ISTANBUL TECHNICAL UNIVERSITY ★ GRADUATE SCHOOL OF SCIENCE ENGINEERING AND TECHNOLOGY

ON MAHARAM OPERATORS

M.Sc. THESIS
Zeynep ERCAN

Department of Mathematics

Mathematics Engineering Programme

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Zeynep Ercan, a **M.Sc.** student of ITU **Graduate School of Science Engineering and Technology,** 509111079, successfully defended the **thesis** entitled "**ON MAHARAM OPERATORS",** which she prepared fulfilling requirements specified in the associated legislations, before the jury whose signatures are below.

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FOREWORD

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LIST OF SYMBOLS

x^+ : the positive part of vector x

x^- : the negative part of vector x

$|x|$: the absolute value of vector x

$\hat{x}$: the image of x in its double dual

$\|x\|$: the norm of vector x

$x_\alpha \uparrow x$: the increasing net to x

$x_\alpha \downarrow x$: the decreasing net to x

$x_\alpha \to x$: order convergent to x

$x \vee y$: the supremum of x and y

$x \wedge y$: the infumum of x and y

$x \perp y$: the disjoint vectors

$L \oplus M$: the direct sum of two Riesz spaces

$L \otimes M$: the vector space of finite rank operators

L^+ : the positive cone of L

$L^\sim$: the order dual of L

$L^{\sim\sim}$: the second order dual of L

$L_n^\sim$: the order continuous dual of L

$L_c^\sim$: the σ- order continuous dual of L

$L^\wedge$: the Dedekind completion of L

L^u : the universal completion of L

B_x : the band generated by x

P_x : the projection onto the band B_x

P_B : the projection onto the band B

$|T|$: the modulus of operator T

T_A : the smallest extension of operator T

T_c : the σ- order continuous component of operator T

T_n : the order continuous component of operator T

T_s : the singular component of operator T

T' : the adjoint of operator T

$\mathfrak{C}_T$: the Dedekind complete Boolean algebra of all components of T

A^d : the disjoint complement of set A

A^{dd} : the band generated by set A

$f \otimes u$: rank one operator

$Orth(L)$: orthomorphisms on L

$\varepsilon(T)$: the set of positive extensions of positive operator T

$\mathcal{L}(L, M)$: the vector space of operators from L to M

$\mathcal{L}_b(L, M)$: the order bounded operators from L to M

$\mathcal{L}_c(L, M)$: the σ- order continuous operators from L to M

$\mathcal{L}_n(L, M)$: the order continuous operators from L to M

$\mathcal{L}_r(L, M)$: the regular operators from L to M

$\mathcal{L}_s(L, M)$: the singular operators from L to M

$\mathcal{N}_T$: the null ideal of operator T

N_T : the band projection onto $\mathcal{N}_T$

$\mathcal{C}_T$: the carrier of operator T

C_T : the band projection onto $\mathcal{C}_T$

σ_T : the order continuous Boolean ring homomorphism

$\mathcal{B}(L)$: the complete Boolean algebra of bands in L

$Z(E)$: the collection of all center operators

$\mathcal{L}_n^h(L,M)$: the order continuous operators which satisfies $\Pi T = Th(\Pi)\ \forall \Pi \in Z(M)$

$\mathbb{R}$: the set of real numbers

ON MAHARAM OPERATORS

SUMMARY

In this thesis, it has been worked on positive operators and Maharam operators which is a special kind of positive operators.

In the first section after the introduction, the circumstances in which a Riesz space is Archimedean or Dedekind complete have been given and their results have been discussed. The important fact that an additive mapping between the positive cones of two Riesz spaces has a unique extension to a positive operator between these Riesz spaces has been stated. The definition of order bounded operators has been given and the modulus of an order bounded operator and the infimum and supremum of two order bounded operators have been examined. The extension of positive operators has been discussed in a detailed fashion. The concept of ideal, band and band projection of a Riesz space has been examined. The relationship between order continuous and order bounded operators has been given and related to the band projections. The concept of the null ideal and the carrier of an operator has been discussed. The linear functional, order bounded functinal and order continuous functional has been introduced. The definition of a component of a positive operator has been given. Freudenthal's Spectral Theorem has been proved. The lattice homomorphisms and orthomorphisms have been introduced and the properties of the space of lattice orthomorphisms has been examined in a detailed way. Consequently, f-algebras have been introduced and their properties have been discussed. In the last part of the first section, as an additional information, the concept of the Banach lattices has been given.

In the beginning of the second section, the definition of Maharam operators, the center operators and the center of a Banach lattice has been given. The properties of the Boolean ring homomorphism between the Boolean algebra of the corresponding band projections induced by the Boolean ring homomorphism between the Boolean algebras of bands in two Dedekind complete Riesz spaces have been examined. The properties of f-algebra homomorphisms between the center of Riesz spaces have been discussed in a detailed fashion. The lattice structure of $\mathcal{L}_n^h(L,M)$, the collection of order continuous operators which satisfies $\Pi T = Th(\Pi)\ \forall \Pi \in Z(M)$, has been constructed and proved to be a band in $\mathcal{L}_n(L,M)$. All operators in the band generated by a Maharam operator are proved to be Maharam. The properties of null ideal and the carrier of an operator in $\mathcal{L}_n^h(L,M)$ and the relationship between them have been examined in a detailed fashion.

In the third section, first of all, the concept of f-module, f-submodule and $A-linear$ operator has been introduced. $\mathcal{L}_b^A(E,F)$ and $\mathcal{L}_n^A(E,F)$ have been defined and proved to be a band in $\mathcal{L}_b(E,F)$ and $\mathcal{L}_n(E,F)$, respectively. Some examples of f-modules has been given. In this examples, $\mathcal{L}_n^{Z(M)}(E,F)=\mathcal{L}_n^h(E,F)$, where h is an f-algebra homomorphism between the centers of two Archimedean Riesz spaces where the latter is also Dedekind complete, is defined and its properties have been discussed in a detailed fashion. It is also proved to be a band in $\mathcal{L}_n(L,M)$ and f-module structure over $Z(M)$. The concept of an ideal of $\mathcal{L}_n^{Z(M)}(\mathcal{J},M)$ and it's being an f-submodule over $Z(M)$ have been examined. The circumstances in which the range of an Archimedean Riesz space under an order continuous $Z(M)-linear$ Riesz homomorphism is an ideal of $\mathcal{L}_n^{Z(M)}(\mathcal{J},F)$ or $\mathcal{L}_n^{Z(M)}(\mathcal{J},F)$ itself have been shown.

In the fourth section after introduction, $Z(M)\otimes_{\mathcal{J}}L$, the linear subspace of $\mathcal{L}_n^{Z(M)}(\mathcal{J},M)$ generated by the operators of the form $\Pi\otimes f$ where L is an Archimedean Riesz space and $\mathcal{J}$ is an ideal of $\mathcal{L}_b(L,M)$, and $Z(M)\widetilde{\otimes}_{\mathcal{J}}L$, the ideal generated by $Z(M)\otimes_{\mathcal{J}}L$ in $\mathcal{L}_n^{Z(M)}(\mathcal{J},M)$ and a mapping $\chi_0\colon L\to\mathcal{L}_n^{Z(M)}(\mathcal{J},M)$ given by $\chi_0 f=I\otimes f$ has been defined and the properties of this operator has been discussed in a detailed fashion. The definition of a Maharam extension space (F,χ_F), where F is a Dedekind complete f-module over $Z(M)$ and $\chi_F\colon L\to F$ is a Riesz homomorphism, for an ideal of $\mathcal{L}_n^{Z(M)}(\mathcal{J},M)$, has been given and the circumstances in which a Maharam extension is minimal or maximal has been examined.

In the last section, to extend the subject, two examples have been given. In the first example, the Riesz spaces have been taken as finite dimensional real spaces. In the second example, the Maharam extensions of kernel operators have been discussed.

MAHARAM OPERATÖRLER

ÖZET

Bu tezde, pozitif operatörler ve pozitif operatörlerin özel bir çeşidi olan Maharam operatörler üzerine çalışılmıştır. Temelde bir pozitif operatör, iki sıralı uzay arasında pozitif elemanları yine pozitif elemanlara taşıyan bir lineer operatördür. Benzer şekilde, eğer bir pozitif operatör sıralı sınırlı kümeleri yine sıralı sınırlı kümelere götürüyorsa sıralı sınırlı olarak adlandırılır. Bu argüman devam ettirilerek, Maharam operatörler, modülünün sıralı aralıkları yine sıralı aralıklara taşıdığı, yani sıralı aralık koruma özelliği (interval preserving) bulunduğu bir sıralı sınırlı operatör olarak adlandırılabilir.

Girişten sonraki ilk bölümde, Riesz uzaylarının hangi şartlarda Arşimed veya Dedekind complete olacağı verilmiş ve sonuçları tartışılmıştır. İki Riesz uzayının positif konileri arasında tanımlanan bir toplamsal fonksiyonun bu Riesz uzayları arasındaki bir pozitif operatöre yalnızca bir genişlemesinin bulunacağı ispatlanmıştır. Bu genişleme her eleman için elemanın pozitif ve negatif kısımlarının görüntülerinin farkıdır. Bu şekilde, her pozitif operatörün tanım kümesinin pozitif konisi üzerindeki davranışıyla belirleneceği sonucuna varılmıştır. Riesz uzayının bir alt kümesinin hangi şartlarda sıralı sınırlı olacağı açıklanmış, sıralı sınırlı ve düzenli operatör tanımları verilmiştir. Dedekind complete Riesz uzaylarda bir operatörün modülü ve iki farklı operatörün supremumu ve infumumu tanımlanmış ve tanımın doğruluğu ispat edilmiştir. Sublineer fonksiyon tanımlanmış ve bir positif operatöre nasıl genişleyeceği gösterilmiştir. İdeal, sıralı yoğun Riesz altuzayı, sıralı yoğun ideal anlamına gelen band, band projeksiyonları, sıralı sürekli operatör kavramları verilmiş ve birbirleriyle ilişkileri incelenmiştir. Sıralı sürekli bir operatörün aynı zamanda sıralı sınırlı olacağı ve modülünün de sıralı sürekli olacağı gibi temel teoremler belirtilmiş ve ispatlanmıştır. Buradan sıralı sürekli operatörlerin uzayının sıralı sınırlı operatörlerin uzayının bir bandı olacağı sonucuna varılmıştır. Projeksiyon band kavramı verilmiş ve örneklendirilmiştir. Bir operatörün null ideali ve taşıyıcı uzayı tanımlanmıştır. Lineer fonksiyonel, sıralı sınırlı lineer fonksiyonel, sıralı sürekli fonksiyonel tanımları verilmiş ve sıralı sınırlı lineer fonksiyonellerinin vektör uzayının Riesz uzayındaki noktaları ayırmasının hangi anlama geleceği ifade edilmiştir. Lineer fonksiyoneller ve null idealleri ve taşıyıcıları arasındaki bağıntılar incelenmiştir. Bir Riesz uzayının duali, ikinci dereceden duali ve perfect Riesz uzayı kavramları detaylandırılmış ve her perfect Riesz uzayının aynı zamanda Dedekind complete olması gerektiği belirtilmiştir. Rank operatör kavramı verilmiş ve özellikleri incelenmiştir. Riesz uzaylarının arasındaki sıralı sınırlı bir operatörün adjoint operatörünün özellikleri incelenmiş ve ispatlanmıştır. Bir positif operatörün

bileşen operatörü tanımlanış ve detaylandırılmış ve sıralı projeksiyonlarla bağlantısı gösterilmiştir. x-step fonksiyonu incelenmiş ve buna bağlı olarak Freudenthal'in spektral teoremi verilmiş ve ispatlanmıştır. Latis homomorfizm ve izomorfizmi kavramları tanıtılmış ve incelenmiştir. İzomorfik iki Riesz uzayın eş kabul edeceği belirtilmiştir. Bir operatörün hangi şartlarda aralık koruyucu olacağı (interval preserving) ifade edilmiş ve aralık koruyucu bir operatörün görüntü kümesinin her zaman bir ideal olacağı belirtilmiştir. Bir Riez uzayının Dedekind completion kavramı incelenmiştir. Band koruyucu operatör (band preserving) tanıtılmış ve bu kavramın farklı şartlarda nasıl sonuçlar doğuracağı gösterilmiştir. Ortomorfizm kavramı tanıtılmış ve ortomorfizmlerin uzayının özellikleri incelenmiştir. Bir ortomorfizmin modülünün varlığı ve yine bir ortomorfizm olacağının yanı sıra iki ortomorfizmin supremumunun ve infimumunun da yine bir ortomorfizm olacağı ispatlanmıştır. Ortomorfizmlerin uzayının sıralı sınırlı operatörlerin uzayındaki birim operatörün ürettiği ideal ile çakışacağı gösterilmiştir. Arşimed Riesz uzayında bir ortomorfizmin bu uzayın Dedekind completion uzayında yalnızca bir genişlemesi olacağı ispatlanmıştır. Bir ortomorfizmin kernelinin bir band olacağı gösterilmiştir. f-cebiri kavramı verilmiş ve özellikleri incelenmiştir. Her Arşimed f-cebirinin değişmeli olacağına dikkat çekilmiştir. İlk bölümün sonunda Banach latis kavramı verilerek Maharam operatörler ve genişlemeleri kavramının anlaşılması için gerekli altyapı tamamlanmıştır.

İkinci bölümün başında Maharam operatörler, merkez operatörler ve bir Banach operatörünün merkezi tanıtılmıştır. İki Dedekind complete uzayının bandlarının Boolean cebirleri arasındaki sıralı sürekli Boolean halka homomorfizminin bunlara karşılık gelen band projeksiyonlarının Boolean cebirleri arasında bir sıralı sürekli Boolean homomorfizmi indirgeyeceği gösterilmiştir. Eğer sıralı sürekli Boolean homomorfizminin üzerine kurulduğu operatör bir Maharam operatör ise sonuçların nasıl gelişeceği incelenmiştir. İki Arşimed uzayının, ikincisi Dedekind complete olmak üzere, merkezleri arasında tanımlanan bir f-cebiri homomorfizminin h yardımıyla oluşturulan $\mathcal{L}_n^h(L,M)$, $\Pi T = Th(\Pi)\ \forall \Pi \in Z(M)$ şartını sağlayan sıralı sürekli operatörlerin uzayı, tanımlanmış ve $\mathcal{L}_n(L,M)$ uzayında bir band olduğu kanıtlanmıştır. Bir Maharam operatörün ürettiği banddaki her operatörün de Maharam olacağı gösterilmiştir. Bundan yararlanılarak $\mathcal{L}_n^h(L,M)$ deki disjoint operatörlerin birtakım özellikleri ispatlanmıştır. Örneğin $\mathcal{L}_n^h(L,M)$ de iki operator disjoint ise taşıyıcılarının da disjoint olacağı gösterilmiştir. Sonrasında bu operatörlerin disjoint olmaması durumunda taşıyıcıları arasında nasıl bir bağlantı kurulabileceğine çalışılmıştır. Buradaki ilk uzayın da Dedekind complete alınmasıyla sonuçların nasıl değişeceği gösterilmiştir. $\mathcal{L}_n^h(L,M)$ uzayının Maharam operatörlerden oluşan bir ideali ile f-cebiri homomorfizmleri arasındaki bağlantı incelenmiş ve son olarak Maharam operatörlerden oluşan bir ideal tarafından üretilen bir bandın da yine Maharam operatörlerden oluşacağı ispatlanarak bölüm bitirilmiştir.

Üçüncü bölüme f-modülü, f-altmodülü ve $A-lineer$ operatör kavramları incelenerek başlanmıştır. $\mathcal{L}_b^A(E,F)$ ve $\mathcal{L}_n^A(E,F)$ tanıtılmış ve sırasıyla $\mathcal{L}_b(E,F)$ ve $\mathcal{L}_n(E,F)$ uzaylarının bandı olduğu ispatlanmıştır. Sonrasında f-modüllerinin çeşitli

örnekleri incelenmiştir. Bu örneklerin ilkinde herhangi bir Arşimed Riesz uzayının, merkezinin üzerinde bir f-modülü olarak alınabileceği gösterilmiştir. İkinci örnekte ise $\mathcal{L}_n(E,F)$ uzayının $\mathcal{L}_b(E,F)$ uzayının bir f-altmodülü olduğu ifade edilmiştir. Üçüncü örnekte ise, h, ikincisi aynı zamanda Dedekind complete olarak alınırsa iki Arşimed uzayının merkezleri arasındaki f-cebiri homomorfizmi olmak üzere $\mathcal{L}_n^{Z(M)}(E,F)=\mathcal{L}_n^{h}(E,F)$ tanımlanmış ve özellikleri üçüncü ve dördüncü örneklerde detaylı olarak incelenmiştir. Bu uzayın aynı zamanda $\mathcal{L}_n(L,M)$ uzayının bir bandı ve $Z(M)$ üzerinde bir f-altmodülü olduğu ifade edilmiştir. $\mathcal{L}_n^{Z(M)}(E,M)$ bir idealinin aynı zamanda bir f-altmodülü olduğu sonucuna varılmış ve bu idealden alınmış herhangi bir operatör üzerinde tanımlanan $\mathcal{L}_n^{Z(M)}(E,M)$ uzayına ait bir fonksiyonun özellikleri incelenmiştir. Sıralı sınırlı $Z(M)-lineer$ Riesz homomorfizmi kavramı tanıtılmış ve bir Arşimed Riesz uzayının bir sıralı sürekli $Z(M)-lineer$ Riesz homomorfizmi altındaki görüntüsünün hangi şartlarda $\mathcal{L}_n^{Z(M)}(\mathcal{J},F)$ in bir idealine veya kendisine eşit olacağı gösterilerek ispatlanmıştır.

Girişten sonraki dördüncü bölüm Maharam genişlemelerinin kurulma sürecini anlatmaktadır. Bölüme $\mathbb{k}$ bir cisim olmak üzere, üzerinde kurulan birimsel A modül ve $\mathbb{k}$-lineer gömme kavramları hatırlatılarak başlanmıştır. L bir Arşimed uzayı, M bir Dedekind complete Riesz uzayı ve $\mathcal{J}$, $\mathcal{L}_b(L,M)$ nin bir ideali olmak üzere $Z(M)\otimes_{\mathcal{J}}L$, $\mathcal{L}_n^{Z(M)}(\mathcal{J},M)$ uzayının $\Pi\otimes f$ formundaki operatörler kullanılarak üretilmiş lineer altuzayı, ve $Z(M)\widetilde{\otimes}_{\mathcal{J}}L$, $Z(M)\otimes_{\mathcal{J}}L$ in ürettiği ideal, tanımları verilmiş $\chi_0: L\to\mathcal{L}_n^{Z(M)}(\mathcal{J},M)$ fonksiyonu $\chi_0 f=I\otimes f$ şeklinde tanımlanmıştır. $Z(M)\widetilde{\otimes}_{\mathcal{J}}L$ in $\chi_0(L)$ in $\mathcal{L}_n^{Z(M)}(\mathcal{J},M)$ uzayında ürettiği ideal ile çakıştığı belirtilmiş ve ispatlanmıştır ve bu idealin hangi şartlarda sıralı yoğun olacağı tartışılmıştır. (F,χ_F) ikilisi hangi özellikleri sağlarsa $\mathcal{L}_n^{Z(M)}(\mathcal{J},M)$ uzayında $\chi_0(L)\subseteq E$ şartını sağlayacak tek bir ideal olacağı ve $\gamma\circ\chi_F=\chi_0$ şartını sağlayan tek bir $Z(M)$-lineer Riesz izomorfizmi olacağı gösterilmiştir. Buna bağlı olarak bir ideal için Maharam genişleme uzayı tanımlanmıştır. (F,χ_F) Maharam genişleme uzayında F, $Z(M)$ üzerinde bir Dedekind complete f-modülü ve $\chi_F: L\to F$ bir Riesz homomorfizmidir, idealdeki her T operatörü için $T=\tilde{T}\circ\chi_F$ şartını sağlayan $\mathcal{L}_n^{Z(M)}(F,M)$ de tek bir $\tilde{T}$ operatörü vardır ve $\{\tilde{T}: 0\le T\in\mathcal{J}\}$ kümesi F uzayındaki noktaları ayırır. Bir Maharam genişlemesinin hangi şartlarda minimal veya maksimal olacağı incelenmiştir. Her Maharam genişleme uzayının $\mathcal{L}_n^{Z(M)}(\mathcal{J},M)$ da bir ideal olarak düşünülebileceği sonucuna varılmıştır. Ayrıca, her minimal Maharam genişleme uzayının $\mathcal{J}$ nin minimal Maharam genişleme uzayı olan $Z(M)\widetilde{\otimes}_{\mathcal{J}}L$ e izomorfiktir ve benzer olarak $\mathcal{J}$ nin maksimal Maharam genişleme uzayı izomorfizm yoluyla tek olarak belirlenmiştir ve $Z(M)\widehat{\otimes}_{\mathcal{J}}L$ ile gösterilmiştir. Eğer (E,χ), $\mathcal{J}$ nin bir Maharam genişleme uzayı ise χ in her zaman birebir olmasına gerek olmadığı belirtilmiş ve hangi şartlarda birebir olacağı ve bunun diğer hangi sonuçlara yol açacağı incelenmiştir. Yine χ homomorfizminin sıralı sürekli olma koşulları ispatlanmıştır. Bir idealin minimal Maharam genişleme uzayının aynı zamanda o idealin ürettiği bandın da minimal Maharam genişleme uzayı olacağı gösterilmiş ve detaylandırılmıştır.

Son bölümde, konunun detaylarının daha açık bir şekilde incelenebileceği iki örnek verilmiştir. İlk örnekte, Riesz uzayları m ve n boyutlu reel uzaylar, lineer operatörler n×m boyutlu matrisler ve $Z(M)$ n×n boyutlu diagonal matrisler olarak alınmış ve buna bağlı olarak $Z(M)$ üzerinde f-modülün nasıl şekil alacağı incelenmiştir. İkinci örnekte ise kernel operatörlerin Maharam genişlemeleri incelenmiştir.

Sonuç ve öneriler kısmında, bu tezde incelenen Maharam genişleme uzaylarının cebirsel yapısını yanı sıra topolojik ve Banach latis yapısının da incelenerek farklı sonuçlara varılabileceği belirtilmiştir.

1. INTRODUCTION

Operator theory is located in the intersection of the main subjects of mathematics such as Analysis, Algebra and Topology. Besides the beauty of working on the intersection of different subjects, operator theory is useful on some very striking areas such as it's being one of the most powerful tools in the study of Quantum Mechanics. The concept of positive operators has been worked on since the beginning of the nineteenth century but the importance of this subject has not been understood until the middle of the last century.

A positive operator is actually a special type of a linear operator between two ordered spaces that carries positive elements to positive elements. As a special case, a positive operator is called order bounded if it carries order bounded subsets to order bounded subsets. In the same fashion, a Maharam operator is defined as an order bounded operator whose modulus is interval preserving, that is, the image of an order interval remains in an order interval.

In this work, after the necessary information is given, in the Section 3, it is shown that for any order continuous Maharam operator T between Dedekind complete spaces L and M, there exists an algebra and lattice homomorphism h from $Z(M)$ into $Z(L)$ such that $\Pi T = Th(\Pi)$ for all $\Pi \in Z(M)$. The existence of this homomorphism h proves that Maharam operators are indeed $Z(M)$-linear operators with respect to an appropriate $Z(M)$-module structure on L induced by h. After establishing this result, in the Section 4, some of the properties of f-modules and in particular f-modules over $Z(M)$ for some Dedekind complete space M are given. In this section, for a $Z(M)$-module E, $\mathcal{L}_n^{Z(M)}(E, M)$ denotes the space of all normal $Z(M)$-linear operators from E into M. $\mathcal{L}_n^{Z(M)}(E, M)$ is proved to have many properties in common with the normal dual $L_n^{\sim}$ of a vector lattice L, one of which plays the crucial part in constructing of the Maharam extensions and that is the analogue of the well known Nakano perfectness criterion for vector lattices in the setting of $Z(M)$-modules. In section 5, it has been worked on to construct Maharam extension spaces for the

ideals of operators in $\mathcal{L}_b(L, M)$. The uniqueness of these extensions is discussed and the main properties of them are established.

2.BASİC PROPERTİES OF POSİTİVE OPERATORS

Definition 2.1: A Riesz space L is said to be Archimedean if $\frac{1}{n}x \downarrow 0$ in L for each $x \in L^+$.

