = \mathbb{C}$, then the vector space $L(\mathbb{K})$ is the complex vector space.

To illustrate this definition we will introduce some standard examples of vector spaces.

Example 2.1.3 : 1) Let $(\mathbb{R}, +, .)$ be the field of real numbers and $\mathbb{R}^n = \{(x_1, x_2, \dots, x_n): x_1, x_2, \dots, x_n \in \mathbb{R}\}$. Consider any two elements $X = (x_1, x_2, \dots, x_n)$ and $Y = (y_1, y_2, \dots, y_n)$ of $\mathbb{R}^n$, and define $X + Y = (x_1 + y_1, \dots, x_n + y_n)$, also, the scalar multiplication in $\mathbb{R}^n$ over $\mathbb{R}$ by $\alpha.X = (\alpha.x_1 +, \dots, \alpha.x_n)$ for all $\alpha \in \mathbb{R}$, $X \in \mathbb{R}^n$. Hence, $\mathbb{R}^n$ is a vector (linear) space over $\mathbb{R}$.

2)Let $(\mathbb{C}, +, .)$ be the field of complex numbers and $\mathbb{C}^n = \{(x_1, x_2, \dots, x_n): x_1, x_2, \dots, x_n \in \mathbb{C}\}$. Consider any two elements $X = (x_1, x_2, \dots, x_n)$ and $Y = (y_1, y_2, \dots, y_n)$ of $\mathbb{C}^n$, define $X + Y = (x_1 + y_1, \dots, x_n + y_n)$, also, the scalar multiplication in $\mathbb{R}^n$ over $\mathbb{R}$ by $\alpha.X = (\alpha.x_1 +, \dots, \alpha.x_n)$ for all $\alpha \in \mathbb{C}$, $X \in \mathbb{C}^n$. Hence $\mathbb{C}^n$ is a vector (linear) space over $\mathbb{C}$.

3) Let $C^b\ (R) = \{f: f: R \to R$ is bounded and continuous$\}$ For any $\mathrm{f}, \mathrm{g} \in \mathrm{C^b}(\mathbb{R})$ and for any $\alpha \in \mathbb{R}$, we define addition and scalar multiplication as $(f + g)(x) = f(x) + g(x)$ and $(\alpha f)(x) = \alpha f(x)$, $\forall x \in \mathbb{R}$. Therefore; $\mathrm{C^b}(\mathbb{R})$ is a vector (linear) space over $\mathbb{R}$, also, $C[a.b] = \{f: f: [a, b] \to R$ is continuous$\}$ with same addition and scalar multiplication above is also a vector (linear) space over $\mathbb{R}$.

Some properties of a vector space over the field are given by the following theorem.

Theorem 2.1.4: Let $L(\mathbb{K})$ be a linear space (vector space) with 0_L zero vector. Then

1) for all $\alpha \in \mathbb{K}$, $\alpha.0_L = 0_L$;

2) for all $x \in L$ $x.0 = 0_L$;

3) for all $\alpha \in \mathbb{K}, x \in L$ $\alpha.(-x) = -(\alpha.x)$;

4) for all $\alpha \in \mathbb{K}, x \in L$ $(-\alpha).x = -(\alpha.x)$;

5) for all $\alpha \in \mathbb{K}, x \in L$ $\alpha.(x - y) = \alpha.x - \alpha.y$;

6) If $\alpha.x = 0_L$ then $\alpha = 0$ or $x = 0_L$.

The following theorem is very useful.

Theorem 2.1.5 : Let $L(\mathbb{K})$ be a linear space (vector space) over the filed $\mathbb{K}$.

1) If $x, y \in L$ and $\alpha \neq 0 \in \mathbb{K}$, and $\alpha . x = \alpha . y$, then $x = y$.

2) If $\alpha, \beta \in \mathbb{K}$ and $x \neq 0 \in L$, and $\alpha . x = \beta . x$, then $\alpha = \beta$.

Next, we will introduce the subspace of a vector space.

Definition 2.1.6 : A nonempty subset H of a linear space $L(\mathbb{K})$ is said to be subspace of L if $x, y \in L$ and $\alpha, \beta \in \mathbb{K}$, then $\alpha . x + \beta . y \in L$.

To illustrate this definition, let us consider, $H_1 = \{(0, x_2, x_3): x_2, x_3 \in \mathbb{R}\}$ and $H_2 = \{\mathrm{f} \in \mathrm{C}[\mathrm{a}, \mathrm{b}]: \mathrm{f}(0) = 0\}$. Then H_1 and H_2 are the subspaces of $\mathbb{R}^3$ and $\mathrm{C}[\mathrm{a}, \mathrm{b}]$ under the pointwise operations, respevtively.

2.2 Ordered Vector Spaces

In this section, we introduce a special kind of vector spaces which is called ordered vector space. We start with the following definition.

Definition 2.2.1: Let X be a nonempty set, an order relation on X denoted by $(\preccurlyeq)$ and satisfies the following condition for all $x, y, z \in X$:

i) $x \preccurlyeq x$; (Reflexivity)

ii) if $x \preccurlyeq y$ and $y \preccurlyeq x$ then $x = y$; (Antisymmetry)

iii) if x$\preccurlyeq y$ and $y \preccurlyeq z$ then $x \preccurlyeq z$; (Transitivity)

Then the pair $(X, \preccurlyeq)$ is said to be ordered set. This means the ordered set is a nonempty set with an order relation defined on it.

Remark 2.2.2:

1) The order relation appeared in the above definition is said to be a partial order and a set with a partial order is called a partially ordered set.

2) Another type of order relation is said to be total if it satisfies all conditions in

Definition 2.2.1 with the following additional condition: for all $x, y \in X,\ either\ \ x \preccurlyeq y$ or $y \preccurlyeq x$.

3) It is clear that, every total order is a partial order, but the converse is not necessarily true.

4) The set X is said to be well ordered set if every subset of X has a least element. The set of natural numbers is a well ordered set.

To explain these definitions, we consider the following example .

Examples 2.2.3: 1) The set of real numbers $\mathbb{R}$ with the relation of less than or equal (greater than or equal) is a totally ordered set.

2) The divided relation on the set of all natural numbers $\mathbb{N}$ is a total order, but on theset of all integer numbers $\mathbb{Z}$ is not.

3) Consider the relation of proper subset $(\subset)$ on the power sets of $\{0,1\}$. Obviously, it satisfies the axioms for a partial order but not for the total order.

Now, let us come back to ordered vector spaces and start with the following definition.

Definition 2.2.4: Let L be a real vector space. Then L is said to be an ordered vector space if there is an order relation $(\preccurlyeq)$ on it that is compatible with the algebraic structure of L in the following sense:

1) If $x, y \in L$ with $x \preccurlyeq y$, then $x + z \preccurlyeq y + z$ for all $z \in L$.

2) if $x, y \in E$ with $x \preccurlyeq y$, then $\alpha . x \preccurlyeq \alpha . y$ for each $0 \leq \alpha \in \mathbb{R}$.

Remarks 2.2.5: 1) The element x in L is called positive if $x \geq 0$, and if $x > 0$ is said to be strictly positive.

2) The set $L^+ = \{x \in L : x \geq 0\}$, is called the positive cone of L.

3) Let L be an ordered vector space then for each $x, y \in$ L, one can have;

i) $\sup(x, y) = x \vee y$, may exist in L.

ii) $\inf(x, y) = x \wedge$ y, may exist in L.

iii) $\sup(x,0)=x^+$ may exist in L and it is called the positive part of x.

iv) $\sup(-x,0)=x^-$, may exist in L and it is called the negative part of x.

iiv) $|x|=x\vee(-x)$, may exist in L and it is called the modules of x.

Now, the following theorem gives some properties of the ordered vector spaces.

Theorem 2.2.6: Let L be an ordered vector space, then for all $x,y\in L$ and $\alpha\in\mathbb{R}$, the following axioms are met:

i) $x\neq y$ if and only if $x-y\neq 0$;

ii) $\alpha.x\neq 0$ if and only if $\alpha\neq 0$ and $x\neq 0$.

Proof:

i) Since $x\neq y$, we get $x<y\ or\ x\nleq y$, so $x-y\nleq 0$, then $x-y\neq 0$. Conversely, suppose $x-y\neq 0$, then $x-y\nleq 0$, so $x<y\ or\ x\nleq y$, one can have $x\neq y$.

ii) Let $\alpha.x\neq 0$, so one can have either $\alpha.x\nleq 0$ or $-\alpha.x\nleq 0$ and since $\alpha.x\nleq 0$ implies that $(\alpha>0\ and\ x\nleq 0)\ or\ (\alpha<0\wedge 0$, hence $\alpha\neq 0$ and $x\neq 0$. Conversely, suppose $\alpha\neq 0$ and $x\neq 0$, thus either α or $-\alpha$ is strictly positive, also x or $-x$ exceeds 0, then either $\alpha.x\nleq 0$ or $-\alpha.x\nleq 0$, hence $\alpha.x\neq 0$.

The following theorem was given in [13]. We omit its proof.

