Republic of Türkiye
Harran University
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

DOCTORAL THESIS

SOLUTIONS OF FRACTIONAL ORDER LINEAR AND NONLINEAR PSEUDO-HYPERBOLIC TELEGRAPH PARTIAL DIFFERENTIAL EQUATIONS

SADEQ TAHA ABDULAZEEZ

MATHEMATICS

Şanlıurfa
2024

Republic of Türkiye
Harran University
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

DOCTORAL THESIS

SOLUTIONS OF FRACTIONAL ORDER LINEAR AND NONLINEAR PSEUDO-HYPERBOLIC TELEGRAPH PARTIAL DIFFERENTIAL EQUATIONS

SADEQ TAHA ABDULAZEEZ

DEPARTMENT OF MATHEMATICS

Thesis Supervisor: Assoc. Prof. Dr. MAHMUT MODANLI

Şanlıurfa
2024

TABLE OF CONTENTS

ÖZET

Doktora Tezi

KESIRLI MERTEBEDEN LINEER VE LINEER OLMAYAN PSEUDO-HIPERBOLIK TELEGRAF KISMI DIFERANSIYEL DENKLEMLERIN ÇÖZÜM METOTLARI

SADEQ TAHA ABDULAZEEZ

Harran Üniversitesi
Fen Bilimleri Enstitüsü
Matematik Bölümü

Danışman: Doç. Dr. Mahmut MODANLI
Yıl: 2024, Sayfa: 92

Lineer ve lineer olmayan kesirli mertebeden pseudo hiperbolik telgraf kısmi diferansiyel denklemleri mühendislik, fizik ve finans gibi çeşitli alanlarda önemli uygulamalara sahiptir. Ancak bu tür denklemlerin analitik çözümlerini elde etmek zordur. Bu tezde, Caputo kesirli türevleriyle tanımlanan lineer ve lineer olmayan kesirli mertebeden pseudo hiperbolik telgraf kısmi diferansiyel denklemlerini çözmek için doğru ve etkili sayısal teknikleri araştırır. Başlangıçta, lineer kesirli mertebeden pseudo hiperbolik telgraf kısmi diferansiyel denklemleri için analitik çözümler elde etmek amacıyla değiştirilmiş bir çift Laplace dönüşümü yöntemi geliştirildi. Yöntemin basitliği ve etkinliği bir örnek problem gösterimi yoluyla testedilmiştır. Buna ek olarak, bu çalışma abstract formun lineer olmayan kesirli mertebeden pseudo hiperbolik telgraf kısmi diferansiyel denklemlerine iyi tanımlı çözümlerin belirlenmesi için kapsamlı bir metodoloji sunmaktadır. Bu çözümlerin kararlılığı da araştırılmaktadır. Soyut tekniğin birçok lineer olmama türünde geniş uygulanabilirliğe sahip olduğu gösterilmiştir. Ayrıca, lineer kesirli mertebeden pseudo hiperbolik telgraf kısmi diferansiyel denklemlerinin çözümü için açık sonlu fark yaklaşımı adı verilen özel bir sayısal teknik formüle edilmiş ve incelenmiştir. Kararlılık tahminleri hesaplanır ve yöntem, sayısal ve tam çözümler arasında yüksek düzeyde bir uyum olduğunu gösteren örnek bir problem üzerinde doğrulanır. Ayrıca, lineer olmayan kesirli mertebeden pseudo hiperbolik telgraf kısmi diferansiyel denklemlerini çözmek için açık sonlu fark tekniğinin özel bir uygulamasından yararlanılır. Bu fark şemanın uygulanması MATLAB kullanılarak gerçekleştirilir. Sayısal örnekler, farklı kesirli mertebeli için kararlılık ve yakınsaklığı gösterir ve dalga davranışını ve hafıza etkilerini etkili bir şekilde yansıtır. Özetle, bu tez, lineer ve lineer olmayan kesirli mertebeden pseudo hiperbolik telgraf kısmi diferansiyel denklemlerinin çözümü için yeni sayısal ve analitik yaklaşımlar sunmaktadır. Hem sonlu fark hem de değiştirilmiş çift Laplace dönüşümü yöntemleri geliştirildi ve bunların doğru, kararlı ve hesaplama açısından verimli olduğu gösterildi. Bu teknikler, bilim ve mühendislik disiplinleri genelinde kesirli dinamikler ve hafıza etkileri sergileyen karmaşık sistemlerin daha iyi modellenmesine ve anlaşılmasına olanak sağlayacaktır.

ANAHTAR KELİMELER: Lineer ve lineer olmayan kesirli mertebeden pseudo hiperbolik telgraf denklemleri, Sonlu farklar yöntemi, Değiştirilmiş çift Laplace dönüşümü yöntemi, Caputo kesirli türevi, Analitik çözüm, Sayısal çözüm, Kbararlılık kestirimleri.

ABSTRACT

PhD Thesis

SOLUTIONS OF FRACTIONAL ORDER LINEAR AND NONLINEAR PSEUDO-HYPERBOLIC TELEGRAPH PARTIAL DIFFERENTIAL EQUATIONS

SADEQ TAHA ABDULAZEEZ

Harran University
Graduate School of Natural and Applied Sciences
Department of Mathematics

Supervisor: Assoc. Prof. Dr. Mahmut MODANLI
Year: 2024, Page: 92

Linear and nonlinear fractional-order pseudo-hyperbolic telegraph partial differential equations have important applications in diverse fields including engineering, physics, and finance. However, analytical solutions to such equations are challenging to obtain. This thesis investigates accurate and efficient numerical techniques for solving linear and nonlinear fractional-order pseudo-hyperbolic telegraph partial differential equations defined by Caputo fractional derivatives.

Initially, a modified double Laplace transform method is developed to get analytical solutions for linear fractional-order pseudo-hyperbolic telegraph partial differential equations. The method's simplicity and effectiveness are exemplified through a problem demonstration. In addition, this study presents a comprehensive methodology for identifying mild solutions to abstract form of nonlinear fractional-order pseudo-hyperbolic telegraph partial differential equations. The stability of these solutions is also investigated. The abstract technique is demonstrated to have broad applicability across many types of nonlinearity. Furthermore, a specific numerical technique called explicit finite difference approach is formulated and examined for solving linear fractional-order pseudo-hyperbolic telegraph partial differential equations. Stability estimates are calculated and the method is validated on an example problem, showing a high level of agreement between the numerical and precise solutions. Furthermore, a special application of an explicit finite difference technique is utilised to solve nonlinear fractional-order pseudo-hyperbolic telegraph partial differential equations. The implementation of this scheme is carried out using MATLAB. Numerical experiments demonstrate stability and convergence for different fractional orders, and effectively reflect the wave behaviour and memory effects.

In summary, this thesis presents novel numerical and analytical approaches for solving linear and nonlinear fractional-order pseudo-hyperbolic telegraph partial differential equations. Both finite difference and modified double Laplace transform methods are developed and shown to be accurate, stable, and computationally efficient. These techniques will enable improved modeling and understanding of complex systems exhibiting fractional dynamics and memory effects across science and engineering disciplines.

KEYWORDS: Linear and nonlinear pseudo hyperbolic telegraph equations, Finite difference method, Modified double Laplace transform method, Caputo fractional derivative, Analytical solution, Numerical solution, Stability.

ACKNOWLEDGEMENT

In the name of God, the Most Gracious, the Most Merciful. All praise is due to God who granted me the strength, inspiration, and prudence to complete this PhD thesis. I am deeply grateful to my thesis supervisor, Assoc. Prof. Dr. Mahmut MODANLI, for his guidance throughout this process. I wish to express my heartfelt thanks to my parents, and family for their endless love, support and encouragement. Their patience, understanding, and prayers have carried me through the challenges of pursuing a PhD. I am also thankful for the cooperation of all my lecturers in the Mathematics Department at Harran University. Finally, I sincerely appreciate all of my friends and colleagues who have walked alongside me on this journey. Thank you all.

LIST OF FIGURES

Page No

LIST OF TABLES

Page No

LIST OF SYMBOLS and ABBREVIATIONS

α	Alfa
β	Beta
$\Gamma(.)$	Gamma function
H	Hilbert space
ε	Absolute error
ζ	Zeta
η	Eta
φ	Phi
ψ	Psi
${}_0^C D_t^\alpha w(t,x)$	Caputo fractional derivative
$D^\alpha f(t)$	Riemann-Liouville fractional derivative
$\mathcal{L}$	Laplace transform
$\mathcal{L}_x$	Laplace transform for variable x
$\mathcal{L}_t$	Laplace transform for variable t
$\mathcal{L}^{-1}$	Inverse Laplace transform
PDEs	Partial Differential Equations.
FPHTPDEs	Fractional-Order Pseudo-Hyperbolic Telegraph Partial Differential Equations
DLT	Double Laplace Transform
C-N-G-FE	Crank–Nicolson–Galerkin finite element
A-B	Atangana-Baleanu derivative
VIM	Variational Iteration Method
HPM	Homotopy Perturbation Method
LVIM	Laplace Variational Iteration Method
LADM	Laplace Adominan Decomposition Method
NIM	New Iterative Approach
LBB	Ladyzhenskaya-Babuka-Brezi
SPDMFE	Splitting positive definite mixed finite element
RPSM	Residual power series method
MHB-DQM	Modified Cubic B-Spline Differential Quadrature Method
FPDEs	Fractional Partial Differential Equations

1. INTRODUCTION

1.1. Introduction

Fractional differential equations research has grown in importance during the last several decades. The exploration of fractional calculus originated in the 19th century, nevertheless, its utilization in the context of differential equations has experienced a significant increase in popularity during the 1990s (Miller and Ross, 1993). Fractional differential equations extend conventional differential equations by allowing the order of the derivative to take fractional values rather than solely integer orders. Fractional calculus, a branch of mathematics that focuses on non-integer differentiation and integration, provides a highly useful technique for modeling complex systems exhibiting memory, hereditary characteristics, and long-range interdependence. Fractional derivatives enable more precise modeling of physical processes by considering anomalous dynamics and diffusion phenomena neglected in traditional integer-order models. Fractional differential equations have become valuable tools with broad applicability across diverse scientific and engineering disciplines including physics, biology, engineering, and finance (Diethelm, 2010). Key research priorities include developing fractional modeling approaches for real-world processes such as viscoelastic materials, control theory, and signal processing. Ongoing theoretical advances in analytical and numerical techniques for solving fractional differential equations across various applications suggest this field will continue to expand as researchers discover new uses for fractional models in representing complex dynamical systems.

The fractional-order pseudo-hyperbolic telegraph partial differential equations (FPHTPDEs) are of particular importance. This class of models extends the classical telegraph equation by incorporating fractional derivatives, introducing non-local dynamics. The resulting equation balances between hyperbolic and parabolic behavior, allowing to capture both wave-like dispersive effects as well as diffusive characteristics. Furthermore, the inclusion of nonlinear terms adds another layer of complexity, enabling the description of highly complex phenomena exhibiting memory and nonlinearity.

The FPHTPDEs have garnered substantial attention due to their demonstrated ability to realistically describe processes in electromagnetics, porous media flow, viscoelastic materials, and many other domains. However, analytical solutions to such equations remain challenging to obtain in most cases. Therefore, developing accurate and efficient numerical methods for solving linear and nonlinear FPHTPDEs is of great significance. Advancement of techniques for analyzing FPHTPDEs will lead to enhanced modeling capabilities and new insights into complex fractional-order processes with far-reaching impacts across science and technology.

The purpose of this thesis is to explore innovative analytical and numerical methods for solving FPHTPDEs, both linear and nonlinear, that are formulated using Caputo fractional derivatives. Both finite difference schemes and integral transform methods will be constructed and analyzed. The resulting techniques will provide the means to accurately and efficiently solve a wide class of fractional-order models to better understand the intricate dynamics exhibited by systems with memory and hereditary features.

1.2. Preliminaries

This subsection introduces fundamental concepts and definitions pertaining to fractional derivatives and the technique of double Laplace transform. This establishes the fundamental basis necessary to utilize this approach for resolving the models that emerge in subsequent portions of this thesis. In addition, we explore the characteristics and prerequisites necessary for the effectiveness of this method.

Definition 1.2.1. "*In (Podlubny, 1998), the gamma function is defined as*

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\, dt, \quad Re(z) > 0\text{''}. \tag{1.1}$$

Definition 1.2.2. "*According to (Oldham and Spanier, 1974), the Riemann-Liouville fractional derivative of order* $\alpha > 0$, *for the functions* $f(t)$ *is expressed as*

$$D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-\tau)^{(n-\alpha-1)} f(\tau) d\tau \, , (n-1 < \alpha \leq n) \quad (1.2)$$

where $n \geq \alpha$, *and* $\Gamma(.)$ *is Gamma function*".

Definition 1.2.3. "*The Caputo fractional derivative* ${}_0^C D_t^{\alpha} f(t)$ *is an extension of the conventional integer-order derivative to fractional orders. It is explicitly defined (Podlubny, 1998) as:*

$${}_0^C D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(p)}{(t-p)^{\alpha-n+1}} dp, (n-1 < \alpha \leq n) \quad (1.3)$$

where, ${}_0^C D_t^{\alpha}$ *represents the Caputo fractional derivative of* $\alpha -$ *order,* $f(t)$ *represents the function that has to be differentiated,* $\Gamma(.)$ *represents the Gamma function, and* n *represents the lowest integer that is higher than or equal to* α*".*

Caputo fractional derivative properties include the following (Podlubny, 1999):

1. Linearity: The Caputo fractional derivative meets the following property because it is a linear operator:

   $${}_0^C D_t^{\alpha} (Af(t) + Bg(t)) = A {}_0^C D_t^{\alpha} f(t) + B {}_0^C D_t^{\alpha} g(t),$$

   where A and B are constants.

2. Product Rule: The following product rule is satisfied by the Caputo fractional derivative:

   $${}_0^C D_t^{\alpha} (f(t)g(t)) = f(t) {}_0^C D_t^{\alpha} g(t) + g(t) {}_0^C D_t^{\alpha} f(t),$$

which is equivalent to the classical derivatives' product rule.

3. Additionally, it complies with the following chain rule:

$$ {}_0^C D_t^\alpha \big(f(g(t)\big) = \frac{d^n g(t)}{dt} {}_0^C D_t^\alpha f(g(t)), $$

its comparable to the conventional derivatives chain rule.

4. The index law is satisfied by the Caputo fractional derivative,

$$ {}_0^C D_t^\alpha \left({}_0^C D_t^\beta f(t) \right) = {}_0^C D_t^{\alpha+\beta} f(t). $$

5. The derivative of a constant, as defined by Caputo, is equal to zero, ${}_0^C D_t^\beta C = 0$.

6. It is possible to establish a direct link between the Laplace transform and the Caputo fractional derivative equation. The Caputo fractional derivative's Laplace transform, in particular, for a function $f(t)$, is provided by:

$$ \mathcal{L}\left\{\frac{\partial^\alpha}{\partial t^\alpha} f(t)\right\} = s^\alpha F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(a), $$

where $n-1 < \alpha < n$, and $F(s)$ is the Laplace transform of $f(t)$. When employing Laplace transform methods to solve fractional differential equations, this relationship is helpful.

The fractional derivatives defined by Riemann-Liouville and Caputo are widely recognized techniques for defining fractional derivatives. Both of these techniques are referred to as "fractional derivatives." Every one of them comes with its own set of benefits and drawbacks. Both of them assumed different approaches to dealing with the early situations, which resulted in considerable differences. To define the Riemann-Liouville fractional derivative, a form integral is utilized, as was demonstrated in (1.2).

In contrast, the definition of the Caputo fractional derivative involves an integral that involves the function's higher-order derivatives. The fact that the Caputo fractional derivative includes the beginning conditions in its definition, whereas the Riemann-Liouville fractional derivative does not, makes it an excellent tool for addressing initial value problems. This is a significant outcome of this difference. Because the Caputo fractional derivative naturally accounts for the system's initial state, it is especially helpful for simulating physical systems with fractional order dynamics.

The behavior of constant functions is the subject of a further significant difference. According to (Kilbas et al., 2006), the Riemann-Liouville fractional derivative of a constant function does not necessarily have to be zero all the time, in contrast to the Caputo fractional derivative of a constant function, which is always zero. Applying these derivatives in different mathematical and practical contexts can have important ramifications because of this disparity.

Both varieties of fractional derivatives are employed in various branches of applied mathematics, engineering and physics, and each has advantages of its own. Depending on the particular characteristics and behaviors of the system under study as well as the mathematical methods and instruments being used, there are a few different sorts of fractional derivatives that one can choose to use, including the Riemann-Liouville, Caputo, and other varieties.

Definition 1.2.4. "*In the positive quadrant of the xt-plane, define the function $u(t,x)$ as a function of two variables. In (Dhunde and Waghmare, 2016), the double Laplace transform for the function $u(t,x)$ is specified as follows:*

$$L_x L_t\{u(t,x)\} = \bar{u}(s,p) = \int_0^\infty \int_0^\infty e^{-px-st}\, u(t,x)dxdt, \tag{1.4}$$

whenever the integral is present the numbers p and s are both complex numbers".

We can deduce the following from this definition:

$$L_x L_t\{u(x)g(t)\} = \bar{u}(p)\bar{g}(s) = L_x\{u(x)\}L_t\{g(t)\} \tag{1.5}$$

Definition 1.2.5. (Debnath, 2016) **"***The complex double integral formula is used to define the inverse double Laplace transform $L_x^{-1}L_t^{-1}\{\bar{u}(s,p)\} = u(t,x)$, as follows:*

$$L_x^{-1}L_t^{-1}\{\bar{u}(s,p)\} = u(t,x) = \frac{1}{2i\pi}\left[\int_{c-i\infty}^{c+i\infty} e^{px}dp \int_{d-i\infty}^{d+i\infty} e^{st}\,\bar{u}(s,p)ds\right], \tag{1.6}$$

where c and d are appropriate real constants, and $\bar{u}(s,p)$ must be an analytical function for each p and s in the area denoted by the inequalities $Re(p) \geq c$ and $Re(s) \geq d$".

The double Laplace transform formulas in (Modanli, and Bajjah, 2021), is used to transform the partial derivatives of any integer order.

$$L_x L_t\left\{\frac{\partial^n u(t,x)}{\partial x^n}\right\} = p^n\bar{u}(s,p) - \sum_{i=0}^{n-1} p^{n-1-i} L_t\left\{\frac{\partial^i u(t,0)}{\partial x^i}\right\},$$

$$L_x L_t\left\{\frac{\partial^k u(t,x)}{\partial t^k}\right\} = s^k\bar{u}(s,p) - \sum_{j=0}^{k-1} s^{n-1-i} L_x\left\{\frac{\partial^j u(0,x)}{\partial x^j}\right\},$$

$$L_x L_t\left\{\frac{\partial^{k+n} u(t,x)}{\partial x^n \partial t^k}\right\} = p^n s^k\left[\bar{u}(s,p) - \sum_{i=0}^{n-1} p^{n-1-i} L_t\left\{\frac{\partial^i u(t,0)}{\partial x^i}\right\}\right.$$
$$\left. - \sum_{j=0}^{k-1} s^{n-1-j} L_x\left\{\frac{\partial^j u(0,x)}{\partial x^j}\right\} + \sum_{i=0}^{n-1}\sum_{j=0}^{k-1} p^{-1-i}s^{-1-j} L_x\left\{\frac{\partial^{i+j} u(0,0)}{\partial x^{i+j}}\right\}\right].$$

Definition 1.2.6. **“***According to (Modanli, and Bajjah, 2021), the double Laplace transform formula for Caputo fractional derivative (1.3) is defined as*

$$L_x L_t \left\{ \frac{\partial^\alpha u(t,x)}{\partial t^\alpha} \right\} = s^\alpha \bar{u}(s,p) - \sum_{k=0}^{n-1} s^{\alpha-1-k} \, L_x \left\{ \frac{\partial^k u(0,x)}{\partial t^k} \right\} \text{”}. \tag{1.7}$$

Definition 1.2.7. **“***(Yildirim, 2007) Defines an inner product space as a complex linear space H along with a complex-valued function $\langle .,. \rangle : H \times H \to C$, called the inner product that satisfies certain properties:*

a) $\langle x,x \rangle \geq 0, \langle x,x \rangle = 0.$
b) $\langle x,y \rangle = \overline{\langle x,y \rangle}, \forall x,y \in H.$
c) $\langle \mu x,y \rangle = \mu \langle x,y \rangle, \forall x,y \in H$ *and* $\mu \in C.$
d) $\langle x+y,z \rangle = \langle x,z \rangle + \langle y,z \rangle, \forall x,y,z \in H.$

A Hilbert space H is defined as a complete inner product space. The inner product induces a norm given by $\|x\| = \langle x,y \rangle^{\frac{1}{2}}$. Therefore, inner product spaces contain a norm and Hilbert spaces are a type of Banach space”.

Definition 1.2.8. **“***(Yildirim, 2007) A bounded linear operator $A: H \times H$ on a Hilbert space H is said to be self-adjoint if $\langle Ax,y \rangle = \langle x,Ay \rangle, \forall x,y \in H$”.*

Definition 1.2.9. A self-adjoint operator A is said to be positive if $A \geq 0$, that is $\langle Ax,y \rangle \geq 0, \forall x \in H$.

Example 1.2.1. **“***(Yildirim, 2007) The space $C_2[-1,1]$ [of all defined and continuous functions on a given closed interval $[-1,1]$ is an inner product space with the inner product given by*

$$\langle x,y \rangle = \int_{-1}^{1} x(t)\, \overline{y(t)} dt.$$

The space $L_2[-1,1] = \overline{C_2[-1,1]}$ *with the inner product*

$$\langle x, y \rangle = \int_{-1}^{1} x(t)\,\overline{y(t)}dt,$$

is a Hilbert space".

1.3. Motivation

- Fractional calculus and fractional differential equations offer effective methods for analysing intricate systems characterised by fractional dynamics and memory influences.
- Pseudo-hyperbolic telegraph equations, which are a particular form of fractional-order partial differential equation, have significant relevance in modelling diverse phenomena such as heat conduction, signal transmission, population dynamics, and option pricing.
- Obtaining analytical solutions for fractional pseudo-hyperbolic telegraph equations is difficult, requiring the development of accurate and efficient numerical approaches.
- The primary goal is to formulate a finite difference approach to solve both linear and nonlinear fractional pseudo-hyperbolic telegraph equations. These methods provide a very effective way to obtain numerical approximations.
- The stability estimates are given for both linear and nonlinear models.
- Additionally, analytical methods based on Laplace transforms have been developed to offer closed-form solutions. These methods complement numerical approaches and eliminate the requirement for extensive symbolic manipulations.
- The proposed techniques enhance the modelling and comprehension of intricate fractional-order systems, with potential applications in domains such as physics, engineering, biology, and finance, where fractional dynamics and memory effects perform a significant part.

The thesis is structured as follows:

This thesis thoroughly examines precise and effective numerical methods for solving linear and nonlinear FPHTPDEs with fractional-order Caputo derivatives. The next section continues with a literature review, which offers background information and underscores the significance of the study problem. The materials and methods section presents the main techniques employed in the thesis, such as the modified double Laplace transform method, finite difference method, and the derivation of exact solutions and stability estimates for the abstract form of nonlinear FPHTPDEs. The results and discussion section illustrate the use of these techniques to address linear and nonlinear FPHTPDEs. More precisely, it includes the application of the modified double Laplace Transform Method and the explicit finite difference method to solve linear equations, along with the explicit finite difference approach for solving nonlinear equations. The suggested techniques are validated using numerical experiments and analyses to ensure their accuracy, stability, and convergence. These techniques effectively capture the wave behaviour and memory effects associated with the complex equations. Ultimately, the thesis finishes by briefly summarising the significant discoveries and contributions, and providing suggestions for future study avenues in this domain.

2. LITERATURE REVIEW

Fractional calculus has emerged as a powerful mathematical tool for modeling complex physical phenomena in recent decades. The fractional differential and integral operators provide a more accurate description of systems with non-local and memory effects compared to classical integer-order models (Podlubny, 1998). Numerous studies have demonstrated the benefits of fractional calculus across diverse fields such as physics, biology, engineering, finance, and medicine (Baleanu et al., 2021; Delic et al., 2020; Entezari et al., 2019; Fanwei, 2013; Fedotov et al., 2016; Jajarmi et al., 2022; Kumar & Nisar, 2022; Liu & Hou, 2017; Nisar et al., 2021; Owolabi et al., 2020; Özkan & Kurt, 2020; Peter et al., 2021; Singh et al., 2014; Zada et al., 2020).

