COMPUTATIONS ON COXETER GROUPS

by

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ABSTRACT

COMPUTATIONS ON COXETER GROUPS

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Gröbner-Shirshov bases of certain Coxeter groups are computed. Using Gröbner-Shirshov bases, reduce forms and their multiplications are obtained for these Coxeter groups.

Keywords: Coxeter Groups, Gröbner-Shirshov Bases, Reduced Forms.

ÖZET

COXETER GRUBLARI ÜZERİNE HESABLAMALAR

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Bazı Coxeter gruplarının Gröbner-Shirshov tabanları hesablandı. Gröbner-Shirshov tabanları kullanılarak bu grubların indirgenmiş formları ve onların çarpımları elde edildi.

Anahtar Kelimeler: Coxeter Grubları, Gröbner-Shirshov Tabanları, İndirgenmiş Formlar.

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CHAPTER 1

INTRODUCTION

1.1 Finitely Presented Groups

Definition 1.1. Let A be an alphabet, that is, a set of symbols. Consider $A \cup A^{-1}$, where A^{-1} is the set of symbols a^{-1} disjoint from A, one for each $a \in A$. The free group $F(A)$ consists of all words in $A \cup A^{-1}$ without occurrences of the kind aa^{-1} or $a^{-1}a$ for $a \in A$. Such words will be called reduced. Multiplication on $F(A)$ is given by concatenation followed by removal of all forbidden occurrences so as to obtain a reduced word, and in which empty word 1 is the identity element.

Definition 1.2. A group presentation $< A \mid R >$ is made up of a set A of generators and a set R of relations. Here a relation is an expression of the form $w_1 = w_2$, where w_1 and w_2 are elements of the free group $F(A)$. The group $< A \mid R >$ is then defined as the quotient group of $F(A)$ on the generating symbol from A by the normal subgroup generated by all $w_1 w_2^{-1}$ for each expression $w_1 = w_2$ occurring in R. A group presentation $< A \mid R >$ is said to be finite if both A and R is finite. It is said to be a presentation of G if G is a group isomorphic to $< A \mid R >$. The group is called finitely presented if it has a finite presentation.

Example 1.1. Let $m \in N$. By Dih_{2m} we denote the group with presentation

$$\langle \{a, b\} | a^2 = 1, b^2 = 1, (ab)^m = 1 \rangle$$

Set $c = ab$. It is easy to claim that, as a set

$$Dih_{2m} = \{\ a^i c^j \ |\ i \in \{0, 1\}, j \in \{0, 1, 2, ..., m-1\}\}$$

Since

$$c^k a = (ab)^k a = a(ba)^k = a(b^{-1}a^{-1})^k = a((ab)^{-1})^k = ac^{-k} = ac^{m-k}$$

each element is of the form $a^i c^j$ for some $i, j \in N$. Now, keeping into account that $a^2 = 1$ and $c^m = 1$, We see that $i \in \{0, 1\}$ and $j \in \{0, 1, 2, ..., m-1\}$

1.2 Coxeter Groups

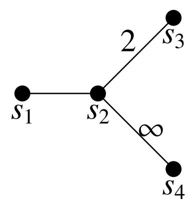
In this section we follow the notation given in [1].

Definition 1.3. A $n \times n$ symmetric matrix M whose elements are positive integers or positive infinity is called a Coxeter matrix if it satisfies $m_{ij} = 1$ if and only if $i = j$.

Definition 1.4. Let $S = \{s_1, s_2, \ldots, s_n\}$. A Coxeter matrix can be presented by a Coxeter graph whose nodes are elements of S and whose edges are unordered pairs $\{s_i, s_j\}$ such that $m_{ij} \geq 3$. The edges with $m_{ij} \geq 4$ are weighted by $m_{ij} - 2$. For instance

$$\begin{pmatrix} 1 & 3 & 2 & 2 \\ 3 & 1 & 4 & \infty \\ 2 & 4 & 1 & 2 \\ 2 & \infty & 2 & 1 \end{pmatrix}$$

$$\Updownarrow$$



Definition 1.5. Let M be a Coxeter matrix. A Coxeter group is a free group with

presentation

$$< S|(s_i s_j)^{m_{ij}} = 1 >$$

Since $m_{ii} = 1$, we have that $s_i^2 = 1$. It is easy to show that for $i \neq j$ and $m_{ij} \neq \infty$, $(s_i s_j)^{m_{ij}} = 1$ is equivalent to

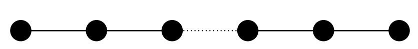
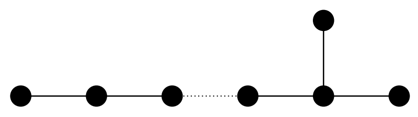
$$s_i s_j s_i s_j \cdots = s_j s_i s_j s_i \cdots$$

where length both sides is m_{ij}.

For example, the group determined by the above Coxeter diagram has a presentation.

$$< \{s_1, s_2, s_3, s_4\}, s_1^2 = s_2^2 = s_3^2 = s_4^2 = 1, s_1 s_2 s_1 = s_2 s_1 s_2, s_1 s_3 = s_3 s_1, s_2 s_3 s_2 s_3 = s_3 s_2 s_3 s_2 >$$

Coxeter groups arise naturally in several areas of mathematics. They are studied in Lie theory, commutative algebra, representation theory, combinatoric, and geometric group theory. In this thesis, we restrict ourselves to the finite Coxeter groups. The finite Coxeter groups where classified in [1], in terms of Coxeter diagrams. They are all represented by reflection groups of finite-dimensional Euclidean spaces. The following table gives graphics of finite Coxeter groups of type A_n, B_n and D_n.

Name	Diagram
A_n $(n \geq 1)$	
B_n $(n \geq 2)$	
D_n $(n \geq 4)$	

1.3 Gröbner-Shirshov Bases

The Gröbner basis theory for commutative algebras was introduced by Buchberger [4]. A parallel theory of Gröbner bases was developed for Lie algebras by Shirshov

[8]. The key ingredient of the theory is the so-called composition lemma which characterizes the leading terms of elements in the given ideal. Later, Bokut noticed that Shirshov's method works for associative algebras as well [2]. Thus this theory in non-commutative setting is called the Gröbner-Shirshov basis theory. In the following, we give some fundamental facts about this important subject.

Definition 1.6. Let S be a linearly ordered set and $F(G) =< S \mid R >$ be a finitely presented group. Each relation $u = v$ can be presented by a polynomial $f = u - v$. We impose the degree lexicographic order (compare two words first by length and the lexicographically) on words. Hence any polynomial f has a leading word $\bar{f}$. We say that f is monic if $\bar{f}$ has coefficient 1. A word s is called R-reduced if $s \neq a\bar{f}b$ for any words a, b and $f \in R$.

Definition 1.7. Let f and g be two monic polynomials. A composition of intersection of the polynomials f and g over some word w is $< f, g >= fb - ag$ where $w = \bar{f}b = a\bar{g}$.

Definition 1.8. Let f and g be two monic polynomials such that $w = \bar{f} = a\bar{g}b$ for some words a and b. A composition of inclusion of the polynomial f and g is $< f, g >_w = f - agb$. The transformation $f \mapsto f - agb$ is called the elimination of the leading word (ELW) of g in f.

Definition 1.9. For two polynomials f and g, we say f is equivalent to g relative to R, write $f \equiv g \mod R$, if g is R-reduced word and it can be obtained from f by a sequence of ELW's. If g is zero polynomial we say f is trivial R.

Definition 1.10. A set of relations R of a finitely presented group $G =< S \mid R >$ is called a Gröbner-Shirshov basis of G if any composition of polynomials from R is trivial relative to R.

At this point, we prove a result about triviality of certain compositions.

Lemma 1.1. *Let $f = \bar{f} - \tilde{f}$ and $g = \bar{g} - \tilde{g}$ be two polynomials. The composition of intersection of f and g $< f, g >_w$ is trivial if $w = \bar{f}\bar{g}$.*

Proof.

$$< f, g >_w = (\bar{f} - \tilde{f})\bar{g} - \bar{f}(\bar{g} - \tilde{g}) = \tilde{f}(\bar{g} - \tilde{g}) - (\bar{f} - \tilde{f})\tilde{g}.$$

Hence we only have to consider the compositions $< f, g >_w$ with $w \neq \bar{f}\bar{g}$. □

Definition 1.11. If R is not a Gröbner-Shirshov basis for $G =< S \mid R >$ then one can add to R all nontrivial compositions of polynomials of R, and continue this process until all compositions are trivial. This procedure is called the Buchberger-Shirshov algorithm.

Definition 1.12. A Gröbner-Shirshov basis R is called reduced if there is no composition of inclusion in R.

If $\langle f, g\rangle_w$ is a composition of inclusion in R, then $R \setminus \{r\}$ is still a Gröbner basis. In this way we can obtain a reduced Gröbner-Shirshov basis.

Gröbner-Shirshhov bases of Coxeter groups of the type A_n, B_n and D_n is obtained by Bokut and Sharer in [3]. They order generating sets as $r_1 < r_2 < \cdots < r_n$. They used famous composition-Diamond Lemma in order to prove the set is a Gröbner-Shirshov basis. We will use only Buchberger-Shirshov algorithm to obtain Gröbner-Shirshov bases. Furthermore, we order generating set as $r_1 > r_2 > \cdots > r_n$ which more natural than their order. Our reduced words will be different than they found because of the order we choose. We claim that our reduced words are more suitable for understanding of structures of these groups. Gröbner-Shirshhov bases of exceptional finite Coxeter groups is found in [7] and [9]. The first example of finding a Gröbner-Shirshov basis of an infinite Coxeter group is given in [10].

The famous Composition Diamond Lemma implies that reduced words with respect to Gröbner-Shirshhov basis is a set of normal forms of the finitely presented group (see [7]). A group can have several set of normal forms. The most common normal form for the elements of a Coxeter group is the lexicographically first reduced word. This normal form is called reduced form. The properties of the reduced form are not only useful for computations but are also interesting and elegant from a combinatorial point of view (see [1]). Because of our choice of order, we will obtain the reduced forms of Coxeter groups. In addition, we obtain some results about multiplication of these reduced forms. Thus, we completely reveal the groups.

