STEREOGRAPHIC COORDINATES FOR UNITARY QUANTUM GROUPS

by

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ABSTRACT

In this thesis, unitary matrix representation of quantum group $SU_q(2)$ is studied in terms of stereographic coordinates z, z^* and its central unitary phase u. It is observed that this parametrization of $SU_q(2)$ can be regarded as a q^{-1}-oscillator representation of $SU_q(2)$ where $q^{-1} > 1$. The fractional linear transformation of z and z^* which gives the action of $SU_q(2)$ on the quantum sphere $SU_q(2)/U(1)$ is studied. Then we extend this action to $SU_q(2)$ and derive the transformation law of u. In the final part of this work, transformation laws for dz and operator ∂ are also studied.

KISA ÖZET

$SU_q(2)$ kuantum grubunun matris temsili stereografik koordinatlar z, z^* ve birim faz u cinsinden çalışıldı. $SU_q(2)$' nin bu parametrizasyonu ile $SU_q(2)$' nin q^{-1}-oscillator temsili şeklinde göz önüne alınabileceği gözlendi. z, z^* ve u için dönüşüm eşitlikleri çıkarıldı. Son olarak da dz ve ∂ operatörü için dönüşüm kuralları incelendi.

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF SYMBOLS

a , a^+	Ladder operator
H	Hamiltonian operator
N	Number operator
u	Central unitary phase
z , z^*	Stereographic coordinates
δ_{nm}	Kronecker delta

1. INTRODUCTION

Classical theoretical physics concerns itself with mathematical procedures for analyzing changes in the observable properties of a system[1]. These observable properties can be described in terms of variables. Classical mechanics is self-consistent and it can be tested by experiments. In some situations it gives the wrong answer. An object moving with a speed comparable to the speed of light is an example of such a situation. Moreover, classical theoretical physics also does not work for phenomena on the atomic scale. These inadequacies of classical mechanics required new theories which include these situations. At the beginning of the twentieth century, quantum theory which became the answer to the questions involving the atomic scale started to form. In quantum mechanics, we work in a Hilbert Space which is a normed complete vector space. In the Hilbert space, we use operators in order to transform one vector into another vector. This transformation can be shown as

$$A|\psi\rangle = |\varphi\rangle . \tag{1.1}$$

All operators which are used in quantum mechanics are linear operators such that they satisfy the following relation[2]

$$A(c_1|\psi_1\rangle + c_2|\psi_2\rangle) = c_1 A|\psi_1\rangle + c_2 A|\psi_2\rangle \tag{1.2}$$

where A is an operator, c_1 and c_2 are constants.

In quantum mechanics, Hermitian operators which have real eigenvalues are used in order to represent observable quantities. The possible results of measurement are the eigenvalues of the corresponding operator. These possible values will occur with definite probabilities that depend on the state being measured. If the state is an eigenstate of the observable, then the result is certain to be the eigenvalue belonging to the state[3].

In the quantum harmonic oscillator[4], we use the operator H to denote the Hamiltonian of the system. This operator H can be expressed in terms of the two canonical observables momentum p and position q as

$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 q^2 \tag{1.3}$$

where hermitian operators p and q satisfy the algebraic rule

$$[p,q] = -i\hbar \, . \tag{1.4}$$

In order to find the eigenvalues and eigenvectors of the Hamiltonian for the quantum harmonic oscillator, it is convenient to introduce new operators a and its hermitian conjugate a^+ as

$$a = \left(\frac{m\omega}{2\hbar}\right)^{1/2}\left(q + i\frac{p}{m\omega}\right) \tag{1.5}$$

$$a^+ = \left(\frac{m\omega}{2\hbar}\right)^{1/2}\left(q - i\frac{p}{m\omega}\right) . \tag{1.6}$$

Here the operator a is called the lowering operator whereas a^+ is called the raising operator. These names refer the fact that they raise or lower the oscillatory energy state of a particle as formulated in the following equalities

$$a^+|n\rangle = (n+1)^{1/2}|n+1\,\rangle \tag{1.7}$$

$$a|n\rangle = n^{1/2}|n-1\,\rangle \, . \tag{1.8}$$

By using the commutation relation between p and q, the commutator of a and a^+ can be obtained as

$$\left[a,a^{+}\right] = aa^{+} - a^{+}a = 1 . \tag{1.9}$$

Also note that the number operator N can be defined as

$$N = a^{+}a \tag{1.10}$$

which satisfies the following equality

$$N|n\rangle = n|n\rangle . \tag{1.11}$$

The commutation relation which is satisfied between a and a^{+} can be generalized by using a parameter q. This can be thought as the q-deformation of the quantum harmonic oscillator which is called q-oscillator. After making such a generalization of commutation relation (1.9), many studies and different approaches about the q-oscillator have been considered[5-9]. One of them can be described by the following relation

$$aa^{+} - qa^{+}a = 1 . \tag{1.12}$$

Here q is a real deformation parameter and it is between 0 and 1 whereas a and a^{+} are annihilation and creation operators respectively. As in quantum harmonic oscillators, we can write the following equations for a and a^{+}

$$a^{+}|n\rangle = |n+1\rangle \tag{1.13}$$

$$a|n\rangle = [n]|n-1\rangle \tag{1.14}$$

where $[n]$ denotes a function of n to be determined by the commutation relation (1.12). Since there is no state below the ground state, the following equality also holds

$$a|0\rangle = 0 \tag{1.15}$$

where $[0]=0$. Thus, we can write that

$$|n\rangle=(a^{+})^{n}|0\rangle \tag{1.16}$$

which helps us to write the following equality

$$\langle n|m\rangle=\langle 0\,|\,a^{n}(a^{+})^{m}|0\rangle=[n]!\,\delta_{nm} \tag{1.17}$$

where

$$[n]!=[0][1]...[n] \tag{1.18}$$

$$[0]!=1. \tag{1.19}$$

By using the number operator in the q-oscillator commutation relation (1.12), we get the following difference equation for $[n]$

$$[n+1]-q[n]=1. \tag{1.20}$$

$$[n]\equiv\frac{1-q^{n}}{1-q}=1+q+...+q^{n-1} \tag{1.21}$$

is the basic integer which is also the solution of (1.20). Note that in order to see this solution, equation (1.15) is also used[10].