Through the application of both classical and fractional calculus approaches, (Baleanu et al. 2021) conducted an investigation of the dynamics of a linear triatomic molecule. The equations of motion were obtained classically using a Lagrangian formulation. Fractional Hamilton equations were also created using the Caputo derivative. A numerical method that is based on the Euler convolution quadrature was developed in order to come up with an accurate solution for the models. Numerical simulations with different fractional orders indicated that fractional calculus allows more flexible modeling to demonstrate novel aspects of real-world processes.

The authors (Delic et al. 2020) an examination of an extended time-fractional telegraph equation was carried out. A category of equations of this kind was shown to have a weak solution, which they showed. A finite difference approach was designed and demonstrated to possess stability in approximating the problem. The study calculated the pace at which convergence occurs in a discrete energetic Sobolev norm. Theoretical results were shown using numerical examples.

(Entezari et al., 2019) proposed a set of normalized Bernstein wavelets and constructed fractional integration and operational matrices for these wavelets. These methods were utilized to solve fractional PDEs. The use of matrices operational in conjunction with collocations methods transformed the fractional issues into algebraic equations. The proposed approach automatically took into account the boundary conditions. The analysis of function approximation error was conducted using normalized Bernstein wavelets. The method's adaptability and application were proved using numerical examples. The numerical results were cross-referenced with existing literature and analytical solutions to validate their accuracy.

Using universal fractional operators, (Jajarmi et al., 2022) provided a novel fractional model for the dynamics of immunogenic tumors. They looked at the model's stability and equilibrium points. There was an introduction of a revised predictor-corrector method for numerical implementation. Experimental data was contrasted with model projections. Capturing the dynamics of complicated biological systems became easier with the new fractional model. To slow the expansion of tumor cells, researchers devised a tracking control mechanism. In many settings, the simulations proved that tumor-free stable states could be successfully tracked. In sum, the research demonstrated a viable method for controlling immunogenic tumors using a generalized fractional modeling methodology.

In their work, (Kumar and Nisar, 2022) presented a linearized fractional (C-N-G-FE) approach for the purpose of solving a nonlinear coupled delay subdiffusion problem. The system exhibited second orders accuracy and circumvented the need for iterative procedures. The analysis focused on the examination of the discrete solution's existence, uniqueness, and a priori bounds. Convergence estimates were determined in the $L_\infty(L_2)$ norm. The scheme's accuracy and efficiency were the results of computations carried out in both one and two dimensions are displayed. In summary, the work introduced a new method using finite element analysis to rigorously examine a significant group of nonlinear fractional models.

In their publication, (Liu and Hou, 2017) developed an implicit finite difference method for the purpose of solving fractional advection-dispersion equations that incorporate boundary conditions with fractional derivatives. Among the characteristics that they displayed were first-order reliability, unconditional stability, solving capacity, and first-order convergence. An efficient iterative algorithm was proposed, which significantly decreased both the storage and computing expenses in comparison to Gaussian elimination. The accuracy and efficiency of the suggested implicit framework were confirmed by numerical examples. In summary, the study presented a successful method for a significant category of fractional differential equations.

The researchers (Nisar et al., 2021) formulated a fractional-order SIRD model for COVID-19 by employing Caputo derivatives. The fundamental reproductive number was calculated and a stability analysis was performed. The utilisation of fixed point theory demonstrated both the existence of solutions and their singularity, so establishing their uniqueness. The numerical approximation employed the fractional Adams-Bashforth approach. The numerical findings demonstrated the patterns of disease transmission and corresponded with the documented occurrences of infection and mortality in Wuhan throughout a span of 67 days. In summary, the work introduced a fractional modeling method for COVID-19 that was thoroughly analyzed and showed promising levels of accuracy.

The authors (Owolabi et al., 2020) examined the utilization of fractal and fractional derivatives for the purpose of modeling a system of PDEs. The work precisely formulates the fractal fractional Schnakenberg system by employing various operators, including the Caputo operator, Caputo-Fabrizio operator, and A-B fractional derivative. Several computational techniques are proposed as potential approaches to the development of algorithms for the Schnakenberg model. The study emphasizes that classical examples are obtained when $\alpha = 1$ and $\beta = 1$. In addition, numerical data are provided for various fractal orders ($\beta \in (0,1)$) and fractional orders ($\alpha \in (0,1)$) to address any questions or concerns that may arise.

According to (Singh et al., 2014), the numerical method that makes use of the homotopy perturbation Sumudu transform methodology is proficient in the process in order to resolve nonlinear fractional PDEs associated with the movement of biological populations across space in animals. In contrast to the Sumudu decomposition method, their findings indicate that this strategy is not only straightforward to use, but also accurate and computer-friendly.

Within the framework of the new iterative approach (NIM), Researchers (Zada et al., 2020) looked on PDEs of fractional order. Using the suggested approach, fractional-order inhomogeneous PDEs and fractional-order Roseau-Hyman equations are analysed. This is accomplished without rearranging the issues in an integer fashion. In addition, there are other approaches like VIM, HPM, LVIM, and LADM, when applied to this problem, have been compared to the findings achieved by the NIM, which demonstrate a better level of accuracy. The accuracy of the method that has been proposed gets better as more iterations are performed.

The pseudo-hyperbolic equation is an important model that finds application in areas like vibration theory, thermoelasticity, nerve conduction, and plasma physics (Aliev and Lichaei, 2010; Fedotov et al., 2016; Guo and Rui, 2007). Several mathematical aspects of this high-order mixed derivative equation have been analyzed, such as existence, uniqueness, stability, and analytical solutions (Kirane and Ragoub, 2015; Liu et al., 2012; Mesloub et al., 2019; Modanli et al., 2021; Zhao and Li, 2019).

The Cauchy problem with higher-order pseudohyperbolic equations was investigated by (Aliev and Lichaei, 2010). To ascertain the presence and non-existence conditions for global solutions, they employed $L_p \to L_q$ estimates for the linear issue that was connected with it. In addition, the authors investigated the circumstances that required the existence of one-of-a-kind smooth global solutions. In addition to this, they examined the response of the solutions as well as the derivatives of those solutions as the independent variable became closer and closer to positive infinity.

Reviewing the progress of mathematical models for longitudinal vibration of bars (Fedotov et al., 2016) provided valuable insights. After starting with classical wave equations, the models expanded to include lateral motion, shear stress, and higher-order derivatives, among other broad ideas. Researchers such as Researchers such as Bishop, Rayleigh, Herrmann, and Mindlin were acknowledged for their contributions to the models' formulation and generalization. The study looked at using generic hyperbolic equation theory to sort problems into groups and judge how well they can be solved. It also looked at how to make higher-order differential equations and how to add lateral displacement. It summarized how vibration models have evolved and how they have been mathematically generalized.

To numerically solve pseudohyperbolic equations, (Guo and Rui, 2007) developed two different finite element mathematical approaches. A least-squares Galerkin formulation is utilized in these approaches, which does not necessitate the fulfilment of the (LBB) condition. The convergence analysis demonstrates that the approaches generate L^2 solutions that are extraordinarily precise. As far as time accuracy is concerned, one technique is first-order, whereas the other is second-order. The LBB-free finite element approaches for pseudohyperbolic equations have been shown to be effective and accurate through the use of numerical experiments.

(Kirane and Ragoub, 2015) provided sufficient conditions in the setting of the nonlinear pseudo-hyperbolic equation, when solutions do not exist in context of the Kohn-Laplace operator on the Heisenberg group. There were no solutions found to the equation. In addition to that, this finding is extended to the situation of a system of the same kind that is $2x2$. In the weak formulation of the answers that are being sought, the proving technique is centered on making careful choices of test functions.

(Liu et al., 2012) presented the SPDMFE techniques for solving second-order pseudo-hyperbolic equation computationally. The authors created two precise SPDMFE schemes by dividing the equation according to physical properties and conducted error analysis on both the semi-discrete and completely discrete formulations. They also demonstrated the presence and individuality yield of the semidiscrete solutions. The main contributions included presenting the SPDMFE approach, assessing correctness using error estimates, and establishing theoretical existence and uniqueness for pseudo-hyperbolic models inside the SPDMFE framework.

(Mesloub et al., 2019) investigated a one-dimensional pseudohyperbolic equations problem with non-local initial and boundary conditions. They proved that a unique strong solution exists for this problem. The authors also developed numerical methods using finite differences to calculate approximate solutions. Their main contributions were theoretically establishing existence and uniqueness of solutions for this non-local pseudohyperbolic model, and creating a finite difference scheme to numerically approximate the solutions.

(Modanli et al. 2021) used the residual power series method (RPSM) to solve pseudohyperbolic partial differential equations subject to non-local conditions. The RPSM was applied to find solutions to these types of pseudohyperbolic PDEs problems with non-local constraints. The proposed technique utilizes a Taylor series formulation that includes a residual error term. The authors developed a novel analytical solution utilizing RPSM to estimate the solutions. To showcase the precision, dependability, and swift convergence of the proposed strategy, they contrasted the RPSM solutions with the precise answers. The results were graphically displayed at various time points to visually represent the solutions derived from the RPSM study. The main contributions included the application of RPSM to pseudohyperbolic equations with nonlocal conditions, the derivation of an analytical solution, validation of its accuracy through comparisons, and the provision of graphical representations of the solutions at different time points.

(Zhao and Li, 2019) introduced a space-time continuous Galerkin approach for solving 2D pseudo-hyperbolic equation with variable coefficients. This method employs finite element discretization in time and space, enabling adaptive algorithms that can adjust time steps and spatial mesh. The authors demonstrated the presence and uniqueness of the approximate solutions without requiring grid requirements. They also calculated a priori error estimate. The scheme's success was confirmed through numerical findings. The primary achievements included the creation of the space-time Galerkin technique, the establishment of theoretical results regarding existence, uniqueness, and error estimates, and the demonstration of its accuracy in solving 2D pseudo-hyperbolic equations with variable coefficients.

Efficient numerical techniques are crucial for solving the fractional pseudo-hyperbolic models arising in real-world problems. Some common methods include Dufort-Frankel, finite difference, finite element, differential transform, theta approach , and decomposition methods (Modanli, 2022; Akgül and Modanli, 2019; Atangana et al., 2020; Hashmi et al., 2022; Khan et al., 2020; Kumar et al., 2019; Modanli, 2018; Özbag and Modanli, 2021; Modanli and Akgül, 2017; Modanli et al., 2022). Stability, convergence, and consistency are important considerations while designing accurate finite difference schemes for such equations.

(Modanli, 2022) examined numerical solutions for hyperbolic telegraph partial differential equations utilizing two distinct fractional derivative methodologies - the Caputo and the Aa-B derivatives. The solutions were derived using the Dufort-Frankel difference scheme approach. The study emphasizes the need of investigating various fractional orders by comparing the answers obtained from two forms of fractional derivatives. The study demonstrated that the Dufort-Frankel method was successful in producing accurate approximate solutions for problems incorporating Caputo and $(A - B)$ fractional derivatives. This was demonstrated using stability calculations and error analysis. The results suggest that the Dufort-Frankel approach is suitable for solving this type of fractional differential equations.

Both Caputo fractional derivative and (A– B) derivative were used in the analysis of a third-order PDEs that was conducted by (Akgül and Modanli, 2019). They demonstrated that stability estimates were accurate for the exact solution. The Crank–Nicholson approach was utilized by the authors to build finite difference schemes, and the Von Neumann method was utilized to verify the stability of the schemes. The efficiency and precision of the method were demonstrated by the numerical results. They also developed a replication kernel function for the issue that they were working on. Obtaining a replicating kernel for the third-order fractional partial differential equation, developing and analyzing stable finite difference schemes, and showing the stability of the solution are the most important contributions. The proposed methods are useful via numerical experiments.

(Atangana et al., 2020) proposed integration and nonlocal differential operators with fractional order and fractal dimension to model complex real-world problems. They utilized these sophisticated operators in a model of advection-dispersion with a velocity of one and three distinct kernels. Comprehensive analyses were conducted, which involved numerical solutions, stability assessments, error evaluations, and simulations. The results showed that these mathematical tools are capable and useful in dealing with complicated systems that go beyond traditional operators. The primary contributions included proposing sophisticated nonlocal operators, applying them to the advection-dispersion model, and conducting thorough numerical evaluations to showcase their modeling capabilities for complicated systems.

(Hashmi et al., 2022) introduced a novel approach, MHB-DQM, for solving the fractional telegraph equation by utilizing the differential quadrature method with cubic B-splines. This method discretizes the Caputo fractional derivative and approximates spatial derivatives by integrating DQM with a modified cubic B-spline. The partial differential equation is simplified into a set of equations to guarantee stability and practicality. Numerical findings showed convergence for small time values and various fractional orders ranging from 1 to 2, confirming the efficacy of MHB-DQM. The main achievements included creating a cubic B-spline-based DQM discretization for the fractional telegraph equation, demonstrating stability and convergence, and presenting numerical examples to highlight correctness across different fractional orders.

Solutions to systems of nonlinear fractional PDEs were introduced by (Khan et al. 2020) using a Laplace decomposition approach. In order to implement this technique, the nonlinear terms are first transformed into Laplace integral form and then the Stehfest numerical inversion procedure is applied. The result is an iteratively solvable sequence of linear fractional differential sub-systems, which breaks down the original system. It is not necessary to linearize or discretize in order to implement the suggested method. Numerical examples of nonlinear time- and space-fractional systems proved the applicability and efficacy of the methodology. The results demonstrated a high level of concordance when contrasted with alternative approaches and exact answers. For nonlinear multi-term fractional models, the Laplace decomposition method offers a powerful analytical tool.

(Kumar et al., 2019) proposed a finite difference method for solving the generalized time-fractional telegraph equation with generalized fractional derivatives. They introduced scale and weight functions to generalize fractional derivatives, showing these reduce to Caputo and Riemann-Liouville cases. They analyzed solution behavior under varying scales/weights, proved the finite difference scheme's convergence and stability, and validated it numerically for the generalized fractional model.

(Özbag and Modanli, 2021) They studied an initial-boundary value problem involving a fractional order telegraph integro-differential equation, utilizing Riemann-Liouville and Caputo derivatives with zero initial conditions. They created first and second order finite difference methods to address the problem numerically and offered stability assessments. Numerical samples with different fractional orders were used to confirm the accuracy in comparison to exact solutions. Graphs illustrating the relationship between solutions and fractional derivatives were provided, along with error graphs indicating improved accuracy with higher fractional orders. The primary contributions included developing finite difference schemes, demonstrating their stability, and verifying their accuracy in solving the fractional telegraph integro-differential equation model using numerical tests.

(Modanli, 2018) provided an accurate solution of a fractional telegraph partial differential equation for a nonlocal boundary value problem. They presented stability estimates for the equation and developed various difference schemes, such as implicit finite difference and Dufort–Frankel finite difference schemes. The stability of these schemes has been demonstrated, and numerical solutions are computed via Caputo fractional derivatives for fractional orders $\alpha = 0.1, 0.5, 0.9$. The proposed approaches are assessed for correctness and efficacy by comparing numerical results with the exact solution.

(Modanli and Akgül, 2017) created finite difference schemes utilizing the theta approach to computationally solve a fractional telegraph differential equation with the Caputo fractional derivative. They examined the schemes for several values of θ and the fractional order α, demonstrating stability using a matrix-based approach. The stability of the precise solution was also demonstrated. The numerical results confirmed the accuracy and effectiveness of the proposed difference schemes in solving the fractional telegraph equation, thus establishing the method's reliability.

(Modanli et al., 2022) investigated numerical solutions of the fractional order pseudo telegraph integro-differential equation. They proposed a first-order finite difference scheme and performed stability analysis, providing theoretical statements with proofs. Additionally, the research validates these theoretical findings through numerical experiments conducted for various fractions of order α.

On the other hand, the nonlinear FPHTPDEs are significant mathematical models used in several scientific and engineering fields (Modanli et al., 2023). The equations expand on the traditional integer-order telegraph equation by including fractional order derivatives, which introduce a non-local aspect to the system's dynamics. Nonlinear terms increase complexity and enable the explanation of various physical events that show nonlinearity (Liaqat et al., 2022). The equation's pseudo-hyperbolic nature indicates a combination of hyperbolic and parabolic traits, allowing it to effectively simulate wave-like processes with both dispersion and diffusion properties (Ahmed et al., 2023). These equations are important for precisely describing complex systems with memory effects, and are significant tools in domains including electromagnetics, viscoelasticity, and fluid dynamics (Osman et al., 2021). It is crucial to comprehend the characteristics, solutions, and numerical techniques related to nonlinear fractional pseudo-hyperbolic telegraph equations in order to enhance our grasp of real-world occurrences and create cutting-edge technologies (Alaroud et al., 2023).

Recent studies have employed a range of methods to address several types of nonlinear fractional PDEs. (Saleh et al. 2019) introduced a mathematical approach to derive precise solutions for specific nonlinear FPDEs and illustrated the solutions graphically. (Mofarreh et al. 2023) numerically solved the fractional order Fokker-Planck equation utilizing iterative and perturbation approaches that involved Caputo derivatives. (Ahmad et al., 2021) utilized the Yang Transform and HPM to solve nonlinear FPDEs. (Xu et al., 2021) developed analytical solutions for the nonlinear Swift-Hohenberg equations in fluid dynamics using the Laplace Adomian decomposition approach. (Liaqat et al., 2022) introduced the conformable fractional power series approach as superior to decomposition methods for solving systems of nonlinear FPDEs. (Bansu and Kumar 2019) resolved fractional telegraph equations by integrating Chebyshev polynomials with cubic radial basis functions.

The double Laplace transform method is an effective way for solving fractional PDEs. It converts the controlling fractional PDEs into the Laplace domain, where the complex equation is simplified and analytical solutions can be obtained. The transform solutions are then reversed back into the original domain. This method produces accurate closed-form answers in a few step, making it more efficient than multi-stage procedures. Researchers have extensively used the double Laplace transform technique for solving a variety of linear and nonlinear fractional PDEs in mathematical physics and engineering. It has been useful for solving equations with variable derivative orders, initial and boundary conditions, and convolution terms.

In recent years, researchers have extensively used the double Laplace transform method, a powerful approach for resolving partial differential equations of fractional order, to address a variety of challenges, including the one listed above. For example, (Debnath, 2016) investigated the properties and applications of double Laplace transforms, which have gotten less attention than single-variable Laplace transforms. The study looked at theorems concerning the general features of double Laplace transforms and their relevance in solving various equations. Furthermore, (Dhunde et al., 2016) demonstrated the effectiveness of the double Laplace transform technique in producing correct solutions to linear and nonlinear fractional telegraph equations. Their approach used Mittag-Leffler functions and took into account various initial and boundary conditions. The study offered illustrated examples that demonstrated the method's adaptability. Furthermore, (Osman et al., 2021) used double Laplace transforms to efficiently solve a novel class of fractional differential equations in a single step. Their investigation demonstrated that this strategy is far more efficient than previous multi-step methods. (Özkan and Kurt, 2020) proposed the fractional Laplace differential transform method, which combines Laplace transforms and the differential transform method to provide approximate analytical solutions for fractional partial differential equations. They demonstrated that this unique method is simple to adopt and yields very precise numerical answers for a variety of fractional differential equations. Moreover, (Khan et al., 2019) concentrated on developing an analytical solution for the time-fractional wave equation. They utilized the double Laplace transform technique in their approach. The researchers achieved the precise solution to the fractional wave equation by applying this strategy. The study emphasized the ease and direct practicality of using the double Laplace transform method. An example problem was used to demonstrate the usefulness of the strategy in reaching an accurate solution. Furthermore, (Modanli and Bajjah, 2021) investigated a time-dependent fractional Schrödinger equation utilizing the double Laplace decomposition method and an implicit finite difference approach. The numerical findings contrasted the solutions' effectiveness and stability. Overall, academics have successfully used methods that incorporate Laplace transforms to efficiently and accurately solve a wide range of fractional differential equations. In this thesis we consider the following linear and nonlinear fractional-order pseudo-hyperbolic telegraph partial differential equations:

An equation that is defined as the linear fractional-order pseudo-hyperbolic telegraph partial differential equation (FPHTPDEs) is as follows:

$$\begin{cases} w_{tt}(t,x) + {}_0^C D_t^{\alpha} w(t,x) + w(t,x) - \lambda w_{txx}(t,x) - w_{xx}(t,x) = f(t,x), \\ 0 < x < L, \qquad t > 0, \lambda > 0, \qquad 0 < \alpha \leq 1. \\ w(0,x) = h(x), \qquad w_t(0,x) = g(x), \quad 0 \leq x \leq L, \\ w(t,0) = w(t,L) = 0, t \geq 0, \end{cases} \tag{2.1}$$

here ${}_0^C D_t^{\alpha} w(t,x)$ is the fractional Caputo derivative of $\alpha -$order with respect to t, and is defined in (Modanli, 2022) as:

$${}_0^C D_t^{\alpha} w(t,x) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{1}{(t-p)^{\alpha-n+1}} \frac{\partial^n w(p,x)}{\partial p^n} dp, (n-1 < \alpha < n), \tag{2.2}$$

here $\Gamma(.)$ is the Gamma function and for $\alpha = n \in N$ defined as:

$${}_0^C D_t^{\alpha} w(t,x) = \frac{\partial^{\alpha} w(t,x)}{\partial t^{\alpha}} = \frac{\partial^n w(t,x)}{\partial t^n}.$$

In the above model, we employ the Caputo derivative (Podlubny, 1998), The Caputo derivative is widely used in fractional calculus literature due to its "memory effect" property and the fact that the derivative of a constant function equals zero under the Caputo definition. In derivatives, it's also common knowledge that fractional operators don't have the same properties as a classical derivative.

An equation that is defined as the nonlinear fractional-order pseudo-hyperbolic telegraph partial differential equation is as follows:

$$\begin{cases} w_{tt}(t,x) + {}_0^C D_t^{\alpha} w(t,x) + w^2(t,x) = \lambda w_{txx}(t,x) + w_{xx}(t,x) + f(t,x), \\ 0 < t < T, 0 < x < L, 0 < \alpha \le 1, \lambda > 0, \\ w(0,x) = \varphi(x), w_t(0,x) = \psi(x), \ \ 0 \le x \le L, \\ w(t,0) = w(t,L) = 0, \quad 0 \le t. \end{cases} \tag{2.3}$$

The model (2.3) is called the nonlinear FPHTPDEs, which describes the propagation of waves in physical and engineering systems that exhibit anomalous diffusion and other complex phenomena. A Caputo fractional derivative of order α, replaces the time derivative in this generalized version of the classical hyperbolic telegraph equation.

In the model (2.3), the fractional order is denoted by $0 < \alpha \le 1$, λ is the constant, the fractional derivative is represented by ${}_0^C D_t^{\alpha} w(t,x)$, and the dependent variable is written as $w(x,t)$. The nonlinear equations include a wide range of nonlinear functions of $w(x,t)$, including but not limited to $w^2(x,t)$, $\sin w(x,t)$, and $e^{w(x,t)}$ and so on.

The linear and nonlinear forms of FPHTPDEs are utilized in a wide range of physical and engineering fields. These models are utilized to simulate wave propagation and memory effects in various materials, including viscoelastic substances like polymers and composites, porous rocks and sediments, and electromagnetic waves within plasmas. Furthermore, wave behavior in systems exhibiting anomalous diffusion or non-exponential relaxation processes can be described by these equations. By incorporating fractional derivatives into these equations, intricate material responses and effects dependent on time can be accounted for, making them highly useful instruments for examining wave phenomena in disciplines including mechanics, acoustics, electromagnetics, heat transfer, and others that deal with disorganized or memory-dependent media.