CHAPTER 2

COXETER GROUPS OF TYPE A_n

Let n be a positive integer. Define the following words

$$r_{ij} = \begin{cases} r_i r_{i+1} \cdots r_j, & 1 \le i < j \le n, \\ r_i r_{i+1} \cdots r_n r_{n-1} \cdots r_{2n-j}, & 1 \le i \le n < j \le 2n, \\ r_{2n-i} r_{2n-(i+1)} \cdots r_{2n-j}, & n \le i < j < 2n, \\ 1, & j < i, \\ r_i, & i = j. \end{cases}$$

Definition 2.1. Coxeter group of type A_n ($n \ge 1$) is generated by $r_1, r_2, \cdots, r_n$ with defining relations:

$R1 : r_i^2 = 1$ where $1 \le i \le n$,

$R2 : r_i r_j = r_j r_i$ where $1 \le i < j - 1 \le n - 1$ and

$R3 : r_i r_{i+1} r_i = r_{i+1} r_i r_{i+1}$ where $1 \le i \le n - 1$.

Hence A_n has a presentation $A_n(r_1, \ldots r_n) = \langle \{r_1, \ldots, r_n\} \mid R1, R2, R3 \rangle$.

2.1 Gröbner-Shirshov Basis of A_n

Proposition 2.1. *The reduced Gröbner-Shirshov basis of Coxeter group A_n contains the following polynomials:*

$f_1^{(i)} = r_i r_i - 1$ *where* $1 \le i \le n$,

$f_2^{(i,j)} = r_i r_j - r_j r_i$ *where* $1 \le i \le j - 1 \le n - 1$,

$f_3^{(i)} = r_i r_{i+1} r_i - r_{i+1} r_i r_{i+1}$ *where* $1 \le i \le n - 1$ *and*

$g^{(i,j)} = r_{ij}r_i - r_{i+1}r_{ij}$ *where* $1 \le i < j-1 \le n-1$.

Proof. The first three polynomials comes from defining relations of A_n. We have to obtain $g^{(i,j)}$ from some compositions of the first three polynomials.

$$< f_3^{(i)}, f_2^{(i,i+2)} > = (r_i r_{i+1} r_i - r_{i+1} r_i r_{i+1}) r_{i+2} - r_i r_{i+1} (r_i r_{i+2} - r_{i+2} r_i)$$
$$= r_i r_{i+1} r_{i+2} r_i - r_{i+1} r_i r_{i+1} r_{i+2}$$
$$= g^{(i,i+2)} \qquad \text{where} \quad 1 \le i \le n-2.$$

$$< g^{(i,j)}, f_2^{(i,j+1)} > = (r_{ij} r_i - r_{i+1} r_{ij}) r_{j+1} - r_{ij} (r_i r_{j+1} r_{j+1} r_i)$$
$$= r_{i,j+1} r_i - r_{i+1} r_{i,j+1}$$
$$= g^{(i,j+1)} \qquad \text{where} \quad 1 \le i < j-1 \le n-2.$$

□

Let $R = \{f_1^{(i)}, f_2^{(i)}, f_3^{(i,j)}, g^{(i,j)}\}$. At this point, we are not claiming that R is a reduced Gröbner-Shirshov basis for A_n. The following results ease to prove this fact.

Lemma 2.1. *If* $1 \le i \le t < j \le n$, *then* $r_{ij} r_t \equiv r_{t+1} r_{ij} \mod R$.

Proof.

$r_{ij} r_t \equiv r_{i,t-1} r_{tj} r_t,$

$\equiv r_{i,t-1} r_{t+1} r_{tj}$ by an ELW of g or f_3,

$= r_{t+1} r_{ij}$ by a sequence of ELWs of f_2.

□

Proposition 2.2. *Let* $1 \le i \le j \le n$ *and* $1 \le k \le l \le n$. *If* $i > k$ *then the word* $r_{ij} r_{kl}$ *is reduced. Otherwise*

$$r_{ij} r_{kl} \equiv \begin{cases} r_{k+1,l+1} r_{ij} & l < j, \\ r_{k+1,l} r_{i,j-1} & k \le j \le l, \\ r_{il} & j = k-1, \\ r_{kl} r_{ij} & j < k-1 \end{cases} \mod R.$$

Proof.

Case 1 : If $l < j$, then $r_{ij}r_{kl} \equiv r_{k+1,l+1}r_{ij}$ by Lemma 2.1.

Case 2 : If $k \leq j \leq l$, then

$$\begin{aligned} r_{ij}r_{kl} &\equiv r_{ij}r_{k,j-1}r_{jl} \\ &\equiv r_{k+1,j}r_{i,j-1}r_{jl} \quad \text{by Case 2} \\ &\equiv r_{k+1,j}r_{i,j-1}r_{j+1,l} \quad \text{by an ELW of } f_1 \\ &\equiv r_{k+1,l}r_{i,j-1} \quad \text{by a sequence of ELWs of } f_2. \end{aligned}$$

Case 3 : If $j = k - 1$, then $r_{ij}r_{j+1,l} \equiv r_{il}$.

Case 4 : If $k > j + 1$, $r_{ij}r_{kl} \equiv r_{kl}r_{ij}$ by a sequence of ELWs of f_2. □

Theorem 2.1. *The set $R = \{f_1^{(i)}, f_2^{(i)}, f_3^{(i,j)}, g^{(i,j)}\}$ is a reduced Gröbner-Shirshov basis for A_n.*

Proof. We have to show that all compositions of $R = \{f_1^{(i)}, f_2^{(i)}, f_3^{(i,j)}, g^{(i,j)}\}$ are trivial relative to R.

$$\begin{aligned} < f_1^{(i)}, f_2^{(i,j)} > &= (r_ir_i - 1)r_j - r_i(r_ir_j - r_jr_i) \\ &= r_ir_jr_i - r_j \\ &= f_2^{(i,j)}r_i - r_jf_1^{(i)} \qquad \text{where} \quad 1 \leq i < j-1 \leq n-1. \end{aligned}$$

$$\begin{aligned} < f_1^{(i)}, f_3^{(i)} > &= (r_ir_i - 1)r_{i+1}r_i - r_i(r_ir_{i+1}r_i - r_{i+1}r_ir_{i+1}) \\ &= r_ir_{i+1}r_ir_{i+1} - r_{i+1}r_i \\ &= f_3^{(i)}r_{i+1} + r_{i+1}r_if_1^{(i)} \qquad \text{where} \quad 1 \leq i \leq n-1. \end{aligned}$$

$$\begin{aligned} < f_2^{(i,j)}, f_1^{(i)} > &= (r_ir_j - r_jr_i)r_j - r_i(r_jr_j - 1) \\ &= -r_jr_ir_j + r_i \\ &= -r_jf_2^{(i,j)} - f_1^{(j)}r_i \qquad \text{where} \quad 1 \leq i \leq j-1 \leq n-1. \end{aligned}$$

$$
\begin{aligned}
< f_3^{(i)}, f_1^{(i)} > &= (r_i r_{i+1} r_i - r_{i+1} r_i r_{i+1}) r_i - r_i r_{i+1} (r_i r_i - 1) \\
&= -r_{i+1} r_i r_{i+1} r_i + r_i r_{i+1} \\
&= -r_{i+1} f_3^{(i)} - f_1^{(i+1)} r_i r_{i+1} \qquad \text{where} \quad 1 \le i \le n-1.
\end{aligned}
$$

$$
\begin{aligned}
< f_1^{(i)}, f_2^{(i,j)} > &= (r_i r_i - 1) r_j - r_i (r_i r_j - r_j r_i) \\
&= r_i r_j r_i - r_j \\
&= f_2^{(i,j)} r_i - r_j f_1^{(i)} \qquad \text{where} \quad 1 \le i < j - 1 \le n-1.
\end{aligned}
$$

$$
\begin{aligned}
< f_2^{(i,j)}, f_3^{(j)} > &= (r_i r_j - r_j r_i) r_{j+1} r_j - r_i (r_j r_{j+1} r_j - r_{j+1} r_j r_{j+1}) \\
&= r_i r_{j+1} r_j r_{j+1} - r_j r_i r_{j+1} r_j \\
&= f_2^{(i,j+1)} r_j r_{j+1} + r_{j+1} f_2^{(i,j)} r_{j+1} + r_{j+1} r_j f_2^{(i,j+1)} - r_j f_2^{(i,j+1)} r_j \\
&\qquad - r_j r_{j+1} f_2^{(i,j)} - f_3^{(j)} r_i \qquad \text{where} \quad 1 \le i < j - 1 \le n-1.
\end{aligned}
$$

$$
\begin{aligned}
< f_3^{(i)}, f_2^{(i,j)} > &= (r_i r_{i+1} r_i - r_{i+1} r_i r_{i+1}) r_j - r_i r_{i+1} (r_i r_j - r_j r_i) \\
&= r_i r_{i+1} r_j r_i - r_{i+1} r_i r_{i+1} r_j \\
&= r_i f_2^{(i+1,j)} r_i + f_2^{(i,j)} r_{i+1} r_i + r_j f_2^{(i)} - r_{i+1} r_i f_3^{(i+1,j)} - r_{i+1} f_2^{(i,j)} r_{i+1} \\
&\qquad - f_2^{(i+1,j)} r_i r_{i+1} \qquad \text{where} \quad 1 \le i \le j - 2 \le n-2.
\end{aligned}
$$

$$
\begin{aligned}
< f_1^{(i)}, g^{(i,j)} > &= (r_i r_i - 1) r_{i+1,j} r_i - r_i (r_{ij} r_i - r_{i+1} r_{ij}) \\
&= r_i r_{i+1} r_{ij} - r_{i+1,j} r_i \qquad \text{where} \quad 1 \le i < j - 1 \le n-1.
\end{aligned}
$$

$$
\begin{aligned}
r_i r_{i+1} r_{ij} &\equiv r_{i+1} r_i r_{i+1} r_{i+1,j} \quad \text{by ELW of } f_3 \\
&= r_{i+1} r_i r_{i+2,j} \quad \text{by ELW of } f_1 \\
&= r_{i+1,j} r_i \quad \text{by a sequence of ELWs of } f_2
\end{aligned}
$$

$$< f_2^{(i,j)}, g^{(j,k)} > = (r_ir_j - r_jr_i)r_{j+1,k}r_j - r_i(r_{jk}r_j - r_{j+1}r_{jk})$$

$$= r_ir_{j+1}r_{jk} - r_jr_ir_{j+1,k}r_j \qquad \text{where} \quad 1 \le i < j-1 < k-2 \le n-2.$$