Theorem 2.2: Let L and M be Riesz spaces with M Archimedean and let $T: L^+ \to M^+$ be an additive mapping, that is, $T(x+y) = T(x) + T(y)$ for all $x, y \in L^+$. Then, T has a unique extension to a positive operator from L to M.

Moreover, the extension which will be denoted by T again is given by

$$T(x) = T(x^+) - T(x^-) \tag{2.1}$$

for all $x \in L$.

Proof: Let $T: L^+ \to M^+$ be an additive mapping and let us define a mapping

$$S: L \to M \tag{2.2}$$

by

$$S(x) = T(x^+) - T(x^-) \tag{2.3}$$

If $x \in L^+$, $S(x) = T(x)$. Therefore, the mapping S extends T to all of L.

Since $x = x^+ - x^-$ for each $x \in L$, S becomes the only possible linear extension of T to all of L.

Hence, it is only left to show that S is linear. For this purpose, we need to show that S is additive and homogeneous.

Let us fix any $x \in L$ and assume that $x = x^+ - x^- = x_1 - x_2$ where $x_1, x_2 \in L^+$. Then, $x^+ + x_2 = x_1 + x^-$. Since T is additive on L^+, it can be written as

$$T(x^+) + T(x_2) = T(x^+ + x_2) = T(x_1 + x^-) = T(x_1) + T(x^-) \tag{2.4}$$

So,

$$S(x) = T(x^+) - T(x^-) = T(x_1) - T(x_2) \tag{2.5}$$

From this property, the additivity of S can be easily established. If $x, y \in L$, then

$$\begin{aligned} S(x+y) &= S\big(x^+ + y^+ - (x^- + y^-)\big) \\ &= T(x^+ + y^+) - T(x^- + y^-) \\ &= T(x^+) + T(y^+) - T(x^-) - T(y^-) \\ &= [T(x^+) - T(x^-)] + [T(y^+) - T(y^-)] \\ &= S(x) + S(y) \end{aligned} \tag{2.6}$$

In particular, the additivity of S implies that $S(\lambda x) = \lambda S(x)$ for all $x \in L$ and all rational numbers λ.

To show that S is homogeneous, first we need to show that S is monotone.

If $x \geq y$, then $x - y \in L^+$, and so by the additivity of S, we obtain that

$$S(x) = S\big((x-y)+y\big) = S(x-y) + S(y) = T(x-y) + S(y) \geq S(y) \tag{2.7}$$

So, if $x \geq y$ in L, then $S(x) \geq S(y)$ in M. It follows that S is monotone.

Now, let $x \in L^+$ and let $\lambda \geq 0$. If we pick two sequences of nonnegative rational numbers $\{r_n\}$ and $\{t_n\}$ such that $r_n \uparrow \lambda$ and $t_n \downarrow 0$, then the inequalities $r_n x \leq \lambda x \leq t_n x$ and the monotonicity of S imply

$$r_n S(x) = S(r_n x) \leq S(\lambda x) \leq S(t_n x) = t_n S(x) \tag{2.8}$$

for each n.

If $\lambda \in \mathbb{R}$ and $x \in L$, then,

$$\begin{aligned} S(\lambda x) &= S(\lambda x^+ + (-\lambda)x^-) = S(\lambda x^+) + S\big((-\lambda)x^-\big) \\ &= \lambda S(x^+) - \lambda S(x^-) = \lambda[T(x^+) - T(x^-)] = \lambda S(x) \end{aligned} \tag{2.9}$$

Therefore, S is homogeneous and the proof is finished.

According to this theorem, we conclude that a positive operator is determined completely by its action on the positive cone of its domain. (Aliprantis and Burkinshaw, 2006).

Definition 2.3: For an operator $T: L \to M$ between two Riesz spaces, its modulus $|T|$ exists if

$$|T| = T \vee (-T) \tag{2.10}$$

exists, that is, $|T|$ is the supremum of the set $\{-T, T\}$ in $\mathcal{L}(L, M)$.

Theorem 2.4 : Let $T: L \to M$ be an operator between two Riesz spaces such that $\sup\{|Ty|: |y| \leq x\}$ exists in M for each $x \in L^+$. Then, the modulus of T exists and

$$|T|(x) = \sup\{|Ty|: |y| \leq x\} \tag{2.11}$$

for all $x \in L^+$.

Proof: Let us define $S: L^+ \to M^+$ by

$$S(x) = \sup\{|Ty|: |y| \leq x\} \tag{2.12}$$

for each $x \in L^+$.

Since $|y| \leq x$ implies $|\pm y| = |y| \leq x$, it can be written as $S(x) = \sup\{Ty: |y| \leq x\}$ for each $x \in L^+$.

We need to show that S is additive. For this purpose, let $u, v \in E^+$. If $|y| \leq u$ and $|z| \leq v$, then $|y + z| \leq |y| + |z| \leq u + v$, so it follows from $T(y) + T(z) = T(y + z) \leq S(u + v)$ that $S(u) + S(v) \leq S(u + v)$. On the other hand, if $|y| \leq u + v$, then there exist y_1 and y_2 with $|y_1| \leq u$, $|y_2| \leq v$ and $y = y_1 + y_2$. Therefore $T(y) = T(y_1) + T(y_2) \leq S(u) + S(v)$ and it follows that $S(u + v) \leq S(u) + S(v)$. Hence, $S(u + v) = S(u) + S(v)$. Now, S can be extended to a positive operator from L to M.

By the definition of S, $T \leq S$ and $-T \leq S$ in $\mathcal{L}(L, M)$. If we assume that $\pm T \leq R$ in $\mathcal{L}(L, M)$, then for a fixed $x \in L^+$ and for all $y \in L$ such that $|y| \leq x$, it can be written as

$$Ty = Ty^+ - Ty^- \leq Ry^+ + Ry^- = R|y| \leq Rx \tag{2.13}$$

Therefore. $S(x) \leq R(x)$ for each $x \in L^+$ and so $S = T \vee (-T)$ in $\mathcal{L}(L, M)$. (Aliprantis and Burkinshaw, 2006).

Definition 2.5:

1) A subset A of a Riesz space is bounded above if there exists some x satisfying $y \leq x$ for all $y \in A$.
2) A subset A of a Riesz space is bounded below if there exists some x satisfying $y \geq x$ for all $y \in A$.
3) A subset A of a Riesz space is called order bounded if it is bounded both above and below.

4) An operator $T: L \to M$ between two Riesz spaces is said to be order bounded if it maps order bounded subsets of L to order bounded subsets of M.

The vector space of all order bounded operators from L to M will be denoted by $\mathcal{L}_b(L, M)$.

Definition 2.6: An operator $T: L \to M$ between two Riesz spaces is said to be regular if it can be written as a difference of two positive operators and the vector space of all regular operators from L to M will be denoted by $\mathcal{L}_r(L, M)$.

Definition 2.7:

1) A Riesz space is called Dedekind complete if every nonempty bounded above sunset has a supremum (or, equivalently, if every nonempty bounded below subset has an infumum) . That is, a Riesz space L is Dedekind complete if and only if $0 \leq x_\alpha \uparrow x$ implies the existence of $\sup\{x_\alpha\}$.

2) A Riesz space is said to be Dedekind σ-complete if every countable subset that is bounded above has a supremum. (or, equivalently if $0 \leq x_n \uparrow x$) implies the existence of $\sup\{x_n\}$. (Gök, 2011)

Theorem 2.8: If L and M are Riesz spaces with M Dedekind complete, then the ordered vector space $\mathcal{L}_b(L, M)$ is a Dedekind complete Riesz space. Its lattice operations satisfy

$$|T|(x) = \sup\{|Ty|: |y| \leq x\} \tag{2.14}$$

$$[S \vee T](x) = \sup\{S(y) + T(z): y, z \in L^+ \text{ and } y + z = x\} \tag{2.15}$$

$$[S \wedge T](x) = \inf\{S(y) + T(z): y, z \in L^+ \text{ and } y + z = x\} \tag{2.16}$$

for all $S, T \in \mathcal{L}_b(L, M)$ and $x \in L^+$.

Proof: Let $T \in \mathcal{L}_b(L, M)$. Since T is order bounded,

$$\sup\{|Ty|: |y| \leq x\} = \sup\{Ty: |y| \leq x\} = \sup T[-x, x] \tag{2.17}$$

exists in M for all $x \in L^+$, so by Theorem 2.4, the modulus of T exists and

$$|T|(x) = \sup\{Ty: |y| \leq x\} \tag{2.18}$$

By using the lattice properties, it can be easily seen that $\mathcal{L}_b(L, M)$ is a Riesz space.

Let $S, T \in \mathcal{L}_b(L, M)$ and $x \in L^+$. $y, z \in L^+$ satisfy $y + z = x$ if and only if there exists $|u| \leq x$ with $y = \frac{1}{2}(x + u)$ and $z = \frac{1}{2}(x - u)$. So,

$$[S\vee T](x) = \frac{1}{2}(Sx + Tx + |S - T|x)$$

$$= \frac{1}{2}(Sx + Tx + \sup\{(S - T)u: |u| \leq x\})$$

$$= \frac{1}{2}\sup\{Sx + Su + Tx - Tu : \ |u| \leq x\}$$

$$= \sup\left\{S\left(\frac{1}{2}(x + u)\right) + T\left(\frac{1}{2}(x - u)\right) : |u| \leq x\right\}$$

$$= \sup\{S(y) + T(z): y, z \in L^+ \ and \ y + z = x\} \qquad (2.19)$$

$[S\wedge T] = \inf\{S(y) + T(z): y, z \in L^+ \ and \ y + z = x\}$ can be proven in the same fashion so $\mathcal{L}_b(L, M)$ is a Riesz space.

Let $0 \leq T_\alpha \uparrow \leq T$ in $\mathcal{L}_b(L, M)$ and let $S(x) = \sup\{T_\alpha(x)\}$ for all $x \in L^+$, so, $T_\alpha(x) \uparrow S(x)$. Since $T_\alpha(x + y) = T_\alpha(x) + T_\alpha(y)$, the mapping $S: L^+ \to M^+$ is additive so S defines a positive operator from L to M. Hence, $T_\alpha \uparrow S$ in $\mathcal{L}_b(L, M)$ and it proves that $\mathcal{L}_b(L, M)$ is Dedekind complete.

Theorem 2.9: If L and M are Riesz spaces with M Dedekind complete, then for all $S, T \in \mathcal{L}_b(L, M)$ and each $x \in L^+$, the followings hold:

(1) $\{\sum_{i=1}^{n} S(x_i) \vee T(x_i) : \ x_i \in E^+ \ and \ \sum_{i=1}^{n} x_i = x\} \uparrow [S\vee T](x)$

(2) $\{\sum_{i=1}^{n} S(x_i) \wedge T(x_i) : \ x_i \in E^+ \ and \ \sum_{i=1}^{n} x_i = x\} \downarrow [S\wedge T](x)$

(3) $\{\sum_{i=1}^{n} T|(x_i)| : \ x_i \in E^+ \ and \ \sum_{i=1}^{n} x_i = x\} \downarrow |T|(x)$

Definition 2.10: A function $p: G \to M$, where G is a real vector space and M is an ordered vector space, is called sublinear if the followings hold

(a) $p(x + y) \leq p(x) + p(y)$

(b) $p(\lambda x) = \lambda p(x)$

for all $x, y \in G$ and for all $\lambda \geq 0$.

Theorem 2.11: Let G be a real vector space and M be a Dedekind complete Riesz space, and let $p: G \to M$ be a sublinear function. If H is a vector subspace of G and $S: H \to M$ is an operator satisfying $S(x) \leq p(x)$ for all $x \in H$, then there exists some operator $T: G \to M$ such that

(1) $T = S$ on H, that is, T is a linear extension of S to all of G.

(2) $T(x) \leq p(x)$ for all $x \in G$.

Proof: If S has a linear extension which satisfies (2) on an arbitrary vector subspace generated by H and one extra vector, then we can guarantee the existence of an extension of S to all of G with the desired properties by using Zorn's lemma.

Let $x \notin H$ and let $V = \{y + \lambda x : y \in H \text{ and } \lambda \in \mathbb{R}\}$. If $T : V \to M$ is a linear extension of S, then

$$T(y + \lambda x) = S(y) + \lambda T(x) \tag{2.20}$$

for all $y \in H$ and $\lambda \in \mathbb{R}$. If we choose $z = T(x)$, it has to be shown the existence of some $z \in M$ such that

$$S(y) + \lambda z \leq p(y + \lambda x) \tag{2.21}$$

for all $y \in H$ and $\lambda \in \mathbb{R}$. For $\lambda > 0$, the inequality (2.21) equals to

$$S(y) + z \leq p(y + x) \tag{2.22}$$

for all $y \in H$ and if $\lambda < 0$, the inequality (2.21) equals to

$$S(y) - z \leq p(y - x) \tag{2.23}$$

for all $y \in H$. The last two inequalities are satisfied by any choice of z for which

$$S(y) - p(y - x) \leq z \leq p(u + x) - S(u) \tag{2.24}$$

for all $y, u \in H$.

To see that, there exists some $z \in M$ satisfying (2.24), the following can be obtained for each $y, u \in H$,

$$\begin{aligned} S(y) + S(u) = S(y + u) \leq p(y + u) &= p\big(y - x + (u + x)\big) \\ &\leq p(y - x) + p(u + x) \end{aligned} \tag{2.25}$$

Therefore,

$$S(y) - p(y - x) \leq p(u + x) - S(u) \tag{2.26}$$

for all $y, u \in H$. This inequality and the Dedekind completeness of M guarantees that both suprema and infima

$$s = \sup\{S(y) - p(y - x) : y \in H\} \quad \text{and} \quad t = \inf\{p(u + x) - S(u) : u \in H\} \tag{2.27}$$

exists in M and satisfy $s \leq t$. Hence any $z \in M$ such that $s \leq z \leq t$ satisfies the inequality (2.24), and so (2.21).

Theorem 2.12: Let $T: L \to M$ be a positive operator between two Riesz spaces with M Dedekind complete. Assume that G is a Riesz subspace of L and that $S: G \to M$ is an operator satisfying $0 \leq Sx \leq Tx$ for all $x \in G^+$. Then S can be extended to a positive operator from L to M such that $0 \leq S \leq T$ in $\mathcal{L}(L, M)$.

Proof: Let us define $p: L \to M$ by $p(x) = T(x^+)$ and so p is sublinear and satisfies $S(x) \leq p(x)$ for all $x \in G$. By the Theorem 2.11, there exists a linear extension of S to all of L which will be denoted by S satisfying $S(x) \leq p(x)$ for all $x \in L$. If $x \in L^+$, then

$$-S(x) = S(-x) \leq p(-x) = T((-x)^+) = T(0) = 0 \tag{2.28}$$

and therefore, $0 \leq S(x) \leq p(x) = T(x)$.

Theorem 2.13: Let L and M be Riesz spaces with M Dedekind complete. If G is a Riesz subspace of L and $T: G \to M$ is a positive operator, then the following statements are equivalent:

(1) T extends to a positive operator from L to M.

(2) T extends to an order bounded operator from L to M.

(3) There exists a monotone sublinear mapping $p: L \to M$ satisfying $T(x) \leq p(x)$ for all $x \in G$.

Proof:

(1)⟹(2) Since every positive operator in a Riesz space is an order bounded operator, it is clear.

(2)⟹(3) Let $S \in \mathcal{L}_b(L, M)$ satisfy $S(x) = T(x)$ for all $x \in G$. So, we can define a mapping $p: L \to M$ by $p(x) = |S|(x^+)$ is monotone, sublinear and satisfies

$$T(x) \leq T(x^+) = S(x^+) \leq |S|(x^+) = p(x) \tag{2.29}$$

for all $x \in G$.

(3)⟹(1) Let $p: L \to M$ be a monotone sublinear mapping satisfying $T(x) \leq p(x)$ for all $x \in G$. Then, $q(x) = p(x^+)$ defines a sublinear mapping from L to M such that

$$T(x) \leq T(x^+) \leq p(x^+) = q(x) \tag{2.30}$$

for all $x \in G$. Therefore, there exists an extension $R \in \mathcal{L}(L, M)$ of T satisfying $R(x) \leq q(x)$ for all $x \in L$. In particular, if $x \in L^+$, then

$$-R(x) = R(-x) \leq q(-x) = p((-x)^+) = p(0) = 0 \qquad (2.31)$$

implies $R(x) \geq 0$. That is, R is a positive linear extension of T to all of L. (Aliprantis and Burkinshaw, 2006)

Definition 2.14:

1) A subset A of a Riesz is called solid if $|x| \leq |y|$ and $y \in A$ imply $x \in A$.
2) A solid vector subspace of a Riesz space is said to be an ideal.

Theorem 2.15: If $T: L \to M$ is a positive operator between two Riesz spaces with M Dedekind complete, then for every ideal A of L

$$T_A(x) = \sup\{T(y): y \in A \; and \; 0 \leq y \leq x\}, \qquad x \in L^+ \qquad (2.32)$$

defines a positive operator from L to M.

Proof: It can be written such that

$$T_A(x) = \sup\{T(x \wedge y): y \in A^+\} \qquad (2.33)$$

for all $x \in L^+$ and if we prove that T_A is additive on L^+, by Theorem 2.2, it extends to all of L.

Let $x, y \in L^+$. If $z \in A^+$, then the inequality

$$(x + y) \wedge z \leq x \wedge z + y \wedge z \qquad (2.34)$$

gives us that

$$T\big((x + y) \wedge z\big) \leq T(x \wedge y) + T(y \wedge z) \leq T_A(x) + T_A(y) \qquad (2.35)$$

So,

$$T_A(x + y) \leq T_A(x) + T_A(y) \qquad (2.36)$$

Also, the inequality

$$x \wedge u + y \wedge v \leq (x + y) \wedge (u + v) \qquad (2.37)$$

yields

$$T_A(x) + T_A(y) \leq T_A(x + y) \qquad (2.38)$$

Therefore,

$$T_A(x + y) = T_A(x) + T_A(y) \qquad (2.39)$$

Hence, T_A is additive on L^+.

If $T: G \to M$ is a positive operator, where G is a vector subspace of an ordered vector space L and M is a Dedekind complete Riesz space. We will denote the collection of all positive extensions of T to all of L by $\varepsilon(T)$:

$$\varepsilon(T) = \{S \in \mathcal{L}(L, M): S \geq 0 \; and \; S = T \; on \; G\} \tag{2.40}$$

Definition 2.16: A positive operator $T: G \to M$, where G is a vector subspace of an ordered vector space E, is said to have a smallest extension if there exists some $S \in \varepsilon(T)$ satisfying $S \leq R$ for all $R \in \varepsilon(T)$ and S is called the smallest extension of T. In other words, T has a smallest extension if and only if $\min \varepsilon(T)$ exists in $\mathcal{L}(L, M)$.

Theorem 2.17: Let L and M be Riesz spaces with M Dedekind complete, let A be an ideal of E, and let $T: A \to M$ be a positive operator. If $\varepsilon(T) \neq \emptyset$, then T has a smallest extension. Moreover, if in this case $S = \min \varepsilon(T)$, then

$$S(x) = \sup\{Ty: y \in A \; and \; 0 \leq y \leq x\} \tag{2.41}$$

for all $x \in L^+$.

Proof: Since $\varepsilon(T) \neq \emptyset$, T has at least one positive extension and

$$T_A(x) = \sup\{T(y): y \in A \; and \; 0 \leq y \leq x\}, \quad x \in L^+ \tag{2.42}$$

defines a positive operator from L to M satisfying $T_A = T$ on A, and so $T_A \in \varepsilon(T)$.

If $S \in \varepsilon(T)$, then $S = T$ on A. Therefore $T_A = S_A \leq S$ and T_A=min $\varepsilon(T)$. (Aliprantis and Burkinshaw, 2006)

Definition 2.18:

1) A Riesz subspace G of a Riez space L is said to be order dense in L if for each $0 < x \in L$, there exists some $y \in G$ such that $0 < y \leq x$.
2) A net $\{x_\alpha\}$ of a Riesz space is said to be order convergent to a vector x, that is, $x_\alpha \longrightarrow x$, if there exists another net $\{y_\alpha\}$ with the same index set satisfying $y_\alpha \downarrow 0$ and $|x_\alpha - x| \leq y_\alpha$ for all indices α :
$$|x_\alpha - x| \leq y_\alpha \downarrow 0 \tag{2.43}$$
3) A subset A of a Riesz space is said to be order closed if $\{x_\alpha\} \in A$ and $x_\alpha \longrightarrow x$ imply $x \in A$.
4) An order closed ideal is called a band. That is, an ideal A is a band if and only if $\{x_\alpha\} \in A$ and $0 \leq x_\alpha \uparrow x$ imply $x \in A$.

5) A band B in a Riesz space L such that $L = B \oplus B^d$ is called a projection band.

6) Let B be a projection band in a Riesz space L. Thus, every vector $x \in L$ has a unique decomposition $x = x_1 + x_2$, where $x_1 \in B$ and $x_2 \in B^d$. Then a positive projection $P_B: L \to L$ defined by $P_B(x) = x_1$ is called a band projection.

7) A vector $e > 0$ in a Riesz L is said to be a weak order unit if the band generated by e satisfies $B_e = L$.

Theorem 2.19: If B is a band in a Dedekind complete Riesz space L, then $L = B \oplus B^d$.

In converse, the set $B = \{f \in C([0,2]): f_{[0,1]} \equiv 0\}$ is a band in $C([0,1])$ but it is not a projection band. (Wickstead, 2004)

Definition 2.20: An operator $T: L \to M$ between two Riesz spaces is said to be

(a) Order continuous if $x_\alpha \to 0$ in L implies $Tx_\alpha \to 0$ in M.

(b) σ-order continuous if $x_n \to 0$ in L implies $Tx_n \to 0$ in M.

The collection of all order continuous operators of $\mathcal{L}_b(L, M)$ will be denoted by $\mathcal{L}_n(L, M)$.

Similarly, the collection of all σ-order continuous operators of $\mathcal{L}_b(L, M)$ will be denoted by $\mathcal{L}_c(L, M)$.

Lemma 2.21: Every order continuous operator is order bounded.

Proof: Let $T: L \to M$ be an order continuous operator and let $x \in L^+$. The order interval $[0, x]$ can be considered as a net $\{x_\alpha\}$, where $x_\alpha = \alpha$ for each $\alpha \in [0, x]$, so, $x_\alpha \downarrow 0$. Since T is an order continuous operator, there exists a net $\{y_\alpha\}$ of M with the same index $[0, x]$ such that $|Tx_\alpha| \leq y_\alpha \downarrow 0$. Therefore, if $\alpha \in [0, x]$, then $|T\alpha| = |Tx_\alpha| \leq y_\alpha \leq y_x$ and it gives us that $T[0, x]$ is an order bounded subset of M.

Theorem 2.22: For an order bounded operator $T: L \to M$ between two Riesz spaces with M Dedekind complete, the followings are equivalent:

1) T is order continuous.

2) If $x_\alpha \downarrow 0$ in L, then $Tx_\alpha \to 0$ in M.

3) If $x_\alpha \downarrow 0$ in L, then $\inf\{|Tx_\alpha|\} = 0$ in M.

4) T^+ and T^- are both order continuous.

5) $|T|$ is order continuous.

Theorem 2.23: If L and M are Riesz spaces with M Dedekind complete, then $\mathcal{L}_n(L,M)$ and $\mathcal{L}_c(L,M)$ are both bands of $\mathcal{L}_b(L,M)$.

Proof: If $|S| \leq |T|$ in $\mathcal{L}_b(L,M)$ with $T \in \mathcal{L}_n(L,M)$, then, by the Theorem 2.22, $S \in \mathcal{L}_n(L,M)$. It means that $\mathcal{L}_n(L,M)$ is an ideal of $\mathcal{L}_b(L,M)$.

Let $0 \leq T_\lambda \uparrow T$ in $\mathcal{L}_b(L,M)$ with $\{T_\lambda\} \subseteq \mathcal{L}_n(L,M)$ and let $0 \leq x_\alpha \uparrow x$ in L. Therefore, for each fixed index

$$0 \leq T(x - x_\alpha) \leq (T - T_\lambda)(x) + T_\lambda(x - x_\alpha) \tag{2.44}$$

And $x - x_\alpha \downarrow 0$. Since $T_\lambda \in \mathcal{L}_n(L,M)$, it is obtained that

$$0 \leq \inf_\alpha\{T(x - x_\alpha)\} \leq (T - T_\lambda)(x) \tag{2.45}$$

for all λ. It follows from $T - T_\lambda \downarrow 0$ that $\inf_\alpha\{T(x - x_\alpha)\} = 0$. Thus, $T(x_\alpha) \uparrow T(x)$ and hence $T \in \mathcal{L}_n(L,M)$.