q-oscillators and quantum groups are deeply related such that with the discovery of quantum groups, studies about the q-oscillator became the center of attention again[11-23]. A quantum group is a q-deformation of a Lie group. Deformations of Lie algebras were first introduced in the context of group contractions[24], where a Lie algebra is deformed into another Lie algebra. The q-oscillator which is the simplest deformation of the oscillator algebra was first introduced by Arik and Coon[6]. Quantum groups and quantum algebras entailing a Hopf Algebra structure were formulated by Faddeev[25], Jimbo[26], Drinfeld[27], Woronowicz[28]. For the quantum algebra $SU_q(2)$, Macfarlane[7] and

Biedeharn[8] achieved a new realization where they constructed the quantum enveloping algebra in terms of two independent new q-deformed harmonic oscillators. They generalized the Schwinger construction used in the quantum theory of SU(2) angular momentum.

In this work, we first review the unitary group SU(2) in the matrix representation. We show that it can be parametrized in terms of a phase and the stereographic variables z and z^*. Then we derive the linear fractional transformation law for z and its unitary phase u. Next we review the unitary quantum group $SU_q(2)$. We consider the 2x2 matrices which are elements of $SU_q(2)$ such that elements of these matrices satisfy bilinear product relations which are q-dependent.

The stereographic (non-commutative) coordinates of $SU_q(2)$ can be obtained by expressing an element of $SU_q(2)$ in the coset representation. This helps us to parametrize the quantum unitary matrix in terms of a commuting phase u and an operator z. Here z is the stereographic quantum coordinate of the quantum sphere $SU_q(2)/U(1)$ and satisfies the following commutation relation;

$$zz^* - q^{-2}z^*z = q^{-2} - 1, \qquad 0 < q < 1. \tag{1.22}$$

The relationship between this commutation relation and q-oscillator commutation relation is shown. Then we give the action of $SU_q(2)$ on the quantum sphere by a fractional transformation on z. This is analogous to the classical case. We also find the quantum linear fractional transformation equations for z^*, z^*z, and u respectively. Finally we study the differential structures on S_q^2.

2. STEREOGRAPHIC COORDINATES AND SU(2)

2.1 Review of the Stereographic Coordinates

In this section, we mention some basic concepts which are used in this study. Firstly, let us introduce the stereographic projection. The stereographic projection of the sphere onto plane means a point (θ,ϕ) on the sphere is sent to the point with the polar coordinates (r,ϕ) in the plane as shown in figure 1 [29].

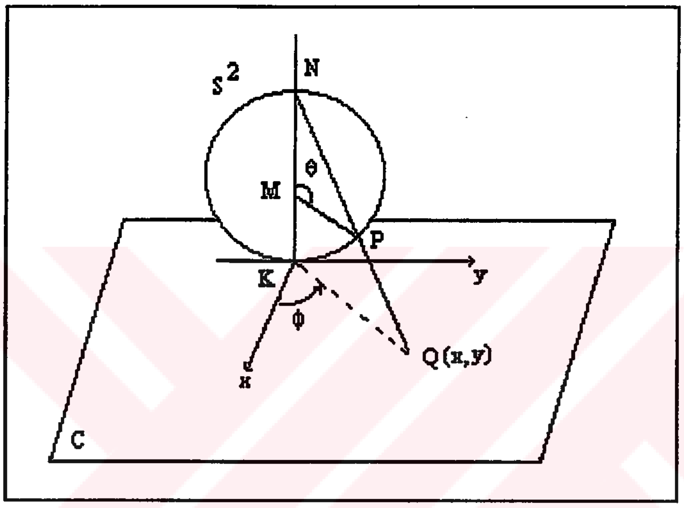

FIGURE 2.1. Geometric interpretation of the stereographic projection of S^2 to complex plane C.

Here the sphere is tangent to the plane and every point except one on the sphere S^2 has a sufficiently small neighborhood which has a one to one map onto an extended complex plane C[30]. In order to see how this mapping is defined, a line is drawn from the point N on the sphere diametrically opposite the point of tangency as in figure 1. This line intersects the sphere at P and the complex plane at Q. When P is mapped to Q, the coordinates of Q in the complex plane can be used in place of the coordinates of P in S^2. Except only one point this map is one to one. Trouble point is at N. Because, when the point P approaches N, the line from N becomes horizontal and the point Q goes to infinity. No matter in what direction in complex plane the point Q goes to infinity, the point P always approaches N. So N is mapped into all of "infinity". But this does not violate the formal equivalence between the two coordinate systems. In order to see this equality between spherical coordinate variables (θ,ϕ) and complex coordinate variables (r,ϕ), let us look at the triangle NMP in figure 1. Since triangle NMP is an isosceles triangle, the measure of angle MNP is ($90-\theta/2$). It is also known that any point in the complex plane can be written as folllows:

$$z = x + iy \tag{2.1}$$

or

$$z = re^{i\phi} \tag{2.2}$$

where $r = (x^2 + y^2)^{1/2}$, $\phi = tan^{-1}(y/x)$.

Therefore only by looking at the right triangle NKQ, following equality which gives the relationship between the sphere S^2 and the complex plane C can be written.

$$r = 2\,a\,cot\frac{\theta}{2} \tag{2.3}$$

$$z = 2\,a\,cot\frac{\theta}{2}e^{i\phi} \tag{2.4}$$

where a is the radius of sphere. For the unit sphere, it is clear that equation (2.4) can be rewritten as follows:

$$z = 2\ cot\frac{\theta}{2}e^{i\phi}\ . \tag{2.5}$$

2.2 Groups and Matrix Representation of Unitary Group SU(2)

After such an information about stereographic projection, let us give some general knowledge about groups and matrices. The most familiar point of view about groups is to consider them as a collection of transformations. Transformations of space are assumed invertible, and every closed collection of invertible transformation is, inevitably, a group. This is the role of group as symmetries. If we want to describe a group in a more mathematical way, the following can be written.