Corollary 1.3.6: Let L be an ordered vector space. Then for all $x,y\in L$ and $\alpha,\beta\in\mathbb{R}$ we have the following conditions:

i) if $x_1+y_1\neq x_2+y_2$ then $x_1\neq x_2\ or\ y_1\neq y_2$.

ii) if $\alpha.x\neq\beta.y$ then $\alpha\neq\beta\ or\ x\neq y$.

2.3 Vector Lattices

In this section, we introduce another special kind of ordered vector space, which is called Riesz space or a vector lattice and illustrate some basic examples, also some important properties from appearing theorems.

Definition 2.3.1: Order vector space L is said to be Riesz space or a vector lattice if it satisfies the following properties: $x \vee y$ and $x \wedge y$ exist in L for all $x, y \in L$. To explain this definition we consider several examples.

Example 2.3.2: 1) Let $L = C[a, b]$ be the vector space of all real-valued continuous functions defined on the closed interval $[a, b]$ with addition and scalar multiplication as appeared in Example 2.1.3.(3) and partially ordered pointwisely that is, $f(x) \leq g(x)$ for all $x \in [a, b]$ if and only if $f \leq g$, therefore ; L is a Riesz space, also $C^b(\mathbb{R})$ the vector of all bounded continuous function and $C(X)$, the vector of all continuous real function defined on a topological space X over the field $\mathbb{R}$ are Riesz spaces by the same way.

2) The Euclidean vector space $\mathbb{R}^n$ with addition and scalar multiplication as in Example 2.1.3(1)) is a Riesz space under the partial order $x \leq y$,whenever $x_k \leq y_k$ for each $k = 1,2,3, \dots, n$. The special case of this real vector space is $\mathbb{R}$ which is also a Riesz space. The vector space $\mathbb{R}^2$ is also a Riesz space with respect to the partial order which is called lexicographical order defined by $(x_1, x_2) \leq (y_1, y_2)$, if and only if either $x_1 < y_1$ or ($x_1 = y_1$ and $x_2 \leq y_2$).

Remark 2.3.3: 1) Every Riesz space is a partially ordered set but not necessarily vice-versa. To illustrate this, consider the vector space $L = \{f : f : [0,1] \to \mathbb{R}$ is a polynomial$\}$, Then L is an ordered vector space under the usual pointwise operations appeared in Example 2.1.3(3) but not a Riesz space.

2) Let $\{L_\alpha ; \alpha \in I\}$, be a family of Riesz spaces, with ordered as coordinatewise (for all $(x_\alpha), (y_\alpha) \in L$, if $x_\alpha \leq y_\alpha$, then $(x_\alpha) \leq (y_\alpha)$), then the Cartesian product $L = \prod L_\alpha$ is Riesz space with this ordering since for each $(x_\alpha), (y_\alpha) \in L$,$(x_\alpha) \vee (y_\alpha) =$
$(x_\alpha \vee y_\alpha)$, $(x_\alpha) \wedge (y_\alpha) = (x_\alpha \wedge y_\alpha)$, exist in L.

3) The subspace of a Riesz space need not to be a Riesz space in its own right. To illustrate this, we consider $L_1 = \{f: f: [a, b] \to \mathbb{R}, f(x) = ax + b\}$ which is a subspace of $L = C[a, b]$, but not a Riesz space.

Now, the following theorem gives us some facts about Riesz spaces.

Theorem 2.3.4: Let L be a Riesz space. Then for all x and y in L, we have the following:

1)If x^+ and $x^- \in L^+$, then $(-x)^+ = x^-$, $(-x)^- = x^+$ and $|-x| = |x|$.

2)$x = x^+ - x^-$, $x^+ \wedge x^- = 0$, and $|x| = x^+ + x^-$.

3) $0 \leq x^+ \leq |x|$, $0 \leq x^- \leq |x|$, $x^- \leq x \leq x^+$ and $|x| = 0$ if and only if $x = 0$.

4) $x \leq y$ if and only if $x^+ \leq y^+$ and $x^- \geq y^-$.

Proof:1)Trivial.

2)Since $x^+ - x = (x \vee 0) - x = 0 \vee (-x) = x^-$, then $x^+ - x^- = x$, and $0 = -x^- + x^- = (x \wedge 0) + x^- = (x + x^-) \wedge x^- = x^+ \wedge x^-$. Finally $|x| = x \vee (-x) = \{(2x) \vee 0\} - x = 2x^+ - (x^+ - x^-) = x^+ + x^-$.

3)$0 < x^+ \leq |x|$ and $0 \leq x^- \leq |x|$ follow from $x^+ + x^- = |x|$. The others are now obvious.

4)Let $x \leq y$, thus, $x^+ = x \vee 0 \leq v \vee 0 = y^+$ by some way one can have $x^- \geq y^-$. Conversely, suppose $x^+ \leq y^+$ and $x^- \geq y^-$, then $x = x^+ - x^- \leq x^+ - x^- = x$.

Definition 2.3.5: Let L be a Riesz space, the elements x and y in L are called disjoint, if $|x| \wedge |y| = 0$, and denoted by $x \perp y$.

Remark 2.3.6: Let L be a Riesz space and $A \subseteq L$, then

1) $A^d = \{x \in L: x \perp y \text{ for all } y \in A\}$ is called the disjoint complement of A.

2) The disjoint complement $A^{dd} = \{x \in L: x \perp y \text{ for all } y \in A^d\}(A^d)^d$ of A^d is called the second disjoint complement of A.

Next, we recall some important definitions about Riesz spaces in which we will need in this thesis.

Definition 2.3.7: If L is a Riesz space, and a subset A of L is inherited the order from L, then

1) The linear subspace H of L is called a Riesz subspace of L if for all members x and y$\in H$, then $x \vee y$ and $x \wedge y$ are the members of H.

2) The subset A of L is called solid if it follows from $x \in A$ and $|y| \leq |x|$ that we get $y \in A$.

3) We say a subset A of L is an ideal (order ideal) in L if A is a solid linear subspace of L. It is totally different from an algebraic ideal in the Ring Theory.

4) The set $L_x = \{y \in E: \exists \lambda > 0 \text{ with } |y| \leq \lambda |x|\}$ is called the principal ideal generated by the element x.

5) The ideal A in L is called a band if for every subset $B \subseteq A$ and $u =$ sup$B$ in L exist in B, ($u \in$ B).

Remark 2.3.8:

1) Every subset of A of a Riesz space L contained in at least one band is the smallest band, and it is said to be the band generated by A and is denoted by $\{A\}$.

2) A principal band is a band generated by a singleton subset and if it is generated by x will be denoted by A_x.

3) A projection band A of a Riesz space L satisfies the decomposition $L = A \oplus A^d$, and projection element is any element whose generating principal band is a projection band.

4) An element $0 < e$ of a Riesz space L is called a weak order unit if $B_e = L$.

The following example explains some relations on Riesz spaces.

Examples 2.3.9:

1) The set $H = \{\alpha.1: \alpha \in \mathbb{R}\}$ is a Riesz subspace of a Riesz space $L = C[0,1]$, but not an ideal of L.

2) Let $S = \{\{a_n\}_i \ , i = 1,2, \dots\}$ be the set of all real sequences and H the Riesz space of real null sequences, the space l_1 (the space of sequences whose series is absolutely convergent,) is a proper ideal in H and the space H is a proper in l_∞ (the space of bounded sequences.) and l_∞ is a proper ideal in S. None of these ideals is a band.

Definition 2.3.10: The Riesz space L is said to be Archimedean if $inf\{n^{-1}x: n = 1,2,3, \dots.\} = 0$ for all $x \in L^+$.

To illustrate that, The Riesz spaces $C(X)$ and $\mathbb{R}^n$, with coordinatewise ordering are Archimedean, but $L = \mathbb{R}^2$ with the lexicographical ordering is non-Archimedean Riesz space, since the element (0,1) in L is a lower bound of the sequence (n^{-1}, n^{-1}). So,$x = (1,1)$ does not agree with $\inf\{n^{-1}x: n = 1,2,3, \dots.\} = 0$, indeed the sequence of all $n^{-1}x$ does not have an infimum at all in this case.

The following theorem gives us some characterizations of Archimedean property.

Theorem 2.3.11: Let L be a Riesz space, then

1)L is Archimedean if and only if for all $x \in L^+$, $\inf\{\epsilon_n x: n = 1,2,3, \dots\} = 0$, where (ϵ_n) is a sequence of non-negative real numbers such that $\epsilon_n \to 0$.

2) L is Archimedean if and only if, given $x, y \in L^+$, such that $0 < ny \le x$ for $n = 1,2, \dots$,implies that $y = 0$ that is, for all $y \ge 0, y \ne 0$ the sequence (ny) is unbounded.

3) If L is Archimedean, then every Riesz subspace of L is also Archimedean. In particular, ideals and bands in E are Archimedean Riesz spaces on their own rights.

2.4 **Topological Vector Spaces**

In this subsection, we give the definition of topological vector spaces with some examples to explain this concept and some important remarks and theorems about it.

