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AUTOMORPHISM GROUPS OF PURE INTEGRAL OCTONIONS AND APPLICATIONS IN PHYSICS

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ABSTRACT

The automorphism groups of the pure integer octonions are studied. The seven-dimensional matrix representation of the automorphism group of pure integer octonions constituting the root system of E_7 is constructed. It is shown that $G_2(2)$ which is a finite subgroup of the exceptional Lie group G_2, of order 12096, has four maximal subgroups. The structure of $G_2(2)$ and its maximal subgroups are identified and their possible implications in physics are discussed. The maximal subgroups of $G_2(2)$ having the orders 432,192,192 and 336 preserve the octonionic root systems of E_6, SO(12), $SU(2)^3 \times SO(8)$ and SU(8), respectively.

Matrix representation of the automorphism group of the octonionic imaginary units $\pm e_i$ (i=1,...7) is also constructed. The group $2^3 \cdot PSL_2(7)$ of order 1344 is the non-split extension of the elementary Abelian group of order 8 by Klein's simple group $PSL_2(7)$ of order 168. The maximal subgroups of $2^3 \cdot PSL_2(7)$ and their structure are analysed. Seven-dimensional matrix representation of another group $2^3{:}PSL_2(7)$ of order 1344 which is the split extension of the elementary Abelian group of order 8 by $PSL_2(7)$ is constructed. Its maximal subgroups are studied.

The character tables and the conjugacy classes of the groups $G_2(2)$, $2^3 \cdot PSL_2(7)$, $2^3{:}PSL_2(7)$ and all maximal and Sylow subgroups are computed.

ÖZ

Oktonyonların otomorfizma grupları çalışıldı. E_7 Lie cebirinin köklerini teşkil eden oktonyonların otomorfizma grubu oluşturuldu ve yedi boyutlu matris temsilleri bulundu. Dört maksimal altgruba sahip olan $G_2(2)$ grubu aynı zamanda G_2 Lie grubunun sonlu bir alt grubudur. $G_2(2)$ grubunun ve maksimal altgruplarının yapıları incelenerek, fizikteki uygulamaları tartışıldı. $G_2(2)$ grubunun maksimal altgruplarının eleman sayıları 432, 192, 192′ ve 336 olup, bunların sırasıyla E_6, $SO(12)$,$SU(2)^3 \times SO(8)$ ve $SU(8)$′ in oktonyonlardan oluşturulan kök sistemlerini koruduğu gösterildi.

Ayrıca oktonyonların $\pm e_i$ $(i=1,\dots,7)$ otomorfizma gruplarının matris temsilleri oluşturuldu. Elde edilen $2^3.PSL_7(7)$ grubunun eleman sayısı 1344 olup 8 elemanlı temel Abelian bir grup ile basit bir grup olan $PSL_2(7)$′ nin ayrışmadan (non-split) genişletilmesiyle elde edilebileceği gösterildi. Maksimal altgrupları ve yapıları çalışıldı. Eleman sayısı yine 1344 olan fakat yapısı farklı olan $2^3{:}PSL_7(7)$ grubunun yedi boyutlu matris temsili elde edilerek, maksimal altgruplarıyla beraber yapısı ortaya konuldu.

$G_2(2)$, $2^3PSL_2(7)$, $2^3{:}PSL_2(7)$ grupları ve maximal altgrupları ile Sylow altgruplarının karakter tabloları oluşturuldu.

CHAPTER 1

INTRODUCTION

Symmetries have scored great success in the simplification of the laws of physics as well as in the description of the physical states. Group theory has been applied in many different fields of physics such as, classification of eigenfunctions, analysis of different types of phase transitions in solids, classifications of energy levels of atomic, molecular, solid-state and nuclear systems. It is also widely used in the symmetry considerations of nuclear forces and the classification of elementary particles and the forces combining them.

The standard, model $SU(3)_C \times SU(2)_L \times U(1)$ is the unique manifestation of the fundamental forces such as the strong interaction and the electroweak interaction. Group theoretical approach has always shaped our perception of the nature.

In physics, Lie groups and the finite subgroups of the rotation group in three dimensions have played important roles. Further unification of the standard model in the larger groups like $SU(5) \approx E_4$, $SO(10) \approx E_5$ and E_6[1, 10, 11, 17] changed our understanding not only of the fundamental interactions of the fundamental particles besides the gravitation but also led to a deeper insight of the modern cosmology. Unification of all interactions including the gravity manifested itself in the superstring theory with an $E_8 \times E_8$ gauge symmetry[12,14]. It turned out that all Lie groups in the E-series ($E_1 \approx SU(2)$,

$E_2 \approx SU(2)\times SU(2)$, $E_3 \approx SU(3)\times SU(2)$, $E_4 \approx SU(5)$, $E_5 \approx SO(10)$, E_6, E_7 and E_8) have been proposed as models toward the solutions of problems of particle physics.

The principal features of the Lie algebras associated with the Lie groups are encoded in the root systems of the Lie algebras. Some of the selected Lie algebras such as SU(2), SO(4), SO(5) , SO(8), F_4, E_8 have peculiar root systems which can be described by the discrete elements of the division algebras[18, 23]. For instance the root system of SO(8) can be described by 24 quaternions which themselves form a finite subgroup of SU(2). Similarly the 240 roots of E_8 can be described by the integer octonions which are closed under octonion multiplication.

The root system of E_8 is also constructed with icosians. Koca has worked out the details of this construction and confronted it with the root system of E_8 obtained by the integer octonions[24]. Koca has also proposed a model where three families of leptons and the electroweak bosons are assigned to the irreducible representations of the binary tetrahedral group of order 24, a discrete subgroup of SU(2)[25].

On the other hand finite subgroups of SO(3) are either used for molecule symmetries or the crystal symmetries together with the discrete translations[33]. Liquid crystals further require the lifting of these symmetries to SU(2)[32]. Ordinary Lie groups or super Lie groups had great impact both in nuclear and particle physics as classification of symmetries and/or gauge symmetries.

Simple finite groups have, for nearly a century, been a central research area in mathematics and their classifications have recently been completed[5]. They are the Lie type groups, alternating groups and the 26 sporadic groups. In 1980 McKay [31, 36] has noted an interesting correspondence between the finite subgroups of SU(2) and the A-E-D classifications of the Lie algebras. In brief, the eigenvectors of the Cartan matrices of the affine Lie algebras of the A-D-E series constitute the character table of the associated finite subgroups of SU(2). It seems that there exists an interesting interplay between certain Lie algebras and the finite subgroups of SU(2).

The $SL_2(7)$ group of order 336, which is the double cover of the finite subgroup of SU(3) of order 168, has some relations with the conformal field theory based on the G_2 affine Lie algebra[4]. A detailed study regarding the group $PSL_2(7)$ of order 168, which is isomorphic to the projective modular group on the finite field with seven elements, has been extensively worked out by Bauer and Itzykson[2].

It is all well known that the automorphism group of the octonion algebra is the exceptional Lie group G_2[26]. For the root system of E_8 can be described by the integer octonions its automorphism group is of great interest. This problem seems to be rather involved but the automorphism group of the pure imaginary octonionic roots corresponding to the root system of E_7 can be handled relatively easily[22].

The group $2^3{\cdot}PSL_2(7)$ is first discussed by Coxeter[7]. The structure of the group $2^3{:}PSL_2(7)$ is worked out by Conway[6] proving its extraordinary structure related to the behaviour of its two non-conjugate maximal subgroups $PSL_2(7)$. The groups $PSL_2(7)$ and $2^3{:}PSL_2(7)$ arise also as the automorphism group of the binary Hamming codes of the types E_7 and E_8 respectively[35]. Some of the subgroups of $2^3{\cdot}PSL_2(7)$ and $2^3{:}PSL_2(7)$, of orders 24,48, and 32 turned out to be the Demazure[8] and Tits[37] subgroups of the Lie groups SU(3), G_2, and SO(5). Their relevance to physics has been studied by Michel, Patera and Sharp[34].

In this thesis we focus mostly on the automorphism group of the 126 pure imaginary octonions constituting the root system of E_7. We determine its order and give an explicit 7×7 irreducible representation of the group of concern. It turns out that it is the Chevalley's group $G_2(2)$[3]. We notice that the maximal subgroups of the group $G_2(2)$ preserve the octonionic root systems of the maximal Lie algebras of E_7 with one exception[27].

Another interesting group $2^3{\cdot}L_2(7)$ is the one with the order of 1344 which is the automorphism group of the set of octonions $\pm e_i$ ($i=1,\ldots,7$). The group $2^3{\cdot}L_2(7)$ and $G_2(2)$ share the same maximal subgroup $4{\cdot}S_4{:}2$ of order 192. We also note that understanding the structure of the group $2^3{\cdot}L_2(7)$ also makes it easier to analyse the structure of the group $2^3{:}L_2(7)$ which is also of the same order and has the same character of the group $2^3{\cdot}L_2(7)$. Many interesting features of these groups have been worked out and their character tables with the relevant irreducible representations have been constructed[28].

We organise the thesis as follows. In chapter 2 we introduce all the relevant group theoretical concepts of finite groups. We do not give the proofs of theorems but explain most of them with examples to make them self contained. This chapter can be regarded as a kind of group theoretical dictionary.

In chapter 3 we introduce the octonion algebra and construct the root system of E_8 and then decompose the roots under its maximal Lie algebra $SU(2)\times E_7$ where pure imaginary octonions constitute the root system of E_7.

Chapter 4 constitutes the main topic of the thesis where we worked out the details of the automophism group $G_2(2)$ of the octonionic roots of E_7. Chapter 5 is devoted to the study of the groups $2^3{\cdot}L_2(7)$ and $2^3{:}L_2(7)$. Chapter

6 involves the conclusive remarks where we briefly discuss our results and make comments on the possible use of these groups in particle physics. We tabulate the character tables in Appendix A and briefly discuss the computer programmes, which we have used in the analysis of the group structure, in Appendix B.

CHAPTER 2

FINITE GROUPS

2.1. Introduction

We start with the basic properties of the finite groups and introduce some related systems that are necessary for our latter applications. Since the finite matrix groups and their applications are the main topic of this work, we had better discuss them in detail[15].

2.2. Group Postulates

A set of symmetry transformations on a physical system has the mathematical properties that are associated with a group. The rotations, translations in space and time, and most symmetry transformations on physical systems have the group properties. Symmetries are described using a branch of mathematics known as group theory. A certain definition of a group is as follows: A **group** is a set of elements (often in practice represented by matrices) that can be combined together with the following four postulates:

- **Closure Law.** The group is closed under multiplication

$$(g_1 \in G,\ g_2 \in G) \Rightarrow g_1 g_2 \in G \tag{2.1}$$

- **Associative Law.** The elements of the group G are combined together under multiplication, that is,

$$g_1(g_2g_3) = (g_1g_2)g_3, \quad g_i \in G \tag{2.2}$$

- **Unit element.** Among the group elements there exist one and only one unit element g_0(or I or 1) which has the property,

$$g_0g_i = g_ig_0 = g_i, \quad g_0, g_i \in G \tag{2.3}$$

- **Inverse element.** For every element g_i of the group G there exists an inverse element g_i^{-1} such that

$$g_ig_i^{-1} = g_i^{-1}g_i = 1 \tag{2.4}$$

A group is called **finite** if it has a finite number of elements. The number of elements of a finite group is called the **order** of the group and is denoted by the symbol $|G|$. If the elements of a group are denumerable infinite, then the group is called **infinite discrete** group. If the elements of a group form a continuum, then the group is said to be a **continuous** group. Every finite group is isomorphic to a suitable group of $n \times n$ non-singular matrices.

There are a number of special groups. We mention just a few here that will be used later. If all elements of a group commute with each other ($g_ig_j = g_jg_i$), then it is called an **Abelian** group. If a group can be generated by repeated applications of one element, then it is called a **cyclic** group. All cyclic groups are Abelian, since each element commutes with itself.

There are various ways to construct the group and to understand the structure of a given group. The subgroups play important roles in understanding the structure of the group. For a subgroup in a group is often useful for the construction of the group and its character table, we will first discuss the subgroups and their roles in a group.

2.3. Subgroups

A subset H of a given group G is called a subgroup if the elements of H form a group under group postulates. Every group has at least two **trivial** subgroups, the group itself and the identity element. A subgroup different from G and the unit element is called a **proper** subgroup. A proper subgroup H is called a **maximal** subgroup if there is no subgroup of G containing H and different from H and G. A subgroup H of G is called a **normal** subgroup if the following condition holds:

$$g_i H g_i^{-1} \in H \text{ for all } g_i \in G \tag{2.5}$$

A group which has no proper normal subgroup is called a **simple** group, otherwise a **composite** group. If a group contains no invariant Abelian subgroup besides the identity element, it is said to be a **semisimple** group.

In this work subgroups, normal subgroups, maximal groups and structure of the related groups are determined by the use of computer. In the latter chapters we will explain how the computer programs are implemented to obtain certain properties of the groups of concern.

2.3.1. Basic Theorems

Consider a finite group G with a subgroup H of G. The fundamental question at this point is about the orders of H and G. Lagrange's theorem partially answers this question. **Lagrange's Theorem** states that the order of a subgroup of a finite group is always a divisor of the order of the parent group. The order of the group divided by the order of the its subgroup |G|/|H| always gives an integer number. This integer number is called **index** of the subgroup H. In spite of this theorem, there is no guarantee that G will contain a subgroup of order n which is a divisor of the order of the group. But if n is a prime divisor of the order of G, then G contains a subgroup of order n. Since the order of an element is equal to the order of the subgroup generated by that element, then the order of every element of G is a divisor of the order of G.

Three remarkable theorems of **Sylow** deal with the existence of certain subgroups in a finite group and with their properties in relation to the group.

- **Sylow's first theorem.** Let p^n be the highest power of the prime number p that divides the order of the group G. Then, G contains at least one Sylow p-subgroup of order p^n.

- **Sylow's second theorem.** In a finite group the Sylow p-subgroups (for a fixed prime p) is isomorphic to each other.

- **Sylow's third theorem.** The number of Sylow p-subgroups of a finite group G is of the form 1+kp and is a divisor of the order of G.

2.3.2. Properties of Maximal Subgroups

In this work, we consider only finite groups, therefore the maximal subgroups always exist. It is hard to calculate all maximal subgroups of a group. Especially, nilpotent and supersolvable groups can be characterised by properties of their maximal subgroups. Huppert's two remarkable theorems that are about the supersolvable and nilpotent groups, are important at this point.

- The elements $g_1^{-1}g_2^{-1}g_1g_2$ of a group G is called commutator of g_1 and g_2, denoted (g_1, g_2). From its definition $(g_1, g_2)=1$ if, and only if, $g_1g_2=g_2g_1$. The subgroup H of G generated by all commutators (g_i, g_j) is called **derived** group. Clearly, H is a fully invariant subgroup of G. A group G is said to be **solvable** if the sequence $G \supseteq H^1 \supseteq H^2 \supseteq \ldots \supseteq \ldots \supseteq H^{(i)} = 1$, where each group $H^{(i)}$ is the derived group of the preceding, group which terminate in the identity in a finite number of steps. If each factor group $H^{(i-1)}/H^{(i)}$ of a solvable group is cyclic, then the group is said to be **supersolvable.** If the factor set $H^{(i-1)}/H^{(i)}$ is in the center of $G/H^{(i)}$, then the group is said to be **nilpotent** group.

- **Huppert's theorem.** If the index of every maximal subgroup of a finite group G is a prime number, then G is **supersolvable.** The other theorem is about the nilpotent group. If a maximal subgroup of a group is normal then the group is called **nilpotent.**

The intersection of all maximal subgroups of a group G is called the **Frattini** subgroup. This is also called **characteristic** subgroup of a group G. It is obvious that if H is a proper subgroup of the group G (not maximal in G), then there exists a maximal subgroup of G containing H. The maximal subgroups of a group play important roles in the group extension together with the normal subgroups of the group.

2.3.3. Cosets and the Properties of Normal Subgroups

Consider a group G and a subgroup H. The set of elements gH, $g \in G$ is called a **left coset** of H. Similarly the set of elements Hg, $g \in G$ is called a **right coset** of H. If G is finite then we can decompose G in terms of cosets (say left cosets)

$$G = H + g_1H + g_2H + \cdots + g_nH \tag{2.6}$$

We state this relation in the form

$$\#\ \text{cosets of}\ H = \frac{\#\ \text{elements in a group G}}{\#\ \text{elements in subgroup H}} \tag{2.7}$$

The right cosets or left cosets of the subgroup H in the group G form a group if, and only if, H is a normal subgroup. At this point it is appropriate to define the factor group. The **factor** group of the group is the group formed by the cosets of the normal subgroup H. Determination of normal subgroups of a group G is rather easy, since the set of generators of the normal subgroup is the union of complete conjugate classes of the elements of G. Hereafter we will denote by N the normal subgroup of G.

2.4. The Concept of Homomorphy

A very fundamental concept of modern mathematics is that of a mapping of a set S into a set T. A single valued mapping of the elements in S onto a certain subset of T is called **homomorphy**, if the product of two elements is mapped onto the product of the image elements. The homomorphy is called **homomorphism** if every element of T is an image element of an element of S, so that the set T is said to be homomorphic to the set S. Two groups are said to be **homomorphic** if there is a many-to-one correspondence between the elements of two groups. This statement implies that a given group G is homomorphic to its factor group H, if $H \approx G/N$. In this type of homomorphism N is called the **kernel** of homomorphism.

2.4.1. The Isomorphy Concept

If a group G is mapped onto the group G' homomorphically, then multiplication in G' parallels that in G. A one-to-one mapping $G \Leftrightarrow G'$ of the elements of group G and G′ is called an **isomorphism** if whenever $g_1 \Leftrightarrow g_1'$ and $g_2 \Leftrightarrow g_2'$ then $g_1 g_2 \Leftrightarrow g_1' g_2'$. Thus, the orders of the two groups must be the same. Then from the abstract group theoretical point of view, the two groups are identical even though the meaning of elements in the two groups may be different.

2.4.2. Automorphism of a Group

An isomorphic mapping of a group onto itself is called an **automorphism.** If α: $g \Leftrightarrow g^{\alpha}$ is a one-to-one mapping of a group G onto itself, α will be an automorphism if, and only if, $g_i g_j = g_k$ implies that $g_i^{\alpha}\, g_j^{\alpha} = g_k^{\alpha}$. The transformation of elements of a group G by a fixed element r is an automorphism. Formally we can write

$$\alpha:\ r g_i r^{-1} = g_i' \ \text{and} \ r g_j r = g_j' \ \Rightarrow \ r g_i g_j r^{-1} = g_i' g_j' \qquad (2.8)$$

where g_i, $g_j \in$ G and g_i', $g_j' \in$ G and r is a fixed element of G. This is called the **inner automorphism.** All of the automorphisms of a group G, different from inner automorphism are called **outer automorphism.**

2.4.3 Extensions of Groups

There are a number of ways for the group extensions. Generally the groups are extended from their subgroups. Although we will mostly consider the extension of a group from its normal subgroup N in our constructions, let us first briefly discuss the possible group extensions.

2.4.3.1. Direct Product

The groups are constructed by the direct product of two groups as follows. Let H and H′ be two groups with h_i, $h_j \in$ H and h_i', $h_j' \in H'$. The multiplication is defined by

$$(h_i,h_j)(h_i',h_j') = (h_ih_j,\ h_i'h_j') \tag{2.9}$$

Since h_i and h_j are elements of H and h_i' and h_j' elements of H′ then (h_ih_j, $h_i'h_j'$) is an element of H×H′. It is clear that H×H′ is isomorphic to H′×H. The order of H×H′ is the product of the orders of H and H′. Now H and H′ are both subgroups of H×H′. A group is extended from its subgroups, in a way such that

$$G = H_1 \times H_2 \times \cdots \times H_m \tag{2.10}$$

if every element of each subgroup commutes with every element of all the other subgroups, and each element g_i of G can be expressed in a unique way in the form $g_i = h_1h_2 \cdots h_m$ where $h_i \in H_i$.

Direct product of the groups defined in Equation (2.9) is applicable for matrix representations of the groups as follows: if H is a group of n×n matrices and H′ is a group of m×m matrices then we can represent H×H′ by matrices

$$(h,h') = h \oplus h' = \begin{bmatrix} h & 0 \\ 0 & h' \end{bmatrix} \tag{2.11}$$

Then the matrix multiplication gives the operation described above on H×H′, such as

$$(h_i,h_j)(h_i',h_j') = \begin{bmatrix} h_ih_j & 0 \\ 0 & h_i'h_j' \end{bmatrix} \tag{2.12}$$

2.4.3.2. Semi-Direct Product (Split Extension)

Consider two groups H and N and for every element $h \in H$ and an automorphism of $k \in N$, then the symbol [k,h] form a group under the product rule,

$$[k_1, h_1]\cdot[k_2, h_2] = [k_1^{h_2}k_2, h_1h_2]; \quad k_i^{h_j} = h_jk_ih_j^{-1} \tag{2.13}$$

called the **semi-direct** or **normal** product of N by H.

Given two groups N and H, with $k \in N$ and $h \in H$ respectively, if G is semi-direct product of N and H, symbolically N:H then the elements [k, 1] form a normal subgroup isomorphic to N and the elements [1, h] of G form a subgroup isomorphic to H. In general G is the semi-direct product of N by H if, and only if N is a normal subgroup of G and H is the factor group with intersection of H and N being only the unit element. This is an example of the split extension. If H, in the above construction, is a factor group but not a subgroup of G then G is the **non-split** extension of N by H.

2.5. Representations of Groups by Matrices

The matrix representations of groups are important, because from a practical point of view, many groups have their natural representations in terms of matrices. If there is a correspondence between a set of square matrices and the elements of a group, then the set of $n \times n$ non-singular matrices form a group under matrix multiplication. Suppose that g_1 and g_2 are two elements of the group G. There exist associated $m \times m$ non-singular matrices with the following properties:

$$A(g_1) = a_{ij}(g_1), \quad (i,j=1,2,\dots,m),$$
$$A(g_2) = a_{ij}(g_2), \quad (i,j=1,2,\dots,m), \tag{2.14}$$

where a_{ij} is the member of real or complex numbers. If they satisfy the relation

$$A(g_1)A(g_2) = A(g_1g_2), \; (g_1, g_2, g_1g_2 \in G). \tag{2.15}$$

Note that we have $A(1) = I_m$ where I_m is $m\times m$ unit matrix and $A(x^{-1}) = [A(x)]^{-1}$ is the matrix inverse. There are a number of ways to construct the matrix representation of a group. Generally groups are formed by symmetry operations. In order to construct any matrix representation of a group, we choose a vector space (carrier space) and then define the operations of G on this space.