Then

$r_ir_{j+1}r_{jk} \equiv r_{j+1}r_{jk}r_i$ by a sequence of ELWs of f_2

and

$r_jr_ir_{j+1,k}r_j \equiv r_{jk}r_jr_i$ by a sequence of ELWs of f_2

$\equiv r_{j+1}r_{jk}r_i$ by ELW of g.

$$< f_3^{(i)}, g^{(i,j)} > = (r_ir_{i+1}r_i - r_{i+1}r_ir_{i+1})r_{i+1,j}r_i - r_ir_{i+1}(r_{ij}r_i - r_{i+1}r_{ij})$$

$$= r_ir_{i+1}r_{i+1}r_{ij} - r_{i+1}r_ir_{i+1}r_{i+1,j}r_i \qquad \text{where} \quad 1 \le i < j-1 \le n-1.$$

Then

$r_ir_{i+1}r_{i+1}r_{ij} \equiv r_ir_{ij}$ by an ELW of f_1

$\equiv r_{i+1,j}$ by an ELW of f_1

and

$r_{i+1}r_ir_{i+1}r_{i+1,j}r_i \equiv r_{i+1}r_ir_{i+2,j}r_i$ by an ELW of f_1

$\equiv r_{i+1,j}r_ir_i$ by a sequence of ELWs of f_2

$\equiv r_{i+1,j}$ by an ELW of f_1.

$$< g^{(i,j)}, f_1^{(i)} > = (r_{ij}r_i - r_{i+1}r_{ij})r_i - r_{ij}(r_ir_i - 1)$$

$$= r_{i+1}r_{ij}r_i + r_{ij}$$

$$= -r_{i+1}g^{(i,j)} - f_1^{(i+1)}r_{ij} \qquad \text{where} \quad 1 \le i < j-1 \le n-1.$$

$$< g^{(i,j)}, f_3^{(i)} > = (r_{ij}r_i - r_{i+1}r_{ij})r_{i+1}r_i - r_{ij}(r_ir_{i+1}r_i - r_{i+1}r_ir_{i+1})$$

$$= r_{ij}r_{i+1}r_ir_{i+1} - r_{i+1}r_{ij}r_{i+1}r_i$$

$$= r_ig^{(i+1,j)}r_ir_{i+1} + r_{i+2}g^{(i,j)}r_{i+1} + r_{i+2}r_{i+1}r_ig^{(i+1,j)} + r_{i+2}r_{i+1}f_2^{(i,i+2)}r_{i+1,j}$$

$$-r_{i+1}r_ig^{(i+1,j)}r_i - r_{i+1}f_2^{(i,i+2)}r_{i+1,j}r_i - r_{i+1}r_{i+2}g^{(i,j)} - f_3^{(i+1)}r_{ij}$$

where $1 \leq i < j - 1 \leq n - 1.$

$$< g^{(i,j)}, f_2^{(i,k)} > = (r_{ij}r_i - r_{i+1}r_{ij})r_k - r_{ij}(r_ir_k - r_kr_i)$$

$$= r_{ij}r_kr_i - r_{i+1}r_{ij}r_k \quad \text{where} \quad 1 \leq i < j - 1 \text{ and } 1 \leq i < k - 1.$$

We have several cases to consider. The case $k = j+1$ has already been investigated. If $k = j$, then $r_{ij}r_jr_i - r_{i+1}r_{ij}r_j = r_{i,j-1}f_1^{(j)} - r_{i+1}r_{i,j-1}f_1^{(j)} + g^{(i,j-1)}$. If $k < j$, then

$r_{ij}r_kr_i \equiv r_{k+1}r_{ij}r_i$ by Lemma 2.1

$\equiv r_{k+1}r_{i+1}r_{ij}$ by an ELW of g

and

$r_{i+1}r_{ij}r_k \equiv r_{i+1}r_{k+1}r_{ij}$ by Lemma 2.1

$\equiv r_{k+1}r_{i+1}r_{ij}$ by an ELW of f_2.

If $k > j + 1$, then

$r_{ij}r_kr_i \equiv r_kr_{ij}r_i$ by a sequence of ELWs of f_2

$\equiv r_kr_{i+1}r_{ij}$ by an ELW of g

and

$r_{i+1}r_{ij}r_k \equiv r_{i+1}r_kr_{ij}$ by a sequence of ELWs of f_2

$\equiv r_kr_{i+1}r_{ij}$ by an ELW of f_2.

$$< g^{(i,j)}, g^{(i,k)} > = (r_{ij}r_i - r_{i+1}r_{ij})r_{i+1,k}r_i - r_{ij}(r_{ik}r_i - r_{i+1}r_{ik})$$

$$= r_{ij}r_{i+1}r_{ik} - r_{i+1}r_{ij}r_{i+1,k}r_i \quad \text{where} \quad 1 \leq i < j - 1 \text{ and } 1 \leq i < k - 1$$

If $k < j$, then

$r_{ij}r_{i+1}r_{ik} \equiv r_{i+2}r_{ij}r_{ik}$ by Lemma 2.1

$\equiv r_{i+2}r_{i+1,k+1}r_{ij}$ by case 1 of Proposition 2.2

and

$r_{i+1}r_{ij}r_{i+1,k}r_i \equiv r_{i+1}r_{i+2,k+1}r_{ij}r_i$ by case 1 of Proposition 2.2

$\equiv r_{i+1,k+1}r_{i+1}r_{ij}$ by an ELW of g

$\equiv r_{i+2}r_{i+1,k+1}r_{ij}$ by an ELW of g.

If $j \leq k$, then

$r_{ij}r_{i+1}r_{ik} \equiv r_{i+2}r_{ij}r_{ik}$ by Lemma 2.1

$\equiv r_{i+2}r_{i+1,k}r_{i,j-1}$ by case 2 of Proposition 2.2

and

$r_{i+1}r_{ij}r_{i+1,k}r_i \equiv r_{i+1}r_{i+2,k}r_{i,j-1}r_i$ by case 2 of Proposition 2.2

$\equiv r_{i+1,k}r_{i+1}r_{i,j-1}$ by an ELW of g

$\equiv r_{i+2}r_{i+1,k}r_{i,j-1}$ by an ELW of g.

□

2.2 Reduced Forms for A_n

Since $R = \{f_1^{(i)}, f_2^{(i)}, f_3^{(i,j)}, g^{(i,j)}\}$ is a Gröbner-Shirshov basis for A_n, the R-reduced words are in fact reduced forms for A_n.

Theorem 2.2. *Any word in A_n can be represented in a form*

$$r_{nj_n}r_{n-1,j_{n-1}}\cdots r_{2j_2}r_{1j_1}$$

where $i - 1 \leq j_i \leq n$ for all i.

Proof. We can claim that the word $r_{i,j_i}r_{k,j_k}$ is reduced if and only if $i < k$. The necessary part is directly given in the Proposition 2.2. If $i \geq k$, the same proposition implies that the word $r_{i,j_i}r_{k,j_k}$ is not reduced and furthermore it is equivalent to a reduced word in the desired form. Hence any word in A_n can be presented in a form

$$r_{nj_n} r_{n-1,j_{n-1}} \cdots r_{2j_2} r_{1j_1}$$

where $i - 1 \leq j_i \leq n$ for all i. □

The word r_{i,j_i} has the generating function $1+x+\cdots+x^{n-i+1}$. That means if $j_i = i-1$, we have empty word whose length is zero. If $j_i = i$, we have the word r_i of length 1. If $j_i = i + 1$, we hat the word $r_i r_{i+1}$ of length 2 and so on. Hence the generation function of all reduced words is

$$(1 + x)(1 + x + x^2) \cdots (1 + x + ... + x^n).$$

This generating function is well known (see [6]). That means number of elements of A_n for each length is known but its forms have not been known until Bokut and Shiao gave a form of them in [3]. Here, we gave another form of them in Theorem 3.2. The form we are given is more natural one, because A_{n-1} can be embedded to A_n by the map $r_i \mapsto r_{i-1}$. After expanding the generating function and taking the sum of coefficients, then one can show that the number of reduced forms in A_n is $(n + 1)!$ [1]. For example, A_3 has the following 24 reduced words according to Theorem 3.2

$$\{1, r_1, r_2, r_3, r_{12}, r_2 r_1, r_{23}, r_3 r_1, r_3 r_2, r_{13}, r_2 r_{12}, r_{23} r_1, r_3 r_{12}, r_3 r_{23},$$
$$r_3 r_2 r_1, r_2 r_{13}, r_{23} r_{12}, r_3 r_{13}, r_3 r_2 r_{12}, r_3 r_{23} r_1, r_{23} r_{13}, r_3 r_2 r_{13}, r_3 r_{23} r_{12}, r_3 r_{23} r_{13}\}.$$

Using Proposition 2.2 one can find complete multiplication tables for A_n. Let us give an example for $n = 3$.

$$
\begin{aligned}
(r_{23}r_{12})(r_3r_{13}) &\equiv r_{23}(r_{12}r_3)r_{13} \\
&\equiv r_{23}r_{13}r_{13} \quad \text{by case 3 of Proposition 2.2} \\
&\equiv r_{23}r_{23}r_{12} \quad \text{by case 2 of Proposition 2.2} \\
&\equiv r_3r_2r_{12} \quad \text{by case 2 of Proposition 2.2.}
\end{aligned}
$$

The following is the multiplication table of A_3.