A set G of elements a, b,... is called a group, if the following four conditions are satisfied[31]:

1. For every pair of elements a, b; a product $a \circ b$ is defined as an element of G.
2. Associative law;

$$(a \circ b) \circ c = a \circ (b \circ c) . \tag{2.6}$$

3. A unit e (or 1) is defined within G, such that for every a in G

$$a \circ e = e \circ a = a . \tag{2.7}$$

4.To every a there is an element a^{-1} in G such that

$$a \circ a^{-1} = a^{-1} \circ a = e . \tag{2.8}$$

The group is called Abelian if all its elements commute:

$$a \circ b = b \circ a \tag{2.9}$$

for every a and b.

If the elements of the group are linear transformations or permutations, and if the product $a \circ b$ is defined as the transformation or permutation obtained by applying first b

and next a, the associative law is automatically is fulfilled. In order to fulfill 3 and 4, we have to define e as the identity and a^{-1} as the inverse transformation to a. Hence we have:

A non-empty set of linear transformations or permutations is a group, if it contains together with any two elements a and b, their product $a \circ b$ and with every element a, its inverse a^{-1}. Let us give some examples:

i) The relations of the real 3-dimensional space leaving fixed the origin O form a non-abelian group; the rotation group O(3) or SO(3,R). The letters have the following meaning:

SO=Special Orthogonal.

R=Field of Real Numbers.

Here, "Orthogonal" means that if two lines are originally at right angles, after a rotation they are still at right angles and this is true for all rotations. "Special" means its determinant is equal to one.

ii) The rotations about a fixed axis form an Abelian group, which may be denoted by O(2) or SO(2,R).

iii) The Lorentz group consists of those non-singular transformations of a real 4-dimensional vector space, which leave the quadratic form $x^2 + y^2 + z^2 - c^2t^2$ unchanged and which do not interchange past and future.

iv) The Special Linear Group SL(n,R) and SL(n,C) are the groups of all linear transformations of determinant one, with real or complex coefficients.(R=Real Fields; C=Complex Field.)

v) The Special Unitary Group SU(n) is the group of unitary transformations of a complex n-dimensional vector space with determinant one.

vi) The symmetric group S_n consists of all one to one permutations of n objects.

In this study, we are interested with SU(2) and q-deformation of it which is called SU_q (2).Therefore, now let us examine SU(2) in detail. As it is mentioned above, SU(2) is a group of all unitary transformations of 2 complex variables with determinant 1 whereas SL(2) is a group of all linear transformations of 2 complex variables with determinant 1. If we consider the complex vector space formed by the linear forms $\alpha_1 u_1 + \alpha_2 u_2$ in two indeterminates u_1, u_2 with complex coefficients α_1, α_2. The transformations of the Special Linear Group SL(2) transform the basic vectors u_1, u_2 of this vector space into

$$u_1' = u_1 a + u_2 c \tag{2.10}$$

$$u_2' = u_1 b + u_2 d\,. \tag{2.11}$$

The corresponding transformations of the coefficients α_1, α_2 of a vector $\alpha_1 u_1 + \alpha_2 u_2$ are,

$$\alpha_1' = a\alpha_1 + b\alpha_2 \tag{2.12}$$

$$\alpha_2' = c\alpha_1 + d\alpha_2 \tag{2.13}$$

and the matrices of these transformations are

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad ad - bc = 1\,. \tag{2.14}$$

By direct verification one sees that the inverse matrix of A is;

$$A^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}. \tag{2.15}$$

Unitary transformations A are those which leave invariant a positive Hermitian form. They form the special unitary group SU(2).

It may be always assumed that the Hermitian form is the unit form $\overline{\alpha}_1\alpha_1 + \overline{\alpha}_2\alpha_2$. In this case, the condition for A to be unitary is

$$A^{-1} = A^{+} . \tag{2.16}$$

From this equality, it can be said that a unitary transformation A always has an inverse A^{-1}, hence any unitary transformation is non-singular. Let us rewrite above equality by considering their elements;

$$\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} \overline{a} & \overline{c} \\ \overline{b} & \overline{d} \end{pmatrix} \tag{2.17}$$

which implies $\overline{a} = d$ and $\overline{b} = -c$. Hence SU(2) consists of all matrices

$$\begin{pmatrix} a & -\overline{c} \\ c & \overline{a} \end{pmatrix} \tag{2.18}$$

with

$$a\overline{a} + c\overline{c} = 1 . \tag{2.19}$$

From (2.18) one sees that a unitary transformation transforms the pair (α_1,α_2) in the same way as it transforms the pair $(-\overline{\alpha}_2,\overline{\alpha}_1)$.

2.3 Complex Projective Space CP(1) and Coset Representation of Unitary Matrix A

Up to now the stereographic projection and the SU(2) group were introduced. In order to see the relationship between them, the following steps can be taken. Firstly, we write matrix A as a product of two matrices:

$$A = UZ. \tag{2.20}$$

Writing matrix A such a multiplication form is called the coset representation of A. Here matrix U is the phase part and it is identified with subgroup U(1). Matrix Z is the canonical representative of the coset SU(2)/U(1)=CP(1). Therefore A can be written as

$$A = UZ = \begin{pmatrix} \bar{u} & 0 \\ 0 & u \end{pmatrix} \begin{pmatrix} ua & -r \\ r & \overline{ua} \end{pmatrix} \tag{2.21}$$

where $a,u \in C;\ r \geq 0;\ u\bar{u} = 1$.

If we write a' instead of ua, and $\bar{a}'$ instead of $\overline{ua}$; then this matrix A can be rewritten as follows:

$$A = UZ = \begin{pmatrix} \bar{u} & 0 \\ 0 & u \end{pmatrix} \begin{pmatrix} a' & -r \\ r & \bar{a}' \end{pmatrix} \tag{2.22}$$

where $a',u \in C;\ r \geq 0;\ u\bar{u} = 1$. Here, the matrix Z is also a unitary matrix. Therefore, its elements also satisfy the following equality;

$$\bar{a}'a' + r^2 = 1, \qquad r \geq 0 \tag{2.23}$$

which describes the upper half of a sphere.