Historically, group theory began with finite permutation groups. If we have n objects, then there are n! arrangements which form a group S_n which is called the **symmetric** group. At this point consider permutations of n objects and a set of finite number of variables $x_1, x_2,\dots,x_n$. A general linear transformation on n variables replaces each variable $x_1, x_2,\dots,x_n$, by a linear combination of the following form

$$A: \quad x_i \rightarrow x'_i = a_{1i}x_1 + a_{2i}x_2 + \cdots + a_{ni}x_n \tag{2.16}$$

The coefficients a_{ij} are real or complex numbers in general. It is possible to write the last Equation in the form

$$\begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix} \rightarrow \begin{bmatrix} x'_1 \\ x'_2 \\ \cdots \\ x'_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix} \tag{2.17}$$

(a) All finite groups are the subgroups of a permutation group and it is possible to write the matrix representation of a permutation group, by linear substitution $x_i \to x_j$. The corresponding matrix gives a matrix representation of the permutation group.

(b) For any finite group there are infinitely many matrix representations. Proof of this statement is easy. Consider a matrix representation A(g) of a group G and any non-singular matrix T. We can define $B(g) = TA(g)T^{-1}$. Then

$$\begin{aligned} B(g_1g_2) &= TA(g_1g_2)T^{-1} = TA(g_1)A(g_2)T^{-1} \\ &= TA(g_1)T^{-1}TA(g_2)T^{-1} = B(g_1)B(g_2) \end{aligned} \tag{2.18}$$

where B is also a matrix representation of G. This transformation is known as the **similarity** transformation, and the representations which are obtained by similarity transformations are called **equivalent** representation.

(c) Let A(g) and B(g) are matrix representations of the group G of dimensions n and m respectively, then the matrix

$$C(g) = \begin{bmatrix} A(g) & 0 \\ 0 & B(g) \end{bmatrix} \tag{2.19}$$

of degree n+m also form a matrix representation of G. The matrix C(g) is defined above is called the direct sum of A(g) and B(g), denoted by $C(g) = A(g) \oplus B(g)$. The representation C(g) is said to be decomposable and A(g) and B(g) are its components.

As an example, we take the group S_3. The symmetric group S_3 can be obtained by permuting three objects that is denoted by $(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$ of the components of a vector **x**. The carrier space of this representation is three dimensional Euclidean space. The permutations of elements are as follows (the bottom row of each element is the result of permutation):

$$\rho_0 = \begin{pmatrix} 123 \\ 123 \end{pmatrix};\ \rho_1 = \begin{pmatrix} 123 \\ 312 \end{pmatrix};\ \rho_2 = \begin{pmatrix} 123 \\ 231 \end{pmatrix};\ \rho_3 = \begin{pmatrix} 123 \\ 132 \end{pmatrix};\ \rho_4 = \begin{pmatrix} 123 \\ 321 \end{pmatrix};\ \rho_5 = \begin{pmatrix} 123 \\ 213 \end{pmatrix}$$

$$\begin{array}{llllll}
x_1' \to x_1 & x_1' \to x_3 & x_1' \to x_2 & x_1' \to x_1 & x_1' \to x_3 & x_1' \to x_2 \\
x_2' \to x_2 & x_2' \to x_1 & x_2' \to x_3 & x_2' \to x_3 & x_2' \to x_2 & x_2' \to x_1 \\
x_3 \to x_3 & x_3' \to x_2 & x_3' \to x_1 & x_3' \to x_2 & x_3' \to x_1 & x_3' \to x_3
\end{array} \tag{2.20}$$

We can construct transformation matrices, as indicated in Equation (2.17), so they take the forms

$$\begin{gathered}
A(\rho_0) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix};\ A(\rho_1) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix};\ A(\rho_2) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}; \\
A(\rho_3) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix};\ A(\rho_4) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix};\ A(\rho_5) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix};
\end{gathered} \tag{2.21}$$

Each matrix element transforms the coordinate x_i into x_i' and they altogether form the group S_3 under group postulates.

We can find other representations of S_3 by considering the set of symmetries of an equilateral triangle which lies in a plane. We chose one unit vector α_1 along the x-axis and the second α_2 along the y-axis. The matrices which are then obtained by making symmetry operations, as in Figure (2.1), form the symmetric group S_3.

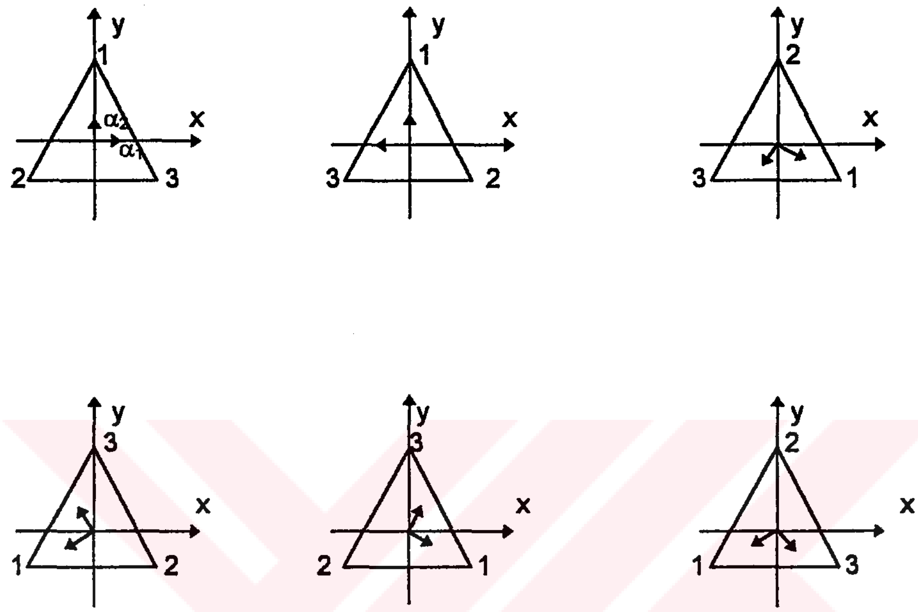


Figure 2.1 Symmetries of the Equilateral Triangle.

The matrices $A(g_i)$ take the forms

$$A(g_0)=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix};\ A(g_1)=\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix};\ A(g_2)=\frac{1}{2}\begin{bmatrix}-1 & -\sqrt{3}\\ \sqrt{3} & -1\end{bmatrix};$$

(2.22)

$$A(g_3)=\frac{1}{2}\begin{bmatrix}-1 & \sqrt{3}\\ -\sqrt{3} & -1\end{bmatrix};\ A(g_4)=\frac{1}{2}\begin{bmatrix}1 & \sqrt{3}\\ \sqrt{3} & -1\end{bmatrix};\ A(g_5)=\frac{1}{2}\begin{bmatrix}1 & -\sqrt{3}\\ -\sqrt{3} & -1\end{bmatrix};$$

The matrices $A(g_i)$ form also a matrix representation of S_3. The matrix representation $A(g)$ of a group G is **reducible** over a given field if there exists a non-singular matrix T such that

$$C(g) = T^{-1}A(g)T = \begin{bmatrix}B(g) & 0\\ E(g) & D(g)\end{bmatrix}. \tag{2.23}$$

Here B(g) and D(g) are representations of G. If E(g) is a null matrix, then the representation A(g) is completely reduced.

2.6. Construction of Character Tables and Conjugate Classes

The conjugacy classes and the character tables involve numbers that are known as the characters of the given representation. The character χ of a matrix representation A(g) is defined by

$$\chi(g) = Tr[A(g)] = \sum_{i=1}^{i=n} a_{ii}(g) \qquad (2.24)$$

Thus the characters are the elements of the given field. All equivalent representations have the same characters over which the matrix representations are defined. It is easy to see that if the dimension of the representation is 1, then $\chi = A$.

2.6.1. Conjugate Classes of a Group

Consider a set S of a group G with elements $s \in S$ and $g \in G$. If $g_1^{-1}Sg_1=S$, then $S=g_1Sg_1^{-1}$. If also $g_2^{-1}Sg_2=S$, then $S=(g_1g_2)^{-1}S(g_1g_2)$. Hence the set of $g_i \in G$, such that $S= g_i^{-1}Sg_i$, is a subgroup of G which we call the **normalizer** of S in G, and we designate this by $N_G(S)$. Again the set of $g_i \in G$ such that $s= g_i^{-1}sg_i$, for all $s \in S$, may similarly be shown to be a subgroup of G which we call the **centralizer** of S in G and designate it by $C_G(S)$. It is obvious that if S consists of a single element, the centralizer and normalizer are identical; moreover, always $C_G(S) \subseteq N_G(S)$. The centralizer of G in G is called the **center** of G. Formally the sets $N_G(S)$ and $C_G(S)$ are obtained by transforming the elements of the set S such that

$$\begin{aligned} N_G(S) &= \{g \in G \mid gs = sg \text{ for all } s \in S\} \\ C_G(S) &= \{g \in G \mid gS = Sg\}. \end{aligned} \tag{2.25}$$

The center is clearly intersection of all the normalizers of elements in G. By definition the center is an Abelian normal subgroup.

Consider a set S obtained as follows: Let G be a group and the set S of elements of the form $g_f^{-1} g_i g_f$. Both g_i and g_f are elements of G and g_f is a fixed element. Then we call the set S a **class of conjugation** of the group G. If the whole group G is partitioned into disjoint sets, they are then called **classes of conjugate** elements or **conjugate classes** or **conjugacy classes** of the group G. Formally we can write

$$C_i(p_i) = \{g \in G \mid h_i = g_f^{-1} g_i g_f\} \tag{2.26}$$

In our latter applications a number of notations designating the conjugate classes, character tables, etc. of a group will be demonstrated, in general, in the form of Table(2.1). The first row of the table corresponds to the name of the class, which is denoted by $C_i(p_i)$ and p_i represents the order of each element. The second and third rows of the table correspond to the number of elements in a class and the characters of each element, respectively.

Table 2.1. The Conjugate Classes of a Given group G

Class	$C_1(1)$	$C_2(p_2)$	...	$C_1(p_n)$
# Elements	h_1	h_2	...	h_n
Characters	χ_n	χ_n	...	χ_n

After this brief definition let us obtain the conjugacy classes of the group S_3 by using matrix representations given in Equations (2.21) and (2.22). The first class contains only unit element of 3×3 matrices. If we chose $A(\rho_1)$ as a fixed element, then the set of matrices obtained by using Equation (2.26) are $A(\rho_3)$, $A(\rho_4)$ and $A(\rho_5)$ which are of order 2 and character(trace) 1. The other class members obtained by choosing $A(\rho_4)$ as a fixed element, are $A(\rho_1)$ and $A(\rho_2)$ of order 3 and trace 0. They are tabulated in Table (2.2). Similarly the conjugate classes of S_3 is obtained for the 2×2 matrix representation, the character values are demonstrated in the fourth row of Table (2.2).

Table 2.2. Conjugacy Classes of the Group S_3.

Class	$C_1(1)$	$C_2(3)$	$C_3(2)$
# Elements	1	2	3
Characters(3×3)	3	0	1
Characters(2×2)	2	-1	0

Since each class is obtained by the similarity transformation, then trace and orders of each element in a class are equal to each other. The number of elements in a class, which divides the order of the group G is called the **index** of a class. We can say that if C_i is a class of conjugate elements of a group G, then C_i^{-1} (inverse of elements of C_i) form also a class of conjugate elements.

2.6.2. Relations on Irreducible Characters

The number of inequivalent irreducible representations of the group G, is equal to the number of classes of the group G. Consider a group of order |G| and let $n_1, n_2, \cdots, n_l$ be the degrees of the inequivalent irreducible representations $\chi^1, \chi^2, \cdots, \chi^l$, then

$$|G| = n_1^2 + n_2^2 + \cdots + n_l^2 \tag{2.27}$$

In our applications, the character tables are demonstrated in the form of Table (2.3). In Table (2.3) $C_i(p_i)$ denotes the name of the class and p_i represents the order of each element with $C_1(1)=1$ which is the unit element. The second row corresponds to the number of elements h_i in a class C_i. The third row represents the order of centralizer or index which is obtained by dividing the order of the group by the number of elements in the corresponding class. χ^1 is the identity representation, and $\chi^2, \ldots, \chi^r$ are the absolutely irreducible representations. χ_i^a is the character of an element of the i^{th} class in a^{th} representation and n_i are the degrees of the irreducible representations.

Table 2.3. The Character Table of a Group G

Class	$C_1(1)$	$C_2(p_2)$	...	$C_i(p_i)$	...	$C_r(p_r)$
# Elements	h_1	h_2	...	h_i	...	h_r
Index	$\|G\|/h_1$	$\|G\|/h_2$	...	$\|G\|/h_3$	...	$\|G\|/h_r$
χ^1	n_1	1	...	1	...	1
χ^2	n_2	χ_2^2	...	χ_i^2	...	χ_r^2
...	...	...	...	...	...	...
χ^s	n_s	χ_2^s	...	χ_i^s	...	χ_r^s
...	...	...	...	...	...	...
χ^r	n_r	χ_2^r	...	χ_i^r	...	χ_r^r

Without proof, we give the following relations on characters: The orthogonality relations satisfied by the irreducible characters of a group G are given successively by the relations:

$$\sum_{i=1}^{i=r} h_i(\chi_i^a)(\overline{\chi}_i^b) = \delta_{ab} g \tag{2.28}$$

on rows, and

$$\sum_{a=1}^{a=r} (\chi_i^a)(\overline{\chi}_j^a) = \frac{g}{h_i}\delta_{ij} \tag{2.29}$$

on columns. The relations within each row is given by the Equation

$$\frac{h_i\chi_i^a}{n_a}\frac{h_j\chi_j^a}{n_a} = \sum_k c_{ijk}\frac{h_k\chi_k^a}{n_a}, \quad (a = 1,2, \cdots, r) \tag{2.30}$$

where c_{ijk} are non-negative integers, n_a and h_i correspond to the degree of irreducible a^{th} representation and the number of elements in i^{th} class respectively.

Consider two irreducible representations of a group G, with characters χ_i^a and χ_i^b corresponding to the same class. Then there is a relation between the irreducible characters of a group G satisfying

$$\chi_i^a\chi_i^b = \sum g_{abc}\chi_i^c \tag{2.31}$$

where g_{abc} are non-negative integers. The relations from (2.27) to (2.31) hold for all irreducible representations of the group G. A reducible representation χ

of a group G can be written in the form of compositions of the irreducible representations, such as,

$$\chi_i = \sum_a k_a \chi_i^a \tag{2.32}$$

where k_a is an integer number χ^a is the character of an irreducible representation of the group G. The integer coefficient k_a is obtained from the relation

$$k_a = \frac{1}{|G|}\sum_i h_i \bar{\chi}_i \chi_i^a \tag{2.33}$$

where χ_i are the characters of the reducible representation and χ_i^a are the characters of the irreducible representations. After this brief summary on the character relations, the construction of character tables will be discussed in the following section.

2.6.3. Construction of Character Tables

In the construction of a character table of a group G the multiplication table of the conjugacy classes plays an important role. Let Y_i be the sum of the elements of the conjugacy class C_i (i=1,...r), where r is the number of conjugacy classes of the group G. Then we can construct the following set of Equations:

$$\begin{aligned} &[Y_1]^2 - 1 = 0 \\ &[Y_i]^2 - \sum_{i,j=2}^{r} m_{ij} Y_j = 0 \\ &Y_i Y_j - \sum_{i,j,k=2}^{r} d_{ijk} Y_k = 0, \quad i \neq j \end{aligned} \tag{2.34}$$

where m_{ij} and d_{ijk} are non-negative integers. We have $\frac{(r-1)(r-2)+2r}{2}$ Equations, then we obtain r independent set of solutions for Y_i. If we denote each set of solutions of Y_i by y_i, then from Equations (2.28), (2.29) and (2.30) we obtain an equation

$$y_i = \frac{h_i}{n_a}\chi_i^a \text{ or } \chi_i^a = \frac{n_a}{h_i} y_i \tag{2.35}$$

Note that it is hard to solve the Equation (2.34). Some computer programs are used to solve them. Now let us construct the character table of symmetric group S_3. The elements of the classes C_1, C_2, and C_3, obtained in Table (2.2), are $\{\rho_0\}$, $\{\rho_3, \rho_4, \rho_5\}$ and $\{\rho_1, \rho_2\}$, respectively. By using the elements and Equation (2.34) we obtain

$$\begin{aligned} &[Y_1]^2 - 1 &&= 0 \\ &[Y_2]^2 - 2Y_1 - Y_2 &&= 0 \\ &[Y_3]^2 - 3Y_1 - 3Y_2 &&= 0 \\ &Y_2Y_3 - 4Y_3 &&= 0 \end{aligned} \tag{2.36}$$

The set of solutions can be obtained by solving Equation (2.36) such that,

$$\begin{aligned} y_1 &= \{1, 2, 3\} \\ y_2 &= \{1, 2, -3\} \\ y_3 &= \{1, -1, 0\} \end{aligned} \tag{2.37}$$

By using the Equations (2.28 and 2.35) we obtain the character Table (2.4) of the group S_3,

Table 2.4. The Character Table of the Symmetric Group S_3

Class	$C_1(1)$	$C_2(3)$	$C_3(2)$
# Elements	1	2	3
Index	6	3	2
χ^1	1	1	1
χ^2	1	1	-1
χ^3	2	-1	0

The 3×3 and 2×2 matrix representations of S_3 were obtained earlier and given by the Equations (2.21) and (2.22) respectively. Conjugacy classes of each representation are obtained in Table (2.2). Let us investigate whether the given representations are irreducible or reducible. Remember the Table (2.2) that involves the characters of the 3×3 and 2×2 matrix representations. By using Equations (2.32-33) and character table of the S_3, we find the relations for 3×3 matrix representations such that

$$\begin{aligned} k_1 &= \tfrac{1}{6}(1\times 3\times 1 + 3\times 1\times 1 + 2\times 0\times 1) = 1 \\ k_2 &= \tfrac{1}{6}(1\times 3\times 1 + 3\times 1\times(-1) + 2\times 0\times 1) = 0 \\ k_1 &= \tfrac{1}{6}(1\times 3\times 2 + 3\times 1\times 0 + 2\times 0\times(-1)) = 1 \end{aligned} \tag{2.38}$$

Then from Equation (2.32) we can write

$$\chi = \chi^1 + \chi^3 \tag{3.39}$$

Similarly for 2×2 matrix representations of S_3 we obtain the Equations

$$k_1 = \tfrac{1}{6}(1\times 2\times 1 + 3\times 0\times 1 + 2\times(-1)\times 1) = 0$$

$$k_2 = \tfrac{1}{6}(1\times 2\times 1 + 3\times 0\times(-1) + 2\times 1\times(-1)) = 0 \qquad (3.40)$$

$$k_1 = \tfrac{1}{6}(1\times 2\times 2 + 3\times 0\times 0 + 2\times(-1)\times(-1)) = 1$$

Then it is obvious that 2×2 matrix representation of S_3 is irreducible, while 3×3 matrix representation is reducible.

Equation (2.39) implies that there is a non-singular matrix T which transforms all 3×3 matrix elements of S_3 into 1×1 and 2×2 block-diagonal forms. As an example the matrix T is given as

$$T = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \text{ and } T^{-1} = \tfrac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \qquad (2.41)$$

Then by the similarity transformation we can transform the 3-dimensional reducible representation to the block-diagonal form

$$B(\rho_i) = TA(\rho_i)T^{-1} \qquad (2.42)$$

They are

$$A(\rho_0) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \quad A(\rho_1) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}; \quad A(\rho_2) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 0 \end{bmatrix};$$

$$(2.43)$$

$$A(\rho_3) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 0 \end{bmatrix}; \quad A(\rho_4) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & -1 \end{bmatrix}; \quad A(\rho_5) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & -1 & 1 \end{bmatrix};$$

The first block corresponds to the trivial representation and the 2×2 block matrices correspond to the 2-dimensional irreducible representation of S_3.

2.6.4. Determination of Normal Subgroups from the Character Table

The presence of a normal subgroup of a group G can be easily understood from the character Table. The condition that if the characters of a conjugate class either equal to the order of the representation or related by $|h_i\chi_i^a| = n_a$ for all irreducible representations, then the elements of the corresponding conjugate classes form the subgroup N. This group is normal subgroup in the group G.

2.7. Generators and Generation Relations

The set of elements $g_1, g_2, \ldots, g_m$, of a group G, are called **generators,** if every elements of G is expressible as a finite product of their powers. It is conveniently denoted by the symbol $\{g_1, g_2, \ldots g_m\}$. It is clear that if m=1 then the group is **cyclic** group. If a given set of generators are abstractly defined then it is called the **abstract** definition or the **presentation** of the group G. An element expressed as a product of generators and their inverses is said to be a **word.** Suppose that X is a set of generators for G, and let Y be another subset for G. Then if Y contains X or if every element of X can be expressed as a word in the elements of Y, then Y is also a set of generators for G.

In our work, the generation relations of the groups of interest will be given without any further discussion. The generation relation for symmetric group S_3 is

$$X = \langle \rho_1, \rho_2 \mid \rho_1^3 = \rho_3^2 = 1, (\rho_1\rho_3)^2 = 1 \rangle \tag{2.44}$$

All the elements of the group can be obtained from the given generators. In other words all the remaining elements of the group can be expressed in terms of the given generators. The elements of S_3 can be written as:

$$\{1, \rho_1, \rho_1^2, \rho_3, \rho_3\rho_1, \rho_1\rho_3\}$$

CHAPTER 3

OCTONIONS AND THE CONSTRUCTION OF THE ROOT SYSTEM OF E_8

3.1. Introduction

The development of the theory of Lie groups is characterised by a number of distinct approachs. Although the Lie groups may be formulated by using the properties of invariant matrices, generally, physicists followed a path which made by the use of the roots associated with the corresponding Lie algebras. The root systems of the E_8 Lie algebras can be constructed by integral octonions.

3.2. Octonions

Due to the **Hurwitz theorem** there are only four division algebras formed by **real** numbers, **complex** numbers, **quaternions**, and **octonions.** Quaternions appear everywhere in physics and has been studied by many authors. The octonion algebra and especially automorphism groups of integral octonions has much less been studied. In this section, the concept of octonions and some of their properties will be introduced[9].