This thesis focuses on implementing and analyzing the explicit finite difference method to obtain approximate numerical solutions for linear and nonlinear FPHTPDEs using the Caputo fractional derivative of order α. We analyze the stability of this finite difference scheme using matrix forms for linear FPHTPDEs. In mathematics, it is crucial to comprehend the accuracy, reliability, and theoretical benefits of numerical methods such as the finite difference technique, particularly with the constant emergence of new schemes. The MATLAB application generates approximate numerical solutions for linear and nonlinear fractional-order pseudo-hyperbolic telegraph partial differential equations. A general methodology is presented for finding mild solutions to FPHTPDEs in abstract form, together with stability analysis. The abstract technique is demonstrated to be broadly applicable to many types of nonlinearity. The current thesis aims to address a research gap by conducting a thorough numerical analysis to evaluate the accuracy and stability of the method. Furthermore, this thesis implements the modified double Laplace transform method for obtaining analytical solutions for this type of differential equation. This method is known for producing closed exact solutions efficiently, especially when dealing with pseudo-hyperbolic telegraph equations that are based on the Caputo fractional derivative. The present work offers novel theoretical insights and practical tools for addressing a significant category of fractional differential equations.

3. MATERIAL and METHOD

3.1. Material

In order to augment the comprehensiveness of this thesis, pertinent scholarly publications, journals, books, and theses were referenced. Comprehensive literature searches were performed utilizing academic databases and search engines. This process ensured a comprehensive and extensive exploration of current knowledge and recent advancements in the subject.

3.2. Method

In this thesis, two distinct numerical methods are used in this research to solve the linear and nonlinear FPHTPDEs. The first method involves the modified double Laplace transform methodology, whereas the second approach uses an explicit finite difference method.

The modified double Laplace transform method is a potent analytical tool designed for solving fractional differential equations. This method entails applying the Laplace transform on the fractional-order equation in terms of spatial and temporal variables, converting the initial partial differential equation into an algebraic equation in the Laplace domain. The algebraic problem is solved first, and then the solution is transformed back to the original space-time domain using the inverse Laplace transform. The modified double Laplace transform method is effective for dealing with non-integer order derivatives in fractional equations, offering accurate analytical or semi-analytical solutions, although it requires more computational complexity than finite difference methods.

On the other hand, the explicit finite difference method is a commonly used numerical approach for solving partial differential equations. The spatial and temporal derivatives in the FPHTPDEs are estimated using finite difference quotients in this method. An iterative method is used to solve the system of algebraic equations, providing an approximation solution to the original differential equation at specific spatial and temporal intervals. This method is advantageous due to its simple concept and efficient computation, especially for linear problems. Ensuring stability and convergence for nonlinear equations or higher-order fractional derivatives can be challenging and may necessitate precise selection of discretization parameters.

Our study seeks to create strong and effective solution methods for linear and nonlinear FPHTPDEs by using two different numerical techniques. These equations are commonly used to represent wave propagation and memory effects in many physical and engineering systems. The explicit finite difference method is computationally efficient, while the modified double Laplace transform method offers an analytical closed form specifically designed for fractional-order equations.

Through the use of numerical examples, a comparison is made between the outcomes of the two approaches about the accuracy and computing efficiency of the results. Detailed information regarding the implementation of both techniques is presented in the following subsections.

3.2.1. Modified double Laplace transform method

This subsection introduces the double Laplace Transform strategy to solve the linear fractional-order pseudo-hyperbolic telegraph equation in the Caputo sense. This subsection develops the key concept of the double Laplace decomposition approach for the proposed problem.

For this purpose, we examine the general form of the pseudo hyperbolic telegraph partial differential equations (2.1) of fractional order as follows:

$$w_{tt}(t,x) = \lambda w_{txx}(t,x) - {}_{0}^{C}D_t^{\alpha} w(t,x) + w_{xx}(t,x) - w(t,x) + f(t,x), \tag{3.1}$$

$$0 < x < L, t > 0,$$

depending on the initial and boundary conditions

$$\begin{matrix} w(0,x) = h(x), \\ w_t(0,x) = g(x), \end{matrix} \quad 0 \le x \le L \tag{3.2}$$

$$w(t,0) = w(t,L) = 0, t \geq 0, \tag{3.3}$$

where, $\lambda > 0$, and ${}_0^C D_t^\alpha w(t,x)$ is Caputo fractional derivative of order $0 < \alpha \leq 1$, while $f(t,x)$ is the given function.

The double Laplace decomposition method utilized here offers an effective analytical technique for managing the fractional-order variables and deriving a series solution. The solution process requires breaking down the initial problem into two differential sub-equations using Laplace transforms. The equations can be solved in a recursive manner to determine the terms of the series solution.

This approach offers the advantage of circumventing intricate fractional calculus calculations. The Laplace domain is a useful framework for dealing with fractional derivatives and nonlinear terms. The solutions are presented in series forms that are readily computable.

To obtain the general solution of equation (3.1), firstly we use double Laplace transform to equation (3.1), we get

$$\begin{aligned} s^2\bar{w}(s,p) - s\bar{w}(0,p) - \bar{w}_t(0,p) \\ &= \mathcal{L}_x\mathcal{L}_t\{\lambda w_{txx}(t,x) - {}_0^C D_t^\alpha w(t,x) + w_{xx}(t,x) - w(t,x)\} \\ &+ \bar{f}(s,p), \end{aligned} \tag{3.4}$$

where $\bar{f}(s,p) = \mathcal{L}_x\mathcal{L}_t\{f(t,x)\}$.

After that, by applying a single Laplace transform to the initial conditions (3.2) and boundary condition (3.3), we obtain

$$\begin{aligned} &\bar{w}(0,p) = \bar{h}(p), \\ &\bar{w}_t(0,p) = \bar{g}(p), \qquad 0 \leq x \leq L, \qquad t \geq 0. \\ &\bar{w}(s,0) = \bar{w}(s,L) = 0, \end{aligned} \tag{3.5}$$

by substituting formula (3.5) into formula (3.4) and simplifying, we obtain the following formula

$$\bar{w}(s,p) = \frac{1}{s^2}\left[\mathcal{L}_x\mathcal{L}_t\{\lambda w_{txx}(t,x) - {}_0^C D_t^\alpha w(t,x) + w_{xx}(t,x) - w(t,x)\} + \bar{f}(s,p) + s\bar{h}(p) + \bar{g}(p)\right]. \tag{3.6}$$

When we apply the inverse double Laplace transform for equation (3.6), yields

$$w(t,x) = \mathcal{L}_x^{-1}\mathcal{L}_t^{-1}\left\{\frac{1}{s^2}\left[\mathcal{L}_x\mathcal{L}_t\left(\lambda w_{txx}(t,x) - {}_0^C D_t^\alpha w(t,x) + w_{xx}(t,x) - w(t,x)\right)\right] + \frac{1}{s^2}\bar{f}(s,p) + \frac{1}{s}\bar{h}(p) + \frac{1}{s^2}\bar{g}(p)\right\}. \tag{3.7}$$

The general solutions to equation (3.7), can be written as an infinite series of terms, as shown below

$$w(t,x) = \sum_{n=0}^{\infty} w_n(t,x). \tag{3.8}$$

Consequently, equation (3.7) becomes

$$\sum_{n=0}^{\infty} w_n(t,x) = \mathcal{L}_x^{-1}\mathcal{L}_t^{-1}\left\{\frac{1}{s^2}\mathcal{L}_x\mathcal{L}_t\left\{\lambda w_{txx}\sum_{n=0}^{\infty} w_n(t,x) + w_{xx}\sum_{n=0}^{\infty} w_n(t,x) - {}_0^C D_t^\alpha \sum_{n=0}^{\infty} w_n(t,x) - \sum_{n=0}^{\infty} w_n(t,x)\right\}\frac{1}{s^2}\bar{f}(s,p) + \frac{1}{s}\bar{h}(p) + \frac{1}{s^2}\bar{g}(p)\right\}. \tag{3.9}$$

The general formula solution of equation (3.1) is defined as follows:

$$w_0(t,x) = \mathcal{L}_x^{-1}\mathcal{L}_t^{-1}\left\{\frac{1}{s^2}\bar{f}(s,p) + \frac{1}{s}\bar{h}(p) + \frac{1}{s^2}\bar{g}(p)\right\}, \tag{3.10}$$

$$w_{n+1}(t,x) = \mathcal{L}_x^{-1}\mathcal{L}_t^{-1}\left\{\frac{1}{s^2}\mathcal{L}_x\mathcal{L}_t\left\{\lambda w_{txx}\sum_{n=0}^{\infty} w_n(t,x) + w_{xx}\sum_{n=0}^{\infty} w_n(t,x)\right.\right.$$
$$\left.\left. - {}_0^C D_t^{\alpha}\sum_{n=0}^{\infty} w_n(t,x) - \sum_{n=0}^{\infty} w_n(t,x)\right\}\right\}, n \geq 0. \tag{3.11}$$

We suppose that each term on the right side of (3.11) has an inverse double Laplac/e transform.

3.2.2. Finite difference method

This subsection explains the necessary ideals and definitions for implementing the difference scheme method to solve the linear and nonlinear fractional-order pseudo-hyperbolic telegraph problem in the Caputo sense.

The difference scheme method is a potent numerical technique used to approximate solutions to differential equations, including those with fractional derivatives. This approach replaces continuous derivatives with finite difference approximations on a mesh grid. This process transforms the initial fractional differential equation into an algebraic system that may be solved iteratively.

We will now implement the difference scheme strategy for general form of the linear pseudo-hyperbolic telegraph equations with Caputo derivatives. The discrete versions of the fractional-order spatial and temporal derivatives will replace the continuous ones in the differential equation. This leads to an algebraic system that connects the answer at the present and prior time steps.

Firstly, suppose that $h = \frac{L}{M}$ for x-axis and $\tau = \frac{T}{N}$ for t-axis as grid mesh in the rectangular domain $w^h \times w^\tau$, for the difference scheme method, then we have

$$x_n = nh, n = 1,2,\cdots,M, \quad t_k = k\tau, k = 1,2,\cdots,N.$$

We can write the problem (2.1) at the points $(t_k, x_n) \in w^h \times w^\tau$, as

$$w_{tt}(t_k, x_n) + {}_0^C D_t^\alpha w(t_k, x_n) + w(t_k, x_n) - \lambda w_{txx}(t_k, x_n) - w_{xx}(t_k, x_n) \\ = f(t_k, x_n). \quad (3.12)$$

Now, in the following definition we explain how to construct the difference scheme method for the proposed problem. We shall start by defining the finite difference scheme for the Caputo fractional derivative.

Definition 3.2.2.1. The first order difference scheme method for the Caputo fractional derivative (2.2) is given by the following formulas (Karatay, et. al., 2014):

$${}_0^C D_t^\alpha w(t_k, x_n) \cong g_{\alpha,\tau} \sum_{j=0}^{k} q_j{}^{(\alpha)} \left(w_n^{k-j+1} - w_n^{k-j}\right), \quad (3.13)$$

where,

$$g_{\alpha,\tau} = \frac{1}{\Gamma(2-\alpha)\tau^\alpha} \text{ and } q_j{}^{(\alpha)} = (\mathrm{j}+1)^{1-\alpha} - \mathrm{j}^{1-\alpha}. \quad (3.14)$$

Using the last values of equation (3.13), then we have the following approximation:

$${}_0^C D_t^\alpha w(t_k, x_n) \cong g_{\alpha,\tau} \times \left[q_1 w_n^k - q_k w_n^0 + \sum_{j=1}^{k-1} \left(q_{k-j+1} - q_{k-j}\right) w_n^j \right]. \quad (3.15)$$

For constructing the first order difference scheme method for another parts of (2.1), we applying the Taylor expansion series with respect to x and t, as follows:
the Taylor expansion for $w_{tt}(t_k, x_n)$ with respect to t is given as follows:

$$w(t_{k+1}, x_n) = w(t_k, x_n) + \frac{\tau}{1!} w_t(t_k, x_n) + \frac{\tau^2}{2!} w_{tt}(t_k, x_n) + \frac{\tau^3}{3!} w_{ttt}(t_k, x_n) \\ + O(\tau^4), \quad (3.16)$$

$$w(t_{k-1},x_n) = w(t_k,x_n) - \frac{\tau}{1!}w_t(t_k,x_n) + \frac{\tau^2}{2!}w_{tt}(t_k,x_n) - \frac{\tau^3}{3!}w_{ttt}(t_k,x_n) + O(\tau^4). \quad (3.17)$$

By adding the formulas (3.16) and (3.17), we get

$$w(t_{k+1},x_n) - 2w(t_k,x_n) + w(t_{k-1},x_n) = \tau^2 w_{tt}(t_k,x_n) + O(\tau^4). \quad (3.18)$$

From the formula (3.18) we have

$$\frac{w(t_{k+1},x_n) - 2w(t_k,x_n) + w(t_{k-1},x_n)}{\tau^2} = w_{tt}(t_k,x_n) + O(\tau^4), \quad (3.19)$$

neglecting the small term from formula (3.19), we get

$$w_{tt}(t_k,x_n) \cong \frac{w_n^{k+1} - 2w_n^k + w_n^{k-1}}{\tau^2}, \quad (3.20)$$

The Taylor expansion for $w_{xx}(t_k,x_n)$ with respect to x is given as follows:

$$w(t_k,x_{n+1}) = w(t_k,x_n) + \frac{h}{1!}w_x(t_k,x_n) + \frac{h^2}{2!}w_{xx}(t_k,x_n) + \frac{h^3}{3!}w_{xxx}(t_k,x_n) + O(h^4), \quad (3.21)$$

$$w(t_k,x_{n-1}) = w(t_k,x_n) - \frac{h}{1!}w_x(t_k,x_n) + \frac{h^2}{2!}w_{xx}(t_k,x_n) - \frac{h^3}{3!}w_{xxx}(t_k,x_n) + O(h^4). \quad (3.22)$$

By adding the formulas (3.21) and (3.22), we get

$$w(t_k,x_{n+1}) - 2w(t_k,x_n) + w(t_k,x_{n-1}) = h^2 w_{xx}(t_k,x_n) + O(h^4). \quad (3.23)$$

From the formula (3.23) we have

$$\frac{w(t_k, x_{n+1}) - 2w(t_k, x_n) + w(t_k, x_{n-1})}{h^2} = w_{xx}(t_k, x_n) + O(h^4), \tag{3.24}$$

neglecting the small term from formula (3.24), we get

$$w_{xx}(t_k, x_n) \cong \frac{w_{n+1}^k - 2w_n^k + w_{n-1}^k}{h^2}. \tag{3.25}$$

Finally, the Taylor expansion for $w_{txx}(t_k, x_n)$ with respect to t and x is defined by (Jachimavičienė, et. al.,2014) and resulted as follows:

$$w_{txx}(t_k, x_n) \cong \frac{w_{n-1}^k - w_{n-1}^{k-1}}{h^2\tau} - 2\frac{w_n^{k+1} - w_n^k}{h^2\tau} + \frac{w_{n+1}^k - w_{n+1}^{k-1}}{h^2\tau}. \tag{3.26}$$

By using the formulas (3.15), (3.20), (3.25) and (3.26), the difference schemes formula for the equation (2.1), is as follow

$$\begin{cases} \dfrac{w_n^{k+1} - 2w_n^k + w_n^{k-1}}{\tau^2} + g_{\alpha,\tau}\left[q_1 w_n^k - q_k w_n^0 + \displaystyle\sum_{j=1}^{k-1}(q_{k-j+1} - q_{k-j})w_n^j\right] \\ +w_n^k - \lambda \times \left(\dfrac{w_{n-1}^k - w_{n-1}^{k-1}}{h^2\tau} - 2\dfrac{w_n^{k+1} - w_n^k}{h^2\tau} + \dfrac{w_{n+1}^k - w_{n+1}^{k-1}}{h^2\tau}\right) \\ -\dfrac{w_{n+1}^k - 2w_n^k + w_{n-1}^k}{h^2} = f(t_k, x_n) = f_n^k \\ 1 \le k \le N-1, \quad 1 \le n \le M-1, \quad \lambda > 0, \quad 0 < \alpha \le 1. \\ w_n^0 = h(x_n), \quad \dfrac{w_n^1 - w_n^0}{\tau} = g(x_n), \quad 0 \le n \le M, \\ w_0^k = w_M^k = 0, \quad 0 \le k \le N, \end{cases} \tag{3.27}$$

here $g_{\alpha,\tau} = \frac{1}{\Gamma(2-\alpha)\tau^{\alpha}}$ and ${q_j}^{(\alpha)} = (j+1)^{1-\alpha} - j^{1-\alpha}$.

We can rewrite the formula (3.27), as

$$\begin{cases} \left(\frac{2\lambda}{h^2\tau}+\frac{1}{\tau^2}\right)w_n^{k+1}+\left(-\frac{\lambda}{h^2\tau}-\frac{1}{h^2}\right)w_{n-1}^{k}+\left(1-\frac{2}{\tau^2}+g_{\alpha,\tau}q_1-\frac{2\lambda}{h^2\tau}+\frac{2}{h^2}\right)w_n^k \\ +\left(-\frac{\lambda}{h^2\tau}-\frac{1}{h^2}\right)w_{n+1}^{k}+\left(\frac{\lambda}{h^2\tau}\right)w_{n-1}^{k-1}+\left(\frac{1}{\tau^2}\right)w_n^{k-1}+\left(\frac{\lambda}{h^2\tau}\right)w_{n+1}^{k-1} \\ +g_{\alpha,\tau}\sum_{j=1}^{k-1}\left(q_{k-j+1}-q_{k-j}\right)w_n^j-g_{\alpha,\tau}q_k w_n^0=f_n^k, \\ 1<k<N-1, \quad 1<n<M-1, \lambda>0, \quad 0<\alpha\le 1. \\ w_n^0=h(x_n), \quad \frac{w_n^1-w_n^0}{\tau}=g(x_n), \quad 0\le n\le M, \\ w_0^k=w_M^k=0, \quad 0\le k\le N, \end{cases} \tag{3.28}$$

by using initial conditions, we can write the difference scheme (3.28) in matrix form as:

$$\begin{cases} w^1=w^0+\tau g(x_n), \\ Aw^{k+1}=Bw^k+Cw^{k-1}-g_{\alpha,\tau}\left(\sum_{j=1}^{k-1}\left(q_{k-j+1}-q_{k-j}\right)w^j-q_k w_n^0\right)+f_n^k, \end{cases} \tag{3.29}$$

where $f_n^k=\left[f_0^k,f_1^k,\cdots,f_M^k\right]^T$ for $1\le n\le M$, $1\le k\le N-1$ and $u^k=\left[w_0^k,w_1^k,\cdots,w_M^k\right]^T$. And here A , B and C are defined to be symmetric tridiagonal matrices.

$$A=\begin{bmatrix} a & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 & \cdots & 0 & 0 & 0 & 0 \\ & \vdots & & & \ddots & & & \vdots & \\ 0 & 0 & 0 & 0 & \cdots & 0 & a & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & a & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & a \end{bmatrix}_{(N-1)\times(N-1)}$$

where, $a=\frac{2\lambda}{h^2\tau}+\frac{1}{\tau^2}$.

$$B = \begin{bmatrix} c & b & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ b & c & b & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ & \vdots & & & \ddots & & & \vdots & \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & b & c & b \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & b & c \end{bmatrix}_{(N-1)\times(N-1)}$$

where, $b = -\frac{\lambda}{h^2\tau} - \frac{1}{h^2}$ and $c = 1 - \frac{2}{\tau^2} + g_{\alpha,\tau}q_1 - \frac{2\lambda}{h^2\tau} + \frac{2}{h^2}$.

$$C = \begin{bmatrix} d & e & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ e & d & e & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ & \vdots & & & \ddots & & & \vdots & \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & e & d & e \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & e & d \end{bmatrix}_{(N-1)\times(N-1)}$$

where, $d = \frac{\lambda}{h^2\tau}$ and $e = \frac{1}{\tau^2}$.

We write

$$\|\mathrm{A}\| = \|\mathrm{A}\|_\infty = \max_{1\le \mathrm{i} \le (\mathrm{N}-1)} \left\{ \sum_{\mathrm{i}=1}^{\mathrm{N}-1} \|a_{im}\| \right\},$$

where $\mathrm{A} = [\mathrm{a}_{\mathrm{km}}]_{(\mathrm{N}-1)\times(\mathrm{N}-1)}$, and I is unit matrix.

lemma 3.2.2.1.

i. If $\frac{2}{\tau^2} + \frac{2\lambda}{h^2\tau} - 1 - \frac{2}{h^2} - g_{\alpha,\tau}q_1 > 0$,

ii. $-\frac{1}{\tau^2} - \frac{2\lambda}{h^2\tau} + 1 + g_{\alpha,\tau}q_1 > 0$.

If the above conditions are satisfies, then

$$\|\mathrm{A}^{-1}B\| \le 1.$$

Proof:

$$\|A^{-1}B\| \le \|A^{-1}\|\|B\| \le \frac{1}{\min\limits_{1\le k\le N-1}\left\{a_{kk} - \sum_{\substack{m\neq k\\ m=1}}^{N-1}|a_{km}|\right\}} \cdot \|B\|$$

$$\le \frac{\left|\frac{2}{\tau^2} + \frac{2\lambda}{h^2\tau} - 1 - \frac{2}{h^2} - g_{\alpha,\tau}q_1\right| + \left|-\frac{\lambda}{h^2\tau} - \frac{1}{h^2}\right| + \left|-\frac{\lambda}{h^2\tau} - \frac{1}{h^2}\right|}{\left|\frac{2\lambda}{h^2\tau} + \frac{1}{\tau^2}\right|}$$

$$= \frac{\frac{2}{\tau^2} + \frac{2\lambda}{h^2\tau} - 1 - \frac{2}{h^2} - g_{\alpha,\tau}q_1 + \frac{2\lambda}{h^2\tau} + \frac{2}{h^2}}{\frac{2\lambda}{h^2\tau} + \frac{1}{\tau^2}}$$

$$= \frac{\frac{2}{\tau^2} + \frac{4\lambda}{h^2\tau} - 1 - g_{\alpha,\tau}q_1}{\frac{2\lambda}{h^2\tau} + \frac{1}{\tau^2}} = 2 - \frac{g_{\alpha,\tau}q_1 + 1}{\frac{2\lambda}{h^2\tau} + \frac{1}{\tau^2}} \le 1.$$

lemma 3.2.2.2. $\|A^{-1}C\| \le 1.$

Proof:

$$\|A^{-1}C\| \le \|A^{-1}\|\|C\| \le \frac{1}{\min\limits_{1\le k\le N-1}\left\{|a_{km}| - \sum_{\substack{m\neq k\\ m=1}}^{N-1}|a_{km}|\right\}} \cdot \|C\|$$

$$\le \frac{\left|-\frac{1}{\tau^2}\right| + \left|-\frac{\lambda}{h^2\tau}\right| + \left|-\frac{\lambda}{h^2\tau}\right|}{\left|\frac{2\lambda}{h^2\tau} + \frac{1}{\tau^2}\right|} = \frac{\frac{2\lambda}{h^2\tau} + \frac{1}{\tau^2}}{\frac{2\lambda}{h^2\tau} + \frac{1}{\tau^2}} = 1.$$

Theorem 3.2.2.1. The difference scheme (3.28) is stable.

Proof: To prove the conditional stability of difference scheme (3.28), let V_n^k and W_n^k are the exact and the approximate solutions of (3.28) with interval value V_n^0 and W_n^0, respectively.

We show that the corresponding error is given by: $\varepsilon_n^k = V_n^k - W_n^k$ and $\varepsilon^k = \left[\varepsilon_1^k, f_2^k, \cdots, \varepsilon_{M-1}^k\right]^t$ where $(0 \leq n \leq M, 1 \leq k \leq N)$. Then ε^ksatisfies if $k = 0$, then

$$A\varepsilon^1 = B\varepsilon^0.$$

If $k > 0$, then

$$A\varepsilon^{k+1} = B\varepsilon^k + C\varepsilon^{k-1} - q_1\varepsilon^k + \sum_{j=1}^{k-1}\left(q_{k-j+1} - q_{k-j}\right)\varepsilon^j + q_k\varepsilon^0.$$

Let us prove $\|\varepsilon^k\| \leq \|\varepsilon^0\|$, $k = 0, 1,2, \cdots$, by induction.