2.4 Unitary Matrix Z in terms of Stereographic Variables

Now; let us try to write the canonical representation of A in terms of stereographic variables z and $\bar{z}$.

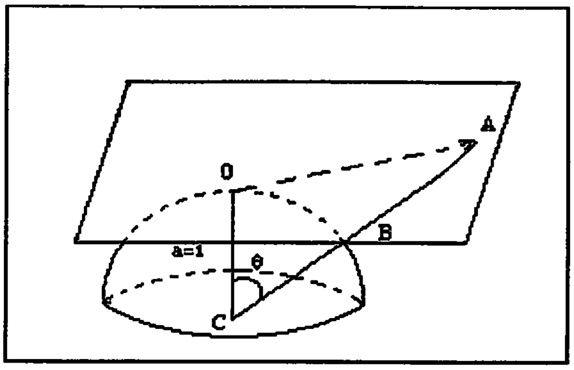

FIGURE 2.2. Projection of upper half of the sphere onto complex plane.

Here the complex plane is placed on the upper half of the sphere. By looking at the triangle AOC in figure 2; the equality for the distance between origin and any point in the complex plane in terms of spherical variables can be written as

$$|OA| = |z| = tan\,\theta. \tag{2.24}$$

From figure 2; the following equality can also be written

$$sin\,\theta\, e^{i\phi} = z(1+z\bar{z})^{-1/2}\ . \tag{2.25}$$

As is seen in the above equalities, the magnitude of projection is independent from ϕ whereas the position of projection depends on θ and ϕ. Now, let us write elements of Z in terms of stereographic variables z and $\bar{z}$ by leaving equation (2.23) invariant.

$$a = z(1+z\bar{z})^{-1/2} \tag{2.26}$$

$$r = (1+z\bar{z})^{-1/2}\ . \tag{2.27}$$

If matrix Z is rewritten by considering the above equalities, it can be seen

$$Z=(1+z\bar{z})^{-1/2}\begin{pmatrix} z & -1 \\ 1 & \bar{z} \end{pmatrix} \tag{2.28}$$

such that here Z describes an element of S^2.

In order to write a linear fractional transformation for z; let us consider the unitary matrix M. Since this matrix M is unitary matrix, it can be written as

$$M=\begin{pmatrix} \alpha & -\bar{\gamma} \\ \gamma & \bar{\alpha} \end{pmatrix} \tag{2.29}$$

where $\alpha,\gamma \in C;\ \ \alpha\bar{\alpha}+\gamma\bar{\gamma}=1$.

When this matrix M is multiplied by the matrix A which can be shown as $N = MA$, new matrix N is also an element of SU(2). This can be easily seen from following equations;

$$N = MA=\begin{pmatrix} \alpha & -\bar{\gamma} \\ \gamma & \bar{\alpha} \end{pmatrix}\begin{pmatrix} a & -\bar{c} \\ c & \bar{a} \end{pmatrix} \tag{2.30}$$

$$N=\begin{pmatrix} \alpha a-\bar{\gamma}\, c & -\alpha\bar{c}-\bar{\gamma}\,\bar{a} \\ \gamma a+\bar{\alpha}\, c & -\gamma\bar{c}+\bar{\alpha}\,\bar{a} \end{pmatrix}=\begin{pmatrix} \beta & -\bar{\delta} \\ \delta & \bar{\beta} \end{pmatrix}. \tag{2.31}$$

As expected, $\beta\bar{\beta}+\delta\bar{\delta}=1$ also holds. Now, to see the transformation law for z; we equate this matrix N to matrix A' where

$$A'=\begin{pmatrix} z'(1+z'\bar{z}')^{-1/2} & -(1+z'\bar{z}')^{-1/2} \\ (1+z'\bar{z}')^{-1/2} & (1+z'\bar{z}')^{-1/2}\bar{z}' \end{pmatrix}\begin{pmatrix} u' & 0 \\ 0 & \bar{u}' \end{pmatrix} \tag{2.32}$$

$$N=\begin{pmatrix} \alpha & -\bar{\gamma} \\ \gamma & \bar{\alpha} \end{pmatrix}\begin{pmatrix} z(1+z\bar{z})^{-1/2} & -(1+z\bar{z})^{-1/2} \\ (1+z\bar{z})^{-1/2} & (1+z\bar{z})^{-1/2}\bar{z} \end{pmatrix}\begin{pmatrix} u & 0 \\ 0 & \bar{u} \end{pmatrix}. \tag{2.33}$$

Since $N = A'$ holds, corresponding elements of these matrices are equal to each other. If (1,1) element and (2,1) element of these matrices are considered, the following equalities are hold.

$$z'(1+z'\bar{z}')^{-1/2}u' = (\alpha z-\bar{\gamma})(1+z\bar{z})^{-1/2}u \tag{2.34}$$

$$(1+z'\bar{z}')^{-1/2}u' = (\gamma z+\bar{\alpha})(1+z\bar{z})^{-1/2}u\,. \tag{2.35}$$

If (2.34) is divided by (2.35), then we get the fractional linear transformation equation for z;

$$z' = \frac{\alpha z-\bar{\gamma}}{\gamma z+\bar{\alpha}}\,. \tag{2.36}$$

By considering matrix N to be equal to matrix A'; we can also derive the transformation law for u. If we take the (2,1) and (1,2) elements of matrices, then we have the following equalities;

$$(1+z'\bar{z}')^{-1/2}u' = (\gamma z+\bar{\alpha})(1+z\bar{z})^{-1/2}u \tag{2.37}$$

$$(1+z'\bar{z}')^{-1/2}\bar{u}' = (\alpha+\bar{\gamma}\bar{z})(1+z\bar{z})^{-1/2}\bar{u}\,. \tag{2.38}$$