3.2.1. Basic Definitions

An algebra which consists of a system of elements, may be combined together by addition or multiplication or both. Let **V** be a vector space over a field $\Im$, for all $\alpha \in \Im$ and **x, y, z** $\in$ **V**. The following conditions hold:

(1) Multiplication relation with field element

$$(\alpha x)y = \alpha(xy) = x(\alpha y) \tag{3.1}$$

(2) Distribution Laws

$$(x + y)z = xz + yz \tag{3.2}$$

$$x(y + z) = xy + xz \tag{3.3}$$

These three conditions are sufficient for an algebra. If in addition, we have

(3) Associative Law

$$(xy)z = x(yz) \tag{3.4}$$

then the algebra is called an associative algebra over the field $\Im$.

An algebra over a field $\Im$ is called **division algebra** over $\Im$, if for every element of V there exists a unit element 1 with the property

(4) $$1x = x1 = x \tag{3.5}$$

and for every element of V there exists an inverse element such that

(5) $$x^{-1}x = xx^{-1} = 1 \tag{3.6}$$

3.2.2. Division Algebras with Integral Elements

Under the above axioms we will construct the octonion algebra. Let X denote an element over the field $\Im$. Complex numbers, quaternions and octonions obey the equation

$$X^2 - (X + \overline{X})X + X\overline{X} = 0 \tag{3.7}$$

where $X + \overline{X}$ is twice the scalar part and $X\overline{X} = \overline{X}X$ is the norm of X, $\overline{X}$ being the conjugate of X. Integer elements of Hurwitz algebras obey equation (3.7) with integer coefficients, so twice the scalar part of X and norm of X are both integers[7,18, 23].

We may define complex numbers by taking a pair of real numbers. Similarly quaternions can be defined as pairs of complex numbers and octonions can be defined as pairs of quaternions. Let [P, Q] be a pair of such elements then they can also be defined by

$$[P, Q] = P + eQ \tag{3.8}$$

where e is an imaginary number, $e^2 = -1$. The product of two pairs are defined by

$$[P, Q][R, S] = [RR - S\overline{Q}, RQ + \overline{P}S] \tag{3.9a}$$

The conjugate of such a pair is defined by

$$\overline{[P, Q]} = [\overline{P}, -\overline{Q}] \tag{3.9b}$$

Then the norm of [P, Q] takes the form

$$[P, Q]\overline{[P, Q]} = \overline{[P, Q]}\,[P, Q] = [P\overline{P}, Q\overline{Q}, 0] = P\overline{P} + Q\overline{Q} \tag{3.10}$$

Let us now find the relation between this description and those with explicit imaginary units. Let x_0 and x_1 be two numbers, then $[x_0, x_1]$ is a complex number which can be written as

$$[x_0, x_1] = x_0 + e_1 x_1 = C \tag{3.11}$$

where $e_1^2 = -1$ and $\overline{e}_1 = -e_1$. e_1 corresponds to the usual imaginary number i. Similarly, we can take two complex numbers as pairs

$$C = x_0 + e_1 x_1 \text{ and } C' = x_3 + e_1 x_2 \tag{3.12}$$

where x_0, x_1, x_2 and x_3, are real numbers. By combining two complex numbers C and C' in the usual manner we obtain a quaternion Q given by

$$\begin{aligned} Q &= [C, C'] = x_0 + e_1 x_1 + e_3(x_3 + e_1 x_2) \\ &= x_0 + e_1 x_1 + e_2 x_2 + e_3 x_3 \end{aligned} \tag{3.13}$$

where the imaginary units $e_3^2 = -1, \overline{e}_3 = -e_3$ and $e_2 = e_3 e_1, = -e_1 e_3$. Similarly we take a pair of quaternions Q and Q' to obtain an octonion x by multiplying Q' by an imaginary unit e_7 on the left

$$\begin{aligned} x &= [Q, Q'] = x_0 + x_1 e_1 + x_2 e_2 + x_3 e_3 + e_7(x_4 e_1 + x_5 e_2 + x_6 e_3 + x_7) \\ &= \sum_{i=0}^{7} x_i e_i \end{aligned} \tag{3.14}$$

with the definitions $e_4 = e_7 e_1 = -e_1 e_7$, $e_5 = e_7 e_2 = -e_2 e_7$, $e_6 = e_7 e_3 = -e_3 e_7$. Koca[29] has used this method to construct the algebra of integral elements of quaternions and octonions. Now we will continue to investigate the properties of octonionic units.

We denoted octonions by a letter e_i (i=0,1,...,7), where e_0=1 corresponds to the unit element. Now we will give some properties of octonions. Consider an octonion **x** that can be written as before

$$x = (x_0, x_1, \dots, x_7) = x_0 e_0 + x_1 e_1 + \cdots + x_7 e_7 \tag{3.15}$$

which are added like vectors and multiplied according to the rules

$$e_0 e_i = e_i e_0 = e_i, \quad (i = 0,1,2, \dots, 7) \tag{3.16a}$$

$$e_i e_j = -\delta_{ij} + \psi_{ijk} e_k, \quad (i, j, k = 1, 2, \dots, 7) \tag{3.16b}$$

where the structure constant ψ_{ijk} are completely antisymmetric and takes values 1 and 0 for the cycles (ijk) = (123), (246), (435), (367), (651), (527), (741) and for i = j respectively. By using equations (3.15) and (3.16) we can construct a multiplication table for octonions as illustrated in Table (3.1).

Table 3.1. Multiplication Table of Octonions

$\rightarrow$	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_0	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	e_1	$-e_0$	e_3	$-e_2$	e_7	$-e_6$	e_5	$-e_4$
e_2	e_2	$-e_3$	$-e_0$	e_1	e_6	e_7	$-e_4$	$-e_5$
e_3	e_3	e_2	$-e_1$	$-e_0$	$-e_5$	e_4	e_7	$-e_6$
e_4	e_4	$-e_7$	$-e_6$	e_5	$-e_0$	$-e_3$	e_2	e_1
e_5	e_5	e_6	$-e_7$	$-e_4$	e_3	$-e_0$	$-e_1$	e_2
e_6	e_6	$-e_5$	e_4	$-e_7$	$-e_2$	e_1	$-e_0$	e_3
e_7	e_7	e_4	e_5	e_6	$-e_1$	$-e_2$	$-e_3$	$-e_0$

The conjugate elements of octonions are defined as

$$\bar{x} = x_0 - x_i e_i, \quad (i = 1,2, \ldots, 7) \tag{3.17}$$

then the scalar and vector parts of octonions can be written as

$$\begin{aligned} Sc(x) &= \tfrac{1}{2}(x + \bar{x}) = x_0 \\ Vec(x) &= \tfrac{1}{2}(x - \bar{x}) = x_i e_i \end{aligned} \tag{3.18}$$

respectively.

The scalar product is defined by

$$< x, y > = \frac{1}{2}(x\bar{y} + y\bar{x}) = \frac{1}{2}(\bar{x}y + \bar{y}x) \tag{3.19}$$

In terms of octonionic units this definition can be written as

$$< e_i, e_j > = \frac{1}{2}(\bar{e}_i e_j + \bar{e}_j e_i) = \frac{1}{2}(e_i \bar{e}_j + e_j \bar{e}_i) = \delta_{ij} \tag{3.20}$$

and the norm N(**x**) of an octonion is its scalar product with itself:

$$N(x) = \bar{x}x = x\bar{x} = x_i x_i \tag{3.21}$$

From the definitions, one can easily show that the octonion algebra is neither commutative nor associative. In terms of octonionic units, we can write the commutator $[e_i, e_j]$ of two octonions in the form

$$[e_i, e_j] = e_i e_j - e_j e_i \tag{3.22}$$

From equations (3.15) and (3.16) we can find a relation such that

$$[e_0, e_i] = 0 \quad \text{and} \quad [e_i, e_j] = 2\psi_{ijk} e_k, \quad (i, j, k = 1, 2, \ldots, 7) \tag{3.23}$$

Equation (3.23) implies that the octonions are non-commutative as quaternions. The associator $[e_i, e_j, e_k]$ of three octonions e_i, e_j, e_k are defined as

$$[e_i, e_j, e_k] = (e_i e_j) e_k - e_i (e_j e_k) \tag{3.24}$$

The associator of octonions can also be written by defining a completely anti-symmetric 4-index object ϕ_{ijkl}, then the associator of octonions becomes

$$[e_i, e_j, e_k] = 2\phi_{ijkl} e_l \tag{3.25}$$

The values of ϕ_{ijkl} are determined by using equation (3.24) and table (3.1). But in practice diagrammatic multiplication table is more useful rather than multiplication Table (3.1). Figure (3.1) illustrates diagrammatic representation of the octonionic multiplication table that is composed of a circle with 7 points representing the imaginary octonionic units and a triangle forming the associative triads.

In figure (3.1) we can read the whole multiplication table by rotating the triangle. It thus gives $e_1 e_2 = e_3$, $e_2 e_4 = e_6$, $e_4 e_3 = e_5$, $e_3 e_6 = e_7$, etc.

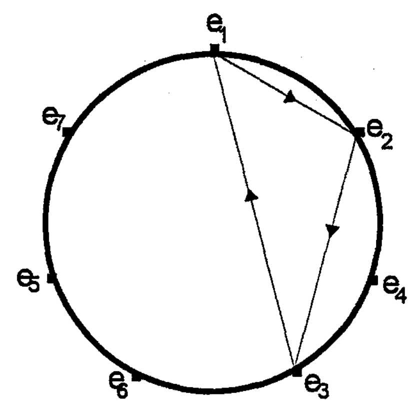


Figure 3.1. Diagrammatic Representation of Octonionic Multiplication Table

The other diagrammatic representation relating the associators of three octonions can be read in a similar manner from the Figure (3.2) by rotating the triangle. For example the triangle defined by the vertices $[e_1, e_2, e_4]$ and the lines joining the vertices of the triangle to e_5 form an associator. By rotating the quadrilateral in Figure (3.2) by the angle $\frac{2\pi}{7}$ one obtains the associators

$[e_1, e_4, e_2]=2e_5$, $[e_2, e_3, e_4]=2e_7$, $[e_4, e_6, e_3]=2e_1$, $[e_3, e_5, e_6]=2e_2$, $[e_5, e_7, e_5]=2e_4$, $[e_5, e_1, e_7]=2e_3$, $[e_7, e_2, e_1]=2e_6$

The quaternions in physics are usually introduced as Pauli σ-matrices which yield a realisation of the algebra of quaternions, in connection with the rotation group in 3-dimensions.

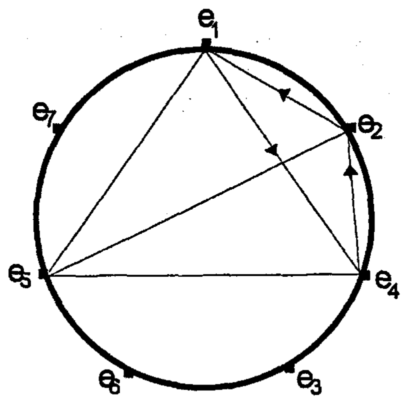


Figure 3.2. Diagrammatic Representations of Associator $[e_i, e_j, e_k]$ of Three Octonions.

The product rule for quaternions can be obtained from octonionic multiplication table, then the quaternionic unit e_i satisfy the relations

$$e_i e_j = -\delta_{ij} + \varepsilon_{ijk} e_k, \quad (i, j, k = 1, 2, 3) \tag{3.26}$$

where δ_{ij} and ε_{ijk} are the Kronecker and Levi-Civita symbols, respectively. Commutation relations of quaternions are

$$[e_i, e_j] = e_i e_j - e_j e_i = 2\varepsilon_{ijk}, \quad (i, j, k = 1, 2, 3) \tag{3.27}$$

Although the algebra of octonions are non-associative, the quaternions are associative.

3.2.3. Matrix Representations of Quaternions and Quaternion Group

The integral quaternionic units e_i can be represented in terms of Pauli spin matrices σ_i(i=1, 2, 3):

$$e_1 = i\sigma_1, \; e_2 = i\sigma_2, \; e_3 = -i\sigma_3, \tag{3.28}$$

where

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{3.29}$$

It can be directly verified that the Pauli matrices in Equation (3.29) satisfy the quaternion algebra. The matrices Q: $\{\pm 1, \pm e_1, \pm e_2, \pm e_3\}$ form a finite subgroup of SU(2) which is called the **quaternion** group.

The quaternion group is the smallest dicyclic group which has the generation relation

$$q = \langle e_1, e_2 \mid e_1^4 = e_2^4 = (e_1 e_2)^4 = 1 \rangle \tag{3.30}$$

The quaternion group Q is a non- abelian group but all of its proper subgroups are abelian and normal. The conjugate classes of Q are given in table (3.2) is obtained from the Pauli matrices. The character table of Q is illustrated in Table (3.3).

Table 3.2. Conjugate Classes of Quaternion Group Q

Class	$C_1(1)$	$C_2(2)$	$C_3(4)$	$C_4(4)$	$C_5(4)$
# Elements	1	1	2	2	2
Characters	2	-2	0	0	0

The quaternion group has 5 conjugacy classes and the 2×2 dimensional representation is irreducible. The elements of the classes C_i(i=1, 2,..., 5) are 1, -1, $\pm e_1$, $\pm e_2$, $\pm e_3$, respectively.

Table 3.3. Character Table of the Quaternion Group Q

Class	$C_1(1)$	$C_2(2)$	$C_3(4)$	$C_4(4)$	$C_5(4)$
# Elements	1	1	2	2	2
Index	8	8	4	4	4
χ^1	1	1	1	1	1
χ^2	1	1	1	-1	-1
χ^3	1	1	-1	1	-1
χ^4	1	1	-1	-1	1
χ^5	2	-2	0	0	0

3.2.4. Split Octonions

In the construction of root systems, sometimes we need to rewrite the octonionic units in the split-base. In general octonions can be written in the split-base such as[16]

$$u_0 = \frac{1}{2}(1+e_7), \quad u_j = \frac{1}{2}(e_j + ie_{j+3}), \quad (j = 1, 2, 3) \tag{3.31}$$

and their complex conjugates. The split octonion units u_j satisfy the multiplication table given in Table (3.4).

The anti-commutation relation of split octonions have the interesting property,

$$\{u_j, u_k\} = \{u_j^*, u_k^*\} = 0, \quad \{u_j, u_k^*\} = -\delta_{jk} \tag{3.32}$$

Table 3.4. Multiplication of the Split Octonionic Units

$\rightarrow$	u_0	u_0^*	u_1	u_2	u_3	u_1^*	u_2^*	u_3^*
u_0	u_0	0	u_1	u_2	u_3	0	0	0
u_0^*	0	u_0^*	0	0	0	u_1^*	u_2^*	u_3^*
u_1	0	u_1	0	u_3^*	$-u_2^*$	$-u_0$	0	0
u_2	0	u_2	$-u_3^*$	0	u_1^*	0	$-u_0$	0
u_3	0	u_3	u_2^*	$-u_1^*$	0	0	0	$-u_0$
u_1^*	u_1^*	0	u_0^*	0	0	0	u_3	$-u_2$
u_2^*	u_2^*	0	0	u_0^*	0	$-u_3$	0	u_1
u_3^*	u_3^*	0	0	0	u_0^*	u_2	$-u_1$	0

Although, in general, the split octonions are defined in equation (3.31), we can also partially split octonions, such that the quaternionic units remain the same and the remaining octonions can be written as

$$u=\frac{1}{\sqrt{2}}(e_4+ie_5), \quad \text{and} \quad d=\frac{1}{\sqrt{2}}(e_6+ie_7) \tag{3.33}$$

and the complex conjugates of u and d.

3.3. Construction of the Root System of E_8 with Integral Octonions

Root systems of the Exceptional Lie group E_8 and its subgroups SU(2), SU(2)×SU(2), SO(8), SU(2)×E_7, and SU(3)×E_6 may be constructed with integral elements of four division algebras. Dynkin has shown that the simple roots may be represented by two dimensional diagrams referred as **Dynkin**

diagrams, from which the complete set of root vectors may be obtained as well as all information concerning root lengths and angles.

3.3.1. Properties of the Simple Roots and Dynkin Diagrams

All groups of continuous operators that are called **Lie groups** are outside our main work, but we are dealing with automorphism groups of integral octonions that are strongly related by the root systems of Lie groups. The infinitesimal generators X_τ of the Lie group satisfy the condition:

$$[X_\mu, X_\nu] = f_{\mu\nu\lambda} X_\lambda \tag{3.34}$$

where $f_{\mu\nu\lambda}$ are the set of numerical constants called structure constants. Since the commutators are linearly independent, then the structure constant satisfy $f_{\mu\nu\lambda}=-f_{\nu\mu\lambda}$. The commutators satisfy the Jacobi identity

$$[X_\mu, [X_\nu, X_\lambda]] + [X_\nu, [X_\lambda, X_\mu]] + [X_\lambda, [X_\mu, X_\nu]] = 0 \tag{3.35}$$

Under the above conditions the infinitesimal operators X_τ form the **Lie algebra** of the corresponding Lie group. For an r-element Lie algebra there are at most r roots. However degeneracies of the roots may exist. The root vectors have the following properties:

(1) If α is a root vector, then $-\alpha$ is also root vector.

(2) If α and β are root vectors then

$$2\frac{(\alpha, \beta)}{(\alpha, \alpha)} = \text{integer} \tag{3.36}$$

(3) Two root vectors α and β give a third one such that

$$\beta - 2\alpha\frac{(\alpha, \beta)}{(\alpha, \alpha)} = \gamma \tag{3.37}$$

is also root vector. From the definition of scalar product the angle ϑ between two roots α and β is given by

$$\mathrm{Cos}\vartheta = \frac{(\alpha, \beta)}{\sqrt{(\alpha, \alpha)(\beta, \beta)}} \tag{3.38}$$

or

$$\mathrm{Cos}^2\vartheta = \frac{(\alpha, \beta)(\alpha, \beta)}{(\alpha, \alpha)(\beta, \beta)} = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, \text{ or } 1 \tag{3.39}$$

Thus we obtain the angles between the roots α and β as 0^0, 90^0, 120^0, 135^0 or 150^0.

It is obvious that the simple roots either all have the same length or all have one of two lengths.

We consider a **root diagram**, which is the graphical representation of the root vectors in an r-dimensional space. In this approach it is hard to draw root diagrams in three dimensions. We give here only the root diagram for SU(3), as an example. The root diagram of SU(3) is presented as follows: Suppose α is a root vector with coordinate being (1,0). Since the angle between α and the other root vector β, possessing the same length as α, is 60^0 Equation (3.36) leads to the coordinates $(1/2, \sqrt{3}/2)$ for β. From equation (3.37) another root vector α-β is obtained and it lies at the point $(1/2, -\sqrt{3}/2)$. It follows that if α and β are roots, then $-\alpha$ and $-\beta$ are also roots, so that we can draw the resultant root diagram that is the hexagon as in figure (3.3). The roots for su(3) Lie algebra are listed as follows:

$$(1,0);\ (\frac{1}{2},\ \frac{\sqrt{3}}{2});\ (\frac{1}{2},-\frac{\sqrt{3}}{2});\ (-1,0);\ (-\frac{1}{2},-\frac{\sqrt{3}}{2});\ (-\frac{1}{2},\frac{\sqrt{3}}{2}) \qquad (3.40)$$

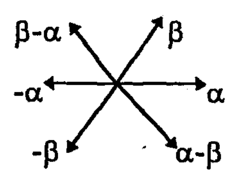
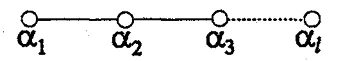
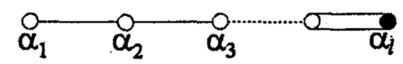
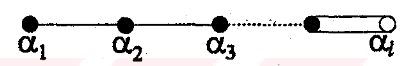
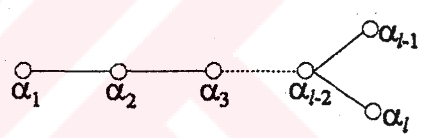
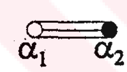
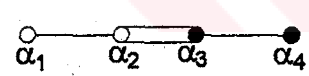
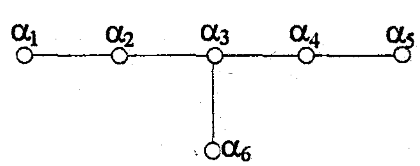
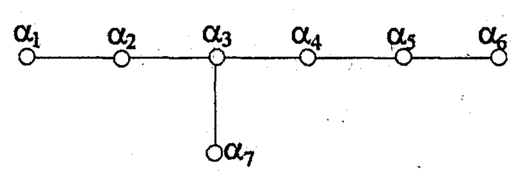
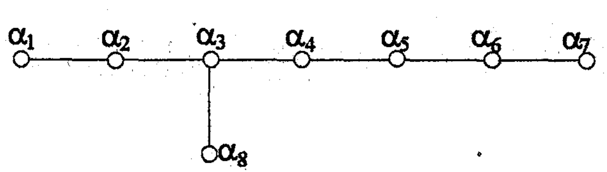


Figure 3.3. Root Diagram of SU(3) Algebra

For the Lie algebra of rank r>2 it is hard to draw a diagram as in figure (3.3). A shorthand method has been developed by Dynkin. Each simple root is represented by a small circle. If the simple root is a long one the circle is white-o; if it is a short one the circle is made black-•. The angle between any two roots is denoted by the number of lines joining the circles which represent them. We join the circles by one, two, or three lines depending on the angles between the corresponding simple roots are 120^0, 135^0, or 150^0, respectively. No line is drawn if two roots are perpendicular to each other.

All the Lie algebras of simple compact groups have been classified by Cartan(1933). The Dynkin diagram of all the simple Lie algebras are illustrated in Table (3.5). According to the Cartan's classification no other diagrams are possible.

Table 3.5. Dynkin Diagrams and Root Structure of the Simple Lie Algebras

Order	Cartan label	Group label	Dynkin Diagram
$l(l+1)$	A_l	$SU(l+1)$	
$l(2l+1)$; $l \geq 2$	B_l	$SO(2l+1)$	
$l(2l+1)$; $l \geq 3$	C_l	$Sp(2l)$	
$l(2l-1)$; $l \geq 4$	D_l	$SO(2l)$	
14	G_2	G_2	
52	F_4	F_4	
78	E_6	E_6	
133	E_7	E_7	
248	E_8	E_8	

3.2.2. Weights

The concept of weight and weight space play an important role in the theory and application of semisimple Lie groups. For an N-dimensional representation of a group of rank r there are r self commuting n×n matrices H_{α_i} (i=1, ..., r). Then we may construct a set of eigenvectors |u> of the matrices H_{α_i}, thus

$$H_{\alpha_i}|u\rangle = m_i|u\rangle \quad (i = 1, \dots, r) \tag{3.41}$$

The eigenvalue m is called a **weight**. We shall now give some of the properties of the weights associated with the simple Lie groups.

