SOME MATHEMATICAL PROBLEMS FOR THE STOCHASTIC NAVIER STOKES EQUATIONS

by

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Abstract

This thesis collects three interrelated projects related to time-wise approximation, bound estimates and control of stochastic Navier-Stokes equations in a smooth non-periodic bounded domain $\mathcal{O} \subset \mathbb{R}^2$ with a multiplicative noise.

First, we show that in an open bounded domain $\mathcal{O}$, we have

$$\lim_{n\to\infty} \mathbb{E}[\sup_{t\in[0,T]} \phi_1(\|(u(t) - u^n(t))\|_V^2)] = 0$$

for any deterministic time $T > 0$, for a specified moment function $\phi_1(x) = \log(1+x)^{1-\epsilon}$ with $0 < \epsilon < 1$, where $u^n(t,x)$ corresponds to the Galerkin approximation of the solution $u(t,x)$. Similarly, we prove that

$$\lim_{n\to\infty} \mathbb{E}[\sup_{t\in[0,T]} \phi_2(\|(u(t) - u^n(t))\|_H^p)] = 0$$

for any $p > 0$, for a specific function $\phi_2(x) = x^{1-\epsilon}$ with $0 < \epsilon < 1$ and for any deterministic time $T > 0$. Finally, we show that

$$\mathbb{E}\big[\sup_{0\le t\le T} \exp\big(\|u\|_H/K\big)\big] < \infty,$$

for a constant K with specified regularity assumptions on the initial data.

Second, we show that a special linearized scheme $\{u_n\}_{n\geq 1}$ for the convergence of the SNSE gives the same convergence results that we have achieved through the Galerkin approximation. Namely, we show that

$$\lim_{n\to\infty} \mathbb{E}[\sup_{t\in[0,T]} \phi_1(\|(u(t) - u_n(t))\|_V^2)] = 0$$

for any deterministic time $T > 0$, for a specified moment function $\phi_1(x)$. Moreover, we prove that

$$\lim_{n\to\infty} \mathbb{E}[\sup_{t\in[0,T]} \phi_2(\|(u(t) - u_n(t))\|_H^p)] = 0$$

for any $p > 0$, for a specific function $\phi_2(x)$ and for any deterministic time $T > 0$, where

$$\phi_1(x) = (\log(1 + x))^{1-\epsilon}$$

$$\phi_2(x) = x^{1-\epsilon},$$

with $0 < \epsilon < 1$.

Third, we solve an optimal control problem verifying the existence of feedback controls that are optimal in $\sup_{t\in[0,T]}$ sense. Namely, we show that there exists an optimal feedback control ϕ for a specific types of cost functional

$$J(\phi) = \mathbb{E} \sup_{t\in[0,T]} (\varphi(\mathcal{L}[t, u_\phi(t), \phi(t)])),$$

of the SNSE in 2D on an open bounded nonperiodic domain $\mathcal{O}$ given that the control set $\mathcal{U}$ is compact, where $\varphi(x) = \log(1+x)^{1-\epsilon}$ with $0 < \epsilon < 1$.

Chapter 1

Notation and Functional Setting

1 Introduction

We consider the stochastic Navier-Stokes equations (SNSE) in 2D in a smooth non-periodic bounded domain $\mathcal{O} \subset \mathbb{R}^2$ with a multiplicative white noise

$$\partial_t u + (u \cdot \nabla) u - \nu \Delta u + \nabla p = f + g(u) d\mathcal{W} \tag{1.1}$$

$$\nabla \cdot u = 0 \tag{1.2}$$

$$u(0) = u_0 \tag{1.3}$$

[BKL, CG, C, CP, DD, FG, FR, GV, M, MR2, MS, O, S] with $u_0 \in L^4(\Omega; H) \cap L^2(\Omega; V)$ and Dirichlet boundary condition $u = 0$ on $[0, \infty) \times \partial\mathcal{O}$. Here $u = (u_1, u_2)$ represents the velocity field, p represents the pressure, and ν stands for the viscosity, whereas f stands for the deterministic force. Moreover, $g(u)\mathcal{W} = \sum_k g_k(u) e_k W_k$ stand for the infinite dimensional Brownian motion, where each W_k is the standard one dimensional Brownian motion and $g_k(u)$ are the corresponding Lipschitz coefficients. First, we recall the deterministic and probabilistic framework used throughout the thesis.

2 Deterministic Framework

Let $\mathcal{O}$ be a smooth bounded open connected subset of $\mathbb{R}^2$, and let $\mathcal{V} = \{u \in C_0^\infty(\mathcal{O}) : \nabla \cdot u = 0\}$. Denote by H and V the closures of $\mathcal{V}$ in $L^2(\mathcal{O})$ and $H^1(\mathcal{O})$ respectively. The spaces H and V are identified by

$$H = \{u \in L^2(\mathcal{O}) : \nabla \cdot u = 0, u \cdot N|_{\partial\mathcal{O}} = 0\}, \tag{2.1}$$

$$V = \{u \in H_0^1(\mathcal{O}) : \nabla \cdot u = 0\} \tag{2.2}$$

(cf. [CF2, T]). Here N is the outer pointing normal to $\partial\mathcal{O}$. On H we take the $L^2(\mathcal{O})$ inner product and the norm as

$$\begin{aligned} \langle u, v\rangle &= \int_{\mathcal{O}} u \cdot v dx \\ \|u\|_H &= \sqrt{\langle u, u\rangle}. \end{aligned} \tag{2.3}$$

Let $\mathcal{P}_H$ be the Leray-Hopf projector of $L^2(\mathcal{O})$ onto H. Recall that for $u \in L^2(\mathcal{O})$ we have $\mathcal{P}_H u = (1 - \mathcal{Q}_H)u$ where $\mathcal{Q}_H u = \nabla\pi_1 + \nabla\pi_2$ and $\pi_1, \pi_2 \in H^1(\mathcal{O})$ are solutions of the problems

$$\begin{aligned} \Delta\pi_1 &= \nabla \cdot u \text{ in } \mathcal{O} \\ \pi_1 &= 0 \text{ on } \partial\mathcal{O} \end{aligned} \tag{2.4}$$

and

$$\Delta\pi_2 = 0 \text{ in } \mathcal{O}$$
$$\nabla\pi_2 \cdot N = u - \nabla\pi_1 \text{ on } \partial\mathcal{O} \tag{2.5}$$

([CF2, T]). Let

$$A = -\mathcal{P}_H \Delta \tag{2.6}$$

be the Stokes operator with the domain $\mathcal{D}(A) = V \cap H^2(\mathcal{O})$. The dual of $V = \mathcal{D}(A^{1/2})$ with respect to H is denoted by $V' = \mathcal{D}(A^{-1/2})$. Here A is defined as a bounded, linear map from V to V' via

$$\langle Au, v\rangle = \int_{\mathcal{O}} \nabla u \cdot \nabla v dx, \qquad u, v \in V,$$

with the corresponding norm defined as

$$\|u\|_V^2 = \langle Au, u\rangle = \langle A^{1/2}u, A^{1/2}u\rangle, \qquad u \in V.$$

By the theory of symmetric, compact operators for A^{-1}, there exists an orthonormal basis $\{e_k\}$ for H consisting of eigenfunctions of A. The corresponding eigenvalues $\{\lambda_k\}$ form an increasing, unbounded sequence

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots$$

We also define the nonlinear term as a bilinear mapping $V \times V$ to V' via

$$B(u, v) = \mathcal{P}_H(u \cdot \nabla v).$$

The deterministic force f is assumed to be bounded with values in H. Note that the cancellation property $\langle B(u, v), v\rangle = 0$ holds for $u, v \in V$.

3 Notation

Let $(S, \|\cdot\|_S)$ be a Banach space. We denote by $\mathcal{L}^2_S(\Omega)$ the linear space of all functions $u :\to S$ that are $\mathcal{F}$-measurable and $\mathbb{E}\|u\|_S^2 < \infty$. We also denote by $\mathcal{L}^2_S(\Omega \times [0, T])$ the linear space of all processes $u : \Omega \times [0, T] \to S$ that are $\mathcal{F} \times \mathcal{B}_{[0,T]}$ measurable, adapted to the filtration $(\mathcal{F}_{t\in[0,T]})$ and $\mathbb{E} \int_0^T \|u\|_S^2 dt < \infty$. Weak convergence is denoted by $\rightharpoonup$.

4 Stochastic Framework

In this section, we recall the necessary background material for stochastic analysis in infinite dimensions needed in this paper (cf. [DZ, DGT, F, PR]). Fix a stochastic basis $\mathcal{S} = (\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}, \mathcal{W})$, which consists of a complete probability space $(\Omega, \mathbb{P})$, equipped with a complete right-continuous filtration $\mathcal{F}_t$, and a cylindrical Brownian motion $\mathcal{W}$, defined on a separable Hilbert space U adapted to this filtration.

Given a separable Hilbert space X, we denote by $L_2(U, X)$ the space of Hilbert-Schmidt operators from U to X, equipped with the norm $\|G\|_{L_2(U,X)} = (\sum_k \|G\|_X^2)^{1/2}$

[DZ]. For an X-valued predictable process $G \in L^2(\Omega; L^2_{\mathrm{loc}}([0,\infty]); L_2(U,X))$, we define the Itô stochastic integral

$$\int_0^t G d\mathcal{W} = \sum_k \int_0^t G_k dW_k \tag{4.1}$$

which lies in the space $\mathcal{O}_X$ of X-valued square integrable martingales. We also recall the Burkholder-Davis-Gundy inequality: For all $p \geq 1$ we have

$$\mathbb{E}\left[\sup_{t\in[0,T]} \left\| \int_0^t G d\mathcal{W} \right\|_X^p\right] \leq C\mathbb{E}\left[\int_0^T \|G\|^2_{L_2(U,X)}\right]^{p/2} \tag{4.2}$$

for some $C = C(p) > 0$. Given a pair of Banach spaces X and Y, we denote by $\mathrm{Lip}_u(X,Y)$ the collection of continuous functions $h\colon [0,\infty) \times X \to Y$ which are sublinear

$$\|h(t,x)\|_Y \leq K_Y(1+\|x\|_X), t \geq 0, x \in X \tag{4.3}$$

and Lipschitz

$$\|h(t,x) - h(t,y)\|_Y \leq K_Y\|x-y\|_X, t \geq 0, x,y \in X \tag{4.4}$$

for some constant $K_Y > 0$ independent of t. The noise term $g(u)d\mathcal{W}$, which is defined by

$$g = \{g_k\}_{k\geq 1}\colon [0,\infty) \times H \to L_2(U,H) \tag{4.5}$$

satisfies

$$\|g(t,x)\|_{L_2(U,\mathcal{D}(A^{j/2}))} \leq K_j(1+\|x\|_{\mathcal{D}(A^{j/2})}) \text{ for } j \in \{0,1,2\} \tag{4.6}$$

and

$$\|g(t,x)-g(t,y)\|_{L_2(U,\mathcal{D}(A^{j/2}))} \leq K_j\|x-y\|_{\mathcal{D}(A^{j/2})} \text{ for } j \in \{0,1,2\}. \tag{4.7}$$

In particular, we have

$$g \in \mathrm{Lip}_u(H, L^2(U,H)) \cap \mathrm{Lip}_u(V, L_2(U,V)) \cap \mathrm{Lip}_u(\mathcal{D}(A), L_2(U,\mathcal{D}(A))). \tag{4.8}$$

Given $u \in L^2(\Omega; L^2([0,T];H))$ and g as above, the stochastic integral $\int_0^t g(u)d\mathcal{W}$ is a well-defined H-valued Itô stochastic integral that is predictable and is such that

$$\left\langle \int_0^t g(u)d\mathcal{W}, v \right\rangle = \sum_k \int_0^t \langle g_k(u), v\rangle d\mathcal{W}_k$$

holds for all $v \in H$.

We consider *strong pathwise solutions* in the PDE sense, i.e., solutions bounded in time with values in V, square integrable in time with values in $\mathcal{D}(A)$, and *strong* in the probabilistic sense, i.e., the driving noise and the filtration are given in advance.

Definition 1.4.1. *Let g be as in* (4.8) *predictable, and let $f \in L^1(\Omega; L^4([0,T];V'))$ be predictable. Assume that the initial data $u_0 \in L^4(\Omega;H) \cap L^2(\Omega;V)$ is $\mathcal{F}_0$ measurable.*

We call a process $(u(t))_{t\geq 0}$ *in* $\mathcal{L}^2_V(\Omega\times[0,T])$ *with* $\mathbb{E}\|u(t)\|^2_H<\infty$ *for all* $t\in[0,T]$ *a solution of the SNSE if it satisfies the equation*

$$\begin{aligned}\langle u(t),v\rangle+\int_0^t\langle Au(s),v\rangle ds=\langle u_0,v\rangle+\int_0^t\langle B(u,u),v\rangle ds\\ +\int_0^t\langle f,v\rangle ds+\int_0^t\langle g(u),v\rangle dW_s,\end{aligned}\tag{4.9}$$

for all $v\in V$ *and* $t\in[0,T]$ *and* $\omega\in\Omega$ *a.s., where the stochastic integral is understood in the Ito sense.*

Definition 1.4.2. Let g be as in (4.8) predictable, and let $f\in L^1(\Omega;L^4([0,T);V'))$ be predictable. Assume that the initial data $u_0\in L^4(\Omega;H)\cap L^2(\Omega;V)$ is $\mathcal{F}_0$ measurable. The pair (u,τ) is called a pathwise strong solution of the system if τ is a strictly positive stopping time, $u(\cdot\wedge\tau)$ is a predictable process in H such that

$$u(\cdot\wedge\tau)\in L^2(\Omega;C([0,T];V))\tag{4.10}$$

with

$$u\mathbf{1}_{t\leq\tau}\in L^2(\Omega;L^2([0,T];\mathcal{D}(A)))\tag{4.11}$$

and if

$$\langle u(t\wedge\tau),v\rangle+\int_0^{t\wedge\tau}\langle \nu Au+B(u,u)-f,v\rangle dt=\langle u_0,v\rangle+\sum_k\int_0^{t\wedge\tau}\langle g_k(u),v\rangle dW_k\tag{4.12}$$

holds for every $v \in H$. Moreover, (u, ξ) is called a maximal pathwise strong solution if ξ is a strictly positive stopping time and there exists a non-decreasing sequence of stopping times τ_n such that $\tau_n \to \xi$ and (u, τ_n) is a local strong solution and

$$\sup_{t\in[0,\tau_n]} \|u\|_V^2 + \nu \int_0^{\tau_n} \|Au\|_H^2 dt \geq n \tag{4.13}$$

on the set $\{\xi < \infty\}$. Such a solution is called global if $\mathbb{P}(\xi < \infty) = 0$.

We proceed with the definition of the Galerkin system.

Definition 1.4.3. An adapted process u^n in $C([0,T]; H_n)$, where $H_n = \mathcal{L}\{e_1, \dots, e_n\}$, is a solution to the Galerkin system of order n if for any v in H_n

$$\begin{aligned} &d\langle u^n, v\rangle + \langle \nu A u^n + B(u^n), v\rangle dt = \langle f, v\rangle dt + \sum_{k=1}^{\infty} \langle g_k(u^n), v\rangle dW_k \\ &\langle u^n(0), v\rangle = \langle u_0, v\rangle. \end{aligned} \tag{4.14}$$

We may also rewrite 4.14 as equations in H_n, i.e.,

$$\begin{aligned} &du^n + (\nu A u^n + P_n B(u^n))dt = P_n f dt + \sum_{k=1}^{\infty} P_n g_k(u^n) dW_k \\ &u^n(0) = P_n u_0 = u_0^n. \end{aligned}$$

Chapter 2

Background Work and Summary

In this chapter, we briefly outline the background work leading to our results and summarize our results proved in the subsequents chapters. The parallel works that have dealt with $\mathcal{L}^2_V([0,T]\times\Omega)$ criteria convergence and control criteria rather than investigating the supremum in the whole time interval $[0,T]$ are [B, B3, B2, B4]. We briefly explain these results.

In [B3], the main results are as follows

Theorem 2.0.1. *The equation 1.4.1 has a solution in the space $\mathcal{L}^2_V(\Omega\times[0,T])$. The solution is unique almost surely and has in H almost surely continuous trajectories.*

Theorem 2.0.2. *Let u and u_n be the corresponding solution of the SNSE in equation 1.4.1 and Galerkin approximation in equation 4.15 respectively. Given that $\mathbb{E}\|u_0\|^4_H<\infty$ and the initial data are regular enough , then, for each fixed time T, the following convergence holds:*

$$\mathbb{E}\int_0^T \|u-u_n\|^2_V dt \to 0, \tag{0.1}$$

as $n\to\infty$.

Secondly, in [B2], a linearized scheme is proposed. Namely, for each $n = 0, 1, 2, 3, ...$, we consider the linear evolution equation with multiplicative noise

$$\begin{aligned} \langle u_n(t), v\rangle + \int_0^t \langle Au_n(s), v\rangle ds = \langle u_0, v\rangle ds \\ + \int_0^t \langle B(u_{n-1}(s), u_n(s)), v\rangle ds + \int_0^t \langle \phi, v\rangle ds + \int_0^t \langle g(u_{n-1}, v)\rangle dW_s, \end{aligned} \tag{0.2}$$

for all $v \in H$, $t \in [0, T]$, for almost every $\omega \in \Omega$, where we let $u_0(t) \equiv u_0$ be an H-valued $\mathcal{F}_0$ measurable random variable with $u_0 \in L^4(\Omega; H) \cap L^2(\Omega; V)$. The reason to work on this evolution scheme is that the Galerkin method is useful to prove the *existence* of the solution u of the SNSE, but from numerical perspective, it is complicated to implement it due to the the nonlinear terms. Since $B(u_{n-1}, u_n)$ is linear as opposed to the Galerkin approximation in this scheme, this linearized scheme seems to be more efficient than the Galerkin approximation. The main results in [B2] are as follows.

Theorem 2.0.3. *For each $n \in \mathbb{N}_0$ equation 0.2 has an almost surely unique solution $u_n \in \mathcal{L}^2_V(\Omega \times [0, T])$ with almost surely continuous trajectories in H.*

Theorem 2.0.4. *The following convergences hold*

$$\lim_{n\to\infty} \mathbb{E} \int_0^T \|u - u_n\|_V^2 ds = 0 \tag{0.3}$$

and

$$\lim_{n\to\infty} \mathbb{E}\|u - u_n\|_H^2 = 0, \tag{0.4}$$

for all $t \in [0, T]$.

Thirdly, [B] considers the SNSE controlled by linear and continuous feedback controls as well as bounded controls. Denoting $\mathcal{U}$ as the set of all admissible linear and continuous controls, which is assumed to be the set of all functions $\phi : [0, T] \times H \to H$ satisfying the following conditions: For all $t \in [0, T]$, we have $\phi(t, .) \in \mathcal{L}(H)$ where $\mathcal{L}(H)$ being the space of continuous linear feedback controls and

$$\|\phi(t_1, x_1) - \phi(t_2, x_2)\|_H^2 \leq \alpha|t_1 - t_2|^2 + \mu\|x_1 - x_2\|_H^2,$$

for all $t_1, t_2 \in [0, T], x_1, x_2 \in H$, where $\alpha, \mu > 0$ are given constants. In [B], the solution u_ϕ of the SNSE via the corresponding control force ϕ reads as

$$\begin{aligned} &\langle u(t), v\rangle + \int_0^t \langle Au(s), v\rangle ds = \langle u_0, v\rangle ds \\ &\quad + \int_0^t \langle B(u(s), u(s)), v\rangle ds + \int_0^t \langle \phi(t, u_\phi), v\rangle ds + \int_0^t \langle g(u, v)\rangle dW_s, \end{aligned} \tag{0.5}$$

for all $v \in V$, $t \in [0, T]$. a.e. $\omega \in \Omega$, by the feedback controls $\phi \in \mathcal{U}$. The following cost functional in [B] is taken into consideration

$$J(\phi) = \mathbb{E} \int_0^T \mathcal{L}[s, u_\phi, \phi(s, u_\phi)]ds + \mathbb{E}\mathcal{K}[u_\phi(T)], \tag{0.6}$$

with $\phi \in \mathcal{U}$, where $\mathcal{L} : [0,T] \times V \times H \to \mathbb{R}_+$, and $\mathcal{K} : H \to \mathbb{R}_+$ satisfying

$$|\mathcal{L}(t, x_1, y_1) - \mathcal{L}(t, x_2, y_2)| \leq C(\|x_1 - x_2\|_H^2 + \|y_1 - y_2\|_H^2) \tag{0.7}$$

$$|\mathcal{K}(x_1) - \mathcal{K}(x_2)| \leq C\|x_1 - x_2\|_H^2, \tag{0.8}$$

for all $t \in [0,T]$, $x_1, x_2, y_1, y_2 \in H$, where C is a constant. We state the following result from [B4]

Lemma 2.0.5. *Let $\{\phi_n\}_{n\geq 1}$ be a sequence in $\mathcal{U}$ and let $\phi \in \mathcal{U}$ be such that*

$$\lim_{n\to\infty} \int_0^T \|\phi_n - \phi\|_H^2 dt = 0, \tag{0.9}$$

then we have $\lim_{n\to\infty} J(\phi_n) = J(\phi)$.

To extend these results to *time-wise* namely in $\sup_{t\in[0,T]}$ case, we first need to have that the expression $\sup_{t\in[0,T]}\|u\|_V^2$ is well defined. For that, the global existence of SNSE in 2D up to a sequence of stopping times $\{\tau_n\}_{n\geq 0}$ in the sense of Definition 1.4.1 is proven in [GZ]. There, they have shown that those stopping for any of those stopping times τ

$$\mathbb{E}[\sup_{t\in[0,\tau]} \|u\|_V^2] < \infty.$$

$$\mathbb{E}[\sup_{t\in[0,\tau]} \|u - u_n\|_V^2] \to 0,$$

as $n \to \infty$. They left as an open question, whether it is possible to replace the stopping time τ above with the deterministic time T. Namely, whether we could have

$$\mathbb{E}[\sup_{t\in[0,T]} \|u\|_V^2] < \infty,$$

$$\mathbb{E}[\sup_{t\in[0,T]} \|u - u_n\|_V^2] \to 0,$$

for a deterministic time T. The importance to study this problem is that using a deterministic time T, such moment bounds can used to study statistical properties of the long time behavior of the SNSE. A partial answer to bound $\mathbb{E}[\sup_{t\in[0,T]} \|u\|_V^2]$ has been given by [KV], where the authors have shown that it holds

$$\mathbb{E}[\sup_{t\in[0,T]} \log(1 + \log(1 + \|u\|_V^2))] < \infty.$$

The main difficulty in bounding $\mathbb{E}[\sup_{t\in[0,T]} \|u\|_V^2]$ lies in estimating the nonlinear term $B(u, u)$. As opposed to the periodic domain, on a bounded domain, we do not have $\langle B(u, u), Au\rangle = 0$ for $t > 0$. In our work [KUZ] as well as in Chapter 3, we replace the estimating function $\varphi(x) = \log(1 + \log(1 + x))$ with the stronger function $\varphi(x) = \log(1 + x)$. We show, moreover, via a uniform integrability argument that the

Galerkin approximations $\{u_n\}_{n\geq 1}$ converge to the solution u of the SNSE in 2D with $\varphi(x) = \log(1+x)^{1-\epsilon}$ as well. Namely, we have proved that

$$\mathbb{E}[\sup_{t\in[0,T]} \log(1+\|u\|_V^2)] < \infty$$

$$\mathbb{E}[\sup_{t\in[0,T]} \log(1+\|u-u_n\|_V^2)^{1-\epsilon}] \to 0.$$

as $n\to\infty$ for any deterministic time T and for $0<\epsilon<1$.

Furthermore, when we work with the H-norm for u, $\|u\|_H$, by using $\langle B(u,u),u\rangle = 0$ and increasing the initial data regularity accordingly, we have also shown that

$$\mathbb{E}[\sup_{t\in[0,T]} \|u-u_n\|_H^p] \to 0$$

for any $p>0$, where u_n corresponds to the Galerkin approximation of u. Finally, we have proved that

$$\mathbb{E}\left[\sup_{0\leq t\leq T} \exp\left(\frac{\|u\|_H}{K}\right)\right] < \infty,$$

for a constant K, if the initial data is essentially bounded as well.

Secondly, we adapt the linearized alternative scheme in [B2] to our framework. We prove the analogue results of [B2]. Namely, using this linearized scheme instead of the Galerkin approximation, we show the global existence of SNSE in 2D up to a sequence of stopping times $\{\tau_n\}_{n\geq 0}$, which converges to infinity almost surely. Moreover, we prove analogously to our previous work that

$$\mathbb{E}[\sup_{t\in[0,T]} \log(1+\|u-u_n\|_V^2)^{1-\epsilon}] \to 0.$$

as $n\to\infty$, and by increasing the initial data regularity, we have shown that

$$\mathbb{E}[\sup_{t\in[0,T]} \|u-u_n\|_H^p] \to 0$$

for any $p>0$.

Our third work in this direction is the investigation of the existence of optimal feedback controls for the minimization of the vorticity in the SNSE, controlled by different external forces. Namely, similar to [B], we consider the SNSE with multiplicative noise

$$\begin{aligned}\langle u(t),v\rangle + \int_0^t \langle Au(s),v\rangle ds = \langle u_0,v\rangle ds \\ + \int_0^t \langle B(u(s),u(s)),v\rangle ds + \int_0^t \langle \phi(t,u_\phi),v\rangle ds + \int_0^t \langle g(u,v)\rangle dW_s,\end{aligned} \tag{0.10}$$

for all $v \in V$, $t \in [0,T]$, for almost every $\omega \in \Omega$, by the feedback controls $\phi \in \mathcal{U}$.,where we assume that the initial data $u_0 \in L^4(\Omega; H) \cap L^2(\Omega; V)$ is $\mathcal{F}_0$ measurable. Moreover, we also suppose that the set of feedback controls $\phi \subset \mathcal{U}$ satisfy:

$$\|\phi(\omega,t)\|_V \leq K, \text{ for all } \omega \times t \in \Omega \times [0,T].$$

$$\|\phi(t,x_1) - \phi(t_2,x_2)\|_V^2 \leq C_1|t_1 - t_2|^2 + C_2\|x_1 - x_2\|_V^2,$$

for all $t \in [0,T]$ and $x_1, x_2 \in V$. We consider the following cost functional

$$J(\phi) = \mathbb{E} \sup_{t\in[0,T]} (\varphi(\mathcal{L}[t, u_\phi(t), \phi(t)]))$$

where $\mathcal{L} : [0,T] \times V \times H \to \mathbb{R}_+$ satisfies the followings:

$$|\mathcal{L}(t,x_1,y_1) - \mathcal{L}(t,x_2,y_2)| \leq C(\|x_1 - x_2\|_V^2 + \|y_1 - y_2\|_H^2)$$

denoting

$$\varphi(x) = \log(1+x)^{1-\epsilon},$$

with $0 < \epsilon < 1$. Based on these assumptions and framework, we have proved the existence of an optimal control, namely we have shown that there exists an optimal feedback control ϕ^* satisfying

$$J(\phi^*) = \min_{\phi\in\mathcal{U}} J(\phi)$$

Chapter 3

Timewise Galerkin Approximations and Norm Estimates

1 Introduction

This chapter is based on [KUZ]. We address the convergence properties of the Galerkin approximation to the stochastic Navier-Stokes equations and obtain new estimates on the convergence in the strong norm. Namely, the goal of this chapter is to address the convergence of the Galerkin pointwise in time for the V norm in the case of a non-periodic domain with Dirichlet boundary conditions. In this case, it is easy to obtain results in this direction up to a suitable stopping time. However, the expected value of the second moment of the norm $\|u(t)\|_V^2$ for any fixed *non-random* time t is an open problem. By the same token, it is not known whether the expected value of $\|u(t) - u_n(t)\|_V^2$ converges to 0 as $n \to \infty$. A step toward establishing the finiteness of the expected value of $\|u(t)\|_V^2$ for $t > 0$ was obtained in [KV], where it was proven that

$$\mathbb{E}\left[\sup_{0\leq t\leq T} \tilde{\phi}(\|u\|_V^2)\right] < \infty \tag{1.1}$$

where

$$\tilde{\phi}(\tau) = \log(2 + \log(2 + \tau)), \tau \in (0, \infty) \tag{1.2}$$

The aim of this chapter is twofold. First, we strengthen the main result in [KV] by showing that (1.1) holds with

$$\phi(\tau) = \log(2 + \tau) \tag{1.3}$$

instead of $\tilde{\phi}$ (cf. Theorem 3.3.4 below). The second goal is to obtain convergence of the Galerkin approximation in the V norm. Namely, we prove that

$$\mathbb{E}\left[\sup_{[0,T]} \phi(\|u - u^n\|_V^2)^{1-\epsilon}\right] \to 0 \tag{1.4}$$

as $n \to \infty$ for all $\epsilon > 0$. In addition, we obtain two results on the 2D SNSE of independent interest. The first result states that

$$\mathbb{E}[\sup_{t\in[0,T]} \|(u(t) - u^n(t))\|_H^q] \to 0$$

for all $q \geq 2$ while the second concerns finiteness of the Zygmund type norm

$$\mathbb{E}\left[\sup_{0\leq t\leq T} \exp\left(\frac{\|u\|_H}{K}\right)\right] \tag{1.5}$$

for K sufficiently large.