If (2.37) is divided by (2.38), then we will get

$$u'^2 = (\gamma z+\bar{\alpha})(\alpha+\bar{\gamma}\bar{z})^{-1}u^2 \tag{2.39}$$

which implies

$$u' = ((\gamma z+\bar{\alpha})(\alpha+\bar{\gamma}\bar{z})^{-1})^{1/2}u\,. \tag{2.40}$$

3. STEREOGRAPHIC COORDINATES FOR UNITARY QUANTUM GROUPS

3.1 Quantum Matrix Groups

In this section; firstly we introduce quantum groups briefly, then we give a brief information about $GL_q(2)$ and $SU_q(2)$. Quantum groups can be identified as the name given to the "geometrical object" which lies behind a non-commutative Hopf algebra[32]. On the other hand, they can be also thought as sets of matrices whose elements obey various commutation relations[33]. Here, we study the quantum group $GL_q(2,C)$. It consist of 2x2 matrices such that their elements do not commute among themselves but obey certain q-dependent commutation relations. Before giving more detail about $GL_q(2,C)$, let us say something about q-dependent commutation relations and quantization. On the algebraic side, quantization means deforming a commutative algebra to a non-commutative algebra. For example, if $xy = yx$ is deformed by using a parameter q, then it may become $xy = qyx$. We usually deal with $q = e^{\hbar/\hbar_0}$ rather than $\hbar$. Thus, this non-commutative algebra reduces to the classical one as $\hbar \to 0$ which means $q \to 1$. This is consistent with the procedure which we use to reduce quantum mechanics to classical mechanics.

Let us return to the deformed form of the classical group $GL(2,C)$ which is $GL_q(2,C)$. If matrix A is the element of $GL_q(2,C)$, then

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{3.1}$$

and the matrix elements a, b, c, d obey the following relations[34]

$$ab = qba \tag{3.2}$$

$$ac = qca \tag{3.3}$$

$$bd = qdb \tag{3.4}$$

$$cd = qdc \tag{3.5}$$

$$bc = cb \tag{3.6}$$

$$ad - da = (q - \frac{1}{q})bc\,. \tag{3.7}$$

These relations can be considered as the relations which define $GL_q(2,C)$. For these commutation relations; the following figure can be used as mnemonic:

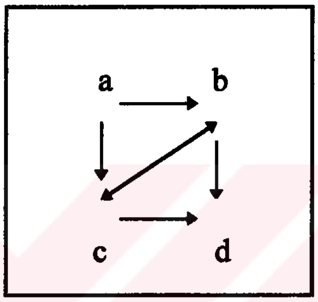

FIGURE 3.1. Mnemonic for q-dependent commutation relation of the elements of $SU_q(2)$

In this table, direction of arrow indicates which side of the commutation relation equation has the parameter q.

Determinant of this matrix A can be written as

$$D = det_q\, A = ad - qbc = da - \frac{1}{q}bc = (a - bd^{-1}c)d\,. \tag{3.8}$$

Here D is central which means that it commutes with a, b, c, d. This can be checked by using the equations (3.2-7).

Since determinant D commutes with all the elements of matrix A, the matrix elements may be normalized such that

$$det_q\, A = 1\,. \tag{3.9}$$

Such a normalization procedure restricts the quantum group to $SL_q(2)$ in analogy with the classical restriction to the special linear group[35]. In addition to this, if A is restricted to be unitary, then A becomes the element of $SU_q(2)$ such that its elements satisfy

$$A^{+} = A^{-1} \tag{3.10}$$

in addition to relations for $SL_q(2)$. Therefore, if $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is an element of $SU_q(2)$, in addition to $GL_q(2)$ commutation relations, the equalities $d = a^*$, $b = -qc^*$ and $ad - qbc = 1$ hold. Thus, matrix A can be rewritten as

$$A = \begin{pmatrix} a & -qc^* \\ c & a^* \end{pmatrix} \tag{3.11}$$

where $q \in R$.

As is known, elements of matrix A are non-commuting quantities and they satisfy the following commutation relations:

$$ac = qca \tag{3.12}$$

$$c^* a^* = qa^* c^* \tag{3.13}$$

$$ac^* = qc^* a \tag{3.14}$$

$$ca^* = qa^* c \tag{3.15}$$

$$cc^* = c^* c \tag{3.16}$$

$$aa^* + q^2 c^* c = 1 \tag{3.17}$$

$$a^* a + cc^* = 1\,. \tag{3.18}$$

These relations can also be derived only by using (3.11) and the following one

$$AA^{+} = A^{+} A = 1\,. \tag{3.19}$$

As is known, for a quantum matrix group, the elements of a matrix do not commute among themselves but obey various bilinear commutation relations. However, when the "group product" is taken, the elements of different matrices commute[36]. Let us consider following matrices

$$M_1 = \begin{pmatrix} a_1 & -qc_1{}^* \\ c_1 & a_1{}^* \end{pmatrix} \tag{3.20}$$

and

$$M_2 = \begin{pmatrix} a_2 & -qc_2{}^* \\ c_2 & a_2{}^* \end{pmatrix} \tag{3.21}$$

where $M_1, M_2 \in SU_q(2)$. Here $(a_1, a_1{}^*, c_1, c_1{}^*)$ pairwise commute with $(a_2, a_2{}^*, c_2, c_2{}^*)$. Thus, M_1M_2 and M_2M_1 are both $SU_q(2)$ matrices. Also note that the following equality

$$Det_q(M_1M_2) = Det_q(M_2M_1) = (Det_q M_1)(Det_q M_2) = 1 \tag{3.22}$$

reinforces above statement with determinant. Therefore if M is defined as the product of matrix M_1 and M_2,

$$M = \begin{pmatrix} a_1a_2 - qc_1{}^*c_2 & -q(a_1c_2{}^* + c_1{}^*a_2{}^*) \\ c_1a_2 + a_1{}^*c_2 & -qc_1c_2{}^* + a_1{}^*a_2{}^* \end{pmatrix} = \begin{pmatrix} a & -qc^* \\ c & a^* \end{pmatrix} \tag{3.23}$$