To prove that $\mathcal{L}_c(L,M)$ is a band of $\mathcal{L}_b(L,M)$ can be done in the same fashion.

Let L and M be Riesz spaces with M Dedekind complete. The band of all operators in $\mathcal{L}_b(L,M)$ that are disjoint from $\mathcal{L}_c(L,M)$ will be denoted by $\mathcal{L}_s(L,M)$, that is, $\mathcal{L}_s(L,M) = \mathcal{L}_c^d(L,M)$, and its nonzero members will be called singular operators. Since $\mathcal{L}_b(L,M)$ is a Dedekind complete Riesz space, $\mathcal{L}_c(L,M)$ becomes a projection band and it can be written as

$$\mathcal{L}_b(L,M) = \mathcal{L}_c(L,M) \oplus \mathcal{L}_s(L,M) \tag{2.46}$$

In particular, each operator $T \in \mathcal{L}_b(L,M)$ has a unique decomposition $T = T_c + T_s$, where $T_c \in \mathcal{L}_c(L,M)$ and $T_s \in \mathcal{L}_s(L,M)$. The operator T_c is called the σ-order continuous component of T and T_s is called the singular component of T.

Similarly,

$$\mathcal{L}_b(L,M) = \mathcal{L}_n(L,M) \oplus \mathcal{L}_\sigma(L,M) \tag{2.47}$$

where $\mathcal{L}_\sigma(L,M) = \mathcal{L}_n^d(L,M)$. So, in a similar way each operator $T \in \mathcal{L}_b(L,M)$ has a unique decomposition $T = T_n + T_\sigma$ where $T_n \in \mathcal{L}_n(L,M)$ and $T_\sigma \in \mathcal{L}_\sigma(L,M)$. The operator T_n is called the order continuous component of T.

Definition 2.24: Let $T: L \to M$ be an order operator between two Riesz spaces with M Dedekind complete.

1) The null ideal $\mathcal{N}_T$ of T is defined by

$$\mathcal{N}_T = \{x \in L: |T|(|x|) = 0\} \tag{2.48}$$

2) The disjoint complement of $\mathcal{N}_T$ is called the carrier of T, denoted by $\mathcal{C}_T$, and is defined by

$$\mathcal{C}_T = \mathcal{N}_T^d = \{x \in L : x \perp \mathcal{N}_T\} \tag{2.49}$$

When an order bounded operator is also order continuous, its null ideal is a band.

An operator $T \in \mathcal{L}_b(L, M)$ is said to have zero carrier if $\mathcal{C}_T = 0$. It is clear that the zero operator is the only continuous operator with zero carrier. (Aliprantis and Burkinshaw, 2006)

We will denote the band projections in L onto $\mathcal{C}_T$ and $\mathcal{N}_T$ by C_T and N_T, respectively. (Luxemburg and De Pagter, 2001).

Definition 2.25:

1) Let L be a Riesz space. A linear functional f is called positive if $f(x) \geq 0$ for all $x \in L^+$.

2) A linear functional called order bounded if f maps order bounded subsets of L to order bounded subsets of $\mathbb{R}$.

3) The vector space $L^{\frown}$ of all order bounded linear functionals on L is called order dual of L, that is, $L^{\frown} = \mathcal{L}_b(L, \mathbb{R})$.

If we consider the order relation $f, g \in L^{\frown}$ such that $f \geq g$ in $L^{\frown}$ means $f(x) \geq g(x)$ for all $x \in L^+$, $L^{\frown}$ becomes a Dedekind complete Riesz space itself.

4) $L^{\frown}$ seperates the points of L means that for each $x \neq 0$, there exists some $f \in L^{\frown}$ with $f(x) \neq 0$.

5) A positive linear functional on L is order continuous if and only if $x_\alpha \downarrow 0$ in L implies $f(x_\alpha) \downarrow 0$ in $\mathbb{R}$. Similarly, f is σ-order continuous if and only if for every sequence $\{x_n\}$ with $x_n \downarrow 0$, $f(x_n) \downarrow 0$ in $\mathbb{R}$

6) The vector space $\mathcal{L}_n(L, \mathbb{R})$ of all order continuous linear functionals on L will be denoted by $L_n^{\sim}$. Similarly, the vector space $\mathcal{L}_c(L, \mathbb{R})$ of all σ-order continuous linear functionals on L will be denoted by $L_c^{\sim}$.

Theorem 2.26: If L is an Archimedean Riesz space, then for all $f, g \in L_n^{\sim}$, the followings are equivalent:

1) $f \perp g$

2) $\mathcal{C}_f \subseteq \mathcal{N}_g$

3) $\mathcal{C}_g \subseteq \mathcal{N}_f$

4) $\mathcal{C}_f \perp \mathcal{C}_g$

Proof: Without loss of generality, it may be assumed that $0 \leq f, g \in L_n^{\sim}$.

(1)⟹(2) Let $0 \leq x \in \mathcal{C}_f = \mathcal{N}_f^d$ and let $\varepsilon > 0$. Since $f \wedge g = 0$, a sequence $\{x_n\} \in L^+$ can be found such that

$$0 \leq x_n \leq x \qquad \text{and} \qquad f(x_n) + g(x - x_n) < 2^{-n}\varepsilon \qquad (2.50)$$

Let us pick $y_n = \bigwedge_{i=1}^{n} x_i$. If $0 \leq y \leq y_n$ for all n, then $0 \leq f(y) \leq f(y_n) < 2^{-n}\varepsilon$ for all n and hence $f(y) = 0$. So, $y \in \mathcal{C}_f \cap \mathcal{N}_f$ and it follows that $y = 0$. Therefore $y_n \downarrow 0$.

Now, since $0 \leq g \in L_n^{\sim}$, we obtain that $g(x - y_n) \uparrow g(x)$. Also,

$$0 \leq g(x - y_n) = g\left(\bigvee_{i=1}^{n}(x - x_i)\right) \leq \sum_{i=1}^{n} g(x - x_i) < \varepsilon \qquad (2.51)$$

So, $0 \leq g(x) \leq \varepsilon$ for all $\varepsilon > 0$. Hence $g(x) = 0$ and it follows that $\mathcal{C}_f \subseteq \mathcal{N}_g$.

(2)⟹(3) As we know that $\mathcal{N}_f$ is a band, it yields that $\mathcal{C}_f = \mathcal{N}_f^d \subseteq \mathcal{N}_g$. Hence,

$$\mathcal{C}_g = \mathcal{N}_g^d \subseteq \mathcal{N}_f^{dd} = \mathcal{N}_f \qquad (2.52)$$

(3)⟹(4) Since $\mathcal{C}_g \subseteq \mathcal{N}_f$ and also $\mathcal{N}_f \perp \mathcal{C}_f$, it gives us that $\mathcal{C}_f \perp \mathcal{C}_g$.

(4)⟹(1) Since $\mathcal{C}_f \perp \mathcal{C}_g$, $\mathcal{C}_g \subseteq \mathcal{C}_f^d = \mathcal{N}_f^{dd} = \mathcal{N}_f$. So, f $0 \leq x = y + z \in \mathcal{N}_g \oplus \mathcal{C}_g$ and it follows that

$$0 \leq [f \wedge g](x) = [f \wedge g](y) + [f \wedge g](z) \leq g(y) + f(z) = 0 \quad (2.53)$$

Hence, $f \wedge g = 0$ on $\mathcal{N}_g \oplus \mathcal{C}_g$. Since $f \wedge g \in L_n^{\sim}$, $[f \wedge g](x) = 0$ for all $x \in L$.

By this theorem, it results that two linear functionals in $L_n^{\sim}$ are disjoint if and only if their carriers are disjoint sets.

Definition 2.27:

1) Let L be a Riesz space and $L^{\sim}$ be its order dual. The Riesz space of all order bounded linear functionals on $L^{\sim}$ is called the second order dual of L and denoted by $L^{\sim\sim}$. It is also the order dual of $L^{\sim}$.

2) For all $x \in L$, an order bounded linear functional $\hat{x}$ can be defined on $L^{\sim}$ by

$$\hat{x}(f) = f(x), \quad f \in L^{\sim} \tag{2.54}$$

Here, if $x \geq 0$, then $\hat{x} \geq 0$. Also, as we know that $f_\alpha \downarrow 0$ in $L^{\sim}$ if and only if $\hat{x}(f_\alpha) = f_\alpha(x) \downarrow 0$ for all $x \in L^+$, it follows that each $x \in L$ defines an order continuous linear functional on $L^{\sim}$. So, a positive operator $x \to \hat{x}$ can be defined from L to $L^{\sim\sim}$. This operator called the canonical embedding of L into $L^{\sim\sim}$. The canonical embedding always preserves finite suprema and infima and when $L^{\sim}$ separates the points of L, it is also injective.

3) Let L be a Riesz space and let A be an ideal of $L^{\sim}$. Then, for each $x \in L$, the restriction of $\hat{x}$ to A defines an order continuous linear functional on A. Hence, there exists a natural embedding $x \to \hat{x}$ of L into $A_n^{\sim}$ defined by

$$\hat{x}(f) = f(x), \quad f \in A \tag{2.55}$$

The natural embedding $x \to \hat{x}$ from L into $A_n^{\sim}$ is lattice preserving and also is injective if and only if the ideal A separates the points of L.

When A consists of order continuous linear functionals, $x \to \hat{x}$ preserves arbitrary suprema and infima.

4) A Riesz space L is called perfect if the natural embedding $x \to \hat{x}$ from L to $(L_n^{\sim})_n^{\sim}$ is one-to-one and onto.

Every perfect Riesz space must be Dedekind complete.

5) If $f \in L^{\sim}$ and $u \in M$, then $f \otimes u$ will denote the order bounded operator of $\mathcal{L}_b(L, M)$ defined by

$$[f \otimes u](x) = f(x)u \tag{2.56}$$

for each $x \in L$. Every operator of the form $f \otimes u$ is called a rank one operator.

If $f \in L_n^{\sim}$, then $f \otimes u$ is an order continuous operator.

Every operator $T: L \to M$ of the form $T = \sum_{i=1}^{n} f_i \otimes u_i$, where $f_i \in L^{\sim}$ and $u_i \in M$ $(i = 1,2, \dots, n)$, is called a finite rank operator.

In general, if G is a vector subspace of $L^{\sim}$, then we define

$$G \otimes F = \left\{ \begin{matrix} T \in \mathcal{L}(L, M) : \exists n, f_i \in G, u_i \in M (1 \leq i \leq n) \, with \\ T = \sum_{i=1}^{n} f_i \otimes u_i \end{matrix} \right\} \tag{2.57}$$

Hence, $G \otimes F$ is a vector subspace of $\mathcal{L}_b(L, M)$.

Theorem 2.28: Let L and M be Riesz spaces Then, the followings hold:

1) If $0 \leq f \in L^{\sim}$ and $u, v \in M$, then $(f \otimes u) \vee (f \otimes v)$ and $(f \otimes u) \wedge (f \otimes v)$ both exist in $\mathcal{L}(L, M)$ and

$$(f\otimes u)\vee(f\otimes v)=f\otimes(u\vee v) \tag{2.58}$$

$$(f\otimes u)\wedge(f\otimes v)=f\otimes(u\wedge v) \tag{2.59}$$

2) If $0\leq u\in M$ and $f,g\in L^{\sim}$, then $(f\otimes u)\vee(g\otimes u)$ and $(f\otimes u)\wedge(g\otimes u)$ both exist in $\mathcal{L}(L,M)$ and

$$(f\otimes u)\vee(g\otimes u)=(f\vee g)\otimes u \tag{2.60}$$

$$(f\otimes u)\wedge(g\otimes u)=(f\wedge g)\otimes u \tag{2.61}$$

3) If $f\in L^{\sim}$ and $u\in M$, then the modulus of $f\otimes u$ exists in $\mathcal{L}(L,M)$ and

$$|f\otimes u|=|f|\otimes|u| \tag{2.62}$$

Proof:

1) Let $0\leq f\in L^{\sim}$ and $u,v\in M$. So, $f\otimes u\leq f\otimes(u\vee v)$ and $f\otimes v\leq f\otimes(u\vee v)$ and then $f\otimes(u\vee v)$ is an upper bound for $f\otimes u$ and $f\otimes v$.

On the other hand, if $T\in\mathcal{L}(L,M)$ satisfies $f\otimes u\leq T$ and $f\otimes v\leq T$, then for each $x\in L^{+}$ we obtain that

$$\begin{aligned}[f\otimes(u\vee v)\,](x)&=f(x)(u\vee v)=[f(x)u]\vee[f(x)v]\\&\leq T(x)\vee T(x)=T(x)\end{aligned} \tag{2.63}$$

So, $f\otimes(u\vee v)\leq T$ in $\mathcal{L}_b(L,M)$ and hence $f\otimes(u\vee v)$ is the least upper bound of $f\otimes u$ and $f\otimes v$ in $\mathcal{L}(L,M)$.

The other equality can be proven in the same fashion.

2) Let $u\in M^{+}$ and $f,g\in L^{\sim}$. Then, $f\otimes u\leq(f\vee g)\otimes u$ and $g\otimes u\leq(f\vee g)\otimes u$, so, $(f\vee g)\otimes u$ is an upper bound for $f\otimes u$ and $g\otimes u$.

On the other hand, if $y,z\in L^{+}$ such that $y+z=x$

$$[f\otimes u](y)+[g\otimes u](z)\leq T(x)+T(y)=T(x) \tag{2.64}$$

So, for all $x\in L^{+}$ we obtain that

$$\begin{aligned}[(f\vee g)\otimes u](x)&=[(f\vee g)(x)].u\\&=[\sup\{f(y)+g(z)\colon y,z\in L^{+}\ and\ y+z=x\}].u\\&=\sup\{f(y)u+g(z)u\colon y,z\in L^{+}\ and\ y+z=x\,\}\\&=\sup\{[f\otimes u](y)+[g\otimes u](z)\colon y,z\in L^{+}\ and\ y+z=x\}\\&\leq T(x)\end{aligned} \tag{2.65}$$

Thus, $(f \vee g)\otimes u$ is the least upper bound of $f\otimes u$ and $g\otimes u$ in $\mathcal{L}(L,M)$.

The other equality can be proven in the same fashion.

3) For all $x \in L^+$, $\pm[f\otimes u](x) = \pm[f(x).u] \leq |f(x)u| = |f(x)||u| \leq |f|(x)|u| = [|f|\otimes|u|](x)$ and so, $\pm[f\otimes u] \leq |f|\otimes|u|$.

On the other hand if $T \in \mathcal{L}(L,M)$ satisfies $f\otimes u \leq T$ and $-[f\otimes u] \leq T$.

Let $x \in L^+$. If $f(x) < 0$, then $[f\otimes|u|](x) \leq T(x)$, and if $f(x) \geq 0$, then

$$[f\otimes|u|](x) = f(x)|u| = [f(x)u] \vee [-f(x)u] \leq T(x) \tag{2.66}$$

So, $f\otimes|u| \leq T$ and by the same way we obtain that $(-f)\otimes|u| \leq T$. Using (2), we see that

$$|f|\otimes|u| = [f\otimes|u|] \vee [(-f)\otimes|u|] \leq T \tag{2.67}$$

Hence, $|f|\otimes|u|$ is the least upper bound of $f\otimes u$ and $-f\otimes u$. Therefore, $|f\otimes u| = |f|\otimes|u|$ in $\mathcal{L}(L,M)$.

Definition 2.29: Let $T: L \to M$ be an operator between two Riesz spaces and let us define an operator $T^*: M^\sim \to L^\frown$ by $[T^*f](x) = f(Tx)$ for all $f \in M^\sim$ and $x \in L$. If A is an order bounded subset of L and $f \in M^\sim$, then it follows from $[T^*f](A) = f\big(T(A)\big)$ that $[T^*f](A)$ is a bounded subset of $\mathbb{R}$ and so $T^*f \in L^\frown$. The restriction of T^* to $M^\sim$ is called the (order) adjoint of T and denoted by T'. That is, $T': M^\sim \to L^\frown$ satisfies that

$$\langle T'f, x\rangle = \langle f, Tx\rangle \tag{2.68}$$

for all $f \in M^\sim$ and $x \in L$.

If T is a positive operator, then its adjoint T' is also a positive operator.

Theorem 2.30: If $T: L \to M$ is an order bounded operator between two Riesz spaces, then its adjoint $T': M^\sim \to L^\frown$ is order bounded and order continuous.

Lemma 2.31: $T: L \to M$ is an order bounded operator between two Riesz spaces, then for all $0 \leq f \in M^\frown$ and all $x \in L^+$,

$$\langle f, |Tx|\rangle \leq \langle |T'|f, x\rangle \tag{2.69}$$

Proof: Let $0 \leq f \in M^\frown$ and all $x \in L^+$. Then, there exists some $g \in M^\frown$ with $|g| \leq f$ and $\langle f, |Tx|\rangle = \langle g, Tx\rangle$. So,

$$\langle f, |Tx| \rangle = \langle g, Tx \rangle = \langle T'g, x \rangle \leq \langle |T'||g|, x \rangle \leq \langle |T'|f, x \rangle \tag{2.70}$$

Theorem 2.32: If $T: L \to M$ is an order bounded operator between two Riesz spaces with M Dedekind complete, then

$$|T'|f = |T|'f \tag{2.71}$$

for all $f \in F_n^{\sim}$.

Proof: Let $0 \leq f \in F_n^{\sim}$. Then, $|T'|f \leq |T|'f$.

On the other hand, if $0 \leq x \in L$

$$\begin{aligned} \langle |T'|f, x \rangle &= \langle f, |T|x \rangle \\ &= \langle f, \sup\{\textstyle\sum_{i=1}^{n} |Tx_i| : x_i \in L^+ \text{ and } \sum_{i=1}^{n} x_i = x\} \rangle \\ &= \sup\{\textstyle\sum_{i=1}^{n} \langle f, |Tx_i| \rangle : x_i \in L^+ \text{ and } \sum_{i=1}^{n} x_i = x\} \\ &\leq \{\textstyle\sum_{i=1}^{n} \langle |T'|f, x_i \rangle : x_i \in L^+ \text{ and } \sum_{i=1}^{n} x_i = x\} \\ &= \langle |T|'f, x \rangle \end{aligned} \tag{2.72}$$

Hence, $|T|'f \leq |T'|f$, and so $|T'|f = |T|'f$ for all $f \in F_n^{\sim}$.

Theorem 2.33: For a Riesz space L whose order dual separates the points of L, the followings are equivalent:

- **(a)** $L_c^{\sim} = L^{\sim}$, that is, every positive linear functional on L is σ-order continuous.
- **(b)** Whenever $T: L \to L$ is a positive operator and a sequence $\{T_n\}$ of positive operators from L to L such that $T_n(x) \uparrow T(x)$ in L for all $x \in L^+$, then $T_n^2(x) \uparrow T^2(x)$ in L for all $x \in L^+$.

Proof:

(1)⟹(2) Let $0 \leq T_n(x) \uparrow T(x)$ for all $x \in L^+$ and let $y \in L^+$. So, $0 \leq T_n^2(y) \uparrow T^2(y)$ in L. To see that $T^2(y)$ is the least upper bound of the sequence $\{T_n^2(y)\}$, let $T_n^2(y) \leq z$ in L for all n. So, for all $0 \leq f \in L^{\sim}$, $f\left(T_n^2(y)\right) \leq f(z)$ for all n.

On the other hand, for all $0 \leq f \in L^{\sim}$, the sequence $\{f \circ T_n\} \subseteq L^{\sim} = L_c^{\sim}$ satisfies that $0 \leq f \circ T_n \uparrow f \circ T$ in $L^{\sim}$. Therefore,

$$f\left(T_n^2(y)\right) = [f \circ T_n](T_n y) \uparrow [f \circ T](Ty) = f\left(T^2(y)\right) \tag{2.73}$$

It follows that $f\left(T^2(y)\right) \leq f(z)$ for all $0 \leq f \in L^{\sim}$. Since $L^{\sim}$ separates the points of L, $T^2(y) \leq z$. Hence, $T_n^2(y) \uparrow T^2(y)$ in L for all $y \in L^+$.

(2)⟹(1) Let $0 \leq f \in L^{\sim}$ and let $0 \leq x_n \uparrow x$ in L. Hence $0 \leq [f \otimes x_n](y) \uparrow [f \otimes x](y)$ for all $y \in L^+$ and it follows that

$$[f \otimes x_n]^2(y) = f(x_n)[f(y)x_n] \uparrow [f \otimes x]^2(y) = f(x)[f(y)x] \qquad (2.74)$$

for all $y \in L^+$. Hence $f(x_n) \uparrow f(x)$ and it follows that f is σ-order continuous. Therefore, $L_c^{\sim} = L^{\sim}$.

Definition 2.34:

1) Let $T: L \to M$ be a positive operator between two Riesz spaces with M Dedekind complete. Then a positive operator $S: L \to M$ is a component of T if $S \wedge (T - S) = 0$ and $\mathfrak{C}_T$ will denote the Dedekind complete Boolean algebra of all components of T:

$$\mathfrak{C}_T = \{S \in \mathcal{L}_b(L, M): S \wedge (T - S) = 0\} \qquad (2.75)$$

2) Let P be an order projection on L and Q be an order projection on the Dedekind complete Riesz space M. Then, the operator $\Pi: \mathcal{L}_b(L, M) \to \mathcal{L}_b(L, M)$ defined by $\Pi(\mathrm{Q}) = \mathrm{QTP}$ satisfies $0 \leq \Pi(\mathrm{T}) \leq \mathrm{T}$ for all $T \in \mathcal{L}_b^+(L, M)$ and $\Pi^2 = \Pi$. Therefore Π is an order projection on $\mathcal{L}_b(L, M)$. In particular, if T is a positive operator, then each order projection Q on M and each order projection P on L, the operator QTP is a component of T. Any component of the form QTP is called an elementary component of T. Any component of the form $\bigvee_{i=1}^{n} Q_i T P_i$ is called a simple component of T. The collection of all simple components of T will be denoted by S_T.

For two order projections Π_1, Π_2 on a Riesz space,it can be written as

$$\Pi_1 u \wedge \Pi_2 v = \Pi_1 \Pi_2 (u \wedge v) \qquad (2.76)$$

for all $u, v \geq 0$. So, if Q_1, Q_2 are order projections on M and P_1, P_2 are order projections on L, then for $\Pi_1(T) = Q_1 S P_1$ and $\Pi_2(T) = Q_2 T P_2$ on $\mathcal{L}_b(L, M)$,

$$(Q_1 S P_1) \wedge (Q_2 T P_2) = Q_1 Q_2 (S \wedge T) P_1 P_2 \qquad (2.77)$$

for all $S, T \in \mathcal{L}_B^+(L, M)$. In particular, if either $Q_1 Q_2 = 0$ or $P_1 P_2 = 0$, then $Q_1 S P_1 \perp Q_2 T P_2$ for all $\mathcal{L}_B^+(L, M)$. It shows that

$$T - QTP = (I - Q)T + QT(I - P) = [(I - Q)T] \vee [QT(I - P)] \qquad (2.78)$$

Let $T: L \to M$ be a positive operator between two Riesz spaces with M Dedekind complete. If $\bigvee_{i=1}^{n} Q_i T P_i$ and $\bigvee_{j=1}^{m} R_j T S_j$ are two simple components of T, then we obtain that

$$\left(\bigvee_{i=1}^{n} Q_i T P_i\right) \vee \left(\bigvee_{j=1}^{m} R_j T S_j\right) \in S_T \tag{2.79}$$

$$\left(\bigvee_{i=1}^{n} Q_i T P_i\right) \wedge \left(\bigvee_{j=1}^{m} R_j T S_j\right) = \bigvee_{i=1}^{n} \bigvee_{j=1}^{m} Q_i R_j T P_i S_j \in S_T \tag{2.80}$$

$$T - \bigvee_{i=1}^{n} Q_i T P_i = \bigwedge_{i=1}^{n} (T - Q_i T P_i) = \bigwedge_{i=1}^{n} [(I - Q_i)T \vee Q_i T(I - P_i)] \in S_T \tag{2.81}$$

Hence, S_T is a Boolean subalgebra of the (Dedekind complete) Boolean of all components $\mathfrak{C}_T$ of T.