Definition 2.4.1:

Let X be a non-empty set. Then the collection $\mathcal{J}$ of subsets of X is said to be a topology on X if it satisfies the following axioms:

1) X and $\emptyset$ belong to $\mathcal{J}$;

2) The intersection of two members in $\mathcal{J}$ belongs to $\mathcal{J}$;

3) The union of any family of sets in $\mathcal{J}$ belongs to $\mathcal{J}$;

The ordered pair $(X, \mathcal{J})$ is said to be a topological space.

Now, the following remark gives us some types of topological spaces defined on a set X.

Remark 2.4.2: 1) Let X be a nonempty set and $\mathcal{J}$ be a collection of all subsets of X then (X, $\mathcal{J}$) is called a discrete topological space.

2) Let X be a nonempty set and $\mathcal{J} = \{X, \emptyset\}$ then $(X, \mathcal{J})$ is called indiscrete topological space.

3) Let X be a nonempty set and $\mathcal{J}_{co}$ be the collection of subsets $U \subset X$ whose complements $X - U$ is finite, together with the empty set $U = \emptyset$, then $(X, \mathcal{J})$ is called co-finite topological space.

4) Let $X = \mathbb{R}$ and $\mathcal{J}_u$ be the collection of subsets of $\mathbb{R}$ which can be expressed as a union of open intervals, then $\mathcal{J}_u$ forms a topology. ($\mathbb{R}$, $\mathcal{J}_u$) is called usual topological space.

It's clear that, every metric space is a topological space, but not necessarily vice-versa, and any member in a topology is called an open set and its complement is called a closed set.

Definition 2.4.3: Let X be a linear space over the scalar field $\mathbb{K}$, then X is said to be a topological vector space and written shortly as $(\zeta\nu s)$ if X together with a topology $\mathcal{J}$ satisfies the following conditions:

i) The map of addition $(x, y) \to x + y$, $X \times X \to X$, is a continuous mapping;

ii) The map of scalar multiplication $(\alpha, x) \rightarrow \alpha x$, $\mathbb{K} \times X \rightarrow X$, a is continuous mapping for all $x \in X$, $\alpha \in \mathbb{K}$.

To illustrate this definition, we consider the following examples.

Example 2.4.4: 1) It is clear that, every vector space X over the filed $\mathbb{K}$ with the indiscrete topology is a $\zeta v s$.

2) The vector space X not singleton with discrete topology is $\zeta v s$. Otherwise, it is not a $\zeta v s$.

Some information to explain $\zeta v s$ is given by the following remark.

Remark 2.4.5: 1) We know that every metric space is a topological space, but this fact is not necessary to be true to be a $\zeta v s$. This means the metric vector space is not necessarily a $\zeta v s$. In general, there exist metric spaces with mappings of sum and scalar multiplications are discontinuous.

2) The intersection of any family of $\zeta v s$ on X also contains $\zeta v s$ on X, but the union of this collection may be not $\zeta v s$ on X.

3) On $\zeta v s$, translation mappings from X onto X such that $y \rightarrow x + y$ and $x \rightarrow \alpha x$ for some $x \in X, 0 \neq \alpha \in \mathbb{K}$ are homomorphisms.

Now, the following proposition introduces some properties of $\zeta v s$.

Proposition 2.4.6: Let X be a $\zeta v s$ and G, H and E are subsets of X.Then:

1) $\overline{x + G} = x + \bar{G}$ for all $x \in X$, where $\bar{G}$ is closure of a set G;

2) $\overline{G + H} \supseteq \bar{G} + \bar{H}$;

3) $G + E$ is open for any open set ;

4) If G and H are compact sets, then $G + H$ is compact, but not conversely. Also if G is compact and H is closed then $G + H$ is closed;

5) The closure of a subset G can be defined as $\bar{G} = \cap \{(G + E): E$ neigh. of 0$\}$;

6) If G is convex (subspace, balanced) set then $\bar{G}$ is convex (subspace, balanced) set

respectively;

7) If G is compact, E neigh. of G, then there exists a neigh. H of 0 such that $G + E \subseteq H$.

More properties of a ζvs are given below.

Proposition 2.2.7: Let X be a ζvs and G, H and E subsets of X. Then:
1) If E is a neigh. of 0 there exists a balanced neigh. G of 0 such that $G \subseteq E$;

2) If G is a balanced neigh. of 0 then G^0, the interior of G is balanced;

3) Every convex neigh. of 0 contains a closed, balanced, convex neigh. of 0;

4) Every convex neigh. of 0 contains a closed, balanced, convex neigh. of 0.

2.5 Local Base for Topological Vector Spaces

In this subsection, we are dealt with the base and local base in a topological vector space with some basic definitions and properties of these concepts. Let us start with the following definition.

Definition 2.5.1 : Let $\mathcal{J}$ be a topological vector space. Then local base $\mathcal{B}$ of 0 is a collection requiring that each $B \in \mathcal{B}$ is open and every neigh. of 0 contains a set in $\mathcal{B}$. To illustrate this definition, see next remark.

Remark 2.5.2: 1) Clearly the topology $\mathcal{J}$ is generated by the sets

$$E = \bigcup_{\alpha,\beta} (x_\alpha + B_\beta) \ where \ x_\alpha \in X, \qquad B_\beta \in \mathcal{B}.$$

2) In topology $\mathcal{J}$ a base at 0 is called a local base.

3) Every ζvs has a balanced local base.

The following introduces some properties of a local base.

Proposition 2.5.3: Let X be a ζvs and G, H and E are subsets of X. Then:
i) If G is any neigh. of $0, G$ contains a closed balanced neigh. of 0. In other words,

the closed balanced neigh. form a local base at 0.

ii) $\mathcal{J}$ is Hausdorff if and only if $\{0\}$ is closed if and only if $\{0\} = \cap \{E: E \in H\}$, for any local base H.

Proof: i) There exists a balanced neigh. E of 0 such that $E + E \subseteq G$. Now $\bar{E}$ is also balanced and $\bar{E} \subseteq E + E \subseteq G$.

ii) Obvious.

Theorem 2.5.4 : Let $\mathcal{B}$ be a local base at 0 of topological vector space X. Then for any subsets G, H and E, one can have:

1) If $G, H \in \mathcal{B}$, then there exists $E \in \mathcal{B}$ such that $E \subseteq G \cap H$;

2) If $G \in \mathcal{B}$, then there exists $H \in \mathcal{B}$ such that $H + H \subseteq G$;

3) If $G \in \mathcal{B}$, then there exists $H \in \mathcal{B}$ such that $\lambda H \subseteq G$ for all $\lambda \in \mathbb{K}$ satisfying $|\lambda| \leq 1$;

4) If $x \in X$, then there exists $\delta > 0$ such that $\lambda x \in G$ for all $\lambda \in \mathbb{K}$ satisfying $|\lambda| \leq \delta$.

Other concepts of a ζvs are introduced by the following definitions.

Definition 2.5.5 : 1) A topological vector space X is called locally convex if there exists a local base at 0 consisting of convex sets.

2) A function $f: X \to \mathbb{R}$ is said to be sublinear if f satisfies

$i)$ $f(x + y) \leq f(x) + f(y)$ for all $x, y \in X$;

$ii)$ $f(\alpha x) = \alpha f(x)$ for all $x \in X$, and $\alpha \geq 0$.

From the definition above, one can have other types of functions on a ζvs.

Remark 2.5.6:

1) Let X be ζvs, if f satisfies

i) $f(x + y) \leq f(x) + f(y)$, for all $x, y \in X$, and

ii) $f(\alpha x) = |\alpha| f(x)$, for all $x, y \in X$ and $\alpha \geq 0$, then f is said to be a seminorm of ζvs.

2) Let X be a ζvs. If $f(x) = 0, x = 0$ for all $x \in X$, then f is said to a norm on ζvs.
Next, the following theorem shows that the family of seminorms on a vector space generates a locally convex topology.

Theorem 2.5.7: Let X be a vector space and the family $\{f_i : i \in I\}$ be seminorms on X. If $\mathcal{U}$ is the class of all finite intersections of sets of the form $\{x \in X : f_i(x) < \delta_i, i \in I, \delta_i > 0.\}$, then $\mathcal{U}$ is a local base for a topology $\mathcal{J}$ on X and X is a locally convex ζvs.

Theorem 2.5.8: Let X be a vector space and the family $\{f_i : i \in I\}$ be seminorms on X. Then X has the weakest ζvs if all the f_i continuous, and for a net $\{x_\alpha\} \subseteq X, x_\alpha \to x$ in $\mathcal{J}$ if and only if $f_i(x_\alpha - x) \to 0$ for all $i \in I$.

Next, we will recall the Hausdorff topology (sometimes denoted by $T_2 -$ space) on topological vector spaces.