In fact, if $k = 0$, then we have

$$\varepsilon^1 = A^{-1}B\varepsilon^0,$$

from the above formula, we have

$$\|\varepsilon^1\| = \|A^{-1}B\varepsilon^0\| \leq \|A^{-1}B\|\|\varepsilon^0\| \leq \|\varepsilon^0\|.$$

Since $\|A^{-1}B\| \leq 1$, from the lemma 3.2.1, we have

$$\|\varepsilon^1\| \leq \|\varepsilon^0\|.$$

If $k = 1$, then we get

$$\varepsilon^2 = A^{-1}(B - q_1I)\varepsilon^1 + q_1A^{-1}\varepsilon^0 + A^{-1}C\varepsilon^0.$$

From the last equation, we obtain

$$\|\varepsilon^2\| = \|A^{-1}(B - q_1I)\varepsilon^1\| + \|q_1A^{-1}\varepsilon^0\| + \|A^{-1}B\varepsilon\|$$

$$\leq \|A^{-1}(B - q_1 I)\|\|\varepsilon^1\| + \|q_1 A^{-1}\|\|\varepsilon^0\| + \|A^{-1}C\|$$

$$\leq \|A^{-1}(B - q_1 I)\|\|\varepsilon^0\| + \|q_1 A^{-1}\|\|\varepsilon^0\| + \|A^{-1}C\|$$

$$\leq \{\|A^{-1}(B - q_1 I)\| + \|q_1 A^{-1}\| + \|A^{-1}C\|\}\|\varepsilon^0\|$$

$$\leq \{\|A^{-1}B\| + \|A^{-1}C\|\}\|\varepsilon^0\|.$$

Since $\|\mathrm{A}^{-1}B\| \leq 1$ and $\|\mathrm{A}^{-1}C\| \leq 1$, then we have

$$\|\varepsilon^2\| \leq \|\varepsilon^0\|.$$

Now, assume that $\|\varepsilon^k\| \leq \|\varepsilon^0\|$ for $s \leq k$. We will prove that it is also true for $s = k + 1$,

$$\|\varepsilon^{k+1}\| = \left\|A^{-1}\left(B\varepsilon^k - q_1\varepsilon^k + \sum_{j=1}^{k-1}\left(q_{k-j} - q_{k-j+1}\right)\varepsilon^j + q_k\varepsilon^0\right)\right\| + \|A^{-1}C\varepsilon^k\|$$

$$\leq \|A^{-1}(B - q_1 I)\|\|\varepsilon^k\| + \sum_{j=1}^{k-1}\left(q_{k-j} - q_{k-j+1}\right)\|A^{-1}\|\left\|\varepsilon^j\right\| + q_k\|A^{-1}\|\|\varepsilon^0\| + \|A^{-1}C\|\|\varepsilon^0\|$$

$$\leq \|A^{-1}(B - q_1 I)\|\|\varepsilon^0\| + \sum_{j=1}^{k-1}\left(q_{k-j} - q_{k-j+1}\right)\|A^{-1}\|\|\varepsilon^0\| + q_k\|A^{-1}\| + \|A^{-1}C\|\|\varepsilon^0\|$$

$$\leq \|A^{-1}(B - q_1 I)\|\varepsilon^0 + \{(q_{k-1} - q_k) + (q_{k-2} - q_{k-1}) + \cdots + (q_1 - q_2) + q_k\}\|A^{-1}\|\|\varepsilon^0\| + \|A^{-1}C\|\|\varepsilon^0\|$$

$$\leq (\|A^{-1}(B - q_1 I)\| + q_1\|A^{-1}\| + \|A^{-1}C\|)\|\varepsilon^0\|$$

$\leq \|\varepsilon^0\|.$

Therefore under the condition $\frac{2}{\tau^2}+\frac{2\lambda}{h^2\tau}-1-\frac{2}{h^2}-g_{\alpha,\tau}q_1>0$, the stability inequality obtained.

Here we give the first order difference scheme for the nonlinear PHTPDEs (2.3):

similar to the prior method, we will develop a difference scheme strategy to the nonlinear pseudo-hyperbolic telegraph PDEs (2.3), including Caputo fractional derivatives.

Firstly, we will write the problem (2.3) at the points $(t_k, x_n) \in w^h \times w^\tau$, as follows

$$\begin{aligned} &w_{tt}(t_k,x_n)+{}_0^CD_t^\alpha w(t_k,x_n)+w^2(t_k,x_n)\\ &\qquad=\lambda u_{txx}(t_k,x_n)+u_{xx}(t_k,x_n)+f(t_k,x_n). \end{aligned} \tag{3.30}$$

By using the formulas (3.15), (3.20), (3.25), (3.26), and ${w_n}^2=w^2(t_k,x_n)$, we can construct the difference schemes formula for the equation (3.30), is as follow

$$\begin{cases} \frac{w_n^{k+1}-2w_n^k+w_n^{k-1}}{\tau^2}+\mathrm{S}_{\alpha,\tau}(w_n^{k+1}-w_n^k)\\ +\mathrm{g}_{\alpha,\tau}\sum_{j=1}^{k}q_j^{(\alpha)}\left(w_n^{k-j+1}-w_n^{k-j}\right)+{w_n}^2\\ =\frac{\lambda}{\tau}\left[\frac{w_{n+1}^k-2w_n^k+w_{n-1}^k}{h^2}-\frac{w_{n+1}^{k-1}-2w_n^{k-1}+w_{n-1}^{k-1}}{h^2}\right]\\ +\frac{w_{n+1}^k-2w_n^k+w_{n-1}^k}{h^2}+f_n^k,\\ 1\leq k\leq \mathrm{N}-1, 1\leq n\leq \mathrm{M}-1,\\ w_n^0=\varphi(x_n),\ \frac{w_n^1-w_n^0}{\tau}=\psi(x_n),\ 0\leq n\leq \mathrm{M}\\ w_0^k=w_M^k=0\,, \qquad 0\leq k\leq \mathrm{N}. \end{cases} \tag{3.31}$$

Numerical simulations will be used to showcase the precision and effectiveness of the suggested difference method for equations (2.1) and (2.3). An analysis will be conducted to examine how the fractional order terms impact the solution behavior. The difference scheme provides a cost-effective alternative to expensive numerical approaches for solving intricate fractional differential equations.

3.2.3. Exact solution and stability estimates for abstract form for nonlinear FPHTPDEs

This section provides a comprehensive analysis of the exact solution and stability estimations for the fractional order nonlinear telegraph PDEs. Among its many applications in physics, engineering, and telecommunications, the nonlinear Telegraph Equation is a significant mathematical model. Understanding this equation's exact solutions and stability properties is crucial for understanding how dynamic systems it governs behave. We want to clarify the complex interplay between nonlinearity and wave propagation, by conducting a comprehensive investigation and utilizing mathematical methods, we want to highlight the fundamental properties underlying this important family of Partial Differential Equations (PDEs).

First, we consider the following fractional order nonlinear telegraph PDEs

$$\begin{cases} w_{tt}(t) + Aw_t(t) + Aw(t) + w^2(t) + {}_0^C D_t^\alpha w(t) = f(t), \\ w(0) = \varphi, w'(0) = \psi. \end{cases} \tag{3.32}$$

for the telegraph equation, we look into a Cauchy problem

$$\begin{cases} w_{tt}(t) + A\lambda w_t(t) + Aw(t) = G(t), 0 \le t \le T \\ w(0) = \varphi, \qquad w'(0) = \psi. \end{cases} \tag{3.33}$$

In this instance, $\delta > 0$ contains a self-adjoint positive definite operator A in a Hilbert space H and $A \ge \delta I$.

The initial-value problem for a system of first-order differential equations is identical to equation (3.33) and can be written in this way

$$\begin{cases} w'(t) + \dfrac{A\lambda}{2} w(t) + iB^{\frac{1}{2}} w(t) = z(t), 0 \le t \le T \\ z'(t) + \dfrac{A\lambda}{2} z(t) - iB^{\frac{1}{2}} z(t) = G(t), \end{cases} \tag{3.34}$$

here,

$$G(t) = f(t) - w^2(t). \tag{3.35}$$

Integrating the equation (3.34), we get

$$w(t) = e^{-\left(\frac{A\lambda}{2}+iB^{\frac{1}{2}}\right)t} w(0) + \int_0^t e^{-\left(\frac{A\lambda}{2}+iB^{\frac{1}{2}}\right)(t-s)} z(s)\, ds,$$

$$z(t) = e^{-\left(\frac{A\lambda}{2}-iB^{\frac{1}{2}}\right)t} z(0) + \int_0^t e^{-\left(\frac{A\lambda}{2}+iB^{\frac{1}{2}}\right)(t-s)} G(s)\, ds.$$

Applying the initial condition $z(0) = w'(0) + \left(\frac{A\lambda}{2} + iB^{\frac{1}{2}}\right) w(0)$, then we have

$$w(t) = e^{-\left(\frac{A\lambda}{2}+iB^{\frac{1}{2}}\right)t} w(0) + \int_0^t e^{-\left(\frac{A\lambda}{2}-iB^{\frac{1}{2}}\right)(t-s)} \int_0^s e^{-\left(\frac{A\lambda}{2}-iB^{\frac{1}{2}}\right)(s-p)} G(p) dp ds$$
$$+ \int_t^s e^{-\left(\frac{A\lambda}{2}+iB^{\frac{1}{2}}\right)(t-s)} . e^{-\left(\frac{A\lambda}{2}+iB^{\frac{1}{2}}\right)t} ds \left(w'^{(0)} + \left(\frac{A\lambda}{2} + iB^{\frac{1}{2}}\right) w(0) \right).$$

By switching the order of integration, the above expression can be expressed as follows:

$$w(t)=\left[e^{-\left(\frac{A\lambda}{2}+iB^{\frac{1}{2}}\right)t}+\left(\frac{A\lambda}{2}+iB^{\frac{1}{2}}\right)\int_0^t e^{-\left(\frac{A\lambda}{2}+iB^{\frac{1}{2}}\right)(t-s)}e^{-\left(\frac{A\lambda}{2}-iB^{\frac{1}{2}}\right)s}ds\right]w(0)$$
$$+\int_0^t e^{-\left(\frac{A\lambda}{2}+iB^{\frac{1}{2}}\right)(t-s)}e^{-\left(\frac{A\lambda}{2}-iB^{\frac{1}{2}}\right)s}dsw'^{(0)}$$
$$+\int_0^t e^{-\frac{A\lambda}{2}(t-s)}B^{-\frac{1}{2}}\frac{e^{i(t-s)B^{\frac{1}{2}}}-e^{-i(t-s)B^{\frac{1}{2}}}}{2i}G(s)ds$$

$$w(t)=e^{-\frac{A\lambda}{2}t}\left[\frac{e^{itB^{\frac{1}{2}}}-e^{-itB^{\frac{1}{2}}}}{2}+\frac{A\lambda}{2}B^{-\frac{1}{2}}\frac{e^{itB^{\frac{1}{2}}}-e^{-itB^{\frac{1}{2}}}}{2i}\right]w(0)$$
$$+e^{-\frac{A\lambda}{2}t}\left[B^{-\frac{1}{2}}\frac{e^{itB^{\frac{1}{2}}}-e^{-itB^{\frac{1}{2}}}}{2i}\right]w'(0)$$
$$+\int_0^t e^{-\frac{A\lambda}{2}(t-s)}B^{-\frac{1}{2}}\frac{e^{i(t-s)B^{\frac{1}{2}}}-e^{-i(t-s)B^{\frac{1}{2}}}}{2i}G(s)ds. \quad (3.36)$$

We can obtain the following from formula (3.36):

$$w(t)=e^{-\frac{A\lambda}{2}t}\left[\left(c(t)+s(t)\right)w(0)+s(t)w'(0)\right]$$
$$+\int_0^t e^{-\frac{A\lambda}{2}(t-s)}s(t-z)G(z)dz, \quad (3.37)$$

here, $c(t)=\frac{e^{itB^{\frac{1}{2}}}+e^{-itB^{\frac{1}{2}}}}{2}$, $s(t)=B^{-\frac{1}{2}}\frac{e^{itB^{\frac{1}{2}}}-e^{-itB^{\frac{1}{2}}}}{2i}$, and $B=A-\frac{A^2}{4}$.

Utilizing equation (3.34), we have

$$w(t)=e^{-\frac{A\lambda}{2}t}\left[\left(c(t)+\frac{A\lambda}{2}s(t)\right)w(0)+s(t)w'(0)\right]+\int_0^t e^{-\frac{A\lambda}{2}(t-z)}s(t-z)F(z)dz$$
$$-\int_0^t e^{-\frac{A\lambda}{2}(t-z)}s(t-z)w^2(z)dz.$$

From the last formula, we obtain

$$w(t)+\int_0^t e^{-\frac{A\lambda}{2}(t-z)}s(t-z)w^2(z)dz$$
$$=e^{-\frac{A\lambda}{2}t}\left[\left(c(t)+\frac{A\lambda}{2}s(t)\right)w(0)+s(t)w'(0)\right]$$
$$+\int_0^t e^{-\frac{A\lambda}{2}(t-z)}s(t-z)F(z)dz,$$

here,

$$F(z)=f(z)-{}_0^CD_t^\alpha(t). \tag{3.38}$$

Using the formula (3.35), we can get the mild solution of equation (3.32) as follows

$$w(t)+\int_0^t e^{-\frac{A\lambda}{2}(t-z)}s(t-z)w^2(z)dz+\int_0^t e^{-\frac{A\lambda}{2}(t-z)}s(t-z){}_0^CD_t^\alpha w(z)dz$$
$$=e^{-\frac{A\lambda}{2}t}\left[\left(c(t)+\frac{A\lambda}{2}s(t)\right)w(0)+s(t)w'(0)\right]$$
$$+\int_0^t e^{-\frac{A\lambda}{2}(t-z)}s(t-z)F(z)dz. \tag{3.39}$$

We shall now demonstrate the following theorem about formula (3.39).

Theorem 3.2.3.1. Assuming that $\varphi\in D(A)$, $\psi\in D(A^{\frac{1}{2}})$ and $f(t)$ are functions that are continuously differentiable over the interval $[0,T]$, it may be concluded that there is a singular solution to a problem (3.39) and satisfies the following stability inequalities

$$\max_{0\le t\le T}\|w(t)\|_H + \max_{0\le t\le T}\left\|A^{-\frac{1}{2}}w^2(t)\right\|_H + \max_{0\le t\le T}\left\|A^{-\frac{1}{2}}{}_0^C D_t^{\alpha}(t)\right\|_H$$

$$\le M\left[\|\varphi\|_H + \left\|A^{-\frac{1}{2}}\psi\right\|_H + \max_{0\le t\le T}\left\|A^{-\frac{1}{2}}f(t)\right\|\right].$$

where M depend on λ, δ, but not depends on φ, ψ, and $f(t,x)$.

Proof: Using the formula (3.39), $A \ge \delta I > 0$ and the following estimates

$\|c(t)\|_{H\to H} \le 1$, $\left\|B^{\frac{1}{2}}s(t)\right\|_{H\to H} \le 1$, $\left\|e^{-\frac{A\lambda}{2}(t-z)}\right\|_{H\to H} \le 1$, $\left\|e^{-\frac{A\lambda}{2}}\varphi\right\|_{H\to H} \le \frac{1}{\delta}\|\psi\|_H$,

and $\left\|\frac{A}{2}B^{-\frac{1}{2}}\right\|_{H\to H} \le M(\delta)$.

We can write the following inequalities

$$\|w(t)\|_H + \int_0^t \left\|e^{-\frac{A\lambda}{2}(t-z)}\right\|_{H\to H}\left\|B^{\frac{1}{2}}s(t-z)\right\|_{H\to H}\left\|A^{\frac{1}{2}}B^{-\frac{1}{2}}\right\|_{H\to H}\left\|A^{-\frac{1}{2}}w^2\right\|_{H\to H}dz$$
$$+\int_0^t \left\|e^{-\frac{A\lambda}{2}(t-z)}\right\|_{H\to H}\left\|B^{\frac{1}{2}}s(t-z)\right\|_{H\to H}\left\|A^{\frac{1}{2}}B^{-\frac{1}{2}}\right\|_{H\to H}\left\|A^{-\frac{1}{2}}{}_0^C D_t^{\alpha}(t)\right\|_{H\to H}dz$$

$$\le \|s(t)\|_{H\to H}\left\|e^{-\frac{A\lambda}{2}}\varphi\right\|_{H\to H} + \left\|B^{\frac{1}{2}}s(t)\right\|_{H\to H}\left\|A^{\frac{1}{2}}B^{-\frac{1}{2}}\right\|_{H\to H}\times\left|\frac{A}{2e^{\frac{A}{2}}}\right|\left\|e^{-\frac{A}{2}}\varphi\right\|_{H\to H}$$
$$+\left\|B^{\frac{1}{2}}s(t)\right\|_{H\to H}\left\|A^{\frac{1}{2}}B^{-\frac{1}{2}}\right\|_{H\to H}e^{-\frac{A\lambda}{2}t}\left\|e^{-\frac{A\lambda}{2}}\psi\right\|_{H\to H}$$
$$+\int_0^t \left\|e^{-\frac{A\lambda}{2}(t-z)}\right\|_{H\to H}\left\|B^{\frac{1}{2}}s(t-z)\right\|_{H\to H}\left\|A^{\frac{1}{2}}B^{-\frac{1}{2}}\right\|_{H\to H}$$
$$\times\left\|A^{-\frac{1}{2}}f(s)\right\|_{H\to H}ds, \text{for any } t\in[0,T].$$

Then, we obtain

$$\max_{0\le t\le T}\|w(t)\|_H + \max_{0\le t\le T}\left\|A^{-\frac{1}{2}}w^2(t)\right\|_H + \max_{0\le t\le T}\left\|A^{-\frac{1}{2}}{}_0^C D_t^{\alpha} w(t)\right\|_H$$
$$\le M(\delta)\left[\|\varphi\|_H + \left\|A^{-\frac{1}{2}}\psi\right\|_H + \max_{0\le t\le T}\left\|A^{-\frac{1}{2}}f(t)\right\|\right].$$

We will now investigate the applications of Theorem 3.2.3.1. in further detail after proving it. First, we shall address the fractional order telegraph equation boundary value problem. Due to its numerous applications, including heat conduction in porous media and the modeling of the dynamics of viscoelastic materials, this subject is significant.

The proposed theorem will be utilized to create a series of approximations that will lead to the exact solution in order to solve this problem. Along with various numerical examples, we will also talk about the solution's uniqueness.

Additionally, we will study numerous applications of Theorem 3.2.3.1, such as the solution of fractional order integral equations and the starting value problem for fractional order differential equations. These applications will be in addition to the boundary value problem for fractional order telegraph equations.

In conclusion, after solving the initial value problem (3.32) in the Hilbert space $H = L_2[0, L]$ using the supplied operator A^x, we are able to locate the singular smooth solution to problem (2.3). The reason for this is that problem (2.3) has a singular smooth solution $w(t, x)$, for $x \in (0, L)$, and $\delta > 0$. This is because $\varphi(x)$ is smooth and $\psi(x)$ is smooth. By utilizing this particular formula, we are able to simplify the problem to the initial value (3.32) within a Hilbert space, which is characterized by a self-adjoint positive definite operator A^x, given by (Ashyralyev and Modanli, 2015). As follows

$$A^x w(x) = -w_{xx}(x) + \delta w(x), \tag{3.40}$$

we can write the following expression $(A^x w_{xx})_t = -w_{xxt} = -w_{txx}$, for the continuous $w(t, x)$, with the domain

$$D(A^x) = \{w(x): w, w_x, w_{xx} \in L_2[0, L], w(0) = w(L), w'(0) = w'(L)\},$$

here, $f(t) = f(t, x)$, $w(t) = w(t, x)$, and ${}_0^C D_t^\alpha w(t) = {}_0^C D_t^\alpha w(t, x)$ are known and unknown functions defined on $[0, L]$ and taking values in the $H = L_2[0, L]$ with the norm

$$\|\mathrm{f}\|_{\mathrm{L}_2[0,\mathrm{L}]} = \left\{\int |\mathrm{f}(\mathrm{x})|^2 \mathrm{dx}\right\}^{\frac{1}{2}}.$$

3.2.4. Difference scheme and its stability for abstract form nonlinear FPHTPDEs

In this part, we introduce the formula for the difference scheme and discuss its stability for the abstract form of the nonlinear FPHTPDEs (2.3) when α approaches 1. When $\alpha \to 1$, the abstract form of equation (2.3) can be expressed as:

$$\begin{cases} w_{tt}(t) + (\lambda A + 1) w_t(t) + w^2(t) = f(t), \\ w(0) = \eta, w'(0) = \zeta. \end{cases} \tag{3.41}$$

The difference scheme formula for the differential equation (3.41) can be written in the following formulas:

$$\begin{cases} \dfrac{w_{k+1} - 2w_k + w_{k-1}}{\tau^2} + (\lambda A + 1)\dfrac{w_{k+1} - w_k}{\tau} + {w_k}^2 = f_k, \\ w_0 = \eta, \dfrac{w_1 - w_0}{\tau} + \big(1 + \tau(\lambda A + 1)\big) w_0 = \zeta. \end{cases} \tag{3.42}$$

Taking $F_k = f_k - {w_k}^2 - A w_k$, and rewrite the formula (3.42), then we have

$$\begin{cases} w_{k-1} + \big(2 + \tau(\lambda A + 1)\big) w_k + \big(1 + \tau(\lambda A + 1)\big) w_{k+1} = \tau^2 F_k, \\ 1 \le k \le N - 1, \\ w_0 = \eta, w_1 = \eta + \big(1 + \tau(\lambda A + 1)\big)^{-1} = \zeta. \end{cases} \tag{3.43}$$

Theorem 3.2.3.2. Assume that $\eta \in D(A)$, $\zeta \in D(A^{\frac{1}{2}})$. Then the difference scheme (42) is stable, and the following inequalities holds

$$\|w_1\|_H \leq [\|\eta\|_H + \|\zeta\|_H], \tag{3.44}$$

$$\max_{1\leq k\leq N}\left\|\frac{w_{k+1}-w_k}{\tau}\right\|_H + \max_{1\leq k\leq N}\sum_{j=0}^{k-1}\left\|{w_{j+1}}^2\right\|_H$$
$$\leq M\left[\|\eta\|_H + \sum_{j=0}^{k-1}\tau\left\|f_{j+1}\right\|_H + \|\zeta\|_H\right], \tag{3.45}$$

where M does not depend on the τ, η, ζ and f_k.

Proof. Using the triangle inequality for the formula (3.44), we have

$$\|w_1\|_H = \left\|\eta + \left(1+\tau(\lambda A+1)\right)^{-1}\zeta\right\|_H$$

$$\leq \|\eta\|_H + \left\|\left(1+\tau(\lambda A+1)\right)^{-1}\right\|_H \|\zeta\|_H$$

$$\leq \|\eta\|_H + \|\zeta\|_H.$$

Here, $\left(1+\tau(\lambda A+1)\right)^{-1} \leq 1$, since $A \geq \delta I > 0$, and $\lambda > 0$.

Now, we will estimate formula (3.45) for $k \geq 2$. Using the formula (3.42), we can obtain the following formulas

$$w_0 = \eta,$$

$$w_1 = \eta + \frac{1}{R}\tau\zeta,$$

$$w_2 = \eta + \frac{R+1}{R^2}\tau\zeta + \frac{R+1}{R^2}\tau\zeta + \frac{1}{R}\tau^2 F_1,$$

$$w_3 = \eta + \frac{R^2 + R + 1}{R^3}\tau\zeta + \frac{R+1}{R^2}\tau^2 F_1 + \frac{1}{R}\tau^2 F_2,$$

$$w_4 = \eta + \frac{R^3 + R^2 + R + 1}{R^4}\tau\zeta + \frac{R^2 + R + 1}{R^3}\tau^2 F_1 + \frac{R+1}{R^2}\tau^2 F_2 + \frac{1}{R}\tau^2 F_3,$$

$$w_5 = \eta + \frac{R^4 + R^3 + R^2 + R + 1}{R^5}\tau\zeta + \frac{R^3 + R^2 + R + 1}{R^4}\tau^2 F_1 + \frac{R^2 + R + 1}{R^3}\tau^2 F_2 + \frac{R+1}{R^2}\tau^2 F_3 + \frac{1}{R}\tau^2 F_4,$$

$$\vdots$$

$$w_k = \eta + \frac{R^{k-1} + R^{k-2} + \cdots + R + 1}{R^k}\tau\zeta + \frac{R^{k-2} + R^{k-1} + \cdots + R + 1}{R^{k-1}}\tau^2 F_1 + \frac{R^{k-3} + R^{k-4} + \cdots + R + 1}{R^{k-2}}\tau^2 F_2 + \cdots + \frac{R+1}{R^2}\tau^2 F_{k-2} + \frac{1}{R}\tau^2 F_{k-1}, \quad (3.46)$$

$$w_{k+1} = \eta + \frac{R^{k-2} + R^{k-1} + \cdots + R + 1}{R^{k+1}}\tau\zeta + \frac{R^{k-1} + R^{k-2} + \cdots + R + 1}{R^k}\tau^2 F_1 + \frac{R^{k-2} + R^{k-3} + \cdots + R + 1}{R^{k-1}}\tau^2 F_2 + \cdots + \frac{R+1}{R^2}\tau^2 F_{k-1} + \frac{1}{R}\tau^2. \qquad (3.47)$$

Here, $\mathrm{R} = \big(1 + \tau(\lambda + \mathrm{A})\big)$.