Let $T: L \to M$ be a positive operator between two Riesz spaces with M Dedekind complete. Now let us consider the positive operator T_A defined by

$$T_A(x) = \sup\{T(y): y \in A \text{ and } 0 \le y \le x\}, \quad x \in L^+ \tag{2.82}$$

If A is a ideal of L, then, a mapping $T \to T_A$ defined from $\mathcal{L}_B^+(L, M)$ to $\mathcal{L}_B^+(L, M)$ satisfies

$$(S + T)_A(x) = S_A(x) + T_A(x) \tag{2.83}$$

for all $S, T \in \mathcal{L}_B^+(L, M)$ and $x \in L^+$. It shows that the mapping $T \to T_A$ is additive and so it extends to a positive operator from $\mathcal{L}_b(L, M)$ to $\mathcal{L}_b(L, M)$.

Theorem 2.35: If M is Dedekind complete and A is a ideal of L, then the operator $T \to T_A$, from $\mathcal{L}_b(L, M)$ to $\mathcal{L}_b(L, M)$, is an order projection.

In particular, if $T: L \to M$ is a positive operator, then T_A is a component of T for all ideal A of L.

Proof: Since $0 \le T_A \le T$ for all $T \in \mathcal{L}_B^+(L, M)$, $T \to T_A$ is bounded by the identity operator of $\mathcal{L}_b(L, M)$. Also, it follows from $T_A = T$ that $(T_A)_A = T_A$. Since every projection which satisfies $0 \le T \le I$ is an order projection, so is the operator $T \to T_A$.

Definition 2.36: Let L be a Riesz space and a vector $0 \le x \in L$. An x-step function is any vector $s \in L$ for which there exists pairwise disjoint components $x_1, x_2, \dots, x_n$ of x with $x_1 + + \cdots + x_n = x$ and real numbers $\alpha_1, \alpha_2, \dots, \alpha_n$ such that

$$s = \sum_{i=1}^{n} \alpha_i x_i \tag{2.84}$$

It is clear to see that every component of x is an x-step function. Also, every x-step function belongs to the ideal generated by x.

If u and v are x-step functions such that

$$u = \sum_{i=1}^{n} \alpha_i x_i \quad \text{and} \quad v = \sum_{j=1}^{m} \beta_j y_j \tag{2.85}$$

then

$$u = \sum_{i=1}^{n}\sum_{j=1}^{m} \alpha_i (x_i \wedge y_j) \quad \text{and} \quad v = \sum_{i=1}^{n}\sum_{j=1}^{m} \beta_j (x_i \wedge y_j) \tag{2.86}$$

are also representations of u and v as x-step functions.So, it can be expressed as follows

$$u + v = \sum_{i=1}^{n}\sum_{j=1}^{m} (\alpha_i + \beta_j)(x_i \wedge y_j) \tag{2.87}$$

$$\lambda u = \sum_{i=1}^{n} \lambda \alpha_i x_i \tag{2.88}$$

$$|u| = \sum_{i=1}^{n} |\alpha_i| x_i \tag{2.89}$$

Hence, the collection of all x-step functions is a Riesz space.

Theorem 2.37: (Freudenthal's Spectral Theorem): Let E be a Riesz space with the principal projection property (that is, every projection band is a principal projection) and let $0 < x \in L$. Then for every $y \in L_x$, there exists a sequence $\{u_n\}$ of x-step functions such that

$$0 \le y - u_n \le \frac{1}{n} x \quad \text{for each n and} \quad u_n \uparrow y \tag{2.90}$$

Proof: Without loss of generality, we may assume that $0 \le y \le x$. For all $0 \le \alpha \le 1$, let us denote the order projection of L onto the band generated by $(y - \alpha x)^+$ by P_α. The family $\{P_\alpha : 0 \le \alpha \le 1\}$ of order projections has the following properties:

(a) $P_0 = P_y$ and $P_1 = 0$

(b) If $0 \le \alpha \le \beta \le 1$, then $0 \le P_\beta \le P_\alpha \le I$

(c) If $0 \le \alpha \le \beta \le 1$, then

$$\alpha(P_\alpha - P_\beta)x \le (P_\alpha - P_\beta)y \le \beta(P_\alpha - P_\beta)x \tag{2.91}$$

Indeed, from

$$\begin{aligned}(P_\alpha - P_\beta)y - \alpha(P_\alpha - P_\beta)x &= (I - P_\beta)P_\alpha(y - \alpha x) \\ &= (I - P_\beta)(y - \alpha x)^+ \ge 0\end{aligned} \tag{2.92}$$

It follows that $\alpha(P_\alpha - P_\beta)x \leq (P_\alpha - P_\beta)y$. On the other hand, from

$$\begin{aligned}(P_\alpha - P_\beta)y - \beta(P_\alpha - P_\beta)x &= P_\alpha(y - \beta x) - P_\beta(y - \beta x) \\ &= P_\alpha(y - \beta x) - (y - \beta x)^+ \\ &\leq P_\alpha(y - \beta x)^+ - (y - \beta x)^+ \leq 0 \quad (2.93)\end{aligned}$$

we obtain that $(P_\alpha - P_\beta)y \leq \beta(P_\alpha - P_\beta)x$.

(d) If $0 \leq \alpha \leq \beta \leq \alpha_1 \leq \beta_1 \leq 1$, then $(P_\alpha - P_\beta)x$ and $(P_{\alpha_1} - P_{\beta_1})x$ are two disjoint components of x.

Indeed,

$$0 \leq (P_\alpha - P_\beta)x \wedge (P_{\alpha_1} - P_{\beta_1})x \leq (x - P_\beta x) \wedge P_\beta x = 0 \qquad (2.94)$$

and

$$\begin{aligned}0 &\leq (P_\alpha - P_\beta)x \wedge [x - (P_\alpha - P_\beta)x] \\ &\leq P_\alpha x \wedge (x - P_\alpha x) + (x - P_\beta x) \wedge P_\beta x = 0 \qquad (2.95)\end{aligned}$$

Let $\varepsilon > 0$ and let $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_n = 1$ be a partition of $[0,1]$ with less than ε. Pick $x_i = (P_{\alpha_{i-1}} - P_{\alpha_i})x$ for all $i = 1,2,\dots,n$ and $x_1 + x_2 + \cdots + x_n \leq x$ and $x_i \wedge x_j = 0$ for $i \neq j$. Also, by using (c), we obtain that the x-step function $u = \sum_{i=1}^n \alpha_{i-1}x_i$ satisfies

$$\begin{aligned}u = \sum_{i=1}^n \alpha_{i-1}x_i &\leq \sum_{i=1}^n (P_{\alpha_{i-1}} - P_{\alpha_i})y = y \leq \sum_{i=1}^n \alpha_i x_i \\ &\leq \sum_{i=1}^n (\alpha_{i-1}\varepsilon)x_i \leq u + \varepsilon x \qquad (2.96)\end{aligned}$$

and hence $0 \leq y - u \leq \varepsilon x$. (Aliprantis and Burkinshaw, 2006)

Definition 2.38: An operator $T: L \to M$ between two Riesz spaces is said to be lattice (Riesz) homomorphism if $T(x \vee y) = T(x) \vee T(y)$ for all $x, y \in L$.

If $x \in L^+$, then

$$T(x) = T(x \vee 0) = T(x) \vee T(0) = [T(x)]^+ \geq 0 \qquad (2.97)$$

in M.Hence, every lattice homomorphism $T: L \to M$ is also a positive operator.Also, the range of a lattice homomorphism is a Riesz subspace. (Wickstead, 2004)

Theorem 2.39: For an operator $T: L \to M$ between two Riesz spaces, the followings are equivalent:

1) T is a lattice homomorphism.

2) $T(x^+) = (Tx)^+$ for all $x \in L$.

3) $T(x \wedge y) = T(x) \wedge T(y)$ for all $x, y \in L$.

4) If $x \wedge y = 0$ in L, then $T(x) \wedge T(y) = 0$ in M.

5) $T(|x|) = |T(x)|$ for all $x \in L$.

It follows from this theorem that every order projection is an order continuous lattice homomorphism. Also, the kernel of every lattice homomorphism is an ideal.

Definition 2.40: A lattice homomorphism which is also injective is called a lattice (Riesz) isomorphism.

Two Riesz spaces L and M are called Riesz isomorphic if there exists a lattice isomorphism from L onto M. If two Riesz spaces L and M are lattice isomorphic, then from the Riesz space point of view, L and M are considered as identical.

Definition 2.41: An operator $T: L \to M$ between two Riesz spaces is called interval preserving if T is a positive operator and $T[0, x] = T[0, Tx]$ for all $x \in L^+$. That is, if $0 \leq w \leq T$ in M, then there exists $0 \leq v \leq u$ in L such that $w = Tv$.

Therefore, the range of an interval preserving operator is an ideal. Also, every positive linear functional is interval preserving.

Definition 2.42: A sequence $\{x_n\}$ in a Riesz space is said to be relatively uniformly convergent to x if there exist some $u > 0$ and a sequence $\{\varepsilon_n\}$ of real numbers with $\varepsilon_n \downarrow 0$ such that $|x_n - x| \leq \varepsilon_n u$ for all n. A subset A of a Riesz space is said to be uniformly closed if for each sequence $\{x_n\} \subseteq A$ which is relatively uniformly convergent to some x, then $x \in A$.

Definition 2.43: A Dedekind complete Riesz space M is called a Dedekind completion of the Riesz space L if L is Riesz isomorphic to a majorizing order dense Riesz subspace of M and denoted by $L^{\wedge}$.

It is obvious that only Archimedean Riesz spaces can have Dedekind completions. Also, any two Dedekind completions of a Riesz space are necessarily Riesz isomorphic.

Definition 2.44:

1) A Riesz space L is said to have the σ-order continuity property if every positive operator from L to an arbitrary Archimedean Riesz space is σ-order continuous.
2) A Riesz space L has the order continuity property if every positive operator L to an Archimedean Riesz space is order continuous.

Theorem 2.45: For a Riesz space L, the followings are equivalent:

1) L has the σ-order continuity property.
2) Every lattice homomorphism from L to an Archimedean Riesz space is σ-order continuous.
3) Every uniformly closed ideal of L is a σ-ideal.

Definition 2.46: A Riesz space L is said to have the countable sup property if $\sup A \in L$, then there exists an at most countable subset B of A with $\sup B = \sup A$.

Theorem 2.47: An Archimedean Riesz space has the countable sup property if every σ-ideal is a band.

Theorem 2.48: For an Archimedean Riesz space L, the followings are equivalent:

1) L has the order continuity property.
2) Every lattice homomorphism from L into any Archimedean Riesz space is order continuous.
3) Every uniformly closed ideal of L is a band.
4) L has the σ-order continuity property and the countable sup property.

Theorem 2.49: Let G be an order dense Riesz subspace of an Archimedean Riesz space L. If G is Dedekind complete itself, then G is an ideal of L.

Proof: Let $0 \leq x \leq y \in G$ and $x \in L$. Since L is Archimedean and G is order dense in L, there exists a net $\{x_\alpha\} \subseteq G$ with $0 \leq x_\alpha \uparrow x$ in L. On the other hand, by the Dedekind completeness of G, we obtain that $0 \leq x_\alpha \uparrow z$ in G for some $z \in G$.Hence $x_\alpha \uparrow z$ in L. So, $x = z \in G$ and it shows that G is an ideal of L.

Definition 2.50: A Riesz space is called laterally complete if every subset of pairwise disjoint positive vectors has a supremum.

Definition 2.51: An operator $T: L \to L$ on a Riesz space is called band preserving if T leaves all the bands of L invariant, that is, if $T(B) \subseteq B$ for all B of L.

Theorem 2.52: For an operator $T: L \to L$ on Archimedean Riesz space the followings are equivalent:

1) T is band preserving.

2) If $x \perp y$, then $Tx \perp y$

3) For all $x \in L$, $Tx \in B_x$, where B_x is the band generated by x.

Proof:

(1)⟹(2) Let $x \perp y$. Then $x \perp B_x$. Since $T(B_x) \subseteq B_x$, $y \perp T(B_x)$. It follows that $Tx \perp y$.

(2)⟹(3) Let $x \in L$. So, for all $y \in B_x^d$, $x \perp y$, and hence $Tx \perp y$ for all $y \in B_x^d$. Then, $Tx \in B_x^{dd} = B_x$.

(3)⟹(1) Let B be a band of L. If $x \in B$, then $B_x \subseteq B$ and so $Tx \in B_x \subseteq B$. Hence $T(B) \subseteq B$ and T is band preserving.

Theorem 2.53: If L has the principal projection property, then an operator $T: L \to L$ is band preserving if and only if T commutes with every order projection of L.

Proof: Assume that T commutes with every order projection. Then for all $x \in L$,

$$T(x) = TP_x(x) = P_xT(x) \in B_x \tag{2.98}$$

and by the Theorem 2.52, T is band preserving.

For the converse, assume that T is band preserving, and let P_B be an order projection. If $x \in L$, the it can be written as $x = y + z \in B \oplus B^d$. So if $T(y) \in B$ and $T(z) \in B^d$. Therefore,

$$P_BT(x) = P_BTy + P_BTz = Ty = TP_B(x) \tag{2.99}$$

for all $x \in L$ and hence $P_BT = TP_B$.

Definition 2.54: An operator $T: L \to M$ between two Riesz spaces is said to preserve disjointness if $x \perp y$ in L implies $Tx \perp Ty$ in M.

Therefore, every band preserving operator preserves disjointness.

Theorem 2.55: If an order bounded operator $T: L \to M$ between two Riesz spaces with M Archimedean preserves disjointness, then its modulus exists and

$$|T|(|x|) = |T(|x|)| = |T(x)| \tag{2.100}$$

for all $x \in L$.

Proof: Let $T: L \to M$ be an order bounded and disjointness preserving operator between two Riesz spaces with M Archimedean. If $0 \leq y \leq x$, then

$$\begin{aligned} T(x) &= T(x-y) + T(y) \\ &= \left[\left(T(x-y)\right)^{+} + (Ty)^{+}\right] - \left[\left(T(x-y)\right)^{-} + (Ty)^{-}\right] \end{aligned} \tag{2.101}$$

and also since $(Tx)^{+} \wedge (Ty)^{-} = 0$ for all $x, y \in L^{+}$, we obtain that

$$\begin{aligned} |Tx| &= \left(T(x-y)\right)^{+} + (Ty)^{+} + \left(T(x-y)\right)^{-} + (Ty)^{-} \\ &= |T(x-y)| + |Ty| \geq |Ty| \end{aligned} \tag{2.102}$$

Let $|y| \leq x$ in L. Since $Ty^{+} \perp Ty^{-}$, it gives us that

$$|Ty| = |Ty^{+} - Ty^{-}| = |Ty^{+} + Ty^{-}| = |T(|y|)| \leq |Tx| \tag{2.103}$$

Then, $|Tx| = \sup\{|Ty|: |y| \leq x\}$ for all $x \in L^{+}$. So, the modulus of T exists and

$$|T|(x) = |T(x)| \tag{2.104}$$

for all $x \in L^{+}$.

On the other hand, if $x \in L$, then $T(x^{+}) \perp T(x^{-})$ and hence

$$\begin{aligned} |T(|x|)| = |T|(|x|) &= |T|(x^{+}) + |T|(x^{-}) = |T(x^{+})| + |T(x^{-})| \\ &= |\,T(x^{+}) - T(x^{-})| = |T(x)| \end{aligned} \tag{2.105}$$

Definition 2.56:

1) An orthomorphism is a band preserving operator that is also order bounded.

It also means that an order bounded $T: L \to L$ is an orthomorphism if and only if $x \perp y$ implies $Tx \perp y$. In this case, every orthomorphism preserves disjointness.

2) If an orthomorphism is also a positive operator is called a positive orthomorphism.

3) The vector space which is formed by the collection of all orthomorphisms in a Riesz space L is denoted by $Orth(L)$, that is,

$$Orth(L) = \{T \in \mathcal{L}_b(L): x \perp y \text{ implies } Tx \perp y\} \quad (2.106)$$

$Orth(L)$ is a vector subspace of $\mathcal{L}_b(L)$ and $Orth(L)$ with the ordering inherited from $\mathcal{L}_b(L)$ is an ordered vector space itself.

4) Let $T: L \to M$ be an operator between two Riesz spaces. Then the set

$$\{x \in L: T[0, |x|] \text{ is order bounded}\} \quad (2.107)$$

is nonempty and solid. This set is also a vector subspace of L and so it is an ideal. Hence it becomes the largest ideal on the order bounded operator T and is called the ideal of order boundness of T.

Theorem 2.57: If L is an Archimedean Riesz space, then $Orth(L)$ is an Archimedean Riesz space under the pointwise algebraic and lattice operations. That is, if $S, T \in Orth(L)$, then

$$[S \vee T](x) = S(x) \vee T(x) \quad \text{and} \quad [S \wedge T](x) = S(x) \wedge T(x) \quad (2.108)$$

for all $x \in L^+$.

Proof: Let $T: L \to L$ be an orthomorphism. So the modulus of T exists and satisfies $|T|(x) = |Tx|$ for all $x \in L^+$. Since $|T|$ is an orthomorphism, $Orth(L)$ is a Riesz space. The followings can be obtained from the lattice identities

$$S \vee T = \frac{1}{2}(S + T + |S - T|) \quad \text{and} \quad S \wedge T = \frac{1}{2}(S + T - |S - T|) \quad (2.109)$$

Theorem 2.58: Every orthomorphism on an Archimedean Riesz space is order continuous.

Proof: Let $T: L \to L$ be a positive orthomorphism on an Archimedean Riesz space and let $x_\alpha \downarrow 0$ in L. We may assume that $0 \leq x_\alpha \leq x$ for all α and some $x \in L$.

Suppose that some $y \in L^+$ satisfies $0 \leq y \leq T(x_\alpha) \leq T(x)$ for all α. Since $(x_\alpha - \varepsilon x)^+ \wedge (x_\alpha - \varepsilon x)^- = 0$ for all $\varepsilon > 0$, we obtain that

$$0 \leq (y - \varepsilon Tx)^+ \wedge (x_\alpha - \varepsilon x)^- \leq T(x_\alpha - \varepsilon x)^+ \wedge (x_\alpha - \varepsilon x)^- = 0 \quad (2.110)$$

Hence

$$0 = (y - \varepsilon Tx)^+ \wedge (x_\alpha - \varepsilon x)^- \uparrow_\alpha (y - \varepsilon Tx)^+ \wedge \varepsilon x \quad (2.111)$$

and so $(y - \varepsilon Tx)^+ \wedge x = 0$ for all α. Since L is Archimedean, $y \wedge x = 0$. Hence, $y = y \wedge T(x) = 0$ and so $T(x_\alpha) \downarrow 0$ in L and it proves that T is order continuous.

Since each orthomorphism on L is the difference of two positive orthomorphisms, every orthomorphism is order continuous.

Theorem 2.59: If L is a Dedekind complete Riesz space, then $Orth(L)$ coincides with the band generated by the identity operator in $\mathcal{L}_b(L)$.

Proof: Let B_I be the band generated by the identity operator I. So, $B_I \in Orth(L)$.

For the inverse, let $0 \leq T \in Orth(L)$. Since I is a weak order unit in the Archimedean Riesz space $Orth(L)$, we see that $T \wedge nI \uparrow T$. It follows from $\{T \wedge nI\} \subseteq B_I$ that $T \in B_I$. Hence $Orth(L) \subseteq B_I$ and we obtain that $Orth(L) = B_I$.

Theorem 2.60: If $T: L \to L$ is an orthomorphism on Archimedean Riesz space, then T extends uniquely to an orthomorphism $L^{\wedge}$.

Proof: We need to show that every positive orthomorphism on L extends to a positive orthomorphism $L^{\wedge}$. Therefore, let $T: L \to L$ be a positive orthomorphism. Then T is order continuous and

$$T^*(x) = \sup\{T(y): y \in L \text{ and } 0 \leq y \leq x\}, \quad 0 \leq x \in L^+ \tag{2.112}$$

defines an order continuous positive operator on L^+. Since L is order dense in L^+, T^* becomes the only linear order continuous extension of T to $L^{\wedge}$.

Let $x \wedge y = 0$ in $L^{\wedge}$. If $u, v \in L$ satisfy $0 \leq u \leq x$ and $0 \leq y \leq v$, then $u \wedge v = 0$ in L and hence $u \wedge Tv = 0$. Therefore,

$$\begin{aligned} x \wedge T^*y &= [\sup\{u \in L: 0 \leq u \leq x\}] \wedge [\sup\{Tv: v \in L \text{ and } 0 \leq v \leq y\}] \\ &= \sup\{u \wedge Tv: u, v \in L, 0 \leq u \leq x \text{ and } 0 \leq v \leq y\} = 0 \end{aligned} \tag{2.113}$$

Hence, T^* is an orthomorphism.

For each $T \in Orth(L)$, we will denote the unique orthomorphism in $Orth(L^{\wedge})$ by T^* which extends T linearly to all of $L^{\wedge}$. It means that a mapping $T \to T^*$ defined from $Orth(L)$ to $Orth(L^{\wedge})$ is linear and one-to-one. The image $T \to T^*$ is the Riesz subspace of $Orth(L^{\wedge})$ consisting of all orthomorphisms of $Orth(L^{\wedge})$ which leave L invariant. So, if we identify $Orth(L)$ with its image in $Orth(L^{\wedge})$ under the lattice isomorphism $T \to T^*$, then

$$Orth(L) = \{T \in Orth(L^{\wedge}): T(E) \subseteq E\} \tag{2.114}$$

Theorem 2.61: The kernel of an orthomorphism is a band. In particular, if two orthomorphisms agree on a set, then they also agree on the band generated by that set.

Proof: Let $T: L \to L$ be an orthomorphism. Then, $Ker(T) = \mathcal{N}_T$. Since T is order continuous, the ideal $\mathcal{N}_T$ is a band and so is $Ker(T)$.

If two isomorphisms S and T agree on a set A, then the orthomorphism $R = S - T$ is zero on A, and so $A \subseteq Ker(R)$. Since $Ker(R)$ is a band, the band generated by A is in $Ker(R)$. Hence S and T agree on the same band.

Definition 2.62:

1) A Riesz space L under an associative multiplication is called a Riesz algebra if the multiplication forms L as an algebra and additionally, with the property that if $x, y \in L^+$, then $xy \in L^+$.
2) A Riesz algebra L is called an f-algebra if $x \wedge y = 0$ implies $(xz) \wedge y = (zx) \wedge y = 0$ for all $x \in L^+$.

   Let L be an f-algebra. Then for each $y \in L^+$, the mappings $x \to xy$ and $x \to yx$ are positive orthomorphisms. In particular, for all $y \in L$, the mappings

   $$x \to xy = xy^+ - xy^- \quad \text{and} \quad x \to yx = y^+x - y^-x \qquad (2.115)$$

   are both orthomorphisms on L.

Theorem 2.63: In any f-algebra if $x \perp y$, then $xy = yx = 0$.

Proof: Let $x \wedge y = 0$. Then $(xy) \wedge y = 0$ and hence $(xy) \wedge (xy) = 0$. So, if $x \perp y$,

$$xy = (x^+ - x^-)(y^+ - y^-) = x^+y^+ - x^-y^+ - x^+y^- + x^-y^- = 0 \quad (2.116)$$

In the same way, $yx = 0$.

Corollary 2.64: If x is a vector in an f-algebra, then $x^2 = (x^+)^2 + (x^-)^2 \geq 0$

Theorem 2.65: Every Archimedean f-algebra is commutative.

Proof: Let $y \in L$. Then, $T(x) = xy$ and $S(x) = yx$ define orthomorphisms on L. Since $S(y) = T(y) = y^2$, $S = T$ on L_y. Also, if $x \in E_y^d$, then $S(x) = T(x) = 0$. So, $S = T$ on the order dense ideal $E_y \oplus E_y^d$ and also on L. That is $xy = yx$ for all $x, y \in L$.

Let L be an Archimedean f-algebra with a multiplicative unit vector e. Since $e = e^2 \geq 0$, e has to be a positive vector. Also, since $e \wedge x = 0$ implies $x = x \wedge x = (xe) \wedge x = 0$, it concludes that e is a weak order unit and so $x \wedge ne \uparrow x$ for all $x \in L^+$.