Remark 2.5.9 : A ζvs $(X, \mathcal{J})$ is a Hausdorff topology if and only if the family $\{f_i : i \in I\}$ is separating. That means when $x \neq 0$, there exists at least f_i such that $f_i(x) \neq 0$. To illustrate this, consider $X = C^\infty[a, b]$, the vector space of all infinitely differentiable functions defined on the closed bounded interval $[a, b]$ and the seminorm $p_n(f) = \sup\{|f^{(n)}(t)| : t \in [a, b]\}$ where $f^{(n)}$ is the n-th derivative of f.
In the topology defined by the p_n, convergence means uniform convergence of all derivatives.

2.6 Locally Convex Topological Vector Spaces

In this subsection, we study and discuss the locally convex topological vector spaces with some properties of this concept. Let us firstly study the convex set in a vector space X over the field $\mathbb{K}$.

Definition 2.6.1: Let X be a vector space over the field $\mathbb{K}$. A subset A of X is convex if A contains the segment of a straight line joining any two points x and y in A.
It was well known that every open interval in the real vector space $\mathbb{R}$ is a convex set.

Another properties of a convex set in a vector space are given by the following theorem.

Theorem 2.6.2: Let X be a vector space. Then

1) $\emptyset$ and X are convex sets;

2) The intersection of any convex subsets in X is convex set;

3) The addition of two convex subsets in X is convex set;

4) every subspace of a vector space X is convex set.

Remark 2.6.3: In any vector space X, the unions of convex sets are generally not convex. To illustrate that consider, $A = (3,7) \cup (7,12)$ which is not a convex set.

Now, we are about to say something on convex set in a ζvs and start by the following proposition.

Proposition 2.6.4: The closure and the interior of convex sets in a ζvs X are convex sets.

Definition 2.6.5: Let X be a ζvs , a subset A ⊆ X is said to be radial at x if A contains a line segment passing through x in any direction. That means, for every $y \in X$ there is $\delta > 0$ such that $x + \lambda y \in A$ for all $\lambda \in [0, \delta]$ and x is called an internal point of A.

From above, one can recall the Minkowski functional of A is defined as $p_A(x) = inf\{r > 0: x \in rA\}$, some properties of Minkowski functional are given by the following theorem.

Theorem 2.6.6: Let A be convex and radial at 0. Then

1) p_A is sublinear;

2) $\{x \in X: p_A(x) < 1\} = \{x \in A: A\ is\ radial\ at\ x\} \subseteq A \subseteq \{x: p_A(x) \leq 1\}$;

3) If A is balanced, $\mathrm{p_A}$ is a seminorm;

4) If X is a ζvs and $0 \in A^\circ$, then p_A is continuous; Since, $\bar{A} = \{p_A \leq 1\}, A^\circ = \{p_A < 1\}$, hence $\{p_A = 1\} = \partial A$, the boundary of A.

Now, we give the definition of locally convex topological vector space.

Definition 2.6.7: Let X be a ζvs. Then X is said to be locally convex and denoted by (L.C.) if there is a basis of neigh. of the origin in X consisting of convex sets.

Locally convex spaces are very important classes of ζvs so thatwe focus on the properties of neigh. of locally convex spaces.

Proposition 2.6.8: A locally convex ζvs has a basis of neigh. of the origin consisting of open absolutely convex subsets.

Theorem 2.6.9: If X is a locally convex ζvs then its topology is generated by a family $\mathcal{P}$ of seminorms.

Definition 2.6.10: A subset A of ζvs Y is bounded if, for every neigh. U of origin in Y, there exists a positive real number r such that $A \subseteq rU$.

Theorem 2.6.11: Let X be locally convex and satisfy condition in Theorem 2.6.9. Then $A \subseteq X$ is bounded if and only if every $p \in \mathcal{P}$ is bounded on A.

To illustrate the above theorem we consider the following example.

Example 2.6.12: Every compact set is bounded in a Hausdorff ζvs but no subspace other than $\{0\}$ is bounded in ζvs.

Remark 2.6.13: Assume that $\mathcal{P} = \{h_i : i = 1,2, \dots\}$ is a countable separated collection of seminorms on a linear space X generating a ζvs $\mathcal{J}$. Then there exists a translation-invariant metric compatible with $\mathcal{J}$, define$d\ by\ d(x,y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{h_i(x-y)}{1+h_i(x-y)}$.

It is strange that the closed balls $\{x : d(0,x) \leq r\}$ need not be convex (they are balanced) see the following example:

Example 2.6.14 :Let $B = \{(x_n)_{n=1}^{\infty} : x_n \in \mathbb{K}\ \textit{for all}\ n \geq 1\}$, the space of all sequences. Topology of pointwise convergence is described by the seminorms $p_k, (k \geq 1), p_k\big((x_n)\big) = |x_k|$ and the metric is $d(x,y) = \sum \frac{1}{2^n} \frac{|x_n - y_n|}{1+|x_n-y_n|}$, $x = (x_n)$, $y = (y_n)$ The ball $B_{\frac{1}{4}} = \left\{x : d(0,x) \leq \frac{1}{4}\right\}$ is not convex, since $(1,0,0, \dots)$, $(0,1,0,0, \dots) \in B_{\frac{1}{2}}$, but $\frac{3}{4}(1,0,0, \dots) + \frac{1}{4}(0,1,0, \dots) = \left(\frac{3}{4}, \frac{1}{4}, 0,0,0, \dots\right) \notin B_{\frac{1}{4}}$.

Theorem 2.6.15 : If $(X, \mathcal{J})$ is a (Hausdorff) ζvs with a countable local base, then there is a metric d on X such that

1) d is compatible with the topology $\mathcal{J}$;

2) The balls $\{x: d(0,x) \leq r\}$ are balanced;

3) d is translation-invariant: $d(x+z, y+z) = d(x,y)$ for $x, y, z \in X$. If, in addition, X is locally convex then d can be chosen so that all open balls $\{x: d(0,x) < r\}$ are convex.

3.GENERALIZATIONS AND PROPERTIES OF SOLID SEQUENCES FROM AN INFINITE MATRIX AND SOLID SEQUENCE SPACE

In this chapter, we study the spaces of sequences denoted by $sld - A^{-1}(\lambda)$ and $A^{-1}(\lambda)$, where λ is a solid space and A is an infinite matrix with positive entiries and no zero columns. We must say that these type of spaces were introduced in (Johnson and Mohapatra 1985). Actually these type of spaces are the generalizations of ces_p and vector lattices with the coordinatewise order.

3.1 The Spaces $sld - A^{-1}(\lambda)$ and $A^{-1}(\lambda)$

In this subsection, we will give the definitions and some properties of the sequence spaces $\boldsymbol{sld - A^{-1}(\lambda)}$ and $\boldsymbol{A^{-1}(\lambda)}$, for a solid set $\boldsymbol{\lambda}$ and an infinite matrix $\boldsymbol{A}$. We start with a definition before giving the definitions of the spaces above.

Definition 3.1.1 : Let L be a vector space such that $a \in L, N \subseteq L$, then

1) The solid hull of a is denoted by $S(a)$ and is defined by $\mathrm{S}(a) = \{b \in L: |b_n| \leqslant |a_n|, n \in Z^+\}$.

2) The solid hull of N is denoted by $N(S)$ and is defined by $S(N) = \bigcup_{a \in N} S(a)$.

Remark 3.1.2:

If $S(N) = N$, then N is said to be a solid set. Equivalently, the subset N of L is called solid if it follows from $x \in N$ and $|y| \leq |x|$ that we get $y \in N$. In some literature, we note that normal set is used instead of solid set, but we prefer using solid term instead of normal term.

Definitions 3.1.3: 1) Let $A = (a_{mn})$ $(m, n = 1,2, \dots)$ be an infinite matrix and $\mathbb{F}^{\mathbb{N}}$ be the space of all scalar sequences, then domain of A denoted by $domain(A)$, is defined as $domain(A) = \left\{ x \in \mathbb{F}^{\mathbb{N}} \mid \sum_{n=1}^{\infty} a_{mn} x_n \text{ converges for } m \in Z^+ \right\}$.

2) The operator $(Ax)_m = \sum_{n=1}^{\infty} a_{mn} x_n$ is said to be A -transform of x.

3) If $S \subseteq domain(A)$, then $AS = \{Ax \mid x \in S\}$.

Next, we are going to give the definitions of the spaces $sld - A^{-1}(\lambda)$ and $A^{-1}(\lambda)$.

Notation 3.1.4 :

Let $S \subseteq \mathbb{F}^{\mathbb{N}}$, then

1) $A^{-1}(S) = \{x \in domain(A) : Ax \in S\}$, where A is an infinite matrix;

2) For the invertible infinite matrix A and $S \subseteq domain(A^{-1})$ we write $A^{-1}S = \{A^{-1}x : x \in S\}$;

3) $sld - A^{-1}(S) = \{x \in \mathbb{F}^{\mathbb{N}} : |x| \in A^{-1}(S)\} = \{x \in \mathbb{F}^{\mathbb{N}} : |x| \in domain(A)$ and $A|x| \in S\}$.

Remark 3.1.5 : 1) $sld - A^{-1}(S)$ is the largest normal subset of $A^{-1}(S)$.