(1) For any weight **m** and root ρ there is a relationship

$$\frac{2(m.\rho)}{(\rho,\rho)} = \text{integer} \quad \text{and} \quad m' = m - \frac{2(m.\rho)\rho}{\rho,\rho)} \tag{3.42}$$

where **m′** is also a weight as **m.**

It is obvious that if we have a weight m, then by reflecting it in the hperplane perpendicular to ρ, we obtain the new weight **m′**.

(2) Cartan has proved a theorem states that, for every simple group of rank r, there are r dominant weights called fundamental dominant weights denoted by $M^{(i)}$, (i=1,...r), such that any other dominant weight M is a linear combination of the fundamental dominant weights

$$M = \sum_{i=1}^{r} p_i M^{(i)} \tag{3.43}$$

where p_i are nonnegative integers.

3.3.3. Construction of the Root System of the Exceptional Lie group E_8 and its Subgroups SU(2), SU(2)×SU(2),SO(8), F_4, and E_7[18,23]

During the construction of the octonion algebra, we have already given the properties of an element X of the division algebra. Let us try to find Hurwitz integers for real numbers, complex numbers, quaternions and octonions. For real numbers the Hurwitz integer r of unit norm is only X=±1. These correspond to the roots of SU(2). The Gaussian integers of unit norm satisfy

$$X = m + in \tag{3.44}$$

where m and n are ordinary integers. Its solutions are X=±1 and X=±i, since $X\overline{X}$ and $X+\overline{X}$ must be integer and half odd integer, respectively. Four units of Gaussian integers represent the four roots of SO(4)≈SU(2)×SU(2).

For the integer quaternions we have

$$\begin{aligned} &X = x_0 + e_1x_1 + e_2x_2 + e_3x_3, \text{ with} \\ &X + \overline{X} = x_0 \text{ and } X\overline{X} = x_0^2 + x_1^2 + x_2^2 + x_3^2 = 1 \end{aligned} \tag{3.45}$$

If x_0=integer then we obtain 8 solutions for equation (3.43), that is

$$X = \{\pm 1,\ \pm e_1,\ \pm e_2,\ \pm e_3\} \tag{3.46}$$

where e_1, e_2 and e_3 are the imaginary quaternion units. If x_0 = half odd integer then 16 linear combinations are obtained as the solution of the Equation (3.45), these are,

$$X = \tfrac{1}{2}(\pm 1 \pm e_1 \pm e_2 \pm e_3) \tag{3.47}$$

The set of solutions in Equations (3.46) and (3.47) which together form the 24 roots of SO(8). The Coxeter-Dynkin diagram of SO(8) corresponding to the so(8) Lie algebra is depicted in Figure (3.4)

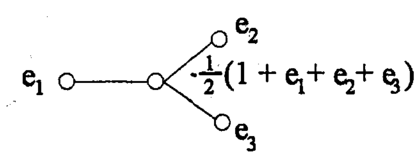


Figure 3.4. Coxeter-Dynkin Diagram of SO(8)

We define the integral quaternions

$$\begin{aligned} S_0 &= \tfrac{1}{2}(1 + e_1 + e_2 + e_3)\,;\; S_1 = \tfrac{1}{2}(1 + e_1 - e_2 - e_3)\,; \\ S_2 &= \tfrac{1}{2}(1 - e_1 + e_2 - e_3)\,;\; S_3 = \tfrac{1}{2}(1 - e_1 - e_2 + e_3)\,; \end{aligned} \tag{4.48}$$

The roots in equation (3.46) are the roots of the SO(4)×SO(4) subgroup of SO(8) and the roots in equation (3.47) represent the roots of the coset $\dfrac{SO(8)}{SO(4)\times SO(4)}$. The roots that are constructed by integral quaternions form a group that is called binary tetrahedral group denoted by <3, 3, 2> meaning that its generation relation satisfy the equation

$$S = \langle S_0,\, S_1 \mid S_0^3 = S_1^3 = (S_0S_1)^2 = Z,\, Z^2 = 1\,\rangle. \tag{3.49}$$

For the quaternionic units to be represented by the Pauli matrices, the 24 roots of SO(8), represented by unitary matrices must form a discrete subgroup of SU(2). The elements of the binary tetrahedral group are compactly written as

$$\pm 1,\ \pm e_1,\ \pm e_2,\ \pm e_3,\ \pm S_a,\ \pm \overline{S}_a,\ (a = 0,1,2,3) \qquad (3.50)$$

The roots of the SO(4)×SO(4) that are { ± 1, $\pm e_1$, $\pm e_2$, $\pm e_3$} form a subgroup of the binary tetrahedral group of order 8 which is the quaternion group that has been discussed before.

The action of the group element P on a root of SO(8) can be defined

$$Q \to Q' = PQP^{-1} = PQ\overline{P} \qquad (3.51)$$

which corresponds to a rotation of an angle π or $2\pi/3$ in a certain hyperplane depending on the choice of the group element P. The transformations by $\pm e_i$ (i=1,2,3) and $\frac{1}{2}(\pm 1 \pm e_1 \pm e_2 \pm e_3)$ correspond to angles π and $2\pi/3$ respectively. The scalar part of a quaternion is left invariant under the binary tetrahedral group. The imaginary units e_1, e_2 and e_3 transform under the action of group elements, such that

$$\pm e_1 : \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} \to \begin{bmatrix} e_1 \\ -e_2 \\ -e_3 \end{bmatrix};\ \pm e_2 : \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} \to \begin{bmatrix} -e_1 \\ e_2 \\ -e_3 \end{bmatrix};\ \pm e_3 : \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} \to \begin{bmatrix} -e_1 \\ -e_2 \\ e_3 \end{bmatrix}$$

(3.52)

$$\pm S_0 = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} \to \begin{bmatrix} e_2 \\ e_3 \\ e_1 \end{bmatrix}; \pm S_1 = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} \to \begin{bmatrix} -e_3 \\ -e_1 \\ e_2 \end{bmatrix}; \pm S_2 = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} \to \begin{bmatrix} e_3 \\ -e_1 \\ -e_2 \end{bmatrix}; \pm S_3 = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} \to \begin{bmatrix} -e_3 \\ e_3 \\ -e_{21} \end{bmatrix};$$

These transformations form the 3-dimensional irreducible representation of the tetrahedral group of order 12 isomorphic to the even permutations A_4 of the four objects. It is clear that the quaternion algebra is left invariant under the binary tetrahedral group. Now we calculate the weights of three spinor representations of SO(8) in terms of quaternionic units. The usual notations are 8_v= (1 0 0 0), 8_s=(0 0 0 1) and 8_c=(0 0 1 0) for the vector, spinor and antispinor representations respectively. The weights for three 8-dimensional representations of SO(8) can be taken as

$$\begin{array}{ccc} A_1 & A_2 & A_3 \\ \hline 8_v : \frac{1}{2}(\pm 1 \pm e_1) & 8_c : \frac{1}{2}(\pm 1 \pm e_2) & 8_s : \frac{1}{2}(\pm 1 \pm e_3) \\ \frac{1}{2}(\pm e_2 \pm e_3) & \frac{1}{2}(\pm e_3 \pm e_1) & \frac{1}{2}(\pm e_1 \pm e_2) \end{array} \qquad (3.53)$$

If we multiply the set of weights A_1, A_2 and A_3 by $\sqrt{2}$ and combine them with 24 integral quaternions we obtain a new discrete subgroup of SU(2) of order 48 which admits the binary tetrahedral group as a subgroup. This group, denoted by <4, 3, 2>, is called the binary octahedral group. The set of weights A_1, A_2, A_3 of SO(8) with 24 integral quaternions that are roots of SO(8) form the root system of F_4 with 24 short and 24 long roots respectively. The Coxeter-Dynkin diagram of F_4 illustrated in figure (3.5) with simple roots are associated with integral quaternions. We will denote the roots of SO(8) by A_0.

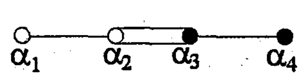


Figure 3.5. Coxeter-Dynkin Diagram of F_4. The roots α_i correspond to the quaternions that are $\alpha_1=\frac{1}{2}(1-e_1-e_2-e_3)$, $\alpha_2=e_2$, $\alpha_3=\frac{1}{2}(e_3-e_2)$ and $\alpha_4=\frac{1}{2}(e_1-e_3)$

The root system of E_8 can be decomposed in terms of the units of octonions by taking the pairs of the sets of F_4 root system. By pairing the roots of F_4 with each other and A_0 with zero root we obtain the octonionic representation of E_8 roots:

$$[A_0, 0] = A_0 \ , \quad [A_1, A_1] = A_1 + e_7 A_1 \ , [A_3, A_2] = A_3 + e_7 A_2$$
$$[0, A_0] = e_7 A_0 \ , [A_2, A_3] = A_2 + e_7 A_3 \tag{3.54}$$

Thus we obtain a decomposition of E_8 under $SO(8)\times SO(8)$:

$$\underline{248} = (28,\ 1) + (1,\ 28) + (\underline{8}_v,\ \underline{8}_v) + (\underline{8}_s,\ \underline{8}_c) + (\underline{8}_c,\ \underline{8}_s) \tag{3.55}$$

The non-zero roots of E_8 are represented by the following integral octonions,

$$(24,\ 1) : \pm 1,\ \pm e_1,\ \pm e_2,\ \pm e_3,\ \tfrac{1}{2}(\pm 1 \pm e_1 \pm e_2 \pm e_3)$$

$$(1,24) : \pm e_7,\ \pm e_4,\ \pm e_5,\ \pm e_6,\ \tfrac{1}{2}(\pm e_7 \pm e_4 \pm e_5 \pm e_6) \tag{3.56a}$$

$$(\underline{8}_v,\ \underline{8}_v) : [\tfrac{1}{2}(\pm 1 \pm e_1),\ \tfrac{1}{2}(\pm 1 \pm e_1)] = \tfrac{1}{2}(\pm 1 \pm e_1 \pm e_4 \pm e_7)$$
$$[\tfrac{1}{2}(\pm 1 \pm e_1),\ \tfrac{1}{2}(\pm e_3 \pm e_2)] = \tfrac{1}{2}(\pm 1 \pm e_1 \pm e_5 \pm e_6)$$
$$[\tfrac{1}{2}(\pm e_3 \pm e_2),\ \tfrac{1}{2}(\pm 1 \pm e_1)] = \tfrac{1}{2}(\pm e_2 \pm e_3 \pm e_4 \pm e_7) \tag{3.56b}$$
$$[\tfrac{1}{2}(\pm e_3 \pm e_2),\ \tfrac{1}{2}(\pm e_3 \pm e_2)] = \tfrac{1}{2}(\pm e_2 \pm e_3 \pm e_5 \pm e_6)$$

$$(\underline{8}_s,\ \underline{8}_c) : [\tfrac{1}{2}(\pm 1 \pm e_3),\ \tfrac{1}{2}(\pm 1 \pm e_2)] = \tfrac{1}{2}(\pm 1 \pm e_3 \pm e_5 \pm e_7)$$
$$[\tfrac{1}{2}(\pm 1 \pm e_3),\ \tfrac{1}{2}(\pm e_3 \pm e_1)] = \tfrac{1}{2}(\pm 1 \pm e_3 \pm e_4 \pm e_6)$$
$$[\tfrac{1}{2}(\pm e_1 \pm e_2),\ \tfrac{1}{2}(\pm 1 \pm e_2)] = \tfrac{1}{2}(\pm e_1 \pm e_2 \pm e_5 \pm e_7) \tag{3.56c}$$
$$[\tfrac{1}{2}(\pm e_1 \pm e_2),\ \tfrac{1}{2}(\pm e_3 \pm e_1)] = \tfrac{1}{2}(\pm e_1 \pm e_2 \pm e_4 \pm e_6)$$

$$
\begin{aligned}
(\underline{8}_c, \underline{8}_s):\ & [\tfrac{1}{2}(\pm 1 \pm e_2), \tfrac{1}{2}(\pm 1 \pm e_3)] = \tfrac{1}{2}(\pm 1 \pm e_2 \pm e_6 \pm e_7) \\
& [\tfrac{1}{2}(\pm 1 \pm e_2), \tfrac{1}{2}(\pm e_1 \pm e_2)] = \tfrac{1}{2}(\pm 1 \pm e_2 \pm e_4 \pm e_5) \\
& [\tfrac{1}{2}(\pm e_3 \pm e_1), \tfrac{1}{2}(\pm 1 \pm e_3)] = \tfrac{1}{2}(\pm e_1 \pm e_3 \pm e_6 \pm e_7) \\
& [\tfrac{1}{2}(\pm e_3 \pm e_1), \tfrac{1}{2}(\pm e_1 \pm e_2)] = \tfrac{1}{2}(\pm e_1 \pm e_3 \pm e_4 \pm e_5)
\end{aligned}
\qquad (3.56d)
$$

The extended Coxeter-Dynkin diagram of E_7 including the roots in terms of octonionic units are given in Figure (3.6). Several properties of octonionic roots can be mentioned:

i) Real roots M : ± 1 (roots of SU(2))

ii) Pure imaginary roots N: (126 roots of E_7)

iii) Mixed roots R: (112 roots of $\dfrac{E_8}{SU(2)\times E_7}$

These roots satisfy $M^2 = 1$, $N^2 = -1$ and $R^3 = \pm 1$.

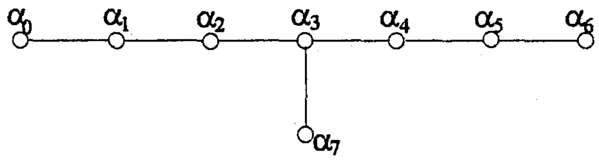


Figure 3.6. Extended Coxeter-Dynkin Diagram of E_7. The roots are $\alpha_0=e_1$, $\alpha_1=\frac{1}{2}(-e_1+e_3+e_4+e_5)$, $\alpha_2=\frac{1}{2}(-e_2-e_3-e_4+e_7)$, $\alpha_3=\frac{1}{2}(e_2+e_3-e_5+e_6)$, $\alpha_4=-e_6$, $\alpha_5=\frac{1}{2}(e_2-e_3+e_5+e_6)$, $\alpha_6=\frac{1}{2}(-e_2+e_3-e_4-e_7)$, $\alpha_7=\frac{1}{2}(-e_2-e_3+e_4-e_7)$.

The most natural representation of the root system of E_7 can be made with pure imaginary octonionic roots. In this case ± 1 represents the roots of SU(2).

CHAPTER 4

AUTOMORPHISM GROUPS OF PURE INTEGRAL OCTONIONS

4.1. Introduction

In the previous section, octonionic presentation of the root systems of E_8 and E_7 have been discussed. It is known that the automorphism group of the octonion algebra is the exceptional group G_2. Its finite subgroup $G_2'(2)$ of order 6048 preserves the discrete octonionic algebra of order 240. $G_2'(2) \approx U_3(3) \approx PSU(3,3)$ (Projective Special Unitary group) was first discovered by Dickson in 1901[38]. $G_2'(2)$ is the maximal subgroup of the sporadic Janko group j4. The adjoint Chevalley group $G_2(2)$ of order 12096, is the automorphic extension of the finite simple group $G_2'(2)$. The seven-dimensional matrix representation of $G_2(2)$ was constructed. In this section we work out the structure of $G_2(2)$, and find the Lie subalgebra structures of E_7 corresponding to the maximal subgroups of $G_2(2)$.

4.2. Seven Dimensional Irreducible Representation of $G_2'(2)$

The root system of E_8 have been described by integral octonions which form a closed non-associative algebra of order 240. There is a natural classification of the octonionic roots decomposing ± 1, pure imaginary integral octonions, and the octonions with non-zero scalar parts respectively. They describe the roots of SU(2), E_7 and the coset space $E_8/E_7\times SU(2)$[22]:

$$\frac{SU(2)}{\pm 1} \qquad \frac{E_7}{\pm e_i,\ \frac{1}{2}(\pm e_j \pm e_j \pm e_l \pm e_m)} \qquad \frac{E_8\,/\,E_7 \times SU(2)}{\frac{1}{2}(\pm 1 \pm e_n \pm e_p \pm e_q)} \tag{4.1}$$

where the indices take the values

$$\begin{aligned} &(i = 1,\ldots,7) \\ &(jklm:\ 1246,\ 1257,\ 1345,\ 1367,\ 2356,\ 2347,\ 4567) \\ &(npq:\ 123,\ 147,\ 165,\ 245,\ 267,\ 346,\ 357) \end{aligned} \tag{4.2}$$

The set of roots in Equation (4.1) is closed under the octonionic multiplication of the roots.

In order to construct the automorphism group of the pure octonions in Equation (4.1) let us choose the three roots

$$\begin{aligned} p_1 &= \tfrac{1}{2}(1 + e_1 + e_2 + e_3) \\ p_2 &= \tfrac{1}{2}(1 - e_1 + e_2 + e_3) \\ p_3 &= \tfrac{1}{2}(1 + e_1 + e_5 + e_6) \end{aligned} \tag{4.3}$$

from the coset space $E_8/E_7\times SU(2)$ and define 7×7 matrices corresponding to the transformations

$$e_i' = (\pm p_a) e_i (\pm \bar{p}_a) \equiv \sum_{j=1}^{7} (P_\alpha)_{ij}\, e_j \quad (\alpha = 1,2,3) \tag{4.4}$$

where e_i' satisfy the octonion algebra described by e_i. From Equation (4.4) all matrix elements of the group $G_2'(2)$ can not be obtained. But whole group can be generated by considering the matrix elements which are obtained from Equation (4.4),as generators. The matrices corresponding to the transformations of these roots with their conjugates can be calculated using Equation (4.4)[16].

$$P_1=\frac{1}{2}\begin{bmatrix}0&0&0&0&0&-2&0\\0&-1&-1&1&0&0&1\\0&1&-1&1&0&0&-1\\0&-1&-1&-1&0&0&-1\\2&0&0&0&0&0&0\\0&0&0&0&-2&0&0\\0&-1&1&1&0&0&-1\end{bmatrix}\quad P_2=\frac{1}{2}\begin{bmatrix}0&0&-2&0&0&0&0\\-2&0&0&0&0&0&0\\0&2&0&0&0&0&0\\0&0&0&-1&-1&1&-1\\0&0&0&1&-1&1&1\\0&0&0&-1&-1&-1&1\\0&0&0&1&-1&-1&-1\end{bmatrix}$$

$$P_3=\frac{1}{2}\begin{bmatrix}0&2&0&0&0&0&0\\0&0&2&0&0&0&0\\2&0&0&0&0&0&0\\0&0&0&-1&-1&1&1\\0&0&0&1&-1&-1&1\\0&0&0&-1&1&-1&1\\0&0&0&-1&-1&-1&-1\end{bmatrix} \tag{4.5}$$

They satisfy the relation

$$P_1^3=P_2^3=P_3^3=(P_1P_2P_3)^7=I \tag{4.6}$$

where I is the 7×7 unit matrix. It is straightforward to check that these matrices leave the root system of E_7 invariant. The matrices $P_\alpha(\alpha=1,2,3)$ generate a group of order 6048 which is the simple group $U_3(3)=G_2'(2)$. The generation relation given in Equation (4.6) is not a minimal generation relation. The matrices Q_1 and Q_2 that can be expressed in terms of P_α, also generates whole group. The minimal generation relation for G'(2) are given by[20]

$$G = \langle Q_1^2 = Q_2^6 = (Q_1Q_2)^7 = (Q_1Q_2^2)^3(Q_1Q_2^{-2})^3 = (Q_1Q_2Q_1Q_2^{-2})^3 Q_1Q_2(Q_1Q_2^{-1})^2 = 1\rangle \qquad (4.7)$$

where Q_1 and Q_2 are

$$Q_1 = \frac{1}{2}\begin{bmatrix} 0 & -1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 & -1 \\ -1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & -1 & 0 \end{bmatrix} \quad Q_2 = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix} \qquad (4.8)$$

4.3. Structure of The Group $G_2'(2)$

The simple group $G_2'(2)$ which has 14 conjugacy classes and 4 maximal subgroups, is the automorphism group of octonion algebra. The character tables of the groups and its maximal and Sylow subgroups will be given in Appendix A. The group $G_2'(2)$ involves many interesting subgroups such as the binary tetrahedral group $2A_4$, symmetric group S_4, and so on. For instance the matrices P_1 and P_2 generate a group of order 24 which is isomorphic to the binary tetrahedral group $2A_4$. The character values of the seven-dimensional irreducible representation of $G_2'(2)$ are given in Appendix A (χ^3 in Table A2).

One can easily check that the seven-dimensional matrix representation of $G_2'(2)$ is irreducible. The maximal subgroups of $G_2'(2)$

and their structures are computed by a computer program the flow-chart of which will be given in Appendix B. The results are

Group	Order	Structure
H_1	216	$3_+^{1+2}{:}8$
H_2	168	$PSL_2(7)$
H_3	96	$4{\cdot}S_4$
H_4	96	$4^2{:}S_3$

which can be contrasted with those in the ATLAS of Finite groups[5]. The character tables of each maximal subgroup H_1, H_2, H_3 and H_4 are given in appendix A, (Tables (A7, A10, A8 and A9), respectively. The seven-dimensional representations of the generators of the group of interest are the matrices;

$$Q_3=\begin{bmatrix}1&0&0&0&0&0&0\\0&0&0&1&0&0&0\\0&0&0&0&0&0&1\\0&0&0&0&0&1&0\\0&0&1&0&0&0&0\\0&1&0&0&0&0&0\\0&0&0&0&1&0&0\end{bmatrix}\quad Q_4=\begin{bmatrix}-1&0&0&0&0&0&0\\0&0&0&0&-1&0&0\\0&0&0&0&0&-1&0\\0&0&0&0&0&0&-1\\0&0&-1&0&0&0&0\\0&1&0&0&0&0&0\\0&0&0&-1&0&0&0\end{bmatrix}\tag{4.9a}$$

$$Q_5=\tfrac{1}{2}\begin{bmatrix}0&-2&0&0&0&0&0\\2&0&0&0&0&0&0\\0&0&2&0&0&0&0\\0&0&0&-1&1&1&-1\\0&0&0&-1&-1&-1&-1\\0&0&0&1&1&-1&-1\\0&0&0&1&-1&1&-1\end{bmatrix}\quad Q_6=\begin{bmatrix}-1&0&0&0&0&0&0\\0&1&0&0&0&0&0\\0&0&-1&0&0&0&0\\0&0&0&0&0&0&-1\\0&0&0&0&0&1&0\\0&0&0&0&1&0&0\\0&0&0&-1&0&0&0\end{bmatrix}\tag{4.9b}$$

The generators of the maximal subgroups H_1, H_2, H_3 and H_4 can be chosen as $<Q_1, Q_4'>$, $<Q_3, Q_5>$, $<Q_2, Q_3>$ and $<Q_2, Q_5>$, respectively. The conjugacy classes of each maximal subgroup obtained for 7×7 matrix representations are reducible except the maximal subgroup $PSL_2(7)$. Decomposition of reducible representations are as follows:

Group	Decomposition	Order of Irreps.
H_1	$\chi^2 + \chi^{10}$	1+6
H_2	χ^5(irreducible)	7
H_3	$\chi^1 + \chi^{10}$	1+6
H_4	$\chi^8 + \chi^{10} + \chi^{11}$	2+2+3

where χ^i are given in tables, in Appendix A. The group $G_2'(2)$ has three Sylow p-subgroup of orders 32, 27 and 7. The character tables of the Sylow p-groups are given in Appendix A, (Tables A11, A12 and A13), respectively.