then it can be easily seen that elements of matrix M satisfy all the commutation relations for $SU_q(2)$ matrices. Such a product of matrices is a map $M(2x2)xM(2x2) \to M(2x2)$. Instead of (3.23), this can be also represented as follows:

$$M = \Delta \begin{pmatrix} a & -qc^* \\ c & a^* \end{pmatrix} = \begin{pmatrix} \Delta(a) & -q\Delta(c^*) \\ \Delta(c) & \Delta(a^*) \end{pmatrix} \tag{3.24}$$

where

$$\Delta(a) = a_1 a_2 - q c_1{}^* c_2 \tag{3.25}$$

$$\Delta(c) = c_1 a_2 + a_1{}^* c_2 \tag{3.26}$$

$$\Delta(a^*) = -q c_1 c_2{}^* + a_1{}^* a_2{}^* \tag{3.27}$$

$$\Delta(c^*) = a_1 c_2{}^* + c_1{}^* a_2{}^* \ . \tag{3.28}$$

Note that here elements of matrix M which have different indices satisfy the following commutation relation

$$[1,2] = 0 \tag{3.29}$$

which means

$$a_1 a_2{}^* = a_2{}^* a_1 \tag{3.30}$$

$$a_1 c_2 = c_2 a_1 \tag{3.31}$$

$$\vdots$$

3.2 Unitary Quantum Group $SU_q(2)$ and q-oscillator

Now, let us return to matrix A. As is known, its elements a and a^* do not commute with each other. Their commutation law can be derived by using (3.16-18) such that

$$aa^* + q^2 c^* c = 1 = a^* a + cc^* \tag{3.32}$$

$$cc^* = c^* c \ . \tag{3.33}$$

If we eliminate $c^* c$ from equation (3.32), then we will get the following one

$$aa^* - q^2 a^* a = 1 - q^2, \qquad 0 < q < 1 \tag{3.34}$$

which is the commutation relation for a and a^*. Also note that this equation can be taken to define the q-oscillator relation. When a is rescaled by $(1-q^2)^{-1/2}$ which means

$$a(1-q^2)^{-1/2} \to b \tag{3.35}$$

$$a^*(1-q^2)^{-1/2} \to b^* \tag{3.36}$$

then we will get precisely the q - oscillator commutation relation such that

$$bb^* - q^2 b^* b = 1. \tag{3.37}$$

This is the form of the one dimensional q-oscillator. Also note that although for $0 < q < 1$ equations (3.34) and (3.37) are related by just rescaling, in the $q \to 1$ limit they are fundamentally different. As q goes to 1, (3.34) reduces to a trivial commutation relation such that a and a^* can be taken as commuting complex numbers

$$[a, a^*] = 0. \tag{3.38}$$

whereas (3.37) reduces to the ordinary oscillator relation such that

$$bb^* - b^* b = 1. \tag{3.39}$$

3.3 Unitary Quantum Groups in terms of Stereographic Coordinates

As in section 2; now let us try to write elements of matrix A in terms of stereographic coordinates z and z^*. Firstly, let us consider the element a such that it can be written as

$$a = z(1+z^*z)^{-1/2} \tag{3.40}$$

where stereographic coordinates z and z^* are non-commuting quantities. Due to analiticity, this equality can also be written as follows:

$$a = (1+zz^*)^{-1/2}z. \tag{3.41}$$

Here, (3.40) and (3.41) are equivalent such that to see how their equivalence holds, let us take the following steps:

$$a = z(1+z^*z)^{-1/2} = z\sum_n f_n(z^*z)^n \tag{3.42}$$

$$a = z\sum_n f_n \underbrace{(z^*zz^*z...z^*)}_{(z^*z)^n}. \tag{3.43}$$

Here z can be inserted to summation. Thus (3.43) becomes

$$a = \sum_n f_n(zz^*zz^*...zz^*z) \tag{3.44}$$

$$a = \sum_n f_n(zz^*)^n z = (\sum_n f_n(zz^*)^n)z. \tag{3.45}$$

Then, as expected a can be written as

$$a = (1+zz^*)^{-1/2}z. \tag{3.46}$$

Before finding the commutation relation between z and z^*, let us try to write c in terms of stereographic coordinates. From (3.18), it can written

$$(1+z^*z)^{-1/2}z^*z(1+z^*z)^{-1/2} + c^*c = 1. \tag{3.47}$$

Since $[z^*z, z^*z] = 0$, above equality becomes

$$(1+z^*z)^{-1}z^*z + c^*c = 1 \tag{3.48}$$

which gives

$$c^*c = (1+z^*z)^{-1} . \tag{3.49}$$

Therefore c can be written in terms of z and z^* as follows:

$$c = e^{i\gamma}(1+z^*z)^{-1/2} \tag{3.50}$$

where $\gamma = \gamma^*$. Here $e^{i\gamma}$ commutes with everything[18] which means that the following equalities hold

$$[e^{i\gamma}, z] = 0 \tag{3.51}$$

$$[e^{i\gamma}, z^*] = 0 . \tag{3.52}$$

3.4 Commutation Relation Between Stereographic Coordinates z, z* and Its Relation with the q-oscillator Commutation Relation

Now, let us return to commutation relation between z and z^*. To find this commutation law, a is written in terms of stereographic coordinates z and z^* in equation (3.34). Since

$$aa^* = zz^*(1+zz^*)^{-1} \tag{3.53}$$

$$a^*a = z^*z(1+z^*z)^{-1} \tag{3.54}$$

hold, equation (3.34) becomes

$$zz^*(1+zz^*)^{-1} - q^2 z^* z(1+z^* z)^{-1} = 1 - q^2 \,. \tag{3.55}$$

If both sides of above equation are multiplied with $(1+zz^*)(1+z^*z)$ and rearrange it, then we get

$$zz^* - q^{-2} z^* z = q^{-2} - 1 \,. \tag{3.56}$$

Before showing the relationship of this commutation relation with the q-oscillator commutation relation, let us say something about one important point. By looking at equation (3.54), the following equation can be written

$$z^* z = a^* a(1 - a^* a)^{-1}. \tag{3.57}$$

This equality says that when a^*a equals 1, z^*z goes to infinity which means that we have no solution for a^*a equal to 1. Thus, the representation of (3.34) with $a^*a = 1$ are not present in the representation of (3.56). These representations are precisely the representation which are not q - oscillator-like.