Let L be an Archimedean f-algebra. Then a mapping $u \to T_u$ from L to $Orth(L)$ defined by $T_u(x) = ux$ is a multiplication preserving operator. If L has a multiplicative unit, then $u \to T_u$ is an onto f-isomorphism.

Theorem 2.66: Let L be an Archimedean f-algebra with a multiplicative unit. Then $u \to T_u$ is an f-isomorphism from L onto $Orth(L)$, and so, according to this f-isomorphism, $Orth(L) = L$.

In particular, for all Archimedean Riesz space L, $Orth\big(Orth(L)\big) = Orth(L)$.

Proof: Let $e > 0$ be the multiplicative unit vector of L. Also, let $T \in Orth(L)$. Then the orthomorphism $S(x) = T(x) - T(e)x$ satisfies $S(e) = 0$. Since e is a weak order unit, $B_e = L$ and so $S = 0$ on L. Hence $T(x) = T(e)x$ for all $x \in L$. It means that $T = T_u$ for $u = T(e)$ and hence $u \to T_u$ is onto.

Since $T_u \geq 0$ if and only if $u \geq 0$, it can be obtained that $u \to T_u$ is one to one.

Definition 2.67:

1) A norm $\|.\|$ on a Riesz space is said to be a lattice norm if $|x| \leq |y|$ implies $\|x\| \leq \|y\|$.
2) A Riesz space equipped with a lattice norm is called a normed Riesz space.
3) If a normed Riesz space is also norm complete, it is called Banach lattice.

In a normed Riesz space $\|x\| = \||x|\|$ for all x.

Theorem 2.68: The norm dual of a normed Riesz space is a Banach lattice.

Proof: Let L be a normed Riesz space. Since E' is an ideal of $E^{\sim}$, E' is a Riesz space itself. And as E' is also a Banach space, we need to show that the norm of E' is a lattice norm.

Let $|x'| \leq |y'|$ in E'. So,

$$|x'(x)| \leq |x'|(|x|) \leq |y'|(|x|) = \sup\{|y'(y)|: |y| \leq |x|\} \leq \|y'\|.\|x\| \quad (2.117)$$

Then, $\|x'\| \leq \|y'\|$. (Aliprantis and Burkinshaw, 2006)

3.MAHARAM OPERATORS

Definition 3.1: Let$L$and$M$be ArchidemeanRiesz spaces with$M$Dedekind complete. Anoperator$T \in \mathcal{L}_b(L, M)$is a Maharam operator if$|T|$is interval preserving, that is, if $0 \leq w \leq |T|u$ in M implies that $0 \leq v \leq u$ in L such that $w = |T|v$.(Luxemburg andBurkinshaw, 2001)

Let $L$and$M$be ArchidemeanRiesz spaces with$M$Dedekind complete and let a Boolean ring homomorphism $\sigma_T: \mathcal{B}(M) \to \mathcal{B}(L)$where$\mathcal{B}(L)$and $\mathcal{B}(M)$are the complete Boolean algebra of bands in $L$and M, respectively, be defined such as

$$\sigma_T(B) = \{f \in L: |T|(|f|) \in B\} \cap \mathcal{C}_T \text{for} B \in \mathcal{B}(M) \quad (3.1)$$

$\sigma_T(B) \in \mathcal{B}(L)and\sigma_T(B)$is a band.

Lemma 3.2: The mapping$\sigma_T: \mathcal{B}(M) \to \mathcal{B}(L)$is an order continuous Boolean ring homomorphism.

Proof: Since$\sigma_T = \sigma_{|T|}$it follows that$T \geq 0$. Also, the following results can be obtained from the definition ofσ_T:

$$\sigma_T(\{0\}) = \{0\} \quad (3.2)$$

$$\sigma_T(B_1 \cap B_2) = \sigma_T(B_1) \cap \sigma_T(B_2) \quad (3.3)$$

$$\sigma_T(B_1) \vee \sigma_T(B_2) \subseteq \sigma_T(B_1 \vee B_2) \quad (3.4)$$

To prove that$\sigma_T$is a Boolean ring homomorphism, we need to show that

$$\sigma_T(B_1 \vee B_2) \subseteq \sigma_T(B_1) \vee \sigma_T(B_2) \text{for} B_1, B_2 \in \mathcal{B}(M) \quad (3.5)$$

If we take $0 \leq f \in \sigma_T(B_1 \vee B_2)$, then$Tf \in B_1 \vee B_2$ .Since M is Dedekind complete, $B_1 \vee B_2 = B_1 + B_2$and it follows that$Tf = w_1 + w_2$for$0 \leq w_j \in B_j (j = 1,2)$. At this point,if we use the Maharam property of the operator$T$, we obtain that there exists $0 \leq u_1 \leq f$in$L$such that$Tu_1 = w_1$.Let us choose$u_2 = f - u_1$, then $u_j \in \sigma_T(B_j)(j = 1,2)$, and so$f = u_1 + u_2 \in \sigma_T(B_1) + \sigma_T(B_2) \subseteq \sigma_T(B_1) \vee \sigma_T(B_2)$. If$f \in$

$\sigma_T(B_1)\vee\sigma_T(B_2)$whenever $0 \le f \in \sigma_T(B_1\vee B_2)$, it is obvious that$\sigma_T(B_1)\vee\sigma_T(B_2) \subseteq \sigma_T(B_1\vee B_2)$, and so$\sigma_T(B_1)\vee\sigma_T(B_2) = \sigma_T(B_1\vee B_2)$

Therefore,$\sigma_T$is a Boolean ring homomorphism.

It is only left to show that$\sigma_T$is order continuous. Since $T\big(\sigma_T(B)\big) \subseteq B$for all$B \in \mathcal{B}(M)$,$\sigma_T(B_\tau) \downarrow 0$whenever $B_\tau \downarrow 0$in $\mathcal{B}(M)$. So,$\sigma_T$is order continuous.(Luxemburg andBurkinshaw, 2001)

Definition 3.3:

1) An operator $T: L \to L$ on a Riesz space is called center if and only if there exists some scalar $\lambda > 0$ such that $|Tx| \le \lambda x$ for all $x \in L$.
2) Thecollection of allcenteroperators is calledthecenter of theBanachlattice$L$ and denoted by $Z(E)$:
   $Z(E) = \{T \in \mathcal{L}_r(L): |T| \le \lambda I \; for \; some \; \lambda \ge 0\}$ (3.6)

Every center operator is regular and every positive center operator is a lattice homomorphism so $Z(E)$ is an ideal of $\mathcal{L}_r(L)$. Actually, the center operators are a special kind of orthomorphisms. (Pestil, 2000)

In this situation, if$L$is also Dedekind complete, the Boolean algebra $\mathcal{B}(L)$of bands in$L$isomorphic to the Boolean algebra$\mathcal{P}(L)$of band projections in$L$. The same argument applies also to $\mathcal{B}(M)$and $\mathcal{P}(M)$. So, the Boolean ring homomorphismσ_T, defined as above, induces a corresponding Boolean ring homomorphism$h_T: \mathcal{P}(M) \longrightarrow \mathcal{P}(L)$. Here,$h_T(I) = C_T$ and it follows that$Th_T(I) = T$. In addition,$PTh_T(P) = Th_T(P)$for all$P \in \mathcal{P}(M)$since $T\big(\sigma_T(B)\big) \subseteq B$for all$B \in \mathcal{B}(M)$. If we replace Pbe$I - P$in this equation and use that$Th_T(I) = T$, it is obtained that$PT = Th_T(P)$and $h_T(P) \le C_T$for all$P \in \mathcal{P}(M)$. If we use Freudenthal's spectral theorem, it gives us that $h_T$extends to an order continuous$f$-algebrahomomorphism from$Z(M)$into$Z(L)$which we denote by$h_T$again.

By using Lemma 3.2, it follows that $\Pi T = Th_T(\Pi)$and$h_T(\Pi)C_T = h_T(\Pi)$for all$\Pi \in Z(M)$.

Proposition 3.4: Let$L$and$M$be Dedekind complete Riesz spaces and$T \in \mathcal{L}_n(L, M)$.

(1) If $T$is a Maharam operator, then there exists a unique$f$-algebra homomorphism$h: Z(M) \longrightarrow Z(L)$satisfying $\Pi T = Th_T(\Pi)$and$h_T(\Pi)C_T = h_T(\Pi)$for all$\Pi \in Z(M)$.

(2) If there exists an$f$-algebra homomorphism$h: Z(M) \longrightarrow Z(L)$such that$\Pi(T) = Th_T(\Pi)$for all$\Pi \in Z(M)$, then$T$is a Maharam operator.

Proof:

(1) Since we have already established the existence, we only need to prove that such an$f$-algebra homomorphism is unique.If we take$\Pi_1, \Pi_2 \in Z(L)$such that$T\Pi_1 = T\Pi_2$and $\Pi_j C_T = \Pi_J$ $(j = 1,2)$, then it follows that $|T|.|\Pi_1 - \Pi_2| = 0$whenever $T(\Pi_1 - \Pi_2) = 0$. So, $|\Pi_1 - \Pi_2| = |\Pi_1 C_T - \Pi_2 C_T| = |\Pi_1 - \Pi_2|C_T = 0$,and, it shows that $\Pi_1 = \Pi_2$.

(2) Let us define$S \to Sh(\Pi)$and $S \to \Pi S$for $\Pi \in Z(M)$as center operators in$\mathcal{L}_n(L, M)$. It follows that $\{S \in \mathcal{L}_n(L, M): \Pi S = Sh(\Pi)\}$is a band in$\mathcal{L}_n(L, M)$. So, by the hypothesis on$T$, we obtain that$\Pi|T| = |T|h(\Pi)$ for all$\Pi \in Z(M)$. Now, let us choose$0 \leq u \in L$and $0 \leq w \leq T|u|$in$M$. Since $M$is Dedekind complete, there exists$0 \leq \Pi \leq I$ in $Z(M)$ such that$w = \Pi|T|u = |T|(h(\Pi)u)$. In addition, as$0 \leq h(\Pi) \leq h(I) \leq I$, $0 \leq h(\Pi)u \leq u$, we conclude that $|T|$is a Maharam operator, so is$T$.

Definition 3.5 : Let $L$and Mbe ArchidemeanRiesz spaces with$M$Dedekind complete and let$h: Z(M) \to Z(L)$be an$f$-algebra homomorphism. Let us define$\mathcal{L}_n^h(L, M)$such that

$$\mathcal{L}_n^h(L, M) = \{T \in \mathcal{L}_n(L, M): \Pi T = Th(\Pi)\ \forall \Pi \in Z(M)\} \qquad (3.7)$$

Lemma 3.6: $\mathcal{L}_n^h(L, M)$ is a band in$\mathcal{L}_n(L, M)$.

Proof: Let us define the operators$L_\Pi, R_\sigma \in Z\big(\mathcal{L}_n(L, M)\big)$ for$\Pi \in Z(M)$and $\sigma \in Z(L)$ by$L_\Pi(T) = \Pi T$and $R_\sigma(T) = T\sigma$respectively for all$\mathcal{L}_n(L, M)$. Therefore,

$$\mathcal{L}_n^h(L, M) = \cap \{ker\big(L_\Pi - R_{h(\Pi)}\big): \Pi \in Z(M)\} \qquad (3.8)$$

As we know that the kernel of any center operator is a band, it concludes that $\mathcal{L}_n^h(L, M)$is a band in$\mathcal{L}_n(L, M)$.

Corollary 3.7: If L and M are Dedekind complete Riesz spaces and $T \in \mathcal{L}_n(L,M)$ is a Maharam operator, then all operators $S \in \{T\}^{dd}$ are Maharam.

Proof: If $h = h_T$ is the f-algebra homomorphism associated with T as in Proposition 3.4 (1), then $T \in \mathcal{L}_n^h(L,M)$ and it follows that $\{T\}^{dd} \subseteq \mathcal{L}_n^h(L,M)$ since $\mathcal{L}_n^h(L,M)$ is a band. If we consider the second part of Proposition 3.4, it concludes that all operators in $\mathcal{L}_n^h(L,M)$ are Maharam.

If we choose $M = \mathbb{R}$, $Z(M) = \mathbb{R}$ and then $L_n^{\sim} = \mathcal{L}_n(L,M) = \mathcal{L}_n^h(L,M)$, where $h(\alpha) = \alpha I$ for all $\alpha \in \mathbb{R}$. In this situation, $L_n^{\sim}$ has new properties which do not hold for $\mathcal{L}_n(L,M)$ in general. For example, $C_\psi = C_\varphi$ whenever $\psi \perp \varphi$ in $L_n^{\sim}$ and there is a natural correspondence between bands in $L_n^{\sim}$ and in L.

Lemma 3.8: Let L and M are Dedekind complete Riesz spaces and $h: Z(M) \to Z(L)$ an f-algebra homomorphism. Suppose that $S, T \in \mathcal{L}_n^h(L,M)$ with $S \wedge T = 0$. If $0 \le u \in L$ and $0 \le w \in M$ such that $Tu \in \{w\}^{dd}$, then there exists $0 \le v \le u$ in L such that $Tv + S(u-v) \le w$.

Proof: Let us define

$$\mathcal{Q} = \{Q \in \mathcal{P}(M): \exists 0 \le v \le u \text{ in } L \text{ such that } Q[Tv + S(u-v)] \le w\} \quad (3.9)$$

If $I \in \mathcal{Q}$, the proof will be done.

We need to use the following observations to complete the proof:

(i) Let $0 \neq P \in \mathcal{P}(M)$, $0 \neq Q \in \mathcal{Q}$ such that $Q \le P$.

Proof: If $Pw = 0$, then $PTu = 0$ and so $P \in \mathcal{Q}$.

Now let $Pw \ge 0$. Then, if $S \wedge T = 0$, there exists $0 \le v \le u$ such that $Z = \{Pw - [Tv + S(u-v)]\}^T > 0$. Here, if Q is a band projection in M onto $\{z\}^{dd}$, then $0 \neq Q \le P$ and $Q[Tv + S(u+v)] \le Pw \le w$. So we reach the conclusion that $Q \in \mathcal{Q}$ and the claim is proved.

(ii) Let $\{Q_\alpha : \alpha \in A\}$ be a disjoint system in $\mathcal{Q}$, so $Q = \bigvee_\alpha Q_\alpha \in \mathcal{Q}$.

Proof: Let $0 \le v_\alpha \le u$ be such that $Q_\alpha[Tv_\alpha + S(u-v_\alpha)] \le w$ and also let us define $= \bigvee_\alpha h(Q_\alpha)v_\alpha$. Since h is an f-algebra homomorphism, $\{h(Q_\alpha): \alpha \in A\}$ is a disjoint system in $\mathcal{P}(L)$. So, $h(Q_\alpha)v = h(Q_\alpha)v_\alpha$. It follows that

$$Q_\alpha[Tv + S(u-v)] = Th(Q_\alpha)v_\alpha + S(h(Q_\alpha)u - h(Q_\alpha)v_\alpha)$$

$= Q_\alpha[Tv_\alpha + S(u - v_\alpha)] \leq w$ (3.10)

for all$\alpha \in A$. Therefore,$Q[Tv + S(u - v)] \leq w$and $Q \in \mathcal{Q}$.

Now, to prove the Lemma 3.8, if we take a maximal disjoint system$\{Q_\alpha\}$in $\mathcal{Q}$, we obtain$\{Q_\alpha\}$as a maximal disjoint system in$\mathcal{P}(M)$by using (i). Therefore,$\bigvee_\alpha Q_\alpha = I$and by using (ii), it concludes that $I \in \mathcal{Q}$.

Proposition 3.9: Let$L$and Mbe ArchidemeanRiesz spaces with$M$Dedekind complete and let$h: Z(M) \to Z(L)$be an$f$-algebra homomorphism. For $S, T \in \mathcal{L}_n^h(L, M)$, the following statements are equivalent:

(1) $S \perp T$

(2) $\mathcal{C}_S \perp \mathcal{C}_T$

Proof:

(1)⟹(2): Let$S, T \geq 0$. In addition, first, let us take$L$as Dedekind complete. If we take $0 \leq u \in \mathcal{C}_T$and $w = Tu$, then by using Lemma 3.8, we obtain that there exists$0 \leq v_n \leq u$such that$Tv_n + S(u - v_n) \leq 2^{-n}w$. Then,$0 \leq Tz_n \leq \sum_{k=n}^{\infty} 2^{-k}w = 2^{-n+1}w$since $T$is order continuous.

Here, if we take $z_n = inf_n z_n$, then$z_n \downarrow z$, so$Tz_n \downarrow Tz$and $TZ = 0$. So,$0 \leq z \in \mathcal{C}_T$whenever $0 \leq z \in u$and it follows that$z = 0$, i.e. , $z_n \downarrow 0$. Hence,$0 \leq u - z_n \uparrow u$and it gives us that $S(u - z_n) \uparrow S$. Since$0 \leq S(u - z_n) \leq S(u - v_n) \leq 2^{-n}w$for all$= 1,2, \ldots$, we obtain that$Su = 0$, i.e. ,$u \in \mathcal{N}_S$and it proves that$\mathcal{C}_T \subseteq \mathcal{N}_S$and so$\mathcal{C}_S \perp \mathcal{C}_T$.

Now let us take$L$as only an Archimedean Riesz space and let$L^\wedge$be the Dedekind completion of$L$. Every operator$R \in \mathcal{L}_n(L^\wedge, M)$ has a unique extension$R^\wedge \in \mathcal{L}_n(L^\wedge, M)$ and the mapping $R \to R^\wedge$is a Riesz isomorphism from$\mathcal{L}_n(L, M)$onto $\mathcal{L}_n(L^\wedge, M)$. Similarly, every$\Pi \in Z(L)$has a unique extension$\Pi^\wedge \in Z(L^\wedge)$and the mapping$\Pi \to \Pi^\wedge$defines an$f$-algebra isomorphism embedding of$Z(L)$into$Z(L^\wedge)$. So, we can consider the$f$-algebra homomorphism $h: Z(M) \to Z(L)$as an$f$- algebra homomorphism from$Z(M)$into $Z(L^\wedge)$and so the above isomorphism$R \to R^\wedge$maps $\mathcal{L}_n^h(L, M)$onto $\mathcal{L}_n^h(L^\wedge, M)$. Moreover,$\mathcal{N}_R = \mathcal{N}_{R^\wedge} \bigcap L$and $\mathcal{C}_R = \mathcal{C}_{R^\wedge} \bigcap L$for all $R \in \mathcal{L}_n(L, M)$. If$S, T \in \mathcal{L}_n^h(L, M)$ such that$S \perp T$, then$S^\wedge \perp T^\wedge$in $\mathcal{L}_n^h(L^\wedge, M)$and it follows that $\mathcal{C}_{S^\wedge} \perp \mathcal{C}_{T^\wedge}$and so$\mathcal{C}_S \perp \mathcal{C}_T$.

(2)⟹(1): Let$\mathcal{C}_S \perp \mathcal{C}_T$, then$\mathcal{C}_S \subseteq \mathcal{N}_T$and so the order continuous operator$|S| \wedge |T|$vanishes on the order dense ideal$\mathcal{C}_S \oplus \mathcal{N}_S$. It follows that$|S| \wedge |T| = 0$and hence$S \perp T$.

Let us denote$E = \mathcal{L}_n^h(L, M)$, where $L, M$and $h$are as in the Proposition 3.9. So, we define the absolute null ideal of $B$by

$\mathcal{N}_B = \{f \in L: |T|(|f|) = 0 \ \forall T \in B\}$(3.11)

for any band $B \subseteq E$. Here,$\mathcal{N}_B$is a band and$\mathcal{N}_B = \wedge \{\mathcal{N}_T: T \in B\}$in $\mathcal{B}(L)$.

Similarly, the carrier of$B$; $\mathcal{C}_B = \mathcal{N}_B^d$is defined by

$\mathcal{C}_B = \vee \{\mathcal{C}_T: T \in B\}$ (3.12)

in $\mathcal{B}(L)$. So, if $B = \{T\}^{dd}$for some$T \in E$, then$\mathcal{N}_B = \mathcal{N}_T$ and$\mathcal{C}_B = \mathcal{C}_T$.

Proposition 3.10: If the same notation is used as above, the following holds:

- **(1)** The mapping$\tau: \mathcal{B}(E) \to \mathcal{B}(L)$defined by $\tau(B) = \mathcal{C}_B$, is an injective Boolean ring homomorphism with$\tau\big(\mathcal{B}(E)\big) = \mathcal{B}(\mathcal{C}_E)$
- **(2)** For all$B \in \mathcal{B}(E)$, we have$B = \{T \in \mathcal{L}_n^h(L, M): \mathcal{C}_T \subseteq \mathcal{C}_B\}$

In particular, if$S, T \in \mathcal{L}_n^h(L, M)$, then$S \in \{T\}^{dd}$if and only if$\mathcal{C}_S \subseteq \mathcal{C}_T$.

Proof:

(1) If $B_1, B_2 \in \mathcal{B}(E)$, then$\mathcal{N}_{B_1 \vee B_2} = \mathcal{N}_{B_1} \cap \mathcal{N}_{B_2}$and so$\mathcal{C}_{B_1 \vee B_2} = \mathcal{C}_{B_1} \vee \mathcal{C}_{B_2}$which implies that$\tau(B_1 \vee B_2) = \tau(B_1) \vee \tau(B_2)$. Let us take$B_1, B_2 \in \mathcal{B}(E)$such as $B_1 \cap B_2 = \{0\}$. So,$S \perp T$for all$S \in B_1$and$T \in B_2$and hence$\mathcal{C}_S \perp \mathcal{C}_T$by Proposition 3.9. It follows that

$\mathcal{C}_{B_1} = \vee \{\mathcal{C}_S: S \in B_1\} \perp \vee \{\mathcal{C}_T: T \in B_2\} = \mathcal{C}_{B_2}$ (3.13)

and so,$\tau(B_1) \wedge \tau(B_2) = 0$. From these properties ofτ, we conclude that$\tau$is a Boolean ring homomorphism.

If $B \in \mathcal{B}(E)$and $\tau(B) = 0$, then$\mathcal{N}_B = L$. Hence$B = \{0\}$and it gives us that$\tau$is injective. Now let us take$A \in \mathcal{B}(\mathcal{C}_E)$and define$B = \{T \in \mathcal{L}_n^h(L, M): \mathcal{C}_T \subseteq A\}$. So,$B \in \mathcal{B}(E)$and $\tau(B) \subseteq A$. If$A_1 = A \cap \tau(B)^d$ and$B_1$is defined by$B_1 = \{T \in \mathcal{L}_n^h(L, M): \mathcal{C}_T \subseteq A_1\}$. Then$B_1 \in \mathcal{B}(E)and\tau(B_1) \subseteq A_1$. Therefore,$\tau(B_1 \cap B_2) = \tau(B_1) \cap \tau(B_2) = \{0\}$and it follows that$B_1 \cap B = \{0\}$. Here,if $A_1 \subseteq A$, then$B_1 \subseteq$

$B$and so $B_1 = \{0\}$. Now let$A_1 \neq \{0\}$. $A_1 \subseteq \mathcal{C}_E$and so there exists$0 \leq T \in \mathcal{L}_n^h(L, M)$such that$T_{|A|} \neq 0$. If$T_1$is the minimal positive extension of the restriction of TofA_1, then $T_1 \neq 0$and $A_1^d \subseteq \mathcal{N}_T$, so$\mathcal{C}_{T_1} \subseteq A_1$. It follows that$0 \neq T_1 \in B_1$, which is a contradiction. Therefore,$A_1 = \{0\}$and $A \subseteq \tau(B)$. So, $\tau(B) = A$. It gives us that$\tau\big(\mathcal{B}(E)\big) = \mathcal{B}(\mathcal{C}_E)$.

(2) Let $B \in \mathcal{B}(E)$and B_0be defined by

$$B_0 = \{T \in \mathcal{L}_n^h(L, M) : \mathcal{C}_T \subseteq \mathcal{C}_B\} \qquad (3.14)$$

If we apply the second part of the proof of (1) above to$A = \mathcal{C}_B$, it gives us that$B_0 \in \mathcal{B}(E)$ satisfies$\tau(B_0) = \mathcal{C}_B$. It follows that$B = B_0$ since $\tau$is injective.

Remark 3.11: Let us take$L$as Dedekind complete in the above argument. For$B \in \mathcal{B}(E)$, let us denote by$N_B$and$\mathcal{C}_B$, the band projections in$L$onto $\mathcal{N}_B$ and$\mathcal{C}_B$respectively. Here, we obtain that $T = T\mathcal{C}_B$for all$T \in B$and that$\mathcal{C}_T \leq \mathcal{C}_E \leq h(I)$for all $T \in E = \mathcal{L}_n^h(L, M)$.