	1	r_1	r_2	r_3	r_{12}	r_2r_1	r_{23}	r_3r_1
1	1	r_1	r_2	r_3	r_{12}	r_2r_1	r_{23}	r_3r_1
r_1	r_1	1	r_{12}	r_3r_1	r_2	r_2r_{12}	r_{13}	r_3
r_2	r_2	r_2r_1	1	r_{23}	r_2r_{12}	r_1	r_3	$r_{23}r_1$
r_3	r_3	r_3r_1	r_3r_2	1	r_3r_{12}	$r_3r_2r_1$	r_3r_{23}	r_1
r_{12}	r_{12}	r_2r_{12}	r_1	r_{13}	r_2r_1	1	r_3r_1	r_2r_{13}
r_2r_1	r_2r_1	r_2	r_2r_{12}	$r_{23}r_1$	1	r_{12}	r_2r_{13}	r_{23}
r_{23}	r_{23}	$r_{23}r_1$	r_3r_{23}	r_2	$r_{23}r_{12}$	$r_3r_{23}r_1$	r_3r_2	r_2r_1
r_3r_1	r_3r_1	r_3	r_3r_{12}	r_1	r_3r_2	$r_3r_2r_{12}$	r_3r_{13}	1
r_3r_2	r_3r_2	$r_3r_2r_1$	r_3	r_3r_{23}	$r_3r_2r_{12}$	r_3r_1	1	$r_3r_{23}r_1$
r_{13}	r_{13}	r_2r_{13}	r_3r_{13}	r_{12}	$r_{23}r_{13}$	$r_3r_2r_{13}$	r_3r_{12}	r_2r_{12}
r_2r_{12}	r_2r_{12}	r_{12}	r_2r_1	r_2r_{13}	r_1	r_2	$r_{23}r_1$	r_{13}
$r_{23}r_1$	$r_{23}r_1$	r_{23}	$r_{23}r_{12}$	r_2r_1	r_3r_{23}	$r_3r_{23}r_{12}$	$r_{23}r_{13}$	r_2
r_3r_{12}	r_3r_{12}	$r_3r_2r_{12}$	r_3r_1	r_3r_{13}	$r_3r_2r_1$	r_3	r_1	$r_3r_2r_{13}$
$r_3r_2r_1$	$r_3r_2r_1$	r_3r_2	$r_3r_2r_{12}$	$r_3r_{23}r_1$	r_3	r_3r_{12}	$r_3r_2r_{13}$	r_3r_{23}
r_3r_{23}	r_3r_{23}	$r_3r_{23}r_1$	r_{23}	r_3r_2	$r_3r_{23}r_{12}$	$r_{23}r_1$	r_2	$r_3r_2r_1$
r_2r_{13}	r_2r_{13}	r_{13}	$r_{23}r_{13}$	r_2r_{12}	r_3r_{13}	$r_3r_{23}r_{13}$	$r_{23}r_{12}$	r_{12}
$r_{23}r_{12}$	$r_{23}r_{12}$	$r_3r_{23}r_{12}$	$r_{23}r_1$	$r_{23}r_{13}$	$r_3r_{23}r_1$	r_{23}	r_2r_1	$r_3r_{23}r_{13}$
r_3r_{13}	r_3r_{13}	$r_3r_2r_{13}$	r_{13}	r_3r_{12}	$r_3r_{23}r_{13}$	r_2r_{13}	r_{12}	$r_3r_2r_{12}$
$r_3r_2r_{12}$	$r_3r_2r_{12}$	r_3r_{12}	$r_3r_2r_1$	$r_3r_2r_{13}$	r_3r_1	r_3r_2	$r_3r_{23}r_1$	r_3r_{13}
$r_3r_{23}r_1$	$r_3r_{23}r_1$	r_3r_{23}	$r_3r_{23}r_{12}$	$r_3r_2r_1$	r_{23}	$r_{23}r_{12}$	$r_3r_{23}r_{13}$	r_3r_2
$r_{23}r_{13}$	$r_{23}r_{13}$	$r_3r_{23}r_{13}$	r_2r_{13}	$r_{23}r_{12}$	$r_3r_2r_{13}$	r_{13}	r_2r_{12}	$r_3r_{23}r_{12}$
$r_3r_2r_{13}$	$r_3r_2r_{13}$	r_3r_{13}	$r_3r_{23}r_{13}$	$r_3r_2r_{12}$	r_{13}	$r_{23}r_{13}$	$r_3r_{23}r_{12}$	r_3r_{12}
$r_3r_{23}r_{12}$	$r_3r_{23}r_{12}$	$r_{23}r_{12}$	$r_3r_{23}r_1$	$r_3r_{23}r_{13}$	$r_{23}r_1$	r_3r_{23}	$r_3r_2r_1$	$r_{23}r_{13}$
$r_3r_{23}r_{13}$	$r_3r_{23}r_{13}$	$r_{23}r_{13}$	$r_3r_2r_{13}$	$r_3r_{23}r_{12}$	r_2r_{13}	r_3r_{13}	$r_3r_2r_{12}$	$r_{23}r_{12}$

	r_3r_2	r_{13}	r_2r_{12}	$r_{23}r_1$	r_3r_{12}	$r_3r_2r_1$	r_3r_{23}	r_2r_{13}
1	r_3r_2	r_{13}	r_2r_{12}	$r_{23}r_1$	r_3r_{12}	$r_3r_2r_1$	r_3r_{23}	r_2r_{13}
r_1	r_3r_{12}	r_{23}	r_2r_1	r_2r_{13}	r_3r_2	$r_3r_2r_{12}$	r_3r_{13}	$r_{23}r_1$
r_2	r_3r_{23}	r_2r_{13}	r_{12}	r_3r_1	$r_{23}r_{12}$	$r_3r_{23}r_1$	r_3r_2	r_{13}
r_3	r_2	r_3r_{13}	$r_3r_2r_{12}$	$r_3r_{23}r_1$	r_{12}	r_2r_1	r_{23}	$r_3r_2r_{13}$
r_{12}	r_3r_{13}	$r_{23}r_1$	r_2	r_3	$r_{23}r_{13}$	$r_3r_2r_{13}$	r_3r_{12}	r_{23}
r_2r_1	$r_{23}r_{12}$	r_3	r_1	r_{13}	r_3r_{23}	$r_3r_{23}r_{12}$	$r_{23}r_{13}$	r_3r_1
r_{23}	1	$r_{23}r_{13}$	$r_3r_{23}r_{12}$	$r_3r_2r_1$	r_2r_{12}	r_1	r_3	$r_3r_{23}r_{13}$
r_3r_1	r_{12}	r_3r_{23}	$r_3r_2r_1$	$r_3r_2r_{13}$	r_2	r_2r_{12}	r_{13}	$r_3r_{23}r_1$
r_3r_2	r_{23}	$r_3r_2r_{13}$	r_3r_{12}	r_1	$r_3r_{23}r_{12}$	$r_{23}r_1$	r_2	r_3r_{13}
r_{13}	r_1	$r_{23}r_{12}$	$r_3r_{23}r_{13}$	$r_3r_2r_{12}$	r_2r_1	1	r_3r_1	$r_3r_{23}r_{12}$
r_2r_{12}	$r_{23}r_{13}$	r_3r_1	1	r_{23}	r_3r_{13}	$r_3r_{23}r_{13}$	$r_{23}r_{12}$	r_3
$r_{23}r_1$	r_2r_{12}	r_3r_2	$r_3r_{23}r_1$	$r_3r_{23}r_{13}$	1	r_{12}	r_2r_{13}	$r_3r_2r_1$
r_3r_{12}	r_{13}	$r_3r_{23}r_1$	r_3r_2	1	$r_3r_{23}r_{13}$	r_2r_{13}	r_{12}	r_3r_{23}
$r_3r_2r_1$	$r_3r_{23}r_{12}$	1	r_3r_1	r_3r_{13}	r_{23}	$r_{23}r_{12}$	$r_3r_{23}r_{13}$	r_1
r_3r_{23}	r_3	$r_3r_{23}r_{13}$	$r_{23}r_{12}$	r_2r_1	$r_3r_2r_{12}$	r_3r_1	1	$r_{23}r_{13}$
r_2r_{13}	r_2r_1	r_3r_{12}	$r_3r_2r_{13}$	$r_3r_{23}r_{12}$	r_1	r_2	$r_{23}r_1$	$r_3r_2r_{12}$
$r_{23}r_{12}$	r_2r_{13}	$r_3r_2r_1$	r_3r_{23}	r_2	$r_3r_2r_{13}$	r_{13}	r_2r_{12}	r_3r_2
r_3r_{13}	r_3r_1	$r_3r_{23}r_{12}$	$r_{23}r_{13}$	r_2r_{12}	$r_3r_2r_1$	r_3	r_1	$r_{23}r_{12}$
$r_3r_2r_{12}$	$r_3r_{23}r_{13}$	r_1	r_3	r_3r_{23}	r_{13}	$r_{23}r_{13}$	$r_3r_{23}r_{12}$	1
$r_3r_{23}r_1$	$r_3r_2r_{12}$	r_2	$r_{23}r_1$	$r_{23}r_{13}$	r_3	r_3r_{12}	$r_3r_2r_{13}$	r_2r_1
$r_{23}r_{13}$	$r_{23}r_1$	$r_3r_2r_{12}$	r_3r_{13}	r_{12}	$r_3r_{23}r_1$	r_{23}	r_2r_1	r_3r_{12}
$r_3r_2r_{13}$	$r_3r_2r_1$	r_{12}	r_2r_{13}	$r_{23}r_{12}$	r_3r_1	r_3r_2	$r_3r_{23}r_1$	r_2r_{12}
$r_3r_{23}r_{12}$	$r_3r_2r_{13}$	r_2r_1	r_{23}	r_3r_2	r_2r_{13}	r_3r_{13}	$r_3r_2r_{12}$	r_2
$r_3r_{23}r_{13}$	$r_3r_{23}r_1$	r_2r_{12}	r_{13}	r_3r_{12}	$r_{23}r_1$	r_3r_{23}	$r_3r_2r_1$	r_{12}