If z is rescaled by $(q^{-2}-1)^{1/2}$ and this rescaled z is called as w which means $z/(q^{-2}-1)^{1/2} \to w$, equation (3.56) becomes

$$ww^* - q^{-2} w^* w = 1 \,. \tag{3.58}$$

As is known, here q is between 0 and 1. Therefore $q^{-2} - 1 > 0$ which implies q^{-2} is bigger than 1. In equation (3.58), if q' is written instead of q^{-2}, it is seen that this equation also shows a q-oscillator relation, but here it is a $q' > 1$ q-oscillator. Thus, it is obvious that the commutation relation which is satisfied by the stereographic coordinates z and z^* can be written in the form of the q-oscillator commutation relation

$$ww^* - q'w^*w = 1, \qquad q' > 1. \tag{3.59}$$

As with the case in (3.34) and (3.37), equation (3.56) and (3.58) are just rescaled form of each other. They show exactly same thing except when q goes to 1. As $q \to 1$, (3.56) and (3.58) respectively become

$$[z, z^*] = 0 \tag{3.60}$$

$$[w, w^*] = 1. \tag{3.61}$$

These equations are fundamentally different equations such that (3.60) is the trivial commutation relation, whereas (3.61) is the harmonic oscillator commutation relation.

Up to now, we expressed an element of $SU_q(2)$ in terms of stereographic coordinates and we found the commutation relation between the elements of $SU_q(2)$. Now, before writing $SU_q(2)$ in the coset representation, let us write z and z* in terms of a, a*, c, c*. By inserting equation (3.49) into equation (3.54), we have the following one

$$a^*a = z^*z(cc^*) \tag{3.62}$$

which means

$$z^*z = a^*a(cc^*)^{-1}. \tag{3.63}$$

If this equality is put into (3.40), we get

$$z = a(1 + z^*z)^{1/2} = a(1 + a^*a(cc^*)^{-1})^{1/2} \tag{3.64}$$

$$z = a((cc^* + a^*a)(cc^*)^{-1})^{1/2}. \tag{3.65}$$

Since $a^*a + cc^* = 1$, the following equation is obtained from (3.65)

$$z = a(cc^*)^{-1/2} . \tag{3.66}$$

By using the same procedure or by taking the hermitian conjugate of the above equation, the inverse "coordinate transformation" equation for z* can be easily obtained as

$$z^* = (c^* c)^{-1/2} a^* . \tag{3.67}$$

3.5 Coset $SU_q(2)/U(1)=CP_q(1)$ and Quantum Linear Fractional Transformations for z, z*, z*z and Its Unitary Phase u.

In order to find the coset representation of matrix A, firstly elements of matrix A are written in terms of stereographic coordinates z and z^*

$$A = \begin{pmatrix} a & -qc^* \\ c & a^* \end{pmatrix} = \begin{pmatrix} z(1+z^*z)^{-1/2} & -qe^{-i\gamma}(1+z^*z)^{-1/2} \\ e^{i\gamma}(1+z^*z)^{-1/2} & (1+z^*z)^{-1/2} z^* \end{pmatrix} . \tag{3.68}$$

Then, as in section 2, this matrix A can be expressed in the coset representation by separating it into two factors where the second factor is the commuting phase part,

$$A = ZU = \begin{pmatrix} u^* z(1+z^*z)^{-1/2} & -q(1+z^*z)^{-1/2} \\ (1+z^*z)^{-1/2} & (1+z^*z)^{-1/2} z^* u \end{pmatrix} \begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix} \tag{3.69}$$

where $u = e^{i\gamma}$.

In the above equality, matrix U is the phase part and it is an element of U(1) which is a subgroup of $SU_q(2)$. At the same time, matrix Z is the element of $CP_q(1)$ which can be also represented as $SU_q(2)/U(1)$. This Z is called the canonical representative of the coset.

Now, let us redefine z and z^* in the following form:

$$z \rightarrow u^* z \tag{3.70}$$

$$z^* \rightarrow z^* u . \tag{3.71}$$

These redefinitions remain invariant commutation relation (3.56) and they allow us to write Z as

$$Z = \begin{pmatrix} z(1+z^*z)^{-1/2} & -q(1+z^*z)^{-1/2} \\ (1+z^*z)^{-1/2} & (1+z^*z)^{-1/2} z^* \end{pmatrix} . \tag{3.72}$$

By using only the commutation relation between z and z^*, (3.72) becomes

$$Z = \begin{pmatrix} z(1+z^*z)^{-1/2} & -(1+zz^*)^{-1/2} \\ (1+z^*z)^{-1/2} & z^*(1+zz^*)^{-1/2} \end{pmatrix} = \begin{pmatrix} z & -1 \\ 1 & z^* \end{pmatrix} \begin{pmatrix} (1+z^*z)^{-1/2} & 0 \\ 0 & (1+zz^*)^{-1/2} \end{pmatrix} . \tag{3.73}$$

Thus, (3.69) can be written as

$$A = ZU = \begin{pmatrix} z & -1 \\ 1 & z^* \end{pmatrix} \begin{pmatrix} (1+z^*z)^{-1/2} & 0 \\ 0 & (1+zz^*)^{-1/2} \end{pmatrix} \begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix} . \tag{3.74}$$

If the above matrix A is multiplied by another matrix which is also an element of $SU_q(2)$ and is put equal to A', then all transformation laws can be obtained from this equality. In order to see $z \xrightarrow{SU_q(2)} \tilde{z}$, let us take the following steps;

$$A' = MA \tag{3.75}$$

where $M = \begin{pmatrix} a & -qc^* \\ c & a^* \end{pmatrix} \in SU_q(2)$.