Let $B \in \mathcal{B}(E)$and $h_B : Z(M) \longrightarrow Z(L)$be defined by$h_B(\Pi) = h(\Pi)\mathcal{C}_B$for all$\Pi \in Z(M)$. It follows that$h_B$is an$f$-algebra homomorphism and that$B \subseteq \mathcal{L}_n^{h_B}(L, M)$. Here $B = \mathcal{L}_n^{h_B}(L, M)$. Indeed, if$T \in \mathcal{L}_n^{h_B}(L, M)$, then $\mathcal{C}_T \leq h_B(I) = \mathcal{C}_B$and hence$T \in B$by (2) of the Proposition 3.10.

Proposition 3.12: Let $L$and $M$are Dedekind complete Riesz spaces and$h : Z(M) \longrightarrow Z(L)$ an$f$-algebra homomorphism. Suppose that$0 \leq T \in \mathcal{L}_n^h(L, M)$.

- **(1)** If S is a component of T, then $S = T\mathcal{C}_S$.
- **(2)** If $0 \leq S \leq T$, then there exists$0 \leq \Pi \leq I$in $Z(L)$such that$S = T\Pi$

Proof:

(1) We have that$\mathcal{C}_S\mathcal{C}_{T-S} = 0$by Proposition 3.9. Hence$S = \mathcal{C}_S$and$-S = (T - S)\mathcal{C}_{T-S}$. It follows that$T\mathcal{C}_S = S\mathcal{C}_S + (T - S)\mathcal{C}_S = S + (T - S)\mathcal{C}_{T-S}\mathcal{C}_S = S$

(2) If we combine the results from (1) with the Freudenthal Spectral Theorem, the proof is complete.

In particular, if we take$T \in \mathcal{L}_n^h(L, M)$in the Proposition 3.12, it gives us that

$$T^+ = |T|\mathcal{C}_{T^+} = T\mathcal{C}_{T^+} \quad , \quad T^- = |T|\mathcal{C}_{T^-} = T\mathcal{C}_{T^-} \qquad (3.15)$$

Proposition 3.13: Let $L$and$M$be Dedekind complete Riesz spaces and suppose that$\mathcal{J} \subseteq \mathcal{L}_n(L,M)$ is an ideal which consists of Maharam operators. Then there exists an f-algebra homomorphism $h: Z(M) \longrightarrow Z(L)$such that$\mathcal{J} \subseteq \mathcal{L}_n^h(L,M)$.

Proof: Before proving the proposition, we need to point out the next three observations:

Let $S, T \in \mathcal{J}$.

- **(i)** If $S \perp T$, then$C_S C_T = 0$
- **(ii)** If $C_S \leq C_T, S \in \{T\}^{dd}$
- **(iii)** $SC_T$is a component of Sin $\{T\}^{dd}$.

Let $\{T_\alpha\}$be a maximal disjoint system in$\mathcal{J}$and let the carrier projection ofT_α be denoted byC_α.

From the observation (i), we have that$C_\alpha C_\beta$whenever $\alpha \neq \beta$. By Proposition 3.5 and Lemma 3.6, for eachα, there exists an$f$-algebra homomorphism$h_\alpha: Z(M) \longrightarrow Z(L)$which satisfies$\Pi T = T h_\alpha(\Pi)$for all $T \in \{T\}^{dd}$ and all$\Pi \in Z(M)$. Without loss of generality, we may assume that$h_\alpha(I) = C_\alpha$. Therefore $h_\alpha(\Pi) \perp h_\beta(\Pi)$ for all$\Pi \in Z(M)$if $\alpha \neq \beta$. Now let uschoose $0 \leq \Pi \in Z(M)$, and so we have$0 \leq h_\alpha(\Pi) \leq \lambda C_\alpha$for all αsince $0 \leq \Pi \in \lambda I$for some$0 \leq \lambda \in \mathbb{R}$. It follows that$h(\Pi) = \vee_\alpha h_\alpha(\Pi)$is well defined in$Z(L)$. It defines a mapping $h: Z(M)^+ \longrightarrow Z(L)^+$such that$h$extends to an$f$-algebra homomorphism$h: Z(M) \longrightarrow Z(L)$. So,$\Pi T = Th(\Pi)$for all $\Pi \in Z(M)$ and all$T \in \mathcal{J}$. To prove this, it is sufficient to show it for$0 \leq \Pi \in Z(M)$and $0 \leq T \in \mathcal{J}$. If we use the observation (iii) and$\{T_\alpha\}$'s being a maximal disjoint system in $\mathcal{J}$, we obtain that$TC_\alpha \in \{T_\alpha\}^{dd}andT = \vee_\alpha TC_\alpha$. Since left and right multiplication in$\mathcal{L}_n(L,M)$ by center operators in$Z(M)$ and $Z(L)$respectively are itself operators in$\mathcal{L}_n(L,M)$, it follows that

$$\Pi T = \Pi \vee_\alpha TC_\alpha = \vee_\alpha \Pi(C_\alpha) = \vee_\alpha (TC_\alpha) h_\alpha(\Pi)$$

$$= \vee_\alpha (TC_\alpha) h(\Pi) = [\vee_\alpha TC_\alpha] h(\Pi) = Th(\Pi) \quad (3.16)$$

This proves that $\Pi T = Th(\Pi)$ for all$\Pi \in Z(M)$ and all$T \in \mathcal{J}$and the proof of the proposition is complete.

Corollary 3.14: Let $L$and Mbe Dedekind complete Riesz spaces and suppose that$\mathcal{J} \subseteq \mathcal{L}_n(L, M)$ is an ideal which consists of Maharam operators. Then the band generated by $\mathcal{J}$ in $\mathcal{L}_n(L, M)$also consists of Maharam operators.

Remark 3.15: Let$\mathcal{J}$be an ideal in$\mathcal{L}_n(L, M)$ consisting of Maharam operators as in Proposition 3.13, and let$h: Z(M) \longrightarrow Z(L)$be the f-algebra homomorphism as constructed in the proof of the proposition. If $B = \mathcal{J}^{dd}$is the band generated by $\mathcal{J}$, then $\mathcal{J} \subseteq B \subseteq \mathcal{L}_n^h(L, M)$. Since$\{T_\alpha\}$is a maximal disjoint system in$B$as well, it follows that$C_B = \vee_\alpha C_\alpha$. So,$h(I) = \vee_\alpha C_\alpha = C_B$. This implies that $B = \mathcal{L}_n^h(L, M)$.(Luxemburg andBurkinshaw, 2001).

4. f- MODULES

Let A be an Archimedean f-algebra and E an Archimedean Riesz space.

Definition 4.1: E is called a left f-module over A if:

- **(i)** E is a left-module over A with respect to a left multiplication $(a, f) \to af$ from $A \times E$ into E, i.e. , $(a+b).f = a.f + b.f$, $a.(f+g) = a.f + a.g$, $a.(b.f) = (ab).f$ and $a(\lambda f) = (\lambda a).f = \lambda(a.f)$ for all $a, b \in A$, $f, g \in E$ and $\lambda \in \mathbb{R}$;
- **(ii)** $a.f \geq 0$ whenever $0 \leq a \in A$ and $0 \leq f \in E$.
- **(iii)** If $f \perp g$ in E, $af \perp g$ in E for all $a \in A$.

The right f-module can be defined in a similar fashion.

Remark 4.2: Let E be an f-module over the f-algebra A and for $a \in A$ let us define $\Pi_\alpha(f) = a.f$ for all $f \in E$. From the Definition 4.1, we have that $\Pi_\alpha \in Orth(E)$. The mapping $h: A \to Orth(E)$ is a positive algebra homomorphism and so h is a Riesz homomorphism. (Indeed, $a \perp b$ in A implies that $ab = 0$, so $\Pi_\alpha \Pi_\beta = 0$ and hence $\Pi_\alpha \perp \Pi_\beta$ in $Orth(E)$).

Conversely, if E is an Archimedean Riesz space, A is an Archimedean f-algebra and $h: A \to Orth(E)$ is an f-algebra homomorphism, then h induces an f-module structure over A on E by setting $a.f = h(a)f$ for all $f \in E$ and all $a \in A$.

Lemma 4.3: Let E be an f-module over A.

- **(1)** $(a \vee b)f = (af) \vee (bf)$ and $(a \wedge b)f = (af) \wedge (bf)$ for all $a, b \in A$, $0 \leq f \in E$.
- **(2)** $|af| = |a|.|f|$ for all $a \in A$, $f \in E$.
- **(3)** If $a \perp b$ in A, then $af \perp bg$ for all $f, g \in E$.

In definition, it is stressed that even if f-algebra A has a unit element e, we do not assume that $e.f = f$ for all $f \in E$. It follows that if $e \in A$ is a unit element, then the

mapping $f \to e.f$ in E is a band projection. If $e.f = f$ for all $f \in E$, then E is called a unital f-module.

In addition, if $\mathcal{J} \subseteq E$ is a band, or more generally uniformly closed ideal in the f-module E over A, then $af \in \mathcal{J}$ for all $a \in A$ whenever $f \in \mathcal{J}$ and hence $\mathcal{J}$ inherits the f-module structure such that $\mathcal{J}$ is an f-submodule of E. If A has a unit element $0 < e \in A$ which is also a strong order unit in A, then every ideal $\mathcal{J} \subseteq E$ is an f-submodule.

Definition 4.4: : Let A be an Archimedean f-algebra and let E and F be f-modules over A. A linear mapping $T: E \to F$ is called $A - linear$ if $T(a.f) = a.Tf$ for all $f \in E$ and all $a \in A$.

Then, $\mathcal{L}_b^A(E,F)$ and $\mathcal{L}_n^A(E,F)$ can be defined as

$$\mathcal{L}_b^A(E,F) = \{T \in \mathcal{L}_b(E,F): T \text{ is } A - linear\} \tag{4.1}$$

$$\mathcal{L}_n^A(E,F) = \mathcal{L}_b^A(E,F) \cap \mathcal{L}_n(E,F) \tag{4.2}$$

Lemma 4.5: If E and F are f-modules over A, with F Dedekind complete, then $\mathcal{L}_b^A(E,F)$ is a band in $\mathcal{L}_b(E,F)$, and $\mathcal{L}_n^A(E,F)$ is a band in $\mathcal{L}_n(E,F)$.

Proof: It is sufficient to prove that $\mathcal{L}_b^A(E,F)$ is a band in $\mathcal{L}_b(E,F)$. For $a \in A$, let us define $\Pi_a \in Orth(E)$ by $\Pi_a f = af$ for all $f \in E$ and $\sigma_a \in Orth(F)$ by $\sigma_a g = ag$ for all $g \in F$. Also, if we define the operators R_a and L_a from $\mathcal{L}_b(E,F)$ into itself by

$$R_a(T) = T\,\Pi_a \ , \ L_a(f) = \sigma_a T \qquad \forall T \in \mathcal{L}_b(E,F) \tag{4.3}$$

then $R_a, L_a \in Orth\big(\mathcal{L}_b(L,M)\big)$ and hence

$$\mathcal{L}_b^A(E,F) = \{T \in \mathcal{L}_b(E,F): (R_a - L_a)(T) = 0 \ \ \forall a \in A\}$$

$$= \bigcap\{ker(R_a - L_a): \forall a \in A\} \tag{4.4}$$

Since the kernel $ker(R_a - L_a)$ of the orthomorphism $R_a - L_a$ is a band, it is obtained that $\mathcal{L}_b^A(E,F)$ is also a band.

Let us give some examples of f-modules and examine the relations with the above arguments:

Examples 4.6:

(i) Let L be an Archimedean Riesz space. Then L has a natural f-module structure $Orth(L)$ if we set $\Pi.f = \Pi(f)$ for all $f \in L$ and $\Pi \in Orth(L)$. In the same way, we obtain that L is an f-module over $Z(L)$. If it is not stated otherwise, we will consider a Riesz space L as an f-module over $Z(L)$ in this natural way.

(ii) Let L be an Archimedean Riesz space and F be a Dedekind complete f-module over the f-algebra A. Then $\mathcal{L}_b(L,F)$ has a natural f-module structure and defines $a.T$ by

$$(a.T)f = a.Tf \quad \forall f \in L \tag{4.5}$$

for $a \in A$ and $T \in \mathcal{L}_b(L,F)$. It follows from $\mathcal{L}_n(L,F)$'s being a band $\mathcal{L}_b(L,F)$ that $\mathcal{L}_n(L,F)$ is an f-submodule of $\mathcal{L}_b(L,F)$. In particular, $\mathcal{L}_b(L,M)$ and $\mathcal{L}_n(L,M)$ have natural f-module structures over $Z(M)$ if M is a Dedekind complete Riesz space. In the same way, if E and F are f-modules over A, with F Dedekind complete, then $\mathcal{L}_b^A(E,F)$ and $\mathcal{L}_n^A(E,F)$ are similar f-modules over A.

(iii) Let L and M are Archimedean Riesz spaces with M Dedekind complete. If $h: Z(M) \longrightarrow Z(L)$ is an f-algebra homomorphism, h induces on L an f-module structure over $Z(M)$ with $\Pi.f = h(\Pi)f$ for all $f \in L$ and $\Pi \in Z(M)$. So, $\mathcal{L}_n^{Z(M)}(E,F) = \mathcal{L}_n^h(E,F)$.

Conversely, let E be an f-module over $Z(M)$ and let $h: Z(M) \longrightarrow Orth(E)$ be the corresponding f-algebra homomorphism as in Remark 4.3. Since $h(I)$ is a band projection in E, h maps $Z(M)$ onto $Z(E)$ and so $\mathcal{L}_n^{Z(M)}(E,M) = \mathcal{L}_n^h(E,M)$. Hence the results of Propositions 3.9, 3.10 and 3.12 can be applied to $\mathcal{L}_n^{Z(M)}(E,M)$. Furthermore, all operators in $\mathcal{L}_n^{Z(M)}(E,M)$ are Maharam operators.

(iv) Let L and M be Dedekind complete Riesz spaces and let $T \in \mathcal{L}_n(L,M)$ be a Maharam operator. So, there exists an f-module structure over $Z(M)$ on L such that $T \in \mathcal{L}_n^{Z(M)}(L,M)$ by Proposition 3.4. In fact, we obtain from Proposition 3.13 that there exists an f-module structure on L such that $B = \mathcal{L}_n^{Z(M)}(L,M)$ if $B \subseteq \mathcal{L}_n(L,M)$ is a band consisting of Maharam operators.

Let M be a Dedekind complete Riesz space and E be an f-module over $Z(M)$. By Lemma 4.5, $\mathcal{L}_n^{Z(M)}(E,M)$ is a band in $\mathcal{L}_n(E,M)$ and as it is observed above, $\mathcal{L}_n^{Z(M)}(E,M)$ is an f-module over $Z(M)$, where for $\Pi \in Z(M)$ and $T \in \mathcal{L}_n^{Z(M)}(E,M)$, the product $\Pi.T$ is defined by $(\Pi.\mathrm{T})f = \Pi(Tf)$ for all $f \in E$. If $\mathcal{J}$ is an ideal in $\mathcal{L}_n^{Z(M)}(E,M)$, then $\mathcal{J}$ is an f-submodule of $\mathcal{L}_n^{Z(M)}(E,M)$. For $f \in E$, the mapping $f^{\sim}(T) = Tf$ for all $T \in \mathcal{J}$.

Lemma 4.7: $f^{\sim} \in \mathcal{L}_n^{Z(M)}(\mathcal{J},M)$ for all $f \in E$.

Proof: As we know that $f^{\sim} = (f^{+})^{\sim} - (f^{-})^{\sim}$ and $(f^{+})^{\sim}, (f^{-})^{\sim} \geq 0$ and so $f^{\sim} \in \mathcal{L}_b(\mathcal{J},M)$. If we take that $T_\tau \downarrow 0$ in $\mathcal{J}$, then $T_\tau \downarrow 0$ in $\mathcal{L}_n(E,M)$ and hence $T_\tau|f| \downarrow 0$ in M $inf_\tau |f^{\sim}(T_\tau)| = 0$ since $|f^{\sim}(T_\tau)| \leq T_\tau|f|$. Therefore, $f^{\sim} \in \mathcal{L}_n(\mathcal{J},F)$. In addition, if $\Pi \in Z(M)$ and $T \in \mathcal{J}$, then $f^{\sim}(\Pi T) = (\Pi T)f = \Pi f^{\sim}(T)$ and hence $f^{\sim}$ is $Z(M) - linear$.

Lemma 4.8: Let the mapping $\alpha: E \to \mathcal{L}_n^{Z(M)}(\mathcal{J},M)$ be defined by $\alpha(f) = f^{\sim}$ for all $f \in E$. Then α is an order continuous $Z(M) - linear$ Riesz homomorphism from E into $\mathcal{L}_n^{Z(M)}(\mathcal{J},M)$.

Proof: By the way of defining α, α is a positive linear mapping. If $f_\tau \downarrow 0$ in E and $0 \leq T \in \mathcal{J}$, then $Tf_\tau \downarrow 0$ in M since T is order continuous. So, $\alpha(f_\tau) \downarrow 0$ in $\mathcal{L}_n(\mathcal{J},F)$ and $\alpha(\Pi f)(T) = T(\Pi f) = \Pi Tf = \Pi\alpha(f)(T)$ for $f \in E$, $\Pi \in Z(M)$ and for all $T \in \mathcal{J}$. Hence $\alpha(\Pi f) = \Pi\alpha(f)$ and it gives us that α is $Z(M) - linear$ and so α is a Riesz homomorphism.

α is injective if and only if $\mathcal{J}$ separates the points of E, that is, for every $0 \neq f \in E$, there exists $T \in \mathcal{J}$ with $Tf \neq 0$. But, if α is injective, then E and $\alpha(E)$ are isomorphic f-modules over (M).

Lemma 4.9: If $0 \leq f \in E$ and $0 \leq \lambda \in \mathcal{L}_n^{Z(M)}(\mathcal{J},M)$ are such that $0 < \lambda \leq f^{\sim}$, then there exists $0 \leq g \in E$ such that $0 < g^{\sim} \leq \lambda$.

Proof: As $0 < \lambda \leq f^{\sim}$, there exists $0 < \varepsilon \in \mathbb{R}$ such that $\Lambda_0 = (\Lambda - \varepsilon \mathrm{f}^{\sim})^{+} > 0$. Now, if $\Lambda_0 > 0$, then the carrier $\mathrm{e}_{\Lambda_0} \neq \{0\}$ since Λ_0 is order continuous. Let us take $0 < T_o \in \mathrm{e}_{\Lambda_0}$, $0 < \Lambda_0 < \Lambda \leq f^{\sim}$ and so

$$T_0 f = f^{\sim}(T_0) \geq \Lambda_0(T_0) > 0 \qquad (4.6)$$

Hence, the ideal $e_{T_0} \oplus N_{T_0}$ is order dense in E. It follows from $T_0 \in \mathcal{J}$ that T_0 is order continuous and since $T_0 f > 0$, there exists $f_0 \in e_T$ such that $0 < f_0 < f$. As $\mathcal{J}$ itself is Dedekind complete, $\mathcal{J} = e_{\Lambda_0} \oplus N_{\Lambda_0}$. Let us choose any $0 \leq T \epsilon\, \mathcal{J}$ and take

$T = T_1 + T_2$ with $0 < T_1 \in \mathrm{e}_{\Lambda_0}$ and $0 < T_2 \in \mathrm{N}_{\Lambda_0}$. As we know that, in $\mathcal{L}_n^{Z(M)}(\mathcal{J}, M)$

$$(\Lambda - \varepsilon f \wedge\sim)^- \perp (\Lambda - \varepsilon f \wedge\sim)^+ = \Lambda_0 \tag{4.7}$$

it is obtained that $e_{\Lambda_0} \subseteq N_{(\Lambda - \varepsilon f \wedge\sim)^-}$ by Proposition (3.9). As a consequence, $(\Lambda - \varepsilon f^{\sim})(T_1) = (\Lambda - \varepsilon f^{\sim})^+(T_1) \geq 0$, that is, $\varepsilon f^{\sim}(T_1) \leq \Lambda(T_1)$. Also, since $0 < T_2 \in \mathrm{N}_{\Lambda_0}$ and $0 < T_0 \in \mathrm{e}_{\Lambda_0}$, it yields that $\mathrm{e}_{\mathrm{T}_0} \subseteq \mathrm{N}_{\mathrm{T}_2}$. So, in particular , $T_2 f_0 = 0$, and $Tf_0 = T_1 f_0$. Therefore ,

$$\Lambda(T) \geq \Lambda(T_1) \geq \varepsilon f^{\sim}(T_1) = \varepsilon T_1 f_0 = \varepsilon T f_0 = (\varepsilon f_0{}^{\sim})(T) \tag{4.8}$$

Here, if we take $\mathsf{g}= \varepsilon f_0$, we have $0 \leq g^{\sim}(\mathsf{T}) \leq \Lambda(T)$ for all $0 \leq T \in \mathcal{J}$, that is, $0 \leq g^{\sim} \leq \Lambda$ in $\mathcal{L}_n^{Z(M)}(\mathcal{J}, M)$. We conclude that $0 \leq g^{\sim} \leq \Lambda$ since $g^{\sim}(T_0) = \varepsilon T_0 f_0$.

Theorem 4.10: Let M be a Dedekind complete Riesz space, E be an f-module over $Z(M)$ and $\mathcal{J}$ be an ideal in $\mathcal{L}_n^{Z(M)}(\mathcal{J}, M)$ separating the points of E.

(1) The band generated by $\alpha(E)$ in $\mathcal{L}_n^{Z(M)}(\mathcal{J}, M)$ is equal to $\mathcal{L}_n^{Z(M)}(\mathcal{J}, M)$

(2) $\alpha(E)$ is an ideal in $\mathcal{L}_n^{Z(M)}(\mathcal{J}, M)$ if and only if E is Dedekind complete.

(3) $\alpha(E) = \mathcal{L}_n^{Z(M)}(\mathcal{J}, M)$ if and only if $0 < f_T \uparrow$ in E with $sup_z T f_z$ existing in M implies that there exists $0 \leq f \in E$ such that $0 < f_T \uparrow f$.

Proof:

1) Let $\Lambda \in \mathcal{L}_n^{Z(M)}(\mathcal{J}, M)$ be such as $\Lambda \perp \alpha(f) = f^{\sim}$ for all $f \in E$. So, $e_{\Lambda} \subseteq C_{f^{\sim}}$, that is, $e_{\Lambda} \subseteq N_{f^{\sim}}$ for all $f \in E$. It gives us that if $0 \leq T \in e_{\Lambda}$, then $Tf = f^{\sim}(T) = 0$ for all $0 \leq f \in E$ and hence $T = 0$. Therefore, $e_{\Lambda} = \{0\}$ and $\Lambda = 0$.

2) Let $\alpha(E)$ be an ideal in $\mathcal{L}_n^{Z(M)}(\mathcal{J}, M)$, then $\alpha(E)$ becomes Dedekind complete. As E and $\alpha(E)$ are Riesz isomorphic, we obtain that E is also Dedekind complete. If $0 \leq \Lambda \leq \alpha(f)$ in

$\mathcal{L}_n^{Z(M)}(\mathcal{J}, M)$ for some $0 \leq f \in E$, we may define

$$g = \sup\{0 \leq h \in E : 0 \leq \alpha \leq \Lambda\} \tag{4.9}$$

$0 \leq \alpha(h) \leq \Lambda \leq \alpha(f)$ implies that $0 \leq h \leq f$ and so it is well defined. By Lemma 4.9, α is an order continuous Riesz homomorphism. And so $0 \leq \alpha(g) \leq \Lambda$.

Assume $\alpha(g) < \Lambda$. Hence $0 \leq \Lambda - \alpha(g) \leq \alpha(f - g)$. So, there exists $0 < g_0 \in E$ satisfying $0 \leq \alpha(g_0) \leq \Lambda - \alpha(g)$. Therefore, $0 < \alpha(g + g_0) \leq \Lambda$ and by the definition, it gives us that $g + g_0 \leq g$ and it is an contradiction. Hence, $\alpha(g) = \Lambda$ and $\alpha(E)$ becomes an ideal in $\mathcal{L}_n^{Z(M)}(\mathcal{J}, M)$.