2) If λ is a ζvs with some topology τ, then $A^{-1}(\lambda)$ $is\ a\ \zeta vs$ with the induced topology $A^{-1}(\tau)$;

3) Every open set in $A^{-1}(\lambda)$ is in the form of $A^{-1}(U)\ where\ U \in \tau$;

4) $sld -\ A^{-1}(\lambda)$ is a subspace of topological space $(A^{-1}(\lambda), A^{-1}(\tau))$;

5) If $sld -\ A^{-1}(\tau)$, is a topological space, then the matrix A generates from a ζvs (λ, τ) , two other ζvs $(A^{-1}(\lambda), A^{-1}(\tau))$ and $(sld - A^{-1}(\lambda), sld - A^{-1}(\tau))$.

We know that a topological space may be coarser than the relative topological space, but this fact is not necessarily to be true in a ζvs in $sld - A^{-1}(\lambda)$, such as T Induces a topology on $sld - A^{-1}(\lambda)$ which is not coarser than the relative topology on $sld - A^{-1}(\lambda)$.

To illustrate all the concepts above, we consider the following examples.

Examples 3.1.6:

1) Let A be identity infinite matrix, $\lambda = l_p$, we can consider T as the usual norm topology in l_p, then $A^{-1}(\lambda)$ and $sld - A^{-1}(\lambda)$ become l_p in this case.

2) Let A be the Cesàro matrix $C = (c_{nk})$ with $c_{nk} = \begin{cases} \frac{1}{n}; & k \leqslant n \\ 0; & k > n; \end{cases}$ $Cx = \left(\frac{1}{n}\sum_{k=1}^{n} x_k\right)_n$, $\lambda = l_p$, then $sld - C^{-1}(l_p)$ is the sequence space denoted by $cesr_p$.

Some properties of the sequence space $cesr_p$, are given by the following proposition.

These properties are obtained by G. Leibowitz (Leibowitz 1971).

Proposition 3.1.7:

Let $cesr_p$ be the sequence space appeared in above example, then

1) If $0 < p \leqslant 1$, then the only zero sequence exists in $cesr_p$ and so it is trivial;

2) If $1 < p \leqslant \infty$, then $cesr_p$contains l_p as a proper linear subspace;

3) The sequence space $cesr_p$ is a normed linear subspace of $\mathbb{F}^{\mathbb{N}}$, with the norm $\|x\|_{cesr_p} = \|C|x|\|_p$, and complete under this norm.

Now, let us come back to convex hull of a solid set by the following theorem. This result is really a very well-known fact in Riesz space theory.

Theorem 3.1.8 : The convex hull of a solid set of sequences is solid.

Proof: Let $S \subseteq \mathbb{F}^{\mathbb{N}}$. We define $\psi(S) = \{tx + (1-t)y \mid x, y \in S, t \in [0,1]\}$. But the convex hull of S is $\bigcup_{n=1}^{\infty} \psi^n(S)$ by Definition 3.1.1 and a union of solid sets is solid. Therefore, it remains to prove that $\psi(S)$ is solid whenever S is. Let $x, y \in S, t \in [0,1]$, and $|z_n| \leqslant |tx_n + (1-t)y_n|$ for all n. Also, if $tx_n + (1-t)y_n = 0$, we will define $a, b \in L$ by $a_n = b_n = 0$, otherwise define $a_n = z_n x_n (tx_n + (1-t)y_n)^{-1}$, $b_n =$

$z_n y_n (t x_n + (1-t) y_n)^{-1}$. Hence $a \in N(x) \subseteq S$ and $b \in N(y) \subseteq S$,
so $z = ta + (1-t)b \in \psi(S)$. Therefore; $\psi(S)$ is solid and this completes the proof.

The following result belongs to P.D. Johnson and R.N. Mohapatra (Johnson and Mohapatra 1985) and contains a remarkable property of $sld - A^{-1}(\lambda)$.

<u>Proposition 3.1.9:</u>

Let λ be a solid subspace of $\mathbb{F}^{\mathbb{N}}$ and A an infinite matrix with non-negative entries, then:

1) $sld - A^{-1}(\lambda)$ is solid subspace $\mathbb{F}^{\mathbb{N}}$;

2) $sld - A^{-1}(\lambda) \subseteq A^{-1}(\lambda)$;

3) $sld - A^{-1}(\lambda)$ is the largest solid set of sequences contained in $A^{-1}(\lambda)$.

<u>Proof:</u>

1) Let $x \in sld - A^{-1}(\lambda)$ and $y \in S(x)$, then the non-negativity of entries of A implies that $A|y| \in S(A|x|)$. Thus, $sld - A^{-1}(\lambda)$ is solid subset of λ. If $x, y \in sld - A^{-1}(\lambda)$, thus from the non-negativity of a_{mn} and the triangle inequality we get that $A|x+y| \in S(A|x| + A|y|)$, then so that it is a subspace of $\mathbb{F}^{\mathbb{N}}$. This completes the proof of part 1).

2) This is clear from the non-negativity of the terms a_{mn}, and the fact that if $x \in sld - A^{-1}(\lambda)$, then |x| $\in N(A|x|)$.

3) It is obvious from (1) and (2).

Now, note that the following example shows us that Proposition 3.1.9 may be valid even if λ is not solid.

<u>Example 3.1.10</u>:

Take $\beta = \{(-1)^n\}_{n \geqslant 1}$ and λ as the linear span of β and c_0 (the space of null sequences). Let A be the identity matrix. Thus, $sld - A^{-1}(\lambda) = c_0$ and properties in Proposition 3.1.9 are satisfied, but λ is not solid.

<u>Example 3.1.11</u>:

Let C be a Cesàro matrix and $\lambda = c$ (the set of convergent sequences.), then the

Proposition 3.1.9 are not valid. To clarify that, take $e = \{1,1,\dots\}$, then $e \in sld - C^{-1}(c)$. But there exists sequences of 0 's and l's that c is not mapped into c. Also let X be a sequence of $\pm$ l's such that $x + e$, a sequence of 2's and 0's, is not mapped from c into c, so (1) is not satisfied. Now, one can construct a sequence of ±1 's which C does not map c into c. So (2) does not hold. Finally the failure of (3) is upon part (1).

Notation 3.1.12:
For each positive integer k, let e_k denote the $k - th$ coordinate sequence, with 1 in the $k - th$ entry and zeros elsewhere.

We have the following result which gives us the relation between c_{00} (finitely non zero sequences) and $sld - A^{-1}(\lambda)$.

Proposition 3.1.13 :
If λ is a solid subspace of $\mathbb{F}^{\mathbb{N}}$ and an infinite matrix $A = (a_{mn})$ has non-negative entries, then $c_{00} \cap sld - A^{-1}(\lambda)$ =span$\{e_k$: the k^{th} column of A is in $\lambda\}$.

Proof: Ae_k is the k^{th} column of the matrix A, then by using Proposition 3.1.9 it exists in λ if and only if $e_k \epsilon$ sld- $A^{-1}(\lambda)$. So from hypothesis, $sld - A^{-1}(\lambda)$ is trivial if only if no column of A is in λ, also $c_{00} \subseteq sld - A^{-1}(\lambda)$ if and only if every column of A is in λ. This completes the proof.

Example 3.1.15: Suppose $A = C$ and $\lambda = l_1$. Thus $sld - A^{-1}(\lambda)$ is trivial.
Also, $e_1 - e_2 \in C^{-1}(l_1)$, so $c_{00} \cap A^{-1}(\lambda)$ is non-empty.

Remark 3.1.16 : From above in this section, we assume that A will denote an infinite matrix with non-negative entries and λ will be a solid subspace of L. Now we can put other assumptions on A and λ by the following:
$P(1)$: A sends $A^{-1}(\lambda)$ injectively into λ ;

$P(2)$: A sends $A^{-1}(\lambda)$ injectively onto λ;

$P(3)$: A and λ satisfy $P(2)$, and the inverse linear transformation, from λ onto $A^{-1}(\lambda)$, is matricial.

It is clear that if A is a lower triangular infinite matrix with positive entries on the main diagonal, then for any solid subspace λ of $\mathbb{F}^{\mathbb{N}}$, $P(3)$ is satisfied.

Next, we introduce the space of sequence in case λ is a $\zeta\nu s$ with some topology τ.

Normally, the space $A^{-1}(\lambda)$ becomes a topological vector space with the induced topology $A^{-1}(\tau)$ and the members are open sets $A^{-1}(U)$, such that $U \in \tau$.

Remark 3.1.17:

1) If $P(1)$ is not satisfied, then the $\zeta\nu s$ $A^{-1}(\lambda)$ will not be Hausdorff.

2) If the property $P(2)$ is valid, then $(A^{-1}(\lambda), A^{-1}(T))$ is isomorphic as $\zeta\nu s$ to (λ, τ).

3) In general, τ will induce a topology on $sld - A^{-1}(\lambda)$ which is no coarser than the relative topology on $sld - A^{-1}(\lambda)$ as a subspace of $(A^{-1}(\lambda), A^{-1}(T))$.