	$r_{23}r_{12}$	r_3r_{13}	$r_3r_2r_{12}$	$r_3r_{23}r_1$	$r_{23}r_{13}$	$r_3r_2r_{13}$	$r_3r_{23}r_{12}$	$r_3r_{23}r_{13}$
1	$r_{23}r_{12}$	r_3r_{13}	$r_3r_2r_{12}$	$r_3r_{23}r_1$	$r_{23}r_{13}$	$r_3r_2r_{13}$	$r_3r_{23}r_{12}$	$r_3r_{23}r_{13}$
r_1	$r_{23}r_{13}$	r_3r_{23}	$r_3r_2r_1$	$r_3r_2r_{13}$	$r_{23}r_{12}$	$r_3r_{23}r_1$	$r_3r_{23}r_{13}$	$r_3r_{23}r_{12}$
r_2	r_3r_{12}	$r_{23}r_{13}$	$r_3r_{23}r_{12}$	$r_3r_2r_1$	r_3r_{13}	$r_3r_{23}r_{13}$	$r_3r_2r_{12}$	$r_3r_2r_{13}$
r_3	$r_3r_{23}r_{12}$	r_{13}	r_2r_{12}	$r_{23}r_1$	$r_3r_{23}r_{13}$	r_2r_{13}	$r_{23}r_{12}$	$r_{23}r_{13}$
r_{12}	r_3r_2	$r_{23}r_{12}$	$r_3r_{23}r_{13}$	$r_3r_2r_{12}$	r_3r_{23}	$r_3r_{23}r_{12}$	$r_3r_2r_1$	$r_3r_{23}r_1$
r_2r_1	r_3r_{13}	r_3r_2	$r_3r_{23}r_1$	$r_3r_{23}r_{13}$	r_3r_{12}	$r_3r_2r_1$	$r_3r_2r_{13}$	$r_3r_2r_{12}$
r_{23}	$r_3r_2r_{12}$	r_2r_{13}	r_{12}	r_3r_1	$r_3r_2r_{13}$	r_{13}	r_3r_{12}	r_3r_{13}
r_3r_1	$r_3r_{23}r_{13}$	r_{23}	r_2r_1	r_2r_{13}	$r_3r_{23}r_{12}$	$r_{23}r_1$	$r_{23}r_{13}$	$r_{23}r_{12}$
r_3r_2	r_{12}	$r_3r_{23}r_{13}$	$r_{23}r_{12}$	r_2r_1	r_{13}	$r_{23}r_{13}$	r_2r_{12}	r_2r_{13}
r_{13}	$r_3r_2r_1$	$r_{23}r_1$	r_2	r_3	$r_3r_{23}r_1$	r_{23}	r_3r_2	r_3r_{23}
r_2r_{12}	r_3r_{23}	r_3r_{12}	$r_3r_2r_{13}$	$r_3r_{23}r_{12}$	r_3r_2	$r_3r_2r_{12}$	$r_3r_{23}r_1$	$r_3r_2r_1$
$r_{23}r_1$	$r_3r_2r_{13}$	r_3	r_1	r_{13}	$r_3r_2r_{12}$	r_3r_1	r_3r_{13}	r_3r_{12}
r_3r_{12}	r_2	$r_3r_{23}r_{12}$	$r_{23}r_{13}$	r_2r_{12}	r_{23}	$r_{23}r_{12}$	r_2r_1	$r_{23}r_1$
$r_3r_2r_1$	r_{13}	r_2	$r_{23}r_1$	$r_{23}r_{13}$	r_{12}	r_2r_1	r_2r_{13}	r_2r_{12}
r_3r_{23}	r_2r_{12}	$r_3r_2r_{13}$	r_3r_{12}	r_1	r_2r_{13}	r_3r_{13}	r_{12}	r_{13}
r_2r_{13}	$r_3r_{23}r_1$	r_3r_1	1	r_{23}	$r_3r_2r_1$	r_3	r_3r_{23}	r_3r_2
$r_{23}r_{12}$	1	$r_3r_2r_{12}$	r_3r_{13}	r_{12}	r_3	r_3r_{12}	r_1	r_3r_1
r_3r_{13}	r_2r_1	$r_3r_{23}r_1$	r_3r_2	1	$r_{23}r_1$	r_3r_{23}	r_2	r_{23}
$r_3r_2r_{12}$	r_{23}	r_{12}	r_2r_{13}	$r_{23}r_{12}$	r_2	r_2r_{12}	$r_{23}r_1$	r_2r_1
$r_3r_{23}r_1$	r_2r_{13}	1	r_3r_1	r_3r_{13}	r_2r_{12}	r_1	r_{13}	r_{12}
$r_{23}r_{13}$	r_1	$r_3r_2r_1$	r_3r_{23}	r_2	r_3r_1	r_3r_2	1	r_3
$r_3r_2r_{13}$	$r_{23}r_1$	r_1	r_3	r_3r_{23}	r_2r_1	1	r_{23}	r_2
$r_3r_{23}r_{12}$	r_3	r_2r_{12}	r_{13}	r_3r_{12}	1	r_{12}	r_3r_1	r_1
$r_3r_{23}r_{13}$	r_3r_1	r_2r_1	r_{23}	r_3r_2	r_1	r_2	r_3	1

CHAPTER 3

COXETER GROUPS OF TYPE B_n

Definition 3.1. Coxeter group of type B_n ($n \geq 2$) is generated by $r_1, r_2, \cdots, r_n$ with defining relations:

$R1 : r_i^2 = 1$ where $1 \leq i \leq n$,

$R2 : r_i r_j = r_j r_i$ where $1 \leq i < j - 1 \leq n - 1$,

$R3 : r_i r_{i+1} r_i = r_{i+1} r_i r_{i+1}$ where $1 \leq i \leq n - 2$ and

$R4 : r_{n-1} r_n r_{n-1} r_n = r_n r_{n-1} r_n r_{n-1}$.

3.1 Gröbner-Shirshov Basis of B_n

Proposition 3.1. *The reduced Gröbner-Shirshov basis of Coxeter group B_n contains the following polynomials:*

$f_1^{(i)} = r_i r_i - 1$ *where* $1 \leq i \leq n$,

$f_2^{(i,j)} = r_i r_j - r_j r_i$ *where* $1 \leq i \leq j - 1 \leq n - 1$,

$f_3^{(i)} = r_i r_{i+1} r_i - r_{i+1} r_i r_{i+1}$ *where* $1 \leq i \leq n - 2$,

$f_4 = r_{n-1} r_n r_{n-1} r_n - r_n r_{n-1} r_n r_{n-1}$,

$g^{(i,j)} = r_{ij} r_i - r_{i+1} r_{ij}$ *where* $1 \leq i < j - 1 \leq 2n - (i + 2)$ *and*

$h^{(i)} = r_{i,2n-i} r_{i+1} - r_{i+1} r_{i,2n-i}$ *where*$1 \leq i < n - 1$.

Proof. Let us first look at the possible compositions of f_4.

$$< f_1^{(n-1)}, f_4 > = (r_{n-1}r_{n-1} - 1)r_n r_{n-1} r_n - r_{n-1}(r_{n-1}r_n r_{n-1} r_n - r_n r_{n-1} r_n r_{n-1})$$
$$= r_{n-1} r_n r_{n-1} r_n r_{n-1} - r_n r_{n-1} r_n$$
$$= f_4 r_{n-1} + r_n r_{n-1} r_n f_1^{(n-1)}.$$

$$< f_2^{(i,n-1)}, f_4 > = (r_i r_{n-1} - r_{n-1} r_i) r_n r_{n-1} r_n - r_i (r_{n-1} r_n r_{n-1} r_n - r_n r_{n-1} r_n r_{n-1})$$
$$= r_i r_n r_{n-1} r_n r_{n-1} - r_{n-1} r_i r_n r_{n-1} r_n$$
$$= f_2^{(i,n)} r_{n-1} r_n r_{n-1} + r_n f_2^{(i,n-1)} r_n r_{n-1} + r_n r_{n-1} f_2^{(i,n)} r_{n-1}$$
$$+ r_n r_{n-1} r_n f_2^{(i,n-1)} - r_{n-1} f_2^{(i,n)} r_{n-1} r_n - r_{n-1} r_n f_2^{(i,n-1)} r_n$$
$$- r_{n-1} r_n r_{n-1} f_2^{(i,n)} - f_4 r_i \qquad \text{where } 1 \leq i < n-2.$$

$$< f_4, f_1^{(n)} > = (r_{n-1} r_n r_{n-1} r_n - r_n r_{n-1} r_n r_{n-1}) r_n - r_{n-1} r_n r_{n-1} (r_{n-1} r_{n-1} - 1)$$
$$= -r_n r_{n-1} r_n r_{n-1} r_n + r_{n-1} r_n r_{n-1}$$
$$= -r_n f_4 - f_1^{(n)} r_{n-1} r_n r_{n-1}.$$

The polynomial $g^{(i,j)}$ is same as in A_n for $1 \leq i < j-1 \leq n-1$. Notice that we had the polynomial $f_3^{(n-1)}$ in A_n but it is not exist in B_n. Hence we can not use it in B_n. This forces us to recalculate the following compositions.

$$< g^{(i,n)}, f_2^{(i,n-1)} > = (r_{in} r_i - r_{i+1} r_{in}) r_{n-1} - r_{in}(r_i r_{n-1} - r_{n-1} r_i)$$
$$= r_{in} r_{n-1} r_i - r_{i+1} r_{in} r_{n-1}$$
$$= g^{(i,n+1)} \qquad \text{where } 1 \leq i < n-1.$$

$$< g^{(i,j)}, f_2^{(i,2n-(j+1))} > = (r_{ij} r_i - r_{i+1} r_{ij}) r_{2n-(j+1)} - r_{in}(r_i r_{2n-(j+1)} - r_{2n-(j+1)} r_i)$$
$$= r_{ij} r_{2n-(j+1)} r_i - r_{i+1} r_{ij} r_{2n-(j+1)}$$
$$= r_{i,j+1} r_i - r_{i+1} r_{i,j+1}$$
$$= g^{(i,j+1)} \qquad \text{where } n < j \leq 2n-i, 1 \leq i < n.$$

The polynomial $h^{(i)}$ can be obtained as follows.

$$
\begin{aligned}
< g^{(i,2n-(i+2))}, f_3^{(i)} > &= (r_{i,2n-(i+2)}r_i - r_{i+1}r_{i,2n-(i+2)})r_{i+1}r_i - r_{i,2n-(i+2)}(r_ir_{i+1}r_i - r_{i+1}r_ir_{i+1}) \\
&= r_{i,2n-(i+2)}r_{i+1}r_ir_{i+1} - r_{i+1}r_{i,2n-(i+2)}r_{i+1}r_i \\
&= r_{i,2n-i}r_{i+1} - r_{i+1}r_{i,2n-i} \\
&= h^{(i)} \qquad \text{where } 1 \le i < n-1.
\end{aligned}
$$

□

Let $R = \{f_1^{(i)}, f_2^{(i)}, f_3^{(i,j)}, f_4, g^{(i,j)}, h^{(i)}\}$. The following results ease to prove that R is in fact is a reduced Gröbner-Shirshov basis for B_n.