3) First let us choose $\alpha(E) = \mathcal{L}_n^{Z(M)}(\mathcal{J}, M)$ and let $0 \leq f_T \uparrow$ in E satisfying $sup_z \mathsf{T} f_z$ exists for all $0 \leq T \in \mathcal{J}$. If it is defined as

$$\Lambda(T) = sup_\tau T f_\tau \tag{4.10}$$

for all $0 \leq T \in \mathcal{J}$, then $\Lambda: \mathcal{J}^+ \to M^+$ is additive and hence Λ extends uniquely to a positive linear operator $\Lambda: \mathcal{J} \to M$. By the definition of Λ, we obtain that $0 < f_T \uparrow f\Lambda$ in $\mathcal{L}_b(\mathcal{J}, M)$. Since $f_z{}^{\sim} \in \mathcal{L}_n^{Z(M)}(\mathcal{J}, M)$ and $\mathcal{L}_n^{Z(M)}(\mathcal{J}, M)$ is a band in $\mathcal{L}_b(\mathcal{J}, M)$, it follows that $0 \leq \Lambda \in \mathcal{L}_n^{Z(M)}(\mathcal{J}, M)$. Our hypothesis gives us that there exists $0 \leq f \in E$ such that $\Lambda = f^{\sim} = \alpha(f)$. As α is a Riesz isomorphism, $\alpha(f_\tau) \uparrow \alpha(f)$ implies that $f_\tau \uparrow f$ in E.

Conversely, by using (2), we obtain that $\alpha(E)$ is an ideal in $\mathcal{L}_n^{Z(M)}(\mathcal{J}, M)$. Let us pick $0 \leq \Lambda \in \mathcal{L}_n^{Z(M)}(\mathcal{J}, M)$. As we know that $\alpha(E)$ coincides with $\mathcal{L}_n^{Z(M)}(\mathcal{J}, M)$, then there exist $0 \leq f_\tau \uparrow$ in E satisfying $0 \leq \widetilde{f_\tau} \uparrow \Lambda$ in $\mathcal{L}_n^{Z(M)}(\mathcal{J}, M)$. So,

$$0 \leq T f_\tau = \widetilde{f_\tau}(T) \uparrow \Lambda(T) \tag{4.11}$$

for all $0 \leq T \in \mathcal{J}$. Hence, $sup_\tau T f_\tau \in M$ exists. The hypothesis gives us that there exists $0 \leq f \in E$ such that $f_\tau \uparrow f$. So, $0 \leq \widetilde{f_\tau} \uparrow \tilde{f}$ in $\mathcal{L}_n^{Z(M)}(\mathcal{J}, M)$ and $\Lambda = \tilde{f} = \alpha(f)$. It concludes that $\mathcal{L}_n^{Z(M)}(\mathcal{J}, M) = \alpha(E)$. (Luxemburg and De Pagter, 2001)

5. THE CONSTRUCTION OF MAHARAM EXTENSIONS

Let $\Bbbk$ be a field, V be a vector space over $\Bbbk$, A be an algebra with unit element e over $\Bbbk$ and let the tensor product $A\otimes V$ of A and V as vector spaces over $\Bbbk$.

1) $A\otimes V$is a unital A module with$a.(b\otimes v)=(ab)\otimes v$ for all $a,b\in A$ and $v\in V$.

2) $\chi_0: V\to A\otimes V$defined by $\chi_0(v)=e\otimes v$ for all $v\in V$ is a $\Bbbk$-linear embedding of V into $A\otimes V$.

Assume that M is a unital A-module. Since $\Bbbk$ is embedded in A by the mapping $\lambda\to\lambda e$, M is also a $\Bbbk$-vector space.

Let $T: V\to M$ be a $\Bbbk$-linear mapping. Then, the mapping $(a,v)\to a.Tv$ is $\Bbbk$-linear and hence there exists a unique $\Bbbk$-linear mapping $\overline{T}$ from $A\otimes V$ into M such that

$$\overline{T}(a\otimes v)=a.Tv \qquad (5.1)$$

for all $a\in A$ and all $v\in V$. Here, $\overline{T}$ is A-linear and that $T=\overline{T}_0\chi_0$ (that is, $\overline{T}$extends T). Also, $\overline{T}$ is unique. The A-module $A\otimes V$ is said to be the extension of Vover A.

Let L be an Archimedean Riesz space and M be a Dedekind complete Riesz space and consider M as an f-module over $Z(M)$. Then, $\mathcal{L}_b(L,M)$ is an also an f-module over $Z(M)$ with $\Pi.T=\Pi\circ T$ for all $\Pi\in Z(M)$and $T\in\mathcal{L}_b(L,M)$. Let us pick an ideal $\mathcal{J}\in\mathcal{L}_b(L,M)$. Via the operations induced by $\mathcal{L}_b(L,M)$, this ideal is an f-module over $Z(M)$.

For $\Pi\in Z(M)$and $f\in L$, let us define the mapping

$$\Pi\otimes f:\mathcal{J}\to M \qquad (5.2)$$

by$(\Pi\otimes f)(T)=\Pi(Tf)$ for all $T\in\mathcal{J}$. So, $\Pi\otimes f\in\mathcal{L}_n^{Z(M)}(\mathcal{J},M)$.

And, also, let us define the mapping

$$\psi: Z(M)\times L\to\mathcal{L}_n^{Z(M)}(\mathcal{J},M) \qquad (5.3)$$

by $\psi(\Pi,f)=\Pi\otimes f$.

Lemma 5.2: ψ is a Riesz bimorphism, that is, ψ is bilinear and $|\psi(\Pi, f)| = \psi(|\Pi|, |f|)$ for all $\Pi \in Z(M)$and $f \in L$.

Proof: The bilinearity is obvious.

Let $\Pi \in Z(M)$, $f \in L$ and $0 \leq T \in \mathcal{J}$. So,

$$|\Pi \otimes f|(T) = \sup\{|(\Pi \otimes f)(S)|: S \in \mathcal{J}, |S| \leq T\}$$

$$= \sup\{|\Pi(Sf)|: S \in \mathcal{J}, |S| \leq T\}$$

$$= \sup\{|\Pi|(|Sf|): S \in \mathcal{J}, |S| \leq T\}$$

$$= |\Pi| \sup\{|Sf|: S \in \mathcal{J}, |S| \leq T\}$$

$= |\Pi|(T|f|) = (|\Pi| \otimes |f|)(T)$ (5.4)

Therefore, $|\Pi \otimes f| = |\Pi| \otimes |f|$in $\mathcal{L}_b^{Z(M)}(\mathcal{J}, M)$.

Definition 5.3: We will denote by $Z(M) \otimes_{\mathcal{J}} L$ the linear subspace of $\mathcal{L}_n^{Z(M)}(\mathcal{J}, M)$ generated by the operators of the form $\Pi \otimes f$, that is,

$Z(M) \otimes_{\mathcal{J}} L = \{\sum_{i=1}^{n} \Pi_i \otimes f_i : \Pi_i \in Z(M), f_i \in L, i = 1, \dots, n; n \in \mathbb{N}\}$ (5.5)

Moreover, $Z(M) \widetilde{\otimes}_{\mathcal{J}} L$ will denote the ideal generated by $Z(M) \otimes_{\mathcal{J}} L$in $\mathcal{L}_n^{Z(M)}(\mathcal{J}, M)$. The mapping

$\chi_0 : L \to \mathcal{L}_n^{Z(M)}(\mathcal{J}, M)$ (5.6)

defined by $\chi_0 f = I \otimes f$ for all $f \in L$ is a Riesz homomorphism and so χ_0 maps L into $Z(M) \widetilde{\otimes}_{\mathcal{J}} L$. Hence, it can be considered $\chi_0 : L \to Z(M) \widetilde{\otimes}_{\mathcal{J}} L$.

Lemma 5.4: $Z(M) \widetilde{\otimes}_{\mathcal{J}} L$ coincides with the ideal generated by $\chi_0(L)$in $\mathcal{L}_n^{Z(M)}(\mathcal{J}, M)$, that is,

$$Z(M) \widetilde{\otimes}_{\mathcal{J}} L = \left\{\theta \in \mathcal{L}_n^{Z(M)}(\mathcal{J}, M): |\theta| \leq I \otimes u \; for \; some \; 0 \leq u \in L\right\} \quad (5.7)$$

Proof: Let $\theta \in \mathcal{L}_n^{Z(M)}(\mathcal{J}, M)$. Then there exist $\Pi_1, \dots, \Pi_n \in Z(M)$ and $f_1, \dots, f_n \in L$ such that

$|\theta| \leq |\sum_{i=1}^{n} \Pi_i \otimes f_i|$ (5.8)

Let $u = \sum_{i=1}^{n} |f_i|$ and $\Pi = \sum_{i=1}^{n} |\Pi_i|$. $0 \leq \Pi \in Z(M)$ and so there exists $0 \leq k \in \mathbb{R}$ such that $0 \leq \Pi \leq kI$. Therefore, $|\theta| \leq |\sum_{i=1}^{n} \Pi_i \otimes f_i| \leq \sum_{i=1}^{n} |\Pi_i| \otimes |f_i| \leq \Pi \otimes u \leq (kI) \otimes u = I \otimes (ku) = \chi_0(ku)$

Let us pick any ideal E in $\mathcal{L}_n^{Z(M)}(\mathcal{J}, M)$ such that

$$Z(M) \widetilde{\otimes}_{\mathcal{J}} L \subseteq E \subseteq \mathcal{L}_n^{Z(M)}(\mathcal{J}, M) \qquad (5.9)$$

Actually, E is Dedekind complete and an f-submodule of $\mathcal{L}_n^{Z(M)}(\mathcal{J}, M)$. Moreover, if $T \in \mathcal{J}$ satisfies $\theta(T) = 0$ for all $\theta \in E$, then $Tf = (I \otimes f)(T) = 0$ for all $f \in L$, hence $T = 0$. It concludes that E separates the points of $\mathcal{J}$.

Let us define the mapping

$$\alpha_E : \mathcal{J} \to \mathcal{L}_n^{Z(M)}(E, M) \qquad (5.10)$$

by $\alpha_E(T) = \hat{T}$, where $\hat{T}(\theta) = \theta(T)$ for all $\theta \in E$. Then α_E is a Riesz isomorphism and $\alpha_E(\mathcal{J})$ is an order dense ideal in $\mathcal{L}_n^{Z(M)}(E, M)$. Here, $\hat{T}(\Pi \otimes f) = \Pi(Tf)$ for all $\Pi \in Z(M)$ and $f \in L$. Also, $\hat{T}(\chi_0 f) = \hat{T}(I \otimes f) = Tf$ for all $f \in L$, that is, $T = \hat{T}_0 \chi_0$.

Theorem 5.5: Let $L, M, \mathcal{J} \subseteq \mathcal{L}_b(L, M)$ and $E \subseteq \mathcal{L}_n^{Z(M)}(\mathcal{J}, M)$ be defined as above. Then

(1) E is a Dedekind complete f-module over $Z(M)$.

(2) $\chi_0 : L \to E$ is a Riesz homomorphism such that $\chi_0(L)$ is order dense in E, that is, $\chi_0(L)^d = \{0\}$.

(3) For every $T \in \mathcal{J}$, the operator $\hat{T} = \alpha_E(T)$ is the unique operator in $\mathcal{L}_n^{Z(M)}(E, M)$ satisfying $T = \hat{T}_0 \chi_0$ and α_E is a Riesz isomorphism from $\mathcal{J}$ onto an order dense ideal in $\mathcal{L}_n^{Z(M)}(E, M)$.

(4) $\{\hat{T} : 0 \leq T \in \mathcal{J}\}$ separates the points of E.

Furthermore, if $\mathcal{J}$ is a band in $\mathcal{L}_b(L, M)$, then α_E is a Riesz isomorphism from $\mathcal{J}$ onto $\mathcal{L}_n^{Z(M)}(E, M)$.

Proof:

(1) It has been observed above.

(2) Assume that $\theta \in E$satisfies $\theta \in \chi_0(L)^d$. Then, $\theta \perp \chi_0 f$ in $\mathcal{L}_n^{Z(M)}(\mathcal{J}, M)$ for all $f \in L$. So, $\mathcal{C}_\theta \perp \mathcal{C}_{\chi_0 f}$, that is, $\mathcal{C}_\theta \subseteq \mathcal{N}_{\chi_0 f}$ in $\mathcal{J}$ for all $f \in L$. Therefore, if $T \in \mathcal{C}_\theta$, then $Tf = (\chi_0 f)(T) = 0$ for all $f \in L$. Hence, $\mathcal{C}_\theta = \{0\}$and $\theta = 0$.

(3) We only need to show that the operator $\hat{T} = \alpha_E(T)$ is unique.

Let us define the operator

$R: \mathcal{L}_n^{Z(M)}(E, M) \to \mathcal{L}_b(L, M)$ (5.11)

by$R(\Lambda) = \Lambda \circ \chi_0$ for all $\Lambda \in \mathcal{L}_n^{Z(M)}(E, M)$ which is positive and order continuous. Also, $R(\hat{T}) = T$ for all $T \in \mathcal{J}$.

Assume that $\Lambda_1 \wedge \Lambda_2 = 0$in $\mathcal{L}_n^{Z(M)}(E, M)$. As we know that $\alpha_E(\mathcal{J})$ is order dense in $\mathcal{L}_n^{Z(M)}(E, M)$, there exist $0 \leq T_\tau \uparrow$ and $0 \leq S_\sigma \uparrow$ in $\mathcal{J}$ satisfying $0 \leq \hat{T}_\tau \uparrow \Lambda_1$ and $0 \leq \widehat{S_\sigma} \uparrow \Lambda_2$ in $\mathcal{L}_n^{Z(M)}(E, M)$. Also,

$0 \leq \alpha_E(T_\tau \wedge S_\sigma) = \hat{T}_\tau \wedge \widehat{S_\sigma} \leq \Lambda_1 \wedge \Lambda_2 = 0$ (5.12)

Hence, $\alpha_E(T_\tau \wedge S_\sigma) = 0$ and since α_E is one-to-one, it gives us that $T_\tau \wedge S_\sigma = 0$ for all τ, σ. Since R is order continuous, $0 \leq T_\tau \uparrow R(\Lambda_1)$ and $0 \leq S_\sigma \uparrow R(\Lambda_2)$in $\mathcal{L}_b(L, M)$. So, $R(\Lambda_1) \wedge R(\Lambda_2) = 0$ and R is a Riesz homomorphism.

Assume that $\Lambda \in \mathcal{L}_n^{Z(M)}(E, M)$satisfies $R(\Lambda) = 0$. $R(|\Lambda|) = 0$as$R$ is a Riesz homomorphism. So, $|\Lambda|(\chi_0 f) = 0$ for all $0 \leq f \in L$. Since $\chi_0(L)$ is order dense in E, $\Lambda = 0$. Hence, if $T \in \mathcal{J}$ and $\Lambda \in \mathcal{L}_n^{Z(M)}(E, M)$satisfies $T = \hat{T}_0 \circ \chi_0 = \Lambda \circ \chi_0$, then $\Lambda = T$. So, R is injective.

(4) It follows directly from the definition.

Let $\mathcal{J}$ be a band in $\mathcal{L}_b(L, M)$. If $0 \leq T_\tau \uparrow$ in $\mathcal{J}$ such that $sup_\tau \theta(T_\tau) \in M$ exists for all $0 \leq \theta \in E$, by taking $\theta = I \otimes f$, $0 \leq f \in L$, we obtain that $sup_\tau T_\tau f \in M$ exists for all $0 \leq f \in L$. Let us define $Tf = sup_\tau T_\tau f$ for all $0 \leq f \in L$ and extends T to a positive operator $0 \leq T \in \mathcal{L}_b(L, M)$. So, $0 \leq T_\tau \uparrow T$ in $\mathcal{L}_b(L, M)$ and by $\mathcal{J}'$s being a band, we obtain that $0 \leq T_\tau \uparrow T \in \mathcal{J}$ and hence $\alpha_E$is surjective.

Proposition 5.6: Let L and M be Archimedean Riesz spaces with M Dedekind complete and let $\mathcal{J} \subseteq \mathcal{L}_b(L, M)$ be an ideal. Suppose that the pair (F, χ_F) satisfies

(1) $F$is a Dedekind complete f-module over $Z(M)$ and $\mathcal{L}_n^{Z(M)}(F, M)$ separates the points of F.

(2) $\chi_F$is a Riesz homomorphism from L into F.

(3) For each $T \in \mathcal{J}$, there exists a unique $\tilde{T} \in \mathcal{L}_n^{Z(M)}(F, M)$ satisfying $T = \tilde{T} \circ \chi_F$ and the mapping ω_F defined by $\omega_F(T) = \tilde{T}$ is a Riesz isomorphism from $\mathcal{J}$ onto an order dense ideal in $\mathcal{L}_n^{Z(M)}(F, M)$.

Then, there exists a unique ideal E in $\mathcal{L}_n^{Z(M)}(\mathcal{J}, M)$ satisfying $\chi_0(L) \subseteq E$ and a unique $Z(M)$-linear Riesz isomorphism γ from F onto E such that $\gamma \circ \chi_F = \chi_0$.

Proof: Since ω_F is an injective Riesz homomorphism and $\omega_F(\mathcal{J})$ is an ideal, ω_F becomes order continuous.

For $\mu \in \mathcal{L}_n^{Z(M)}\left(\mathcal{L}_n^{Z(M)}(F, M), M\right)$, let us define $\omega_F{}^*\mu: \mathcal{J} \to M$ by $(\omega_F{}^*\mu)(T) = \mu(\omega_F T)$ for all $T \in \mathcal{J}$. Since ω_F is order continuous and $Z(M)$-linear, it gives us that $\omega_F{}^*\mu \in \mathcal{L}_n^{Z(M)}(\mathcal{J}, M)$. So a linear mapping can be defined such as

$$\omega_F{}^*: \mathcal{L}_n^{Z(M)}\left(\mathcal{L}_n^{Z(M)}(F, M), M\right) \to \mathcal{L}_n^{Z(M)}(\mathcal{J}, M) \qquad (5.13)$$

Since ω_F is a Riesz homomorphism and $\omega_F(\mathcal{J})$ is an order dense ideal in $\mathcal{L}_n^{Z(M)}(F, M)$, we obtain that $\omega_F{}^*$ is a Riesz homomorphism which is Maharam, injective and $Z(M)$-linear.

As we know that F is Dedekind complete and $\mathcal{L}_n^{Z(M)}(F, M)$ separates the points of F, then it gives us that the mapping

$$j_F: F \to \mathcal{L}_n^{Z(M)}\left(\mathcal{L}_n^{Z(M)}(F, M), M\right) \qquad (5.14)$$

defined by $(j_F h)(\Lambda) = \Lambda(h)$ for all $\Lambda \in \mathcal{L}_n^{Z(M)}(F, M)$, $h \in F$, is a $Z(M)$-linear Riesz isomorphism onto an order dense ideal. Also, let us define $E = (\omega_F{}^* \circ j_F)(F)$. Hence E is an ideal in $\mathcal{L}_n^{Z(M)}(\mathcal{J}, M)$ and also $\gamma = \omega_F{}^* \circ j_F$ is a $Z(M)$-linear Riesz isomorphismfrom Fonto E. It follows that $\gamma \circ \chi_F = \chi_0$ and so $\chi_0(L) \subseteq E$.

Now assume that E is an ideal in $\mathcal{L}_n^{Z(M)}(\mathcal{J}, M)$ with $\chi_0(L) \subseteq E$ and that $\beta: E \to \mathcal{L}_n^{Z(M)}(\mathcal{J}, M)$ is $Z(M)$-linear and order continuous satisfying $\beta \circ \chi_0 = \chi_0$. Then, for all $\theta \in E$, $\beta(\theta) = \theta$. Let us denote $\hat{T} = \alpha_E(T) \in \mathcal{L}_n^{Z(M)}(E, M)$ for $T \in \mathcal{J}$ and define $\Lambda_T \in \mathcal{L}_n^{Z(M)}(E, M)$ by $\Lambda_T(\theta) = \hat{T}(\beta\theta)$ for all $\theta \in E$. Hence, $\Lambda_T(\chi_0 f) = \hat{T}(\beta\chi_0 f) = \hat{T}(\chi_0 f) = Tf$ for all $f \in L$. It gives us that $\Lambda_T = \hat{T}$ and so $(\beta\theta)(T) = \theta(T)$ for all $T \in \mathcal{J}$ and all $\theta \in E$. Therefore $\beta(\theta) = \theta$ for all $\theta \in E$ and hence E and γ becomes unique.

Definition 5.7: A Maharam extension space for $\mathcal{J}$ is a pair (F, χ_F) satisfying

(1) $F$is a Dedekind complete f-module over $Z(M)$ and $\chi_F: L \to F$ is a Riesz homomorphism.

(2) For each $0 \leq T \in \mathcal{J}$, there exists a unique $0 \leq \tilde{T} \in \mathcal{L}_n^{Z(M)}(F, M)$ such that $T = \tilde{T} \circ \chi_F$

(3) $\{\tilde{T}: 0 \leq T \in \mathcal{J}\}$separates the points of F.

Also, (F, χ_F) is said to be minimal if the ideal generated by $\chi_F(L)$ coincides with F and it is called maximal if $0 \leq h_\tau \uparrow$ in F with $sup_\tau \tilde{T} h_\tau \in M$ exists for all $0 \leq T \in \mathcal{J}$ implies that $0 \leq h_\tau \uparrow h \in F$.

Remark 5.8: It is clear that (E, χ_0) is a Maharam extension space for any ideal E in $\mathcal{L}_n^{Z(M)}(\mathcal{J}, M)$with $\chi_0(L) \subseteq E$. Also, $Z(M)\widetilde{\otimes}_{\mathcal{J}} M$ is a minimal Maharam extension space and $\mathcal{L}_n^{Z(M)}(\mathcal{J}, M)$ is maximal. To this end, assume that $0 \leq \theta_\tau \uparrow$ in $\mathcal{L}_n^{Z(M)}(\mathcal{J}, M)$ such that $sup_\tau \hat{T}(\theta_\tau)$ for all $0 \leq T \in \mathcal{J}$. Hence $0 \leq \theta \in \mathcal{L}_n^{Z(M)}(\mathcal{J}, M)$and $0 \leq \theta_\tau \uparrow \theta$.

Proposition 5.9: Let (F, χ_F) be a Maharam extension space for the ideal $\mathcal{J} \in \mathcal{L}_b(L, M)$. Let us define

$$R: \mathcal{L}_n^{Z(M)}(F, M) \to \mathcal{L}_b(L, M) \qquad (5.15)$$

by$R(\Lambda) = \Lambda \circ \chi_0$ for all $\Lambda \in \mathcal{L}_n^{Z(M)}(F, M)$.

(1) $R$is a Riesz isomorphism (into) and $\left\{\Lambda \in \mathcal{L}_n^{Z(M)}(F, M): R(\Lambda) \in \mathcal{J}\right\}$ is an order dense ideal in $\mathcal{L}_n^{Z(M)}(F, M)$.

(2) For each $T \in \mathcal{J}$, there exists a unique $\tilde{T} \in \mathcal{L}_n^{Z(M)}(F, M)$ such that $T = \tilde{T} \circ \chi_F$ and the mapping $T \to \tilde{T}$ is a Riesz isomorphism from $\mathcal{J}$ onto some order dense ideal in $\mathcal{L}_n^{Z(M)}(F, M)$.

Proof: Since (F, χ_F) is a Maharam extension space,$\{\tilde{T}: 0 \leq T \in \mathcal{J}\}$ separates the points of F. Assume that $0 \leq \Lambda \in \mathcal{L}_n^{Z(M)}(F, M)$ satisfies that $\Lambda \wedge \tilde{T} = 0$ for all $0 \leq T \in \mathcal{J}$. Then $\mathcal{C}_\Lambda \perp \mathcal{C}_{\tilde{T}}$, that is, $\mathcal{C}_\Lambda \subseteq \mathcal{N}_{\tilde{T}}$ for all $0 \leq T \in \mathcal{J}$. It follows that $\mathcal{C}_\Lambda = 0$ and so $\Lambda = 0$. Hence, $\left\{\hat{T}: 0 \leq T \in \mathcal{J}\right\}^d = \{0\}$in $\mathcal{L}_n^{Z(M)}(F, M)$. Then R is a Riesz isomorphism and $\left\{\Lambda \in \mathcal{L}_n^{Z(M)}(F, M): R(\Lambda) \in \mathcal{J}\right\}$ is an order dense ideal in

$\mathcal{L}_n^{Z(M)}(F,M)$. It also requires that for each $T \in \mathcal{J}$, there exists a unique $\tilde{T} \in \mathcal{L}_n^{Z(M)}(F,M)$ such that $T = \tilde{T} \circ \chi_F$ and the mapping $T \to \tilde{T}$ is a Riesz isomorphism from $\mathcal{J}$ onto some order dense ideal in $\mathcal{L}_n^{Z(M)}(F,M)$.