4.3. $G_2(2)$ as the Automorphic Extension of the Group $G'_2(2)$

The seven-dimensional irreducible representation of $G_2'(2)$ involves three diagonal matrices other than the unit matrix and are given by

$$N_1 = (1,\ 1,\ 1,\ -1,\ -1,\ -1,\ -1)$$
$$N_2 = (1,\ -1,\ -1,\ 1,\ -1,\ -1,\ 1) \qquad (4.11)$$
$$N_3 = N_1N_2 = N_2N_1 = (1,\ -1,\ -1,\ 1,\ -1,\ -1,\ 1)$$

An inspection proved that the diagonal matrix $N_7 = (-1,\ 1,\ -1,\ 1,\ -1,\ 1,\ -1)$, $N_7^2 = I$ also preserves the octonionic roots of E_7 in Equation (4.1). The matrices N_1, N_2, N_7 generate an elementary

Abelian group of order 8 where we define the other elements of the group by

$$N_4 = N_7N_1 \qquad N_5 = N_7N_2 \qquad N_6 = N_7N_3. \qquad (4.12a)$$

They satisfy

$$N_iN_j = N_jN_i = N_k \quad (ijk = 123,\ 147,\ 165,\ 246,\ 257,\ 354,\ 367) \quad (4.12b)$$

One can readily show that the generators of $G_2'(2)$ and N_7 satisfy the relations

$$N_7Q_1N_7^{-1} = Q_1' \qquad N_7Q_2N_7^{-1} = Q_2' \qquad (4.13)$$

where Q_1, Q_2, $Q_1', Q_2' \in G_2'(2)$.

This proves that N_7 constitutes the outer automorphism of $G_2'(2)$, and $G_2(2)$ is obtained by adjoining N_7 to the generators Q_1 and Q_2 of $G_2'(2)$. The product of the elements of $G_2'(2)$ with N_7 from left or right define the new sets of elements of the extended group $G_2(2)$ of order 12096, denoted as $G_2'(2){:}2$. Therefore the group $G_2(2)$ is the split extension of the $G_2'(2)$ by the Abelian group of order 2. Thus, we obtain the seven-dimensional irreducible representation of the adjoint Chevalley group $G_2(2)$ which is one of the maximal subgroups of the Weyl group of E_7 with index 240. Indeed, the Weyl group of E_7 is the direct product of Chevalley group $SO_7(2)$ with the inversion group of order 2. Therefore, $G_2(2)$ are the orthogonal matrices of determinant +1. The character table of $G_2(2)$ is computed and given in Appendix A, (Table A1).

4.4. Maximal Subgroups of $G_2(2)$ and the Related Subalgebras of E_7

The simple group $G_2'(2)$ has four maximal subgroups of orders $216(3_+^{1+2}:8)$, $96(4{\cdot}S_4)$, $96'(4^2{\cdot}S_3)$ and $168(PSL_2(7)$ which have been mentioned[5]. The notation in the parentheses are the names of the groups and will be clarified in what follows. Double covers of four respective groups of orders 432, 192, 192′ and 336 preserve the root systems of E_6, $SO(12)$, $SU(2)^3{\times}SO(8)$ and $SU(8)$.

We discuss each case separately below.

4.4.1. The Group $3_+^{1+2}:8:2$ of Order 432 and the Octonionic Roots of E_6

$E_6{\times}U(1)$ is one of the maximal Lie algebras of E_7. $U(1)$, being in the Cartan subalgebra, is represented by the zero root. 126 non-zero roots of E_7 decompose as $126 = 72+27+27^*$, where the 72 non-zero roots of E_6 are given by

$$\begin{array}{ll}
\pm e_2, \pm e_3, \pm e_5, \pm e_6 & \frac{1}{2}(\pm e_2 \pm e_3 \pm e_5 \pm e_6) \\
\pm\frac{1}{2}(e_1 \pm e_2 - e_4 \pm e_6) & \pm\frac{1}{2}(e_1 \pm e_3 - e_4 \pm e_5) \\
\pm\frac{1}{2}(e_1 \pm e_2 \pm e_5 - e_7) & \pm\frac{1}{2}(e_1 \pm e_3 \pm e_6 - e_7) \\
\pm\frac{1}{2}(\pm e_2 \pm e_3 \pm e_4 - e_7) & \pm\frac{1}{2}(e_4 \pm e_5 \pm e_6 - e_7)
\end{array} \qquad (4.14)$$

The 24 roots in the top line of Equation (4.14) correspond to the roots of $SO(8)$ subalgebra of E_6. The matrices $G_2'(2)$ which leave the set of roots in Equation (4.14) form a group of order 216. This group has 13 conjugacy classes. The notation 3_+^{1+2}:8 used for the group 216 has the following meaning. It is the semi-direct product of two groups: one is the cyclic group of order 8 and the other is an extraspecial group of order 27. The E_6 root system is also left invariant by the matrix Q_6, which is not an element of $G_2'(2)$. The order of Q_6 is 2 since Q_6^2=I. By taking the semi-direct product of 3_+^{1+2}:8 with the cyclic group of order 2 we obtain the group 3_+^{1+2}:8:2 of order 432. We have also checked that 3_+^{1+2} is a normal subgroup of 3_+^{1+2}:8:2 of index 16.

The Weyl group of E_6 is the automorphic extension of $SO_6(2) \equiv U_4(2) \equiv {}^2A_3(2)$ with the cyclic group of order 2. Here $SO_6(2)$, for example, is generated by 6×6 matrices over the field with two elements preserving a non-singular quadratic form $x_1^2 + x_1x_2 + x_2^2 + x_3x_4 + x_5x_6$.

It is interesting to note that $|W(E_6)|/|3_+^{1+2}:8:2|=120$. We have analysed the maximal subgroups of the group of order 432 and obtained three different maximal subgroups, each of order 216. In addition to the usual maximal subgroup of $G_2'(2)$ having 13 conjugacy classes, we obtained the other two groups with 13 and 16 conjugacy classes. The second group of order 216, although possessing 13 conjugacy classes, is different from the one in $G_2'(2)$. The character table of the group 3_+^{1+2}:8:2 are shown in Appendix A (Table A7).

4.4.2. The Group $4{\cdot}S_4{:}2$ of Order 192 and the Root System of $SU(2)\times SO(12)$

The Lie algebra $SU(2)\times SO(12)$ is maximal in E_7. In fact the isotropy group of the highest root of E_7 is the Weyl group of $SO(12)$. The E_7 roots decompose under $SU(2)\times SO(12)$ roots as follows:

$$
\begin{array}{ll}
SU(2): & \pm e_1 \\
SO(12): & \pm e_2, \pm e_3, \pm e_4, \pm e_5, \pm e_6, \pm e_7 \\
 & \frac{1}{2}(\pm e_2 \pm e_3 \pm e_5 \pm e_6) \\
 & \frac{1}{2}(\pm e_2 \pm e_3 \pm e_4 \pm e_7) \\
 & \frac{1}{2}(\pm e_4 \pm e_5 \pm e_6 \pm e_7)
\end{array}
\tag{4.15}
$$

The remaining roots of E_7 transform as the weights of the direct product of spinor representation [2,32] of $SU(2)\times SO(12)$. Here we are concerned with the transformations which stabilise $\pm e_1$. The matrices of $G_2'(2)$ preserving roots in Equation (4.15) form a group of order 96 with 10 conjugacy classes whose generators can be taken as the matrices Q_3, Q_4 whose elements consist of only $\pm 1, 0$. The other maximal subgroups of $G_2(2)$ generated by the other matrices also include elements $\pm\frac{1}{2}$ in addition to $\pm 1, 0$. Being a subgroup of $G_2'(2)$, the $4{\cdot}S_4$ consisting of all matrices with elements $\pm 1, 0$ involves the diagonal matrices N_i (i=1,2,3). We note that N_7 also preserves the root system in Equation (4.15) so that the group can be extended to $4{\cdot}S_4{:}2$ of order 192 possessing 14 conjugacy classes of all matrices with $\pm 1, 0$ in $G_2(2)$. These 192 matrices are of the form 7=1+6 so that the 6×6 blocks act on the octonionic units e_i(i=2,...,6) and can be further reduced to two sets of 3×3 matrix blocks. This is then a discrete subgroup of SU(3). The matrices generating S_4 can further be reduced to 7=1+3+3', two blocks of 3×3 matrices acting on the sets (e_2, e_4, e_5) and (e_3, e_6, e_7) which correspond to two different three-dimensional irreducible representations of S_4. In fact, the whole group involving S_4 (finite subgroup of SO(3)) can be reduced to 7=1+3+3'. The SO(12) roots in Equation (4.15) can be regarded as the union of three $SO(8)_a$ (a=1, 2, 3) roots. One of the S_3 subgroups of $4{\cdot}S_4{:}2$ permutes these root systems amongst themselves. We have checked that the group of order 192 has three maximal subgroups of orders 96. In addition to the usual one which is involved in $G_2(2)$ there are two more groups with 10 conjugacy classes. However, they are not isomorphic to each other.

4.4.3. The Group $4^2{:}S_3{:}2=4^2{:}D_6$ of Order 192 and The Root System $SU(2)^3\times SO(8)$

A computer program gave two maximal groups of order 96 in $G_2'(2)$. The one with elements ±1,0 which we have already discussed was obtained while searching for the transformations preserving the root system of SO(12). The second one was given by computer calculation. This group has 16 conjugacy classes, and the matrices can be reduced to block diagonal forms of 3×3 and 4×4 matrices where the upper 3×3 matrices possess elements ±1,0 only acting on (e_1, e_2, e_3) and the lower 4×4 matrices involve the additional elements $\pm\frac{1}{2}$ and transform the octonionic units (e_4, e_5, e_6, e_7) among themselves. The upper block of 3×3 matrices in fact transforms the roots of SU(2)×SU(2)×SU(2) where each SU(2) is represented by one octonionic unit $\pm e_i$ (i=1,2,3) whereas the lower 4×4 matrices preserve the root system of an SO(8) Lie algebra whose roots ore the linear combination of e_i's (i=4,5,6,7). Therefore, the group of order 96 with 16 conjugacy classes leaves the root system of $SU(2)^3 \times SO(8)$ invariant where the roots are given by

$$\begin{aligned}
&SU(2): \ \pm e_1 \\
&SU(2): \ \pm e_2 \\
&SU(2): \ \pm e_3 \qquad\qquad (4.16) \\
&SU(8): \ \pm e_4, \pm e_5, \pm e_6, \pm e_7 \quad \tfrac{1}{2}(\pm e_4 \pm e_5 \pm e_6 \pm e_7)
\end{aligned}$$

The matrices of $G_2'(2)$ which preserve this root system are generated by the matrices Q_2 and Q_5. Again noting that N_7 preserves the same root system, the group is easily extended to a group of order 192 which is maximal in $G_2(2)$ and has 17 conjugacy classes, the character table is shown in appendix A, (Table (A4). The $SU(2)^3 \times SO(8)$ is not a maximal subalgebra as it is involved in $SU(2) \times SO(12)$. However, we should emphasise that while the SU(2) in the case of $SU(2) \times SO(12)$ were invariant, here three SU(2) roots are permuted. Once again we notice that the group 192' involves three maximal subgroup of order 96 but with different structures. They have, respectively 16, 13 and 19 conjugacy classes, and only the one with 16 classes is involved in $G_2'(2)$. The group of order 96 with 13 conjugacy classes has the structure $4^2 \cdot S_3$ where 4^2 denotes the direct product of two cyclic groups of order 4 and S_3 is the permutation group of 3-objects. It is understood that 4^2 is invariant in $4^2 \cdot S_3$. An extension of this to 192' is made by the 2-group, so the structure is given by $4^2 \cdot D_6$ where D_6 is the dihedral group of order 12. In fact, D_6 is the automorphism group of the root system of G_2. The $4^2 \cdot D_6$ group has many interesting subgroups such as the binary tetrahedral group $2A_4$ of order 24, and dihedral groups of various orders. With a unitary transformation

$$(e_1, e_2, e_3, e_4, e_5, e_6, e_7) \to (e_1, e_2, e_3, u, d, d^*, -u^*) \tag{4.17}$$

where u, d are the split octonions

$$\begin{aligned} u &= \tfrac{1}{\sqrt{2}}(e_4 + ie_5) \qquad & u^* &= \tfrac{1}{\sqrt{2}}(e_4 - ie_5) \\ d &= \tfrac{1}{\sqrt{2}}(e_6 + ie_7) \qquad & d^* &= \tfrac{1}{\sqrt{2}}(e_6 - ie_7) \end{aligned} \tag{4.18}$$

the generators of 192′ can be written in the form

$$\left[\begin{array}{c|c} R_i & \\ \hline & S_i \end{array}\right] \tag{4.19}$$

where

$$\begin{aligned} R_1 &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \qquad S_1 = \tfrac{1}{\sqrt{2}}(i\sigma_2 + i\sigma_3) \otimes (I + i\sigma_1) \\ R_2 &= \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad S_2 = \tfrac{1}{\sqrt{2}}(I + i\sigma_1) \otimes (I + i\sigma_3) \end{aligned} \tag{4.20}$$

and σ_i (i=1,2,3) are the 2×2 Pauli matrices and ⊗ stands for the direct product of S_1 and S_2. If all possible products of R_1 and R_2 were allowed then we would have generated the direct products of two binary octahedral groups so that the order would be 48×48=2304 instead of 192.

We have checked that the 3×3 matrices R_1 and R_2 generate a group of order 24 isomorphic to S_4 with five conjugacy classes. This is the octahedral group where $\pm e_i$ (i=1,2,3) denote the vertices of octahedron. However, the lower 4×4 blocks of matrices S_1 and S_2 generate a group of order 192 which unifies the two two-dimensional irreducible representations of D_6 in the four-dimensional representation of the group 192′.

4.4.4 $PSL_2(7)$ and the Root System of SU(8)

SU(8) is a maximal subalgebra of E_7 where the E_7 roots decompose under SU(8) as

$$126 = 56 + 70 \tag{4.21}$$

Here, 56 represents the non-zero roots of SU(8) and 70 the weights of the 70-dimensional representation. We are interested in the subgroup of $G_2'(2)$ which preserves this decomposition. Octonionic roots of SU(8) are given by

$$\begin{aligned}
&\pm e_1, \pm e_2, \pm e_4, \pm e_6 \\
&\pm\tfrac{1}{2}(\pm e_1 \pm e_2 + e_4 + e_7) \\
&\pm\tfrac{1}{2}(\pm e_1 + e_3 \pm e_6 + e_7) \\
&\pm\tfrac{1}{2}(\pm e_1 + e_3 \pm e_4 + e_5) \\
&\pm\tfrac{1}{2}(\pm e_2 + e_3 - e_5 \pm e_6) \\
&\pm\tfrac{1}{2}(\pm e_4 + e_5 \pm e_6 - e_7).
\end{aligned} \tag{4.22}$$

The matrices Q_1 and Q_5 of $G_2'(2)$ preserve this system and generate a group of order 168 with six conjugacy classes (the well known finite subgroup of SU(3)). The 7×7 dimensional representation is irreducible and the character table is given in appendix A (Table A10).

The group 168 has the maximal subgroups S_4 and a group of order 21 with the structure 3:7. We will see that the group 21 will also play an important role in the construction of the group 1344. We should emphasise that the group 168 which preserves the octonionic roots of SU(8) is the finite subgroup of SU(3) which is isomorphic to $PSL_2(7)$. Its class structure and the characters of the seven dimensional irreducible representation are the same as for $PSL_2(7)$. The matrix N_7 also preserves the root system of SU(8). Therefore, $PSL_2(7)$ can be extended to the group $PSL_2(7)$:2 of order 336 with nine conjugacy classes. The character table of the group 336 will be given in appendix A, (Table A6). Hence the generators of the group 336 are Q_1, Q_5 and N_7. This group is related to the conformal field theory of G_2 affine Lie algebra.

Another interesting observation here is that the group 336 is a subgroup of the Weyl group of SU(8) with index 120.

CHAPTER 5

OCTONIONS, THE GROUPS OF ORDER 1344

5.1. Introduction

In this chapter the irreducible representation of the automorphism group of the octonionic imaginary units $\pm e_i$ ($i=1,\ldots,7$) and the group of order 1344 possessing the same character table are constructed by 7×7 matrices. Although the two groups have the same character table their structures are different. Two groups are the extensions of the elementary Abelian group 2^3 of order 8 by the Klein's simple group isomorphic to $PSL_2(7)$ of order 168. One of the groups $2^3{\cdot}PSL_2(7)$ which is the non-split extension of the elementary Abelian group by the Klein's group of order 168 turns out to be the automorphism group of the octonionic imaginary units $\pm e_i$ ($i=1,\ldots,7$). The other group $2^3{:}PSL_2(7)$ is the split extension displaying exceptional features for it admits two non-conjugate subgroups of the Klein's group of order 168.

It is interesting to note that the group $2^3{\cdot}PSL_2(7)$ occurs naturally as a subgroup in various sporadic groups. The group $2^3{\cdot}PSL_2(7)$ is first discussed by Coxeter. The structure of the group $2^3{:}PSL_2(7)$ is worked out by Conway proving its extraordinary structure related to the behaviours of its two non-conjugate maximal subgroups $PSL_2(7)$. The groups $PSL_2(7)$ and $2^3{:}PSL_2(7)$ arise also as the automorphism groups of the binary Hamming codes of the types E_7 and E_8 respectively[39].

We observe that the group $2^3{\cdot}PSL_2(7)$ can be derived from one of the maximal subgroups of order 192 of the adjoint Chevalley group $G_2(2)$ of order 12096.

5.2. Octonions and the Group of Order 21

The octonion algebra has been studied in chapter 3. We are searching the group that its elements preserve the multiplication table of octonions under transformation operations. Since there are seven imaginary octonionic units, the dimension of the matrix elements of the group must be 7×7. We are dealing with the transformations of only integral octonions(not linear combinations of octonions), so only one element of the rows or columns of the matrix is ± 1, all the others are zero. Let us consider a matrix α which operates on the imaginary units e_i $(i=1,\ldots,7)$ in the cyclic order $(e_1e_2e_4e_3e_6e_5e_7)$ so that $\alpha^7 = 1$. Similarly let β fix e_1 and permutes the associative triad $(e_4e_6e_2)$ and the anti-associative triad $(e_3e_7e_5)$ in the indicated orders. These two transformations preserve the multiplication table of octonions permuting the set of seven imaginary units e_i among themselves and form a group 7:3 of order 21 with 5 conjugacy classes. The matrix representations of α and β acting on the basis $(e_1,\ldots,e_7)$ are given by[28]

$$\alpha = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \beta = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \tag{5.1}$$

They satisfy the relations

$$\alpha^7 = \beta^3 = 1, \quad \beta\alpha\beta^{-1} = \alpha^4 \tag{5.2}$$

indicating that α forms an invariant subgroup of the 7:3. The conjugacy classes are given by

$$C_1(1)=1 \qquad C_2(7)=\{\alpha,\alpha^2,\alpha^4\} \qquad C_3(7)=\{\alpha^3,\alpha^5,\alpha^6\}$$
$$C_4(3)=\{\alpha^a\beta\} \qquad C_5(3)=\{\alpha^a\beta^2\} \qquad (a=0,1,\ldots,6) \tag{5.3}$$

The character table of the group of order 21 is given in Appendix A, (Table A7). The seven dimensional representation of 7:3 is reducible and the character values are given in Table (5.1). The 7-dimensional reducible representation can be decomposed as the direct sum of its irreducible representations

$$7 = 1 + 3 + 3' \tag{5.4}$$

The group 7:3 is the largest group preserving the octonion multiplication without changing the sign of e_i (i=1,.....,7). Had we allowed the change of signs of e_i, in other words, if we had considered the transformations preserving the octonionic set of imaginary units $\pm e_i$(i=1,...,7) we could have generated a larger group.

Table 5.1. The Conjugacy Classes of the Group 7:3

CLASS	$C_1(1)$	$C_2(3)$	$C_3(3)$	$C_4(7)$	$C_5(7)$
ELEMENTS	1	7	7	3	3
CHARACTERS	7	1	1	0	0

5.3. The group $2^3 \cdot L_2(7)$ as the Automorphism Group of the Octonionic Set $\pm e_i$(i=1,...,7)

In the previous chapter the seven-dimensional matrix representation of the group $G_2(2)$ was constructed. We mentioned that there was seven diagonal matrices besides the unit matrix preserving the octonion algebra. Those matrices playing an important role in the construction of the group $2^3{\cdot}L_2(7)$, are the matrices N_i ($i=1,\ldots,7$) given by the equations (4.11 and 4.12)..

Now we consider the matrix transformations α: $N_j \rightarrow \alpha N_i \alpha^{-1}$ and β: $N_j \rightarrow \beta N_i \beta^{-1}$ where α and β are the matrices given in Equation (5.1) and N_i are the diagonal matrices defined in Equations (4.11 and 4.12) . It is rather trivial to show that α and β permutes the set of matrices N_i in the cyclic order defined by

$$\begin{aligned} &\alpha : N_i \rightarrow (N_1N_2N_4N_3N_6N_5N_7) \\ &\beta : N_i \rightarrow (N_3N_2N_1)(N_4N_6N_5)(N_7) \end{aligned} \tag{5.5}$$

This proves that the group 7:3 acts as an automorphism group on the elementary Abelian group of the matrices N_i. (We will complete the proof in the next section that full automorphism group of the group 2^3 is the Klein's simple group of order 168 isomorphic to the group $L_2(7)$).