Lemma 3.1. *If* $1 \le i \le t < j < 2n-(t+1)$, *then* $r_{ij}r_t \equiv r_{t+1}r_{ij}$.

Proof.

$$
\begin{aligned}
r_{ij}r_t &\equiv r_{i,t-1}r_{tj}r_t \\
&\equiv r_{i,t-1}r_{t+1}r_{tj} \qquad \text{by ELW of } g \text{ or } f_3, \\
&= r_{t+1}r_{ij} \qquad \text{by a sequence of ELWs of } f_2.
\end{aligned}
$$

□

Lemma 3.2. *If* $1 \le i \le t < n < 2n-t < j \le 2n-1$, *then* $r_{ij}r_t \equiv r_tr_{ij}$.

Proof.

$$
\begin{aligned}
r_{ij}r_t &\equiv r_{i,2n-(t-1)}r_tr_{2n-(t-2),j} \quad \text{by a sequence of ELWs of } f_2 \\
&\equiv r_{i,t-2}r_{t-1,2n-(t-1)}r_tr_{2n-(t-2),j} \\
&\equiv r_{i,t-2}r_tr_{t-1,2n-(t-1)}r_{2n-(t-2),j} \quad \text{by ELW of } h \text{ or } f_4 \\
&\equiv r_tr_{ij} \quad \text{by a sequence of ELWs of } f_2
\end{aligned}
$$

□

Proposition 3.2. *Let* $1 \le i, k \le n$, $i \le j \le 2n-i$ *and* $k \le l \le 2n-k$. *If* $i > k$ *then the word* $r_{ij}r_{kl}$ *is reduced. Otherwise*

$$r_{ij}r_{kl} \equiv \begin{cases} r_{k+1,l+1}r_{ij} & (l < n) \wedge (l < j < 2n-(l+1)) \\ r_{k+1,l}r_{i,j-1} & (j < n) \wedge (k \le j \le l \le 2n-(j+1)) \\ r_{k+1,l}r_{i,j+1} & (j \ge n) \wedge (2n-(j+1) \le l \le j < 2n-k) \\ r_{k+1,l-1}r_{ij} & (l > n) \wedge (2n-(l+1) < j < l) \\ r_{kl}r_{ij} & (j < k-1) \vee (j > 2n-k) \\ r_{il} & j = k-1 \\ r_{i,2n-(l+1)} & j = 2n-k \end{cases} \quad mod\, R.$$

Proof.

Case 1 : If $l < n$ and $l < j < 2n-(l+1)$, then $r_{ij}r_{kl} \equiv r_{k+1,l+1}r_{ij}$ by Lemma 3.1.

Case2 : If $j < n$ and $k \le j \le l \le 2n-(j+1)$, then

$$\begin{aligned} r_{ij}r_{kl} &\equiv r_{ij}r_{k,j-1}r_{jl} \\ &\equiv r_{k+1,j}r_{i,j-1}r_{jl} \quad \text{by Case 1} \\ &\equiv r_{k+1,j}r_{i,j-1}r_{j+1,l} \quad \text{by an ELW of } f_1 \\ &\equiv r_{k+1,l}r_{i,j-1} \quad \text{by a sequence of ELWs of } f_2. \end{aligned}$$

Case3 : Let $j \ge n$ and $2n-(j+1) \le l \le j < 2n-k$.

If $j = 2n-(k+1)$, then

$$\begin{aligned} r_{ij}r_{kl} &\equiv r_{i,k-1}r_{k,2n-(k+1)}r_kr_{k+1}r_{k+2,l} \\ &\equiv r_{i,k-1}r_{k+1}r_{k,2n-k}r_{k+2,l} \quad \text{by an ELW of } h \\ &\equiv r_{k+1}r_{i,2n-k}r_{k+2,l} \quad \text{by a sequence of ELWs of } f_2 \\ &\equiv r_{k+1,l}r_{i,2n-k} \quad \text{by Lemma 3.2.} \end{aligned}$$

If $j < 2n-(k+1)$, then

$$\begin{aligned} r_{ij}r_{kl} &\equiv r_{ij}r_{k,2n-(j+2)}r_{2n-(j+1),l} \\ &\equiv r_{k+1,2n-(j+1)}r_{ij}r_{2n-(j+1),l} \quad \text{by Lemma 3.1} \\ &\equiv r_{k+1,2n-(j+1)}r_{2n-j,l}r_{i,j+1} \quad \text{by Case 2.1} \\ &\equiv r_{k+1,l}r_{i,j+1}. \end{aligned}$$

Case 4 : Let $l > n$ and $2n-(l+1) < j < l$.

If $j \leq n$, then

$$r_{ij}r_{kl} \equiv r_{ij}r_{k,j-1}r_{jl}$$

$$\equiv r_{k+1,j}r_{ij}r_{jl} \quad \text{by Lemma 3.1}$$

$$\equiv r_{k+1,j}r_{i,j-1}r_{j+1,l} \quad \text{by an ELW of } f_1$$

$$\equiv r_{k+1,2n-(j+1)}r_{i,j-1}r_{2n-j,l} \quad \text{by a sequence of ELWs of } f_2$$

$$\equiv r_{k+1,2n-(j+1)}r_{ij}r_{2n-(j-1),l}$$

$$\equiv r_{k+1,l-1}r_{ij} \quad \text{by Lemma 3.1.}$$

If $j > n$, then

$$r_{ij}r_{kl} \equiv r_{ij}r_{kj}r_{j+1,l}$$

$$\equiv r_{k+1,j}r_{i,j+1}r_{j+1,l} \quad \text{by Case 3}$$

$$\equiv r_{k+1,j}r_{ij}r_{j+2,l} \quad \text{by an ELW of } f_1$$

$$\equiv r_{k+1,l-1}r_{ij} \quad \text{by Lemma 3.1.}$$

Case5 : Let $j < k-1$ or $j > 2n-k$.

If $j < k-1$, then $r_{ij}r_{kl} \equiv r_{kl}r_{ij}$ by a sequence of ELWs of f_2.

If $j > 2n-k$, then $r_{ij}r_{kl} \equiv r_{kl}r_{ij}$ by Lemma 3.2.

Case 6 : If $j = k-1$, then $r_{ij}r_{j+1,l} \equiv r_{il}$.

Case 7 : If $j = 2n-k$, then $r_{i,2n-k}r_{kl} = r_{i,2n-(l+1)}$ by a sequence of ELWs of f_2.

□

Theorem 3.1. *The set $R = \{f_1^{(i)}, f_2^{(i)}, f_3^{(i,j)}, f_4, g^{(i,j)}, h^{(i)}\}$ is a reduced Gröbner-Shirshov basis for B_n.*

Proof. Since all polynomials except $h^{(i)}$ is also in Gröbner-Shirshov basis of A_n, we only consider compositions involving $h^{(i)}$.

$$< f_1^{(i)}, h^{(i)} > = (r_ir_i - 1)r_{i+1,2n-i}r_{i+1} - r_i(r_{i,2n-i}r_{i+1} - r_{i+1}r_{i,2n-i})$$

$$= r_ir_{i+1}r_{i,2n-i} - r_{i+1,2n-i}r_{i+1} \qquad \text{where} \quad 1 \leq i < n-1.$$

Then

$$r_i r_{i+1} r_{i,2n-i} \equiv r_{i+1} r_i r_{i+1} r_{i+1,2n-i} \quad \text{by an ELW of } f_3$$

$$\equiv r_{i+1} r_i r_{i+2,2n-(i+2)} r_{i+1} r_i \quad \text{by an ELW of } f_1$$

$$\equiv r_{i+1,2n-(i+2) r_i r_{i+1} r_i} \quad \text{by case 5 of Proposition 3.2}$$

$$\equiv r_{i+1,2n-i} r_{i+1} \quad \text{by an ELW of } f_3.$$

$$< f_2^{(i,j)}, h^{(j)} > = (r_i r_j - r_j r_i) r_{j+1,2n-j} r_{j+1} - r_i (r_{j,2n-j} r_{j+1} - r_{j+1} r_{j,2n-j}$$

$$= r_i r_{j+1} r_{j,2n-j} - r_j r_i r_{j+1,2n-j} r_{j+1} \qquad \text{where} \quad 1 \le i < j-1 \le n-1.$$

Then

$$r_i r_{j+1} r_{j,2n-j} \equiv r_{j+1} r_{j,2n-j} r_i \quad \text{by a sequence of ELWs of } f_2,$$

and

$$r_j r_i r_{j+1,2n-j} r_{j+1} \equiv r_{j,2n-j} r_{j+1} r_i \quad \text{by a sequence of ELWs of } f_2$$

$$= r_{j+1} r_{j,2n-j} r_i \quad \text{by an ELW of } h.$$

$$< f_3^{(i)}, h^{(i)} > = (r_i r_{i+1} r_i - r_{i+1} r_i r_{i+1}) r_{i+1,2n-i} r_{i+1} - r_i r_{i+1} (r_{i,2n-i} r_{i+1} - r_{i+1} r_{i,2n-i})$$

$$= r_i r_{i+1} r_{i+1} r_{i,2n-i} - r_{i+1} r_i r_{i+1} r_{i+1,2n-i} r_{i+1} \quad \text{where} \quad 1 \le i < n-1.$$