Subtracting (3.46) from (3.47), we get

$$\frac{w_{k+1}-w_k}{\tau}=\left(\frac{1}{R}\right)^{k+1}\zeta+\left(\frac{1}{R}\right)^{k}\tau F_1+\left(\frac{1}{R}\right)^{k-1}\tau F_2+\left(\frac{1}{R}\right)^{k-2}\tau F_3+\cdots+\left(\frac{1}{R}\right)^{2}\tau F_{k-1}$$

$$+\left(\frac{1}{R}\right)\tau F_k=\left(\frac{1}{R}\right)^{k+1}\zeta+\tau\sum_{j=0}^{k-1}\left(\frac{1}{R}\right)^{k-j}F_{j+1}. \qquad (3.48)$$

If we write $F_k=f_k-{w_k}^2$ into formula(3.48), we obtain

$$\frac{w_{k+1}-w_k}{\tau}+\tau\sum_{j=0}^{k-1}{w^2}_{j+1}=\left(\frac{1}{R}\right)^{k+1}\zeta+\sum_{j=0}^{k-1}f_{j+1}. \qquad (3.49)$$

Using the triangle inequality, formula (3.49) becomes

$$\left\|\frac{w_{k+1}-w_k}{\tau}\right\|_H+\tau\sum_{j=0}^{k-1}\left\|{w^2}_{j+1}\right\|_H+\|Aw_k\|_H$$

$$\leq\left|\left(\frac{1}{R}\right)^{k+j}\right|_H.\|\zeta\|_H+\tau\sum_{j=0}^{k-1}\left|\left(\frac{1}{R}\right)^{k-j}\right|_H\left\|f_{j+1}\right\|_H$$

$$\leq\|\zeta\|_H+\tau\sum_{j=0}^{k-1}\left\|f_{j+1}\right\|_H.$$

Here, $\left|\left(\frac{1}{R}\right)^k\right|=\left|\left(\frac{1}{(1+\tau(\|\zeta\|_H))}\right)^k\right|\leq 1.$

Here, the Hilbert space $L_{2h}=L_2([0,X]_h)$ of the grid functions $\varphi^h(x)=\{\varphi_n\}_0^\eta$ defined on the $[0,X]_h$ provided with the norm

$$\|\varphi^h\|_{L_{2h}}=\left(\sum_{x\in[0,X]_h}|\varphi(x)|^2h\right)^{\frac{1}{2}}.$$

4. RESULTS and DISCUSSION

In this section, we will utilize two specific techniques: the modified double Laplace transform method and the finite difference scheme. These methodologies were explained in subsections 3.2.1, 3.2.2, and 3.2.3, respectively. Using the modified double Laplace transform technique, our objective is to get the analytical solutions for the linear FPHTPDEs (2.1). This analytical methodology not only serves as a standard for verifying practical solutions, but also gives essential observations into the natural behavior and properties of the underlying equations.

In addition to the analytical framework, we will use the finite difference technique to create numerical solutions for the for the linear and nonlinear FPHTPDEs, denoted as (2.1) and (2.3). The numerical approach employed in this study involves discretizing the fractional derivatives and the governing equations, so converting them into algebraic systems that may be easily implemented in computational simulations. By carefully choosing suitable discretization techniques and iterative solvers, we may accurately represent the intrinsic complexities of the fractional-order equations while taking use of the efficiency and flexibility of numerical methods.

4.1. Modified double Laplace transform for solving linear FPHTPDEs

In this sub section, we provide an illustration and applicability of how to solve the linear FPHTPDEs using the provided approach.

Example 4.1.1. We consider the following linear fractional-order pseudo-hyperbolic telegraph PDEs:

$$\begin{aligned} w_{tt}(t,x) &= \lambda w_{txx}(t,x) - {}_0^C D_t^{\alpha} w(t,x) + w_{xx}(t,x) - w(t,x) \\ &\quad + \left(2 + \frac{2t^{2-\alpha}}{\Gamma(3-\alpha)} + 2\lambda t + 2(t^2+1)\right)\sin x\,, 0 \le x \le \pi, 0 \le t \le 1, \lambda \\ &\quad > 0, 0 < \alpha \le 1, \end{aligned} \tag{4.1}$$

subject to the initial and boundary conditions

$$\begin{aligned} &w(0,x) = sinx, \\ &w_t(0,x) = 0, \end{aligned} \quad 0 \leq x \leq \pi \tag{4.2}$$

$$w(t,0) = w(t,\pi) = 0, t \geq 0. \tag{4.3}$$

Taking the double Laplace Transform (DLT) to both sides of (4.1) and by using the initial conditions (4.2) and simplifying, we get

$$\begin{aligned} \overline{w}(s,p) &= \mathcal{L}_x\mathcal{L}_t\{w(t,x)\} \\ &= \left[\frac{1}{s} + \frac{2}{s^3} + \frac{2}{s^{5-\alpha}} + \frac{2\lambda}{s^4} + \frac{4}{s^5} + \frac{2}{s^3}\right]\frac{1}{p^2+1} \\ &+ \frac{1}{s^2}\mathcal{L}_x\mathcal{L}_t\{\lambda w_{txx}(t,x) - {}_0^C D_t^\alpha w(t,x) + w_{xx}(t,x) - w(t,x)\}. \end{aligned} \tag{4.4}$$

Equation (4.4), when transformed using the inverse double Laplace method, yields

$$\begin{aligned} w(t,x) &= \left(1 + 2t^2 + \frac{2}{\Gamma(5-\alpha)}t^{4-\alpha} + \frac{\lambda}{3}t^3 + \frac{1}{6}t^4\right)sinx \\ &+ \mathcal{L}_x^{-1}\mathcal{L}_t^{-1}\left\{\frac{1}{s^2}\mathcal{L}_x\mathcal{L}_t\{\lambda w_{txx}(t,x) - {}_0^C D_t^\alpha w(t,x) + w_{xx}(t,x) \right. \\ &\left. - w(t,x)\}\right\}. \end{aligned} \tag{4.5}$$

When the decomposition series is applied to equation (4.5) for $w(t,x)$, then we have

$$\sum_{n=0}^{\infty} w_n(t,x) = \left(1 + 2t^2 + \frac{2}{\Gamma(5-\alpha)} t^{4-\alpha} + \frac{\lambda}{3} t^3 + \frac{1}{6} t^4\right) sinx$$
$$+ \mathcal{L}_x^{-1}\mathcal{L}_t^{-1}\left\{\frac{1}{s^2}\mathcal{L}_x\mathcal{L}_t\left\{\lambda \frac{\partial^3}{\partial x^2 \partial t}\sum_{n=0}^{\infty} w_n(t,x) + \frac{\partial^2}{\partial x^2}\sum_{n=0}^{\infty} w_n(t,x)\right.\right.$$
$$\left.\left. - \frac{\partial^\alpha w(t,x)}{\partial t^\alpha}\sum_{n=0}^{\infty} w_n(t,x) - \sum_{n=0}^{\infty} w_n(t,x)\right\}\right\}. \tag{4.6}$$

The following recursive relationship can be deduced from equation (4.6):

$$w_0(t,x) = \left(1 + 2t^2 + \frac{2}{\Gamma(5-\alpha)} t^{4-\alpha} + \frac{\lambda}{3} t^3 + \frac{1}{6} t^4\right) sinx, \tag{4.7}$$

$$w_{n+1}(t,x) = \mathcal{L}_x^{-1}\mathcal{L}_t^{-1}\left\{\frac{1}{s^2}\mathcal{L}_x\mathcal{L}_t\left\{\lambda \frac{\partial^3}{\partial x^2 \partial t}\sum_{n=0}^{\infty} w_n(t,x) + \frac{\partial^2}{\partial x^2}\sum_{n=0}^{\infty} w_n(t,x)\right.\right.$$
$$\left.\left. - \frac{\partial^\alpha w(t,x)}{\partial t^\alpha}\sum_{n=0}^{\infty} w_n(t,x) - \sum_{n=0}^{\infty} w_n(t,x)\right\}\right\}, n \geq 0. \tag{4.8}$$

If $n = 0$, in equation (4.8), then we get

$$w_1(t,x) = \mathcal{L}_x^{-1}\mathcal{L}_t^{-1}\left\{\frac{1}{s^2}\mathcal{L}_x\mathcal{L}_t\left\{\lambda \frac{\partial^3}{\partial x^2 \partial t} w_0(t,x) + \frac{\partial^2}{\partial x^2} w_0(t,x) - \frac{\partial^\alpha w(t,x)}{\partial t^\alpha} w_0(t,x)\right.\right.$$
$$\left.\left. - w_0(t,x)\right\}\right\}, n \geq 0. \tag{4.9}$$

Substituting $w_0(t,x)$ in equation (4.9), then we get $w_1(t,x)$ as follows

$$w_1(t,x) = -\left(\frac{2}{\Gamma(5-\alpha)} t^{4-\alpha} + \frac{2}{\Gamma(7-2\alpha)} t^{6-2\alpha} + \frac{4\lambda}{\Gamma(6-\alpha)} t^{5-\alpha} + \frac{8}{\Gamma(7-\alpha)} t^{6-\alpha}\right.$$
$$\left. + t^2 + \frac{1}{3} t^4 + \frac{\lambda}{30} t^5 + \frac{1}{90} t^6 + \frac{2\lambda}{3} t^3 + \frac{\lambda^2}{12} t^4 + \frac{\lambda}{10} t^5\right) sinx,$$

If $n = 1$, in equation (4.8), then we get $w_2(t,x)$ as follows

$$w_2(t,x) = \left(\frac{6\lambda}{\Gamma(6-\alpha)}t^{5-\alpha} + \frac{6\lambda}{\Gamma(8-2\alpha)}t^{7-2\alpha} + \frac{6\lambda^2}{\Gamma(7-\alpha)}t^{6-\alpha} + \frac{32\lambda}{\Gamma(8-\alpha)}t^{7-\alpha} + \frac{\lambda}{3}t^3 + \frac{16\lambda}{120}t^5 + \frac{\lambda^2}{36}t^6 + \frac{56\lambda}{\Gamma(8)}t^7 + \frac{\lambda^2}{6}t^4 + \frac{\lambda^3}{60}t^5 + \frac{12}{\Gamma(7-\alpha)}t^{6-\alpha} + \frac{12}{\Gamma(9-2\alpha)}t^{8-2\alpha} + \frac{24}{\Gamma(9-\alpha)}t^{8-\alpha} + \frac{1}{6}t^4 + \frac{16}{\Gamma(7)}t^6 + \frac{16}{\Gamma(9)}t^8 + \frac{2}{\Gamma(7-2\alpha)}t^{6-2\alpha} + \frac{2}{\Gamma(9-3\alpha)}t^{8-3\alpha}\right) sinx.$$

If $n = 2$, in equation (4.8), then we get $w_3(t,x)$ as follows

$$w_3(t,x) = -\left(\frac{6\lambda}{\Gamma(6-\alpha)}t^{5-\alpha} + \frac{10\lambda^2}{\Gamma(7-\alpha)}t^{6-\alpha} + \frac{8\lambda^3}{\Gamma(8-\alpha)}t^{7-\alpha} + \frac{40\lambda}{\Gamma(8-\alpha)}t^{7-\alpha} + \frac{8\lambda}{\Gamma(8-2\alpha)}t^{7-2\alpha} + \frac{64\lambda^2}{\Gamma(9-\alpha)}t^{8-\alpha} + \frac{24}{\Gamma(9-\alpha)}t^{8-\alpha} + \frac{12\lambda^2}{\Gamma(9-2\alpha)}t^{8-2\alpha} + \frac{16}{\Gamma(9-2\alpha)}t^{8-2\alpha} + \frac{144\lambda}{\Gamma(10-\alpha)}t^{9-\alpha} + \frac{56\lambda}{\Gamma(10-2\alpha)}t^{9-2\alpha} + \frac{8}{\Gamma(10-3\alpha)}t^{9-3\alpha} + \frac{48}{\Gamma(11-\alpha)}t^{10-\alpha} + \frac{24}{\Gamma(11-2\alpha)}t^{10-2\alpha} + \frac{16}{\Gamma(11-3\alpha)}t^{10-3\alpha} + \frac{2}{\Gamma(11-4\alpha)}t^{4-2\alpha} + \frac{\lambda^2}{12}t^4 + \frac{\lambda}{15}t^5 + \frac{\lambda^3}{30}t^5 + \frac{\lambda^4}{360}t^6 + \frac{\lambda^2}{90}t^6 + \frac{1}{90}t^6 + \frac{\lambda^3}{210}t^7 + \frac{\lambda}{105}t^7 + \frac{\lambda}{720}t^8 + \frac{\lambda^2}{7.12^2}t^8 + \frac{1}{7.60^2}t^8 + \frac{\lambda}{280.9^2}t^9 + \frac{t^{10}}{56.5^2.9^2}\right) sinx.$$

⋮

The general solution of equation (4.1) can be expressed as an infinite series as follows

$$w(t,x) = \sum_{n=0}^{\infty} w_n(t,x) = w_0(t,x) + w_1(t,x) + w_2(t,x) + \cdots.$$

$$w(t,x) = \Bigg[(t^2+1) + \left(\frac{2}{\Gamma(5-\alpha)} - \frac{2}{\Gamma(5-\alpha)}\right)t^{4-\alpha} + \left(\frac{\lambda}{3} - \frac{2\lambda}{3} + \frac{\lambda}{3}\right)t^3$$
$$+ \left(\frac{1}{6} - \frac{1}{3} + \frac{1}{6}\right)t^4 + \left(-\frac{\lambda^2}{12} + \frac{\lambda^2}{6} - \frac{\lambda^2}{12}\right)t^4$$
$$+ \left(-\frac{2}{\Gamma(7-2\alpha)} + \frac{2}{\Gamma(7-2\alpha)}\right)t^{6-2\alpha} + \cdots\Bigg] sinx.$$

As a result, we get the exact solution

$$w(t,x) = (t^2+1)\sin x.$$

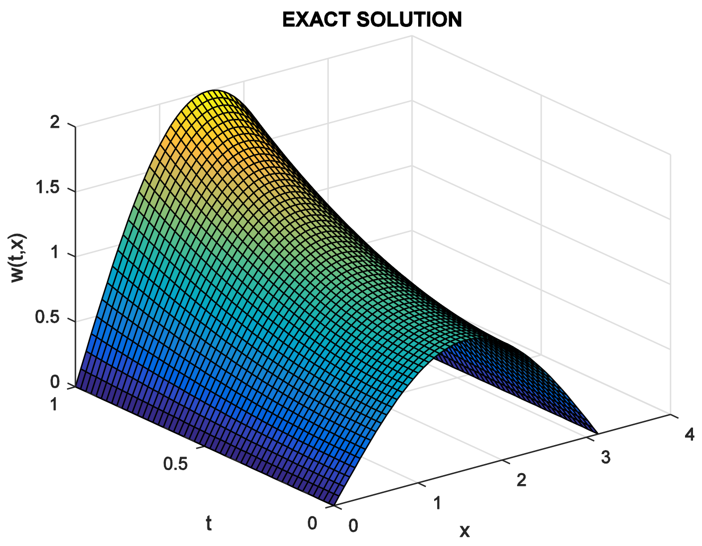

Figure 4.1. Exact solution graph of example 4.1, for $0 \leq x \leq \pi$, $0 \leq t \leq 1$.

In conclusion, from the obtained result we can show that the Modified Double Laplace Transform technique is accurate and don't needed more steps to get an exact solution as shown in Figure 4.1.

4.2. Explicit finite difference method for solving linear FPHTPDEs

In this subsection, we will utilize the explicit finite difference approach to solve linear FPHTPDEs. The explicit finite difference approach is a robust technique that converts the continuous problem domain into a discrete form, converting the governing differential equations into a set of algebraic equations that can be easily implemented in computer systems. By carefully estimating the fractional derivatives and the spatial and temporal derivatives in the equations, we may create a numerical method that accurately represents the fundamental fractional-order dynamics.

The explicit scheme nature allows for the direct computation of the solution at a specific time step using the known solutions from earlier time steps, eliminating the need for iterative techniques or nonlinear solvers. The explicit finite difference method is computationally efficient due to this property, especially for linear problems. In such cases, the algebraic system can be solved directly using matrix operations or other linear algebraic techniques.

Now, we will solve the **Example 4.1.1.** by the explicit finite difference method:

To find the exact solution of equation (4.1), we have used the Laplace transform method, so the exact solution of (4.1) is ($w(t,x) = (t^2+1)\sin x$). For calculating the maximum norm of the error analysis between the exact and approximate solutions, We have used the procedures of the modified Gauss elimination method for equation (3.28), using the following formula

$$\varepsilon = \max_{\substack{n=0,1,\cdots,M \\ k=0,1,\cdots,N}} |w(t,x) - w_n^k|, \tag{4.9}$$

where ε is the error estimates, $w(t,x)$ and w_n^k are the exact solution and the approximate solution, respectively.

For the approximate solutions linear fractional-order pseudo-hyperbolic telegraph PDEs (4.1), we suppose that small parameters $h = \frac{\pi}{M}$ for x-axis and $\tau = \frac{1}{N}$ for t-axis as grid mesh in the rectangular domain $[0,\pi]_h \times [0,1]_\tau$, for the difference scheme method, then we have

$$[0,\pi]_h \times [0,1]_\tau = \{x_n, t_k\}: x_n = nh, n = 1,2,\cdots, M, hM = 1,$$

$$t_k = k\tau, k = 1,2,\cdots, N, \tau N = 1.$$

We can write the problem (4.1) at the points $(t_k, x_n) \in [0,\pi]_h \times [0,1]_\tau$, as

$$w_{tt}(t_k, x_n) + {}_0^C D_t^\alpha w(t_k, x_n) + w(t_k, x_n) - \lambda w_{txx}(t_k, x_n) - w_{xx}(t_k, x_n) \\ = f(t_k, x_n). \quad (4.10)$$

By applying the following formulas

$${}_0^C D_t^\alpha w(t_k, x_n) \cong g_{\alpha,\tau} \sum_{j=0}^{k} q_j{}^{(\alpha)} \left(w_n^{k-j+1} - w_n^{k-j}\right),$$

where,

$$g_{\alpha,\tau} = \frac{1}{\Gamma(2-\alpha)\tau^\alpha},$$

and

$$q_j{}^{(\alpha)} = (j+1)^{1-\alpha} - j^{1-\alpha},$$

$$\mathrm{w_{tt}}(t_k, x_n) = \frac{w_n^{k+1} - 2w_n^k + w_n^{k-1}}{\tau^2},$$

$$w_{xx}(t_k, x_n) = \frac{w_{n+1}^k - 2w_n^k + w_{n-1}^k}{h^2},$$

$$w(t_k, x_n) = w_n^k,$$

$$w_{txx}(t_k, x_n) = \frac{w_{n-1}^k - w_{n-1}^{k-1}}{h^2\tau} - 2\frac{(w_n^{k+1} - w_n^k)}{h^2\tau} + \frac{w_{n+1}^k - w_{n+1}^{k-1}}{h^2\tau},$$

and

$$w_n^0 = h(x_n), \frac{w_n^1 - w_n^0}{\tau} = g(x_n), 0 \le n \le M$$

$$w_0^k = w_M^k = 0, 0 \le k \le N,$$

and by employing the first-order accuracy in the time (t) variable of an explicit difference scheme for the linear fractional-order pseudo-hyperbolic telegraph PDEs, one can obtain a difference scheme with first-order accuracy in t and first-order accuracy in the spatial variable x for approximating solutions to the given linear fractional-order pseudo-hyperbolic telegraph PDEs (4.10), then we have

$$\begin{cases} \dfrac{w_n^{k+1} - 2w_n^k + w_n^{k-1}}{\tau^2} + g_{\alpha,\tau}(w_n^{k+1} - w_n^k) \\ + g_{\alpha,\tau} \sum_{j=1}^{k} q_j{}^{(\alpha)} \left(w_n^{k-j+1} - w_n^{k-j}\right) + w_n^k - \lambda \\ \times \left(\dfrac{w_{n-1}^k - w_{n-1}^{k-1}}{h^2\tau} - 2\dfrac{(w_n^{k+1} - w_n^k)}{h^2\tau} + \dfrac{w_{n+1}^k - w_{n+1}^{k-1}}{h^2\tau} \right) \\ - \dfrac{(w_{n+1}^k - 2w_n^k + w_{n-1}^k)}{h^2} = f_n^k, \\ f_n^k = \left(2 + \dfrac{2(k\tau)^{2-\alpha}}{\Gamma(3-\alpha)} + 2\lambda k\tau + 2(\tau^2 k^2 + 1) \right) sin(nh), \\ Mh = 1, 0 \le n \le M, N\tau = 1, 0 \le k \le N, \lambda > 0, 0 < \alpha \le 1, \\ w_n^0 = sin(nh), 0 \le n \le M, \\ \dfrac{w_n^1 - w_n^0}{\tau} = 0, \quad 0 \le n \le M, \\ w_0^k = w_L^k = 0, \quad 0 \le k \le N, \end{cases} \tag{4.11}$$

here, $g_{\alpha,\tau} = \frac{1}{\Gamma(2-\alpha)\tau^{\alpha}}$ and $q_j{}^{(\alpha)} = (j+1)^{1-\alpha} - j^{1-\alpha}$,

We can rewrite the formula (4.11), as the following coefficient form

$$\begin{cases} \left(\frac{2\lambda}{h^2\tau} + \frac{1}{\tau^2} + g_{\alpha,\tau}\right) w_n^{k+1} + \left(1 - \frac{2\lambda}{h^2\tau} - \frac{2}{\tau^2} + \frac{2}{h^2} - g_{\alpha,\tau}\right) w_n^k \\ + \left(\frac{1}{\tau^2}\right) w_n^{k-1} + \left(\frac{\lambda}{h^2\tau}\right) w_{n+1}^{k-1} + \left(- \frac{\lambda}{h^2\tau} - \frac{1}{h^2}\right) w_{n+1}^k + \left(\frac{\lambda}{h^2\tau}\right) w_{n-1}^{k-1} \\ + \left(- \frac{\lambda}{h^2\tau} - \frac{1}{h^2}\right) w_{n-1}^k + g_{\alpha,\tau} \sum_{j=0}^{k} q_j{}^{(\alpha)} \left(w_n^{k-j+1} - w_n^{k-j}\right) = f_n^k, \\ f_n^k = \left(2 + \frac{2(k\tau)^{2-\alpha}}{\Gamma(3-\alpha)} + 2\lambda k\tau + 2(\tau^2 k^2 + 1)\right) sin(nh), \\ Mh = \pi, 1 \le n \le M-1, N\tau = 1, 1 \le k \le N-1, \lambda > 0, 0 < \alpha \le 1, \\ w_n^0 = sin(nh)\,, 0 \le n \le M, \\ \frac{w_n^1 - w_n^0}{\tau} = 0, \quad 0 \le n \le M, \\ w_0^k = w_M^k = 0, \quad 0 \le k \le N \end{cases} \tag{4.12}$$

and matrix form is as follows

$$\begin{cases} Aw_{n+1} + Bw_n + Cw_{n-1} = DF, \qquad 0 \le n \le M \\ w_0 = 0,\ w_M = 0. \end{cases} \tag{4.13}$$

Where $F = f_n^k = \left[f_0^k, f_1^k, \cdots, f_M^k\right]^T$ for $0 \le n \le M, 1 \le k \le N-1$, and $w^k = \left[w_0^k, w_1^k, \cdots, w_M^k\right]^T$.