We now make the split extension of the group 2^3 by the group of order 21 by adjoining the generator, say N_1, to the generators α and β. This group has then the structure 2^3:7:3 possessing the eight conjugacy classes. This structure is evident from Equation (5.5) indicating that the group 2^3 is the invariant Abelian subgroup of the group 2^3:7:3 where the group 7:3 forms the factor group. Since the matrices α and β themselves generate the group of order 21 it is the split extension. The matrices α, β and N_i acting on the imaginary units e_i preserve the octonion algebra. Therefore the group 2^3:7:3 constitutes an automorphism group of the octonion algebra.

With the use of the character table (in Appendix A, Table A17) of the group of order 21 it is rather straightforward to calculate the character table of the group 2^3 :7:3 which is given in Table (A5) in Appendix A. The 7- dimensional representation generated by the matrices α, β and N_1 is real and irreducible which can be read from Table (5.2)

Table 5.2. Conjugacy Classes of 7-Dimensional Matrix Representation of the Group 2^3:7:3.

CLASS	$C_1(1)$	$C_2(2)$	$C_3(3)$	$C_4(3)$	$C_5(7)$	$C_6(6)$	$C_7(6)$	$C_8(7)$
# ELEMENTS	1	7	28	28	24	28	28	24
CHARACTERS	7	-1	1	1	0	-1	-1	0

The group 2^3:7:3 is not complete all possible transformations of octonions. We note that another transformation T_1 (which cannot be obtained from α,β and N_1)

$$T_1 = \begin{bmatrix} 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} \tag{5.6}$$

preserves the octonion algebra. Adjoining T_1 to α, β and N_1 one can generate a group of order 1344 with 11 conjugacy classes. The representation so obtained is irreducible. One can show that T_1 acts as an automorphism on N_i.

$$T_1 : N_i \rightarrow (N_1N_5)(N_2)(N_3N_7)(N_4)(N_6) \tag{5.7}$$

Hence the elementary Abelian group 2^3 is an invariant subgroup of order 1344. The factor group obtained dividing the group of order 1344 by the group 2^3 is isomorphic to $L_2(7)$. However the coset representatives do not form a subgroup, rather generate the whole group. Therefore the automorphism group of order 1344 of the octonion algebra is the non-split extension of the elementary Abelian group 2^3 by $L_2(7)$. The group $2^3{\cdot}L_2(7)$ is a finite subgroup of the Lie group G_2. It is maximal in the Chavalley group $G_2(3)$. It is also true that the whole group $2^3{\cdot}L_2(7)$ can be generated simply by two generators α and T_1. It turns out that the group $2^3{:}7{:}3$ is a maximal subgroup of the group $2^3{\cdot}L_2(7)$. We have calculated the character table of the group $2^3{\cdot}L_2(7)$ using the character table of its maximal subgroup $2^3{:}7{:}3$ and also compared it with that of obtained from an independent calculation based on the class multiplication. The result is given in Appendix A (Table (A14). It is rather amusing to note that the octonionic set $\pm e_i (i=1,\ldots,7)$ constitute a scaled root system of the Lie algebra $SU(2)^7$ in the octonionic representation of the E_7 root system.

The group $2^3{\cdot}L_2(7)$ has three maximal subgroups two of which are of order 192 and one is the usual group $2^3{:}7{:}3$ of order 168. One of the groups of order 192 has 14 conjugacy classes and can be generated by the matrices T_1 given in Equation (5.8) and T_2

$$T_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad (5.8)$$

The character values of the 7-dimensional representation of the group of order 192 generated by the matrices T_1 and T_2 indicate that it can be decomposed as the sum of two irreducible representations of the group of order 192, namely 7=1+6. This is also obvious from the matrices T_1 and T_2 which leave $\pm e_6$ invariant. This group also occurs as a maximal subgroup in the adjoint Chevalley group $G_2(2)$, of order 12096, the automorphism group of the octonionic roots of E_7 Lie algebra. In fact one can first construct the group of order 192 then make a cyclic extension to the group $2^3{\cdot}L_2(7)$ by the generator α. We have checked that this group of order 192 with 14 conjugacy classes has a structure $4{\cdot}S_4{:}2$ indicating that it has an invariant subgroup of order 96 isomorphic to the group $4{\cdot}S_4$ which is the non-split extension of a cyclic group by the permutation group S_4.

Another interesting subgroup, of order 192, of the group $2^3{\cdot}L_2(7)$ can be generated by the matrices T_3 and T_4

$$T_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \end{bmatrix} \quad T_4 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad (5.9)$$

The 7-dimensional representation generated by matrices T_3 and T_4 has 13 conjugacy classes which can be decomposed as the direct sum 7=3+4. Indeed, we have calculated the character table of this group and checked that it possesses the irreducible representations of dimensions 3 and 4(see Table A18). One can also be convinced by looking at the matrices T_3 and T_4 that they preserve the sets of octonionic units $(\pm e_2, \pm e_4, \pm e_6)$ and $(\pm e_1, \pm e_3, \pm e_5, \pm e_7)$ separately where the first set always form the associative triads while the second anti-associative triads. It is also clear why one can generate the group $2^3{\cdot}L_2(7)$ by adjoining the generator α to either sets of generators T_1 and T_2 in Equations (5.5 and 5.8) or T_3 and T_4 in Equation (5.9) . In the first case $\pm e_6$ is permuted through the other imaginary units of octonions whereas in the second the triad $(\pm e_2, \pm e_4, \pm e_6)$ is cyclically rotated through the 7 sets of associative triads.

The detailed structures of these groups of order 192 are very interesting. The maximal subgroup $4 \cdot S_4$ of order 96 fixes e_6 therefore constitutes a finite subgroup of SU(3). We also note that the subgroup $2^3{:}7$ of the group $2^3{:}7{:}3$ is one of the maximal subgroup of order 56 with 8 conjugacy classes and turns out to be also a maximal subgroup in the simple group of order 504 known as the derived Ree group $R(3) \approx {}^2G_2(3)$.

5.4. The group $2^3{:}L_2(7)$ of Order 1344

We have already discussed in the preceding section that the generators α and β generate an automorphism group, of order 21 of the elementary Abelian group generated by the matrices N_1, N_2, and N_7. This is not the all automorphism. Indeed another generator γ of order 2 [28]

$$\gamma = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \gamma^2 = I \qquad (5.10)$$

permutes the elements N_i of the group 2^3 as follows

$$\gamma : N_i \rightarrow (N_1N_2)(N_3)(N_4N_5)(N_6)(N_7) \qquad (5.11)$$

When γ is adjoined to the generators α and β one generates a simple group of order 168 which is the Klein's simple group isomorphic to the $PSL_2(7)$. This group possesses no invariant subgroup indicating that it is a simple group with 6 conjugacy classes. The generators satisfy the well-known relation of $L_2(7)$:

$$\alpha^7 = \beta^3 = \gamma^2 = \beta\alpha\beta^{-1}\alpha^3 = (\beta\gamma)^2 = (\alpha\gamma)^3 = I \tag{5.12}$$

The character table of $L_2(7)$ given in Appendix A,(Table (A10)) can be computed with the use of the character table of its maximal subgroup of order 21 given in Appendix A (Table (A17)) and/or via the class multiplication. One can also find it in various references.

The 7- dimensional representation generated by the matrices α,β and γ is reducible which can be checked comparing the values in Appendix A (Table A10) with those given in Table (5.3). One can readily see that it decomposes as 7=1+6.

Table 5.3. The Character Values of the 7-Dimensional Representation of the Group $L_2(7)$ Generated by the Matrices α, β, γ.

CLASS	$C_1(1)$	$C_2(3)$	$C_3(7)$	$C_4(7)$	$C_5(4)$	$C_6(2)$
ELEMENTS	1	56	24	24	42	21
CHARACTERS	7	1	0	0	1	3

Interestingly enough one can choose another matrix δ with $\delta^2 = 1$

$$\delta = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{5.13}$$

which acts on the matrices N_i in the same way as that the matrix γ operates on N_i.

$$\delta : N_i \to (N_1N_2)(N_3)(N_4N_5)(N_6)(N_7) \tag{5.14}$$

Therefore it is expected that the generators α, β and δ also generate the group $L_2(7)$. Indeed it does. The character values of the 7-dimensional representation in this case is given by Table (5.4). When compared with the character values in Table 4 it is obvious that the 7-dimensional representation of $L_2(7)$ generated by the matrices α, β and δ is irreducible.

Table 5.4. The character values of 7-dimensional representation of the group $L_2(7)$ generated by the matrices α, β, δ.

CLASS	$C_1(1)$	$C_2(3)$	$C_3(7)$	$C_4(7)$	$C_5(4)$	$C_6(2)$
ELEMENTS	1	56	24	24	42	21
CHARACTERS	7	1	0	0	-1	-1

Knowing that the Klein's simple group $L_2(7)$ generated either by α, β, γ or by α, β, δ is the automorphism group of the elementary Abelian group 2^3 generated by N_1, N_2, and N_7 we can extend the group either by the set of generators (α, β, γ, N_7) or by (α, β, δ, N_7). Both ways one reaches the same group of order 1344 with 11 conjugacy classes. The 7- dimensional representation so obtained is irreducible. It follows from the preceding discussions that the group of order 1344 admits the elementary Abelian group 2^3 as an invariant subgroup. We notice once again that $L_2(7)$ generated by either set of matrices do not involve the diagonal matrices $N_i(i=1,...,7)$ except the unit matrix. Therefore the group $L_2(7)$ not only constitutes the factor group of the group of order 1344 but also a subgroup whose intersection with the group 2^3 is just the unit element. Therefore the group generated this way is the semi-direct product (or split extension) of the group 2^3 by $L_2(7)$ which we can safely denote by the group $2^3{:}L_2(7)$. The group obtained this way does not preserve the octonion algebra.

The surprise in this construction is that the group $2^3{:}L_2(7)$. possesses two maximal subgroups isomorphic to $L_2(7)$ but they are not conjugate in the group $2^3{:}L_2(7)$. This follows from their representations; one is reducible and the other is irreducible. It is also amusing that the generators γ and δ satisfy the relation

$$\delta\gamma = \gamma\delta = N_7 \tag{5.15}$$

Conway[6] attributes this exceptional behaviour to the holomorphy of an elementary Abelian group. The group $2^3{:}L_2(7)$ is one of the maximal subgroups of A_8, the group of even permutations of 8 letters.

The group $2^3{:}L_2(7)$ can also be generated by two matrices, say, α, and N_1, δ. The character table of the group $2^3{:}L_2(7)$ is the same as that of the group $2^3{\cdot}L_2(7)$ and given in Appendix A (Table A14).

The group $2^3{:}L_2(7)$ has five maximal subgroups as compared to the group $2^3{\cdot}L_2(7)$ which has only three maximal subgroups. Of these maximal subgroups only the group $2^3{:}7{:}3$ occurs in both groups. This is obvious from the construction of two groups of order 1344. The remaining four maximal subgroups of the group $2^3{:}L_2(7)$ are of orders 192 and 168. Two subgroups of order 192 have 14 and 13 conjugacy classes respectively, however they are not isomorphic to those in the group $2^3{\cdot}L_2(7)$. These subgroups of order 168 are the non-conjugate Klein's groups which we have already studied to some extend.

The group of order 192 with 14 conjugacy classes is generated by the matrices

$$T_5 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \end{bmatrix} \quad T_6 = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad (5.16)$$

When they act on seven objects x_i($i = 1,\ldots,7$) they leave $\pm x_i$ invariant. We have checked that the representation of the group of the matrices T_5 and T_6 can be decomposed as $7 = 1+6$. Although many things are similar to those of the group of order 192 with 14 conjugacy classes in $2^3{\cdot}L_2(7)$ they are not isomorphic for some of the powers of the elements do not match.

The similar arguments are also valid for the groups of order 192 with 13 conjugacy classes. The subgroup of order 192 with 13 conjugacy classes is generated by the matrices T_6 given by equation (5.16) and T_7

$$T_7 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad (5.17)$$

It is indeed isomorphic to $2^3{:}S_4$ the Weyl group of $D_4 \approx SO(8)$. This group preserves the sets $(\pm x_2, \pm x_4, \pm x_6)$ and $(\pm x_1, \pm x_3, \pm x_5, \pm x_7)$ separately. Therefore the 7-dimensional representation can be decomposed as sum of the irreducible representations 7=3+4. We have checked through the character tables that is true.

5. 5. The Demazure-Tits Subgroups of SU (3), G_2 and SO(5) as Subgroups of the Groups of Order 1344

The Demazure-Tits subgroup DT(G)[8, 37] of a compact Lie group G is a particular finite subgroup of the group G. We will only deal with DT(SU(3)), DT(G_2), and DT(SO(5)) as they naturally occur in the groups we have discussed in the preceding sections. For further information on the DT groups the reader is referred to the paper by Michel, Patera and Sharp and the references therein[34].

The DT(SU(n)) for odd values of n is the semi-direct product of the elementary Abelian group 2^{n-1} by the Weyl group W (SU(n)$\approx S_n$. In our notation it is denoted by DT(SU(n)) $\approx 2^{n-1}:S_n$ (n=odd integers). For SU(3)$\approx 2^2:S_3 \approx S_4$. The DT subgroup of the exceptional Lie group G_2 is isomorphic to DT(G_2)$\approx 2^2 \times W(G_2) \approx 2 \times S_4$. The DT subgroup of SO(5) is the semi-direct product of the quaternion group Q of order 8 by the cyclic group 4 of order 4 which can be denoted by DT(SO(5))$\approx$Q:4

We had already noted in section 2 that the group $2^3 \cdot L_2(7)$ is the subgroup of the Chevalley group $G_2(K)$ with the characteristic of K not equal 2. Indeed the group $2^3 \cdot L_2(7)$ is a maximal subgroup of the $G_2(3)$ of order 4,245,696. Therefore it is quite natural to expect that the Demazure-Tits subgroups of G_2 and SU(3) occur in the group $2^3 \cdot L_2(7)$.

The DT(SU(3))$\approx S_4$ is a group of order 24 with 5 conjugacy classes with the irreducible representations 1, 2', 2, 3, 3'. Since the DT (G_2) is the direct product of S_4 with the 2-group it turns out to be a group of order 48 with 10 conjugacy classes doubling the irreducible representations of S_4 . Therefore the DT(G_2) possesses four singlets, two doublets and four triplets as irreducible representations. The DT(SO(5)) is a group of order 32 with 14 conjugacy classes.

Let us consider the DT groups arising as subgroups in the group $2^3 \cdot L_2(7)$. They are naturally nested in the subgroups of order 192 having both conjugacy classes 13 and 14. For instance, the 7-dimensional representations of DT (G_2) in the group of order 192 with 14 conjugacy classes can be generated by the matrices.

$$T_8 = \begin{bmatrix} 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} \quad T_9 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \end{bmatrix} \qquad (5.18)$$

This 7-dimensional representation of DT(G_2) can be decomposed as the direct sum of one singlet and two triplets representation of the same group. In terms of the notation of reference[34] it can be written as $7 = \Gamma_2 + \Gamma_7 + \Gamma_8$ where Γ_2 is a singlet and Γ_7 and Γ_8 stand for two different irreducible triplet representations. The group of order 192 with 13 conjugacy classes possesses also a DT (G_2) subgroup whose 7-dimensional representation can be decomposed in the same way as above so that they are conjugate in the group $2^3 \cdot L_2(7)$. The DT(SU(3)) is a subgroup of DT (G_2). Therefore it arises as the subgroup of the group DT (G_2) discussed above. However we have checked that it also occurs independently in the group of order 192 with 13 conjugacy classes.

We have checked they are of the form of the semi-direct product of the quaternion group Q with the cyclic group of order 4. One of the 7-dimensional representations of DT(SO(5)) is decomposed as the direct sum of three singlets and two doublets. The group of order 192 with 13 conjugacy classes involves the same two representations of the DT(SO(5)). Since the Lie group G_2 has the group $SU(2)\times SU(2)$ as the maximal subgroup we have also checked whether the group

DT(SU(2)×SU(2)) ≈ 4×4 of order 16 occurs as a subgroup. It arises as subgroups in the both groups of order 192 with 14 and 13 conjugacy classes. We note in passing that no DT groups occur in the subgroup 2^3:7:3 of order 168.

When we turn to the group $2^3:L_2(7)$ we naturally do not expect the DT subgroups of concern because $2^3:L_2(7)$. is not a subgroup of G_2. But they really do exist as subgroups except the DT (SO(5)). The DT (SU(3)) ≈ S_4 is a maximal subgroup of $L_2(7)$ which occurs as two non-conjugate subgroups in the group $2^3:L_2(7)$. Moreover the two groups of order 192 possess the DT (G_2) as subgroups. The situation for the DT(G_2) is very similar to that in $2^3{\cdot}L_2(7)$ although the groups of order 192 are not isomorphic to those in the group $2^3:L_2(7)$. For instance, the group 192 with 14 conjugacy classes admits DT(G_2) as a subgroup with the decomposition 7= $\Gamma_4 + \Gamma_7 + \Gamma_{10}$ where Γ_4 represents a singlet Γ_7 and Γ_{10} stand for two distinct triplets. The same DT(G_2) happens to be a subgroup of order 192 with 13 conjugacy classes but in the form of two non-conjugate groups. One of them has the same decomposition as above whereas the other one is in a different form. Another distinct feature between two groups of order 1344 that the DT(SU(2)×SU(2)) does not arise as a subgroup in the group $2^3:L_2(7)$

5.6. P, CP, and CPT Violations

It is intriguing to note that the operators, charge conjugation C, parity P, and time reversal T when acting on the bilinear Dirac fields generate an elementary Abelian group of order 8. Eigenvalues of the group elements corresponding to the eigenvectors: $\overline{\psi}\Gamma_i\psi$ where Γ_i represent the eigen set of Dirac matrices I, $i\gamma_n$, $i\gamma_5$, γ_4, $i\sigma_{mn}$, $i\gamma_5\gamma_n$, $i\sigma_{4m}$, $\gamma_5\gamma_4$ constitute the character table of the group of concern (see Table 5.5)

Table 5.5. The Character Table of the Group 2^3 Generated by C, P, T and the $\overline{\psi}$ Corresponding Eigenvectors of Hermitian Dirac Bilinears

	C	P	T	CP	CT	PT	CPT	I
$V_1 = i\overline{\psi}\gamma_n\psi$	-1	-1	-1	1	1	1	-1	1
$V_2 = i\overline{\psi}\gamma_5\psi$	1	-1	-1	-1	-1	1	1	1
$V_3 = \overline{\psi}\gamma_4\psi$	-1	1	1	-1	-1	1	-1	1
$V_4 = \overline{\psi}\sigma_{mn}\psi$	-1	1	-1	-1	1	-1	1	1
$V_5 = i\overline{\psi}\gamma_5\gamma_n\psi$	1	1	-1	1	-1	-1	-1	1
$V_6 = i\overline{\psi}\sigma_{4n}\psi$	-1	-1	1	1	-1	-1	1	1
$V_7 = \overline{\psi}\gamma_5\gamma_4\psi$	1	-1	1	-1	1	-1	-1	1
$V_8 = \overline{\psi}\psi$	1	1	1	1	1	1	1	1
	N_4	N_5	N_6	N_3	N_2	N_1	N_7	I

Since the automorphism group $L_2(7)$ of the group 2^3 has three generators α, β, γ with the properties $\alpha^7 = \beta^3 = \gamma^2 = I$ which transform the seven elements of the group 2^3 in the manners of equations (5.5 and 5.11) it is natural to interpret them as parity, CP, and CPT violating transformations. Seven dimensional representations act on the first seven bilinear Dirac fields depicted in the first column of table 5.5. Then it is rather trivial to classify the quadratic forms invariant under the respective actions of γ, β, and α.

Let us first discuss the case of γ which acts on the discrete operators and on the bilinear forms as follows[30]

$$\gamma : \begin{array}{l} (C \leftrightarrow P)\ (CP)\ (CT \leftrightarrow PT)\ (T)\ (CPT) \\ V_1 \to V_1,\ V_2 \leftrightarrow V_4,\ V_3 \leftrightarrow V_7,\ V_5 \to V_5,\ V_6 \to V_6 \end{array} \qquad (5.19)$$

This leaves the following effective Hamiltonian invariant:

$$H = g\sum_{i=1}^{7} V_i V_i + g_1(V_1^2 + V_5^2 + V_6^2) + g_2(V_2V_4 + V_4V_2) + g_3(V_3V_7 + V_7V_3) \qquad (5.20)$$

The Lorentz invariance requires $g_1 = g_3 = 0$ and $g = \pm g_2$. This leads to the usual parity violation of the form V+A or V-A which is consistent with a gauge theory of electroweak interaction $SU(2)_L \times SU(2)_R \times U(1)$.

If one imposes the β-invariance on an effective Hamiltonian we may obtain an interaction violating CP while preserving CPT. Indeed β operates as follows

$$\beta : \begin{array}{l} (C, P, T)\ (CP, PT, CT)\ (CPT) \\ (V_1)\ (V_2V_4V_6)\ (V_3V_7V_5) \end{array} \qquad (5.21)$$

This leaves the following quadratic form invariant

$$g_1(V_1^2 + V_2^2 + \cdots + V_7^2) + g_2V_1^2 + g_3(V_2V_4 + V_4V_6 + V_6V_2) + g_4(V_3V_7 + V_7V_5 + V_5V_3) \qquad (5.22)$$

If we insist on the Lorentz invariance no CP violating term is left over. Consequently, we suggest that, if this approach makes sense, then CP violation will require also the violation of the Lorentz invariance.

The third generator α of the group $L_2(7)$ permutes the seven discrete operators as follows

$$\alpha : (C(CP)TP(CPT)(PT)(CT)) \tag{5.23}$$

which also transforms the Dirac bilinears accordingly

$$\alpha : V_1 \to V_2 \to V_4 \to V_3 \to V_6 \to V_5 \to V_7 \to V_1 \tag{5.24}$$

The quadratic form

$$g(V_1^2 + V_2^2 + \cdots + V_7^2) + g'(V_1V_2 + V_2V_4 + V_4V_3 + V_3V_6 + V_6V_5 + V_5V_7 + V_7V_1) \tag{5.25}$$

is left invariant under α. This is one of the typical cases where all discrete symmetries including CPT are violated which certainly violates also the Lorentz invariance.

According to the present scenario it is tempting to assume that the effective Hamiltonian of particle interactions consists of four parts: The first part preserves all discrete symmetries as in the case of strong and electromagnetic interactions. The second part violates parity, the third part violates CP and finally fourth term violates CPT. Last two parts violate also Lorentz invariance. There are also possibilities that CP and CPT violations can be reconciled with Lorentz invariance.