Then

$$r_i r_{i+1} r_{i+1} r_{i,2n-i} \equiv r_i r_{i,2n-i} \quad \text{by an ELW of } f_1$$

$$\equiv r_{i+1,2n-i} \quad \text{by an ELW of } f_1,$$

and

$$r_{i+1} r_i r_{i+1} r_{i+1,2n-i} r_{i+1} \equiv r_{i+1} r_i r_{i+2,2n-(i+2)} r_{i+1} r_i r_{i+1} \quad \text{by an ELW of } f_1$$

$$\equiv r_{i+1,2n-(i+2)} r_i r_{i+1} r_i r_{i+1} \quad \text{by case 5 of Proposition 3.2}$$

$$\equiv r_{i+1,2n-i} r_{i+1} r_{i+1} \quad \text{by an ELW of } f_3$$

$$\equiv r_{i+1,2n-i} \quad \text{by an ELW of } f_1.$$

$$< g^{(i,j)}, h^{(i)} > = (r_{ij} r_i - r_{i+1} r_{ij}) r_{i+1,2n-i} r_{i+1} - r_{ij} (r_{i,2n-i} r_{i+1} - r_{i+1} r_{i,2n-i})$$

$$= r_{ij} r_{i+1} r_{i,2n-i} - r_{i+1} r_{ij} r_{i+1,2n-i} r_{i+1} \quad \text{where} \quad 1 \le i < j \le 2n-i.$$

Then

$$r_{ij}r_{i+1}r_{i,2n-i} \equiv r_{i+2}r_{ij}r_{i,2n-i} \quad \text{by Lemma 3.1}$$

$$\equiv r_{i+2}r_{i+1,2n-(i+1)}r_{ij} \quad \text{by case 4 of Proposition 3.2,}$$

and

$$r_{i+1}r_{ij}r_{i+1,2n-i}r_{i+1} \equiv r_{i+1}r_{ij}r_{i+1,2n-(i+1)}r_ir_{i+1}$$

$$\equiv r_{i+1}r_{i+2,2n-(i+2)}r_{ij}r_ir_{i+1} \quad \text{by case 4 of Proposition 3.2}$$

$$\equiv r_{i+1}r_{i+2,2n-(i+1)}r_{ij}r_{i+1} \quad \text{by an ELW of } g$$

$$\equiv r_{i+1,2n-(i+1)}r_{i+2}r_{ij} \quad \text{by Lemma 3.1}$$

$$\equiv r_{i+2}r_{i+1,2n-(i+1)}r_{ij} \quad \text{by an ELW of } h.$$

$$< h^{(i)}, h^{(i+1)} > = (r_{i,2n-i}r_{i+1} - r_{i+1}r_{i,2n-i})r_{i+2,2n-(i+1)}r_{i+2} - r_{i,2n-i}(r_{i+1,2n-(i+1)}r_{i+2} - r_{i+2}r_{i+1,2n-(i+1)})$$

$$= r_{i,2n-i}r_{i+2}r_{i+1,2n-(i+1)} - r_{i+1}r_{i,2n-i}r_{i+2,2n-(i+1)}r_{i+2} \quad \text{where} \quad 1 \leq i < n-2.$$

Then

$$r_{i,2n-i}r_{i+2}r_{i+1,2n-(i+1)} \equiv r_{i+2}r_{i,2n-i}r_{i+1,2n-(i+1)} \quad \text{by Lemma 3.2}$$

$$\equiv r_{i+2}r_{i+1,2n-(i+1)}r_{i,2n-i} \quad \text{by case 5 of Proposition 3.2,}$$

and

$$r_{i+1}r_{i,2n-i}r_{i+2,2n-(i+1)}r_{i+2} \equiv r_{i+1}r_{i,2n-i}r_{i+2,2n-(i+2)}r_{i+1}r_{i+2}$$

$$\equiv r_{i+1,2n-(i+2)}r_{i,2n-i}r_{i+1}r_{i+2} \quad \text{by case 5 of Proposition 3.2}$$

$$\equiv r_{i+1,2n-(i+1)}r_{i,2n-i}r_{i+2} \quad \text{by an ELW of } h$$

$$\equiv r_{i+1,2n-(i+1)}r_{i+2}r_{i,2n-i} \quad \text{by Lemma 3.2}$$

$$\equiv r_{i+2}r_{i+1,2n-(i+1)}r_{i,2n-i} \quad \text{by an ELW of } h.$$

$$< h^{(i)}, f_1^{(i+1)} > = (r_{i,2n-i}r_{i+1} - r_{i+1}r_{i,2n-i})r_{i+1} - r_{i,2n-i}(r_{i+1}r_{i+1} - 1)$$

$$= -r_{i+1}r_{i,2n-i}r_{i+1} + r_{i,2n-i}$$

$$= -r_{i+1}h^{(i)} - f_1^{(i+1)}r_{i,2n-i} \quad \text{where} \quad 1 \leq i < n-1.$$

$$< h^{(i)}, f_2^{(i+1,j)} > = (r_{i,2n-i}r_{i+1} - r_{i+1}r_{i,2n-i})r_j - r_{i,2n-i}(r_{i+1}r_j - r_jr_{i+1})$$

$$= r_{i,2n-i}r_jr_{i+1} - r_{i+1}r_{i,2n-i}r_j \quad \text{where} \quad 1 \le i < j - 2 \le n - 2.$$

Then

$$r_{i,2n-i}r_jr_{i+1} \equiv r_jr_{i,2n-i}r_{i+1} \quad \text{by Lemma 3.2}$$

$$\equiv r_jr_{i+1}r_{i,2n-i} \quad \text{by an ELW of } h,$$

and

$$r_{i+1}r_{i,2n-i}r_j \equiv r_jr_{i+1}r_{i,2n-i} \quad \text{by Lemma 3.2.}$$

$$< h^{(i)}, f_3^{(i+1)} > = (r_{i,2n-i}r_{i+1} - r_{i+1}r_{i,2n-i})r_{i+2}r_{i+1} - r_{i,2n-i}(r_{i+1}r_{i+2}r_{i+1} - r_{i+2}r_{i+1}r_{i+2})$$

$$= r_{i,2n-i}r_{i+2}r_{i+1}r_{i+2} - r_{i+1}r_{i,2n-i}r_{i+2}r_{i+1} \quad \text{where} \quad 1 \le i < n - 2.$$

Then

$$r_{i,2n-i}r_{i+2}r_{i+1}r_{i+2} \equiv r_{i+2}r_{i,2n-i}r_{i+1}r_{i+2} \quad \text{by Lemma 3.2}$$

$$\equiv r_{i+2}r_{i+1}r_{i,2n-i}r_{i+2} \quad \text{by an ELW of } h$$

$$\equiv r_{i+2}r_{i+1}r_{i+2}r_{i,2n-i} \quad \text{by Lemma 3.2,}$$

and

$$r_{i+1}r_{i,2n-i}r_{i+2}r_{i+1} \equiv r_{i+1}r_{i+2}r_{i,2n-i}r_{i+1} \quad \text{by Lemma 3.2}$$

$$\equiv r_{i+1}r_{i+2}r_{i+1}r_{i,2n-i} \quad \text{by an ELW of } h$$

$$\equiv r_{i+2}r_{i+1}r_{i+2}r_{i,2n-i} \quad \text{by an ELW of } f_3.$$

$$< h^{(i)}, g^{(i+1,j)} > = (r_{i,2n-i}r_{i+1} - r_{i+1}r_{i,2n-i})r_{i+2,j}r_{i+1} - r_{i,2n-i}(r_{i+1,j}r_{i+1} - r_{i+2}r_{i+1,j})$$

$$= r_{i,2n-i}r_{i+2}r_{i+1,j} - r_{i+1}r_{i,2n-i}r_{i+2,j}r_{i+1} \quad \text{where} \quad 1 \le i < j - 1 < 2n - (i + 1).$$

Then

$$r_{i,2n-i}r_{i+2}r_{i+1,j} \equiv r_{i+2}r_{i,2n-i}r_{i+1,j} \quad \text{by Lemma 3.2}$$

$$\equiv r_{i+2}r_{i+1,j}r_{i,2n-i} \quad \text{by case 5 of Proposition 3.2,}$$

and

$$r_{i+1}r_{i,2n-i}r_{i+2,j}r_{i+1} \equiv r_{i+1}r_{i+2,j}r_{i,2n-i}r_{i+1} \quad \text{by case 5 of Proposition 3.2}$$

$$\equiv r_{i+1,j}r_{i+1}r_{i,2n-i} \quad \text{by an ELW of } h$$

$$\equiv r_{i+2}r_{i+1,j}r_{i,2n-i} \quad \text{by an ELW of } g.$$

□

3.2 Reduced Forms for B_n

Since $R = \{f_1^{(i)}, f_2^{(i)}, f_3^{(i,j)}, f_4, g^{(i,j)}, h^{(i)}\}$, is a Gröbner-Shirshov basis for B_n, the R-reduced words are in fact reduced forms for B_n.

Theorem 3.2. *Any word in B_n can be presented in a form*

$$r_{nj_n}r_{n-1,j_{n-1}}\cdots r_{2j_2}r_{1j_1}$$

where $i-1 \leq j_i \leq 2n-i$ for all i.

Proof. We can claim that the word $r_{i,j_i}r_{k,j_k}$ is reduced if and only if $i < k$. The necessary part is directly given in the Proposition 3.2. If $i \geq k$, the same proposition implies that the word $r_{i,j_i}r_{k,j_k}$ is not reduced and furthermore it is equivalent to a reduced word in desired form. Hence any word in B_n can be presented in a form

$$r_{nj_n}r_{n-1,j_{n-1}}\cdots r_{2j_2}r_{1j_1}$$

where $i-1 \leq j_i \leq 2n-i$ for all i. □

Like in A_n, Proposition 3.2 can be used to inductively compute the normal form of any product of normal forms. Hence one can obtain whole multiplication table for B_n.