2 Main Results in this Chapter

The first main result is related to the convergence of Galerkin approximations in the V norm.

Theorem 3.2.1. *Let $\epsilon \in (0,1)$ and let $T > 0$ be arbitrary. Suppose that u is a solution to the equation as in Definition 1.4.1, and let u^n be the corresponding Galerkin approximation as in Definition 1.4.3. Then we have*

$$\mathbb{E}\left[\sup_{[0,T]} \phi_1(\|u - u^n\|_V^2)\right] \to 0, \tag{2.1}$$

with $\phi_1(x) = (\log(1+x))^{1-\epsilon}$, as $n \to \infty$.

Our second statement gives the convergence in the H norm of the Galerkin approximations u^n on the whole bounded C^∞ domain $\mathcal{O}$.

Theorem 3.2.2. *Let u be the solution as in Definition 1.4.1 and let u^n be the corresponding Galerkin approximation as in 1.4.3. If, additionally, we have $f \in L^{2k}(\Omega; L^{2k}([0,\infty); V'))$ and $u_0 \in L^{2k+2}(\Omega; H) \cap L^2(\Omega; V)$, then*

$$\mathbb{E}\left[\sup_{[0,T]} \|u - u^n\|_H^{2k(1-\epsilon)}\right] \to 0, \tag{2.2}$$

for any deterministic time $T > 0$ and any $\epsilon \in (0,1)$ as $n \to \infty$.

Our third result shows that a Zygmund type norm of $\|u\|_H$ is bounded up to any deterministic time $T > 0$.

Theorem 3.2.3. *Let u be the solution as in Definition 1.4.1. If $f \in L^1(\Omega; L^\infty([0,T]; V'))$ and $u_0 \in L^\infty(\Omega; V)$, then*

$$\mathbb{E}\left[\sup_{0\le t\le T} \exp\left(\frac{\|u\|_H}{K}\right)\right] < \infty \tag{2.3}$$

for any deterministic time $T > 0$, where K is a sufficiently large constant.

The proofs of the above theorems are given in the remaining sections in this chapter.

3 Timewise Galerkin Convergence in V

In this section, we give the proof of the first main result, Theorem 3.2.1. First, we recall a statement from [GZ].

Theorem 3.3.1. *Let $\{u^n\}$ be the sequence of solutions of (4.14) with u being the solution as in Definition 1.4.1 and with g, f, and u_0 as in Definition 1.4.2.*

$$\mathcal{T}_n^{M,T} = \{\tau \le T : (\sup_{t\in[0,\tau]} \|u_n\|_V^2 + \nu \int_0^\tau \|Au_n\|_H^2 dt)^{1/2} \le M\} \tag{3.1}$$

Take

$$\mathcal{T}_{m,n,m-1,n-1}^{M,T} := \mathcal{T}_m^{M,T} \cap \mathcal{T}_n^{M,T} \cap \mathcal{T}_{m-1}^{M,T} \cap \mathcal{T}_{n-1}^{M,T}, \tag{3.2}$$

then we have for any $T > 0$ and $M > 1$

1. For any $T > 0$ and $M > 1$

$$\lim_{n\to\infty} \sup_{m\geq n} \sup_{\tau\in\mathcal{T}^{M,T}_{m,n,m-1,n-1}} \mathbb{E}[\sup_{t\in[0,\tau]} \|u_m - u_n\|_V^2 + \int_0^\tau \|A(u_m - u_n)\|_H^2 dt] = 0 \quad (3.3)$$

2.

$$\lim_{s\to 0} \sup_{n} \sup_{\tau\in\mathcal{T}^{M,T}_n} \mathbb{P}\left(\sup_{t\in[0,\tau\wedge S]} \|u_n\|_V^2 + \int_0^{\tau\wedge S} \|A(u_n)\|_H^2 dt > (M-1)^2 \right) = 0. \quad (3.4)$$

Theorem 3.3.2. *[GZ] Let $\{u^n\}$ be the sequence of solutions of (4.14) with u being the solution to the equation as in Definition 1.4.1 and with g, f, and u_0 and, there exists a global, maximal pathwise strong solution (u,ξ) in the sense of Definition 1.4.2. Namely, there exists an increasing sequence of strictly positive stopping times $\{\tau_m\}_{m\geq 0}$ converging to ξ, for which $\mathbb{P}(\xi < \infty) = 0$.*

We start with the following lemma.

Lemma 3.3.3. *Let u and u^n be defined as in Definitions 1.4.2 and 1.4.3. Then for any deterministic time $T > 0$, the Galerkin approximations u^n converge in probability with respect to the V norm, i.e., for any $\delta > 0$ we have*

$$\mathbb{P}(\sup_{t\in[0,T]} \|u - u^n\|_V^2 \geq \delta) \to 0 \quad (3.5)$$

as $n \to \infty$.

In particular, under the assumptions, we have

$$\mathbb{P}(\sup_{t\in[0,T]}(\log(1+\|u-u^n\|_V^2))^{1-\epsilon}\geq\delta)\to 0 \tag{3.6}$$

for any $\epsilon\in(0,1)$, as well as, by the Poincaré inequality,

$$\mathbb{P}(\sup_{t\in[0,T]}\|u-u^n\|_H^{2k}\geq\delta)\to 0 \tag{3.7}$$

for all $\delta>0$. Both statements shall be used below.

Proof. By assumption, we have $u_0\in L^2(\Omega,V)$. Hence, by Chebyshev theorem we have,

$$\mathbb{P}(\|u_0\|_V^2>s)\to 0 \tag{3.8}$$

as $s\to\infty$. Denoting $\Omega_s=\{\|u_0\|_V^2\leq s\}$, we have $\Omega_s\to\Omega$. Hence, we choose s such that $\mathbb{P}(\Omega_s)>1-\frac{\epsilon}{2}$. Moreover, We know by Theorem 3.3.2 and Lemma 6.0.2, that there exists a sequence of stopping times $\{\tilde{\tau}_{n_l}^M\}_{n_l\geq 1}$ with the corresponding subsequence $\{u_{\phi_{n_l}}\}$ converging monotone decreasing to τ^M. We also know that $\tau^M\to\infty$ a.s. as $M\to\infty$, where M is the constant defined as in Theorem 3.3.1, since the solution is global in the sense of Definition 1.4.2 by Theorem 3.3.2.

Hence, denoting $\{\tau_{n_l}^M=\tilde{\tau}_{n_l}^M\wedge T\}_{n_l\geq 1}$, there exists M_0 such that $\mathbb{P}(\tau^{M_0}<T)\leq\frac{\epsilon}{4}$ and by Lebesgue dominated convergence theorem, we have

$$\lim_{n_l\to\infty}\mathbb{E}\left[\mathbb{1}_{\Omega_s}\sup_{t\in[0,\tau^{M_0}]}\|u_\phi-u_{\phi_{n_l}}\|_V^2\right]=0. \tag{3.9}$$

This implies convergence in probability. Thus,

$$\lim_{n_l\to\infty} \mathbb{P}(\mathbb{1}_{\Omega_s} \sup_{t\in[0,\tau^{M_0}]} \|u_\phi - u_{\phi_{n_l}}\|_V^2 > \delta) = 0, \tag{3.10}$$

for any $\delta > 0$. Hence, we have

$$\begin{aligned}
&\mathbb{P}\big(\mathbb{1}_{\Omega_s} \sup_{t\in[0,T]} \|u_\phi - u_{\phi_{n_l}}\|_V^2 \geq \delta\big) = \mathbb{P}\big(\{ \sup_{t\in[0,T]} \|u_\phi - u_{\phi_{n_l}}\|_V^2 \geq \delta\} \cap \{\tau^{M_0} < T\} \cap \{\omega \in \Omega_s\}\big) \\
&\quad + \mathbb{P}(\{ \sup_{t\in[0,T]} \|u_\phi - u_{\phi_{n_l}}\|_V^2 \geq \delta\} \cap \{\tau^{M_0} = T\} \cap \{\omega \in \Omega_s\}) \\
&\leq \mathbb{P}(\tau^{M_0} < T) + \mathbb{P}(\mathbb{1}_{\Omega_s} \sup_{t\in[0,\tau^{M_0}]} \|u_\phi - u_{\phi_{n_l}}\|_V^2 > \delta) \qquad (3.11)
\end{aligned}$$

Then, we get

$$\begin{aligned}
\mathbb{P}\left(\sup_{t\in[0,T]} \|u_\phi - u_{\phi_{n_l}}\|_V^2 \geq \delta \right) &\leq \mathbb{P}\left(\tau^{M_0} < T\right) + \mathbb{P}\left(\mathbb{1}_{\Omega_s} \sup_{t\in[0,\tau^{M_0}]} \|u_\phi - u_{\phi_{n_l}}\|_V^2 > \delta \right) + \mathbb{P}\left(\Omega_s^c\right) \\
&\leq \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{2} = \epsilon. \qquad (3.12)
\end{aligned}$$

for n_l large enough. Then by taking any subsequence $u_{\phi_{m_l}}$ and by Theorem 3.3.1 and Lemma 6.0.2 repeating the same arguments above, we get that every subsequence $\{u_{\phi_{m_l}}\}$ has a further subsequence that converges in probability to u_ϕ, which implies that the whole sequence $\{u_{\phi_n}\}$ converges in probability to u_ϕ, which concludes the proof. □

The following theorem improves the main result from [KV].

Theorem 3.3.4. *Let u_0, f, and g be as in Definition 1.4.2 and suppose that u is as defined in Definition 1.4.1 . Then we have*

$$\mathbb{E}[\sup_{[0,T]} \phi(\|u\|_V^2)] \leq C(f, g, u_0, T), \tag{3.13}$$

where $\phi(x) = \log(1 + x)$.

Proof. From the infinite dimensional version of Itô's lemma we get

$$\begin{aligned} &d(\phi(\|u\|_V^2)) + 2\nu\phi'(\|u\|_V^2)\|Au\|_H^2 dt \\ &\quad = \phi'(\|u\|_V^2)\Big(2\langle f, Au\rangle - 2\langle B(u,u), Au\rangle + \phi'(\|u\|_V^2)\|g(u)\|_V^2\Big)dt \\ &\qquad + 2\phi''(\|u\|_V^2)\sum_k \big\langle g_k(u), Au\big\rangle^2 dt + 2\phi'(\|u\|_V^2)\langle g(u), Au\rangle dW. \end{aligned} \tag{3.14}$$

We take the supremum up to the stopping time $\tilde{\tau}_m = \tau_m \wedge T$, where τ_m is introduced in Theorem 3.3.2. Denoting $\Omega_m = \{\omega \in \Omega : \tilde{\tau}_m = T\}$, we see that $\Omega_m \uparrow \Omega$ as $m \to \infty$ by Theorem 3.3.2. By taking the expectation on Ω_m and, suppressing $\mathbb{1}_{\Omega_m}$ for simplicity of notation, we get

$$\begin{aligned} &\mathbb{E}[\sup_{[0,\tilde{\tau}_m]} \phi(\|u\|_V^2)] + 2\nu\mathbb{E}\left[\int_0^{\tilde{\tau}_m} \phi'(\|u\|_V^2)\|Au\|_H^2 ds\right] \\ &\quad \leq \phi'(\|u_0\|_V^2) + \mathbb{E}\left[\int_0^{\tilde{\tau}_m} (T_1 + T_2 + T_3 + T_4) ds\right] + 2\mathbb{E}\left[\sup_{s\in[0,\tilde{\tau}_m]} \left|\int_0^s T_0 dW_s\right|\right] \end{aligned} \tag{3.15}$$

where we denoted

$$T_0 = 2\phi'(\|u\|_V^2)|\langle g(u), Au\rangle| \tag{3.16}$$

$$T_1 = 2\phi'(\|u\|_V^2)|\langle B(u,u), Au\rangle| \tag{3.17}$$

$$\begin{aligned} T_2 &= 2\phi'(\|u\|_V^2)|\langle f, Au\rangle| \le 2\phi'(\|u\|_V^2)\|f\|_H\|Au\|_H \\ &\le C\phi'(\|u\|_V^2)\|f\|_H^2 + \frac{\nu}{8}\phi'(\|u\|_V^2)\|Au\|_H^2 \end{aligned} \tag{3.18}$$

$$T_3 = \phi'(\|u\|_V^2)\|g(u)\|_V^2 \le C\phi'(\|u\|_V^2)(1+\|u\|_V^2) \tag{3.19}$$

$$T_4 = 2|\phi''(\|u\|_V^2)\langle g(u), Au\rangle|^2 \le C|\phi''(\|u\|_V^2)|\|u\|_V^2(1+\|u\|_V^2) \tag{3.20}$$

where C is allowed to depend on K_j, for $j = 0,1,2$, and K_Y. Appealing to the BDG inequality, we have

$$\mathbb{E}\left[\sup_{s\in[0,\tau_m]}\left|\int_0^s T_0 dW\right|\right] \le C\mathbb{E}\left[\left(\int_0^{\tau_m}\left|\phi'(\|u\|_V^2)\right|^2\|g(u)\|_V^2\|u\|_V^2 ds\right)^{1/2}\right] \tag{3.21}$$

and thus, using the Lipschitz condition on $g(u)$,

$$\mathbb{E}\left[\sup_{s\in[0,\tau_m]}\left|\int_0^s T_0 dW\right|\right] \le C\mathbb{E}\left[\left(\int_0^{\tau_m}\frac{1}{(1+\|u\|_V^2)^2}(1+\|u\|_V^2)\|u\|_V^2 ds\right)^{1/2}\right] \le C(T). \tag{3.22}$$

Next, we estimate the term T_1 as

$$\begin{aligned} T_1 &= 2\phi'(\|u\|_V^2)|\langle B(u,u), Au\rangle| \\ &\le 2\phi'(\|u\|_V^2)\|u\|_H^{1/2}\|u\|_V^{1/2}\|u\|_V^{1/2}\|Au\|_H^{3/2} \\ &\le C\phi'(\|u\|_V^2)\|u\|_H^2\|u\|_V^4 + \frac{1}{4}\phi'(\|u\|_V^2)\|Au\|_H^2 \\ &\le C\|u\|_H^2\|u\|_V^2 + \frac{1}{4}\phi'(\|u\|_V^2)\|Au\|_H^2, \end{aligned} \tag{3.23}$$

where we note that by Lemma 4.3.9

$$\mathbb{E}\left[\int_0^T \|u\|_V^2\|u\|_H^2 dt\right] \le M(\|u_0\|_H^4, \|f\|_{V'}^4, T). \tag{3.24}$$

By combining all the estimates and writing out $\mathbb{1}_{\Omega_m}$ explicitly, we obtain

$$\mathbb{E}[\mathbb{1}_{\Omega_m} \sup_{[0,\tilde{\tau}_m]} \phi(\|u\|_V^2)] \le C(f,g,u_0,T). \tag{3.25}$$

By letting $m \to \infty$ and appealing to the monotone convergence theorem, we get

$$\mathbb{E}[\sup_{[0,T]} \phi(\|u\|_V^2)] \le C(f,g,u_0,T) \tag{3.26}$$

and the proof is concluded. □

Lemma 3.3.5. *Let u^n be as in Definition 1.4.3. Then we have*

$$\mathbb{E}[\sup_{[0,T]} \log(1+\|u^n\|_V^2)] \le C(f,g,u_0,T) \tag{3.27}$$

and

$$\mathbb{E}[\sup_{[0,T]} \log(1 + \|u - u^n\|_V^2)] \leq C(f, g, u_0, T), \tag{3.28}$$

for all $n \in \mathbb{N}$.

Proof of Lemma 3.3.5. The proof of (3.27) follows the same steps as the proof of Lemma 3.3.4 and it is thus omitted. The inequality (3.28) is a consequence of (3.13) and (3.27). □

Now, we are ready to prove our first main result, Theorem 3.2.1.

Proof. Let $\epsilon \in (0, 1)$. By (3.58), we have

$$\sup_{[0,T]} \log(1 + \|u - u^n\|_V^2)^{1-\epsilon} \to 0 \tag{3.29}$$

in probability as $n \to \infty$. Moreover, using Lemma 3.3.5,

$$\mathbb{E}\left[\sup_{[0,T]}(\log(1 + \|u - u^n\|_V^2)\right] \leq M(u_0, f, g, T). \tag{3.30}$$

Denoting

$$U_n = \sup_{[0,T]} \log(1 + \|u - u^n\|_V^2)^{1-\epsilon} \tag{3.31}$$

we have by (3.30)

$$\mathbb{E}\left[U_n^{1/(1-\epsilon)}\right] \leq M(u_0, f, g, T) \tag{3.32}$$

while (3.58) gives

$$U_n^{1/(1-\epsilon)} \to 0 \tag{3.33}$$

in probability. Using the de la Vallée-Poussin criterion for uniform integrability (see e.g. [D]), we get that $U_n \to 0$ in L^1 as $n \to \infty$ and Theorem 3.2.1 is proven. $\square$

4 Timewise Galerkin Convergence in H

In this section, we prove our second result, Theorem 3.2.2, on convergence of the Galerkin approximations in the H space.

Lemma 3.4.1. *Let u be the solution to the equation as in Definition 1.4.1. Then we have*

$$\mathbb{E}[\sup_{[0,T]}\|u\|_H^{2n}] + \mathbb{E}\left[\int_0^T \|u\|_V^2\|u\|_H^{2n-2}ds\right] \le C(n, \|u_0\|_H^{2n}, \|f\|_{V'}^{2n}, T) \tag{4.1}$$

for any positive integer n.

Proof. We use an induction argument. Let $z(t) = \exp(-Kt)$ for a positive constant K to be specified below. For any positive integer n, we have

$$\begin{aligned} d(z(t)\|u\|_H^{2n}) = -Kz(t)\|u\|_H^{2n}dt - 2n\|u\|_V^2\|u\|_H^{2n-2}z(t)dt + 2nz(t)\|u\|_H^{2n-2}\langle f, u\rangle dt \\ + nz(t)\|u\|_H^{2n-2}\|g(u)\|_H^2 dt \\ + 2nz(t)\|u\|_H^{2n-2}\langle g(u), u\rangle dW_t \end{aligned} \tag{4.2}$$

and thus

$$
\begin{aligned}
&d(z(t)\|u\|_H^{2n}) + 2n\|u\|_V^2\|u\|_H^{2n-2}z(t)dt \\
&\qquad = -Kz(t)\|u\|_H^{2n}dt + 2nz(t)\|u\|_H^{2n-2}\langle f,u\rangle dt + nz(t)\|u\|_H^{2n-2}\|g(u)\|_H^2 dt \\
&\qquad\qquad + 2nz(t)\|u\|_H^{2n-2}\langle g(u),u\rangle dW_t \\
&\leq -Kz(t)\|u\|_H^{2n}dt + 2nz(t)\|u\|_H^{2n-2}|\langle f,u\rangle|dt + nz(t)\|u\|_H^{2n-2}\|g(u)\|_H^2 dt \\
&\qquad + 2nz(t)\|u\|_H^{2n-2}\langle g(u),u\rangle dW_t \\
&\leq -Kz(t)\|u\|_H^{2n}dt + C(n)z(t)\|u\|_H^{2n}dt + C(n)z(t)\|f\|_H^{2n}dt + 2nz(t)\|u\|_H^{2n-2}\langle g(u),u\rangle dW_t.
\end{aligned} \tag{4.3}
$$

Choosing a sufficiently large K so it cancels the second term on the far right side, we get

$$
\begin{aligned}
&d(z(t)\|u\|_H^{2n}) + 2n\|u\|_V^2\|u\|_H^{2n-2}z(t)dt \\
&\qquad \leq C(n)z(t)\|f\|_H^{2n}dt + 2nz(t)\|u\|_H^{2n-2}\langle g(u),u\rangle dW_t.
\end{aligned} \tag{4.4}
$$

We take the supremum up to the stopping times $\{\tau_m\}_{m\geq 1}$ introduced in Theorem 3.3.2 and integrate up to τ_m and take the expectation on Ω_m as in Theorem 3.3.4. By suppressing $\mathbb{1}_{\Omega_m}$ below, we get

$$\begin{aligned}
\mathbb{E}[\sup_{[0,\tau_m]} z(t)\|u\|_H^{2n}] &\leq M(\|u_0\|_H^{2n}, \|f\|_{V'}^{2n}, T) + C(n)\mathbb{E}\left[\sup_{t\in[0,\tau_m]}\left|\int_0^{\tau_m} z(s)\|u\|_H^{2n-2}\langle g(u), u\rangle dW_s\right|\right] \\
&\leq M(\|u_0\|_H^{2n}, \|f\|_{V'}^{2n}, T) + C(n)\mathbb{E}\left[\left(\int_0^{\tau_m} z^2(s)\|u\|_H^{4n-4}|\langle g(u), u\rangle|^2 ds\right)^{1/2}\right] \\
&\leq M(\|u_0\|_H^{2n}, \|f\|_{V'}^{2n}, T) + C(n)\mathbb{E}\left[\left(\int_0^{\tau_m} z^2(s)\|u\|_H^{4n-4}(1+\|u\|_H^2)\|u\|_H^2 ds\right)^{1/2}\right] \\
&\leq M(\|u_0\|_H^{2n}, \|f\|_{V'}^{2n}, T) + C(n)\mathbb{E}\left[\left(\int_0^{\tau_m} z^2(s)\|u\|_H^{4n-2}(1+\|u\|_H^2) ds\right)^{1/2}\right] \\
&\leq M(\|u_0\|_H^{2n}, \|f\|_{V'}^{2n}, T) + C(n)\mathbb{E}\left[\sup_{t\in[0,\tau_m]} z(t)\|u\|_H^{2n-1}\left(\int_0^{\tau_m} (1+\|u\|_H^2) ds\right)^{1/2}\right] \\
&\leq M(\|u_0\|_H^{2n}, \|f\|_{V'}^{2n}, T) + \frac{1}{4}\mathbb{E}\left[\sup_{[0,\tau_m]} z(t)\|u\|_H^{2n}\right] + C(n^{2n})\mathbb{E}\left[\left(\int_0^{\tau_m} (1+\sup_{[0,\tau_m]}\|u\|_H^2) ds\right)^{n}\right]
\end{aligned}$$

where in the last line, we used the ϵ-Young inequality with $p = 2n/(2n-1)$ and $q = 2n$. By observing $e^{-KT} \leq e^{-Kt} \leq 1$ for $0 \leq t \leq \tau_m \leq T$ and using the Gronwall lemma, we obtain

$$\mathbb{E}[\mathbb{1}_{\Omega_m} \sup_{[0,\tau_m]} \|u\|_H^{2n}] \leq M(n^{2n}, \|u_0\|_H^{2n}, \|f\|_{V'}^{2n}, T). \tag{4.5}$$

Moreover, based on the inequality (3.40), we get

$$\mathbb{E}\left[\mathbb{1}_{\Omega_m}\int_0^{\tau_m} \|u\|_V^2\|u\|_H^{2n-2} ds\right] \leq M(n^{2n}, \|u_0\|_H^{2n}, \|f\|_{V'}^{2n}, T). \tag{4.6}$$

We conclude the result using the monotone convergence theorem as in the proof of Theorem 3.3.4. □

Now, we are ready to prove Theorem 3.2.2.

Proof of Theorem 3.2.2. We use that

$$\mathbb{E}[\sup_{[0,T]}\|u-u^n\|_H^{2k}] \leq 2^{2k}(\mathbb{E}[\sup_{[0,T]}\|u\|_H^{2k}] + \mathbb{E}[\sup_{[0,T]}\|u^n\|_H^{2k}]). \tag{4.7}$$

Then, we have using (3.59) and Lemma 3.4.1

$$\mathbb{E}[\sup_{[0,T]}\|u-u^n\|_H^{2k(1-\epsilon)}] \to 0, \tag{4.8}$$

as $n \to \infty$ using the uniform integrability principle. □

5 The Zygmund norm estimate in H

In the final section, we prove Theorem 3.2.3.