Now, we give a basic lemma in which we will need to describe the above topology in the following theorem. For more information, see the book (Schaeffer 1966).

Lemma 3.1.18: If (E, τ) is a $\zeta\nu s$ and $\psi: E \longrightarrow \mathbb{R}^{+} \cup \{0\}$, then ψ determines τ if and only if the sets $U_a = \{x \in E \mid \psi(x) < \alpha\}$, $\alpha > 0$ form a neigh. base at the origin for τ.

Note that if two functions ψ and $\psi^{\backslash}$ both determine τ, then they are equivalent.

Now, we need some essentials of norms, quasinorms, paranorms and pseudonorms.
For more information about these types of norm-like functions, see (Köthe 1960, Maddox 1970, Schaeffer 1966). From the definitions of spaces above, one can have the following relations between these concepts :

1) Every quasinormed or Hausdorff paranormed space is a metric space;

2) A Hausdorff paranormed or quasinormed space is equivalently pseudonormable;

3) A paranorm g determining a Hausdorff $\zeta\nu s$. topology on a vector space E for which each set $\{x \in E \mid g(x) \leqslant \alpha\}$ $(\alpha > 0)$ is balanced is, in fact, a pseudonorm.

Remark 3.1.20: 1) If μ is a solid subspace of $\mathbb{F}^{\mathbb{N}}$, and (μ, Γ) is a $\zeta\nu s$, then Γ is solid if and only if (μ, Γ) possesses a neigh. base at the origin consisting of solid sets.

2) A function $\psi: E \longrightarrow \mathbb{R}^{+} \cup \{0\}$ is solid if and only if all sets $\{x \in \psi; \gamma(x) \leqslant \alpha\}$, where $\alpha > 0$, is solid. Also ψ is solid if $y \in N(x) \subseteq \mu$ leads us to $\psi(y) \leqslant \psi(x)$.

3) if ψ is a solid function, and induce a ζvs topology Γ on μ, then Γ is solid.

Lemma 3.1.21: Assume that μ is a solid subspace of $\mathbb{F}^{\mathbb{N}}$, and Γ is a Hausdorff solid topology on μ. If Γ is induced by a norm(quasinorm, ,paranorm) ϱ , then there is an equivalent norm(quasinorm, pseudonorm) $\hat{\varrho}$, which is solid.

Proof: Assume ϱ is a quasinorm, and $U = \{x \in \mu; \varrho(x) \leqslant 1\}$. By assumption, there exists a solid (also balanced) neigh.of the origin V, such as $V \subseteq U$, the Minkowski functional $\hat{\varrho}$ of V, defined by $\hat{\varrho}(x) = \inf\{r > 0; x \in rV\}$, is a quasinorm on μ equivalent to ϱ. Clearly that $\hat{\varrho}$ is solid.

Now Assume ϱ is a norm and U is convex. Suppose W is a solid neigh.of the origin contained in U, and let V be the convex hull of W. So by using Theorem 3.1.8, one can have $V \subseteq U$, and V is solid. The Minkowski function $\hat{\varrho}$ of V is a norm equivalent to ϱ and $\hat{\varrho}$ is solid.

Now, the following theorem describes some properties of topology appeared in Remark 3.1.8.

Theorem 3.1.22: Assume τ is a solid topology on λ with $\mathcal{U}$ is a neigh. base at the origin in (λ, τ) consisting of normal sets. Then the collection $sld - A^{-1}(\mathcal{U})$ of the solid sets $sld - A^{-1}(U) = \{x \in sld - A^{-1}(\lambda); A|x| \in U\}$, $U \in \mathcal{U}$, is a neigh. base at the origin for a topology $sld - A^{-1}(\tau)$ with the property that $sld - A^{-1}(\lambda)$ is a ζvs. The topology $sld - A^{-1}(\tau)$ is at least as fine as the relative topology induced by $A^{-1}(\tau)$ on $sld - A^{-1}(\lambda)$. In addition

1) If (λ, τ) is Hausdorff, and A has no zero columns, then $(sld - A^{-1}(\lambda), nor - A^{-1}(\tau))$ is also Hausdorff;

2) If (λ, τ) is locally bounded, then $(sld - A^{-1}(\lambda), sld - A^{-1}(\tau))$ is locally bounded;

3) If (λ, τ) is locally convex, then $(sld - A^{-1}(\lambda), sld - A^{-1}(\tau))$ is locally convex;

4) If τ is a metric, quasinorm, or norm topology on λ, and A has no zero columns, then $sld - A^{-1}(\tau)$ is, respectively, a metric, quasinorm, or norm topology on $sld - A^{-1}(\lambda)$;

5) If ϱ is a normal (norm, quasinorm, pseudonorm) determined by τ, and A has no zero columns, then the functional $\hat{e}(x) = \varrho(A|x|)$ on $sld - A^{-1}(\lambda)$ is, respectively, a solid (norm, quasinorm, pseudonorm) and determines the topology $sld - A^{-1}(\tau)$ on $sld - A^{-1}(\lambda)$.

Proof: Since the proof is almost routine, here we give just the outline of the proof without details. The fact that $sld - A^{-1}(\tau)$ is a ζvs follows from Corollary 1.2 in (Schaeffer 1966) by describing a collection of sets in a vector space comprising a neigh. base at the origin for a ζvs on that vector space. We will not need here a ζvs to be Hausdorff._To see that $sld - A^{-1}(\tau)$ is at least as fine as the relative topology induced by $A^{-1}(\tau)$, note that the open sets in $A^{-1}(\tau)$, which is $A^{-1}(U)$ such as $U \in \mathcal{U}$, are a neigh. base at the origin in $(A^{-1}(\lambda), A^{-1}(\tau))$, and from $U \in \mathcal{U}$ leads to $sld - A^{-1}(U) \subseteq A^{-1}(U) \cap sld - A^{-1}(\lambda)$, because $Ax \in S(A|x|)$. From Theorem 3.1.8, one can have, proof of (3), directly. Note that (4) follows from (5), and by using Lemma 3.1.21, one can see that every metric ζvs is pseudonormable.

To analyse the above theorem, see the following remark.

Remark 3.1.23: From Theorem 3.1.22, one can have the following facts:

1) If A has no zero columns, and x is a non-zero sequence in $sld - A^{-1}(\lambda)$, then $A|x|$ is a non-zero element of λ;

2) If A and λ satisfy $P(1)$, then A has no zero columns;

Remark 3.1.24: Another aim after Theorem 3.1.22 is to illustrate the hereditary property from $(sld - A^{-1}(\lambda), sld - A^{-1}(\tau))$ to (λ, τ). This may not necessarily to be true.

To illustrate the hereditary property consider the following example

Example 3.1.25: Let p and q be positive numbers such that $p < 1 < q < 1 + p$. See

$\sum_{n\geqslant m} n^{-q} > \int_m^\infty x^{-q}dx = \frac{m^{1-q}}{q-1} > (m-1)^{-p}$ for sufficiently large m.

Therefore, there is a sequence $\{m_k\}$ of positive integers such that $m_{k+1} > m_k + 2$ and $m_k^{-p} \leqslant \sum_{m_k<n<m_k+1} n^{-q}$ for all k.

Let $M = \{m_k \mid k = 1,2,\dots\}$ be such that $p_n = p$ if $n \in M$, otherwise $p_n = q$ and the set $l(p_n) = \{x \in \omega; \sum_n |x_n|^{p_n} < \infty\}$ thus $l(p_n)$ is a direct sum of l_p and l_q also its topology is induce by the quasinorm ϱ defined by

$\varrho(x) = (\sum_{n\in M} |x_n|^4)^{\frac{1}{q}} + (\sum_{m\in M} |x_n|^p)^{\frac{1}{p}}$. So, $l(p_n) \subseteq l_a$. For a matrix, choose C, the Cesàro matrix. We claim that $sld - C^{-1}(l(p_n)) = cesr_q$. Clearly, $sld - C^{-1}(l(p_n)) \subseteq sld - C^{-1}(l_q) = cesr_q$.

So let $x \in \text{ces}_q$; then, $\sum_n \left(\frac{1}{n}\sum_{k=1}^n |x_k|\right)^q < \infty$, and $a_n = \sum_{k=1}^n |x_k|$, so $\{a_n\}$ is non-negative and non-decreasing. Now we must prove $\sum_n (n^{-1}a_n)^{p_n} < \infty$. It is enough to prove $\sum_{n\in M} (n^{-1}a_n)^p < \infty$. Now, by taking of $\{m_k\}$ such that $(m_k^{-1}a_m)^p \leqslant a_{mk}^p \sum_{m_k<n<m_{k+1}} n^{-q} \leqslant \sum_{m_k<n<m_k+1} n^{-q}\,\text{m}\,\left(1, a_n^q\right)$, since the a_n's are non-decreasing and $p < q$. So by using the comparison test, with comparison by $\sum_k \sum_{m_k<n<m_k+1} n^{-q}\,\text{m}\,\left(1, a_n^q\right)$, one can have that $\sum_{n\in M} (n^{-1}a_n)^p$ converges.