CHAPTER 4

COXETER GROUPS OF TYPE D_n

Definition 4.1. Coxeter group of type D_n ($n \geq 4$) is generated by $r_1, r_2, \cdots, r_n$ with defining relations:

$R1 : r_i^2 = 1$ where $1 \leq i \leq n$,

$R2 : r_i r_j = r_j r_i$ where $1 \leq i < j - 1 \leq n - 1$ except $(i, j) = (n - 2, n)$,

$R3 : r_i r_{i+1} r_i = r_{i+1} r_i r_{i+1}$ where $1 \leq i \leq n - 2$,

$R4 : r_{n-1} r_n = r_n r_{n-1}$ and

$R5 : r_{n-2} r_n r_{n-2} = r_n r_{n-2} r_n$

4.1 Gröbner-Shirshov Basis of D_n

Definition 4.2.

$$\widetilde{r}_{ij} = \begin{cases} r_{ij}, & j < n \\ r_{i,n-2} r_{n,j}, & j \geq n \end{cases}$$

Proposition 4.1. *The reduced Gröbner-Shirshov basis of Coxeter group D_n contains the following polynomials:*

$f_1^{(i)} = r_i r_i - 1$ *where* $1 \leq i \leq n$,

$f_2^{(i,j)} = r_i r_j - r_j r_i$ *where* $1 \leq i \leq j - 1 \leq n - 1$ *except* $(i, j) = (n - 2, n)$,

$f_3^{(i)} = r_i r_{i+1} r_i - r_{i+1} r_i r_{i+1}$ *where* $1 \leq i \leq n - 2$,

$f_4 = r_{n-1} r_n - r_n r_{n-1}$,

$f_5 = r_{n-2} r_n r_{n-2} - r_n r_{n-2} r_n$,

$g^{(i,j)} = \widetilde{r}_{ij} r_i - r_{i+1} \widetilde{r}_{ij}$ *where* $1 \leq i < j - 1 \leq 2n - (i + 2)$,

$h^{(i)} = \widetilde{r}_{i,2n-i} r_{i+1} - r_{i+1}\widetilde{r}_{i,2n-i}$ *where* $1 \leq i < n-1$,

$p = r_{n-2}r_n r_{n-1} r_{n-2} r_n - r_{n-1}r_{n-2}r_n r_{n-1} r_{n-2}$ *and*

$q = r_{n-2}r_n r_{n-1}r_{n-2}r_{n-1} - r_n r_{n-2} r_n r_{n-1} r_{n-2}$.

Proof. The polynomials $g^{(i,j)}$ and $h^{(i)}$ can be obtained by the following compositions similar to B_n. Because of this, we will not give detailed computations.

$< g^{(i,n-2)}, f_2^{(i,n)} >= g^{(i,n)}$ where $1 \leq i < n-2$.

$< g^{(i,j)}, f_2^{(i,2n-(j+1))} >= g^{(i,j+1)}$ where $n \leq j \leq 2n-i$ and $1 \leq i < n-2$.

$< g^{(i,2n-(i+2))}, f_3^{(i,j)} >= h^{(i)}$ where $1 \leq i < n-2$.

We only show that how the polynomials p and q are obtained.

$$< f_3^{(n-2)}, f_5 > = (r_{n-2}r_{n-1}r_{n-2} - r_{n-1}r_{n-2}r_{n-1})r_n r_{n-2} - r_{n-2}r_{n-1}(r_{n-2}r_n r_{n-2} - r_n r_{n-2} r_n)$$

$$= r_{n-2}r_{n-1}r_n r_{n-2} r_n - r_{n-1}r_{n-2}r_{n-1}r_n r_{n-2} - r_{n-2}f_4 + r_{n-1}r_{n-2}f_4 + p.$$

$$< f_5, f_3^{(n-2)} > = (r_{n-2}r_n r_{n-2} - r_n r_{n-2} r_n) r_{n-1} r_{n-2} - r_{n-2} r_n (r_{n-2}r_{n-1}r_{n-2} - r_{n-1}r_{n-2}r_{n-1})$$

$$= r_{n-2}r_n r_{n-1} r_{n-2} r_{n-1} - r_n r_{n-2} r_n r_{n-1} r_{n-2}$$

$$= q.$$

□

The following proposition is similar to Proposition 3.2. This helps to prove the set $R = \{f_1^{(i)}, f_2^{(i)}, f_3^{(i,j)}, f_4, f_5, g^{(i,j)}, h^{(i)}, p, q\}$ is reduced Gröbner-Shirshov basis for D_n and to multiply reduced forms.

Proposition 4.2. *Let* $1 \leq i, k \leq n$, $i \leq j \leq 2n-i$ *and* $k \leq l \leq 2n-k$. *If* $i > k$ *then the word* $r_{ij}r_{kl}$ *is reduced. Otherwise*

$$\widetilde{r}_{ij}\widetilde{r}_{kl} \equiv \begin{cases} \widetilde{r}_{k+1,l+1}\widetilde{r}_{ij} & (l < n) \wedge (l < j < 2n-(l+1)) \wedge (j,l) \neq (n, n-2) \\ \widetilde{r}_{k+1,l}\widetilde{r}_{i,j-1} & (j < n) \wedge (k \leq j \leq l \leq 2n-(j+1)) \wedge (j,l) \neq (n-1, n) \\ \widetilde{r}_{k+1,l}\widetilde{r}_{i,j+1} & (j \geq n) \wedge (2n-(j+1) \leq l \leq j < 2n-k) \wedge (l \neq n-1, n) \\ \widetilde{r}_{k+1,l-1}\widetilde{r}_{ij} & (l > n) \wedge (2n-(l+1) < j < l) \wedge (j \neq n-1, n) \\ \widetilde{r}_{kl}r_{ij} & ((j < k-1) \wedge (j,k) \neq (n-2, n)) \vee ((j > 2n-k) \wedge (l \neq n-1, n)) \\ \widetilde{r}_{il} & (j = k-1) \wedge ((j,k) \neq (n-1, n)) \\ \widetilde{r}_{i,2n-(l+1)} & (j = 2n-k) \wedge ((j,k) \neq (n,n)) \end{cases}$$

The proof of general cases is similar to prof given in Proposition 3.2. Hence we only proof the exceptional cases.

Proof.

$$\begin{aligned} \tilde{r}_{in}\tilde{r}_{k,n-1} &\equiv \tilde{r}_{in}\tilde{r}_{k,n-3}\tilde{r}_{n-2} \\ &\equiv \tilde{r}_{k+1,n-2}\tilde{r}_{in}\tilde{r}_{n-2} \quad \text{by case 1} \\ &\equiv \tilde{r}_{k+1,n-2}\tilde{r}_{i,n-3}\tilde{r}_{n-2}\tilde{r}_{n}\tilde{r}_{n-2} \quad \text{by a sequence of ELWs of } f_2 \\ &\equiv \tilde{r}_{k+1,n-2}\tilde{r}_{i,n-3}\tilde{r}_{n}\tilde{r}_{n-2}\tilde{r}_{n} \quad \text{by an ELW of } f_5 \\ &\equiv \tilde{r}_{k+1,n}\tilde{r}_{i,n-2}. \end{aligned}$$

$$\begin{aligned} \tilde{r}_{ij}\tilde{r}_{k,n-1} &\equiv \tilde{r}_{ij}\tilde{r}_{k,n-2}\tilde{r}_{n-1} \\ &\equiv \tilde{r}_{k+1,n-2}\tilde{r}_{i,j+1}\tilde{r}_{n-1} \quad \text{by case 3} \\ &\equiv \tilde{r}_{k+1,n-2}\tilde{r}_{i,n+2}\tilde{r}_{n-1}\tilde{r}_{n+3,j+1} \quad \text{by a sequence of ELWs of } f_2 \\ &\equiv \tilde{r}_{k+1,n-2}\tilde{r}_{i,n-3}\tilde{r}_{n-2}\tilde{r}_{n}\tilde{r}_{n-1}\tilde{r}_{n-2}\tilde{r}_{n-1}\tilde{r}_{n+3,j+1} \\ &\equiv \tilde{r}_{k+1,n-2}\tilde{r}_{i,n-3}\tilde{r}_{n}\tilde{r}_{n-2}\tilde{r}_{n}\tilde{r}_{n-1}\tilde{r}_{n-2}\tilde{r}_{n+3,j+1} \quad \text{by an ELW of } p \\ &\equiv \tilde{r}_{k+1,n}\tilde{r}_{i,j+1} \quad \text{by a sequence of ELWs of } f_2. \end{aligned}$$

$$
\begin{aligned}
\tilde{r}_{i,n-1}\tilde{r}_{k,l} &\equiv \tilde{r}_{i,n-1}\tilde{r}_{k,n-2}\tilde{r}_{nl} \\
&\equiv \tilde{r}_{k+1,n-1}\tilde{r}_{i,n-1}\tilde{r}_{nl} \quad \text{by case 4} \\
&\equiv \tilde{r}_{k+1,n-1}\tilde{r}_{i,n-2}\tilde{r}_{n}\tilde{r}_{n-1}\tilde{r}_{n+2,l} \quad \text{by a sequence of ELWs of } f_2 \\
&\equiv \tilde{r}_{k+1,n-1}\tilde{r}_{i,n-2}\tilde{r}_{n}\tilde{r}_{n-1}\tilde{r}_{n+2,l} \\
&\equiv \tilde{r}_{k+1,n-1}\tilde{r}_{i,n-3}\tilde{r}_{n}\tilde{r}_{n-2}\tilde{r}_{n}\tilde{r}_{n+3,l} \quad \text{by an ELW of } f_5 \\
&\equiv \tilde{r}_{k+1,n}\tilde{r}_{in}\tilde{r}_{n+3,l} \quad \text{by a sequence of ELWs of } f_2 \\
&\equiv \tilde{r}_{k+1,l-1}\tilde{r}_{in}.
\end{aligned}
$$

□

Theorem 4.1. *The set* $R = \{f_1^{(i)}, f_2^{(i)}, f_3^{(i,j)}, f_4, f_5, g^{(i,j)}, h^{(i)}, p, q\}$ *is a reduced Gröbner-Shirshov basis for* B_n.

Proof. The proof is just the adaptation of the proof of Theorem 3.1. □

4.2 Reduced Forms for D_n

Since $R = \{f_1^{(i)}, f_2^{(i)}, f_3^{(i,j)}, f_4, f_5, g^{(i,j)}, h^{(i)}, p, q\}$, is a Gröbner-Shirshov basis for D_n, the R-reduced words are in fact reduced forms for D_n.

Theorem 4.2. *Any word in* B_n *can be presented in a form*

$$\tilde{r}_{nj_n}\tilde{r}_{n-1,j_{n-1}}\cdots\tilde{r}_{2j_2}\tilde{r}_{1j_1}$$

where $i-1 \leq j_i \leq 2n-i$ *for all* i.

Proof. The proof is just the adaptation of the proof of Theorem 3.2. □

Proposition 4.2 can be used to inductively compute the normal form of any product normal forms. Hence one can obtain whole multiplication table of D_n.

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