Proof of Theorem 3.2.3. The proof is in spirit of [GT, Lemma 7.13]. First, as in Lemma 3.4.1, we apply the infinite dimensional Itô's lemma up to the stopping time τ_m and bootstrap. Hence, we have

$$\begin{aligned} d(\|u\|_H^{2n}) &= -2n\|u\|_V^2\|u\|_H^{2n-2}dt + 2n\|u\|_H^{2n-2}\langle f, u\rangle dt + n\|u\|_H^{2n-2}\|g(u)\|_H^2 dt \\ &\quad + 2n\|u\|_H^{2n-2}\langle g(u), u\rangle dW_t \\ &\leq -2n\|u\|_V^2\|u\|_H^{2n-2}dt + 2n\|u\|_H^{2n-2}\left(C\|f\|_{H'}^2 + \frac{1}{8}\|u\|_H^2\right)dt \\ &\quad + n\|u\|_H^{2n} + 2n\|u\|_H^{2n-2}\langle g(u), u\rangle dW_t \\ &\leq -2n\|u\|_V^2\|u\|_H^{2n-2}dt + \frac{2n}{8}\|u\|_H^{2n}dt \\ &\quad + 2nC\|u\|_H^{2n-2}\|f\|_{H'}^2 dt + n\|u\|_H^{2n}dt + 2n\|u\|_H^{2n-2}\langle g(u), u\rangle dW_t \\ &\leq -2n\|u\|_V^2\|u\|_H^{2n-2}dt + \frac{2n}{8}\|u\|_H^{2n} + \frac{2n}{8}\|u\|_H^{2n} + Cn\|f\|_{H'}^{2n} \\ &\quad + n\|u\|_H^{2n}dt + 2n\|u\|_H^{2n-2}\langle g(u), u\rangle dW_t. \end{aligned} \tag{5.1}$$

Taking the supremum, the expectation, and integrating, we get

$$\mathbb{E}[\sup_{[0,T]}\|u\|_H^{2n}] \leq \mathbb{E}[\|u_0\|_H^{2n}] + Cn\mathbb{E}\left[\int_0^T \|f\|_{V'}^{2n}dt\right] + 2n\mathbb{E}\left[\int_0^T \|u\|_H^{2n-2}\langle g(u), u\rangle dW_t\right]. \tag{5.2}$$

By appealing to the BDG inequality and using the ϵ-Young inequality, we have

$$\mathbb{E}[\sup_{[0,T]}\|u\|_H^{2n}] \leq C\mathbb{E}[\|u_0\|_H^{2n}] + Cn\mathbb{E}\left[\int_0^T \|f\|_{H'}^{2n}dt\right] + Cn^{2n}\mathbb{E}\left[\int_0^T \sup_{[0,T]}\|u\|_H^{2n}dt\right]. \tag{5.3}$$

Then by the assumptions on $\|f\|_{H'}$ and $\|u_0\|_H$ and appealing to the Gronwall lemma we conclude that

$$\mathbb{E}[\sup_{[0,T]}\|u\|_H^{2n}] \leq Cn^{2n}, \tag{5.4}$$

for n sufficiently large. Hence,

$$\mathbb{E}[\sup_{[0,T]}\|u\|_H^{n}] \leq Cn^{n}, n \in \mathbb{N} \tag{5.5}$$

using the Cauchy-Schwartz inequality used for odd n, whence

$$\mathbb{E}\left[\sup_{[0,T]}\frac{\|u\|_H^n}{n^{n+2}}\right] \leq \frac{C}{n^2}. \tag{5.6}$$

Using Sterling's formula, we have $n^{n+2} \leq C^n n!$, for a sufficiently large constant C. Therefore, for N_0 sufficiently large, we have

$$\mathbb{E}\left[\sum_{n=N_0}\sup_{0\leq t\leq T}\frac{1}{n!}\left(\frac{\|u\|_H}{K}\right)^n\right] \leq \sum_{n=1}^{\infty}\frac{C}{n^2}, \tag{5.7}$$

which implies

$$\mathbb{E}\left[\sup_{0\leq t\leq T}\exp\left(\frac{\|u\|_H}{K}\right)\right] \leq C. \tag{5.8}$$

The proof is thus concluded. □

Chapter 4

A Linearized Scheme for Timewise Approximation of the SNSE

1 Introduction

The aim of this chapter is to approximate the solution of the SNSE in 1.4.1 by the solutions of a sequence of linear equations of the form

$$\begin{aligned}\langle u_n(t), v\rangle + \int_0^t \langle Au_n(s), v\rangle ds = \langle u_0, v\rangle ds \\ + \int_0^t \langle B(u_{n-1}(s), u_n(s)), v\rangle ds + \int_0^t \langle f, v\rangle ds + \int_0^t \langle g(u_{n-1}, v)\rangle dW_s,\end{aligned} \tag{1.1}$$

which is a candidate to be more efficient scheme to study in terms of numerical approximations than the Galerkin approximation. The reason for that is in Equation (1.1), given that u_{n-1} is known, the operators depend linearly on u_n and the noise is additive with respect to u_n, whereas in the Galerkin approximation 1.4.3, the bilinear term B depends on u_n in a nonlinear way in addition to that in Galerkin approximation, noise is a multiplicative term in the model.

2 Main Results

We list the main results in this section, that are analogous to Chapter 3 on Galerkin approximations. The first main result gives the linearized approximation in the V-norm.

Theorem 4.2.1. *Let $\epsilon \in [0,1]$ and let $T > 0$ be arbitrary. Suppose that u is a solution to 1.4.1 and let u_n be the corresponding linearized approximation in Equation (1.1). Then, we have*

$$\mathbb{E}\left[\sup_{[0,T]} \phi_1(\|u-u^n\|_V^2)\right] \to 0, \tag{2.1}$$

with $\phi_1(x) = (\log(1+x))^{1-\epsilon}$, as $n \to \infty$.

Our second result gives the convergence in the H-norm of the linearized scheme u_n on the whole bounded domain $\mathcal{O}$.

Theorem 4.2.2. *Let u be the solution to the equation 1.4.1 and let u_n be the corresponding linearized approximation equation 1.1. If, additionally, we have $f \in L^{2k}(\Omega; L^{2k}([0,\infty); V'))$ and $u_0 \in L^{2k+2}(\Omega; H) \cap L^2(\Omega; V)$, then*

$$\mathbb{E}\left[\sup_{[0,T]} \|u-u^n\|_H^{2k(1-\epsilon)}\right] \to 0, \tag{2.2}$$

for any deterministic time $T > 0$ and any $\epsilon \in (0,1)$ as $n \to \infty$.

3 Proof of the Main Results

To prove these theorems, first we borrow the following two results from [B2].

Theorem 4.3.1. *For each $n \in \mathbb{N}_0$, the Equation 1.1 has an almost surely unique solution in $\mathcal{L}^2_V(\Omega \times [0,T])$ with almost surely continuous trajectories in H*

Theorem 4.3.2. *Let u_0, f, g be as defined in Definition 1.4.2, and u as in Definition* 1.4.1*, then we have*

$$\mathbb{E}\int_0^T \|u - u_n\|_V^2 dt \to 0, \tag{3.1}$$

as $n \to \infty$.

Next we continue with the following theorem.

Theorem 4.3.3. *Let u_n be the linearized scheme in Equation (1.1).*

$$\mathcal{T}_n^{M,T} = \{\tau \leq T : (\sup_{t\in[0,\tau]} \|u_n\|_V^2 + \nu \int_0^\tau \|Au_n\|_H^2 dt)^{1/2} \leq M\} \tag{3.2}$$

Take

$$\mathcal{T}_{m,n,m-1,n-1}^{M,T} := \mathcal{T}_m^{M,T} \cap \mathcal{T}_n^{M,T} \cap \mathcal{T}_{m-1}^{M,T} \cap \mathcal{T}_{n-1}^{M,T}, \tag{3.3}$$

then we have for any $T > 0$ and $M > 1$

1. *For any $T > 0$ and $M > 1$*

$$\lim_{n\to\infty} \sup_{m\geq n} \sup_{\tau\in\mathcal{T}_{m,n,m-1,n-1}^{M,T}} \mathbb{E}[\sup_{t\in[0,\tau]} \|u_m - u_n\|_V^2 + \int_0^\tau \|A(u_m - u_n)\|_H^2 dt] = 0 \tag{3.4}$$

2.

$$\lim_{s\to 0}\sup_{n}\sup_{\tau\in\mathcal{T}_n^{M,T}}\mathbb{P}\left(\sup_{t\in[0,\tau\wedge S]}\|u_n\|_V^2+\int_0^{\tau\wedge S}\|A(u_n)\|_H^2dt>(M-1)^2\right)=0. \quad (3.5)$$

Proof. 1. Given $m > n$, we get:

$$du_m + A(u_m)dt = B(u_{m-1}, u_m)dt + f dt + \sum_{k=1}^{\infty} g_k(u_{m-1})dW_t, \quad (3.6)$$

then by Ito:

$$\begin{aligned} &d\|u_m-u_n\|_V^2+2\|A(u_m-u_n)\|_H^2dt \\ &\quad = 2\langle B(u_{m-1},u_m)-B(u_{n-1},u_n),A(u_m-u_n)\rangle dt \\ &\quad + 2\langle g(u_{m-1})-g(u_{n-1}),A(u_m-u_n)\rangle dW_t \\ &\quad + \|g(u_{m-1})-g(u_{n-1})\|_V^2 \end{aligned} \quad (3.7)$$

We treat each term seperately. First, by the Lipschitz assumption on g we have

$$\begin{aligned} \int_0^{\tau}\|g(u_{m-1})-g(u_{n-1})\|_V^2dt &\le K\int_0^{\tau}\|u_{m-1}-u_{n-1}\|_V^2dt \\ &\to 0, \end{aligned} \quad (3.8)$$

as $n, m \to \infty$. Next we estimate

$$\sup_{t\in[0,T]} 2\mathbb{E}|[\int_0^T \langle g(u_{m-1}) - g(u_{n-1}), A(u_m - u_n)\rangle dW_t]|$$
$$\mathbb{E}\bigg(\int_0^T \|g(u_{m-1}) - g(u_{n-1})\|_V^2 \|u_n - u_m\|_V^2 dt\bigg)^{1/2}$$
$$\leq C(M)\mathbb{E}\bigg(\int_0^T \|g(u_{m-1}) - g(u_{n-1})\|_V^2\bigg)^{1/2}$$
$$\leq C(M)\mathbb{E}\bigg(\int_0^T \|u_{m-1} - u_{n-1}\|_V^2\bigg)^{1/2} \tag{3.9}$$

as $m, n \to \infty$. For the nonlinear terms, we treat the terms as follows

$$B(u_{m-1}, u_m) - B(u_{n-1}, u_n)$$
$$= B(u_{m-1}, u_m) - B(u_{n-1}, u_m) + B(u_{n-1}, u_m) - B(u_{n-1}, u_n)$$
$$\leq |B(u_{m-1} - u_{n-1}, u_m) + B(u_{n-1}, u_m - u_n)|$$
$$\leq \|u_{m-1} - u_{n-1}\|_H^{1/2} \|u_{m-1} - u_{n-1}\|_V^{1/2} \|u_m\|_V^{1/2} \|Au_m\|_H^{1/2} \|A(u_n - u_m)\|_H$$
$$\leq \frac{1}{4}\|A(u_m - u_n)\|_H^2 + K\|u_m - u_n\|_H \|u_{m-1} - u_{n-1}\|_V \|u_m\|_V \|Au_m\|_H$$
$$\leq \frac{1}{4}\|A(u_m - u_n)\|_H^2 + \|u_{m-1} - u_{n-1}\|_V^2 + M\|u_m\|_V^2 \|u_m - u_n\|_V^2 \|Au_m\|_H^2, \tag{3.10}$$

The term $\frac{1}{4}\|A(u_m - u_n)\|_H^2$ is absorbed to LHS and for the term $\|u_{m-1} - u_{n-1}\|_V^2 + M\|u_m\|_V^2 \|u_m - u_n\|_V^2 \|Au_m\|_H^2$ we apply the stochastic Gronwall Lemma 6.0.1 by noticing as well that

$$\int_0^T \|u_{n-1} - u_{m-1}\| dt \to 0, \tag{3.11}$$

as $n,m\to\infty$ by [B2] above. For the term

$$
\begin{aligned}
&|\langle B(u_{n-1},u_m-u_n),A(u_m-u_n)\rangle|\\
&\quad\le C\|u_{n-1}\|_H^{1/2}\|u_{n-1}\|_V^{1/2}\|u_m-u_n\|_V^{1/2}\|A(u_m-u_n)\|_H^{1/2}\|A(u_m-u_n)\|_H\\
&\quad\le C\|u_{n-1}\|_V\|u_m-u_n\|_V^{1/2}\|A(u_m-u_n)\|_H^{3/2}\\
&\quad\le \frac{1}{6}\|A(u_m-u_n)\|_H^2+C_\nu\|u_m-u_n\|_V^2\|u_{n-1}\|_V^4, \qquad (3.12)
\end{aligned}
$$

here the term $\frac{1}{6}\|A(u_m-u_n)\|_H^2$ is absorbed to LHS, whereas for the term $C_\nu\|u_m-u_n\|_V^2\|u_{n-1}\|_V^4$ we use the stochastic Gronwall Lemma 6.0.1.

2. Applying Ito, we get

$$
\begin{aligned}
d\|u_n\|_V^2+2\|Au_n\|_H^2dt&=2\langle f,Au_n\rangle dt-2\langle B(u_{n-1},u_n),Au_n\rangle dt\\
&\quad+\sum_{k=1}^{\infty}\|g_k(u_{n-1}),A(u_n)\|_H^2dt\\
&\quad+2\sum_{k=1}^{\infty}\langle g_k(u_{n-1},Au_n)\rangle dW_k. \qquad (3.13)
\end{aligned}
$$

Fix $\tau\in\mathcal{T}_{n,n-1}^{M,T}$ and $s>0$. Taking supremum and integrating, we get

$$
\begin{aligned}
&\sup_{[0,s\wedge\tau]}\|u_n\|_V^2+2\int_0^{s\wedge\tau}\|Au_n\|_H^2dr\\
&\quad\le\int_0^{s\wedge\tau}2|\langle f,Au_n\rangle|dr+\int_0^{s\wedge\tau}|\langle B(u_{n-1},u_n),Au_n\rangle|dr\\
&\quad+\int_0^{s\wedge\tau}\|g_k(u_{n-1})\|_V^2dt+\sup_{r\in[0,S\wedge\tau]}\left|\sum_{k=1}^{\infty}\int_0^r2\langle g_k(u_{n-1}),Au_n\rangle dW_k\right|. \qquad (3.14)
\end{aligned}
$$

We treat the term $\left|\langle B(u_{n-1}, u_n), Au_n\rangle\right|$ as follows.

$$
\begin{aligned}
&\left|\langle B(u_{n-1}, u_n), Au_n\rangle\right| \\
&\qquad \le C\|u_{n-1}\|_H^{1/2}\|u_{n-1}\|_V^{1/2}\|u_n\|_V^{1/2}\|Au_n\|_H^{1/2}\|Au_n\|_H \\
&\qquad \le C\|u_{n-1}\|_H^{1/2}\|u_{n-1}\|_V^{1/2}\|u_n\|_V^{1/2}\|Au_n\|_H^{3/2} \\
&\qquad \le C\|u_{n-1}\|_H^{2}\|u_{n-1}\|_V^{2}\|u_n\|_V^{2} + \frac{1}{4}\|Au_n\|_H^2,
\end{aligned}
\tag{3.15}
$$

Using this and the Lipschitz condition on g, we have

$$
\begin{aligned}
&\mathbb{P}\bigg(\sup_{r\in[0,S\wedge\tau]}\|u_n\|_V^2 + \int_0^{S\wedge\tau}\nu\|Au_n\|_H^2 dr > (M-1)^2\bigg) \\
&\qquad \le \mathbb{P}\bigg(C(M,K_v)\int_0^{S\wedge\tau}\|f\|_V^2 + 1ds > \frac{(M-1)^2}{2}\bigg) \\
&\qquad\qquad + \mathbb{P}\bigg(\sup_{s\in[0,S\wedge\tau]}\bigg|\int_0^{S\wedge\tau}\langle g(u_{n-1}), Au_n\rangle dW_s\bigg| > \frac{(M-1)^2}{2}\bigg)
\end{aligned}
\tag{3.16}
$$

The first term after inequality of 3.16 converges to 0 as $S \to 0$ via Chebyshev's inequality, whereas the second term after inequality of 3.16 is treated as follows:

$$
\begin{aligned}
&\mathbb{P}\bigg(\sup_{s\in[0,S\wedge\tau]}\bigg|\int_0^{S\wedge\tau}\langle g(u_{n-1}), Au_n\rangle dW_s\bigg| > \frac{(M-1)^2}{2}\bigg) \\
&\qquad \le \mathbb{P}\bigg(\int_0^t|\langle g(u_{n-1}), Au_n\rangle|^2 ds > (M-1)^4\bigg) \\
&\qquad\qquad \le \frac{1}{(M-1)^4}\mathbb{E}\bigg[\int_0^{S\wedge\tau}\|g(u_{n-1})\|_V^2\|u_n\|_V^2 ds\bigg],
\end{aligned}
\tag{3.17}
$$

by Lipschitz condition on g and by letting $S \to 0$, we conclude the proof. □

Proposition 4.3.4. *(Uniqueness) [GZ] Let $\tau > 0$ be a stopping time. Suppose that $(u^{(1)}, \tau)$ and $(u^{(2)}, \tau)$ are solutions to the SNSE in the sense of Definition 1.4.2. Let $u_0^{(1)}, u_0^{(2)}$ be the associated initial conditions and assume that*

$$\mathbb{P}(I_{\Omega_0} u_0^{(1)} = I_{\Omega_0} u_0^{(2)}) = 1, \tag{3.18}$$

for some $\Omega_0 \in \mathcal{F}_0$. Then

$$\mathbb{P}\big(I_{\Omega_0} u^{(1)}(t \wedge \tau) = I_{\Omega_0} u^{(2)}(t \wedge \tau), t \in [0, \infty)\big) = 1 \tag{3.19}$$

Proposition 4.3.5. *(Existence) Suppose that $f \in L^2(\Omega; L^2_{\text{loc}}([0, \infty)); H)$. Then there exists a global strong solution (u, τ) in the sense of Definition 1.4.2.*

Proof. Let $w \in V$ be given. Due to Theorem 4.3.3, we apply Lemma 6.0.2 with $B_1 = V$ and $B_2 = D(A)$ and the sequence $\{X^n\} = \{u_n\}$. We infer the existence of a subsequence $\{u_{n'}\}$, and if necessary by appealing to a subsubsequence $\{u_{n'}\}$, a strictly positive stopping time $\tau \leq T$ and a process $u(.) = u(. \wedge \tau)$, continuous in V such that

$$\begin{aligned} &\sup_{t \in [0, \tau]} \|u_{n'} - u\|_V^2 + \nu \int_0^\tau \|A(u_{n'} - u)\|_H^2 ds \to 0, \\ &\sup_{t \in [0, \tau]} \|u_{n'-1} - u\|_V^2 + \nu \int_0^\tau \|A(u_{n'-1} - u)\|_H^2 ds \to 0, \end{aligned} \tag{3.20}$$

a.s. We also have that the conditions of Lemma 6.0.2(ii) is satisfied for any $p \in (1, \infty)$. Thus, we have for any $p > 1$

$$u(. \wedge \tau) \in L^P(\Omega; C([0,T]; V)), \tag{3.21}$$

with

$$u \mathbb{1}_{t \leq \tau} \in L^p(\Omega; L^2([0,T]; D(A))). \tag{3.22}$$

By Lemma 6.0.2(ii), we infer a collection of measurable sets $\Omega_{n'} \in \mathcal{F}$ with

$$\Omega_{n'} \uparrow \Omega \tag{3.23}$$

such that

$$\sup_{n'} \mathbb{E}\left[\sup_{t \in [0,T]} \|u_{n'}(t \wedge \tau) \mathbb{1}_{\Omega_{n'}}\|_V^2 + \nu \int_0^\tau \|A u_{n'} \mathbb{1}_{\Omega_{n'}}\|_H^2 ds \right]^{p/2} < \infty. \tag{3.24}$$

By Lemma 6.0.3, we have

$$\mathbb{1}_{\Omega_{n'}, t \leq \tau} u^{n'} \rightharpoonup \mathbb{1}_{t \leq \tau} u \text{ in } L^p(\Omega; L^2([0,T]; D(A))), \tag{3.25}$$

and

$$\mathbb{1}_{\Omega_{n'}} u^{n' \wedge \tau} \rightharpoonup^* u \text{ in } L^p(\Omega; L^\infty([0,T]; V)). \tag{3.26}$$

For the nonlinear term, we estimate for all $w \in V$ as follows:

$$\begin{aligned}
&\big|\langle B(u_{n'-1}, u_{n'}) - B(u,u), w\rangle\big| \qquad (3.27)\\
&\le \big|\langle B(u_{n'-1}, u_{n'} - u) - B(u,u), w\rangle\big| + \big|\langle B(u_{n'-1} - u, u), w\rangle\big|\\
&\big|\langle B(u_{n'-1}, u_{n'} - u) - B(u,u), w\rangle\big|\\
&\le \|u_{n-1}\|_H^{1/2}\|u_{n'-1}\|_V^{1/2}\|u_{n'} - u\|_V\|w\|_H^{1/2}\|w\|_V^{1/2}\\
&\le \|u_{n'-1}\|_V\|w\|_V^{1/2}\|w\|_V + \|w\|_V\|u_{n'} - u\|_V\\
&\big|\langle B(u_{n'-1} - u, u), w\rangle\big| \le \|u_{n'-1} - u\|_H^{1/2}\|u_{n'-1} - u\|_V^{1/2}\|u\|_V\|w\|_H^{1/2}\|w\|_V^{1/2}\\
&\le \|u_{n'-1} - u\|_V\|u\|_V^{1/2}\|w\|_V
\end{aligned}$$

Hence the nonlinear terms converge to 0 as well, we conclude that given any $v \in V$

$$\mathbb{1}_{t\le\tau}\langle B(u_{n'-1}, u_{n'}), v\rangle \to \mathbb{1}_{t\le\tau}\langle B(u,u), v\rangle, \qquad (3.28)$$

as $n' \to \infty$, for almost every $(\omega, t) \in \Omega \times [0, T]$. □

Moreover, by using the uniform bound with $p = 4$, one finds that

$$\begin{aligned} &\sup_{n'} \mathbb{E}\left(\mathbb{1}_{\Omega_{n'}} \int_0^\tau |B(u_{n'-1}, u_{n'})|^2 ds \right) \\ &\leq C \sup_{n'} \mathbb{E}\left(\mathbb{1}_{\Omega_{n'}} \int_0^\tau \|u_{n'-1}\|_H^2 \|u_{n'}\|_V^2 ds \right) \\ &\leq C \sup_{n'} \mathbb{E}\left(\mathbb{1}_{\Omega_{n'}} \int_0^\tau \|u_{n'-1}\|_H^2 \|u_{n'}\|_H \|Au_{n'}\|_H ds \right) \\ &\leq C \sup_{n'} \mathbb{E}\left(\mathbb{1}_{\Omega_{n'}} \sup_{t\in[0,\tau]} \|u_{n'-1}\|_H^2 \|u_{n'}\|_H \int_0^\tau \|Au_{n'}\|_H ds \right) \\ &\leq C \sup_{n'} \mathbb{E}\left(\mathbb{1}_{\Omega_{n'}} \sup_{t\in[0,\tau]} \|u_{n'-1}\|_H^4 \|u_{n'}\|_H^2 + \left(\int_0^\tau \|Au_{n'}\|_H^2 ds \right)^2 \right) \\ &< \infty \end{aligned} \tag{3.29}$$

Then by Lemma 6.0.3, we have

$$\mathbb{1}_{\Omega_{n'}, t\leq\tau} B(u_{n'-1}, u_{n'}) \rightharpoonup \mathbb{1}_{t\leq\tau} B(u, u) \text{in} \tag{3.30}$$

By Lipschitz condition on g we get

$$\begin{aligned} &\sum \|g_k(u_{n'}) - g_k(u)\|_V^2 \\ &\leq \|u - u_{n'}\|_V^2 \to 0, \end{aligned} \tag{3.31}$$

We have moreover that

$$\begin{aligned}
&\sup_{n'} \mathbb{E}\left[\mathbb{1}_{\Omega_{n'}} \int_0^\tau \|g_k(u_{n'})\|_V^2 ds\right] \\
&\leq C \sup_{n'} \mathbb{E}\left[\mathbb{1}_{\Omega_{n'}} \int_0^\tau 1 + \|u_{n'}\|_V^2 ds\right] \\
&< \infty,
\end{aligned} \tag{3.32}$$

which means that

$$\mathbb{1}_{\Omega_{n'}, t\leq\tau} g(u_{n'}) \to \mathbb{1}_{t\leq\tau} g(u), \tag{3.33}$$

in $L^2(\Omega; L^2([0,T]; l^2(H)))$. Hence, we deduce that for any fixed $v \in H$

$$\begin{aligned}
\mathbb{1}_{\Omega_{n'}} \int_0^{t\wedge\tau} \langle Au_{n'}, v\rangle ds &\rightharpoonup \int_0^{t\wedge\tau} \langle Au, v\rangle ds, \\
\mathbb{1}_{\Omega_{n'}} \int_0^{t\wedge\tau} \langle Bu_{n'}, v\rangle ds &\rightharpoonup \int_0^{t\wedge\tau} \langle B(u), v\rangle ds, \\
\mathbb{1}_{\Omega_{n'}} \sum_k \int_0^{t\wedge\tau} g_k(u_{n'}, v) dW_k &\rightharpoonup \int_0^{t\wedge\tau} \langle g_k(u), v\rangle dW_k,
\end{aligned} \tag{3.34}$$

weakly in $L^2(\Omega \times [0,T])$. If $K \subset \Omega \times [0,T]$ is any measurable set, we have

$$\begin{aligned}
&\mathbb{E}\int_0^T \chi_K(\omega,t)\langle u(t\wedge\tau), v\rangle dt \qquad (3.35)\\
&= \lim_{n'\to\infty} \mathbb{E}\int_0^T \langle \mathbb{1}_{\Omega_{n'}}(\omega), \chi_K(\omega,t)v\rangle dt\\
&= \lim_{n'\to\infty} \Bigg(\mathbb{E}\int_0^T \chi_K(\omega,t)\mathbb{1}_{\Omega_{n'}}(\omega)\langle u_0, v\rangle dt\\
&\quad - \mathbb{E}\int_0^T \chi_K(\omega,t)\mathbb{1}_{\Omega_{n'}}(\omega)\Bigg[\int_0^{t\wedge\tau}\langle \nu A u^{n'} + B(u^{n'} - f, v\rangle ds\Bigg]dt\\
&\quad + \mathbb{E}\int_0^T \chi_K(\omega,t)\mathbb{1}_{\Omega_{n'}}(\omega)\Bigg[\sum_k \int_0^{t\wedge\tau}\langle g_k(u^{n'}), v\rangle dW_s\Bigg]dt\Bigg)\\
&= \mathbb{E}\int_0^T \chi_K(\omega,t)\Bigg[\langle u_0, v\rangle - \int_0^{t\wedge\tau}\langle \nu A u + B(u) - f, v\rangle ds\Bigg]dt\\
&\quad + \mathbb{E}\int_0^T \chi_K(\omega,t)\Bigg[\sum_k \int_0^{t\wedge\tau}\langle g_k(u), v\rangle dW_k\Bigg]dt
\end{aligned}$$

Since ν, K are arbitrary, we conclude that u satisfies the regularity conditions. Removing the restriction that $\|u_0\|_V \le \tilde{M}$ a.s. and the global uniqueness are due to Theorem 4.2 [GZ]. Next, we continue with the following theorem.

Theorem 4.3.6. *Let u be the solution to the Equation 1.4.1 and let u_n be the corresponding linearized approximation Equation 1.1. Then, we have*

$$\mathbb{E}[\sup_{[0,T]}\|u_n\|_H^{2n}] \le C(\|u_0\|_H^{2n}, \|f\|_{V'}^{2n}) \qquad (3.36)$$

$$\mathbb{E}\Bigg[\int_0^T \|u_n\|_V^2\|u_n\|_H^{2n-2}\Bigg] \le M(\|u_0\|_H^{2n}, \|f\|_{V'}^{2n}, T).$$

for any positive integer n.