And in equation (4.13). A, B and C are defined to be symmetric tridiagonal matrices, and D is the identity matrix.

$$\mathrm{A} = \mathrm{C} = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \mathrm{a} & \mathrm{r} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & \mathrm{a} & \mathrm{r} & 0 & \cdots & 0 & 0 & 0 & 0 \\ & \vdots & & & \ddots & & & \vdots & \\ 0 & 0 & 0 & 0 & \cdots & \mathrm{a} & \mathrm{r} & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & \mathrm{a} & \mathrm{r} & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \end{bmatrix}_{(\mathrm{N}+1)\times(\mathrm{N}+1)} ,$$

$$B=\begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ b & c & d & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & b & c & d & \cdots & 0 & 0 & 0 & 0 \\ & \vdots & & & \ddots & & & \vdots & \\ 0 & 0 & 0 & 0 & \cdots & b & c & d & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & b & c & d \\ e & -e & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \end{bmatrix}_{(N+1)\times(N+1)},$$

$$D=\begin{bmatrix} 1 & 0 & 0 & & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & & 0 & 0 & 0 \\ & \vdots & & \ddots & & \vdots & \\ 0 & 0 & 0 & & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & 0 & & 0 & 0 & 1 \end{bmatrix}_{(N+1)\times(N+1)},$$

and

$$F=f_n^k=\begin{cases} 0, n=0 \\ \left(2+\dfrac{2(k\tau)^{2-\alpha}}{\Gamma(3-\alpha)}+2\lambda k\tau+2(\tau^2k^2+1)\right)sin(nh), \\ 1\le k\le N-1, \\ f(t_k,x_n), 1\le k\le N-1, \\ 0, n=M, \end{cases}$$

$$f_n^0=sin(nh),$$

and

$$f_n=\begin{bmatrix} f_n^0 \\ f_n^1 \\ f_n^2 \\ \vdots \\ f_n^N \end{bmatrix},$$

$$w_s=\begin{bmatrix} w_s^0 \\ w_s^1 \\ w_s^2 \\ \vdots \\ w_s^N \end{bmatrix}_{(n+1)\times 1} \quad ,s=n+1,n,n-1.$$

Now, we will introduce a mathematical procedure for solving a matrix equation using a modification to the standard Gaussian elimination process. We are aiming to solve the matrix equation of the following form

$$w_n = \alpha_{n+1} w_{n+1} + \beta_{n+1}, n = M-1, \cdots, 2,1,0,$$

where α_i, β_i, $(i = 1,2,\cdots, M-1)$ are square matrices with size $(N+1)\times(N+1)$. And α_1, β_1 are as follows

$$\alpha_1 = \begin{bmatrix} 0 & 0 & 0 & & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & & 0 & 0 & 0 \\ & \vdots & & \ddots & & \vdots & \\ 0 & 0 & 0 & & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & & 0 & 0 & 0 \end{bmatrix}_{(N+1)\times(N+1)} ,$$

$$\beta_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{(n+1)\times 1} , w_0 = w_M = 0.$$

By using these equalities, $Aw_{n+1} + Bw_n + Cw_{n-1} = DF$, and $w_s = \alpha_{s+1} w_{s+1} + \beta_{s+1}$, $s = n-1,\ n$, a new equation is derived:

$$(A + B\alpha_{n+1} + C\alpha_n\alpha_{n+1})w_{n+1} + (B\beta_{n+1} + C\beta_n + C\alpha_n\beta_{n+1}) = DF,$$

specifies conditions for the above equation to be satisfied:

$$\begin{cases} A + B\alpha_{n+1} + C\alpha_n\alpha_{n+1} = 0, \\ (B\beta_{n+1} + C\beta_n + C\alpha_n\beta_{n+1}) = DF, \\ 1 \le k \le N-1. \end{cases}$$

Finally, formulas for α_{n+1} and β_{n+1} are presented as follows

$$\begin{cases} \alpha_{n+1} = (-B - C\alpha_n)^{-1}A, \\ \beta_{n+1} = (-B - C\alpha_n)^{-1}(-C\beta_n + Df_n) \\ n = M-1, \cdots, 2,1,0. \end{cases}$$

Hence,

$w_0 = 0,$

$w_N = \alpha_{n+1} w_{n+1} + \beta_{n+1}, n = M-1, \cdots, 2,1,0.$

The key algorithm steps for solving linear fractional-order pseudo-hyperbolic telegraph PDEs are:

1. Enter the time generation $\tau = \frac{1}{N}$ and the space generation $h = \frac{\pi}{M}$.
2. Utilize a difference scheme that is accurate to the first order and express it in matrix form as follows:

$$Aw_{n+1} + Bw_n + Cw_{n-1} = DF, \; 0 \le n \le N.$$

3. Find the values of the elements in matrices A, B, C, and D.
4. Determine the values of α_1 and β_1.
5. Calculate the values of α_{n+1} and β_{n+1}.
6. Compute the values of w_n $(0 \le n \le N)$ using the formula:

$$w_n = \alpha_{n+1} w_{n+1} + \beta_{n+1},$$

with $w_M = 0$ as the initial condition.

Table 4.1 shows a comparison of the exact and approximate solutions of problem (4.9) using the explicit finite difference method for various values of α, λ, N and M, for $0 < t < 1$, $0 < x < \pi$, and the simulation is shown in the error analysis column.

Tale 4.1. Numerical results for problem (4.1)

$\tau = \frac{1}{N}, h = \frac{\pi}{M},$	α	λ	Error (ε)
$N = 50, M = 50$	0.99	0.001	4.0114×10^{-4}
$N = 50, M = 50$	0.5	0.001	0.0055
$N = 50, M = 50$	0.2	0.0001	1.7730×10^{-4}
$N = 50, M = 50$	0.01	0.0001	1.8873×10^{-4}
$N = 100, M = 100$	0.99	0.0001	0.0021
$N = 100, M = 100$	0.5	0.0001	7.7470×10^{-4}
$N = 100, M = 100$	0.2	0.00001	1.9086×10^{-4}
$N = 100, M = 100$	0.01	0.00001	1.6654×10^{-5}
$N = 200, M = 200$	0.99	0.00001	0.0014
$N = 200, M = 200$	0.5	0.00001	2.7100×10^{-6}
$N = 200, M = 200$	0.2	0.00001	1.6626×10^{-4}
$N = 200, M = 200$	0.01	0.00001	2.5818×10^{-4}

Here, $\varepsilon = \ Absolute\ Error|w_{Exact} - w_{EFDM}|$.

Since the graphs do not show the predicted link between the exact and approximate solutions, the error analysis table is given. Using data from table 1. Generally, when $N = 50, 100, 200$, and $M = 50, 100, 200$, for $\alpha = 0.5$, the absolute error results produced using the finite difference approach gave the better results according to other intervals, this means that the exact and approximation solutions are close to each other's. when the intervals are increases the computer rounding errors increased.

The solutions are illustrated graphically as follows:

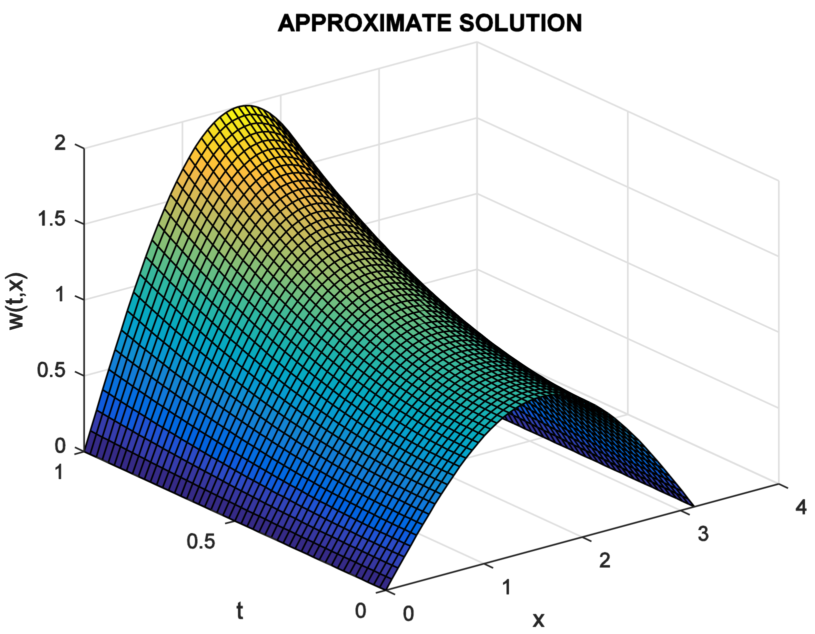

Figure 4.2. Approximate solution graph of example (4.1), where $M = N = 50$, $\alpha = 0.99$, $\lambda = 0.001$, $0 \leq x \leq \pi$, and $0 \leq t \leq 1$, using explicit finite difference method

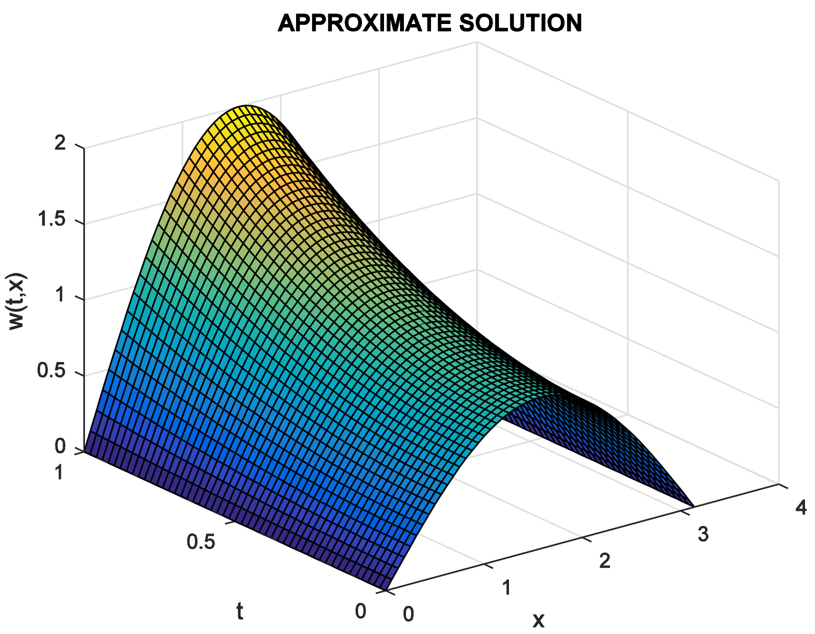

Figure 4.3. Approximate solution graph of example (4.1), where $\mathrm{M} = \mathrm{N} = 50$, $\alpha = 0.2$, $\lambda = 0.0001$, $0 \leq \mathrm{x} \leq \pi$, and $0 \leq \mathrm{t} \leq 1$, using explicit finite difference method

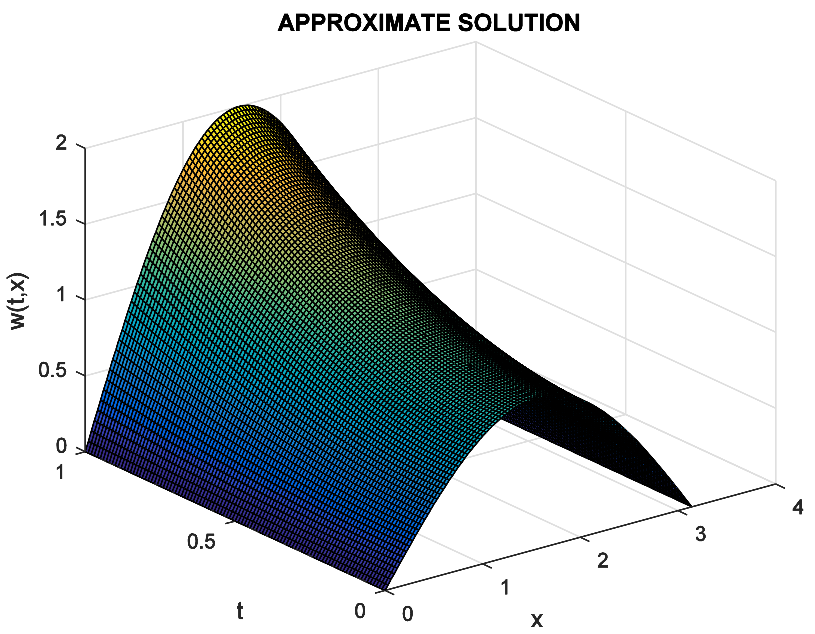

Figure 4.4. Approximate solution graph of example (4.1), where $M = N = 100$, $\alpha = 0.5$, $\lambda = 0.00001$, $0 \leq x \leq \pi$, and $0 \leq t \leq 1$, using explicit finite difference method

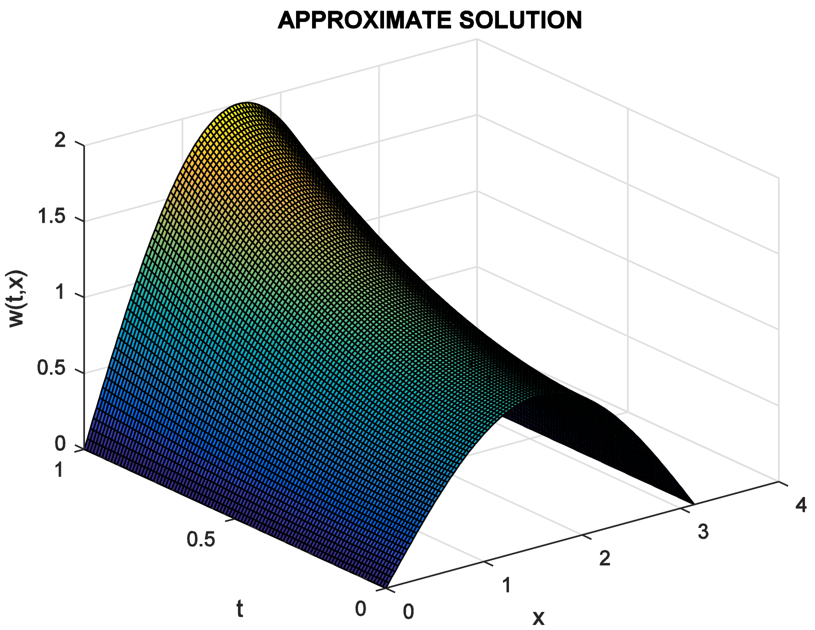

Figure 4.5. Approximate solution graph of example (4.1), where $M = N = 100$, $\alpha = 0.1$, $\lambda = 0.00001$, $0 \leq x \leq \pi$, and $0 \leq t \leq 1$, using explicit finite difference method

When comparing Figure 4.1 (which shows the exact solution) to Figure 4.3 and figure 4.4 (which shows the approximate solution for various parameters), it can be seen that these figures are pretty similar.

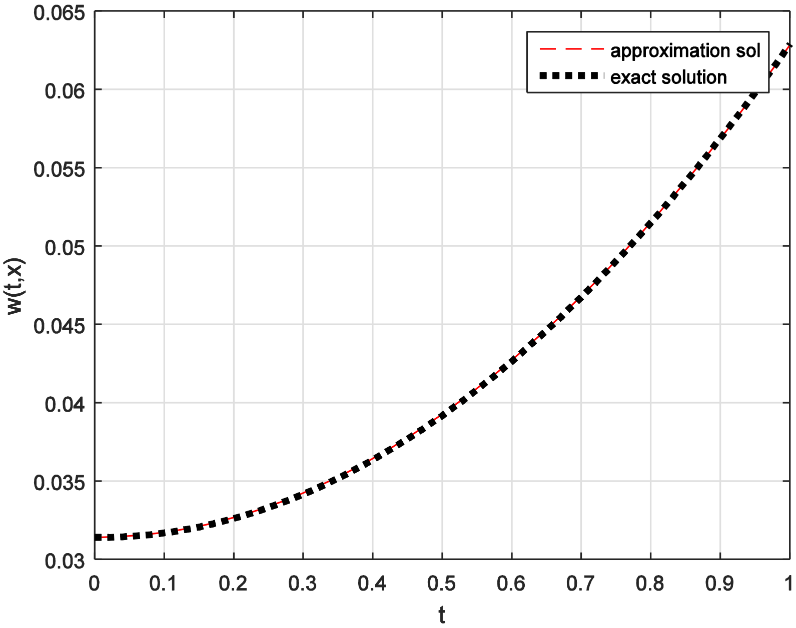

Figure 4.6. Exact and approximate solutions figure of example (4.1), for $\alpha = 0.99$ and $\lambda = 0.0001$, using explicit finite difference method

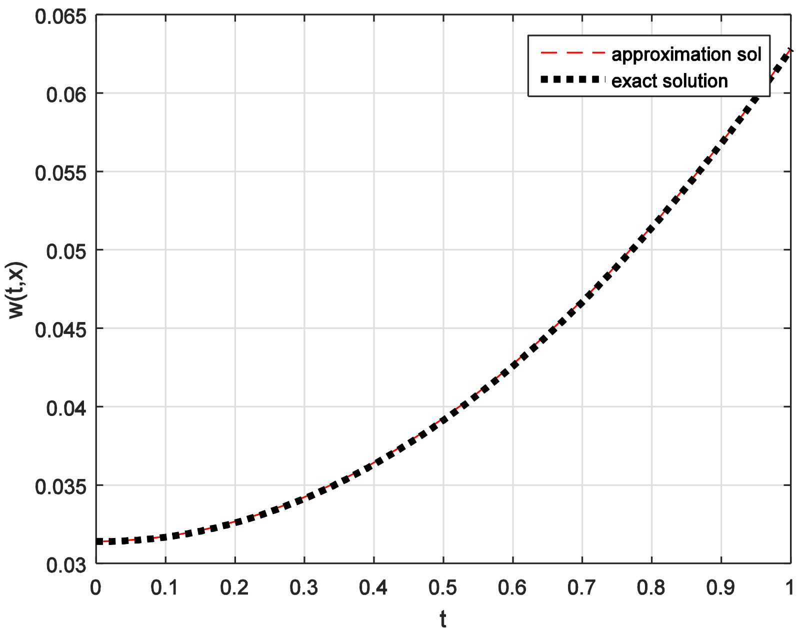

Figure 4.7. Exact and approximate solutions graph of example (4.1), for $\alpha = 0.5$ and $\lambda = 0.0001$, using explicit finite difference method

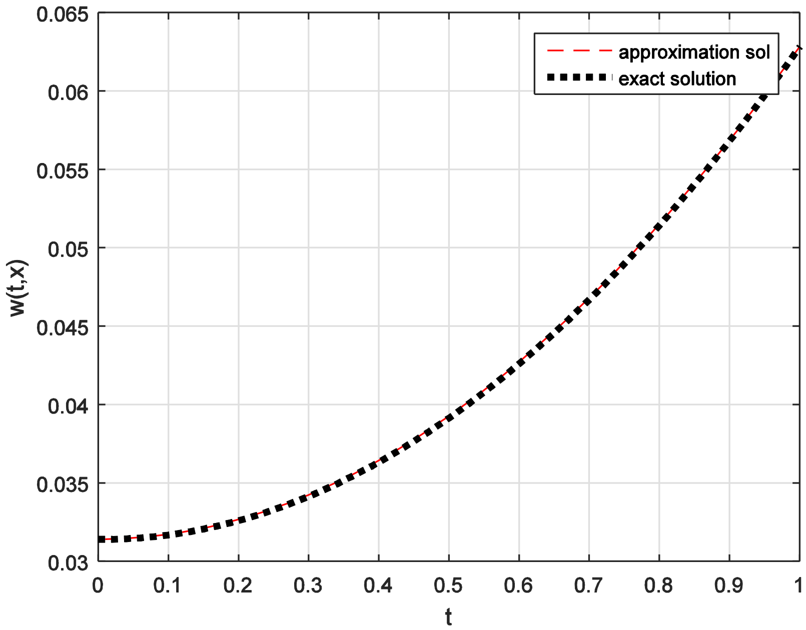

Figure 4.8. Exact and approximate solutions graph of example (4.1), for $\alpha = 0.1$ and $\lambda = 0.00001$, using explicit finite difference method

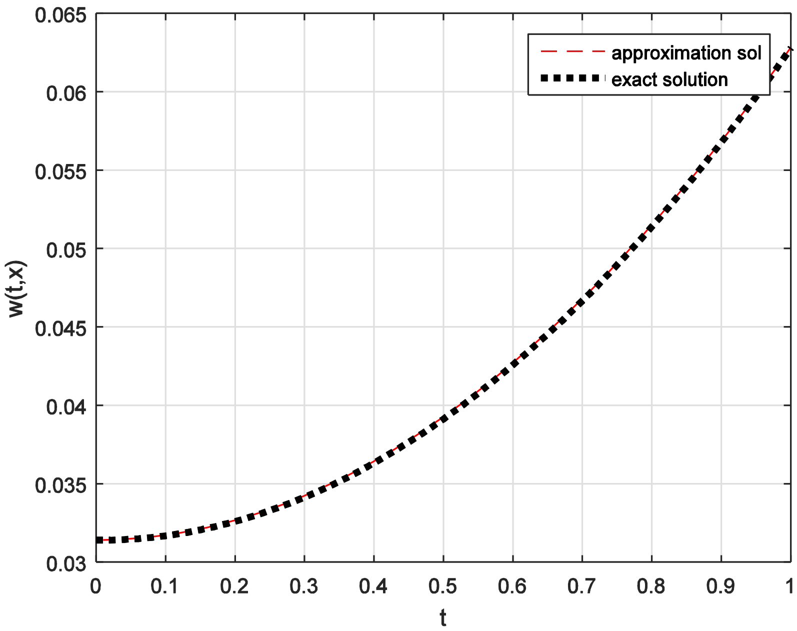

Figure 4.9. Exact and approximate solutions graph of example (4.1), for $\alpha = 0.2$ and $\lambda = 0.0001$, using explicit finite difference method

Exact and approximate solution are compared in figures (4.6,4.7,4.8 and 4.9). The figure of the approximate solution that produced for $\alpha = 0.99$ in figure 4.6 and

$\alpha = 0.5$ in figure 4.7 is significantly close to the exact solution. And also in for $\alpha = 0.1$ in figure 4.8 and $\alpha = 0.2$ in figure 4.9 are close to each other's.

4.3. Explicit finite difference method for solving nonlinear FPHTPDEs

In this subsection, An explicit finite difference method will be employed for solving nonlinear fractional-order pseudo-hyperbolic telegraph PDEs.

Example 4.3.1. We consider the following nonlinear fractional-order pseudo-hyperbolic telegraph PDEs:

$$\begin{cases} w_{tt}(t,x) + {}_0^C D_t^{\alpha} w(t,x) + w^2(t,x) = \lambda w_{txx}(t,x) + w_{xx}(t,x) + f(t,x), \\ f(t,x) = \left(2 + \dfrac{2t^{2-\alpha}}{\Gamma(3-\alpha)} - \dfrac{t^{1-\alpha}}{\Gamma(2-\alpha)} + 2(t^2 - t + 1)^2 \sin x \right. \\ \left. +\lambda(2t-1) + t^2 - t + 1\right) \sin x, \\ 0 < x < \pi, t > 0, \lambda > 0, 0 < \alpha \leq 1, \\ u(0,x) = \sin x, u_t(0,x) = -\sin x, 0 \leq x \leq \pi, \\ u(t,0) = u(t,\pi) = 0, \quad t \geq 0. \end{cases} \tag{4.11}$$

Here, we employ the finite difference scheme (3.31) to compute numerical solutions for the equation (4.11) and compare them to analytical solutions to estimate errors and accuracy. We derive analytical solutions through integration and algebraic manipulation and then implement the finite difference scheme (3.31) to obtain numerical solutions. Then, we estimate relative errors between the numerical and analytical solutions and generate formal error bounds of the numeric to evaluate accuracy. We apply a modified Gauss elimination method to solve the linear systems for better stability and efficiency. Overall, the procedure is to derive analytical solutions, implement numerical simulations, quantify relative errors, derive error bounds, and assess validity to verify the fidelity of our numerical approach precisely. The exact solution of (2.11) is given as

$$w(t,x) = (t^2 - t + 1)\sin x.$$

We have employed the subsequent methodology to compute the error predicts:

$$\varepsilon = \max_{\substack{k=0,1,\ldots,N \\ n=0,1,\ldots,M}} |w(t,x) - w_n^k|. \tag{4.12}$$

The numerical results for example 4.3.1, are present in table 4.2., using the explicit finite difference technique for different value of λ, α, M and N.