CHAPTER 6

CONCLUSIONS

The seven-dimensional irreducible representation of the adjoint Chavalley group $G_2(2)$ constructed by the inner automorphism of the integral octonions has not been obtained elsewhere. The correspondence between the octonionic subroot system of E_7 and its maximal subgroups of $G_2(2)$, for the first time, is made clear in this work. We have observed that there is a close connection between the Weyl groups of these Lie algebras and $G_2(2)$ and their maximal subgroups:

$$\frac{|W(E_7)/Z_2|}{|G_2(2)|}=\frac{|W(E_6)|}{436}=\frac{|W(SU(8))|}{336}=\frac{|W(SO(12))|}{192}=120$$

$G_2(2)$ and the groups of order 1344 arise as maximal subgroups in various groups of Lie type and sporadic groups. For instance, $G_2'(2)$ is one of the maximal subgroup of the Hall-Janko group J_2[19, 21], and the group $2^3{\cdot}PSL_2(7)$ of order 1344 plays an important role in the constructions of various sporadic groups[30]. These features of the automorphism groups of octonions may relate sporadic groups to any physics associated with the exceptional Lie algebras E_8, E_7 and their subalgebras.

We would like to make some remarks on the group structures as to how they can be used as models in physics. Many physicist have made attempts to drag octonions into physics with some partial success. If octonions play any role in physics at all then the groups, associated with their integer elements, which we have discussed in chapters 4 and 5 should show up as relevant symmetries. An immediate concern perhaps is the following obsevation. The elementary Abelian group 2^3 is isomorphic to the discrete group generated by the operators C, P, and T. The extension of the group 2^3 by its automorphism group $L_2(7)$ manifest itself either as a split extension $2^3{:}L_2(7)$ or the non-split extension $2^3 \cdot L_2(7)$. The subgroups of either of these groups of order 1344 can be used for the formulations of the parity, CP and CPT violations. For instance, we have checked that the invariance of a group of order 16 leading to the parity violation necessarily require the form of the weak current of the form V±A. The other subgroups involve CP and/or CPT violations.

Many of the finite subgroups which we have discussed are in use for different purposes. For instance the simple group $L_2(7)$ of order 168 is used in the Lattice Gauge Theory as an approximation for the $SU(3)_C$. From our point of view the roots of SU(8) then constitute the discrete closed quarks which is supposed to be transformed to each other by $SU(3)_C$ on the lattice. We have checked that all the point crystal groups are the subgroups of the groups we have analysed. Therefore we are optimistic that some of these subgroups will show up as models for certain physical phenomena.

Any possible use of the group $2^3 \cdot L_2(7)$ will pave the road for unexpected roles of the sporadic groups in physics for it is naturally embedded in the sporadic groups in a systematic way.

SUMMARY

This thesis contains, two main subjects, one of which is the automorphism group of the octonionic root system of E_7 and the other is the automorphism group of the integer octonions $\pm e_i$ (i=1,2,...,7). First we have introduced some group theoretical concepts of finite groups, the octonion algebra and constructed the root system of E_8, to make the concept and the construction of the automorphism groups of the integer octonions, self contained.

A method is introduced to obtain the seven-dimensional irreducible representation of $G_2(2)$. A computer program is made to generate the matrix group and to find its maximal subgroups. The maximal subgroups of $G_2(2)$ are obtained and their relations to the root systems of the subalgebras of E_7 are discussed.

The full automorphism group of the integer octonions $\pm e_i$(i=1,2,...,7) is $2^3 \cdot L_2(7)$ of order 1344. It is constructed by seven-dimensional matrices. The structure, and the maximal subgroups of the group $2^3 \cdot L_2(7)$ are obtained and their possible implications in physics are discussed. The seven-dimensional matrix representation of the other group 2^3:PSL_2 of the same order as $2^3 \cdot L_2(7)$, but possessing different structure is also constructed. We made remarks on the relations between the groups $G_2(2)$, $2^3 \cdot L_2(7)$ and 2^3:$L_2(7)$.

The character tables of the various finite groups of concern are computed. Finally the results and the possible use of the relevant groups are discussed.

ÖZET

Bu çalışma iki temel konuyu kapsamaktadır. Bunlardan birisi E_7'nin köklerini teşkil eden oktonyonların otomorfizma grubu olan $G_2(2)$'nın yedi-boyutlu matris temsilinin oluşturulmasıdır. Diğeri ise, $\pm e_i(i=1,2,...,7)$, oktonyonların otomorfizma grubunun oluşturulmasıdır. Konunun iyi anlaşılması için, sonlu gruplarla ilgili bazı teoremler, oktonyon cebiri ve E_8'in köklerinin oktonyonlarla nasıl teşkil edilebileceği kısaca ele alınmıştır.

Çalışmamızda, $G_2(2)$ grubunun yedi-boyutlu indirgenemez temsilini elde etmek için bir method ortaya konmuştur. Bir komputer programı hazırlanarak matris grupları oluşturulmuş ve oluşturulan grupların maksimal altgrupları bulunmuştur. $G_2(2)$ grubunun E_7 Lie cebiriyle ilişkisi ortaya konularak, söz konusu grubun maksimal altgruplarının E_7 Lie cebirinin altcebirleriyle ilişkileri tartışılmıştır.

Ayrıca $\pm e_i(i=1,2,...,7)$ oktonyonlarının otomorfizma grubu olan 1344 elemanlı $2^3.L_2(7)$ grubunun yedi-boyutlu matris temsilli elde edilerek fizikteki muhtemel uygulamaları tartışıldı. Grubun yapısı ve maksimal altgrupları elde edildi. Yine 1344 elemanlı fakat yapısı farklı olan $2^3{:}L_2(7)$ grubunun yedi-boyutlu matris temsilleri oluşturuldu. Elde edilen $2^3.L_2(7)$, $2^3{:}L_2(7)$ ve/veya $G_2(2)$ grupları arasındaki ilişkiler bulundu.

İlgilendiğimiz grupların karekter tabloları hesap edildi. Son olarak sonuçlar değerlendirilerek fizikteki uygulamaları tartışıldı.

APPENDIX A

(CHARACTER TABLES)

Table A1. The Character Table of the $G_2(2)$ of Order 12096

ORDERS	1	2	3	3	4	4	6	7	8	12	2	4	6	8	12	12
# ELEMENTS	1	63	56	672	126	378	504	1728	1512	1008	252	252	2016	1512	1008	1008
INDEX	12096	192	216	18	96	32	24	7	8	12	48	48	6	8	12	12
χ^1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ^2	1	1	1	1	1	1	1	1	1	1	-1	-1	-1	-1	-1	-1
χ^3	6	-2	-3	0	-2	2	1	-1	0	1	0	0	0	0	$i\sqrt{3}$	$-i\sqrt{3}$
χ^4	6	-2	-3	0	-2	2	1	-1	0	1	0	0	0	0	$-i\sqrt{3}$	$i\sqrt{3}$
χ^5	7	-1	-2	1	3	-1	2	0	-1	0	1	-3	1	-1	0	0
χ^6	7	-1	-2	1	3	-1	2	0	-1	0	-1	3	-1	1	0	0
χ^7	14	6	-4	2	-2	2	0	0	0	-2	0	0	0	0	0	0
χ^8	14	-2	5	-1	2	2	1	0	0	-1	2	-2	-1	0	1	1
χ^9	14	-2	5	-1	2	2	1	0	0	-1	-2	2	1	0	-1	-1
χ^{10}	21	5	3	0	1	1	-1	0	-1	1	3	-1	0	1	-1	-1
χ^{11}	21	5	3	0	1	1	-1	0	-1	1	-3	1	0	-1	1	1
χ^{12}	42	2	6	0	-6	-2	2	0	0	0	0	0	0	0	0	0
χ^{13}	27	3	0	0	3	-1	0	-1	1	0	3	3	0	-1	0	0
χ^{14}	27	3	0	0	3	-1	0	-1	1	0	-3	-3	0	1	0	0
χ^{15}	56	-8	2	2	0	0	-2	0	0	0	0	0	0	0	0	0
χ^{16}	64	0	-8	-2	0	0	0	1	0	0	0	0	0	0	0	0

Table A2. The Charcter Table of the Group $G_2'(2)$ of Order 6048. Here $\eta=(-1+i\sqrt{7})$.

ORDERS	1	2	3	3	4	4	4	6	7	7	8	8	12	12
# ELEMENTS	1	63	56	672	63	63	378	504	864	864	756	756	504	504
INDEX	6048	96	108	9	96	96	16	12	7	7	8	8	12	12
χ^1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ^2	6	-2	-3	0	-2	-2	2	1	-1	-1	0	0	1	1
χ^3	7	-1	-2	1	3	3	-1	2	0	0	-1	-1	0	0
χ^4	7	3	-2	1	-1-2i	-1+2i	1	0	0	0	i	-i	-1+i	-1-i
χ^5	7	3	-2	1	-1+2i	-1-2i	1	0	0	0	-i	i	-1-i	-1+i
χ^6	14	-2	5	-1	2	2	2	1	0	0	0	0	-1	-1
χ^7	21	5	3	0	1	1	1	-1	0	0	-1	-1	1	1
χ^8	21	1	3	0	-3+2i	-3-2i	-1	1	0	0	i	-i	-i	i
χ^9	21	1	3	0	-3-2i	-3+2i	-1	1	0	0	-i	i	i	-i
χ^{10}	27	3	0	0	3	3	-1	0	-1	-1	1	1	0	0
χ^{11}	28	-4	1	1	4i	-4i	0	-1	0	0	0	0	i	-i
χ^{12}	28	-4	1	1	-4i	4i	0	-1	0	0	0	0	-i	i
χ^{13}	32	0	-4	-1	0	0	0	0	η	$\overline{\eta}$	0	0	0	0
χ^{14}	32	0	-4	-1	0	0	0	0	$\overline{\eta}$	η	0	0	0	0

Table A3. The Character Table of The Group 3_+^{1+2}:8:2 of order 432

ORDERS	1	4	4	2	12	3	12	12	6	6	2	8	8	3
# ELEMENTS	1	36	18	9	36	24	36	36	18	72	36	54	54	2
INDEX	432	12	24	48	12	18	12	12	24	6	12	8	8	216
χ^1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ^2	1	-1	1	1	-1	1	1	-1	1	-1	-1	1	1	1
χ^3	1	1	1	1	1	1	1	1	1	-1	-1	-1	-1	1
χ^4	1	-1	1	1	-1	1	1	-1	1	1	1	-1	-1	1
χ^5	2	0	-2	2	0	2	-2	0	2	0	0	0	0	2
χ^6	2	0	0	-2	0	2	0	0	-2	0	0	$i\sqrt{2}$	$-i\sqrt{2}$	2
χ^7	2	0	0	-2	0	2	0	0	-2	0	0	$-i\sqrt{2}$	$i\sqrt{2}$	2
χ^8	6	2	2	-2	-1	0	-1	-1	1	0	0	0	0	-3
χ^9	6	-2	2	-2	1	0	-1	1	1	0	0	0	0	-3
χ^{10}	6	0	-2	-2	$i\sqrt{3}$	0	1	$-i\sqrt{3}$	1	0	0	0	0	-3
χ^{11}	6	0	-2	-2	$-i\sqrt{3}$	0	1	$i\sqrt{3}$	1	0	0	0	0	-3
χ^{12}	8	0	0	0	0	-1	0	0	0	1	-2	0	0	8
χ^{13}	8	0	0	0	0	-1	0	0	0	-1	2	0	0	8
χ^{14}	12	0	0	4	0	0	0	0	-2	0	0	0	0	-6

Table A4. The Character Table of the Groups $4{\cdot}S_4{:}2$ and $4{:}S_4{:}2$ of order 192.

ORDERS[(1)]	1	4	4	2	8	4	2	3	2	2	4	2	8	6
ORDERS[(2)]	1	2	2	2	4	2	4	3	2	4	2	4	4	6
#ELEMENTS	1	12	6	3	24	6	12	32	4	12	12	12	24	32
INDEX	192	16	32	64	8	32	16	6	48	16	16	16	8	6
χ^1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ^2	1	1	1	1	1	1	1	1	-1	-1	-1	-1	-1	-1
χ^3	1	-1	1	1	-1	1	-1	1	-1	1	1	-1	1	-1
χ^4	1	-1	1	1	-1	1	-1	1	1	-1	-1	1	-1	1
χ^5	2	0	2	2	0	2	0	-1	-2	0	0	-2	0	1
χ^6	2	0	2	2	0	2	0	-1	2	0	0	2	0	-1
χ^7	3	-1	-1	3	1	-1	-1	0	3	-1	-1	-1	1	0
χ^8	3	1	-1	3	-1	-1	1	0	3	1	1	-1	-1	0
χ^9	3	1	-1	3	-1	-1	1	0	-3	-1	-1	1	1	0
χ^{10}	3	-1	-1	3	1	-1	-1	0	-3	1	1	1	-1	0
χ^{11}	6	0	-2	-2	0	2	0	0	0	-2	2	0	0	0
χ^{12}	6	0	-2	-2	0	2	0	0	0	2	-2	0	0	0
χ^{13}	6	2	2	-2	0	-2	-2	0	0	0	0	0	0	0
χ^{14}	6	-2	2	-2	0	-2	2	0	0	0	0	0	0	0

[(1)]Orders of the Group $4{\cdot}S_4{:}2$
[(2)]Orders of the Group $4{:}S_4{:}2$

Table A5. The Character Table of the Group $4^2{:}S_3{:}2$ of Order 192

ORDERS	1	4	4	2	4	8	4	2	2	4	8	2	12	6	12	12	3
# ELEMENTS	1	4	2	1	12	24	12	6	12	6	24	24	16	8	16	16	8
INDEX	192	48	96	192	16	8	16	32	16	32	8	8	12	24	12	12	24
χ^1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ^2	1	-1	1	1	-1	1	-1	1	-1	1	-1	1	1	1	-1	-1	1
χ^3	1	-1	1	1	1	-1	1	1	-1	1	1	-1	1	1	-1	-1	1
χ^4	1	1	1	1	-1	-1	-1	1	1	1	-1	-1	1	1	1	1	1
χ^5	2	-2	2	2	0	0	0	2	-2	2	0	0	-1	-1	1	1	-1
χ^6	2	2	2	2	0	0	0	2	2	2	0	0	-1	-1	-1	-1	-1
χ^7	2	0	-2	2	0	0	0	-2	0	2	0	0	-2	2	0	0	2
χ^8	2	0	-2	2	0	0	0	-2	0	2	0	0	1	-1	$i\sqrt{3}$	$-i\sqrt{3}$	-1
χ^9	2	0	-2	2	0	0	0	-2	0	2	0	0	1	-1	$-i\sqrt{3}$	$i\sqrt{3}$	-1
χ^{10}	3	3	3	3	-1	-1	-1	-1	-1	-1	1	1	0	0	0	0	0
χ^{11}	3	3	3	3	1	1	1	-1	-1	-1	-1	-1	0	0	0	0	0
χ^{12}	3	-3	3	3	-1	1	-1	-1	1	-1	1	-1	0	0	0	0	0
χ^{13}	3	-3	3	3	1	-1	1	-1	1	-1	-1	1	0	0	0	0	0
χ^{14}	4	0	0	-4	2	0	-2	0	0	0	0	0	0	2	0	0	-2
χ^{15}	4	0	0	-4	-2	0	2	0	0	0	0	0	0	2	0	0	-2
χ^{16}	6	0	-6	6	0	0	0	2	0	-2	0	0	0	0	0	0	0
χ^{17}	8	0	0	-8	0	0	0	0	0	0	0	0	0	-2	0	0	2

Table A6. The Character Table of the Group PSL_2:2 of Order 336

ORDERS	1	3	2	8	6	7	8	4	2
#ELEMENTS	1	56	28	42	56	48	42	42	21
INDEX	336	6	12	8	6	7	8	8	16
χ^1	1	1	1	1	1	1	1	1	1
χ^2	1	1	-1	-1	-1	1	-1	1	1
χ^3	6	0	0	0	0	-1	0	2	-2
χ^4	6	0	0	$i\sqrt{2}$	0	-1	$-i\sqrt{2}$	0	2
χ^5	6	0	0	$-i\sqrt{2}$	0	-1	$i\sqrt{2}$	0	2
χ^6	7	1	1	-1	1	0	-1	-1	-1
χ^7	7	1	-1	1	-1	0	1	-1	-1
χ^8	8	-1	2	0	-1	1	0	0	0
χ^9	8	-1	-2	0	1	1	0	0	0

Table A7. The Character Table of The Group 3_+^{1+2}:8 of Order 216. Here $\omega=\frac{1}{\sqrt{2}}(1+i)$.

ORDERS	1	4	4	2	3	12	6	12	8	8	8	8	3
#ELEMENTS	1	9	9	9	24	18	18	18	27	27	27	27	2
INDEX	216	24	24	24	9	12	12	12	8	8	8	8	108
χ^1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ^2	1	1	1	1	1	1	1	1	-1	-1	-1	-1	1
χ^3	1	-i	i	-1	1	i	-1	-i	$-\omega$	ϖ	$-\varpi$	ω	1
χ^4	1	-i	i	-1	1	i	-1	-i	ω	$-\varpi$	ϖ	$-\omega$	1
χ^5	1	i	-i	-1	1	-i	-1	i	ϖ	$-\omega$	ω	$-\varpi$	1
χ^6	1	i	-i	-1	1	-i	-1	i	$-\varpi$	ω	$-\omega$	ϖ	1
χ^7	1	-1	-1	1	1	-1	1	-1	-i	i	i	-i	1
χ^8	1	-1	-1	1	1	-1	1	-1	i	-i	-i	i	1
χ^9	6	-2	-2	-2	0	1	1	1	0	0	0	0	-3
χ^{10}	6	2	2	-2	0	-1	1	-1	0	0	0	0	-3
χ^{11}	6	2i	-2i	2	0	i	-1	-i	0	0	0	0	-3
χ^{12}	6	-2i	2i	2	0	-i	-1	i	0	0	0	0	-3
χ^{13}	8	0	0	0	-1	0	0	0	0	0	0	0	8

Table A8. The Character Table of the Group $4{\cdot}S_4$ of Order 96

ORDERS	1	4	4	2	4	3	2	4	8	8
# ELEMENTS	1	3	3	3	6	32	12	12	12	12
INDEX	96	32	32	32	16	3	8	8	8	8
χ_1	1	1	1	1	1	1	1	1	1	1
χ_2	1	1	1	1	1	1	-1	-1	-1	-1
χ_3	2	2	2	2	2	-1	0	0	0	0
χ_4	3	-1	-1	3	-1	0	-1	-1	1	1
χ_5	3	-1	-1	3	-1	0	1	1	-1	-1
χ_6	3	-1+2i	-1-2i	-1	1	0	-1	1	-i	i
χ_7	3	-1+2i	-1-2i	-1	1	0	1	-1	i	-i
χ_8	3	-1-2i	-1+2i	-1	1	0	-1	1	i	-i
χ_9	3	-1-2i	-1+2i	-1	1	0	1	-1	-i	i
χ_{10}	6	2	2	-2	-2	0	0	0	0	0

Table A9. The Character Table of The Group $4^2{:}S_3$

ORDERS	1	4	4	2	4	4	4	4	2	4	8	8	12	6	3	12
# ELEMENTS	1	1	1	1	6	6	6	6	6	6	12	12	8	8	8	8
INDEX	96	96	96	96	16	16	16	16	16	16	8	8	12	12	12	12
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	1	1	1	-1	-1	-1	-1	1	1	-1	-1	1	1	1	1
χ_3	1	-1	-1	1	i	-i	-i	i	-1	1	-i	i	-1	1	1	-1
χ_4	1	-1	-1	1	-i	i	i	-i	-1	1	i	-i	-1	1	1	-1
χ_5	2	2	2	2	0	0	0	0	2	2	0	0	-1	-1	-1	-1
χ_6	2	-2	-2	2	0	0	0	0	-2	2	0	0	1	-1	-1	1
χ_7	2	2i	-2i	-2	-1-i	1-i	-1+i	1+i	0	0	0	0	i	1	-1	-i
χ_8	2	2i	-2i	-2	1+i	-1+i	1-i	-1-i	0	0	0	0	i	1	-1	-i
χ_9	2	-2i	2i	-2	-1+i	1+i	-1-i	1-i	0	0	0	0	-i	1	-1	i
χ_{10}	2	-2i	2i	-2	1-i	-1-i	1+i	-1+i	0	0	0	0	-i	1	-1	i
χ_{11}	3	3	3	3	1	1	1	1	-1	-1	-1	-1	0	0	0	0
χ_{12}	3	3	3	3	-1	-1	-1	-1	-1	-1	1	1	0	0	0	0
χ_{13}	3	-3	-3	3	-i	i	i	-i	1	-1	-i	i	0	0	0	0
χ_{14}	3	-3	-3	3	i	-i	-i	i	1	-1	i	-i	0	0	0	0
χ_{15}	4	-4i	4i	-4	0	0	0	0	0	0	0	0	i	-1	1	-i
χ_{16}	4	4i	-4i	-4	0	0	0	0	0	0	0	0	-i	-1	1	i

Table A10. The character Table of The Group $PSL_2(7)$ of Order 168

ORDERS	1	3	7	7	4	2
# ELEMENTS	1	56	24	24	42	21
INDEX	168	3	7	7	4	8
χ^1	1	1	1	1	1	1
χ^2	3	0	η	$\overline{\eta}$	1	-1
χ^3	3	0	$\overline{\eta}$	η	1	-1
χ^4	6	0	-1	-1	0	2
χ^5	7	1	0	0	-1	-1
χ^6	8	-1	1	1	0	0

Table A11. The Character Table of the Sylow Subgroup of $G_2'(2)$, of order 32*

ORDERS	1	4	4	2	2	4	8	8	4	4	4	4	4	2
# ELEMENTS	1	1	1	1	4	4	4	4	2	2	2	2	2	2
INDEX	32	32	32	32	8	8	8	8	16	16	16	16	16	16
$\chi 1$	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$\chi 2$	1	1	1	1	-1	-1	1	1	-1	-1	-1	-1	1	1
$\chi 3$	1	1	1	1	1	1	-1	-1	-1	-1	-1	-1	1	1
$\chi 4$	1	1	1	1	-1	-1	-1	-1	1	1	1	1	1	1
$\chi 5$	1	-1	-1	1	-1	1	i	-i	-i	i	i	-i	1	-1
$\chi 6$	1	-1	-1	1	1	-1	-i	i	-i	i	i	-i	1	-1
$\chi 7$	1	-1	-1	1	-1	1	-i	i	i	-i	-i	i	1	-1
$\chi 8$	1	-1	-1	1	1	-1	i	-i	i	-i	-i	i	1	-1
$\chi 9$	2	2	2	2	0	0	0	0	0	0	0	0	-2	-2
$\chi 10$	2	-2	-2	2	0	0	0	0	0	0	0	0	-2	2
$\chi 11$	2	2i	-2i	-2	0	0	0	0	1+i	1-i	-1+i	-1-i	0	0
$\chi 12$	2	2i	-2i	-2	0	0	0	0	-1-i	-1+i	1-i	1+i	0	0
$\chi 13$	2	-2i	2i	-2	0	0	0	0	1-i	1+i	-1-i	-1+i	0	0
$\chi 14$	2	-2i	2i	-2	0	0	0	0	-1+i	-1-i	1+i	1-i	0	0