Proof. Let $z(t) = \exp(-Kt)$ for a positive constant K that is to be specified below. Via an induction argument by bootstrapping Ito lemma, for any positive integer n, we have that

$$\begin{aligned} d(z(t)\|u_n\|_H^{2n}) = &-Kz(t)\|u_n\|_H^{2n}dt - 2n\|u_n\|_V^2\|u_n\|_H^{2n-2}z(t)dt + 2nz(t)\|u_n\|_H^{2n-2}\langle f, u_n\rangle dt \\ &+ nz(t)\|u_n\|_H^{2n-2}\|g(u_{n-1})\|_H^2 dt \\ &+ 2nz(t)\|u_n\|_H^{2n-2}\langle g(u_{n-1}), u_n\rangle dW_t \end{aligned} \tag{3.37}$$

Hence, we have

$$\begin{aligned} &d(z(t)\|u_n\|_H^{2n}) + 2n\|u_n\|_V^2\|u_n\|_H^{2n-2}z(t)dt \\ &\quad\leq -Kz(t)\|u_n\|_H^{2n}dt + Cz(t)\|u_n\|_H^{2n} + C\|f\|_H^{2n} \\ &\qquad + Cz(t)\|u_n\|_H^{2n-2} + Cz(t)\|u_n\|_H^{2n-2}\|u_{n-1}\|_H^2 dt \\ &\qquad + 2nz(t)\|u_n\|_H^{2n-2}\langle g(u_{n-1}), u_n\rangle dW_t \\ &\quad\leq -Kz(t)\|u_n\|_H^{2n}dt + Cz(t)\|u_n\|_H^{2n} + C\|f\|_H^{2n} \\ &\qquad + Cz(t)\|u_n\|_H^{2n-2} + Cz(t)\|u_n\|_H^{2n} + C\|u_{n-1}\|_H^2 dt \\ &\qquad + 2nz(t)\|u_n\|_H^{2n-2}\langle g(u_{n-1}), u_n\rangle dW_t \end{aligned} \tag{3.38}$$

Choosing positive K accordingly cancels the deterministic terms, and so we have

$$\begin{aligned} d(z(t)\|u_n\|_H^{2n}) &\leq C\|f\|_H^{2n} + C + C\|u_{n-1}\|_H^{2n}dt \\ &\quad + 2nz(t)\|u_n\|_H^{2n-2}\langle g(u_{n-1}), u_n\rangle dW_t \end{aligned} \tag{3.39}$$

Integrating, taking supremum, expectation, appealing to the BDG-inequality and by induction step, we get

$$\begin{aligned}
&\mathbb{E}[\sup_{[0,T]} z(t)\|u_n\|_H^{2n}] \leq M(\|u_0\|_H^{2n}, \|f\|_{V'}^{2n}, T) \\
&\qquad + C\mathbb{E}[\sup_{[0,T]} \Big(\int_0^T z^2(s)\|u_n\|_H^{4n-4}(1+\|u_{n-1}\|_H^2)\|u_n\|_H^2\Big)^{1/2}\Big] \\
&\leq M(\|u_0\|_H^{2n}, \|f\|_{V'}^{2n}, T) + \frac{1}{4}\mathbb{E}[\sup_{[0,T]} z(t)\|u_n\|_H^{2n}] + C\mathbb{E}\Big[\Big(\int_0^T 1+\|u_{n-1}\|_H^2\Big)^n\Big]
\end{aligned}$$

By observing $e^{-KT} \leq e^{-Kt} \leq 1$ for $0 \leq t \leq T$ and using the induction step, we conclude the result

$$\mathbb{E}[\sup_{[0,T]}\|u_n\|_H^{2n}] \leq M(\|u_0\|_H^{2n}, \|f\|_{V'}^{2n}, T), \tag{3.40}$$

Moreover, based on inequality 3.40, we immediately also have

$$\mathbb{E}\Big[\int_0^T \|u_n\|_V^2\|u_n\|_H^{2n-2}\Big] \leq M(\|u_0\|_H^{2n}, \|f\|_{V'}^{2n}, T). \tag{3.41}$$

□

Theorem 4.3.7. *Let u be the solution to the equation 1.4.1 and let u_n be the corresponding linearized approximation equation 1.1. Then, we have for any $n \in \mathbb{N}_0$*

$$\mathbb{E}[\sup_{[0,T]}\log(1+\|u_n\|_V^2)] \leq C(f, g, u_0, T), \tag{3.42}$$

Proof. From the infinite dimensional version of Ito lemma we get

$$\begin{aligned}
&d\varphi(\|u_n\|_V^2) + 2\nu\varphi'(\|u_n\|_V^2)\|A(u_n)\|_H^2 dt \\
&\quad = \varphi\prime(\|u_n\|_V^2)\Big(2\langle f, Au_n\rangle \\
&\qquad - 2\langle B(u_{n-1}, u_n), A(u_n)\rangle + \varphi'(\|u_n\|_V^2)\|g(u_{n-1})\|_V^2\Big)dt \\
&\qquad + 2\varphi''(\|u_n\|_V^2)\sum_k \langle g_k(u_{n-1}), A(u_n)\rangle^2 dt \\
&\qquad + 2\varphi'(\|u_n\|_V^2)\langle g(u_{n-1}), A(u_n)\rangle dW_t
\end{aligned} \tag{3.43}$$

Recall here that

$$\begin{aligned}
\varphi(x) &= \log(1+x) \\
\varphi'(x) &= \frac{1}{1+x} \\
|\varphi''(x)| &= \frac{1}{(1+x)^2}
\end{aligned}$$

Integrating, taking the supremum and the expectation, we obtain

$$\begin{aligned}
&\mathbb{E}\sup_{[0,T]}\varphi'(\|u_n\|_V^2) + 2\nu\mathbb{E}\int_0^T \varphi'(\|u_n\|_V^2)\|A(u_n)\|_H^2 ds \\
&\quad \leq \varphi'(\|u_0\|_V^2) + \mathbb{E}\big[\int_0^T (T_1 + T_2 + T_3 + T_4)ds\big] \\
&\qquad + 2\mathbb{E}\sup_{s\in[0,T]}\left|\int_0^s T_0 dW_s\right|
\end{aligned} \tag{3.44}$$

For convenience, we denote

$$T_0 = 2\varphi'(\|u_n\|_V^2)|\langle g(u_{n-1}), Au_n\rangle| \tag{3.45}$$

$$T_1 = 2\varphi'(\|u_n\|_V^2)|\langle B(u_{n-1}, u_n), Au_n\rangle| \tag{3.46}$$

$$\begin{aligned} T_2 &= 2\varphi'(\|u_n\|_V^2)|\langle f, Au_n\rangle| \\ &\leq 2\varphi'(\|u_n\|_V^2)\|f\|_H\|Au_n\|_H \\ &\leq K\varphi'(\|u_n\|_V^2)\|f\|_H^2 + \frac{\nu}{8}\varphi'(\|u_n\|_V^2)\|Au_n\|_H^2 \end{aligned} \tag{3.47}$$

$$\begin{aligned} T_3 &= \varphi'(\|u_n\|_V^2)\|g(u_{n-1})\|_V^2 \\ &\leq K\varphi'(\|u_n\|_V^2)(1 + \|u_{n-1}\|_V^2) \end{aligned} \tag{3.48}$$

$$\begin{aligned} T_4 &= 2|\varphi''(\|u_n\|_V^2)\langle g(u_{n-1}), Au_n\rangle| \\ &\leq K|\varphi''(\|u_n\|_V^2)|\|u_n\|_V^2(1 + \|u_{n-1}\|_V^2) \end{aligned} \tag{3.49}$$

For T_0 term by appealing to the BDG inequality, we have

$$\mathbb{E}\sup_{s\in[0,T]}\left|\int_0^s T_0 dW\right| \leq C\mathbb{E}\left(\int_0^T |\varphi\prime(\|u_n\|_V^2)|^2\|g(u_{n-1})\|_V^2\|u_n\|_V^2 ds\right)^{1/2} \tag{3.50}$$

We thus observe using Lipschitz condition on $g(u)$

$$\begin{aligned} &\mathbb{E}\sup_{s\in[0,T]}|\int_0^s T_0 dW| \\ &\quad\leq C\mathbb{E}\left(\int_0^T \frac{1}{(1+\|u_n\|_V^2)^2}(1 + \|u_{n-1}\|_V^2)\|u_n\|_V^2 ds\right)^{1/2} \\ &\quad\leq C(\|u_0\|_H^2, T) \end{aligned} \tag{3.51}$$

Next, we estimate the term T_1 by appealing to the classical estimate

$$\begin{aligned} T_1 &= 2|\varphi'(\|u_n\|_V^2)\langle B(u_{n-1}, u_n), A(u_n)\rangle| \\ &\le C\varphi'(\|u_n\|_V^2)\|u_{n-1}\|_H^{1/2}\|u_{n-1}\|_V^{1/2}\|u_n\|_V^{1/2}\|Au\|_H^{3/2} \\ &\le C\varphi'(\|u_n\|_V^2)\|u_{n-1}\|_H^2\|u_{n-1}\|_V^2\|u_n\|_V^2 + \frac{\nu}{4}\varphi'(\|u_n\|_V^2)\|Au_n\|_H^2 \\ &\le C\|u_{n-1}\|_H^2\|u_{n-1}\|_V^2 + \frac{\nu}{4}\varphi'(\|u_n\|_V^2)\|Au_n\|_H^2, \end{aligned} \tag{3.52}$$

where we note by 3.41 that

$$\mathbb{E}\int_0^T \|u_{n-1}\|_H^2\|u_{n-1}\|_V^2 dt \le C(\|u_0\|_H^4, \|f\|_{V'}^4, T) \tag{3.53}$$

By combining all the estimates we conclude that

$$\mathbb{E}\sup_{[0,T]}\varphi(\|u_n\|_V^2) \le C(f, g, u_0, T). \tag{3.54}$$

Hence we conclude the proof. □

Next we borrow the following two results from [KUZ].

Theorem 4.3.8. *[KUZ] Let u be the solution in Definition 1.4.1 and let u_n be the corresponding linearized approximation equation in (1.1). Then, we have*

$$\mathbb{E}[\sup_{[0,T]}\varphi_1(\|u\|_V^2)] \le C(f, g, u_0, T), \tag{3.55}$$

where $\varphi_1(x) = \log(1 + x)$.

Theorem 4.3.9. *[KUZ] Let u be the solution to the equation 1.4.1. Then, we have*

$$\mathbb{E}[\sup_{[0,T]}\|u\|_H^{2n}] \leq C(\|u_0\|_H^{2n}, \|f\|_{V'}^{2n}) \tag{3.56}$$
$$\mathbb{E}\left[\int_0^T \|u\|_V^2\|u\|_H^{2n-2}\right] \leq M(\|u_0\|_H^{2n}, \|f\|_{V'}^{2n}, T).$$

for any positive integer n.

Next, we show the convergence in probability of this linearized scheme in Equation (1.1).

Lemma 4.3.10. *Let u be the solution to the equation 1.4.1 and let u_n be defined as in Equation (1.1). Then for any deterministic time $T > 0$, the linear approximations u_n converge in probability with respect to the V norm to the solution of the equation 1.4.1, i.e., for any $\delta > 0$ we have*

$$\mathbb{P}(\sup_{t\in[0,T]}\|u - u_n\|_V^2 \geq \delta) \to 0 \tag{3.57}$$

as $n \to \infty$. In particular, under the assumptions, we have

$$\mathbb{P}(\sup_{t\in[0,T]}(\log(1 + \|u - u^n\|_V^2))^{1-\epsilon} \geq \delta) \to 0 \tag{3.58}$$

for any $\epsilon \in (0,1)$, as well as, by the Poincaré inequality,

$$\mathbb{P}(\sup_{t\in[0,T]}\|u - u^n\|_H^{2k} \geq \delta) \to 0 \tag{3.59}$$

for all $\delta > 0$. Both statements shall be used below.

Proof. By assumption, we have $u_0 \in \mathcal{L}_V^2(\Omega)$. Hence, by Chebyshev theorem we have,

$$\mathbb{P}(\|u_0\|_V^2 > s) \to 0 \tag{3.60}$$

as $s \to \infty$. Denoting $\Omega_s = \{\|u_0\|_V^2 \leq s\}$, we have $\Omega_s \to \Omega$. Hence, we choose s such that $\mathbb{P}(\Omega_s) > 1 - \frac{\epsilon}{2}$. Moreover, We know by Theorem 4.3.3 and Lemma 6.0.2 there exists a sequence of stopping times $\{\tilde{\tau}_{n_l}^M\}_{n_l \geq 1}$ with the corresponding subsequence $\{u_{n_l}\}$ converging monotone decreasing to τ^M. We also know by Theorem 4.3.5 that $\tau^M \to \infty$ a.s. as $M \to \infty$, where M is the constant defined as in Theorem 4.3.3.

Hence, denoting $\{\tau_{n_l}^M = \tilde{\tau}_{n_l}^M \wedge T\}_{n_l \geq 1}$, there exists M_0 such that $\mathbb{P}(\tau^{M_0} < T) \leq \frac{\epsilon}{4}$ and by Lebesgue dominated convergence theorem, we have

$$\lim_{n_l \to \infty} \mathbb{E}\left[\mathbb{1}_{\Omega_s} \sup_{t \in [0, \tau^{M_0}]} \|u - u_{n_l}\|_V^2\right] = 0. \tag{3.61}$$

This implies convergence in probability. Thus,

$$\lim_{n_l \to \infty} \mathbb{P}(\mathbb{1}_{\Omega_s} \sup_{t \in [0, \tau^{M_0}]} \|u - u_{n_l}\|_V^2 > \delta) = 0, \tag{3.62}$$

for any $\delta > 0$. Hence, we have

$$\begin{aligned}
&\mathbb{P}\big(\mathbb{1}_{\Omega_s} \sup_{t\in[0,T]} \|u - u_{n_l}\|_V^2 \geq \delta\big) = \mathbb{P}\big(\{\sup_{t\in[0,T]} \|u - u_{n_l}\|_V^2 \geq \delta\} \cap \{\tau^{M_0} < T\} \cap \{\omega \in \Omega_s\}\big) \\
&\quad + \mathbb{P}(\{\sup_{t\in[0,T]} \|u - u_{n_l}\|_V^2 \geq \delta\} \cap \{\tau^{M_0} = T\} \cap \{\omega \in \Omega_s\}) \\
&\leq \mathbb{P}(\tau^{M_0} < T) + \mathbb{P}(\mathbb{1}_{\Omega_s} \sup_{t\in[0,\tau^{M_0}]} \|u - u_{n_l}\|_V^2 > \delta) \qquad (3.63)
\end{aligned}$$

Then, we get

$$\begin{aligned}
\mathbb{P}\left(\sup_{t\in[0,T]} \|u - u_{n_l}\|_V^2 \geq \delta\right) &\leq \mathbb{P}\left(\tau^{M_0} < T\right) + \mathbb{P}\left(\mathbb{1}_{\Omega_s} \sup_{t\in[0,\tau^{M_0}]} \|u - u_{n_l}\|_V^2 > \delta\right) + \mathbb{P}\left(\Omega_s^c\right) \\
&\leq \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{2} = \epsilon. \qquad (3.64)
\end{aligned}$$

for n_l large enough. Then by taking any subsequence u_{m_l} and by Theorem 4.3.3 and Lemma 6.0.2 repeating the same arguments above, we get that every subsequence $\{u_{m_l}\}$ has a further subsequence that converges in probability to u, which implies that the whole sequence $\{u_n\}$ converges in probability to u, which concludes the proof. □

Lemma 4.3.11. *Let u be the solution to the equation 1.4.1 and let u_n be the corresponding linearized approximation equation 1.1. Then we have*

$$\mathbb{E}[\sup_{[0,T]} \log(1 + \|u - u^n\|_V^2)] \leq M(f, u_0, g, T), \qquad (3.65)$$

where M is independent of n and depends on initial data and time T only.

Proof. We note that $\varphi(x) = \log(1 + x)$ is an increasing concave function. Hence, being concave with $\varphi(x) = 0$, it is also subadditive. By noting that

$$\|u - u^n\|_V^2 \leq 2\|u\|_V^2 + 2\|u^n\|_V^2,$$

we get

$$\begin{aligned}\mathbb{E}[\sup_{[0,T]} \log(1 + \|u - u^n\|_V^2)] \leq \mathbb{E}[\sup_{[0,T]} \log(1 + 2\|u\|_V^2)] \\ + \mathbb{E}[\sup_{[0,T]} \log(1 + 2\|u^n\|_V^2)]\end{aligned} \tag{3.66}$$

By Theorem 4.3.8 above, we conclude the proof. □

Now, we are ready to prove our first result, Theorem 3.2.1.

Proof. We know by Lemma 4.3.10 that

$$\mathbb{P}\left(\sup_{[0,T]} \log(1 + \|u - u^n\|_V^2)^{1-\epsilon}\right) \to 0 \tag{3.67}$$

for $0 < \epsilon < 1$ in probability as $n \to \infty$. Moreover, using 4.3.11, we have that

$$\mathbb{E}\left[\left(\sup_{[0,T]}(\log(1 + \|u - u^n\|_V^2)\right)^{(1-\epsilon)\frac{1}{1-\epsilon}}\right] \leq M(u_0, f, g, T). \tag{3.68}$$

We note here that $g(x) = x^{\frac{1}{1-\epsilon}}$ is a convex function with $\lim_{x\to\infty} \frac{x^{\frac{1}{1-\epsilon}}}{x} = \infty$. Using de La-Vallee-Poussin criteria for uniform integrability (see e.g. [D]) we get that

$$\left\{ \sup_{[0,T]} \left((\log(1 + \|u - u^n\|_V^2) \right)^{1-\epsilon} \right\}_{n\geq 1} \tag{3.69}$$

is uniformly integrable. Uniform integrability and convergence in probability and imply L^1-convergence. Thus, using that $x^{1-\epsilon}$ for $0 < \epsilon < 1$ being increasing and continuous, we get that

$$\mathbb{E}\left[\sup_{[0,T]} \left(\log(1 + \|u - u^n\|_V^2) \right)^{1-\epsilon} \right] \to 0 \tag{3.70}$$

as $n \to \infty$. Hence, Theorem 3.2.1 is proven. □

Next, we prove our second result Theorem 4.2.2.

Proof. We use that

$$\mathbb{E}[\sup_{[0,T]} \|u - u_n\|_H^{2k}] \leq 2^{2k} (\mathbb{E}[\sup_{[0,T]} \|u\|_H^{2k}] + \mathbb{E}[\sup_{[0,T]} \|u_n\|_H^{2k}]). \tag{3.71}$$

Then, we have using Lemma 4.3.10, Theorem 4.3.6 and Theorem 4.3.9

$$\mathbb{E}[\sup_{[0,T]} \|u - u_n\|_H^{2k(1-\epsilon)}] \to 0, \tag{3.72}$$

as $n \to \infty$. This concludes the proof. □

Chapter 5

Timewise Optimal Feedback Control for the SNSE

1 Introduction

The existence of optimal control of stochastic evolution equations has been studied by [PI, G, GR, GS, T, T2] among others by adding linearity or semilinearity assumptions as well as putting boundedness restriction for nonlinearities. On the other hand, more specifically, the literature about the optimal control of SNSE is not very rich, since these assumptions do not apply to the SNSE. The nonlinearity of the SNSE causes the problem to be of non-convex type. We refer the reader the book of Ekeland and Temam about non-convex optimization for further investigations [ET]. Related works about the control of SNSE can be mentioned as follows. Choi et al. investigated the optimal control problem in [CTMK] for the stochastic Burgers equation (one dimensional Navier-Stokes equation) with additive noise. The paper [PD] studies the control of turbulence for the stochastic Burgers equation. Another work in this direction is by Sritharan [S], where the existence of optimal controls is established using techniques for the martingale problem formulation of Stroock and Varadhan [SV] in the context stochastic Navier-Stokes equation. In [B], it is shown that there exist feedback controls for the SNSE of (2.1), which are controlled by different external forces ϕ for a specific cost functional satisfying some regularity conditions. In this chapter, we follow the framework that is studied in [B]. Namely, the SNSE is controlled by deterministic force ϕ. We show using the recent bounds and approximations for the SNSE in 2D in [KUZ], that there exists a feedback control ϕ^* that is optimal for the supremum of SNSE up to a terminal deterministic time T, which is only natural to introduce when we want to control the extreme events on the whole path rather than integrating the path.

2 Main Result

Our purpose is to control the solution u_ϕ of the SNSE

$$\begin{aligned}
&\langle u_\phi(t), v\rangle + \int_0^t \langle A u_\phi(s), v\rangle ds \\
&\quad = \langle u_0, v\rangle + \int_0^t \langle B(u_\phi(s), u_\phi(s)), v\rangle ds + \int_0^t \langle \phi(s, u_\phi), v\rangle ds \\
&\qquad + \int_0^t \langle g(s, u_\phi(s)), v\rangle dW_s,
\end{aligned} \tag{2.1}$$

for all $v \in H$, $t \in [0, T]$, a.e. $\omega \in \Omega$, where $\phi \in \mathcal{U}^b$ and the assumptions on initial data are as in Definition 2.1. We consider the following cost functional

$$J(\phi) = \mathbb{E} \sup_{t\in[0,T]} (\varphi(\mathcal{L}[t, u_\phi(t), \phi(t)])), \tag{2.2}$$

where $\mathcal{L} : [0,T] \times V \times H \to \mathbb{R}_+$ is uniformly locally Lipschitz i.e.

$$|\mathcal{L}(t,x,y) - \mathcal{L}(t,x_2,y_2)|^2 \leq C(\|x_1 - x_2\|_V^2 + \|y_1 - y_2\|_H^2), \tag{2.3}$$

with

$$\varphi(x) = \log(1+x)^{1-\epsilon}, \tag{2.4}$$

and $0 < \epsilon < 1$.

Remark 5.2.1. *Since we require $\mathcal{L}$ to be only uniformly locally Lipschitz, using the concave function $\varphi(x) = \log(1+x)^{1-\epsilon}$ does not imply that, we should check only the end points for the functional $\mathcal{L}$.*

We state now our main result.

Theorem 5.2.2. *Suppose we have a set $\phi \subset \mathcal{U}^b$ of bounded feedback controls that are locally Lipschitz in V-norm, i.e. they satisfy*

$$\sup_{t\in[0,T]} \|\phi(t,\omega)\|_V \leq K, a.s.,$$
$$\|\phi(t_1,x_1) - \phi(t_2,x_2)\|_H^2 \leq C_1|t_1 - t_2|^2 + C_2\|x_1 - x_2\|_H^2, \tag{2.5}$$

where C_1 and C_2 are uniform for the family of controls $\phi \in \mathcal{U}^b$. Then, there exists an optimal feedback control ϕ^* satisfying

$$J(\phi^*) = \min_{\phi\in\mathcal{U}} J(\phi) \tag{2.6}$$

Remark 5.2.3. *In their paper F. Abergel and R. Temam [AT] investigate the deterministic Navier-Stokes equation by controlling the turbulence inside the flow. They give a cost functional regarding the vorticity in the fluid. For our problem, the functional $\mathcal{L}$ would be*

$$\mathcal{L}(t, u_\phi(t), \phi(t)) := \|\nabla \times u_\phi(t)\|_H + \|\phi(t)\|_H^2 \tag{2.7}$$

3 Proof of the Main Result

The proof of Theorem 5.2.2 requires several lemmas and theorems. First, we need the following theorem.

Theorem 5.3.1. *Let $\{\phi_n\}_{n\geq 0}$ be a sequence in $\mathcal{U}^b$ as defined in (3.23) and J is as defined in (3.20). Suppose*

$$\sup_{t\in[0,T]} \|\phi - \phi_n\|_V \to 0 \ a.s. \tag{3.1}$$

Then, we have

$$J(\phi_n) \to J(\phi) \tag{3.2}$$

as $n \to \infty$.

To prove Theorem 5.3.1, we use the following results from [B]

Theorem 5.3.2. *[B] Let $\{\phi_n\}_{n\geq 0}$ and ϕ be a sequence of linear bounded feedback controls with $\mathcal{L}(H)$ being the space of all linear and continuous operators from H to itself and suppose*

$$\mathbb{E}[\int_0^T \|\phi - \phi_n\|^2_{\mathcal{L}(H)}] \to 0, \tag{3.3}$$

then it holds that

$$\mathbb{E}[\int_0^T \|u_\phi - u_{\phi_n}\|^2_V] \to 0, \tag{3.4}$$

as $n \to \infty$.

Proof. See appendix. □

Next, we need the two technical lemmas.

Lemma 5.3.3. *Let $\varphi(x)$ be a concave increasing function with $\varphi(0) = 0$. Then, we have*

$$|\varphi(x_1) - \varphi(x_2)| \leq \varphi(|x_1 - x_2|) \tag{3.5}$$

Proof. Since $\varphi(x)$ is a concave increasing function with $\varphi(0) = 0$, $\varphi(x)$ is subadditive. Hence, we have

$$\varphi(x_1) \leq \varphi(|x_1 - x_2| + x_2)$$

$$\varphi(|x_1 - x_2| + x_2) \leq \varphi(|x_1 - x_2|) + \varphi(x_2)$$

$$\varphi(x_1) - \varphi(x_2) \leq \varphi(|x_1 - x_2|) \tag{3.6}$$

By interchanging x_1 and x_2 we conclude the proof. □

Lemma 5.3.4. *[GZ] Fix $T > 0$. Assume that*

$$X, Y, Z, R : [0, T) \times \Omega \to \mathbb{R} \tag{3.7}$$

are real-valued, non-negative stochastic processes. Let $\tau < T$ be a stopping time so that

$$\mathbb{E} \int_0^\tau (RX + Z) ds < \infty. \tag{3.8}$$

Assume, moreover that for some fixed constant κ we have

$$\int_0^\tau R ds < \kappa, \ \ a.s. \tag{3.9}$$

Suppose that for all stopping times $0 \le \tau_a \le \tau_b \le \tau$

$$\mathbb{E}\Bigg(\sup_{t \in [\tau_a, \tau_b]} X + \int_0^\tau Y ds \Bigg) \le C_0 \mathbb{E}\Bigg(X(\tau_a) + \int_{\tau_a}^{\tau_b} (RX + Z) ds \Bigg), \tag{3.10}$$

where C_0 is a constant independent of the choice of τ_a, τ_b. Then we have

$$\mathbb{E}\Bigg(\sup_{t \in [0, \tau]} X + \int_0^\tau Y ds \Bigg) \le C \mathbb{E}\Bigg(X(0) + \int_0^\tau Z ds \Bigg), \tag{3.11}$$

where C depends on C_0, T and κ.