Now, ces_q included two topologies, one induced by l_q, and the other by $l(p_n)$ such that the first is a norm topology.

Both are metric, and, ces_q is complete with two topologies appeared, also they are comparable and $sld - C^{-1}(l(p_n))$ is locally convex, but $l(p_n)$ is not locally convex.

Example 3.1.26: Let $\lambda = l_\infty$ and A be the matrix whose all entries are 1.

So $sld - A^{-1}(l_\infty)$ is l_1, also the norm $\hat{\varrho} = \| A\,|\cdot|\,\|_\infty$ is the usual l_1 norm, $\|.\|_1$.

Now, $sld - A^{-1}(l_\infty)$ is a non-empty space with a fine topology, $P(1)$ is far from being satisfied. Indeed, A is a rank 1 map from $A^{-1}(l_\infty)$ into l_∞. Let $\langle .\,,\, e_n\rangle$ denote the $n - th$ coordinate projection on $\mathbb{F}^{\mathbb{N}}$ where $\langle x, e_n\rangle = x_n$ for all $x \in \mathbb{F}^{\mathbb{N}}$.

Proposition 3. 1.27: Assume μ is a solid subspace of $\mathbb{F}^{\mathbb{N}}$, with solid topology Γ.

Then $\langle . \ , e_n \rangle$ is continuous on (μ, Γ) for all n if and only if (μ, Γ) is Hausdorff.

Proof: Let (μ, Γ) be Hausdorff. If $x_n = 0$ for all $x \in \mu$, then $\langle , , e_n \rangle$ is the zero linear continuous functional on μ, If $x_n \neq 0$ for some $x \in \mu$ then, since μ is solid and closed under scalar multiplication, so $e_n \in \mu$. Since (μ, Γ) is Hausdorff and solid, there exists a solid neigh. U of the origin in (μ, Γ) such as $e_n \notin U$. Then $x \in U$ and one gets, $|x_n| = |\langle x, e_n \rangle| < 1$, since $|x_n| \geqslant 1$ implies $e_n \in U$, from solidness of U. Therefore; $\langle . , e_n \rangle$ is continuous on (μ, Γ). Conversely, suppose that all $\langle . , e_n \rangle$ is continuous on (μ, Γ). Then Γ is at least as fine as the relative topology of cordinatewise convergence on μ and is therefore a Hausdorff topology.

From above theorem, we get the following corollary.

Corollary 3.1.28: Let τ be a solid topology on λ such that $(\lambda, \tau)$is Hausdorff, and A has no zero columns, then the coordinate projections are continuous on $(sld - A^{-1}(\lambda)$, $sld - A^{-1}(\tau)$).

Proof: Obvious from Theorem 3.1.22 and Theorem 3.1.27.

Notation 3.1.29:

1) The mapping $P_0 \colon \mathbb{F}^{\mathbb{N}} \to \mathbb{F}^{\mathbb{N}}$ is a zero map, $P_0(x) = 0$, for all $x \in \mathbb{F}^{\mathbb{N}}$.

2) The mapping $I \colon \mathbb{F}^{\mathbb{N}} \to \mathbb{F}^{\mathbb{N}}$ is the identity map, $\mathrm{I}(x) = x$, for all $x \in \mathbb{F}^{\mathbb{N}}$.

3) The map $P_n \colon \mathbb{F}^{\mathbb{N}} \to \mathbb{F}^{\mathbb{N}}$ is defined by $P_n(x) = \sum_{k=1}^{n} \langle x, e_k \rangle e_k$, for all $x \in \mathbb{F}^{\mathbb{N}}$.

Thus, let $x \in \mathbb{F}^{\mathbb{N}}$, and any n, $P_n(x), (I - P_n)(x) \in S(x)$, if $y \in S(x)$ then $P_n(y) \in S(P_n(x))$, and $(I - P_n)(y) \in S\big((I - P_n)(x)\big)$.

Let us introduce the definition of Abschnitt-Konvergenz, sectionwise convergence topological vector space.

Deefinition 3.1.30: Let (μ, Γ) be $\zeta\nu s$ and $P_n(\mu) \subseteq \mu$ for all n, then the space (μ, Γ) is called Abschnitt-Konvergenz (or sectionwise convergence) (shortly- AK) if $(I - P_n)(x)$ converges to zero in (μ, Γ) as $n \to \infty$, for each $x \in \mu$.

Now, the following theorem proves that AK is topological property passing from (λ, τ), to ($sld - A^{-1}(\lambda), sld - A^{-1}(\tau)$). For a proof of this result, see (Johnson and Mohapatra 1979).

Theorem 3.1.31: If τ is a solid topology on λ and (λ, T) is AK, then $(sld - A^{-1}(\lambda),\ sld - A^{-1}(T))$ is AK.

Remark 3.1.32: The convers of above theorem does not need to be true in general. To illustrate that consider Example 3.1.26 such that $(sld - A^{-1}(\lambda), sld - A^{-1}(\tau))$ is AK but (λ, τ) is not.

Proposition 3.1.33:

If (μ, Γ) in a solid subspace of $\mathbb{F}^{\mathbb{N}}$, with a solid topology Γ, then the closure of a solid set is solid.

Proof: Suppose (μ, Γ) is a solid subspace of $\mathbb{F}^{\mathbb{N}}$ with a solid topology, S is a normal subset of μ, $\left(y^{(\alpha)}\right)$ is a net in S converging to y in (μ, Γ), and $x \in N(y)$. For all n, in case $y_n \neq 0, a_n = 0$, and take $a_n = x_n y_n^{-1}$, otherwise. Let $a = \{a_n\}$. Then $|a_n| \leqslant 1$ for all n, and $ay = x$, $ay^{(\alpha)} \in S\left(y^{(\alpha)}\right) \subseteq S$, for all α, and $ay^{(\alpha)} - ay = a\left(y^{(\alpha)} - y\right) \in N\left(y^{(\alpha)} - y\right)$. Since Γ is a solid topology, $\left\{ay^{(\alpha)}\right\}$ converges to x in (μ, Γ), so $x \in \bar{S}$, the closure of S.

3.2 LCC – Topological Vector Spaces

In this section, we study another type of topological vector spaces namely, locally coordinate wise closed and denoted by LCC, also hereditary properties which have been given and here we will think of μ as a subspace of linear space $\mathbb{F}^{\mathbb{N}}$, and Γ is a $\zeta\nu s$.

Definition 3.2.1: Let μ be a subspace of $\mathbb{F}^{\mathbb{N}}$ and Γ be a $\zeta\nu s$ topology on μ. Then (μ, Γ) is said to be locally coordinatewise closed (shortly-LCC) , if there is a neigh. base at the origin in (μ, Γ) consisting of closed sets in $(\mu, \mathfrak{F})$, where $\mathfrak{F}$ is topology of coordinatewise convergence on $\mathbb{F}^{\mathbb{N}}$.

It's clear that, $(l_\infty, \|\|_\infty)$ is LCC space, also subspace of an LCC space is LCC with the relative topology, more characterization of LCC are given by the following proposition.

Proposition 3.2.2: Let μ be a solid subspace of $\mathbb{F}^{\mathbb{N}}$, and Γ a solid topology on μ. If (μ, Γ) is AK, then (μ, Γ) is LCC.

Proof: Since (μ, Γ) is AK, then there exists neigh. bases at the origin in (μ, Γ) consisting of solid sets, and closed sets in (μ, Γ). By using Proposition 3.1.33, there is a neigh. base $\mathcal{U}$ at the origin in (μ, Γ) consisting of sets which are normal and closed in (μ, Γ). Now, we claim that each of these sets is closed in $(\mu, \mathfrak{F})$. Suppose $U \in \mathcal{U}$, and $x \in \mu - U$. Then $(x + V) \cap U = \emptyset$ for some $V \in \mathcal{U}$. Also suppose $W \in \mathcal{U}$ such that $W + W \subseteq V$. Since (μ, Γ) is AK, there is an integer n such that $P_n x - x \in W$. Let it be $M = \{k \mid e_k \in \mu\}$, and note that $y_k = 0$ for $k \notin M, y \in \mu$ since μ is a normal subspace of $\mathbb{F}^{\mathbb{N}}$. Since (μ, Γ) is a $\zeta\nu s$, thus $\sum_{\substack{k\in M \\ 1\leqslant k\leqslant n}} c_k e_k \in W$ for all c_k sufficiently small.

Consequently, if x is in the closure of U in $(\mu, \mathfrak{F})$, then there exists $y \in U$ such that $P_n y - P_n x \in W$, and $P_n y = P_n y - P_n x + P_n x - x + x \in W + W + x \subseteq x + V$, as well $P_n y \in N(y) \subseteq U$, since U is solid, which icontradicts $(x + V) \cap U = \emptyset$.
Our next hereditary property of LCC is given in the next theorem.

Theorem 3.2.3: Assume (λ, τ) is a LCC space. Then ($sld - A^{-1}(\lambda), sld - A^{-1}(\tau)$) is an LCC space.