Tale 4.2. Numerical results for example 4.3.1.

$\tau = \frac{1}{N}, h = \frac{\pi}{M},$	Max analytical	Max approximation	Max error	Relative error
$N = 5, M = 25$	0.9511	0.9584	73×10^{-4}	63×10^{-4}
$N = 10, M = 100$	1.0000	1.0020	2×10^{-3}	15×10^{-4}
$N = 15, M = 225$	0.9945	0.9954	9×10^{-4}	7×10^{-4}
$N = 20, M = 400$	1.0000	1.0004	4×10^{-4}	4×10^{-4}
$N = 25, M = 625$	0.9980	0.9983	3×10^{-4}	3×10^{-4}
$N = 30, M = 900$	1.0000	1.0001	2×10^{-4}	2×10^{-4}
$N = 35, M = 1225$	0.9990	0.9993	3×10^{-4}	1×10^{-4}
$N = 40, M = 1600$	0.9990	0.9993	3×10^{-4}	1×10^{-4}

Here, $\varepsilon = \ Absolute\ Error |w_{Exact} - w_{EFDM}|$.

Table 4.2. comprehensive numerical error analysis establishes trust and demonstrates alignment with analytical solutions more robustly than solution graphs, providing critical insights graphs alone cannot. The tables thus supply essential precision and accuracy insights the graphs themselves fail to capture.

The error analysis columns include extensive numerical simulations corresponding to the approximate solutions under various conditions - these simulations serve as the foundation for quantifying accuracy. By simulating across parameters and mesh resolutions, the numerical realizations reveal how approximation errors depend on factors like grid spacing and time steps. The simulations focus specifically on enabling convergence tests and identifying breakdown regimes rather than raw solutions alone. In this way, optimizing the simulations makes the error quantification more credible and thorough by verifying diligence in assessing numerical reliability. Furthermore, tabulating the simulation data facilitates reproducibility so other researchers can independently confirm accuracy.

Note: It is important to note that while dealing with situations that do not have exact solutions, numerical methods are commonly used. Nevertheless, the conditions that meet the requirements of our challenge are initially uncertain. The initial phase involves establishing stability for the problem at hand. Once a state of stability is reached, it is possible to produce numerical findings that are considered acceptable. This method guarantees that the numerical solution precisely reflects the fundamental physical or mathematical system.

The solutions are graphically represents as follows

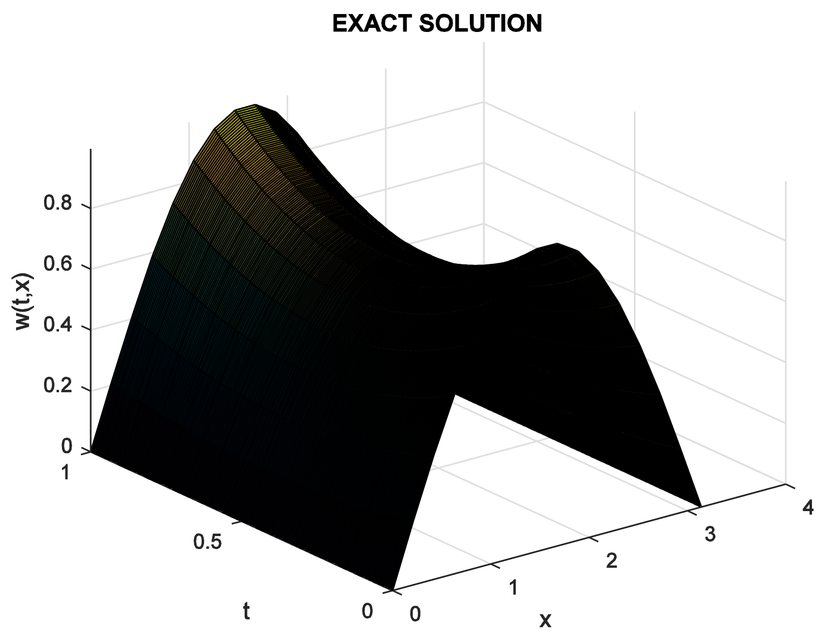

Figure 4.10. Exact and approximate solutions of example (4.2), for $\mathrm{M} = 15, \mathrm{N} = 225$, $\lambda = 5,\ \alpha = 0.9995$, using explicit finite difference method

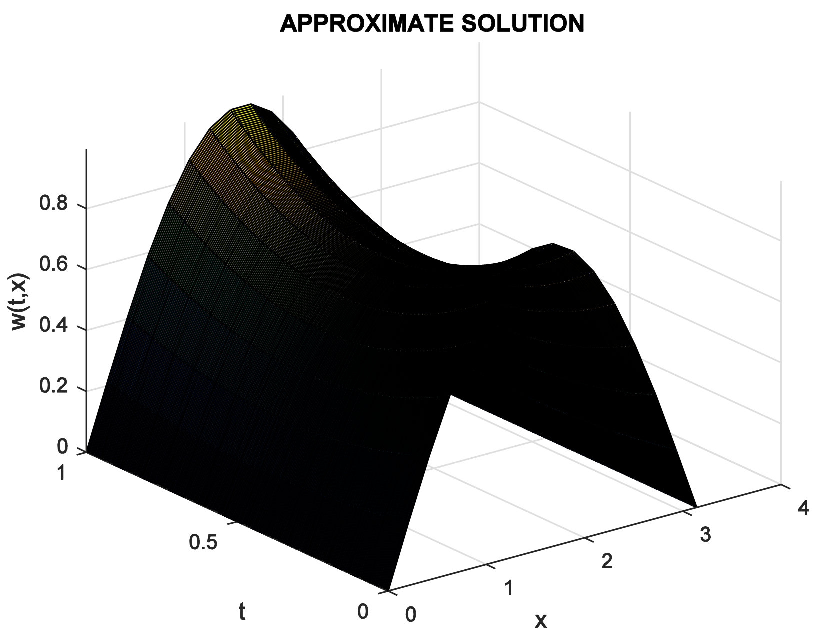

Figure 4.11. Exact and approximate solutions of example (4.3.1), for $M = 15, N = 225$, $\lambda = 5,\ \alpha = 0.9995$, using explicit finite difference method

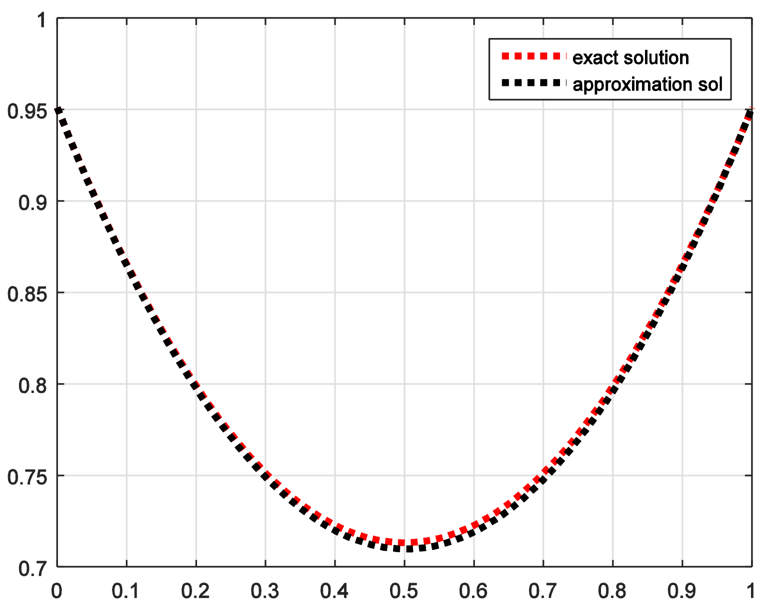

Figure 4.12. Exact and approximate solutions of example (4.3.1), for $M = 10, N = 100$, $\lambda = 5,\ \alpha = 0.95$, using explicit finite difference method

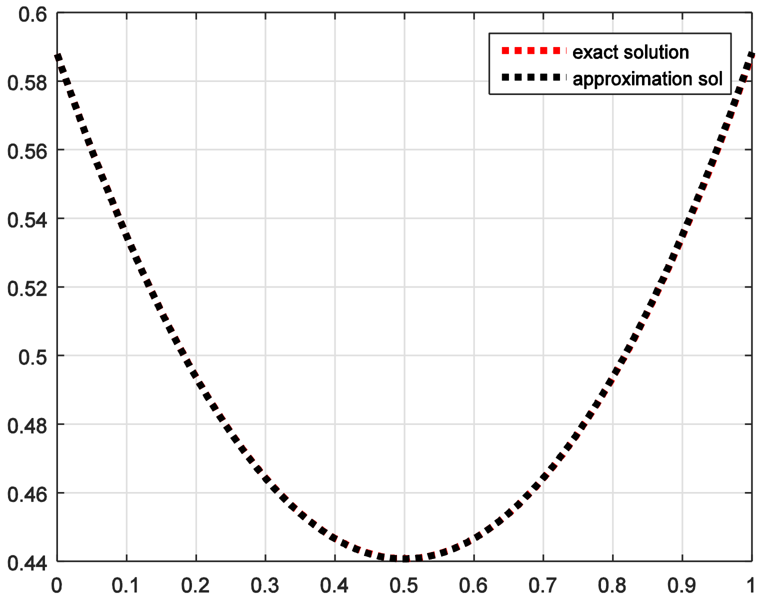

Figure 4.13. Exact and approximate solutions graph of example (4.3.1), for $M = 20, N = 400, \lambda = 5, \alpha = 0.99$, using explicit finite difference method

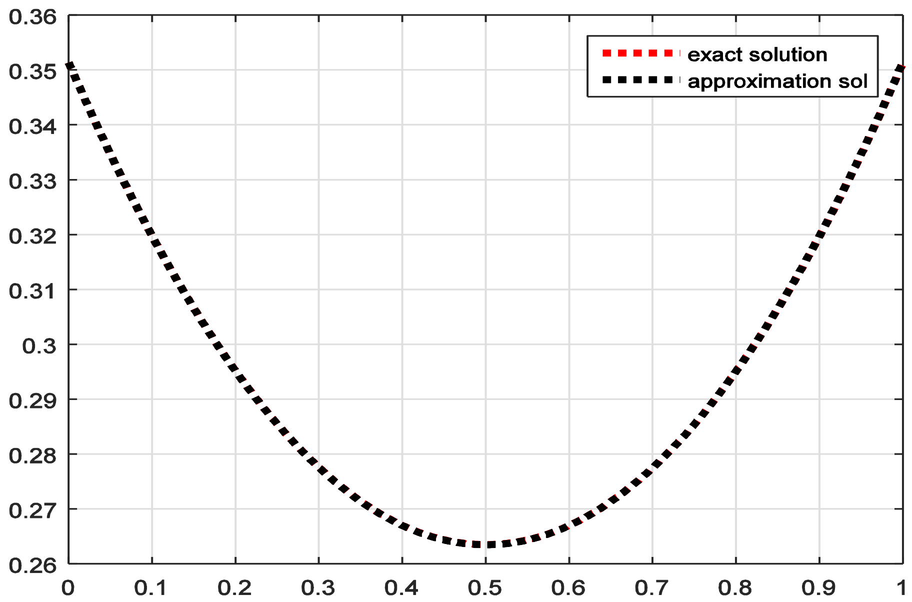

Figure 4.14. Exact and approximate solutions figure of example (4.3.1), for $M = 35, N = 1225, \lambda = 5, \alpha = 0.99$, using explicit finite difference method

Figures (4.10, and 4.11) explained the exact and approximate solutions in the same interval and parameters value for $M = 15, N = 225, \lambda = 5,$ and $\alpha = 0.9995$, using explicit finite difference method for Example 4.2.

The figures (4.12, 4.13, and 4.14) are associated with Example 4.3.1. These figures provide a comparison between exact and approximate solutions using the explicit finite difference approach. The graphs illustrate the efficacy of the technique across several parameters. For all scenarios, the value of λ remains consistent at 5, while α is 0.95 in Figure 4.12 and rises to 0.99 in Figures 4.13 and 4.14. The values of M and N, which presumably correspond to spatial and temporal discretization, incrementally rise from Figure 4.12 ($M = 10$, $N = 100$) to Figure 4.13 ($M = 20$, $N = 400$) and Figure 4.14 ($M = 35$, $N = 1225$). This sequence implies an analysis of how the accuracy of the finite difference approach is influenced by the augmentation of grid points or time steps.

5. CONCLUSION and RECOMMENDATION

5.1. Conclusion

The current study has resulted in major improvements in the numerical and analytical solution of linear and nonlinear fractional-order pseudo-hyperbolic telegraph partial differential equations (FPHTPDEs). Using the modified double Laplace transform (MDLT) method, we have derived analytical solutions for the linear FPHTPDEs based on the Caputo fractional derivative. The MDLT method has demonstrated remarkable efficiency by deriving analytical solutions in a few steps, while alternative methods frequently require additional procedures to obtain similar outcomes. The derived analytical solutions, written as closed-form expressions utilizing established mathematical functions and operations, offer crucial insights into the behavior of these fractional-order equations, which are frequently challenging to solve analytically.

In addition to the analytical technique, we have created and applied a specific numerical method called explicit finite difference scheme to solve linear and nonlinear FPHTPDEs. The suggested approach converts the fractional derivatives and governing equations into algebraic systems that can be easily implemented in computer systems. The proposed method has been extensively tested by numerical experiments and stability assessments, which have shown its dependability, precision, and effectiveness across a wide range of fractional orders. The study emphasizes the critical significance of carefully choosing suitable time intervals, spatial grid dimensions, and fractional orders to guarantee numerical stability and precision.

Furthermore, we have developed an innovative theoretical framework for solving nonlinear FPHTPDEs by creating a general form that is not dependent on the unique properties of nonlinearity. This complete approach has demonstrated its efficacy in resolving a wide spectrum of nonlinear FPHTPDEs, precisely capturing the complex nonlinear dynamics that emerge in many fields of application.

The importance of this work rests in its contributions to enhancing the comprehension and simulation of wave propagation processes with memory effects in several scientific and technical fields. The MDLT approach yields analytical solutions that can be used as benchmarks to validate numerical solutions and provide insights into the inherent behavior of fractional-order equations.

5.2 Recommendation

Given the discoveries and knowledge acquired from this research, we suggest future investigation and expansion of the suggested techniques to address more complex nonlinear fractional-order equations and enhance our comprehension of wave propagation processes in many scientific and technical domains.

Furthermore, we recommend exploring the suitability of the Atangana-Baleanu Caputo fractional derivative operator in relation to the approaches described. This operator has the potential to provide different viewpoints and valuable understanding in the modeling of fractional-order dynamics.

REFERENCES

ASHYRALYEV, A., & MODANLI, M. (2015). An operator method for telegraph partial differential and difference equations. Boundary Value Problems, 2015, 1-17. https://doi.org/10.1186/s13661-015-0302-z

AHMAD, S., ULLAH, A., AKGÜL, A., and DE LA SEN, M. (2021). A novel homotopy perturbation method with applications to nonlinear fractional order KdV and Burger equation with exponential-decay kernel. *Journal of Function Spaces*, 2021, 1-11. https://doi.org/10.1155/2021/6685284

AHMED, S. A., SAADEH, R., QAZZA, A., and ELZAKI, T. M. (2023). Modified conformable double Laplace–Sumudu approach with applications. *Heliyon*, 9(5). https://doi.org/10.1016/j.heliyon.2023.e05465

AKGÜL, A., & MODANLI, M. (2019). Crank–Nicholson difference method and reproducing kernel function for third order fractional differential equations in the sense of Atangana–Baleanu Caputo derivative. *Chaos, Solitons & Fractals*, 120, 75-84. https://doi.org/10.1016/j.chaos.2019.06.011

ALAROUD, M., ALOMARI, A. K., TAHAT, N., AL-OMARI, S., and ISHAK, A. (2023). A novel solution approach for time-fractional hyperbolic telegraph differential equation with Caputo time differentiation. *Mathematics*, 11(9), 2181. https://doi.org/10.3390/math11092181

ALIEV, A. B., and LICHAEI, B. H. (2010). Existence and non-existence of global solutions of the Cauchy problem for higher order semilinear pseudo-hyperbolic equations. *Nonlinear Analysis: Theory, Methods & Applications*, *72*(7-8), 3275-3288. https://doi.org/10.1016/j.na.2009.12.006

ATANGANA, A., AKGÜL, A., and OWOLABI, K. M. (2020). Analysis of fractal fractional differential equations. *Alexandria Engineering Journal*, *59*(3), 1117-1134.

BALEANU, D., SAJJADI, S. S., ASAD, J. H., JAJARMI, A., and ESTIRI, E. (2021). Hyperchaotic behaviors, optimal control, and synchronization of a nonautonomous cardiac conduction system. *Advances in Difference Equations*, 2021(1), 1-19. https://doi.org/10.1186/s13662-021-03320-0

BANSU, H., and KUMAR, S. (2019). Numerical solution of space and time fractional telegraph equation: A meshless approach. *International Journal of*

Nonlinear Sciences and Numerical Simulation, 20(3-4), 325-337. https://doi.org/10.1515/ijnsns-2017-0259

DEBNATH, L. (2016). The double Laplace transforms and their properties with applications to functional, integral and partial differential equations. *International Journal of Applied and Computational Mathematics*, *2*, 223-241. https://doi.org/10.1007/s40819-015-0057-3

DELIC, A., JOVANOVIĆ, B. S., and ŽIVANOVIĆ, S. (2020). Finite difference approximation of a generalized time-fractional telegraph equation. *Computational Methods in Applied Mathematics*, 20(2), 277-293. https://doi.org/10.1515/cmam-2018-0291

DHUNDE, R. R., and WAGHMARE, G. L. (2016). Double Laplace transform method for solving space and time fractional telegraph equations. *International Journal of Mathematics and Mathematical Sciences*, *2016*. https://doi.org/10.1155/2016/1414595

ENTEZARI, M., ABBASBANDY, S., and BABOLIAN, E. (2019). Numerical Solution of Fractional Partial Differential Equations with Normalized Bernstein Wavelet Method. *Applications & Applied Mathematics*, 14(2). https://digitalcommons.pvamu.edu/aam/vol14/iss2/17

FANWEI, M. (2013). A new approach for solving fractional partial differential equation. *Journal of Applied Mathematics*, 2013. https://doi.org/10.1155/2013/830985

FEDOTOV, I., SHATALOV, M., and MARAIS, J. (2016). Hyperbolic and pseudo-hyperbolic equations in the theory of vibration. *Acta Mechanica*, 227(8), 2063-2074. https://doi.org/10.1007/s00707-015-1537-6

GUO, H., and RUI, H. (2007). Least-squares Galerkin procedures for pseudohyperbolic equations. *Applied mathematics and computation*, *189*(1), 425-439. https://doi.org/10.1016/j.amc.2006.11.094

HASHMI, M. S., ASLAM, U., SINGH, J., and NISAR, K. S. (2022). An efficient numerical scheme for fractional model of telegraph equation. *Alexandria Engineering Journal*, *61*(8), 6383-6393. https://doi.org/10.1016/j.aej.2021.11.065

IZADI, M., and SRIVASTAVA, H. M. (2021). Numerical approximations to the nonlinear fractional-order Logistic population model with fractional-order Bessel and Legendre bases. *Chaos, Solitons & Fractals*, 145, 110779. https://doi.org/10.1016/j.chaos.2020.110779

JACHIMAVIČIENĖ, J., SAPAGOVAS, M., ŠTIKONAS, A., & ŠTIKONIENĖ, O. (2014). On the stability of explicit finite difference schemes for a pseudoparabolic equation with nonlocal conditions. *Nonlinear analysis: modelling and control*, 19(2), 225-240. https://doi.org/10.15388/NA.2014.2.6

JAJARMI, A., BALEANU, D., ZARGHAMI VAHID, K., and MOBAYEN, S. (2022). A general fractional formulation and tracking control for immunogenic tumor dynamics. *Mathematical Methods in the Applied Sciences*, 45(2), 667-680. https://doi.org/10.1002/mma.7804

KAI, D. (2010). The analysis of fractional differential equations: An application-oriented exposition using differential operators of Caputo type. In Lecture Notes in Mathematics. Springer. https://doi.org/10.1007/978-3-642-14574-2

KARATAY, I., KALE, N., & ERGUNER, S. R. B. (2014). Stability and convergence of a finite difference scheme for fractional partial differential equations by matrix method. In *International Mathematical Forum* (Vol. 9, No. 36, pp. 1757-1765). http://dx.doi.org/10.12988/imf.2014.411192

KHAN, A., KHAN, T. S., SYAM, M. I., and KHAN, H. (2019). Analytical solutions of time-fractional wave equation by double Laplace transform method. *The European Physical Journal Plus*, *134*(4), 163. https://doi.org/10.1140/epjp/i2019-12499-y

KHAN, H., SHAH, R., KUMAM, P., BALEANU, D., and ARIF, M. (2020). Laplace decomposition for solving nonlinear system of fractional order partial differential equations. *Advances in Difference Equations*, 2020(1), 1-19. https://doi.org/10.1186/s13662-020-02839-y

KILBAS, A. A., SRIVASTAVA, H. M., and TRUJILLO, J. J. (2006). *Theory and applications of fractional differential equations* (Vol. 204). elsevier.

KIRANE, M., and RAGOUB, L. (2015). Nonexistence results for a pseudo-hyperbolic equation in the Heisenberg group. *Electronic Journal of Differential Equations*, *110*, 1-9.

KUMAR, D., and NISAR, K. S. (2022). A novel linearized Galerkin finite element scheme with fractional Crank–Nicolson method for the nonlinear coupled delay subdiffusion system with smooth solutions. *Mathematical Methods in the Applied Sciences*. https://doi.org/10.1002/mma.7858

KUMAR, K., PANDEY, R. K., and YADAV, S. (2019). Finite difference scheme for a fractional telegraph equation with generalized fractional derivative terms. *Physica A: Statistical Mechanics and its Applications*, *535*, 122271. https://doi.org/10.1016/j.physa.2019.122271

LIAQAT, M. I., KHAN, A., and AKGÜL, A. (2022). Adaptation on power series method with conformable operator for solving fractional order systems of nonlinear partial differential equations. *Chaos, Solitons & Fractals*, 157, 111984. https://doi.org/10.1016/j.chaos.2022.111984

LIU, T., and HOU, M. (2017). A fast implicit finite difference method for fractional advection-dispersion equations with fractional derivative boundary conditions. *Advances in Mathematical Physics*, 2017. https://doi.org/10.1155/2017/8716752

LIU, Y., LI, H., WANG, J., and HE, S. (2012). Splitting positive definite mixed element methods for pseudo-hyperbolic equations. *Numerical Methods for Partial Differential Equations*, *28*(2), 670-688. https://doi.org/10.1002/num.20650

MESLOUB, S., ABOELRISH, M. R., and OBAIDAT, S. (2019). Well posedness and numerical solution for a non-local pseudohyperbolic initial boundary value problem. *International Journal of Computer Mathematics*, *96*(12), 2533-2547. https://doi.org/10.1080/00207160.2019.1578349

MILLER, K.S. and ROSS, B. (1993). An introduction to the fractional calculus and fractional differential equations. Wiley.