Proof. Choose a finite sequence of stopping times

$$0 = \tau_0 < \tau_1 < ... < \tau_N < \tau_{N+1} = \tau \tag{3.12}$$

so that

$$\int_{\tau_{k-1}}^{\tau_k} R ds < \frac{1}{2C_0} \text{ a.s.} \tag{3.13}$$

For each pair τ_{k-1}, τ_k take $\tau_a = \tau_{k-1}$ and $\tau_b = \tau_k$ in (0.4). Using (0.7), we have

$$\mathbb{E}\bigg(\sup_{t\in[\tau_{k-1},]} X + \int_{\tau_{k-1}}^{\tau_k} Y ds \bigg) \le C\mathbb{E}X(\tau_{k-1}) + C\mathbb{E}\int_{\tau_{k-1}}^{\tau_k} Z ds. \tag{3.14}$$

By induction we have

$$\mathbb{E}\bigg(\sup_{t\in[0,\tau_j]} X + \int_0^{\tau_j} Y ds \bigg) \le C\mathbb{E}X(0) + C\mathbb{E}\int_0^{\tau_j} Z ds \tag{3.15}$$

then we have

$$\begin{aligned}
\mathbb{E}\bigg(\sup_{t\in[0,\tau_{j+1}]} X + \int_0^{\tau_j} Y ds \bigg) &\le C\mathbb{E}X(0) + C\mathbb{E}\int_0^{\tau_j} Z ds + C\mathbb{E}\bigg(\sup_{t\in[\tau_j,\tau_{j+1}]} X + \int_{\tau_j}^{\tau_{j+1}} Y ds \bigg) \\
&\le C\mathbb{E}X(0) + C\mathbb{E}\int_0^{\tau_{j+1}} Z ds + C\mathbb{E}X(\tau_j) \\
&\le C\mathbb{E}X(0) + C\mathbb{E}\int_0^{\tau_{j+1}} Z ds
\end{aligned} \tag{3.16}$$

Hence, we conclude the proof. □

We continue with the following theorem.

Theorem 5.3.5. *Let $\tilde{M} > 0$ and $M > 1$. Moreover, let $\{u_{\phi_n}\}_{n\ge1}$ be the sequence of solutions of Equation 2.1. Suppose, we have*

$$\|u_0\|_V \le \tilde{M}, \ \textit{a.s.} \tag{3.17}$$

Moreover, assume

$$\sup_{t\in[0,T]} \|\phi_m(t,\cdot)-\phi_n(t,\cdot)\|_V \to 0, \tag{3.18}$$

as $m,n\to\infty$ *almost surely.*

Denote

$$\mathcal{T}_n^{M,T} = \{\tau \le T : (\sup_{t\in[0,\tau]} \|u_{\phi_n}\|_V^2 + \int_0^\tau |Au_{\phi_n}|^2 dt)^{1/2} \le M+\tilde{M}\}. \tag{3.19}$$

Let

$$\mathcal{T}_{m,n}^{M,T} := \mathcal{T}_m^{M,T} \cap \mathcal{T}_n^{M,T}. \tag{3.20}$$

Then

1. *For any* $T>0$, *we have*

$$\lim_{n\to\infty} \sup_{m\ge n} \sup_{\tau\in\mathcal{T}_{m,n}^{M,T}} \mathbb{E}[\sup_{t\in[0,\tau]} \|u_{\phi_m}-u_{\phi_n}\|_V^2 + \int_0^\tau \|A(u_{\phi_m}-u_{\phi_n})\|_H^2 dt] = 0. \tag{3.21}$$

2.

$$\lim_{S\to 0} \sup_n \sup_{\tau\in\mathcal{T}_n^{M,T}} \mathbb{P}\left(\sup_{t\in[0,\tau\wedge S]} \|u_{\phi_n}\|_V^2 + \int_0^{\tau\wedge S} \|A(u_{\phi_n}\|_H^2 dt > \tilde{M}^2 + (M-1)^2 \right) = 0. \tag{3.22}$$

Proof. 1. We have

$$d(u_{\phi_n}-u_{\phi_m}) + A(u_{\phi_n}-u_{\phi_m})dt = (B(u_{\phi_n})-B(u_{\phi_m}))dt + \sum_{k=1}^{\infty}[g_k(u_{\phi_n})-g_k(u_{\phi_m})]dW_k$$
$$+ \phi_n(t,u_{\phi_n}(t)) - \phi_m(t,u_{\phi_n}(t))dt \tag{3.23}$$

Hence by Ito-lemma, we have

$$
\begin{aligned}
& d\|u_{\phi_n} - u_{\phi_m}\|_V^2 + 2\|A(u_{\phi_n} - u_{\phi_m})\|_H^2 dt \\
&\quad = 2\langle B(u_{\phi_n}) - B(u_{\phi_m}), A(u_{\phi_n} - u_{\phi_m})\rangle dt \\
&\qquad + \sum_{k=1}^{\infty} \|g_k(u_{\phi_n}) - g_k(u_{\phi_m})\|_V^2 dt \\
&\qquad + 2\sum_{k=1}^{\infty} \langle g_k(u_{\phi_n}) - g_k(u_{\phi_m})\rangle dW_k \\
&\qquad + 2\langle \phi_n(t, u_{\phi_n}) - \phi_m(t, u_{\phi_m}), A(u_{\phi_n} - u_{\phi_m})\rangle dt
\end{aligned} \tag{3.24}
$$

By taking supremum up to τ, integrating and taking expectation, we get

$$
\begin{aligned}
& \mathbb{E}[\sup_{[0,\tau]} \|u_{\phi_n} - u_{\phi_m}\|_V^2 + 2\int_0^\tau \|A(u_{\phi_n} - u_{\phi_m})\|_H^2 dt] \\
&\quad \leq 2\mathbb{E}\int_0^\tau |\langle B(u_{\phi_m}) - B(u_{\phi_n}), A(u_{\phi_n} - u_{\phi_m})\rangle| dt \\
&\qquad + \mathbb{E}\int_0^\tau \sum_{k=1}^{\infty} \|g_k(u_{u_{\phi_m}}) - g_k(u_{\phi_n})\|_V^2 dt \\
&\qquad + \mathbb{E}[\sup_{r\in[0,\tau]} |2\sum_{k=1}^{\infty} \int_0^r \langle g_k(u_{\phi_m}) - g_k(u_{\phi_n}), A(u_{\phi_n} - u_{\phi_m})\rangle dW_k|] \\
&\qquad + \mathbb{E}[2\int_0^\tau |\langle \phi_n(t, u_{\phi_n}) - \phi_m(t, u_{\phi_m}), A(u_{\phi_n} - u_{\phi_m})\rangle| dt]
\end{aligned} \tag{3.25}
$$

We treat each term above seperately. First

$$
\begin{aligned}
& \mathbb{E}\int_0^\tau \sum_{k=0}^{\infty} \|g_k(u_{\phi_n}) - g_k(u_{\phi_m})\|_V^2 dt \\
&\quad \leq \mathbb{E}\int_0^\tau \|u_{\phi_n} - u_{\phi_m}\|_V^2
\end{aligned} \tag{3.26}
$$

By Poincare lemma, assumption 3.18 implies that

$$\int_0^T \|u_{\phi_n} - u_{\phi_m}\|_H^2 \to 0, \tag{3.27}$$

as $m, n \to \infty$. Then this implies by Theorem 5.3.2

$$\mathbb{E}\int_0^T \|u_{\phi_n} - u_{\phi_m}\|_V^2 \to 0, \tag{3.28}$$

as $n, m \to \infty$. Next we have

$$\begin{aligned}
&\mathbb{E}[2\int_0^\tau |\langle \phi_n(t, u_{\phi_n}) - \phi_m(t, u_{\phi_m}), A(u_{\phi_n} - u_{\phi_m})\rangle| dt] \\
&\leq \int_0^\tau \|\phi_n(t, u_{\phi_n}) - \phi_m(t, u_{\phi_m})\|_V \|u_{\phi_n} - u_{\phi_m}\|_V ds \\
&\quad + C\mathbb{E}\int_0^T \|u_{\phi_n} - u_{\phi_m}\|_V^2 \\
&\leq C\mathbb{E}\int_0^\tau \|\phi_n(t, u_{\phi_n}) - \phi_n(t, u_{\phi_m})\|_V^2 \\
&\quad + C\mathbb{E}\int_0^\tau \|\phi_n(t, u_{\phi_m}) - \phi_m(t, u_{\phi_m})\|_V^2 \\
&\quad + C\mathbb{E}\int_0^T \|u_{\phi_n} - u_{\phi_m}\|_V^2
\end{aligned} \tag{3.29}$$

which goes to 0 as $n, m \to \infty$ by Theorem 5.3.2 as well as by our assumption on ϕ.

Next, we treat the nonlinear term by seperating into two parts as follows.

$$\begin{aligned}
&|\langle B(u_{\phi_m} - u_{\phi_n}), A(u_{\phi_n} - u_{\phi_m})\rangle| \leq |\langle B(u_{\phi_m} - u_{\phi_n}, u_{\phi_m}), A(u_{\phi_n} - u_{\phi_m})\rangle| \\
&+ |\langle B(u_{\phi_n}, u_{\phi_n} - u_{\phi_m}), A(u_{\phi_n} - u_{\phi_m})\rangle|
\end{aligned}$$

For the first term above, we have

$$
\begin{aligned}
&|\langle B(u_{\phi_m} - u_{\phi_n}, u_{\phi_m}), A(u_{\phi_n} - u_{\phi_m})\rangle| \\
&\quad \le \|u_{\phi_n} - u_{\phi_m}\|_V \|u_{\phi_m}\|_V^{1/2} \|Au_{\phi_m}\|_H^{1/2} \|A(u_{\phi_n} - u_{\phi_m})\|_H \\
&\quad \le \frac{1}{6}\nu \|A(u_{\phi_n} - u_{\phi_m})\|_H^2 + C\|u_{\phi_n} - u_{\phi_m}\|_V^2 \|Au_{\phi_m}\|_H, \qquad (3.30)
\end{aligned}
$$

To estimate the term C $\|u_{\phi_n} - u_{\phi_m}\|_V^2 \|Au_{\phi_m}\|_H$ in the second line of inequality, we apply Theorem 6.0.1 with $R = \|Au_{\phi_m}\|_H$, $X = C\|u_{\phi_n} - u_{\phi_m}\|_V^2$ and $Y = \|A(u_{\phi_n} - u_{\phi_m})\|_H^2$ and Z stand for the remaining terms in the right hand side of the equation 3.25, that we prove converging to 0. Moreover, $\frac{1}{6}\nu\|A(u_{\phi_n} - u_{\phi_m})\|_H^2$ is absorbed to the left hand side of the main equation.

Next, we treat the second nonlinear term as

$$
\begin{aligned}
&|\langle B(u_{\phi_n}, u_{\phi_n} - u_{\phi_m}), A(u_{\phi_n} - u_{\phi_m})\rangle| \\
&\quad \le \|u_{\phi_n}\|_V \|u_{\phi_m} - u_{\phi_n}\|_V^{1/2} \|A(u_{\phi_n} - u_{\phi_m})\|_H^{3/2} \\
&\quad \le \frac{1}{6}\|A(u_{\phi_n} - u_{\phi_m})\|_H^2 + C(M, \tilde{M})\|u_{\phi_n} - u_{\phi_m}\|_V^2, \qquad (3.31)
\end{aligned}
$$

where the first term is absorbed to LHS of the equation 3.25, whereas for the second term we have

$$
C(M, \tilde{M})\mathbb{E}[\int_0^\tau \|u_{\phi_n} - u_{\phi_m}\|_V^2] \to 0, \qquad (3.32)
$$

as $m, n \to \infty$ by Theorem 5.3.2. Hence, the first part of the proof is concluded.

2. The proof is identical with [GZ]. First by Ito we have

$$d\|u_\phi\|_V^2 + 2\nu\|Au_\phi\|_H^2 dt = \Big(2\langle \phi - B(u_\phi), Au_\phi\rangle + \|g_k(u_\phi)\|_V^2\Big) + 2\sum_{k=1}^{\infty}\langle g_k(u_\phi), Au_\phi\rangle dW_k \tag{3.33}$$

We fix $\tau \in \mathcal{T}_n^{M,T}$ and $S > 0$. Integrating from 0 to $\tau \wedge S$, we get

$$\begin{aligned}\sup_{r\in[0,S\wedge\tau]}\|u_\phi\|_V^2 + \int_0^{S\wedge\tau} 2\nu\|Au_\phi\|_H^2 ds \le \|u_0\|_V^2 &+ \int_0^{S\wedge\tau} 2|\langle \phi - B(u_\phi), Au_\phi\rangle| dr \\ &+ \int_0^{S\wedge\tau}\|g(u_\phi)\|_V^2 dr \\ &+ \sup_{r\in[0,S\wedge\tau]}\Big|\sum_{k=1}^{\infty} 2\langle g_k(u_\phi), u_\phi\rangle dW_k\Big|. \end{aligned}\tag{3.34}$$

Applying the classical estimate on nonlinear term (see [CF2]), we have

$$\begin{aligned}\Big|\langle B(u_\phi), A(u_\phi)\rangle\Big| &\le \|u_\phi\|_V^{3/2}\|Au_\phi\|_H^{3/2} \\ &\le C\|u_\phi\|_V^6 + \frac{\nu}{4}\|Au_\phi\|_H^2. \end{aligned}\tag{3.35}$$

Using Equation (3.35) and the Lipschitz assumption on g, we get

$$\begin{aligned}&\sup_{r\in[0,S\wedge\tau]}\|u_\phi\|_V^2 + \int_0^{\tau\wedge S}\nu\|Au_\phi\|_H^2 dr \\ &\quad\le \|u_0\|_V^2 + C\int_0^{S\wedge\tau}(\|\phi\|_H^2 + \|u_\phi\|_V^6 + \|u_\phi\|_V^2 + 1)dr \\ &\quad+ \sup_{r\in[0,S\wedge\tau]}\Big|\int_0^r 2\sum_{k=1}^{\infty}\langle g_k(u_\phi), u_\phi\rangle dW_k\Big|. \end{aligned}\tag{3.36}$$

This implies then

$$\begin{aligned}
&\mathbb{P}\bigg(\sup_{s\in[0,\tau\wedge S]}\|u_\phi\|_V^2+\nu\int_0^{\tau\wedge S}\|Au_\phi\|_H^2ds>\|u_0\|_V^2+(M-1)^2\bigg)\\
&\quad\le\mathbb{P}\bigg(C\int_0^{\tau\wedge S}(\|\phi\|_H^2+\|u_\phi\|_V^6+\|u_\phi\|_V^2+1)dr>\frac{(M-1)^2}{2}\bigg)\\
&\quad\le\frac{2C}{(M-1)^2}\mathbb{E}\int_0^{\tau\wedge S}(\|\phi\|_H^2+\|u_\phi\|_V^6+\|u_\phi\|_V^2+1)dr\\
&\quad\le C\mathbb{E}\bigg(\int_0^S\|\phi\|_H^2+1dr\bigg) \qquad (3.37)
\end{aligned}$$

Next, using Doob's inequality for the second term, we get

$$\begin{aligned}
&\mathbb{P}\bigg(\sup_{r\in[0,\tau\wedge S]}\bigg|\sum_{k=1}^{\infty}\int_0^r\langle g_k(u_\phi),u_\phi\rangle dW_k\bigg|>\frac{(M-1)^2}{2}\bigg)\\
&\quad\le\frac{4}{(M-1)^4}\mathbb{E}\bigg(\int_0^{S\wedge\tau}\|u_\phi\|_V^2\sum_{k=1}^{\infty}\|g_k(u_\phi)\|_V^2dr\bigg)\\
&\quad\le C \qquad (3.38)
\end{aligned}$$

By letting $S\to 0$ with the integrability condition imposed on function ϕ, we conclude the proof. □

Theorem 5.3.6. *Given the assumptions on initial data u_0 as in Definition 2.1 and $\phi\in\mathcal{U}^b$, there exists a global strong solution (u_ϕ,τ) in the sense of Definition 2.1 introduced above.*

Proof. Let $w\in H$ be given. Using Theorem 5.3.5 with $\{u_{\phi_n}\}$ be the sequence of solutions of Equation 2.1. Due to 3.21 and 3.22, we apply Lemma 6.0.2 with $B_1=V$ and $B_2=D(A)$ and the sequence $\{X^n\}=\{u_{\phi_n}\}$. We infer the existence of a subsequence $\{u_{\phi'_n}\}$ and a strictly positive stopping time $\tau \le T$ and a process $u_\phi(.) = u_\phi(. \wedge \tau)$, continuous in V such that

$$\sup_{t\in[0,\tau]} \|u_{\phi_{n'}} - u_\phi\|_V^2 + \nu \int_0^\tau \|A(u_{\phi_{n'}} - u_\phi)\|_H^2 ds \to 0, \tag{3.39}$$

a.s. We also have that the conditions of Lemma 6.0.2 (ii) is satisfied for any $p \in (1, \infty)$. Thus, we have for any $p > 1$

$$u_\phi(. \wedge \tau) \in L^p(\Omega; C([0,T]; V)), \tag{3.40}$$

with

$$u_\phi \mathbb{1}_{t\le\tau} \in L^p(\Omega; L^2([0,T]; D(A))). \tag{3.41}$$

By Lemma 6.0.2 (ii) we infer a collection of measurable sets $\Omega_{n'} \in \mathcal{F}$ with

$$\Omega_{n'} \uparrow \Omega \tag{3.42}$$

such that

$$\sup_{n'} \mathbb{E}\left[\sup_{t\in[0,\tau]} \|u_{\phi_{n'}}(t)\mathbb{1}_{\Omega_{n'}}\|_V^2 + \nu \int_0^\tau \|Au_{\phi_{n'}}\mathbb{1}_{\Omega_{n'}}\|_H^2 ds \right]^{p/2} < \infty. \tag{3.43}$$

Using equations 3.39, 3.42, 3.43 and by Lemma 6.0.3 we have

$$\mathbb{1}_{\Omega_{n'}, t\le\tau} u^{n'} \rightharpoonup \mathbb{1}_{t\le\tau} u_\phi \text{ in } L^p(\Omega; L^2([0,T]; D(A))), \tag{3.44}$$

and

$$\mathbb{1}_{\Omega_{n'}} u_\phi^{n'\wedge\tau} \rightharpoonup^* u \text{ in } L^p(\Omega; L^\infty([0,T]; V)). \tag{3.45}$$

For the nonlinear term, we estimate for all $w \in H$ as follows:

$$\begin{aligned}
&\left|\langle B(u_{\phi_{n'}}, u_{\phi_{n'}}) - B(u_\phi, u_\phi), w\rangle\right| \qquad (3.46)\\
&= \left|\langle B(u_{\phi_{n'}} - u_\phi, u_{\phi_{n'}}) + B(u_\phi, u_{\phi_{n'}}) - B(u,u), w\rangle\right|\\
&= \left|\langle B(u_{\phi_{n'}} - u_\phi, u_{\phi_{n'}}) + B(u_\phi, u_{\phi_{n'}} - u_\phi), w\rangle\right|\\
&\leq \left|\langle B(u_{\phi_{n'}} - u_\phi, u_{\phi_{n'}}), w\rangle\right| + \left|\langle B(u_\phi, u_{\phi_{n'}} - u_\phi), w\rangle\right|
\end{aligned}$$

Then we have using the classical estimates [CF]

$$\left|\langle B(u_{\phi_{n'}} - u_\phi, u_{\phi_{n'}}), w\rangle\right| \leq C\|u_{\phi_{n'}} - u_\phi\|_H^{1/2}\|u_{\phi_{n'}} - u_\phi\|_V^{1/2}\|u_{\phi_{n'}}\|_V^{1/2}\|Au_{\phi_{n'}}\|_H^{1/2}\|w\|_H^{1/2} \tag{3.47}$$

$$\left|\langle B(u_\phi, u_{\phi_{n'}} - u_\phi), w\rangle\right| \leq C\|u_\phi\|_H^{1/2}\|u_\phi\|_V^{1/2}\|u_{\phi_{n'}} - u_\phi\|_V^{1/2}\|A(u_\phi - u_{\phi_{n'}})\|_H^{1/2}\|w\|_H^{1/2}$$

Hence the nonlinear terms converge to 0 by 3.39, we conclude that given any $v \in H$

$$\mathbb{1}_{t\leq\tau}\langle B(u_{\phi_{n'-1}}, u_{\phi n'}), v\rangle \to \mathbb{1}_{t\leq\tau}\langle B(u_\phi, u_\phi), v\rangle, \tag{3.48}$$

as $n' \to \infty$, for almost every $(\omega, t) \in \Omega \times [0, T]$. Moreover, by using the uniform bound of Equation 3.39 with $p = 4$, one finds that

$$\begin{aligned}
&\sup_{n'} \mathbb{E}\bigg(\mathbb{1}_{\Omega_{n'}} \int_0^\tau |B(u_{\phi_{n'}}, u_{\phi_{n'}})|^2 ds \bigg) \\
&\leq C \sup_{n'} \mathbb{E}\bigg(\mathbb{1}_{\Omega_{n'}} \int_0^\tau \|u_{\phi_{n'}}\|_H^2 \|u_{\phi_{n'}}\|_V^2 ds \bigg) \\
&\leq C \sup_{n'} \mathbb{E}\bigg(\mathbb{1}_{\Omega_{n'}} \sup_{t \in [0,\tau]} \|u_{\phi_{n'}}\|_H^3 \|Au_{n'}\|_H ds \bigg) \\
&\leq C \sup_{n'} \mathbb{E}\bigg(\mathbb{1}_{\Omega_{n'}} \sup_{t \in [0,\tau]} \|u_{\phi_{n'}}\|_H^4 + \bigg(\int_0^\tau \|Au_{n'}\|_H^2 ds \bigg)^2 \bigg) \\
&< \infty
\end{aligned} \tag{3.49}$$

Using 3.48 and 3.49 and using Lemma 6.0.3 we have

$$\mathbb{1}_{\Omega_{n'}, t \leq \tau} B(u_{\phi_{n'}}, u_{\phi_{n'}}) \rightharpoonup \mathbb{1}_{t \leq \tau} B(u_\phi, u_\phi), \tag{3.50}$$

in $L^2(\Omega; L^2([0, T]; H))$. By Lipschitz condition on g we get

$$\begin{aligned}
&\sum \|g_k(u_{\phi_{n'}}) - g_k(u_\phi)\|_V^2 \\
&\leq \|u_\phi - u_{\phi_{n'}}\|_V^2 \to 0,
\end{aligned} \tag{3.51}$$

by 3.39. We have moreover that

$$\begin{aligned}
&\sup_{n'} \mathbb{E}\Big[\mathbb{1}_{\Omega_{n'}} \int_0^\tau \|g_k(u_{\phi_{n'}})\|_V^2 ds\Big] \\
&\leq C \sup_{n'} \mathbb{E}\Big[\mathbb{1}_{\Omega_{n'}} \int_0^\tau 1 + \|u_{\phi_{n'}}\|_V^2 ds\Big] \\
&< \infty,
\end{aligned} \tag{3.52}$$

which means that

$$\mathbb{1}_{\Omega_{n'}, t\leq\tau} g(u_{\phi_{n'}}) \to \mathbb{1}_{t\leq\tau} g(u_\phi), \tag{3.53}$$

in $L^2(\Omega; L^2([0,T]; l^2(H)))$. Moreover, using $\phi_{n'}$ being bounded as well we deduce that for any fixed $v \in H$

$$\begin{aligned}
\mathbb{1}_{\Omega_{n'}} \int_0^{t\wedge\tau} \langle Au_{\phi_{n'}}, v\rangle ds &\rightharpoonup \int_0^{t\wedge\tau} \langle Au_\phi, v\rangle ds, \\
\mathbb{1}_{\Omega_{n'}} \int_0^{t\wedge\tau} \langle B(u_{\phi_{n'}}), v\rangle ds &\rightharpoonup \int_0^{t\wedge\tau} \langle B(u_\phi), v\rangle ds, \\
\mathbb{1}_{\Omega_{n'}} \int_0^{t\wedge\tau} \langle \phi_{n'}, v\rangle ds &\rightharpoonup \int_0^{t\wedge\tau} \langle \phi, v\rangle ds. \\
\mathbb{1}_{\Omega_{n'}} \sum_k \int_0^{t\wedge\tau} g_k(u_{\phi_{n'}}, v) dW_k &\rightharpoonup \int_0^{t\wedge\tau} \langle g_k(u_\phi), v\rangle dW_k
\end{aligned} \tag{3.54}$$

converges weakly in $L^2(\Omega \times [0,T])$. If $K \subset \Omega \times [0,T]$ is any measurable set, then by 3.54, we have

$$\begin{aligned}
&\mathbb{E}\int_0^T \mathbb{1}_K(\omega,t)\langle u_\phi(t\wedge\tau), v\rangle dt \qquad (3.55)\\
&= \lim_{n'\to\infty} \mathbb{E}\int_0^T \langle \mathbb{1}_{\Omega_{n'}}(\omega), \mathbb{1}_K(\omega,t)v\rangle dt\\
&= \lim_{n'\to\infty} \Bigg(\mathbb{E}\int_0^T \mathbb{1}_K(\omega,t)\mathbb{1}_{\Omega_{n'}}(\omega)\langle u_0, v\rangle dt\\
&\quad - \mathbb{E}\int_0^T \mathbb{1}_K(\omega,t)\mathbb{1}_{\Omega_{n'}}(\omega)\left[\int_0^{t\wedge\tau} \langle \nu A u_{\phi_{n'}} + B(u_{\phi_{n'}} - \phi_{n'}, v\rangle ds\right] dt\\
&\quad + \mathbb{E}\int_0^T \mathbb{1}_K(\omega,t)\mathbb{1}_{\Omega_{n'}}(\omega)\left[\sum_k \int_0^{t\wedge\tau} \langle g_k(u_{\phi_{n'}}), v\rangle dW_s\right] dt\Bigg)\\
&= \mathbb{E}\int_0^T \mathbb{1}_K(\omega,t)\left[\langle u_0, v\rangle - \int_0^{t\wedge\tau} \langle \nu A u_\phi + B(u_\phi) - \phi(s,u_\phi), v\rangle ds\right] dt\\
&\quad + \mathbb{E}\int_0^T \mathbb{1}_K(\omega,t)\left[\sum_k \int_0^{t\wedge\tau} \langle g_k(u_\phi), v\rangle dW_k\right] dt
\end{aligned}$$

Since $v \in H$ and K are arbitrary, we conclude that u_ϕ satisfies the regularity conditions. Hence, we have shown the local existence of the solution u_ϕ. Relaxing the restriction $\|u_0\|_V \leq \tilde{M}$, namely extending to the case $u_0 \in L^2(\Omega, V)$ and the global uniqueness follows the same steps of [GZ] Theorem 4.2. This concludes the proof. □

Next, we borrow the following theorem from [KUZ].

Theorem 5.3.7. *[KUZ] Let u_ϕ, u_0, ϕ, g be as defined in Definition 1.4.2. Then, we have*

$$\mathbb{E}[\sup_{[0,T]} \log(1 + \|u_\phi\|_V^2)] \leq C(\phi, g, u_0, T). \qquad (3.56)$$

We continue with the following lemma.