*It is isomorphic to the Demazure-Tits subgroup of SO(5), DT(SO(5)

Table A12. The Character Table of the Sylow Subgroup of theGroup $G_2'(2)$, of Order 27. Here $\rho=\frac{1}{2}(-1+i\sqrt{3})$

ORDERS	1	3	3	3	3	3	3	3	3	3	3
# ELEMENTS	1	3	3	3	3	3	3	3	3	1	1
INDEX	27	9	9	9	9	9	9	9	9	27	27
χ^1	1	1	1	1	1	1	1	1	1	1	1
χ^2	1	1	1	$\overline{\rho}$	$\overline{\rho}$	$\overline{\rho}$	ρ	ρ	ρ	1	1
χ^3	1	1	1	ρ	ρ	ρ	$\overline{\rho}$	$\overline{\rho}$	$\overline{\rho}$	1	1
χ^4	1	$\overline{\rho}$	ρ	1	$\overline{\rho}$	ρ	$\overline{\rho}$	1	ρ	1	1
χ^5	1	$\overline{\rho}$	ρ	$\overline{\rho}$	ρ	1	1	ρ	$\overline{\rho}$	1	1
χ^6	1	$\overline{\rho}$	ρ	ρ	1	$\overline{\rho}$	ρ	$\overline{\rho}$	1	1	1
χ^7	1	ρ	$\overline{\rho}$	1	ρ	$\overline{\rho}$	ρ	1	$\overline{\rho}$	1	1
χ^8	1	ρ	$\overline{\rho}$	$\overline{\rho}$	1	ρ	$\overline{\rho}$	ρ	1	1	1
χ^9	1	ρ	$\overline{\rho}$	ρ	$\overline{\rho}$	1	1	$\overline{\rho}$	ρ	1	1
χ^{10}	3	0	0	0	0	0	0	0	0	3ρ	$3\overline{\rho}$
χ^{11}	3	0	0	0	0	0	0	0	0	$3\overline{\rho}$	3ρ

Table A13. The Character Table of the Sylow Subgroup of the Group $G_2'(7)$, of Order 7. Here $\tau = e^{\frac{i2\pi}{7}}$

ORDERS	1	7	7	7	7	7	7
# ELEMENTS	1	1	1	1	1	1	1
INDEX	7	7	7	7	7	7	7
χ^1	1	1	1	1	1	1	1
χ^2	1	$\overline{\tau}^2$	$\overline{\tau}$	τ^2	τ^3	τ	$-\tau^3$
χ^3	1	τ^3	$\overline{\tau}^2$	$-\tau^3$	$\overline{\tau}$	τ^2	τ
χ^4	1	τ	$-\tau^3$	$\overline{\tau}$	τ^2	τ^3	$\overline{\tau}^2$
χ^5	1	$\overline{\tau}$	τ^3	τ	$\overline{\tau}^2$	$-\tau^3$	τ^2
χ^6	1	$-\tau^3$	τ^2	τ^3	τ	$\overline{\tau}^2$	$\overline{\tau}$
χ^7	1	τ^2	τ	$\overline{\tau}^2$	$-\tau^3$	$\overline{\tau}$	τ^3

Table A14. The Character Table of the Groups $2^3{\cdot}PSL_2(7)$ and $2^3{:}PSL_2(7)$

ORDERS[(1)]	1	7	7	3	6	2	4	8	4	8	2
ORDERS[(2)]	1	7	7	3	6	2	2	4	2	4	4
# ELEMENTS	1	192	192	224	224	7	42	168	42	168	84
INDEX	1344	7	7	6	6	192	32	8	32	8	16
χ^1	1	1	1	1	1	1	1	1	1	1	1
χ^2	3	η	$\overline{\eta}$	0	0	3	-1	1	-1	1	-1
χ^3	3	$\overline{\eta}$	η	0	0	3	-1	1	-1	1	-1
χ^4	6	-1	-1	0	0	6	2	0	2	0	2
χ^5	7	0	0	1	-1	-1	-1	1	3	-1	-1
χ^6	7	0	0	1	1	7	-1	-1	-1	-1	-1
χ^7	7	0	0	1	-1	-1	3	-1	-1	1	-1
χ^8	8	1	1	-1	-1	8	0	0	0	0	0
χ^9	14	0	0	-1	1	-2	2	0	2	0	-2
χ^{10}	21	0	0	0	0	-3	-3	-1	1	1	1
χ^{11}	21	0	0	0	0	-3	1	1	-3	-1	1

[(1)]Orders of the $2^3{\cdot}PSL_2(7)$

[(2)]Orders of the $2^3{:}PSL_2(7)$

Table A15. The Character Table of the Group 2:3:7. Here $\mu=\frac{1}{2}(-1+i\sqrt{3})$

ORDERS	1	2	3	3	7	6	6	7
# ELEMENTS	1	7	28	28	24	28	28	24
INDEX	168	24	6	6	7	6	6	7
χ^1	1	1	1	1	1	1	1	1
χ^2	1	1	$\overline{\mu}$	μ	1	$\overline{\mu}$	μ	1
χ^3	1	1	μ	$\overline{\mu}$	1	μ	$\overline{\mu}$	1
χ^4	3	3	0	0	η	0	0	$\overline{\eta}$
χ^5	3	3	0	0	$\overline{\eta}$	0	0	η
χ^6	7	-1	1	1	0	-1	-1	0
χ^7	7	-1	μ	$\overline{\mu}$	0	$-\mu$	$-\overline{\mu}$	0
χ^8	7	-1	$\overline{\mu}$	μ	0	$-\overline{\mu}$	$-\mu$	0

Table A16. The Character Table of the Sylow Subgroup of the Groups $2^3{\cdot}PSL_2(7)$ and $2^3{:}PSL_2(7)$, of Order 64

ORDERS[1]	1	4	4	2	4	8	4	2	2	4	2	2	4	2	2	8
ORDERS[2]	1	4	2	2	4	4	2	2	2	2	4	2	2	2	4	4
# ELEMENTS	1	4	2	1	4	8	4	2	4	2	4	4	4	4	8	8
INDEX	64	16	32	64	16	8	16	32	16	32	16	16	16	16	8	8
χ^1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ^2	1	-1	1	1	-1	1	-1	1	-1	1	-1	1	1	-1	1	-1
χ^3	1	-1	1	1	1	-1	1	1	-1	1	-1	1	1	-1	-1	1
χ^4	1	-1	1	1	-1	1	-1	1	-1	1	1	-1	-1	1	-1	1
χ^5	1	-1	1	1	1	-1	1	1	-1	1	1	-1	-1	1	1	-1
χ^6	1	1	1	1	-1	-1	-1	1	1	1	-1	-1	-1	-1	1	1
χ^7	1	1	1	1	1	1	1	1	1	1	-1	-1	-1	-1	-1	-1
χ^8	1	1	1	1	-1	-1	-1	1	1	1	1	1	1	1	-1	-1
χ^9	2	0	-2	2	0	0	0	-2	0	2	0	-2	2	0	0	0
χ^{10}	2	0	-2	2	0	0	0	-2	0	2	0	2	-2	0	0	0
χ^{11}	2	2	2	2	0	0	0	-2	-2	-2	0	0	0	0	0	0
χ^{12}	2	-2	2	2	0	0	0	-2	2	-2	0	0	0	0	0	0
χ^{13}	2	0	-2	2	0	0	0	2	0	-2	-2	0	0	2	0	0
χ^{14}	2	0	-2	2	0	0	0	2	0	-2	2	0	0	-2	0	0
χ^{15}	4	0	0	-4	-2	0	2	0	0	0	0	0	0	0	0	0
χ^{16}	4	0	0	-4	2	0	-2	0	0	0	0	0	0	0	0	0

[1] Orders of the Sylow Subgroup of $2^3{\cdot}PSL_2(7)$
[2] Orders of the Sylow Subgroup of $2^3{:}PSL_2(7)$

Table A17. The Character Table of the Group 7:3 of Order 21

ORDERS	1	3	3	7	7
# ELEMENTS	1	7	7	3	3
INDEX	21	3	3	7	7
χ^1	1	1	1	1	1
χ^2	1	μ	$\overline{\mu}$	1	1
χ^3	1	$\overline{\mu}$	μ	1	1
χ^4	3	0	0	η	$\overline{\eta}$
χ^5	3	0	0	$\overline{\eta}$	η

Table A18. The Character Table of the Groups $2^3{\cdot}S_4$ and $2^3{:}S_4$, of order 192

ORDERS(1)	1	4	2	3	2	4	4	4	8	6	8	2	2
ORDERS(2)	1	2	2	3	4	2	2	2	4	6	4	4	2
# ELEMENTS	1	12	6	32	24	6	12	6	24	32	24	12	1
INDEX	192	16	32	6	8	32	16	32	8	6	8	16	192
χ^1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ^2	1	-1	1	1	-1	1	-1	1	-1	1	-1	1	1
χ^3	2	0	2	-1	0	2	0	2	0	-1	0	2	2
χ^4	3	1	-1	0	-1	-1	1	3	-1	0	1	-1	3
χ^5	3	-1	-1	0	1	-1	-1	3	1	0	-1	-1	3
χ^6	3	1	3	0	1	-1	1	-1	-1	0	-1	-1	3
χ^7	3	1	-1	0	-1	3	1	-1	1	0	-1	-1	3
χ^8	3	-1	-1	0	1	3	-1	-1	-1	0	1	-1	3
χ^9	3	-1	3	0	-1	-1	-1	-1	1	0	1	-1	3
χ^{10}	4	-2	0	1	0	0	2	0	0	-1	0	0	-4
χ^{11}	4	2	0	1	0	0	-2	0	0	-1	0	0	-4
χ^{12}	6	0	-2	0	0	-2	0	-2	0	0	0	2	6
χ^{13}	8	0	0	-1	0	0	0	0	0	1	0	0	-8

(1) Orders of the group $2^3{\cdot}S_4$
(2) Orders of the Group $2^3{:}S_4$

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APPENDIX B

To handle some of the problems of groups such as the generation of a group, determination of its maximal subgroups, calculation of conjugacy classes and the character tables, some computer programs have been written in FORTRAN, in addition to GAP (Groups, Algorithms and Programming). The FORTRAN programs are supported by MATHEMATICA, especially during the calculation of character tables. Here we will give some flowcharts for the prepared programs.

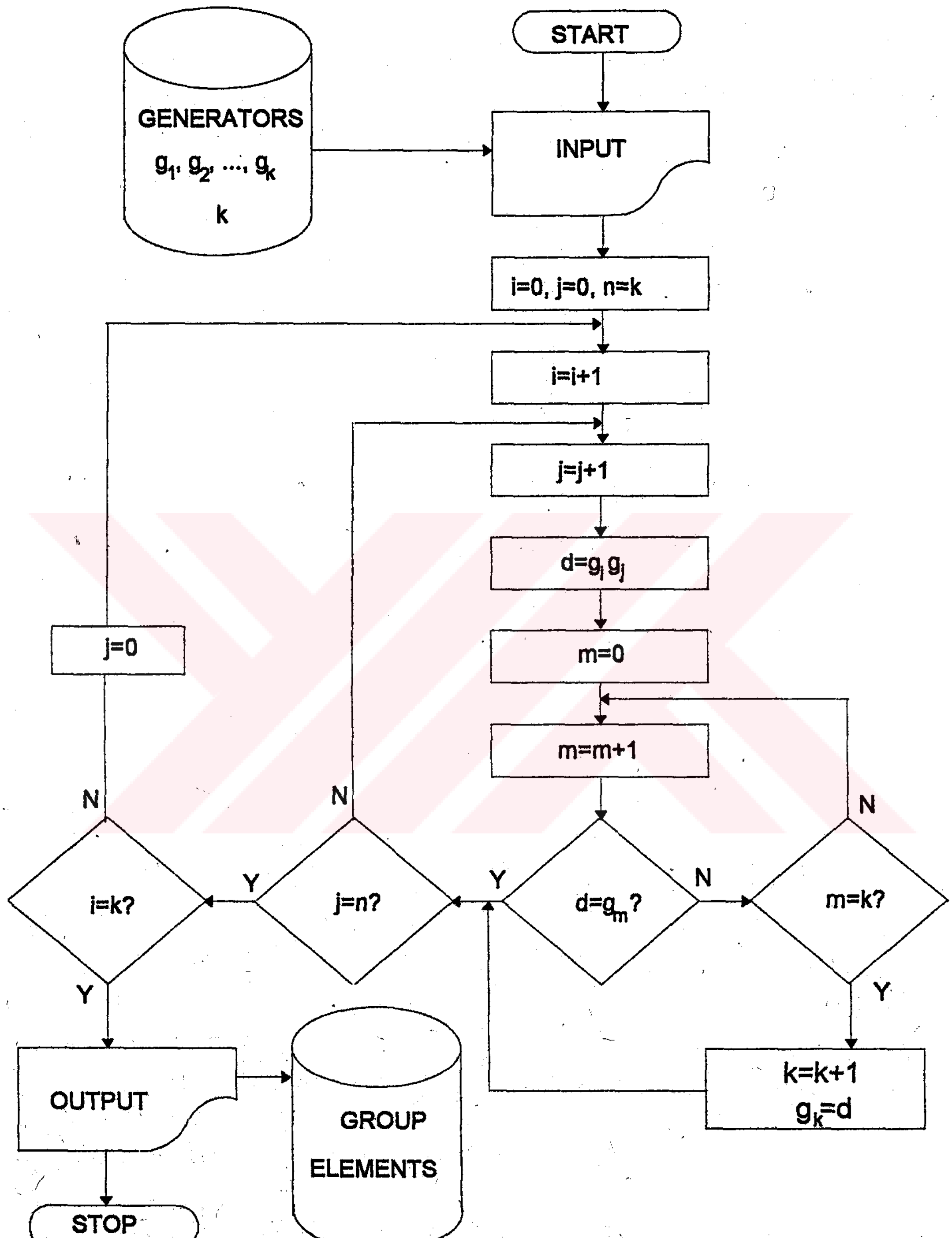

Figure B1. Flow-Chart of the FORTRAN Program that Generates a Matrix Group

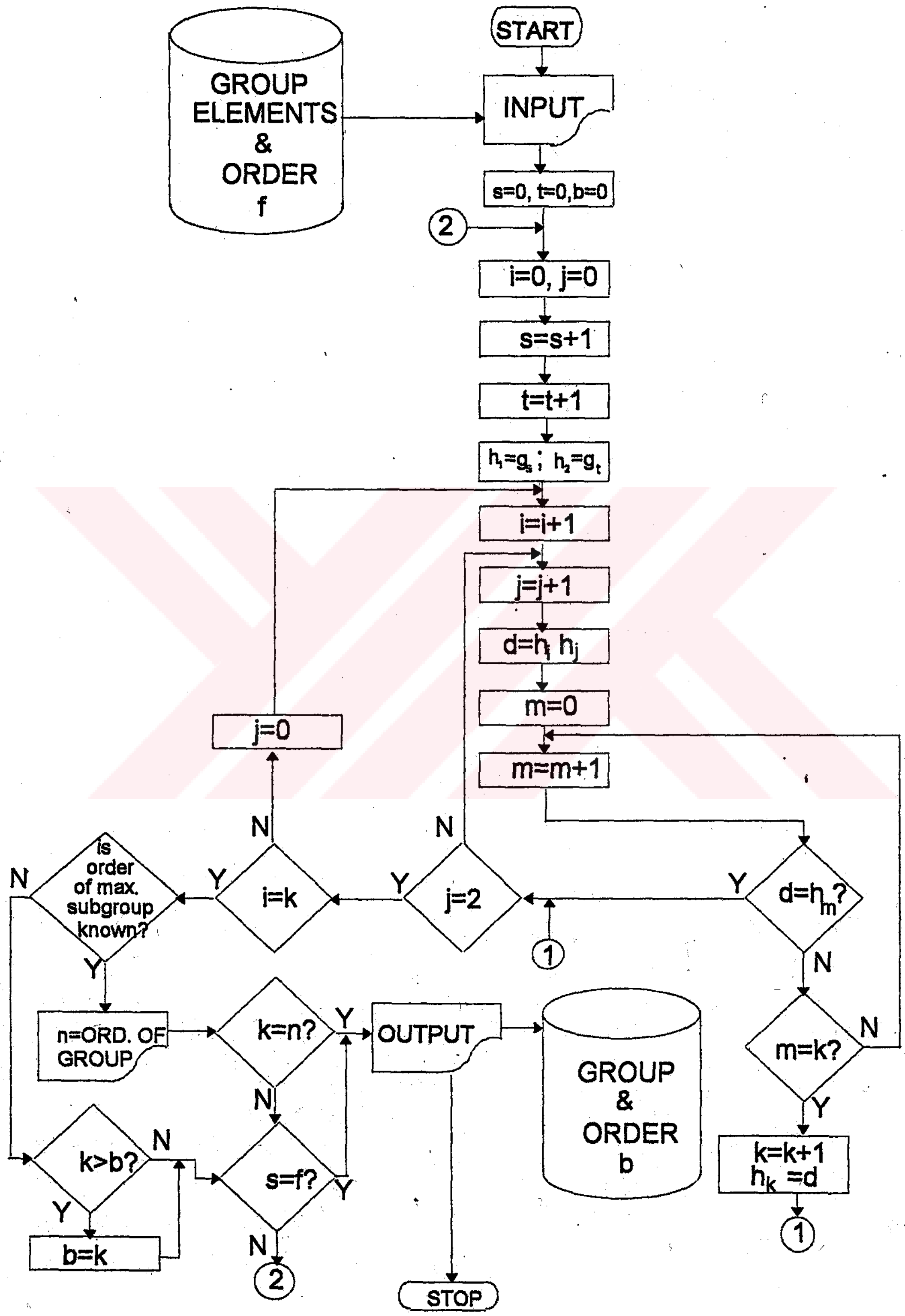

Figure B2. Flow-Chart of the FORTRAN Program to Find Maximal Subgroups of a given Group G.

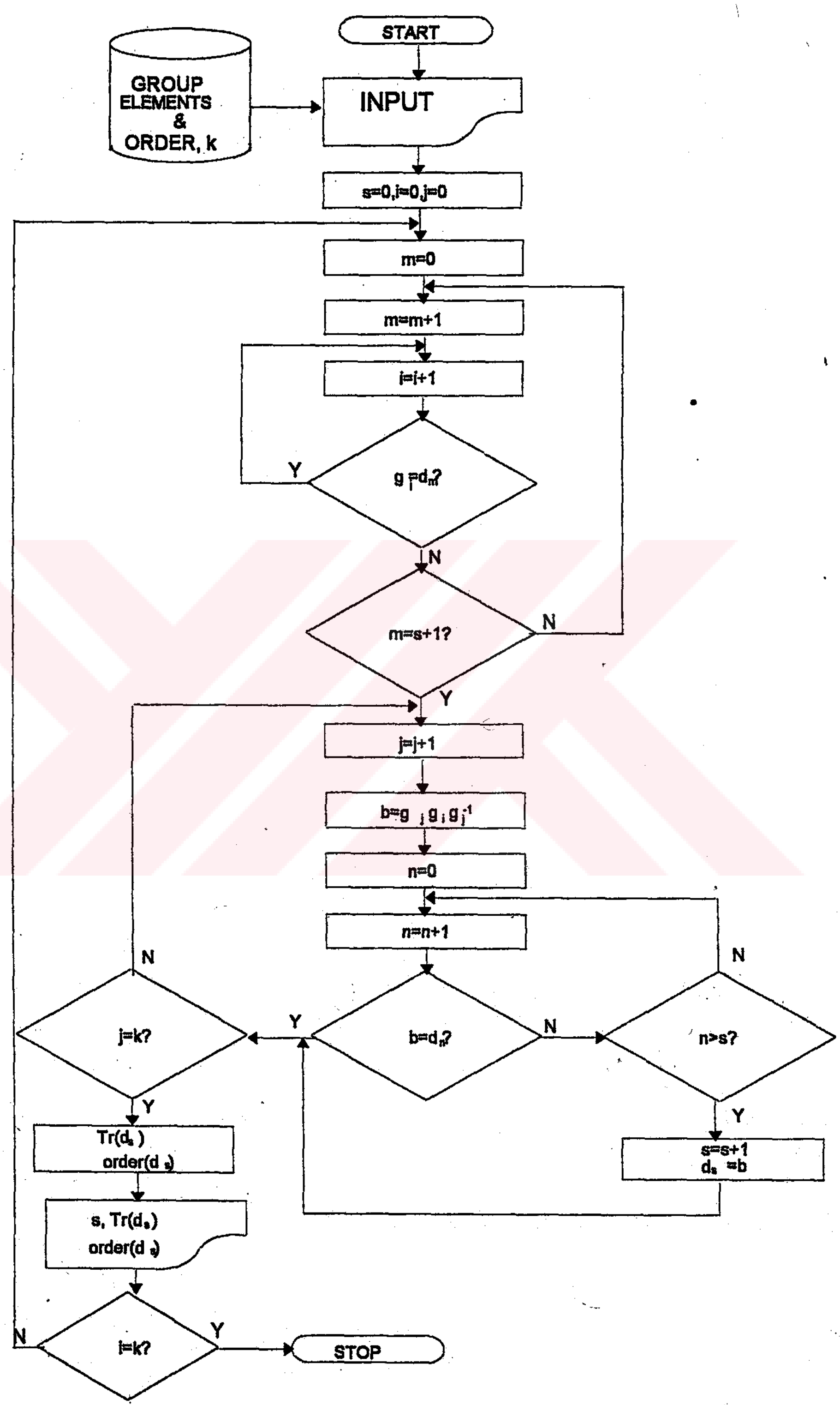

Figure B3. Flow-Chart of the FORTRAN Program to Calculate the Conjugacy Classes Of a Group G.

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