Proof:. Since $\tau, \mathfrak{F}$ are solid topologies, and (λ, τ) is LCC space then by using Proposition 3.1.33, one can have that there exists a neigh. base $\mathcal{U}$ at the origin in (λ, τ) consisting of solid sets and closed in $(\lambda, \mathfrak{F})$. Now, for each $U \in \mathcal{U}$, we must prove $sld - A^{-1}(U) = \{x \in sld - A^{-1}(\lambda)\ A|x| \in U\}$ is closed in $(sld - A^{-1}(\lambda), \mathfrak{F})$. Let $U \in \mathcal{U}$ and $x \in sld - A^{-1}(\lambda) \setminus sld - A^{-1}(U)$. Then $A|x| \in \lambda - U$. Since U is closedin $(\lambda, \mathfrak{F})$, so $\lambda - U$ is open set such that there is a positive integer t, and $\varepsilon > 0$, so that $|\sum_n a_{mn}|x_n| - y_m| > 3\varepsilon$, $m = 1, \dots, t, y \in U$, also from the fact that U is solid, this implies, $\sum_n a_{mn}|x_n| > |y_m| + 3\varepsilon$, $m = 1, \dots, t$, $y \in U$. Next for some $y \in U$, and some $m \in \{1, \dots, t\}$,$0 \leqslant \sum_n a_{mn}|x_n| \leqslant |y_m| + 3\varepsilon$, then we get the sequence $z = \{z_m\}$ by replacing y_m by $max(0, \sum_n a_{mn}|x_n| - 3\varepsilon)$ in $N(y) \subseteq U$, and

$|\sum a_{mn}|x_n|-z_m| \leqslant 3\varepsilon$. Also from other, obviously there exists k satisfying $\sum_{n=1}^{k} a_{mn}|x_n| > |y_m| + 2\varepsilon$, $m = 1, \dots, t, y \in U$, and $\delta > 0$ such that $|x_n - x_n'| < \delta$, $n = 1, \dots, k$ lead to $\sum\ a_{mn}|x_n'| \geqslant \sum_{n=1}^{k}\ a_{mn}|x_n'| > |y_m| + \varepsilon,\ m = 1, \dots, t, y \in U.$ Finally the neigh. $\{x \in L||x_n - x_n'\ |< \delta, n = 1,2, \dots, k\}$ of x in $\mathfrak{F}$ has no elements of $sld-\ A^{-1}(U)$, so $sld-\ A^{-1}(U)$ is a closed set in the topological space $(sld - A^{-1}(\lambda), \mathfrak{F})$. Therefore $(\ sld - A^{-1}(\lambda),\quad sld - A^{-1}(\tau)\ )$ is an LCC space.

Remark 3.2.4: The fact appeared in Theorem 3.2.4 is not necessary to be true in the space of sequence $(A^{-1}(\lambda), A^{-1}(\tau))$, that means when (λ, τ) is an LCC space, $(A^{-1}(\lambda), A^{-1}(T))$ may not be. To illustrate this, consider the following example.

Example 3.2.5: Consider Example 3.1.26 $(A^{-1}(\lambda), A^{-1}(\tau))$ is $\{x \in \mathbb{F}^{\mathbb{N}}: \sum_{n=1}^{\infty} x_n$,converges$\}$ with respect to the seminorm defined by $\|x\| = |\sum_{n=1}^{\infty} x_n|$. The closure of each $A^{-1}(\tau)$ in $(A^{-1}(\lambda), \mathfrak{F})$ neigh. of zero is all of $A^{-1}(\lambda)$. Now, $P(2)$ or $P(3)$ is satisfied, then we cannot say that $(A^{-1}(\lambda), A^{-1}(\tau))$ is LCC whenever $(\lambda, T)$is. In addition, when the infinite matrix A has finite row then if $(\lambda, T) is\ LCC$ one can have $(A^{-1}(\lambda), A^{-1}(T))$ is LCC, only because of the mapping by A preserves coordinatewise convergence in $\mathbb{F}^{\mathbb{N}}$.

3.3 LCC – Hausdorff Topological Vector Spaces

This section is an extension of the study of previous section, where (μ, Γ) is a Hausdorff space in addition the LCC property, and some characterizations of this space have been given. Let us start with the following lemma in which we will need it in the next theorem.

Lemma 3.3.1: Let (μ, Γ) be Hausdorff and LCC space, with μ a solid subspace of $\mathbb{F}^{\mathbb{N}}$, Γ a solid topology on μ. Then if $\{x^{(\alpha)}\}$ is a Cauchy net in (μ, Γ) and convergent to 0 in $(\mu, \mathfrak{F})$, then $\{x^{(\alpha)}\}$ converges to 0 in (μ, Γ).

Proof: Directly from Proposition 3.1.27, we get the fact that (μ, Γ) is Hausdorff leads us to Γ finer than the topology induced on μ by $\mathfrak{F}$. So from Lemma 34.2 in (Treves1967), the result follows.

Theorem 3.3.2: Let (λ,τ) be a Hausdorff, complete and LCC space. If the infinite matrix A has no zero columns, then $(sld - A^{-1}(\lambda), sld - A^{-1}(\tau))$ is complete.

Proof: Since (λ,τ) is Hausdorff, then from Theorem 3.1.22 one can have $(sld - A^{-1}(\lambda), sld - A^{-1}(\tau))$ is Hausdorff and solid, also by using Theorem 3.2.3, it is LCC. In addition, applying Lemma 3.3.1**, we** get $(\mu,\Gamma) = (sld - A^{-1}(\lambda), sld - A^{-1}(\tau))$. Now, let $\{x^{(\alpha)}\}$ be a Cauchy net in $(sld - A^{-1}(\lambda), sld - A^{-1}(\tau))$. Thus for any. x, $y \in \mathbb{F}^{\mathbb{N}}, |x| - |y| \in S(x-y)$, and since $sld - A^{-1}(\tau)$ is a solid topology, consequently $\{|x^{(\alpha)}|\}$ is a Cauchy net in $(sld - A^{-1}(\lambda), sld - A^{-1}(\tau))$. Also the coordinate projections are continuous on $(sld - A^{-1}(\lambda), sld - A^{-1}(\tau))$ from Proposition 3.1.27**.** Hence $\{x^{(\alpha)}\}$ converges to some x in $(\mathbb{F}^{\mathbb{N}},\mathfrak{F})$; and consequently $\{|x^{\alpha}|\}$ converges to $|x|$ in $(\mathbb{F}^{\mathbb{N}},\mathfrak{F})$, and maps $(sld - A^{-1}(\lambda), sld - A^{-1}(\tau))$ continuously into (λ,τ), from Proposition 3.2.27 and assumption $\{A|x^{(x)}|\}$ is Cauchy in (λ,τ). Then $A|x^{(\alpha)}|$ converges to some y in (λ,T), and thus converges coordinatewise to y. Thus, $\{\sum_n a_{mn}|x_n^{(\alpha)}|\}_\alpha$ converges to y_m for all m. Since $\{|x^{(\alpha)}|\}$ converges coordinatewise to $|x|$, it follows that $\sum_n a_{ma}|x_n| \leqslant y_m$ for all m. Thus $A|x| \in S(y) \subseteq \lambda$, so $x \in sld - A^{-1}(\lambda)$. Take $y^{(\alpha)} = x^{(x)} - x$. Then $\{y^{(\alpha)}\}$ is a Cauchy net in $sld - A^{-1}(\lambda)$, with respect to $sld - A^{-1}(\tau)$, and convergent to zero coordinatewise; by Lemma (3.4.1), $\{y^{(\alpha)}\}$ converges to zero in $(sld - A^{-1}(\lambda), sld - A^{-1}(\tau))$. That means that $\{x^{(\alpha)}\}$ converges to x.

Remark 3.3.3:

1) The condition of Hausdorffness appeared in Lemma 3.3.1 and the condition of the infinite matrix A having no zero columns appeared in Theorem 3.3.2, we can dispense with them and modify the details of proof.

2) The conditions λ and T solid and (λ,T) Hausdorff do not guarantee that (λ,T) is an LCC space.

From above theorem one can have the following corollary.

Corollaries (3.3.4):

1) If $(\lambda, \tau) = (l_\infty, \|.\|_\infty)$ and the infinite matrix A has no zero columns, then $(sld - A^{-1}(\lambda), sld - A^{-1}(\tau))$ is complete.

2) If (λ, τ) is AK, Hausdorff and complete and the infinite matrix A has no zero columns, then $(sld - A^{-1}(\lambda), sld - A^{-1}(\tau))$ is complete.

Now, the following example shows that the conditions of completeness in Theorem 3.3.2 is not sufficient.

Example 3.3.5: Let $(\lambda, \tau) = (l_\infty, \mathfrak{F})$, and A be the matrix whose all entry is 1. Then $(sld - A^{-1}(\lambda),\ sld - A^{-1}(\tau))$ is complete although (λ, τ) is not complete, where, in this case, $sld - A^{-1}(\lambda) = l_1$ and $sld - A^{-1}(\tau)$ is the topology generated by the l_1 norm.

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