Lemma 5.3.8. *Given the assumptions on initial data and* $\{\phi_n\}_{n\geq 1}$ *as in Theorem 3.8, we have that for any deterministic time* T

$$\mathbb{P}(\sup_{t\in[0,T]} \|u_\phi - u_{\phi_n}\|_V^2 > \delta) < \epsilon \tag{3.57}$$

for any $n, m \geq N_0$ *for some* N_0*, i.e. solutions with different deterministic force* $\{u_{\phi_n}\}_{n\geq 1}$*, converge in probability to* u_ϕ *as* $n, m \to \infty$*.*

Proof. By assumption, we have $u_0 \in L^2(\Omega, V)$. Hence, by Chebyshev theorem we have,

$$\mathbb{P}(\|u_0\|_V^2 > s) \to 0 \tag{3.58}$$

as $s \to \infty$. Denoting $\Omega_s = \{\|u_0\|_V^2 \leq s\}$, we have $\Omega_s \to \Omega$. Hence, we choose s such that $\mathbb{P}(\Omega_s) > 1 - \frac{\epsilon}{2}$. Moreover, We know by Theorem 5.3.5 and Lemma 6.0.2 that there exists a sequence of stopping times $\{\tilde{\tau}_{n_l}^M\}_{n_l\geq 1}$ with the corresponding subsequence $\{u_{\phi_{n_l}}\}$ converging monotone decreasing to τ^M. We also know by Lemma 6.0.2 that $\tau^M \to \infty$ a.s. as $M \to \infty$, where M is the constant defined as in Lemma 5.3.5, since the solution is global in the sense of Definition 1.4.2.

Hence, denoting $\{\tau_{n_l}^M = \tilde{\tau}_{n_l}^M \wedge T\}_{n_l\geq 1}$, there exists M_0 such that $\mathbb{P}(\tau^{M_0} < T) \leq \frac{\epsilon}{4}$ and by Lebesgue dominated convergence theorem, we have

$$\lim_{n_l\to\infty} \mathbb{E}\left[\mathbb{1}_{\Omega_s} \sup_{t\in[0,\tau^{M_0}]} \|u_\phi - u_{\phi_{n_l}}\|_V^2\right] = 0. \tag{3.59}$$

This implies convergence in probability. Thus,

$$\lim_{n_l\to\infty}\mathbb{P}(\mathbb{1}_{\Omega_s}\sup_{t\in[0,\tau^{M_0}]}\|u_\phi-u_{\phi_{n_l}}\|_V^2>\delta)=0, \tag{3.60}$$

for any $\delta>0$. Hence, we have

$$\begin{aligned}
&\mathbb{P}\big(\mathbb{1}_{\Omega_s}\sup_{t\in[0,T]}\|u_\phi-u_{\phi_{n_l}}\|_V^2\geq\delta\big)=\mathbb{P}\big(\{\sup_{t\in[0,T]}\|u_\phi-u_{\phi_{n_l}}\|_V^2\geq\delta\}\cap\{\tau^{M_0}<T\}\cap\{\omega\in\Omega_s\}\big)\\
&\quad+\mathbb{P}(\{\sup_{t\in[0,T]}\|u_\phi-u_{\phi_{n_l}}\|_V^2\geq\delta\}\cap\{\tau^{M_0}=T\}\cap\{\omega\in\Omega_s\})\\
&\leq\mathbb{P}(\tau^{M_0}<T)+\mathbb{P}(\mathbb{1}_{\Omega_s}\sup_{t\in[0,\tau^{M_0}]}\|u_\phi-u_{\phi_{n_l}}\|_V^2>\delta)
\end{aligned} \tag{3.61}$$

Then, we get

$$\begin{aligned}
\mathbb{P}\left(\sup_{t\in[0,T]}\|u_\phi-u_{\phi_{n_l}}\|_V^2\geq\delta\right)&\leq\mathbb{P}\left(\tau^{M_0}<T\right)+\mathbb{P}\left(\mathbb{1}_{\Omega_s}\sup_{t\in[0,\tau^{M_0}]}\|u_\phi-u_{\phi_{n_l}}\|_V^2>\delta\right)+\mathbb{P}\left(\Omega_s^c\right)\\
&\leq\frac{\epsilon}{4}+\frac{\epsilon}{4}+\frac{\epsilon}{2}=\epsilon.
\end{aligned} \tag{3.62}$$

for n_l large enough. Then by taking any subsequence $u_{\phi_{m_l}}$ and by Theorem 5.3.5 and Lemma 6.0.2 repeating the same arguments above, we get that every subsequence $\{u_{\phi_{m_l}}\}$ has a further subsequence that converges in probability to u_ϕ, which implies that the whole sequence $\{u_{\phi_n}\}$ converges in probability to u_ϕ, which concludes the proof. □

Now we are ready to prove Theorem 5.3.1.

Proof.

$$\begin{aligned}
&\Big|\mathbb{E}\big[\sup_{[0,T]}\varphi(\mathcal{L}(t,u_{\phi_n},\phi_n)) - \sup_{[0,T]}\varphi(\mathcal{L}(t,u_\phi,\phi))\big]\Big| \\
&\quad\le \mathbb{E}\big[\sup_{[0,T]}\big|\varphi(\mathcal{L}(t,u_{\phi_n},\phi_n)) - \varphi(\mathcal{L}(t,u_\phi,\phi))\big|\big] \\
&\quad\le \mathbb{E}\big[\sup_{[0,T]}\big|\varphi(\mathcal{L}(t,u_{\phi_n},\phi_n)) - \varphi(\mathcal{L}(t,u_{\phi_n},\phi))\big|\big] \\
&\qquad + \mathbb{E}\big[\sup_{[0,T]}\big|\varphi(\mathcal{L}(t,u_{\phi_n},\phi)) - \varphi(\mathcal{L}(t,u_\phi,\phi))\big|\big] \\
&\quad\le \mathbb{E}\big[\sup_{[0,T]}\varphi(\big|\mathcal{L}(t,u_{\phi_n},\phi_n)) - \mathcal{L}(t,u_{\phi_n},\phi)\big|\big] \\
&\qquad + \mathbb{E}\big[\sup_{[0,T]}\varphi(\big|\mathcal{L}(t,u_\phi,\phi_n)) - \mathcal{L}(t,u_\phi,\phi)\big|\big] \\
&\quad\le \mathbb{E}\big[\sup_{[0,T]}\varphi(C\|\phi-\phi_n\|_H^2 + C\|u_\phi - u_{\phi_n}\|_V^2)\big] \\
&\quad\le \mathbb{E}\big[\sup_{[0,T]}\varphi(C\|u_\phi - u_{\phi_n}\|_V^2) + \sup_{[0,T]}\varphi(C\|\phi-\phi_n\|_H^2)\big], \qquad (3.63)
\end{aligned}$$

where we appeal to Lemma 5.3.3 in the third inequality and the Lipschitz assumption on $\mathcal{L}(t,u_\phi,\phi)$ in the fourth inequality. We have by boundedness assumption on $\{\phi_n\}$ and assumption (3.26) the followings

$$\sup_{[0,T]}\varphi(C\|\phi-\phi_n\|_H^2) \to 0, \text{ as } n\to\infty$$

$$\mathbb{E}[\sup_{[0,T]}\|\phi-\phi_n\|_H^2] \le M, \qquad (3.64)$$

Hence, we have by uniform integrability and convergence in probability

$$\mathbb{E}[\sup_{[0,T]} \varphi(\|\phi - \phi_n\|_H^2)] \to 0, \text{ as } n \to \infty. \tag{3.65}$$

For the first term in the last line of the Equation 3.63, we have

$$\mathbb{E}\big[\sup_{[0,T]} \varphi(C\|u_\phi - u_{\phi_n}\|_V^2)\big] \leq \mathbb{E}\big[\sup_{[0,T]} \varphi(C\|u_\phi\|_V^2)\big] + \mathbb{E}\big[\sup_{[0,T]} \varphi(C\|u_{\phi_n}\|_V^2)\big] \tag{3.66}$$

By noting that $\log(1+x) \leq x$ for $x > 0$, we have by Lemma 5.3.8 that

$$\mathbb{P}\left(\sup_{[0,T]}(\log(1 + \|u_\phi - u_{\phi_n}\|_V^2)^{1-\epsilon}\right) \to 0 \tag{3.67}$$

for $0 < \epsilon < 1$ in probability as $n \to \infty$. Moreover, using Theorem 5.3.7, we have that

$$\mathbb{E}\left[\left(\sup_{[0,T]}(\log(1 + \|u - u^n\|_V^2)\right)^{(1-\epsilon)\frac{1}{1-\epsilon}}\right] \leq M(u_0, f, g, T). \tag{3.68}$$

We note here that $g(x) = x^{\frac{1}{1-\epsilon}}$ is a convex function with $\lim_{x\to\infty} \frac{x^{\frac{1}{1-\epsilon}}}{x} = \infty$. Using de La-Vallee-Poussin criteria for uniform integrability (see e.g. [D]) we get that

$$\left\{\sup_{[0,T]} \left(\log(1 + \|u_\phi - u_{\phi_n}\|_V^2\right)^{1-\epsilon}\right\}_{n\geq 1} \tag{3.69}$$

is uniformly integrable. Using uniform integrability and by Lemma 3.11 convergence in probability imply L^1-convergence [D]. Thus, using that $x^{1-\epsilon}$ for $0 < \epsilon < 1$ being increasing and continuous, we get that

$$\mathbb{E}\left[\sup_{[0,T]}\left(\log(1+\|u_\phi - u_{\phi_n}\|_V^2\right)^{1-\epsilon}\right] \to 0 \tag{3.70}$$

as $n \to \infty$. Hence, Theorem 5.3.1 is proven. □

Our main result, Theorem 5.2.2 follows from Theorem 5.3.1 above.

Proof. By assumption on $\phi \subset \mathcal{U}^b$ and via Arzela-Ascoli theorem, the set Φ is compact. Then, by generalized Weierstrass theorem (see [Z]), we have that there exists an optimal feedback control $\phi^* \in \mathcal{U}^b$ for $J(\cdot)$. Hence, we conclude the proof. □

Chapter 6

Appendix

In the appendix, we state and prove the critical theorems that are used and not proven during the previous chapters. First, we give two results, first one being the stochastic version of the Gronwall lemma and second one is related to the existence of a process that lives up to a specific stopping time defined below.

Lemma 6.0.1. *[GZ] Fix $T > 0$. Assume that*

$$X, Y, Z, R : [0, T) \times \Omega \to \mathbb{R} \tag{0.1}$$

are real-valued, non-negative stochastic processes. Let $\tau < T$ be a stopping time so that

$$\mathbb{E} \int_0^\tau (RX + Z) ds < \infty. \tag{0.2}$$

Assume, moreover that for some fixed constant κ we have

$$\int_0^\tau R ds < \kappa, \ \ a.s. \tag{0.3}$$

Suppose that for all stopping times $0 \leq \tau_a \leq \tau_b \leq \tau$

$$\mathbb{E}\Big(\sup_{t\in[\tau_a,\tau_b]} X + \int_0^\tau Y ds \Big) \leq C_0 \mathbb{E}\Big(X(\tau_a) + \int_{\tau_a}^{\tau_b} (RX + Z) ds \Big), \tag{0.4}$$

where C_0 is a constant independent of the choice of τ_a, τ_b. Then we have

$$\mathbb{E}\left(\sup_{t\in[0,\tau]} X + \int_0^\tau Y ds\right) \leq C\mathbb{E}\left(X(0) + \int_0^\tau Z ds\right), \tag{0.5}$$

where C depends on C_0, T and κ.

Proof. Choose a finite sequence of stopping times

$$0 = \tau_0 < \tau_1 < ... < \tau_N < \tau_{N+1} = \tau \tag{0.6}$$

so that

$$\int_{\tau_{k-1}}^{\tau_k} R ds < \frac{1}{2C_0} \text{ a.s.} \tag{0.7}$$

For each pair τ_{k-1}, τ_k take $\tau_a = \tau_{k-1}$ and $\tau_b = \tau_k$ in (0.4). Using (0.7), we have

$$\mathbb{E}\left(\sup_{t\in[\tau_{k-1},]} X + \int_{\tau_{k-1}}^{\tau_k} Y ds\right) \leq C\mathbb{E}X(\tau_{k-1}) + C\mathbb{E}\int_{\tau_{k-1}}^{\tau_k} Z ds. \tag{0.8}$$

By induction we have

$$\mathbb{E}\left(\sup_{t\in[0,\tau_j]} X + \int_0^{\tau_j} Y ds\right) \leq C\mathbb{E}X(0) + C\mathbb{E}\int_0^{\tau_j} Z ds \tag{0.9}$$

then we have

$$
\begin{aligned}
\mathbb{E}\Big(\sup_{t\in[0,\tau_{j+1}]} X + \int_0^{\tau_j} Y ds\Big) &\le C\mathbb{E}X(0) + C\mathbb{E}\int_0^{\tau_j} Z ds + C\mathbb{E}\Big(\sup_{t\in[\tau_j,\tau_{j+1}]} X + \int_{\tau_j}^{\tau_{j+1}} Y ds\Big) \\
&\le C\mathbb{E}X(0) + C\mathbb{E}\int_0^{\tau_{j+1}} Z ds + C\mathbb{E}X(\tau_j) \\
&\le C\mathbb{E}X(0) + C\mathbb{E}\int_0^{\tau_{j+1}} Z ds \qquad (0.10)
\end{aligned}
$$

Hence, we conclude the proof. □

Lemma 6.0.2. *[GZ] Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge 0}, \mathbb{P})$ be a fixed filtered probability space. Suppose that B_1 and B_2 are Banach spaces with $B_2 \subset B_1$ with continuous embedding. We denote the associated norms by $|\cdot|_i$. Define*

$$
\mathcal{E}(T) := C([0,T]; B_1) \cap L^2([0,T]; B_2) \qquad (0.11)
$$

with the norm

$$
|Y|_{\mathcal{E}(T)} = \Big(\sup_{t\in[0,T]} |Y(t)|_1^2 + \int_0^T |Y(t)|_2^2 dt\Big)^{1/2}. \qquad (0.12)
$$

Let X_n be a sequence of B_2-valued stochastic process such that for every $T > 0$ we have $X_n \in \mathcal{E}(T)$, a.s. For $M > 1$, $T > 0$ define the collection of stopping times

$$
\mathcal{T}_n^{M,T} := \{\tau \le T : |X_n|_{\mathcal{E}(\tau)} \le M + |X_n(0)|_1\}, \qquad (0.13)
$$

and let $\mathcal{T}_{n,m}^{M,T} : \mathcal{T}_n^{M,T} \cap \mathcal{T}_m^{M,T}$.

1. Suppose that for $M > 1$ and T, we have

$$\lim_{n\to\infty} \sup_{m\geq n} \sup_{\tau in \mathcal{T}_n^{M,T}} \mathbb{E}|X_n - X_m|_{\mathcal{E}(\tau)} = 0 \tag{0.14}$$

and

$$\lim_{S\to 0} \sup_{n} \sup_{\tau\in \mathcal{T}_n^{M,T}} \mathbb{P}[|X_n|_{\mathcal{E}(\tau\wedge S)} > |X_n(0)|_1 + M - 1] = 0. \tag{0.15}$$

Then, there exists a stopping time τ with:

$$\mathbb{P}(0 < \tau \leq T) = 1, \tag{0.16}$$

and a process $X(\cdot) = X(\cdot \wedge \tau \in \mathcal{E}(\tau))$, such that

$$|X_{n_l} - X|_{\mathcal{E}(\tau)} \to 0, \ \ a.s. \tag{0.17}$$

for some subsequence $n_l \uparrow \infty$. Moreover

$$|X|_{\mathcal{E}(\tau)} \leq M + \sup_n |X_n(0)|_1, \ \ a.s. \tag{0.18}$$

2. If, in addition to the conditions imposed above, we also have

$$\sup_n \mathbb{E}|X_n(0)|_1^p < \infty, \tag{0.19}$$

for some $1 \leq p < \infty$, then there exists a sequence of sets $\Omega_l \uparrow \Omega$ such that

$$\sup_l \mathbb{E} I_{\Omega_l} |X_{n_l}|^p_{\mathcal{E}(\tau)} < \infty \tag{0.20}$$

and

$$\mathbb{E}|X|^p_{\mathcal{E}(\tau)} \leq C_q(M^p + \sup_n \mathbb{E}|X_n(0)|^p_1). \tag{0.21}$$

Proof. To find the convergent subsequence, we proceed by induction on l and start with $l = 0$ and $n_0 = 1$. We have by 0.14

$$\sup_{\tau \in \mathcal{T}^{M,T}_{n_{l+1},n_l}} \mathbb{E}|X_{n_l} - X_{n_{l+1}}|_{\mathcal{E}(\tau)} \leq 2^{-2l}. \tag{0.22}$$

Next to find τ in 0.16 and 0.17, we define

$$\tau_l := \inf_{t>0} \{|X_{n_l}|_{\mathcal{E}(t)} > |X_{n_l}(0)|_1 + (M - 1 + 2^{-l})\} \wedge T, \tag{0.23}$$

and let

$$\Omega_N = \bigcap_{j=N}^{\infty} \left\{ |X_{n_j} - X_{n_{j+1}}|_{\mathcal{E}(\tau_j \wedge \tau_{j+1})} < 2^{-(l-2)} \right\} \tag{0.24}$$

Using $\tau_l \wedge \tau_{l+1} \in \mathcal{T}^{M,T}_{n_{l+1},n_l}$, we have

$$\mathbb{P}\big(|X_{n_l} - X_{n_{l+1}}|_{\mathcal{E}(\tau_l \wedge \tau_{l+1})} \geq 2^{-(l+2)}\big) \leq 2^{l+2} \mathbb{E}|X_{n_l} - X_{n_{l+1}}|_{\mathcal{E}(\tau_l \wedge \tau_{l+1})} \leq 2^{-(l-2)}, \tag{0.25}$$

Hence, by Borel-Cantelli lemma, we conclude that

$$\mathbb{P}\Big(\bigcap_{N=1}^{\infty}\bigcup_{j=N}^{\infty}\{|X_{n_j} - X_{n_{j+1}}|\mathcal{E}(\tau_j \wedge \tau_{j+1}) \geq 2^{-(j+2)}\}\Big) = 0 \tag{0.26}$$

and hence $\tilde{\Omega} := \cup_N \Omega_N$ is a set of full measure. Next, we note that

$$\tau_{l+1}(\omega) \leq \tau_l(\omega), \tag{0.27}$$

for every $l \geq N, \omega \in \Omega_N$, since given N and $l \geq N$, take the set $\{\tau_{l+1} > \tau_l\} \cap \Omega_N$. On this set, we have $\tau_l < T$. By continuity of $|X_{n_l}|_{\mathcal{E}(t)}$ in t, that imples

$$|X_{n_l}|_{\mathcal{E}(\tau_l)} = |X_{n_l}(0)|_1 + (M - 1 + 2^{-l}). \tag{0.28}$$

Moreover, we have on Ω_N

$$\begin{aligned} &|X_{n_l}|_{\mathcal{E}(\tau_l \wedge \tau_{l+1})} - |X_{n_{l+1}}|_{\mathcal{E}(\tau_l \wedge \tau_{l+1})} < 2^{-(l+2)} \\ &|X_{n_{l+1}}(0)| - |X_{n_l}(0)| < 2^{-(l+2)}. \end{aligned} \tag{0.29}$$

Hence, we get that

$$
\begin{aligned}
|X_{n_{l+1}}|_{\mathcal{E}(\tau_l \wedge \tau_{l+1})} &> |X_{n_l}|_{\mathcal{E}(\tau_l \wedge \tau_{l+1})} - 2^{-(l+2)} \\
&= |X_{n_l}|_{\mathcal{E}(\tau_l)} - 2^{-(l+2)} \\
&= |X_{n_l}(0)|_1 + (M - 1 + 2^{-l}) - 2^{-(l+2)} \\
&> |X_{n_{l+1}}(0)|_1 + (M - 1 + 2^{-l}) - 2 \cdot 2^{-(l+2)} \\
&= |X_{n_{l+1}(0)}|_1 + (M - 1 + 2^{-l+1}),
\end{aligned} \tag{0.30}
$$

over $\{\tau_{l+1} > \tau_l\} \cap \Omega_N$. Moreover on Ω_N, we have

$$
\begin{aligned}
|X_{n_{l+1}}|_{\mathcal{E}(\tau_l \wedge \tau_{l+1})} &\leq |X_{n_{l+1}}|_{\mathcal{E}(\tau_{l+1})} \\
&\leq |X_{n_{l+1}}(0)| + (M - 1 + 2^{-(l+1)}).
\end{aligned} \tag{0.31}
$$

Hence, we have by 0.30 and 0.31 that $\{\tau_{l+1} > \tau_l \cap \Omega_N\}$ is empty. Hence, by 0.26 and 0.27, we have

$$\tau = \lim_l \tau_l, \text{ a.s.} \tag{0.32}$$

Next, by fixing $\epsilon > 0$ with $T > \epsilon > 0$. We have

$$
\begin{aligned}
\{\tau_l < \epsilon\} &\subset \{|X_{n_l}|_{\mathcal{E}(\tau_l \wedge \epsilon)} = |X_{n_l}(0)|_1 + (M - 1 + 2^{-l})\} \\
&\subset \{|X_{n_l}|_{\mathcal{E}(\tau_l \wedge \epsilon)} > |X_{n_l}(0)|_1 + (M - 1)\}.
\end{aligned} \tag{0.33}
$$

Since

$$\mathbb{P}(\tau < \epsilon) = \mathbb{P}\Big(\bigcap_{l=1}^{\infty}\bigcup_{k=l}^{\infty}\{\tau_k < \epsilon\}\Big) \tag{0.34}$$
$$\leq \limsup_{l} \mathbb{P}(\tau_l < \epsilon)$$
$$\leq \sup_{l} \mathbb{P}(|X_{n_l}|_{\mathcal{E}(\tau_l\wedge\epsilon)} > |X_{n_l}(0)|_1 + (M-1)),$$

by 0.15, we have

$$\mathbb{P}(\tau = 0) = \mathbb{P}(\cap_{\epsilon>0}\{\tau < \epsilon\}) = \lim_{\epsilon\downarrow 0}\mathbb{P}(\tau < \epsilon) = 0. \tag{0.35}$$

Hence, $\tau \leq T$. Next, we show that X_{n_l} is Cauchy in $\mathcal{E}(\tau)$ a.s. By 0.24, for every $\omega \in \tilde{\Omega}$, we can choose $N = N(\omega)$ so that $\omega \in \Omega_N$ and $\tau(w) \leq \tau_{l+1}(w) \leq \tau_l(w)$ whenever $l \geq N$. Hence

$$|X_{n_l}(\omega) - X|_{\mathcal{E}(\tau)} = 0, \text{ a.s.} \tag{0.36}$$

Hence , the first part is proven. Next, we take Ω_l as in 5.15. Hence by

$$\mathbb{1}_{\Omega_l}|X_{n_l}|_{\mathcal{E}(\tau)} \leq 2^{-(l+2)} + \mathbb{1}_{\Omega_l}|X_{n_{l+1}}|_{\mathcal{E}(\tau)} \tag{0.37}$$
$$\leq |X_{n_{l+1}}(0)|_1 + M$$
$$\leq \sup_{n}|X_n(0)|_1 + M,$$

which implies 0.20. Moreover, by 0.37 we have

$$\mathbb{E}\big(\mathbb{1}_{\Omega_l}|X_{n_l}|^p_{\mathcal{E}(\tau)}\big) \leq C_p(M^p + \mathbb{E}|X_{n_l}(0)|^p_1) \tag{0.38}$$

Combining 0.21 and 0.38, we conclude the bound in 0.22. By Fatou's lemma, we conclude the result in 0.21. Hence, the proof is concluded. □

Lemma 6.0.3. *[GZ] Suppose that X is a seperable Banach space and let $D \subset X$ be a dense subset. Let X^* be the dual of X and denote the dual pairing between X and X^* by $\langle \cdot, \cdot \rangle$. Assume that $(E, \mathcal{E}, \mu)$ is a finite measure space and that $p \in (1, \infty)$. Assume that $u, u_n \in L^p(E, X^*)$ with $\{u^n\}$ uniformly bounded in $L^p(E, X^*)$ and*

$$\langle u^n, y \rangle \to \langle u, y \rangle \;\; \mu - a.e. \tag{0.39}$$

for all $y \in D$. Then

$$u^n \rightharpoonup^* u \tag{0.40}$$

in $L^p(E, X^*)$.

Proof. We fix $y \in \mathcal{D}$ and let

$$E_N := \{\omega \in E : |\langle u^m(\omega) - u(\omega), y \rangle| \leq 1, \text{ for every } m \geq N\}. \tag{0.41}$$

Denoting $\mathbb{1}_N$ as the indicator function associated to E_N, by 0.39, we have

$$1 - \mathbb{1}_N \to 0, \tag{0.42}$$

as $N \to \infty$, μ almost surely.

Let $F \in \mathcal{E}$ and $N \geq 1$ be given. By dominated convergence theorem, we have

$$\lim_{n\to\infty} \left| \int_F \mathbb{1}_N \langle u^n - u, y\rangle d\mu \right| = 0. \tag{0.43}$$

Moreover we have

$$\begin{aligned}
&\limsup_{n\to\infty} \left| \int_F \langle u^n - u, y\rangle d\mu \right| && (0.44)\\
&\quad \leq \limsup_{n\to\infty} \left| \int_F (1-\mathbb{1}_N) \langle u^n - u, y\rangle d\mu \right| + \limsup_{n\to\infty} \left| \int_F \mathbb{1}_N \langle u^n - u, y\rangle d\mu \right| \\
&\quad \limsup_{n\to\infty} \left| \int_F (1-\mathbb{1}_N) \langle u^n - u, y\rangle d\mu \right| \\
&\quad \limsup_{n\to\infty} \|y\|_X \Big(\|u^n - u\|_X^p d\mu \Big)^{1/p} \Big(\int_F |1-\mathbb{1}_N|^{p'} d\mu \Big)^{1/p'} \\
&\quad \leq C \Big(\int_F |1-\mathbb{1}_N|^{p'} \Big)^{1/p'},
\end{aligned}$$

by uniform bound on u^n in $L^p(E, X^*)$, we conclude that C can be chosen independently of N. By letting $N \to \infty$ and via dominated convergence theorem with Equation 0.39, we have

$$\lim_{n\to\infty} \left| \int_F \langle u^n - u, y\rangle d\mu \right| = 0. \tag{0.45}$$

By using that

$$\mathcal{S} := \{ s = \sum_{k=1}^{d} y_k : y_k \in \mathcal{D}, F_k \in \mathcal{E}, d < \infty \} \tag{0.46}$$

□

Similarly, we also have the preservation of the weak convergence under continuous linear mappings.

Proposition 6.0.4. *[B3] Let S_1 and S_2 be Banach spaces and let $L : S_1 \to S_2$ be a continuous linear operator. If $\{x_n\}$ is a sequence in S_1 such that $x_n \rightharpoonup x$, where $x \in S_1$, then $L(x_n) \rightharpoonup L(x)$.*

Next, we continue with the critical convergence results in $\mathcal{L}^2_V([0,T] \times \Omega)$ that we extend to the timewise case. We note here that the stopping times $\mathcal{T}^Q_M$ are the first hitting times in the needed sense accordingly.