Theorem 5.10: Let $\mathcal{J} \subseteq \mathcal{L}_b(L,M)$ be an ideal as above and let (F,χ_F) be a Maharam extension space for $\mathcal{J}$. Then there exists a unique ideal E in $\mathcal{L}_n^{Z(M)}(\mathcal{J},M)$ with $\chi_0(L) \subseteq E$ and a unique $Z(M)$-linear Riesz isomorphism γ from F onto Esatisfying $\gamma \circ \chi_F = \chi_0$. Furthermore, (F,χ_F) is minimal if and only if $E = Z(M)\widetilde{\otimes}_{\mathcal{J}}L$ or (F,χ_F) is maximal if and only if $E = \mathcal{L}_n^{Z(M)}(\mathcal{J},M)$.

Proof: By the definition of the Maharam extension space, $\mathcal{L}_n^{Z(M)}(F,M)$ separates the points of F. Also, by Proposition 5.9, there exists a Riesz isomorphism ω_F from $\mathcal{J}$ onto an order dense ideal in $\mathcal{L}_n^{Z(M)}(F,M)$ such that $\tilde{T} = \omega_F(T)$ is the unique operator in $\mathcal{L}_n^{Z(M)}(F,M)$satisfying $T = \tilde{T} \circ \chi_F$. Hence the first part is obtained directly from Proposition 5.6. Clearly, F is minimal if and only if $= Z(M)\widetilde{\otimes}_{\mathcal{J}}L$. Again, let us denote the unique operator in $\mathcal{L}_n^{Z(M)}(E,M)$ such as $T = \hat{T} \circ \chi_0$by $\hat{T}$. Then, $\tilde{T} = \hat{T} \circ \gamma$ for all $T \in \mathcal{J}$. It gives us that (F,χ_F) is maximal if and only if (E,χ_0) is maximal. Hence, now we need to show that (E,χ_0) is maximal if and only if $E = \mathcal{L}_n^{Z(M)}(\mathcal{J},M)$. Since we already know that $\mathcal{L}_n^{Z(M)}(\mathcal{J},M)$ is maximal, let us assume that (E,χ_0) is maximal and pick $0 \le \theta \in \mathcal{L}_n^{Z(M)}(\mathcal{J},M)$. As E is an order dense ideal in $\mathcal{L}_n^{Z(M)}(\mathcal{J},M)$, there exist $\theta_\tau \in E$ satisfying $0 \le \theta_\tau \uparrow \theta$ in $\mathcal{L}_n^{Z(M)}(\mathcal{J},M)$. So, $\theta_\tau(T) \uparrow \theta(T)$, that is, $\hat{T}(\theta_\tau) \uparrow \theta(T)$ in M for all $0 \le T \in \mathcal{J}$. Since E is maximal, $sup_\tau \hat{T}(\theta_\tau) \in M$ exists for all $0 \le T \in \mathcal{J}$, there exists $\theta_0 \in E$ such that $0 \le \theta_\tau \uparrow \theta_0$ in E and so $0 \le \theta_\tau \uparrow \theta_0$ in $\mathcal{L}_n^{Z(M)}(\mathcal{J},M)$ since E is an ideal. Hence $\theta = \theta_0 \in E$ and so $E = \mathcal{L}_n^{Z(M)}(\mathcal{J},M)$.

As a conclusion of Theorem 5.10, we may say that any Maharam extension space can be considered canonically as an ideal in $\mathcal{L}_n^{Z(M)}(\mathcal{J},M)$. Furthermore, any minimal Maharam extension space is canonically isomorphic to $Z(M)\widetilde{\otimes}_{\mathcal{J}}L$, which will be called as the minimal Maharam extension space of $\mathcal{J}$. In the same fashion, the maximal Maharam extension space of $\mathcal{J}$ is uniquely determined via the isomorphism and denoted by $Z(M)\widehat{\otimes}_{\mathcal{J}}L$ and is taken as $Z(M)\widehat{\otimes}_{\mathcal{J}}L = \mathcal{L}_n^{Z(M)}(\mathcal{J},M)$.

Remark 5.11:

(1) Let (E,χ) be a Maharam extension space for the ideal $\mathcal{J} \in \mathcal{L}_b(L,M)$. Then for each $0 \leq T \in \mathcal{J}$, there exists a unique $0 \leq \hat{T} \in \mathcal{L}_n^{Z(M)}(E,M)$ and it satisfies that $T = \hat{T} \circ \chi$. Also, $\hat{T}$ is a Maharam operator and hence $Z(M)$-linear via a $Z(M)$-module structure of the space E.

(2) Let (E,χ) be a Maharam extension space for the ideal $\mathcal{J} \in \mathcal{L}_b(L,M)$, then χ need not be injective. Actually, χ becomes injective if and only if $\mathcal{J}$ separates the points of L. Hence, if $\mathcal{J}$ separates the points of L, then χ is a Riesz isomorphism of L into E and by this way, L can be considered as a Riesz subspace of E. For every $T \in \mathcal{J}$, the corresponding operator $\hat{T} \in \mathcal{L}_n^{Z(M)}(E,M)$ becomes an extension of T. Hence $\hat{T}$ is said to be the Maharam extension of T with respect to (E,χ).

Proposition 5.12: The Riesz homomorphism χ is order continuous if and only if $\mathcal{J} \subseteq \mathcal{L}_n(L,M)$.

Proof: Assume that $\mathcal{J} \subseteq \mathcal{L}_n(L,M)$ and $u_\tau \downarrow 0$in L. Also, let $\theta = inf_\tau \chi(u_\tau)$ in $E$and $\theta \geq 0$. For $T \in \mathcal{J}$, $\hat{T}$ denotes the corresponding operator in $\mathcal{L}_n^{Z(M)}(E,M)$. As $\{\hat{T}: 0 \leq T \in \mathcal{J}\}$ separates the points of E, there exists some $0 \leq T \in \mathcal{J}$ satisfying $\hat{T}\theta > 0$. So, $\hat{T}(\chi u_\tau) = T u_\tau \downarrow 0$in M, but also $\hat{T}(\chi u_\tau) \geq \hat{T}\theta > 0$ and it is a contradiction. It gives us that $\chi$is order continuous.

Conversely, since $T = \hat{T} \circ \chi$ for all $T \in \mathcal{J}$, $\mathcal{J} \subseteq \mathcal{L}_n(L,M)$.

Proposition 5.13: Let $E = Z(M)\breve{\otimes}_{\mathcal{J}} L$, the minimal Maharam extension space of the ideal $\mathcal{J} \subseteq \mathcal{L}_b(L,M)$ and let $\mathfrak{B}$ denote the band generated by $\mathcal{J}$ in $\mathcal{L}_b(L,M)$.

(1) $E$is the minimal Maharam extension space of $\mathfrak{B}$, that is, $Z(M)\widehat{\otimes}_{\mathcal{J}} L = Z(M)\,\widetilde{\otimes}_{\mathfrak{B}}\, L$.

(2) The Riesz isomorphism $T \to \hat{T}$ from $\mathcal{J}$ into $\mathcal{L}_n^{Z(M)}(E,M)$ is sujective if and only if $\mathcal{J} = \mathfrak{B}$.

Proof:

1)Let us consider the mapping $R: \mathcal{L}_n^{Z(M)}(E,M) \to \mathcal{L}_b(L,M)$ defined by $R(\Lambda) = \Lambda \circ \chi_0$ for all $\Lambda \in \mathcal{L}_n^{Z(M)}(E,M)$. Then R is an injective Riesz homomorphism. We need to

show that there exists $0 \leq \Lambda \in \mathcal{L}_n^{Z(M)}(E,M)$ such that $R(\Lambda) = T$ for all $0 \leq T \in \mathfrak{B}$. Indeed, let $0 \leq T \in \mathfrak{B}$ and $T_\tau \in \mathcal{J}$ satisfying $0 \leq T_\tau \uparrow T$in $\mathcal{L}_b(L,M)$. If $0 \leq \theta \in E$, then $0 \leq \theta \leq I \otimes u$ for some $0 \leq u \in L$, hence $\widehat{T_\tau}(\theta) \leq \widehat{T_\tau}(I \otimes u) = T_\tau u \leq Tu$ for all τ. So, $\Lambda(\theta) = \sup_\tau \widehat{T_\tau}(\theta) \in M$ exists for all $0 \leq \theta \in M$ and it defines that $\Lambda \in \mathcal{L}_n^{Z(M)}(E,M)$satisfying $\widehat{T_\tau} \uparrow \Lambda$. It follows from R's being order continuous that $T_\tau = R\left(\widehat{T_\tau}\right) \uparrow R(\Lambda)$ in $\mathcal{L}_b(L,M)$ and therefore $R(\Lambda) = T$.

2) For each$T \in \mathfrak{B}$, there exists a unique$\hat{T} \in \mathcal{L}_n^{Z(M)}(E,M)$such that$T = \hat{T} \circ \chi$ and so the mapping$T \to \hat{T}$is a Riesz isomorphism from$\mathfrak{B}$onto$\mathcal{L}_n^{Z(M)}(E,M)$and it proves the case.

Proposition 5.14: Let$L$and$M$be Archimedean Riesz spaces with$M$Dedekind complete and let Dbe a subset of$\mathcal{L}_b(L,M)^+$. Assume that$E$is a Dedekind complete$f$-module over$Z(M)$and$\chi: L \to E$is a Riesz homomorphism satisfies

(1) For all$T \in D$, there exists a unique$0 \leq \hat{T} \in \mathcal{L}_n^{Z(M)}(E,M)$with$T = \hat{T} \circ \chi$

(2) $\{\hat{T}: T \in D\}$separates the points of E.

Then (E,χ) is a Maharam extension space for the ideal $\mathcal{J}$ generated by Din $\mathcal{L}_b(L,M)$.

Proof: Let us consider the mapping $R: \mathcal{L}_n^{Z(M)}(E,M) \to \mathcal{L}_b(L,M)$ defined by $R(\Lambda) = \Lambda \circ \chi_0$ for all $\Lambda \in \mathcal{L}_n^{Z(M)}(E,M)$. Then R is positive Maharam operator since χ is a Riesz homomorphism. Now, let $0 \leq T_1, T_2 \in \mathcal{L}_b(L,M)$ and so there exists $0 \leq \widehat{T_1}, \widehat{T_2} \in \mathcal{L}_n^{Z(M)}(E,M)$ such that $R\left(\widehat{T_j}\right) = T_j$ $(j = 1,2)$. Also assume that $0 \leq \Lambda \in \mathcal{L}_n^{Z(M)}(E,M)$ such that $R(\Lambda) = T_1 + T_2$. Then $0 \leq T_1 \leq R(\Lambda)$ implies that $T_1 = R(\Lambda_1)$ for some $0 \leq \Lambda_1 \leq \Lambda$. By the way of choosing T_1, we obtain that $\Lambda_1 = \widehat{T_1}$ and so $0 \leq \widehat{T_1} \leq \Lambda$. $\Lambda - \widehat{T_1} \geq 0$and $(\Lambda - T_1) = T_2$, hence $\Lambda - \widehat{T_1} = \widehat{T_2}$ and it proves that $T_1 + T_2$ has the same property with T_1 and T_2.

As a consequence, without loss of generality, we may assume that D is upwards directed. Let $0 \leq S \in \mathcal{J}$. Then there exist $T \in D$ and $n \in \mathbb{N}$ such that $0 \leq S \leq nT = R\left(n\hat{T}\right)$. So, there exists $\hat{S} \in \mathcal{L}_n^{Z(M)}(E,M)$ satisfying that $0 \leq \hat{S} \leq n\hat{T}$and $R\left(\hat{S}\right) = S$. Now assume that $0 \leq \Lambda \in \mathcal{L}_n^{Z(M)}(E,M)$ such that $R(\Lambda) = S$. Then $n\hat{T} + \left(\Lambda - \hat{S}\right) \geq 0$and $R\left[n\hat{T} + \left(\Lambda - \hat{S}\right)\right] = R\left(n\hat{T}\right)$. By combining it with (1), $\Lambda = \hat{S}$.

Corollary 5.15: Let $0 \leq T_0 \in \mathcal{L}_b(L,M)$. Assume that E is a Dedekind complete f-module over $Z(M)$ and $\chi: L \to E$ a Riesz homomorphism such that

(1) There exists a unique $0 \leq \widehat{T_0} \in \mathcal{L}_n^{Z(M)}(E,M)$ satisfying $T_0 = \widehat{T_0} \circ \chi$

(2) $\widehat{T_0}h > 0$ for all $0 < h \in E$

Then, (E,χ) is a Maharam extension space for the ideal $\mathcal{J}_{T_0}$ generated by T_0in $\mathcal{L}_b(L,M)$.

Furthermore, if we choose E as minimal (that is, the ideal generated by $\chi(L)$ equals to E), then (E,χ) is also the minimal extension space for $\{T_0\}^{dd}$, hence there exists a unique $\hat{T} \in \mathcal{L}_n^{Z(M)}(E,M)$ such that $T = \hat{T} \circ \chi$ for all $T \in \{T_0\}^{dd}$, and the mapping $T \to \hat{T}$ is a Riesz isomorphism from $\{T_0\}^{dd}$ onto $\mathcal{L}_n^{Z(M)}(E,M)$.(Luxemburg and De Pagter, 2001)

6. EXAMPLES

1) In this example, we will consider finite dimensional spaces. Let $L = \mathbb{R}^m$ and $M = \mathbb{R}^n$. We will denote the standard bases in $\mathbb{R}^m$ and $\mathbb{R}^n$ by $\{e_1, e_2, \dots, e_m\}$ and $\{f_1, f_2, \dots, f_n\}$, respectively. Every linear operator T from L into M can be represented by $n \times m$ matrix (τ_{ij}) via these basis. Let the center $Z(M)$ correspond to all $n \times n$ diagonal matrices and can be identified with $\mathbb{R}^n$. If $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$, then the corresponding operator $\Pi_\lambda \in Z(M)$ is defined by $\Pi_\lambda = (\lambda_1\eta_1, \lambda_2\eta_2, \dots \lambda_n\eta_n)$ for all $y = (\eta_1, \eta_2, \dots \eta_n) \in M$.

Let $\mathcal{J} = \mathcal{L}_b(L, M) = \mathcal{L}(L, M)$ and $E = \mathbb{R}^{mn}$ with standard basis $\{h_1, \dots, h_{mn}\}$. If $z = (\zeta_1, \dots, \zeta_{mn}) \in M$ and $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n = Z(M)$, then we can define

$$\lambda.z = \left(\lambda_1\zeta_1, \dots, \lambda_1\zeta_m, \lambda_2\zeta_{m+1}, \dots, \lambda_2\zeta_{2m}, \dots, \lambda_n\zeta_{(n-1)m+1}, \dots, \lambda_n\zeta_{nm}\right) \quad (6.1)$$

Then E is an f-module over $Z(M)$. Let us define the Riesz homomorphism $\chi: L \to E$ by

$$\chi(x) = (\xi_1, \dots, \xi_m, \xi_1, \dots, \xi_m, \dots, \xi_1, \dots, \xi_m) \quad (6.2)$$

for all $x = (\xi_1, \dots, \xi_m)$. If $T \in \mathcal{L}_b(L, M)$ has matrix (τ_{ij}), then the linear operator $\hat{T}: E \to M$ with matrix

$$\begin{matrix} \tau_{11} & \dots & \tau_{1m} & 0 & \dots & 0 & \dots\dots & 0 & \dots & 0 \\ 0 & \dots & 0 & \tau_{21} & \dots & \tau_{2m} & \dots\dots & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \dots\dots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & \dots & 0 & \dots\dots & \tau_{n1} & \dots & \tau_{nm} \end{matrix} \quad (6.3)$$

becomes the unique $Z(M)$-linear operator with $T = \hat{T} \circ \chi$. (Luxemburg and De Pagter, 2001)

2) In this example, we will examine the Maharam extensions of kernel operators. Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathfrak{B}, \nu)$ be σ-finite measure spaces. The space of all real valued $\mathcal{A}$-measurable functions on X, with the usual identification of μ-a.e. equal functions will be denoted by $L_0(\mu) = L_0(X, \mathcal{A}, \mu)$. Moreover, $\mathcal{A}\otimes\mathfrak{B}$ is the product σ-algebra of $\mathcal{A}$ and $\mathfrak{B}$ in $X \times Y$ and $\mu\otimes\nu$ will denote the product measure. Now, let us define the

function $g\otimes f$ on $X\times Y$ by $(g\otimes f)(x,y)=g(x)f(y)$ $\mu\otimes\nu$-a.e. for all $g\in L_0(\mu)$ and $f\in L_0(\nu)$.

Assume that $L\subseteq L_0(\nu)$ and $M\in L_0(\mu)$ are order dense ideals (and hence the carriers of L and M are Y and X respectively).We know that a linear operator $T:L\to M$ is said to be an absolute kernel operator if there exists a function $T(x,y)$ in $L_0(\mu\otimes\nu)$ satisfying

$$\int|T(.,y)f(y)|d\nu(y)\in M\quad\forall f\in L \tag{6.4}$$

$$Tf(x)=\int T(x,y)\,f(y)d\nu(y)\ \ \mu\text{-a.e on } X,\ \forall f\in L \tag{6.5}$$

Here, $T\geq 0$ if and only if $T(x,y)\geq 0$ $\mu\otimes\nu$-a.e on $X\times Y$. We will denote the collection of all absolute kernel operators from L into M by $\mathcal{L}_k(L,M)$. Clearly, $\mathcal{L}_k(L,M)\subseteq\mathcal{L}_n(L,M)$ and so $\mathcal{L}_k(L,M)$ is an ideal, also a band in $\mathcal{L}_n(L,M)$.

Without loss of generality, it may be assumed that ${}^{\perp}(L_n^{\sim})=\{0\}$. Also, let us define the ideal E in $L_0(\mu\otimes\nu)$ by

$$E=\left\{h\in L_0(\mu\otimes\nu):|h|\leq \mathbb{I}_\chi\otimes f\ for\ some\ 0\leq f\in L\right\} \tag{6.6}$$

We will identify $Z(M)$ with $L_\infty(\mu)$. Hence $p\in L_\infty(\mu)$ corresponds with $\Pi_p\in Z(M)$ defined by $\Pi_p(g)=pg$ for all $g\in M$. It will be defined as $p.h=(p\otimes\mathbb{I}_\Upsilon)h$ for $p\in L_\infty(\mu)$ and $h\in E$ that gives E the structure of a Dedekind complete f-module over $Z(M)$. The Riesz homomorphism $\chi:L\to M$ is given by $\chi(f)=\mathbb{I}_\chi\otimes f$ for all $f\in L$. For $T\in\mathcal{L}_k(L,M)$, $\hat{T}h$, defined by

$$\left(\hat{T}h\right)(x)=\int T(x,y)h(x,y)\,d\nu(y)\ \ \mu\text{-a.e on } X \tag{6.7}$$

for all $h\in E$ is well defined and includes in M. So, $\hat{T}\in\mathcal{L}_n(E,M)$ and $\hat{T}\circ\chi=T$. Also, if $p\in L_\infty(\mu)$ and $h\in E$, then

$$\begin{aligned}\hat{T}(p.h)(x)&=\int T(x,y)p(x)\,h(x,y)d\nu(y)\\&=p(x)\int T(x,y)h(x,y)\,d\nu(y)=p(x)\left(\hat{T}h\right)(x)\end{aligned} \tag{6.8}$$

μ-a.e on X and it concludes that $\hat{T}\in\mathcal{L}_n^{Z(M)}(E,M)$.

Lemma 6.1: $\left\{\hat{T}:0\leq T\in\mathcal{L}_k(L,M)\right\}$ separates the points of E.

Proof: Let $0<h\in E$ and pick $\{(x,y)\in X\times Y:h(x,y)>0\}$. As the carrier of M is X, there exists pairwise disjoint $A_n\in\mathcal{A}$ $(n=1,2,\dots)$ such that $X=\bigcup_{n=1}^{\infty}A_n$ and

$\mathbb{I}_{A_n} \in M$. Also, since ${}^{\perp}(L_n^{\sim}) = \{0\}$, there exist pairwise disjoint $B_n \in \mathfrak{B}$ $(n = 1,2,\dots)$ such that $Y = \bigcup_{n=1}^{\infty} B_n$ and $\mathbb{I}_{B_n} \in L_n^{\sim}$. If $\mu \otimes \nu(G) > 0$, then there exists n such that $\mu \otimes \nu\big(G \cap (A_n \times B_n)\big) > 0$. Assume that $0 \leq T \in \mathcal{L}_k(L, M)$ is the operator with kernel $T(x, y) = \mathbb{I}_{A_n}(x)\mathbb{I}_{B_n}(y)$, so $\hat{T}h > 0$.

Lemma 6.2: If $\Lambda \in \mathcal{L}_n^{Z(M)}(E, M)$ satisfies $\Lambda \circ \chi = 0$, then $\Lambda = 0$.

Proof: If $A \in \mathcal{A}$ and $B \in \mathfrak{B}$ with $\mathbb{I}_B \in L$, then

$$\Lambda(\mathbb{I}_{A\times B}) = \Lambda(\mathbb{I}_A \otimes \mathbb{I}_B) = \mathbb{I}_A . \Lambda(\chi \mathbb{I}_B) = 0 \tag{6.9}$$

Let $B_0 \in \mathfrak{B}$ with $\mathbb{I}_{B_0} \in L$ and $\mathfrak{G}_0$ be the collection

$$\mathfrak{G}_0 = \{G \in \mathcal{A} \otimes \mathfrak{B} : G \subseteq X \times B_0, \Lambda(\mathbb{I}_G) = 0\} \tag{6.10}$$

Then, it follows that all sets of the form $\bigcup_{i=1}^{n} A_i \times B_i$ with $A_i \in \mathcal{A}$ and $B_i \in \mathfrak{B}$ such that $B_i \subseteq B_0$, includes in $\mathfrak{G}_0$. Also, since Λ is order continuous, $\mathfrak{G}_0$ is a monotone class of subsets of $X \times B_0$. Hence, $\mathfrak{G}_0 = \{G \in \mathcal{A} \otimes \mathfrak{B} : G \subseteq X \times B_0\}$. The carrier of L equals to Y and therefore there exist $B_n \in \mathfrak{B}$ $(n = 1,2,\dots)$ such that $\mathbb{I}_{B_n} \in L$ and $B_n \uparrow B$. If $G \in \mathcal{A} \otimes \mathfrak{B}$ such that $\mathbb{I}_G \in E$, then $\mathbb{I}_{G \cap B_n} \uparrow \mathbb{I}_G$ and so $\Lambda(\mathbb{I}_G) = 0$. By the order continuity of Λ, it concludes that $\Lambda = 0$.

By combining Lemma 6.1 and 6.2 with the Section 5, we obtain that (E, χ) is the minimal Maharam extension space of the ideal $\mathcal{J}$ generated by $\mathcal{L}_k(L, M)^+$. This gives us that $\mathcal{L}_k(L, M)$ is an ideal in $\mathcal{L}_n(L, M)$. (Luxemburg and De Pagter, 2001)

7. RESULTS AND RECOMMENDATIONS

The subject of positive operators and Riesz spaces is a wide area which is very open to new researches and so are Maharam operators and extensions. In this thesis, Maharam operatorsa are introduced and Maharam extensions are constructed not for a single positive operator but for some collections of positive operators.

As we see in Section 6, if the Riesz spaces vary, the specific properties of Maharam extension will also vary according to the choice of the space while the main properties remain same. It is possible to say that working on different spaces may provide more certain grounds and new research areas. Besides that, other than the algebraic construction, the topological or the Banach lattice structure of Maharam extensions may also be discussed and some new properties may come to light in this perspective.

Maharam operators may be connected to the other types of linear operators and the connections between Maharam extension spaces and other types of extension spaces may be constructed via transformations and linear mappings, respectively. These transformations can be examined if they have homomorphism structure. According to the obtained results, new properties of these extension spaces may be found.

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