MODANLI, M. (2018). Two numerical methods for fractional partial differential equation with nonlocal boundary value problem. *Advances in Difference Equations*, *2018*, 1-19. https://doi.org/10.1186/s13662-018-1789-2

MODANLI, M. (2022). Comparison of Caputo and Atangana–Baleanu fractional derivatives for the pseudohyperbolic telegraph differential equations. *Pramana*, *96*(1), 7. https://doi.org/10.1007/s12043-021-02250-6

MODANLI, M., ABDULAZEEZ, S. T., and HUSIEN, A. M. (2021). A residual power series method for solving pseudo hyperbolic partial differential equations with nonlocal conditions. *Numerical Methods for Partial Differential Equations*, *37*(3), 2235-2243. https://doi.org/10.1002/num.22683

MODANLI, M., and AKGÜL, A. (2017). Numerical solution of fractional telegraph differential equations by theta-method. *The European Physical Journal Special Topics*, *226*, 3693-3703. https://doi.org/10.1140/epjst/e2018-00088-6

MODANLI, M., and BAJJAH, B. (2021). Double Laplace decomposition method and finite difference method of time-fractional Schrödinger pseudoparabolic partial differential equation with Caputo derivative. *Journal of Mathematics*, *2021*, 1-10. https://doi.org/10.1155/2021/7113205

MODANLI, M., MURAD, M. A. S., and ABDULAZEEZ, S. T. (2023). A new computational method-based integral transform for solving time-fractional equation arises in electromagnetic waves. *Zeitschrift für Angewandte Mathematik und Physik*, 74(5), 186. https://doi.org/10.1007/s00033-022-01729-5

MODANLI, M., OZBAG, F., and AKGÜL, A. (2022). Finite difference method for the fractional order pseudo telegraph integro-differential equation. *Journal of Applied Mathematics and Computational Mechanics*, *21*(1). DOI: 10.17512/jamcm.2022.1.04

MOFARREH, F., KHAN, A., SHAH, R., and ABDELJABBAR, A. (2023). A comparative analysis of fractional-order Fokker–Planck equation. *Symmetry*, 15(2), 430. https://doi.org/10.3390/sym15020430

Nisar, K. S., Ahmad, S., Ullah, A., Shah, K., Alrabaiah, H., & Arfan, M. (2021). Mathematical analysis of SIRD model of COVID-19 with Caputo fractional derivative based on real data. *Results in Physics*, 22, 103772. https://doi.org/10.1016/j.rinp.2020.103772

OLDHAM, K. B., and SPANIER, J. (1974). The fractional calculus: theory and applications of differentiation and integration to arbitrary order. Academic Press.

OSMAN, W. M., ELZAKI, T. M., and SIDDIG, N. A. (2021). Modified double conformable Laplace transform and singular fractional pseudo-hyperbolic and pseudo-parabolic equations. *Journal of King Saud University - Science*, 33(3), 101378. https://doi.org/10.1016/j.jksus.2020.08.002

OWOLABI, K. M. (2017). Robust and adaptive techniques for numerical simulation of nonlinear partial differential equations of fractional order. *Communications in Nonlinear Science and Numerical Simulation*, 44, 304–317. https://doi.org/10.1016/j.cnsns.2016.09.017

OWOLABI, K. M., ATANGANA, A., and AKGUL, A. (2020). Modelling and analysis of fractal-fractional partial differential equations: application to reaction-diffusion model. *Alexandria Engineering Journal*. https://doi.org/10.1016/j.aej.2020.03.022

OZBAG, F., and MODANLI, M. (2021). On the stability estimates and numerical solution of fractional order telegraph integro-differential equation. *Physica Scripta*, *96*(9), 094008. DOI 10.1088/1402-4896/ac0a2c

ÖZKAN, O., and KURT, A. (2020). A new method for solving fractional partial differential equations. *The Journal of Analysis*, *28*, 489-502. https://doi.org/10.1007/s41478-019-00186-0

PETER, O. J., SHAIKH, A. S., IBRAHIM, M. O., NISAR, K. S., BALEANU, D., KHAN, I., and ABIOYE, A. I. (2021). Analysis and dynamics of fractional order mathematical model of COVID-19 in Nigeria using atangana-baleanu operator. *Computers, Materials, & Continua*, 67(2), 1865-1881. https://doi.org/10.32604/cmc.2020.012314

PODLUBNV, I. (1999). *Fractional differential equations academic press*. San Diego, Boston, 6.

PODLUBNY, I. (1998). *Fractional differential equations: an introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications*. Elsevier.

RASHID, S., ASHRAF, R., and JARAD, F. (2022). Strong interaction of Jafari decomposition method with nonlinear fractional-order partial differential equations arising in plasma via the singular and nonsingular kernels. *AIMS Mathematics*, 7(5), 7936–7963. https://doi.org/10.3934/math.2022508

SALEH, R., KASSEM, M., and MABROUK, S. M. (2019). Exact solutions of nonlinear fractional order partial differential equations via singular manifold method. *Chinese Journal of Physics*, 61, 290-300. https://doi.org/10.1016/j.cjph.2018.12.004

SHAH, F. A., IRFAN, M., NISAR, K. S., MATOOG, R. T., and MAHMOUD, E. E. (2021). Fibonacci wavelet method for solving time-fractional telegraph equations with Dirichlet boundary conditions. *Results in Physics*, 24, 104123. https://doi.org/10.1016/j.rinp.2021.104123

SINGH, J., KUMAR, D., and KILIÇMAN, A. (2014). Numerical solutions of nonlinear fractional partial differential equations arising in spatial diffusion of biological populations. *In Abstract and Applied Analysis* (Vol. 2014). Hindawi. https://doi.org/10.1155/2014/535793

TVERDYI, D., and PAROVIK, R. (2021). Investigation of finite-difference schemes for the numerical solution of a fractional nonlinear equation. *Fractal and Fractional*, 6(1), 23. https://doi.org/10.3390/fractalfract6010023

XU, J., KHAN, H., SHAH, R., ALDERREMY, A. A., ALY, S., and BALEANU, D. (2021). The analytical analysis of nonlinear fractional-order dynamical models. *AIMS Mathematics*, 6(6), 6201-6219. https://doi.org/10.3934/math.2021364

YILDIRIM, Ö. (2007). *Multipoint nonlocal boundary value problems for hyperbolic equations* (Master's thesis, Fen Bilimleri Enstitüsü). https://acikbilim.yok.gov.tr/handle/20.500.12812/616104

ZADA, L., NAWAZ, R., AHSAN, S., NISAR, K. S., and BALEANU, D. (2020). New iterative approach for the solutions of fractional order inhomogeneous partial differential equations. *AIMS Mathematics*, 5(6), 5537-5553. https://doi.org/10.3934/math.2021084

ZHAO, Z., and LI, H. (2019). A continuous Galerkin method for pseudo-hyperbolic equations with variable coefficients. *Journal of Mathematical Analysis and Applications*, *473*(2), 1053-1072. https://doi.org/10.1016/j.jmaa.2019.01.008

Research Fields:

Applied Mathematics, Mathematical Modeling, Numerical Analysis, Fractional Differential Equations, Partial Differential Equations, Mathematical Statistics

PUBLISHED PAPERS:

1. Abdulazeez, S. T., & Modanli, M. (2022). Solutions of fractional order pseudo-hyperbolic telegraph partial differential equations using finite difference method. *Alexandria Engineering Journal*, *61*(12), 12443-12451.
2. Abdulazeez, S. T., & Modanli, M. (2023). Analytic solution of fractional order Pseudo-Hyperbolic Telegraph equation using modified double Laplace transform method. *International Journal of Mathematics and Computer in Engineering*, *1*(1), 105-114.
3. Modanli, M., Karadag, K., & Abdulazeez, S. T. (2023). Solutions of the mobile–immobile advection–dispersion model based on the fractional operators using the Crank–Nicholson difference scheme. *Chaos, Solitons & Fractals*, *167*, 113114.
4. ABDULAZEEZ, S. T., MODANLI, M., & HUSIEN, A. M. (2023). Solutions to nonlinear pseudo hyperbolic partial differential equations with nonlocal conditions by using residual power series method. *Sigma*, *41*(3), 488-492.
5. Rasheed, S. K., Modanli, M., & Abdulazeez, S. T. (2023). Stability analysis and numerical implementation of the third-order fractional partial differential equation based on the Caputo fractional derivative. *J. Appl. Math. Comput. Mech.*, *22*(3), 33-42.
6. Modanli, M., Murad, M. A. S., & Abdulazeez, S. T. (2023). A new computational method-based integral transform for solving time-fractional equation arises in electromagnetic waves. *Zeitschrift für angewandte Mathematik und Physik*, *74*(5), 186.
7. Tenekeci, M. E., Abdulazeez, S. T., Karadağ, K., & Modanli, M. (2024). Edge detection using the Prewitt operator with fractional order telegraph partial differential equations (PreFOTPDE). *Multimedia Tools and Applications*, 1-17.
8. Abdulla, S. O., Abdulazeez, S. T., & Modanli, M. (2023). Comparison of third-order fractional partial differential equation based on the fractional operators using the explicit finite difference method. *Alexandria Engineering Journal*, *70*, 37-44.

9. Modanli, M., & Goksu, H. (2023). SOLUTIONS OF FRACTIONAL ORDER TIME-VARYING LINEAR DYNAMICAL SYSTEMS USING THE RESIDUAL POWER SERIES METHOD. *Honam Mathematical Journal*, *45*(4), 619-628.
10. Kirkpinar, S., Abdulazeez, S. T., & Modanli, M. (2022). Piecewise Modeling of the Transmission Dynamics of Contagious Bovine Pleuropneumonia Depending on Vaccination and Antibiotic Treatment. *Fractals*, *30*(08), 2240217.
11. Abdulazeez, S. T., Modanli, M., & Husien, A. M. (2022). Numerical scheme methods for solving nonlinear pseudo-hyperbolic partial differential equations. *Journal of Applied Mathematics and Computational Mechanics*, *21*(4).
12. Abdulazeez, S. T., Mohammed, S. A., & Modanli, M. (2019). A comparison between applications of the Lyapunov’s second (direct) method and fixed point theory. *Gene Lett. Math*, *7*, 61-78.
13. Mohammed, S. A., Abdulazeez, S. T., & Malo, D. H. (2019). Stability in delay functional differential equation established using the Banach fixed point theorem. *international Journal of advanced Trends in computer Science and Engineering*, *8*(6).
14. Modanli, M., Abdulazeez, S. T., & Husien, A. M. (2021). A residual power series method for solving pseudo hyperbolic partial differential equations with nonlocal conditions. *Numerical Methods for Partial Differential Equations*, *37*(3), 2235-2243.
15. Nisar, K. S., Ilhan, O. A., Abdulazeez, S. T., Manafian, J., Mohammed, S. A., & Osman, M. S. (2021). Novel multiple soliton solutions for some nonlinear PDEs via multiple Exp-function method. *Results in Physics*, *21*, 103769.
16. Malo, D. H., Masiha, R., Murad, M., & Abdulazeez, S. (2021). A new computational method based on integral transform for solving linear and nonlinear fractional systems. *Jurnal Matematika MANTIK*, *7*(1), 9-19.
17. Abdulazeez, S. T., & Hussein, A. M. (2021). The Existence of a Polynomial Inverse Integrating Factors and Studies About the Limit Cycles for Cubic, Quartic and Quintic Polynomial Systems. *Baghdad Science Journal*, *18*(2), 0322-0322.
18. Ilhan, O. A., Abdulazeez, S. T., Manafian, J., Azizi, H., & Zeynalli, S. M. (2021). Multiple rogue and soliton wave solutions to the generalized Konopelchenko–Dubrovsky–Kaup–Kupershmidt equation arising in fluid mechanics and plasma physics. *Modern Physics Letters B*, *35*(23), 2150383.
19. Saeed, D. S., Abdulazeez, S. T., Rasheed, S. K., Masiha, R. Y., & Malo, D. H. (2021). The relationship between petroleum price and real exchange rate: an example of Iraq. *General Letters in Mathematics (GLM)*, *11*(1).
20. Modanli, M., Abdulazeez, S. T., & Akgül, A. (2022). Explicit Finite Difference Analysis of Pseudo-Hyperbolic Partial Differential Equations.

21. Hassan, Z. M., Abdulazeez, S. T., & Rasheed, S. K. (2023). Investigating Solutions of Volterra Integral Equations Using the Successive Approximations. *Mathematical Statistician and Engineering Applications*, *72*(2), 75-82.
22. Masiha, R. Y., Abdulazeez, S. T., & Saeed, D. S. (2021). Investigation of the Relationship Between Economic Growth and use of Fossil and Hydroelectric Energy Resources by ARDL Boundary Test: 1971-2018 Iraq Case. *Jurnal Matematika MANTIK*, *7*(2), 155-164.
23. Abdulla, J. A. W., Rasheed, H. H., Hassan, Z. M., & Abdulazeez, S. T. (2023). Stability Analysis of First Order Integro-Differential Equations With the Successive Approximation Method. Advances in Nonlinear Variational Inequalities, Vol 26, No. 2, 2023.
24. Othman, S. A., Qadir, S. T., & Abdulazeez, S. T. (2023). Comparative Analysis of Scaling Parameter Estimation for the Lindley Distribution in Wait Time Analysis: Simulated and Real Data Applications. *Tuijin Jishu/Journal of Propulsion Technology*, *44*(2), 2023.
25. Othman, S. A., Omar, K. M. T., & Abdulazeez, S. T. (2023). Reliability Analysis and Statistical Fitting for the Transmuted Weibull Model in R. *Mathematical Statistician and Engineering Applications*, *72*(2), 161-177.
26. The stability analysis of the system of integrodifferential equations, (2024). *Journal of Mathematical Analysis*, Vol. 15, No. 1 (2024), PP: 28-39.
27. Othman, S.A., Jameel, H.H., TAHER OMAR, K.M. and Abdulazeez, S.T., 2023. an efficient nonparametric multivariate functional regression with conditional expectation. *Journal of Applied Probability & Statistics*, *18*(3).
28. Othman, S.A., Jameel, H.H. and Abdulazeez, S.T., 2024. Comparative Analysis of Univariate SARIMA and Multivariate VAR Models for Time Series Forecasting: A Case Study of Climate Variables in Ninahvah City, Iraq. *Mathematical Problems of Computer Science*, *61*, pp.27-49.

Conferences:

- First International Conference of Mathematics with Mathematics Department, College of Education and College of Science, University of Sulaimani, Sulaimani-Iraq (24/09/2020 - 25/09/2020).
- International Conference on Data Science, Machine Learning and Statistics 2019 (DMS-2019), Computer Science Research and Application Center, Van Yüzüncü Yıl University, Van-Turkey (26/06/2019 - 29/06/2019).

- 5th International Conference on Mathematics and Related Sciences (ICMRS 2022), Sponsored by Duzce University, Antalya-Turkey (27/10/2022-30/10/2022).

FOREIGN LANGUAGES

Kurdish: Mother tongue

English: Very good

Arabic: Excellent

Turkish: Medium

APPENDICES

MATLAB CODE: Matlab code of a first-order accurate difference scheme, indicating that this approach is connected to solving the linear fractional-order pseudo-hyperbolic telegraph PDEs numerically using finite difference methods with different levels of precision.

```
“% close;close;
% M=400; N=400;
% tau=1/N;
% h=pi/M;
% y=0.2;
% v=0.00001;
% for k=1:N+1;
% m=(tau^(-y)/gamma(2-y));
% end;
% a=(v/(tau*(h^2)));
% r=(-v/(tau*(h^2))-1/(h^2));
% b=(1/(tau^2));
% c=(-2/(tau^2)-m-2*v/(tau*(h^2))+1+2/(h^2));
% d=(2*v/(tau*(h^2))+1/(tau^2)+m);
% for i=2:N; A(i,i-1)=a;A(i,i)=r; end;
% A(N+1,N+1)=0;A;
% C=A;
% K=zeros(N+1);
% for i=1:N+1;f(i)=((i)^(1-y))-((i-1)^(1-y));
% end;
% for i=2:N;K(i,i-1)=b;end;
% for i=2:N;K(i,i)=c;end;
% for i=2:N;K(i,i+1)=d;end;
% K(1,1)=1;K(N+1,1)=-1/tau; K(N+1,2)=1/tau;
% F=zeros(N+1);
% for i=2:N;
% F(i,1)=(-1)*f(i);
```

```
% end;
% for i=2:N;
% F(i,i)=f(2);
% end;
% for i=2:N-1;
% F(i+1,2)=f(i+1)-f(i);
% end;
% for j=3:N;
% for i=j-1:N;
% F(i,j)=F(i-1,j-1);F;
% end;
% end;
% B=(((tau)^(-y))/gamma(2-y))*F+K;
% for i=1:N+1;D(i,i)=1;end;D;
% 'fii(j) finding';
% for j=1:M+1;
% x=((j)*h);
% fii(1,j:j)=sin(x);
% fii(N+1,j:j)=0;
% for k=2:N;
% t=(k)*tau;
%fii(k,j)=sin(x)*(2+2*(t^(2-y))/gamma(3-
y)+2*(t^2+1)+2*v*t);
% end;
% end;
% alpha(N+1,N+1,1)=0;
% betha(N+1,1)=0;
% for j=1:M;
% alpha(:,:,j+1)=inv(B+C*alpha(:,:,j))*(-A);
%betha(:,j+1)=inv(B+C*alpha(:,:,j))*(D*fii(:,j)-
C*betha(:,j));
% end;
% U(N+1,1,M)=0;
% for z=M-1:-1:1;
```

```
% U(:,:,z)=alpha(:,:,z+1)*U(:,:,z+1)+betha(:,z+1);
% end;
% for z=1:M;
% p(:,z+1)=U(:,:,z);
% end;
% 'EXACT SOLUTION OF THIS PROBLEM';
% for j=1:M+1;
% for k=1:N+1;
% x=((j-1)*h);
% t=(tau*(k-1));
% es(k,j)=(t.^2+1)*sin(x);
% end;
% end;
% es;
% 'ERROR ANALSIS';
% maxes=max(max(abs(es)));
% maxapp=max(max(abs(p)));
% maxerror=max(max(abs(maxes-maxapp)));
% relativeerror=max(max(abs(es-p)))/max(max(abs(p)));
% Answer=[maxes,maxapp,maxerror,relativeerror]
% %%%%%%%%%GRAPH OF THE SOLUTION %%%%%%%%%%
%   x =0:tau:1;
%  y2=p(:,N:N);
%  y1=es(:,N:N);
%  plot(x,y1,'--red') %red
% hold on
% plot(x,y2,':k','Linewidth',3) %black
% grid
% legend('approximation sol','exact solution')
% %  x =0:tau:1;
% %  y2=p(:,50:50);
% %  y1=es(:,50:50);
% %  plot(x,y1,'-k','LineWidth',1)
% %  hold on
```

```
% %  plot(x,y2,'-r','LineWidth',1)
% %  grid
% % legend('approximation sol','exact solution')
% % hold on
% % x =0:tau:1;
% % y1=es(:,50:50);
% % y2=p(:,50:50);
% % plot(x,y1,'-r','LineWidth',1.5)
% % grid
% % hold on
% % plot(x,y2,'-g','LineWidth',1.5)
% % xlabel('x axis');ylabel('t axis');
% % title('N=M=100') ;
% hold on
%  U;
%  es;
%  p;
%   [xler,tler]=meshgrid(0:h:pi,0:tau:1);
%  table=[es;p];
%  table(1:2:end,:)=es; table(2:2:end,:)=p;
%  q=min(min(table));
%  w=max(max(table));
%  figure;  surf(xler,tler,es);  title('EXACT  SOLUTION');
set(gca,'ZLim',[q w]);
% rotate3d;XLabel('x axis');YLabel('t axis');
%figure;    surf(xler,tler,abs(p));    title('APPROXIMATE
SOLUTION'); rotate3d ; set(gca,'ZLim',[q w]);
% XLabel('x axis');YLabel('t axis');
% % %%%%%%%%%%%%%%%% END GRAPH %%%%%%%%%%%%%%”.
```

MATLAB CODE: Matlab code for solving the nonlinear fractional-order pseudo-hyperbolic telegraph PDEs numerically using finite difference methods with different levels of precision.

```
‶% close;close;
% N=25;M=5;
% y=0.9;
% z=3;
% tinit=0.;
% tmax=1;
% xinit=0.;
% xmax=pi;
% % Number of Time Steps
% dt =(tmax-tinit)/N;
% dx=(xmax-xinit)/M;
% kat1=((dt)^(-y))/(gamma(2-y));
% % a=(-z/(dt*(dx^2)));
% % r=(z/(dt*(dx^2))+1/(dx^2));
% % for i=2:N; A(i,i-1)=a;A(i,i)=r; end
% % A(N+1,N+1)=0;A;
% % C=A;
% % Initial Condition
% for k=1:N+1;
%for j=1:M;
% x(j)=xinit+(j-1)*dx;
% t(k)=tinit+(k-1)*dt;
% u(1,j)=sin(x(j));
% u(2,j)=(1-dt)*sin(x(j));
% u(k,1)=0;
% u(k,M+1)=0;
% end;
% end;
% for k=1:N+1;
% for j=1:M+1;
% x(j)=xinit+(j-1)*dx;
%  t(k)=tinit+(k-1)*dt;
% end
% end
```

```
% for i=1:N+1;g(i)=((i)^(1-y))-((i-1)^(1-y));
% end;
% G=zeros(N+1);
% for i=2:N;
% G(i,1)=kat1*(-1)*g(i);
% end;
% for i=2:N;
% G(i,i)=kat1*g(2);
% end;
% for i=2:N-1;
% G(i+1,2)=kat1*(g(i+1)-g(i));
% end;
% for j=3:N;
% for i=j-1:N;
% G(i,j)=G(i-1,j-1);G;
% end;
% end;
% w1=(2^(1-y))-1;
% for k=2:N;
% for j=2:M;
%   u(k+1,j)=(1/((1/(dt^2))+kat1))*((((2/dt^2)+kat1-u(k,j)
...
%-w1*kat1-(2*z/(dt*(dx^2)))-(2/dx^2))*u(k,j)+(w1*kat1-
1/(dt^2)+2*z/(dt*dx^2))*u(k-1,j)-G(i,j) ...
+(-z/(dt*dx^2))*(u(k-1,j-1)+u(k-
1,j+1))+(z/(dt*dx^2)+1/(dx^2))*(u(k,j-1)+u(k,j+1))...
%+(2+2*(t(k)^(2-y))/gamma(3-y)-(t(k)^(1-y))/gamma(2-
y)+z*(2*t(k)-1)+((t(k)^2-t(k)+1)^2)...
%    *sin(x(j))+(t(k)^2-t(k)+1))*sin(x(j))));
%   % D=(u(k,j));
% end;
% end;
% %for j=2:M;
% %  syms j;
```

```
% % S=symsum(D,j,2,M);
% % end
% % for k=1:N+1;
% % u(k,M+1)=(S*dx+(3/4)*exp(t(k)))*((1-dx/2)^(-1));
% % end;
% 'EXACT SOLUTION OF THIS PROBLEM';
% for j=1:M+1;
% for k=1:N+1;
% x=((j-1)*dx);
% t=(dt*(k-1));
% es(k,j:j)=(t^2-t+1)*sin(x);
% end;
% end;
% es;
% %'ERROR ANALSIS';
% maxes=max(max(abs(es)));
% maxapp=max(max(abs(u)));
% maxerror=max(max(abs(es-u)));
% relativeerror=max(max((es-u)))/max(max(abs(u)));
% cevap=[maxes,maxapp,maxerror,relativeerror]
% %%%%%%%%%%GRAPH OF THE SOLUTION %%%%%%%%%%
% es;
% u;
% [xler,tler]=meshgrid(0:dx:pi,0:dt:1);
% table=[es;u];
% table(1:2:end,:)=es; table(2:2:end,:)=u;
% q=min(min(table));
% w=max(max(table));
%   %         figure;   surf(xler,tler,es);   title('EXACT
SOLUTION'); set(gca,'ZLim',[q w]);
% % rotate3d;XLabel('x axis');YLabel('t axis');
%       figure;   surf(xler,tler,u);   title('APPROXIMATE
SOLUTION'); set(gca,'ZLim',[q w]);
%  rotate3d ; XLabel('x axis');YLabel('t axis');
```

```
% %%%%%%%%%%%%%% END GRAPH %%%%%%%%%%%%
% % t=0:dt:1;
% %  y2=u(:,5:5);
% %  y1=es(:,5:5);
% % plot(t,y1,':r','Linewidth',3) %red
% % hold on
% % plot(t,y2,':k','Linewidth',3) %black
% % grid
% % legend('exact solution','approximation sol')
% % %%%%%%%%%%%%%%% END GRAPH %%%%%%%%%%%%”.
```