Proposition 6.0.5. *[B4] Let $\{Q(t)_{t\in[0,T]}\}$ be a V-valued process with*

$$\int_0^T \|Q(s)\|_V^2 < \infty,$$

for a.e. $\omega \in \Omega$. For each $M \in \mathbb{N}$, we define the stopping time

$$\mathcal{T}^Q_M = \begin{cases} T, \text{ if } \int_0^T \|Q(s)\|_V^2 ds < M \\ \inf\{t \in [0,T] : \int_0^t \|Q(s)\|_V^2 ds \geq M\}, \text{ otherwise.} \end{cases} \tag{0.47}$$

Hence, we have

$$\int_0^{t\wedge\mathcal{T}^Q_M} \|Q(s)\|_V^2 ds \leq M.$$

Then, we have that

$$\lim_{M\to\infty} \mathbb{P}(\mathcal{T}_M^Q < T) = 0$$

$$\lim_{M\to\infty} \mathcal{T}_M^Q = T, \;\; a.s.$$

Proof. We have that

$$\lim_{M\to\infty} \mathbb{P}(\mathcal{T}_M^Q < T) \leq \lim_{M\to\infty} \mathbb{P}(\int_0^T \|Q(s)\|_V^2 ds) \geq M) \leq \mathbb{P}\Big(\bigcap_{M=1}^{\infty} \Big\{ \int_0^T \|Q(s)\|_V^2 \geq M \Big\} \Big)$$
$$= 0$$

The sequence $(T - \mathcal{T}_M^Q)$ is monotone decreasing a.s. by above, we have that it converges in probability to 0. Hence, $\mathcal{T}_M^Q$ converges to zero a.s. □

Proposition 6.0.6. *[B4] Suppose we have the following assumptions:*

- $k_1, k_2 > 0$ *are real numbers;*
- u_0 *is a* H*-valued* $\mathcal{F}_0$*-measurable random variable. with* $\mathbb{E}\|u_0\|_H^4 < \infty$*.*
- $F_1 \in L^1_{\mathbb{R}}(\Omega \times [0,T])$*, with* $F_2 \in L^2_H(\Omega \times [0,T])$*.*
- $F_3 : [0,T] \times H \to H$ *is a mapping such that for all* $t \in [0,T]$*,* $x \in H$*, we have* $\|F_3(t,x)\|_H \leq C\|x\|_H$ *and* $F_3(\cdot,x) \in L^2_H[0,T]$ *for all* $x \in H$*.*
- $(Q(t))_{t\in[0,T]}$ *is a* V*-valued process with*

$$\int_0^T \|Q(s)\|_V^2 ds < \infty,$$

which satisfies

$$\|Q(t)\|_H^2 + C\int_0^t \|Q(s)\|_V^2 ds \le \|u_0\|_H^2 + C\int_0^t \|Q(s)\|_H^2 ds \qquad (0.48)$$
$$+\int_0^t |F_1(s)|ds + \int_0^t \langle F_2(s) + F_3(s, Q(s)), Q(s)\rangle dW_s,$$

for all $t \in [0, T]$ *and* $w \in \Omega$ *a.s.*

Then, we have

$$\mathbb{E}\sup_{t\in[0,T]}\|Q(t)\|_H^2 + \mathbb{E}\int_0^T \|Q(s)\|_V^2 ds \le C[\mathbb{E}\|u_0\|_H^2 + \mathbb{E}\int_0^T |F_1(s)|ds + \mathbb{E}\int_0^T \|F_2(s)\|_H^2 ds].$$

Moreover, if $\mathbb{E}\int_0^T |F_1(s)|^2 ds < \infty$, *and* $\mathbb{E}\int_0^T \|F_2(s)\|_H^4 ds < \infty$, *then*

$$\mathbb{E}\sup_{t\in[0,T]}\|Q(t)\|_H^4 + \mathbb{E}\big(\int_0^T \|Q(s)\|_V^2 ds\big) \le C[\mathbb{E}\|u_0\|^4 + \mathbb{E}\int_0^T |F_1(s)|^2 ds + \mathbb{E}\int_0^T \|F_2(s)\|_H^4 ds]$$

Proof. By considering the stopping times $\mathcal{T}_M := \mathcal{T}_M^Q$, and $M \in \mathbb{N}$. We have that for all $t \in [0, T]$

$$\sup_{s\in[0,t\wedge\mathcal{T}_M]}\|Q(s)\|_H^2 + C\int_0^{t\wedge\mathcal{T}_M} \|Q(s)\|_V^2 ds \le 2\|u_0\|_H^2 + 2C\int_0^{t\wedge\mathcal{T}_M} \|Q(s)\|_H^2 ds \qquad (0.49)$$
$$+ 2\int_0^{t\wedge\mathcal{T}_M} |F_1(s)|ds + 2\sup_{s\in[0,t\wedge\tau_M]}\left|\int_0^s \langle F_2(r) + F_3(r, Q(r)), Q(r)\rangle dW_r\right|.$$

and

$$\sup_{s\in[0,t\wedge\mathcal{T}_M]}\|Q(s)\|_H^4 + C^2\bigg(\int_0^{t\wedge\mathcal{T}_M}\|Q(s)\|_V^2\bigg)^2 \leq C\|u_0\|_H^4 + C\bigg(\int_0^{t\wedge\mathcal{T}_M}\|Q(s)\|_H^2\bigg)^2$$

$$16\bigg(\int_0^{t\wedge\mathcal{T}_M}|F_1(s)|ds\bigg)^2 + 16\sup_{s\in[0,t\wedge\mathcal{T}_M]}\bigg|\int_0^s\langle F_2(r)+F_3(r,Q(r)),Q(r)\rangle dW_r\bigg|^2. \tag{0.50}$$

By BDG and the Cauchy-Schwartz inequality, we get that

$$\mathbb{E}\sup_{s\in[0,t\wedge\mathcal{T}_M]}\|Q(s)\|_H^2 + C\int_0^{t\wedge\mathcal{T}_M}\|Q(s)\|_V^2 \leq 2\mathbb{E}\|u_0\|_H^2 + 2C\mathbb{E}_0^{t\in[0,t\wedge\mathcal{T}_M]}\|Q(s)\|_H^2ds$$

$$+\,2\mathbb{E}\int_0^{t\wedge\mathcal{T}_M]}|F_1(s)|ds + \frac{1}{2}\mathbb{E}\sup_{s\in[0,t\wedge\mathcal{T}_M]}\|Q(s)\|_H^2 + C\mathbb{E}\int_0^{t\wedge\mathcal{T}_M}\|F_2(s)+F_3(s,Q(s))\|_H^2ds, \tag{0.51}$$

and

$$\mathbb{E}\sup_{s\in[0,t\wedge\mathcal{T}_M]}\|Q(s)\|_H^4 + C\mathbb{E}\bigg(\int_0^{t\wedge\mathcal{T}_M}\|Q(s)\|_V^2ds\bigg)^2 \leq 16\mathbb{E}\|u_0\|_H^4 + C\mathbb{E}\int_0^{t\wedge\mathcal{T}_M}\|Q(s)\|_H^4ds$$

$$C\mathbb{E}\int_0^{t\wedge\mathcal{T}_M}|F_1(s)|^2ds + \frac{1}{2}\mathbb{E}\sup_{s\in[0,t\wedge\mathcal{T}_M]}\|Q(s)\|_H^4 + C\mathbb{E}\int_0^{t\wedge\mathcal{T}_M}\|F_2(s)+F_3(s,Q(s))\|_H^4ds, \tag{0.52}$$

for all $t\in[0,T]$ for some constant C. Hence, for every $t\in[0,T]$, we have

$$\mathbb{E}\sup_{s\in[0,t]}\mathbb{1}_{[0,\mathcal{T}_M]}(s)\|\|Q(s)\|_V^2\| \leq C\|u_0\|_H^2 \tag{0.53}$$

$$+\,C\mathbb{E}\int_0^t\sup_{r\in[0,s]}\mathbb{1}_{[0,\mathcal{T}_M]}(r)\|Q(r)\|_H^2dr + C\int_0^T|F_1(s)|ds + C\mathbb{E}\int_0^T\|F_2(s)\|_H^2ds$$

and

$$\mathbb{E}\sup_{s\in[0,t]}\mathbb{1}_{[0,\mathcal{T}_M]}\|Q(s)\|_H^4+C\mathbb{E}\bigg(\int_0^t\mathbb{1}_{[0,\mathcal{T}_M]}(s)\|Q(s)\|_V^2ds\bigg)^2\leq C\mathbb{E}\|u_0\|_H^4 \tag{0.54}$$
$$+C\mathbb{E}\int_0^t\sup_{r\in[0,s]}\mathbb{1}_{[0,\mathcal{T}_M]}(r)\|Q(r)\|_H^4dr$$
$$+C\mathbb{E}\int_0^T|F_1(s)|^2ds+C\mathbb{E}\int_0^T\|F_2(s)\|_H^4ds$$

By Gronwall lemma we conclude that

$$\mathbb{E}\sup_{s\in[0,T\wedge\mathcal{T}_M]}\|Q(s)\|_H^2+C\mathbb{E}\int_0^{T\wedge\mathcal{T}_M}\|u(s)\|_V^2ds \tag{0.55}$$
$$\leq C[\mathbb{E}\|u_0\|_H^2+\mathbb{E}\int_0^T|F_1(s)|ds+\mathbb{E}\int_0^T\|F_2(s)\|_H^2ds]$$

and

$$\mathbb{E}\sup_{s\in[0,T\wedge\mathcal{T}_M]}\|Q(s)\|_H^4+C\bigg(\mathbb{E}\int_0^{T\wedge\mathcal{T}_M}\|u(s)\|_V^2ds\bigg)^2 \tag{0.56}$$
$$\leq C[\mathbb{E}\|u_0\|_H^4+\mathbb{E}\int_0^T|F_1(s)|^2ds+\mathbb{E}\int_0^T\|F_2(s)\|_H^4ds]$$

□

Theorem 6.0.7. *[B3] Let u and u_n be the corresponding solution of the SNSE and Galerkin approximations in the Definition 1.4.1 and 1.4.3 respectively. Then, for each fixed time T, the following convergence holds:*

$$\mathbb{E}\int_0^T\|u-u_n\|_V^2dt\to 0, \tag{0.57}$$

as $n \to \infty$.

To prove the result, first we give the following lemma.

Lemma 6.0.8. *[B3] There exist a positive constant C such that for all $n \in \mathbb{N}$*

$$\mathbb{E}\|u_n(T)\|_H^2 + 2\nu\mathbb{E}\int_0^T \|u_n(t)\|_V^2 dt \leq C\bigg(\mathbb{E}\|x_0\|_H^2 + \mathbb{E}\int_0^T \|f(t)\|_H^2 dt\bigg)$$

and each of the expressions below

$$\sup_{t \int [0,T]} \mathbb{E}\|u_n(t)\|_H^4, \mathbb{E}\int_0^T \|u_n(t)\|_V^2 \|u_n(t)\|_H^2 dt, \mathbb{E}\bigg(\int_0^T \|u_n(t)\|_V^2 dt\bigg)^2,$$

is less than or equal to $C[\mathbb{E}\|u_0\|_H^4 + \mathbb{E}\int_0^T \|f(t)\|_H^4 dt]$

Proof. Let n be an arbitrary fixed natural number. We rewrite equation Galerkin as

$$\begin{aligned}\langle u_n(t), h_i\rangle + \int_0^t \langle Au_n(s), h_i\rangle ds &= \langle u_0, h_i\rangle + \int_0^t \langle B(u_n(s), u_n(s)), h_i\rangle ds \\ &+ \int_0^t \langle f(s), h_i\rangle ds + \int_0^t \langle g(u_n), h_i\rangle dW_s,\end{aligned}$$

for $i = 1, ..., n$ and $t \int [0, T]$ and a.e. $\omega \in \Omega$.

Let $z(t) = \exp\{-(6\lambda+3)t\}$, so from above and applying Ito lemma, we get

$$z(t)\|u_n(t)\|_H^2 + 2\int_0^t z(s)\langle Au_n(s), u_n(s)\rangle ds = \|u_0\|_H^2 + 2\int_0^t z(s)\langle f(s), u_n(s)\rangle ds \tag{0.58}$$

$$+ \int_0^t z(s)\|g(u_n)\|_H^2 ds - (6\lambda+3)\int_0^t z(s)\|u_n(s)\|_H^2 ds$$
$$+ 2\int_0^t z(s)\langle g(u_n), u_n\rangle dW_s,$$

and by bootstrapping Ito we have

$$z(t)\|u_n(t)\|_H^4 + 4\int_0^t z(s)\langle Au_n(s), u_n(s)\rangle \|u_n(s)\|_H^2 ds = 4\int_0^t z(s)\langle g(u_n), u_n\rangle^2 ds \tag{0.59}$$

$$+ 2\int_0^t z(s)\|g(u_n)\|_H^2\|u_n\|^2 ds - (6\lambda+3)\int_0^t z(s)\|u_n(s)\|_H^4 ds$$
$$+ 4\int_0^t z(s)\langle f(s), u_n(s)\rangle\|u_n(s)\|_H^2 ds + 4\int_0^t z(s)\langle g(u_n), u_n\rangle\|u_n\|^2 dW_s + \|u_0\|^4.$$

Hence, we get

$$\begin{aligned} z(t)\|u_n(t)\|_H^2 &+ 2\nu\int_0^t z(s)\|u_n(s)\|_V^2 ds \\ &\leq \|u_0\|_H^2 + \int_0^t z(s)\|f(s)\|_H^2 ds + 2\int_0^t z(s)\langle g(u_n), u_n\rangle dW_s, \end{aligned} \tag{0.60}$$

and

$$z(t)\|u_0\|_H^4 + 4\nu\int_0^t z(s)\|u_n\|_V^2\|u_n\|_H^2 ds \tag{0.61}$$
$$\leq \|u_0\|_H^4 + \int_0^t z(s)\|f(s)\|_H^4 ds + 4\int_0^t z(s)\langle g(u_n), u_n\rangle \|u_n(s)\|_H^2 dW_s.$$

By squaring both sides of the inequality in 0.60, we obtain

$$z^2(t)\|u_n(t)\|_H^4 + 4\nu^2\bigg(\int_0^t z(s)\|u_n(s)\|_V^2 ds\bigg)^2 \tag{0.62}$$
$$\leq 3\|u_0\|^4 + 3\bigg(\int_0^t z(s)\|f(s)\|_H^2 ds\bigg)^2 + 12\bigg|\int_0^t z(s)\langle g(u_n), u_n\rangle dW_s\bigg|^2$$

Using 0.60, 0.61 and 0.62, we get

$$\mathbb{E}z(t)\|u_n(t)\|_H^2 + 2\nu\mathbb{E}\int_0^t z(s)\|u_n(s)\|_V^2 ds \leq \mathbb{E}\|u_0\|_H^2 + \mathbb{E}\int_0^t z(s)\|f(s)\|_H^2 ds \tag{0.63}$$
$$\mathbb{E}z(t)\|u_n(t)\|_H^4 + 4\nu\mathbb{E}\int_0^t z(s)\|u_n(s)\|_V^2\|u_n\|_H^2 ds$$
$$\leq \mathbb{E}\|u_0\|_H^2 + \mathbb{E}\int_0^t z(s)\|f(s)\|_H^4 ds,$$

and

$$\mathbb{E}z^2(t)\|u_n(t)\|_H^4 + 4\nu^2\mathbb{E}\bigg(\int_0^t z(s)\|u_n\|_V^2 ds\bigg)^2 \tag{0.64}$$
$$\leq 3\mathbb{E}\|u_0\|_H^4 + 3\mathbb{E}\bigg(\int_0^t z(s)\|f(s)\|_H^2 ds\bigg)^2 + 12\bigg|\int_0^t z(s)\langle g(u_n), u_n\rangle dW_s\bigg|^2$$

Hence, we have

$$\mathbb{E}\|u_n(T)\|_H^2 + 2\nu\mathbb{E}\int_0^T \|u_n(s)\|_V^2 ds \le C\big(\mathbb{E}\|u_0\|_H^2 + \mathbb{E}\int_0^T \|f(s)\|_H^2 ds\big), \tag{0.65}$$

$$\sup_{t\in[0,T]} \mathbb{E}\|u_n(t)\|_H^4 + 4\nu\mathbb{E}\int_0^T \|u_n\|_V^2\|u_n\|_H^2 ds$$

$$\le C[\mathbb{E}\|u_0\|_H^4 + \mathbb{E}\int_0^T \|f(s)\|_H^4 ds].$$

By 0.64, we get

$$\mathbb{E}z^2(t)\|u_n(t)\|_H^4 + 4\nu^2\mathbb{E}\bigg(\int_0^t z(s)\|u_n(s)\|_V^2 ds\bigg)^2 \tag{0.66}$$

$$\le 3\mathbb{E}\|u_0\|_H^4 + 3t\mathbb{E}\int_0^t z^2(s)\|f(s)\|_H^4 ds + 12\mathbb{E}\int_0^t z^2(s)\langle g(u_n(s)), u_n(s)\rangle^2 ds$$

$$3\mathbb{E}\|u_0\|_H^4 + 3t\mathbb{E}\int_0^t \|f(s)\|_H^4 ds + 12\lambda\mathbb{E}\int_0^t \|u_n(s)\|_H^4 ds.$$

From 0.65 and 0.66, we conclude that

$$\mathbb{E}\|u_n(T)\|_H^2 + 2\nu\mathbb{E}\int_0^T \|u_n(s)\|_V^2 ds \le C[\mathbb{E}\|u_0\|_H^2 + \mathbb{E}\int_0^T \|f(s)\|_H^2 ds], \tag{0.67}$$

$$\sup_{t\in[0,T]} \mathbb{E}\|u_n(t)\|_H^4 + 4\nu\mathbb{E}\int_0^T \|u_n\|_V^2\|u_n\|_H^2 ds$$

$$\le C[\mathbb{E}\|u_0\|_H^4 + \mathbb{E}\int_0^T \|\|_H^4 ds],$$

and

$$\mathbb{E}\bigg(\int_0^T \|u_n(s)\|_V^2 ds\bigg) \le C[\mathbb{E}\|u_0\|_H^4 + \mathbb{E}\int_0^T \|f(s)\|_H^4 ds]$$

□

Lemma 6.0.9. *[B4] We have the followings.*

- *There exists* $u \in L^2_V(\Omega \times [0,T]), \mathcal{B}^* \in \mathcal{L}^2_{V^*}(\Omega \times [0,T]), g^* \in \mathcal{L}^2_H(\Omega \times [0,T])$ *and a subsequence* $\{n'\}$ *of* $\{n\}$ *such that for* $n' \to \infty$*, we have*

$$u_{n'} \rightharpoonup u \text{ in } \mathcal{L}^2_V(\Omega \times [0,T])$$

$$B(u_{n'}, u_{n'}) \rightharpoonup B^* \text{ in } \mathcal{L}^2_{V^*}(\Omega \times [0,T]),$$

$$g(u_{n'}) \rightharpoonup g^* \text{ in } \mathcal{L}^2_H(\Omega \times [0,T]).$$

- *For all* $v \in V$*,* $t \in [0,T]$ *and* $\omega \in \Omega$ *a.s., the process* $(u(t))_{t\in[0,T]}$ *satisfies the equation:*

$$\langle u, v\rangle + \int_0^T \langle Au, v\rangle ds = \langle u_0, v\rangle + \int_0^t \langle B^*(s), v\rangle ds$$
$$+ \int_0^t \langle f, v\rangle ds + \int_0^t \langle g^*(s), v\rangle dW_s,$$

for all $v \in V$*,* $t \in [0,T]$ *and for a.e.* $\omega \in \Omega$*. The process* $(u(t))_{t\in[0,T]}$ *has in* H *almost surely continuous trajectories. Moreover, the process* $(u(t))_{t\in[0,T]}$ *is a strong solution of the SNSE equation as in Definition* 1.4.1*, and it has almost surely continuous trajectories in* H*. Furthermore, the process* $(u(t))_{t\in[0,T]}$ *is with probability one a unique solution as in Definition* 1.4.1.

Lemma 6.0.10. *[B3] Define the stopping times as follows:*

$$\tau_M = \begin{cases} T, & \text{if } \int_0^T \|u(s)\|_V^2 ds < M \\ \inf\{t \in [0,T] : \int_0^t \|u(s)\|_V^2 ds \geq M\}, & \text{otherwise,} \end{cases} \tag{0.68}$$

then the following convergences hold

$$\lim_{M\to\infty} \mathbb{P}(\tau_M < T) = 0,$$

and for a.e. $\omega \in \Omega$

$$\lim_{M\to\infty} \tau_M = T$$

Proof. From previous lemma, we have

$$\int_0^T \|u(s)\|_H^2 ds < \infty,$$

for a.e. $\omega \in \Omega$. Hence we have

$$\lim_{M\to\infty} \mathbb{P}(\tau_M < T) = \lim_{M\to\infty} \big(\mathbb{P} \int_0^T \|u(s)\|_V^2 ds \geq M\big)$$
$$\leq \mathbb{P}\Bigg(\bigcap_{M=1}^{\infty} \Bigg(\int_0^T \|u(s)\|_V^2 \geq M\Bigg)\Bigg) = 0$$

□

Next we state the following two lemmas from [B3]:

Lemma 6.0.11. *[B3] For each fixed natural number M, there exists a subsequence $\{n_k\}_{k\geq 1}$ such that*

$$\mathbb{E}\int_0^{\tau_M}\|u-u_n\|_V^2 ds \to 0, \tag{0.69}$$

for $n_k \to \infty$ as $n_k \to \infty$.

Lemma 6.0.12. *[B3] There exists a positive constant C such that*

$$\mathbb{E}\left(\int_0^T \|u(s)\|_V^2 ds\right) \leq C[\mathbb{E}\|u_0\|_H^4 + \mathbb{E}\int_0^T \|f(s)\|_H^4]$$

Proof. It is proven that for any constant M

$$\mathbb{E}\int_0^{\tau_M}\|u(s)-u_{n'}(s)\|_V^2 ds \to 0, \tag{0.70}$$

for $n' \to \infty$. By Lemma 3.2, there exists some M_0 such that

$$\mathbb{P}(\tau_{M_0} < T) \leq \frac{\epsilon}{2}, \tag{0.71}$$

but by lemma above convergence, we have

$$\mathbb{E}\int_0^{\tau_{M_0}}\|u(s)-u_{n'}(s)\|_V^2 ds \to 0, \tag{0.72}$$

for $n' \to \infty$. Hence, for given δ and ϵ, we have

$$\begin{aligned}
\mathbb{P}\bigg(\int_0^T \|u - u_{n'}\|_V^2 \geq \delta\bigg) &\leq \mathbb{P}\bigg(\tau_{N_0} < T\bigg) \\
&+ P\bigg(\{\tau_{N_0} = T\} \wedge \bigg\{\int_0^T \|u - u_{n'}\|_V^2 ds \geq \delta\}\bigg) \\
&\leq \frac{\epsilon}{2} + \mathbb{P}\bigg(\int_0^{\tau_{N_0}} \|u - u_{n'}\|_V^2 ds \geq \delta\bigg) \\
&\leq \epsilon, \qquad (0.73)
\end{aligned}$$

but we also have that

$$\mathbb{E}\bigg(\int_0^T \|u_n(t)\|_V^2 dt\bigg)^2 \leq C[\mathbb{E}\|u_0\|_H + \mathbb{E}\int_0^T \|f(t)\|_H^4 dt],$$

which together implies that

$$\mathbb{E}\int_0^T \|u - u_{n'}\|_V^2 ds \to 0, \qquad (0.74)$$

for that subsequence $n' \to \infty$. Since, every subsequence of (u_n) has a further subsequence, which converges in $\mathcal{L}_V^2([0,T] \times \Omega)$ sense to the same limit u, this implies that the whole sequence u_n converges to the same limit u $\mathcal{L}_V^2([0,T] \times \Omega)$ sense, i.e.

$$\mathbb{E}\int_0^T \|u - u_n\|_V^2 ds \to 0, \qquad (0.75)$$

for $n \to \infty$. □

Next, we proceed to prove Theorem 4.3.2.

Lemma 6.0.13. *[B3] There exists a positive constant C depending on ν and T such that each of the following expressions*

$$\sup_{t\in[0,T]} \mathbb{E}\|u_n(t)\|_H^4, \quad \mathbb{E}\bigg(\int_0^T \|u_n(s)\|_V^2 ds\bigg)^2 \tag{0.76}$$

for $n \in \mathbb{N}_0$ is less than or equal to $C(\mathbb{E}\|u_0\|_H^4 + \mathbb{E}\int_0^T \|f(s)\|_H^4 ds)$

Proof. Using $z(t) = e^{-(9\lambda+4)t}$ and applying Ito, we have

$$\begin{aligned} z(t)\|u_n(t)\|_H^2 + 2\int_0^t z(s)\langle Au_n, u_n\rangle ds &= \|u_0\|_H^2 + 2\int_0^t z(s)\langle f(s), u_n(s)\rangle ds \\ + \int_0^t z(s)\|g(u_{n-1})\|_H^2 ds &- (9\lambda+4)\int_0^t z(s)\|u_n(s)\|_H^2 ds \\ + 2\int_0^t z(s)\langle g(u_{n-1}(s), u_n(s))\rangle dW_s & \end{aligned} \tag{0.77}$$

and

$$\begin{aligned} z(t)\|u_n(t)\|_H^4 + 4\int_0^t z(s)\langle Au_n, u_n\rangle \|u_n\|_H^2 ds = \\ = \|u_0\|_H^4 + 2\int_0^t z(s)\|g(u_{n-1})\|_H \|u_n\|_H^2 ds \\ - (9\lambda+4)\int_0^t z(s)\|u_n(s)\|_H^4 \\ + 4\int_0^t z(s)\langle f(s), u_n(s)\rangle \|u_n(s)\|_H^2 ds \\ + 4\int_0^t z(s)\langle g(u_{n-1}), u_n\rangle^2 ds \\ + 4\int_0^t z(s)\langle g(u_{n-1}, u_n)\rangle \|u_n(s)\|_H^2 dW_s. \end{aligned} \tag{0.78}$$

Hence, we have

$$
\begin{aligned}
&z(t)\|u_n\|_H^2 + 2\nu\int_0^t z(s)\|u_n(s)\|_V^2 + 6\lambda\int_0^t z(s)\|u_n\|_H^2 ds \\
&\quad \le \|u_0\|_H^2 + \int_0^t z(s)\|f(s)\|_H^2 \\
&\quad + \lambda\int_0^t z(s)\|u_{n-1}\|_H^2 + 2\int_0^t z(s)\langle g(u_{n-1}), u_n(s)\rangle dW_s
\end{aligned} \tag{0.79}
$$

similarly, we have

$$
\begin{aligned}
&z(t)\|u_n(t)\|_H^4 + 4\nu\int_0^t z(s)\|u_n(s)\|_V^2\|u_n(s)\|_H^2 ds + 6\lambda\int_0^t z(s)\|u_n(s)\|_H^4 ds \\
&\le \|u_0\|_H^4 + \int_0^t z(s)\|f(s)\|_H^4 \\
&+ 3\lambda\int_0^t z(s)\|u_{n-1}(s)\|_H^4 ds + 4\int_0^t z(s)\langle g(u_{n-1}), u_n(s)\rangle\|u_n(s)\|_H^2 dW_s.
\end{aligned} \tag{0.80}
$$

By squaring both sides of the inequality in 0.77 we have

$$
\begin{aligned}
&z^2(t)\|u_n(t)\|_H^4 + 4\nu^2\Big(\int_0^t z(s)\|u_n(s)\|_V^2 ds\Big)^2 + 36\lambda^2\Big(\int_0^t z(s)\|u_n(s)\|_H^2 ds\Big)^2 \\
&4\|u_0\|_H^4 + 4\Big(\int_0^t z(s)\|f(s)\|_H^2\Big)^2 \\
&+ 4\lambda^2\Big(z(s)\|u_{n-1}(s)\|_H^2 ds\Big)^2 + 16\Big|\int_0^t z(s)\langle g(u_{n-1}), u_n(s)\rangle dW_s\Big|^2
\end{aligned} \tag{0.81}
$$

Moreover, we have by 0.78, we have

$$\mathbb{E}z(t)\|u_n(t)\|_H^4 + 6\lambda\mathbb{E}\int_0^t z(s)\|u_n(s)\|_H^4 ds$$
$$\leq \mathbb{E}\|u_0\|_H^4 + \mathbb{E}\int_0^t z(s)\|f(s)\|_H^4 ds + 3\lambda\mathbb{E}\int_0^t z(s)\|u_{n-1}(s)\|_H^4 ds \qquad (0.82)$$

By successive application of 0.82, we obtain

$$\mathbb{E}z(t)\|u_n(t)\|_H^4 + 6\lambda\mathbb{E}\int_0^t z(s)\|u_n(s)\|_H^4 ds$$
$$\leq \left(1 + \frac{1}{2} + \dots + \frac{1}{2^{n-1}}\right)\left[\mathbb{E}\|u_0\|_H^4 + \mathbb{E}\int_0^t z(s)\|f(s)\|_H^4 ds\right]. \qquad (0.83)$$

Hence, we have

$$\sup_{t\in[0,T]} \mathbb{E}\|u_n(t)\|_H^4 + 6\lambda\mathbb{E}\int_0^T \|u_n(s)\|_H^4 ds$$
$$\leq C\left[\mathbb{E}\|u_0\|_H^4 + \mathbb{E}\int_0^T \|f(s)\|_H ds\right]. \qquad (0.84)$$

Using Ito isometry in 0.81, we have

$$36\lambda^2\mathbb{E}\left(\int_0^t z(s)\|u_n(s)\|_V^2 ds\right)^2 \leq 4\mathbb{E}\|u_0\|_H^4 + 4t\mathbb{E}\int_0^t z^2(s)\|f(s)\|_H^4 ds$$
$$+ C\mathbb{E}\int_0^t z^2(s)\|u_{n-1}(s)\|_H^4 ds + 8\mathbb{E}\int_0^t z^2(s)\|u_n(s)\|_H^4 ds. \qquad (0.85)$$

Using 0.84, we conclude that

$$\mathbb{E}\left(\int_0^T \|u_n(s)\|_V^2 ds\right)^2 \leq C\left[\mathbb{E}\|u_0\|_H^4 + \mathbb{E}\int_0^T \|f(s)\|_H^4 ds\right] \qquad (0.86)$$

□

Lemma 6.0.14. *[B3] Let* $y(t) = \exp\Big\{ -\lambda t - 2\beta \int_0^t \|u(s)\|_V^2 ds \Big\}$ *for all* $t \in [0,T]$ *and* $\omega \in \Omega$*, for some constant* β*. Then we have*

$$\mathbb{E}\int_0^T y(s)\|u(s) - u_n(s)\|_V^2 ds \to 0, \tag{0.87}$$

and

$$\mathbb{E}y(t)\|u_n(t) - u(t)\|_H^2 \to 0, \tag{0.88}$$

as $n \to \infty$ *and for all* $t \in [0,T]$*, where*

$$y(t) = \exp\Big\{ -\lambda t - 2\beta \int_0^t \|u(s)\|_V^2 ds \Big\}. \tag{0.89}$$

Proof. Denoting

$$s_N(t) = \sum_{n=1}^N y(t)\|u - u_n\|_H^2, \tag{0.90}$$
$$S_N(t) = \sum_{n=1}^N y(t)\|u - u_n\|_V^2,$$

with N being a natural number, $t \in [0,T]$, $\omega \in \Omega$. Applying Ito, we have

$$
\begin{aligned}
& y(t)\|u(t)-u_n(t)\|_H^2 + 2\int_0^t y(s)\langle Au-u_n, u-u_n\rangle ds \\
& = 2\int_0^t y(s)\langle B(u_{n-1},u_n)-B(u(s),u(s)), u_n(s)-u(s)\rangle ds - 2\beta\int_0^t y(s)\|u(s)\|_V^2\|u-u_n\|_H^2 ds \\
& - \lambda\int_0^t y(s)\|u(s)-u_n(s)\|_H^2 + \int_0^t y(s)\|g(u_{n-1})-g(u)\|_H^2 ds \\
& + 2\int_0^t y(s)\langle g(u_{n-1})-g(u), u-u_n\rangle dW_s, \qquad (0.91)
\end{aligned}
$$

for all $t \in [0,T]$ and ω a.s. Next, we estimate the nonlinear term as follows

$$
\begin{aligned}
& 2\langle B(u_{n-1},u_n)-B(u,u), u_n-u\rangle = -2\langle B(u_{n-1}-u, u_n-u), u\rangle \\
& \leq 2\sqrt{\beta}\|u\|_V\|u_{n-1}-u\|_V^{1/2}\|u_{n-1}-u\|_H^{1/2}\|u_n-u\|_V^{1/2}\|u_n-u\|_V^2 \\
& \leq \frac{\nu}{2}\|u_{n-1}-u\|_V^2 + \frac{\nu}{2}\|u_n-u\|_V^2 \\
& + \beta\|u\|_V^2\|u_{n-1}-u\|_H^2 + \beta\|u\|_V^2\|u-u_n\|_H^2, \qquad (0.92)
\end{aligned}
$$

for all $s \in [0,T]$ and a.s. $\omega \in \Omega$. Hence, we have

$$
\begin{aligned}
& y(t)\|u_n-u\|_H^2 + \frac{3\nu}{2}\int_0^t y(s)\|u_n-u\|_V^2 ds + \lambda\int_0^t y(s)\|u-u_n\|_H^2 ds \\
& \leq \frac{\nu}{2}\int_0^t y(s)\|u-u_{n-1}\|_V^2 ds + 2\int_0^t y(s)\langle g(u_{n-1}g(u)), u_n-u\rangle dW_s \\
& + \beta\int_0^t y(s)\|u\|_V^2(\|u_{n-1}-u\|_H^2 - \|u_n-u\|_H^2) ds \\
& \lambda\int_0^t y(s)\|u_{n-1}-u\|_H^2 ds \qquad (0.93)
\end{aligned}
$$

for all $t \in [0,T]$ and ω a.s. Hence, by summing up these estimates up to arbitrary N, we get

$$
\begin{aligned}
&s_N(t) + \nu \int_0^t S_N(s)ds + \beta \int_0^t y(s)\|u(s)\|_V^2 \|u_N - u\|_H^2 ds \\
&+ \lambda \int_0^t y(s)\|u_n - u\|_H^2 ds \\
&\leq \frac{\nu}{2} \int_0^t y(s)\|u_0 - u\|_V^2 ds + \lambda \int_0^t y(s)\|u_0 - u\|_H^2 ds \\
&+ 2\sum_{n=1}^{N} \int_0^t y(s)\langle g(u_{n-1})g(u), u_n - u\rangle dW_s + \beta \int_0^t y(s)\|u(s)\|_V^2 \|u_0 - u\|_H^2 ds,
\end{aligned} \tag{0.94}
$$

for all $t \in [0,T]$ and ω a.s. By taking expectation, we have

$$
\begin{aligned}
&\mathbb{E}s_N(t) + \nu\mathbb{E}\int_0^t S_N(s)ds \leq \frac{\nu}{2}\mathbb{E}\int_0^t y(s)\|u_0 - u\|_V^2 ds \\
&\lambda\mathbb{E}\int_0^t y(s)\|u_0 - u\|_H^2 ds + \beta\mathbb{E}\int_0^t y(s)\|u(s)\|_V^2 \|u_0 - u\|_H^2 ds,
\end{aligned} \tag{0.95}
$$

for all $t \in [0,T]$ and ω a.s. Hence, we have by Lemma 6.0.12 that there exists a constant C independent of N such that

$$
\begin{aligned}
&\mathbb{E}s_N(t) + \nu\mathbb{E}\int_0^t S_N(s)ds \\
&\leq \mathbb{E}\int_0^T \left(\frac{\nu}{2}\|u(s)\|_V^2 + \lambda\|u(s)\|_H^2 + \beta\|u(s)\|_V^2\|u(s)\|_H^2\right) ds \\
&\leq C.
\end{aligned} \tag{0.96}
$$

But then by letting $N \to \infty$, we conclude that

$$\mathbb{E}\int_0^T y(s)\|u_n(s)-u(s)\|_V^2 ds \to 0, \tag{0.97}$$

and

$$\mathbb{E}y(t)\|u_n(t)-u(t)\|_H^2 \to 0, \tag{0.98}$$

for $n \to \infty$ and for all $t \in [0,T]$. $\square$

We have prepared now to prove the main result in [B2]

Theorem 6.0.15. *[B2] The following convergences hold:*

$$\mathbb{E}\int_0^T \|u_n(s)-u(s)\|_V^2 ds \to 0, \tag{0.99}$$

and for all $t \in [0,T]$ *we have*

$$\mathbb{E}\|u_n(t)-u(t)\|_H^2 \to 0, \tag{0.100}$$

as $n \to \infty$.

Proof. The proof is analogous to Proposition 6.0.6. First, for each $M \in \mathbb{N}_0$, we define the stopping time as

$$\mathcal{T}_M = \begin{cases} T, \text{if } \int_0^T \|u(s)\|_V^2 ds < M \\ \inf\{t \in [0,T] : \int_0^t \|u(s)\|_V^2 ds \geq M\}, \text{ otherwise.} \end{cases} \tag{0.101}$$

Hence, we have immediately that

$$\int_0^{t\wedge\mathcal{T}_M} \|u(s)\|_V^2 ds \le M, \tag{0.102}$$

for each $t \in [0,T]$. But since $u \in \mathcal{L}_V^2(\Omega \times [0,T])$, we have that

$$\lim_{M\to\infty} P(\mathcal{T}_M < T) = 0, \tag{0.103}$$

this means there exists an M_0 such that $\mathbb{P}(\mathcal{T}_{M_0} < T) \le \frac{\epsilon}{2}$. Then, by Lemma 6.0.16, we have that there exists some n_0 such that for all $n \ge n_0$ the inequalities

$$\begin{aligned} &\frac{e^{\lambda T + 2\beta M_0}}{\delta}\mathbb{E}\int_0^T y(s)\|u_n - u\|_V^2 ds \le \frac{\epsilon}{2} \\ &\frac{e^{\lambda T + 2\beta M_0}}{\delta}\mathbb{E} y(t)\|u - u_n\|_H^2 \le \frac{\epsilon}{2}. \end{aligned} \tag{0.104}$$

Hence, for all $n \ge n_0$, we can write

$$\begin{aligned} &\mathbb{P}\left(\int_0^T \|u(s) - u_n(s)\|_V^2 ds \ge \delta\right) \\ &\le \mathbb{P}(\mathcal{T}_{M_0} < T) + \mathbb{P}\left(\{T = \mathcal{T}_{M_0}\} \wedge \left\{\int_0^T \|u - u_n\|_V^2 ds \ge \delta\right\}\right) \\ &\le \frac{\epsilon}{2} + \mathbb{P}\left(\int_0^T y(s)\|u - u_n\|_V^2 ds \ge \delta e^{-\lambda T - 2\beta M_0}\right) \\ &\le \frac{\epsilon}{2} + \frac{e^{\lambda T + 2\beta M_0}}{\delta}\mathbb{E}\int_0^T y(s)\|u - u_n\|_V^2 ds \\ &\le \epsilon. \end{aligned} \tag{0.105}$$

Similarly, we have that

$$\mathbb{P}(\|u - u_n\|_H^2 \geq \delta) \leq \epsilon. \tag{0.106}$$

By 6.0.15, we have uniform integrability as well, hence we have L^1-convergence with respect to ω, and we conclude that

$$\mathbb{E}\int_0^T \|u - u_n\|_V^2 dt \to 0, \tag{0.107}$$

$$\mathbb{E}\|u - u_n\|_H^2 dt \to 0, \text{ for all } t \in [0, T],$$

as $n \to \infty$. □

We proceed to prove Theorem 5.3.2 on Chapter 5.

Theorem 6.0.16. *[B4] Let $\mathcal{U}$ be a set of bounded of continuous linear bounded feedback controls. Let $\{\phi_n\}_{n\geq 1}$ be a sequence in $\mathcal{U}$ and $\phi \in \mathcal{U}$ be such that*

$$\lim_{n\to\infty}\int_0^T \|\phi_n(t, .) - \phi(t, .)\|_{\mathcal{L}(H)}^2 dt = 0 \tag{0.108}$$

where for $t \in [0, T]$ with $x_1, x_2, y_1, y_2 \in H$ we have

$$\mathcal{L} : [0, T] \times H \times H \to \mathbb{R}_+ \tag{0.109}$$

$$\mathcal{K} : H \to \mathbb{R}_+$$

$$|\mathcal{L}(t, x_1, y_1) - \mathcal{L}(t, x_2, y_2)| \leq C\Big(\|x_1 - x_2\|_H^2 + \|y_1 - y_2\|_H^2\Big)$$

$$|\mathcal{K}(x_1) - \mathcal{K}(x_2)| \leq C(\|x_1 - x_2\|_H^2),$$

and $\mathcal{J}(\phi) = \mathbb{E}\int_0^T \mathcal{L}(s, u_\phi, \phi)ds + \mathbb{E}\mathcal{K}(u_\phi(T))$, *then we have*

$$\lim_{n\to\infty} J(\phi_n) = J(\phi). \tag{0.110}$$

Proof. Let $u := u_\phi$ and $e(t) = e^{-\frac{b}{\nu}\int_0^t \|u(s)\|_V^2 ds} \exp\{-(\lambda + 2\sqrt{\mu} + 1)t\}$. Applying Ito we have that

$$\begin{aligned} & e(t)\|u - u_\phi\|_H^2 + 2\int_0^t e(s)\langle A(u - u_{\phi_n}, u - u_{\phi_n})\rangle ds \\ & 2\int_0^t e(s)\langle B(u,u) - B(u_\phi, u_\phi), u - u_{\phi_n}(s)\rangle ds \\ & -\frac{b}{\nu}\int_0^t e(s)\|u(s)\|_V^2\|u - u_{\phi_n}\|_H^2 ds - (\lambda + 2\sqrt{\mu} + 1)\int_0^t e(s)\|u - u_{\phi_n}\|_H^2 ds \\ & +2\int_0^t e(s)\langle \phi(s,u) - \phi_n(s, u_{\phi_n}(s)), u - u_{\phi_n}(s)\rangle ds \\ & +\int_0^t e(s)\|g(u) - g(u_{\phi_n})\|_H^2 ds \\ & +2\int_0^t e(s)\langle g(u) - g(u_{\phi_n}), u - u_{\phi_n}\rangle dW_s. \end{aligned} \tag{0.111}$$

Next, we estimate the nonlinear term as

$$\begin{aligned} & 2\langle B(u,u) - B(u_{\phi_n}, u_{\phi_n}), u - u_{\phi_n}\rangle = 2\langle B(u - u_{\phi_n}, u), u - u_{\phi_n}\rangle \\ & \leq \frac{b}{\nu}\|u\|_V^2\|u - u_{\phi_n}\|_H^2 + \nu\|u - u_{\phi_n}\|_V^2. \end{aligned} \tag{0.112}$$

Hence, we have by the Lipschitz assumption on ϕ and g

$$\begin{aligned}
&\mathbb{E} \sup_{s\in[0,t]} e(s)\|u - u_{\phi_n}\|_H^2 + \nu\mathbb{E}\int_0^t e(s)\|u - u_{\phi_n}\|_V^2 ds \\
&\quad \le 2\mathbb{E}\int_0^t e(s)\|\phi(s,u) - \phi_n(s,u)\|_H^2 ds \\
&\quad + 4\mathbb{E} \sup_{s\in[0,t]} \left| \int_0^s e(r)\langle g(u) - g(u_{\phi_n}), u - u_{\phi_n}\rangle dW_r \right| \\
&\quad \le 2\mathbb{E}\int_0^t e(s)\|\phi(s,u) - \phi_n(s,u)\|_H^2 ds + C\mathbb{E}\int_0^t \sup_{r\in[0,s]} \{\|u - u_{\phi_n}\|_H^2\} ds \\
&\quad + \frac{1}{2}\mathbb{E} \sup_{s\in[0,t]} e(s)\|u - u_{\phi_n}\|_H^2,
\end{aligned} \tag{0.113}$$

where C is a constant and $t \in [0,T]$. By Gronwall's Lemma, we conclude that

$$\begin{aligned}
&\mathbb{E} \sup_{s\in[0,t]} e(s)\|u - u_{\phi_n}\|_H^2 + 2\nu\mathbb{E}\int_0^t e(s)\|u - u_{\phi_n}\|_V^2 ds \\
&\quad \le C\mathbb{E}\int_0^T \|\phi(s,u) - \phi_n(s,u)\|_H^2 ds,
\end{aligned} \tag{0.114}$$

for all $t \in [0,T]$. Next, we apply Proposition 6.0.6 with $t = \mathcal{T}_M^u, \mathcal{T} = T, \mathcal{T}_M = \mathcal{T}_M^u$ and $Q_n(\mathcal{T}) = \|u(\mathcal{T}) - u_{\phi_n}(\mathcal{T})\|_H^2$ and $Q_n(\mathcal{T}) \int_0^{\mathcal{T}} \|u - u_n\|_V^2 ds$, we conclude the proof. □

Bibliography

[AT] F. Abergel and R. Temam, *On Some Control Problems in Fluid Mechanics*, Theoret. Comput.Fluid Dynamics. **1** (1990), 303–325.

[BT] A. Bensoussan and R. Temam, *Équations stochastiques du type Navier-Stokes*, J. Functional Analysis **13** (1973), 195–222.

[B2] H. Breckner, *Approximation of the solution of the stochastic navier-stokes equation*, Optimization **49** (2001), no. 1-2, 15–38.

[B] H. Breckner, *Existence of Optimal and Epsilon-Optimal Controls for the Stochastic Navier-Stokes Equation*, J. Appl. Math. Stochastic Anal. **13** (2000), no. 3, 239–259.

[B3] H. Breckner, *Galerkin approximation and the strong solution of the Navier-Stokes equation*, J. Appl. Math. Stochastic Anal. **13** (2000), no. 3, 239–259.

[B4] H. Breckner, *Approximation and Optimal Control of the Stochastic Navier-Stokes Equation*, Ph.D. Dissertation, Martin-Luther-Universitat-Halle-Wittenberg (2000).

[BKL] J. Bricmont, A. Kupiainen, and R. Lefevere, *Probabilistic estimates for the two-dimensional stochastic Navier-Stokes equations*, J. Statist. Phys. **100** (2000), no. 3-4, 743–756.

[BP] Z. Brzeźniak and S. Peszat, *Strong local and global solutions for stochastic Navier-Stokes equations*, Infinite dimensional stochastic analysis (Amsterdam, 1999), Verh. Afd. Natuurkd. 1. Reeks. K. Ned. Akad. Wet., vol. 52, R. Neth. Acad. Arts Sci., Amsterdam, 2000, pp. 85–98.

[CC] M. Capinski and NJ. Cutland, *Nonstandard Methods for Stochastic Fluid Mechanics*, World Scientific, Singapore, 1995.

[CF2] P. Constantin and C. Foias, *Navier-Stokes equations*, Chicago Lectures in Mathematics, University of Chicago Press, Chicago, IL, 1988.

[CG] M. Capiński and D. Gatarek, *Stochastic equations in Hilbert space with application to Navier-Stokes equations in any dimension*, J. Funct. Anal. **126** (1994), no. 1, 26–35.

[CP] M. Capiński and S. Peszat, *Local existence and uniqueness of strong solutions to 3-D stochastic Navier-Stokes equations*, NoDEA Nonlinear Differential Equations Appl. **4** (1997), no. 2, 185–200.

[CTMK] H. Choi, R. Temam, P. Moin, J. Kim, *Feedback control for unsteady flow and its application to the stochastic Burgers equation*, J. Fluid Mech. **253** (1993) 509–543.

[C] A.B. Cruzeiro, *Solutions et mesures invariantes pour des équations d'évolution stochastiques du type Navier-Stokes*, Exposition. Math. **7** (1989), no. 1, 73–82.

[DD] G. Da Prato and A. Debussche, *Ergodicity for the 3D stochastic Navier-Stokes equations*, J. Math. Pures Appl. (9) **82** (2003), no. 8, 877–947.

[DZ] G. Da Prato and J. Zabczyk, *Stochastic equations in infinite dimensions*, Encyclopedia of Mathematics and its Applications, vol. 44, Cambridge University Press, Cambridge, 1992.

[DGT] A. Debussche, N. Glatt-Holtz, and R. Temam, *Local martingale and pathwise solutions for an abstract fluids model*, Physica D (2011), (to appear).

[D] R. Durrett, *Probability: Theory and Examples*, Cambridge University Press, Cambridge, 2013.

[ET] I. Ekeland, R. Temam, *Convex Analysis and Variational Problems, SIAM Series Classics in Applied Mathematics*, SIAM, Philadelphia, 1999.

[F] F. Flandoli, *An introduction to 3d stochastic fluid dynamics*, SPDE in Hydrodynamic: Recent Progress and Prospects, Lecture Notes in Mathematics, vol. 1942, Springer Berlin / Heidelberg, 2008, pp. 51–150.

[FG] F. Flandoli and D. Gatarek, *Martingale and stationary solutions for stochastic Navier-Stokes equations*, Probab. Theory Related Fields **102** (1995), no. 3, 367–391.

[FR] F. Flandoli and M. Romito, *Partial regularity for the stochastic Navier-Stokes equations*, Trans. Amer. Math. Soc. **354** (2002), no. 6, 2207–2241 (electronic).

[FP] C. Foias and G. Prodi, *Sur le comportement global des solutions non-stationnaires des équations de Navier-Stokes en dimension* 2, Rend. Sem. Mat. Univ. Padova **39** (1967), 1–34.

[FV] A.V. Fursikov and MJ. Vishik, *Mathematical Problems in Statistical Hydromechanics*, 1988,Kluwer, Dordrecht.

[G] D. Gatarek, *Existence of optimal controls for stochastic evolution systems, in: G. Da Prato et al. (Eds.), Control of Partial Differential Equations*, IFIP WG 7.2 Conference, Villa Madruzzo, Trento, Italy, January 4 9, 1993, New York, Marcel Dekker, Inc. Lect. Notes Pure Appl. Math. **165** (1994) 8186.

[GS] D. Gatarek, J. Sobczyk, *On the existence of optimal controls of Hilbert space-valued diffusions*, SIAM Control Optim. **32** (1994) 170–175.

[GT] D. Gilbarg and N.S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer, Berlin, 2001, Reprint of the 1998 edition.

[GV] N. Glatt-Holtz and V. Vicol, *Local and global existence of smooth solutions for the stochastic Euler equations with multiplicative noise*, The Annals of Probability **42** (2014), no. 1, 80–145.

[GZ] N. Glatt-Holtz and M. Ziane, *Strong pathwise solutions of the stochastic Navier-Stokes system*, Advances in Differential Equations **14** (2009), no. 5-6, 567–600.

[GR] W. Grecksch, *Stochastische Evolutionsgleichungen unddderen Steuerung*, BSB B.G. Teubner Verlagsgesellschaft, Leipzig, 1987.

[K] S.B. Kuksin, *Randomly forced nonlinear PDEs and statistical hydrodynamics in 2 space dimensions.* Zurich Lectures in Advanced Mathematics. European Mathematical Society (EMS), Zürich, 2006.

[KUZ] I. Kukavica, K. Uğurlu, M. Ziane, *On the Galerkin approximation and norm estimates of the stochastic Navier-Stokes equations with multiplicative noise*, submitted.

[KV] I. Kukavica and V. Vicol, *On moments for strong solutions of the 2D stochastic Navier-Stokes equations in a bounded domain*, Asymptotic Analysis **90** (2014), no. 3-4, 189–206.

[M] J.C. Mattingly, *The dissipative scale of the stochastics Navier-Stokes equation: regularization and analyticity*, J. Statist. Phys. **108** (2002), no. 5-6, 1157–1179, Dedicated to David Ruelle and Yasha Sinai on the occasion of their 65th birthdays.

[MR] R. Mikulevicius and B.L. Rozovskii, *Stochastic Navier-Stokes equations for turbulent flows*, SIAM J. Math. Anal. **35** (2004), no. 5, 1250–1310.

[MR2] R. Mikulevicius and B.L. Rozovskii, *Global L_2-solutions of stochastic Navier-Stokes equations*, Ann. Probab. **33** (2005), no. 1, 137–176.

[MS] J.-L. Menaldi and S. S. Sritharan, *Stochastic 2-D Navier-Stokes equation*, Applied Mathematics and Optimization, vol. 46, no. 1, pp. 3153, 2002.

[O] C. Odasso, *Spatial smoothness of the stationary solutions of the 3D Navier-Stokes equations*, Electron. J. Probab. **11** (2006), no. 27, 686–699.

[PD] G. Da Prato, A. Debussche, *Control of the stochastic burgers model of turbulence*, SIAM J. Control Optim. **37** (1999) 1123-1149.

[PI] G. Da Prato, A. Ichikawa, *Stability and quadratic control for linear stochastic equations with unbounded coefficients*, Boll. Unione Mat. Ital., VI. Ser. B **4** (1985), 987-1001.

[PR] C. Prévôt and M. Röckner, *A concise course on stochastic partial differential equations*, Lecture Notes in Mathematics, vol. 1905, Springer, Berlin, 2007.

[S] A. Shirikyan, *Analyticity of solutions and Kolmogorov's dissipation scale for 2D Navier-Stokes equations*, Evolution equations (Warsaw, 2001), Banach Center Publ., vol. 60, Polish Acad. Sci., Warsaw, 2003, pp. 49–53.

[S] S.S. Sritharan, *Deterministic and stochastic control of NavierStokes equation with linear, monotone, and hyper viscosities*, Appl. Math. Optim. **41** (2000) 255-308.

[SV] D. W. Stroock and S. R. S. Varadhan, *Multidimensional diffusion processes*, Springer, Berlin, 1979.

[T] R. Temam, *Navier-Stokes equations*, AMS Chelsea Publishing, Providence, RI, 2001, Theory and numerical analysis, Reprint of the 1984 edition.

[T] C. Tudor, *Optimal control for semi-linear evolution equations*, Appl. Math. Optim. **20** (1989) 319–331.

[T2] C. Tudor, *Optimal and optimal control for the stochastic linear-quadratic problem*, Math. Nachr. **145**(1990) 135–149.

[V] M. Viot, *Solutions faibles d'équations aux dérivées partielles non linéaires*, 1976, Thèse, Université Pierre et Marie Curie, Paris.

[Z] E. Zeidler, *Nonlinear Functional Analysis and its Applications, Vol. III: Variational Methods and Optimization*, Springer, New York, 1985.