$$A' = MA = \begin{pmatrix} \tilde{z}(1+\tilde{z}^*\tilde{z})^{-1/2}\tilde{u} & -(1+\tilde{z}\tilde{z}^*)^{-1/2}\tilde{u}^* \\ (1+\tilde{z}^*\tilde{z})^{-1/2}\tilde{u} & \tilde{z}^*(1+\tilde{z}\tilde{z}^*)^{-1/2}\tilde{u}^* \end{pmatrix}$$
$$= \begin{pmatrix} a & -qc^* \\ c & a^* \end{pmatrix} \begin{pmatrix} z(1+z^*z)^{-1/2}u & -(1+zz^*)^{-1/2}u^* \\ (1+z^*z)^{-1/2}u & z^*(1+zz^*)^{-1/2}u^* \end{pmatrix} \tag{3.76}$$

Equation (3.76) is the key to the transformation laws. Since, in order to find transformation laws, we only use this equality by considering corresponding elements of the right hand side and the left hand side matrices. For $z \xrightarrow{SU_q(2)} \tilde{z}$, we consider the (1,1) elements and (2,1) elements of the matrices in the RHS and LHS. Equating these elements gives the following equalities;

$$\tilde{z}(1+\tilde{z}^*\tilde{z})^{-1/2}\tilde{u} = (az - qc^*)(1+z^*z)^{-1/2}u \tag{3.77}$$

$$(1+\tilde{z}^*\tilde{z})^{-1/2}\tilde{u} = (cz + a^*)(1+z^*z)^{-1/2}u\,. \tag{3.78}$$

By dividing the first one by the second one, we obtain the fractional linear transformation equation for z [37];

$$\tilde{z} = (az - qc^*)(cz + a^*)^{-1}\,. \tag{3.79}$$

In the above equality, since z, z^* and a, a^*, c, c^* do not commute among themselves whereas (z, z^*) pairwise commute with (a, a^*, c, c^*), the place of the parenthesis cannot be changed, freely. To change their place, we shall apply a method utilizing c-number coefficient. Let us consider the following equality

$$(az - qc^*)(cz + a^*)^{-1} = (\alpha cz + \beta a^*)^{-1}(\gamma az - \delta qc^*)\,. \tag{3.80}$$

In this equality, $\alpha, \beta, \gamma, \delta$ are the coefficients which we shall determine. In order to find these coefficients, the following steps are taken;

$$(\alpha cz + \beta a^*)(az - qc^*) = (\gamma az - \delta qc^*)(cz + a^*) \tag{3.81}$$

$$\alpha\, caz^2 + \beta\, a^* az - q\alpha\, cc^* z - q\beta\, a^* c^* = \gamma\, acz^2 + \gamma\, aa^* z - q\delta\, c^* cz - q\delta\, c^* a^* \quad (3.82)$$

which means that following equalities

$$\alpha\, caz^2 = \gamma\, acz^2 \quad (3.83)$$

$$q\beta\, a^* c^* = q\delta\, c^* a^* \quad (3.84)$$

$$\beta\, a^* a - q\delta\, cc^* = \gamma\, aa^* - q\delta\, c^* c \quad (3.85)$$

hold. By using (3.12-18), above equalities can be rearranged as

$$\alpha\, caz^2 = \gamma\, qcaz^2 \quad (3.86)$$

$$\beta\, qa^* c^* = \delta\, q^2 a^* c^* \quad (3.87)$$

$$\beta\ (1 - c^* c) - q\alpha\, c^* c = \gamma\ (1 - q^2 c^* c) - q\delta\, c^* c \quad (3.88)$$

which implies the following equalities

$$\alpha = \gamma\, q \quad (3.89)$$

$$\beta = \delta\, q \quad (3.90)$$

$$\delta q = \gamma\ . \quad (3.91)$$

Thus by expressing three of the variables in terms of the fourth, the transformation equation for z becomes

$$\tilde{z} = (az - qc^*)(cz + a)^{-1} = (qcz + a^*)^{-1}(az - c^*)\ . \quad (3.92)$$

This equality gives the action of $SU_q(2)$ on the quantum sphere. In order to see such an equality for z^*, in addition to the method which we used for z, we can find it by taking hermitian conjugate of equation (3.92). Therefore, transformation equation for z^* can be written as

$$\tilde{z}^* = (-c + a^* z^*)(a + qc^* z^*)^{-1} \ . \qquad (3.93)$$

Up to now; we found the linear fractional transformation equations for z and z^*. If these two equations are multiplied, we can also find $z^* z \xrightarrow{SU_q(2)} \tilde{z}^* \tilde{z}$ such that

$$\tilde{z}^* \tilde{z} = (-c + a^* z^*)(a + qc^* z^*)^{-1}(az - qc^*)(cz + a^*)^{-1} \ . \qquad (3.94)$$

To simplify this equality, the following steps are taken

$$(a + qc^* z^*)^{-1}(az - qc^*) = (\alpha az - \beta qc^*)(\gamma a + \delta qc^* z^*)^{-1} \qquad (3.95)$$

$$(az - qc^*)(\gamma a + \delta qc^* z^*) = (a + qc^* z^*)(\alpha az - \beta qc^*) \qquad (3.96)$$

$$\gamma a^2 z + \delta qac^* zz^* - \gamma qc^* a - \delta q^2 c^* c^* z^* = \alpha a^2 z + \alpha qc^* az^* z - \beta qac^* - \beta q^2 c^* c^* z^* \qquad (3.97)$$

which means following equalities hold.

$$\gamma a^2 z = \alpha a^2 z \qquad (3.98)$$

$$\delta q^2 c^* c^* z^* = \beta q^2 c^* c^* z^* \qquad (3.99)$$

$$\delta q^2 c^* azz^* - \gamma qc^* a = -\beta q^2 c^* a + \alpha qc^* az^* z \ . \qquad (3.100)$$

By using (3.56), equation (3.100) becomes

$$\beta q^2 c^* azz^* = -\beta q^2 c^* a + \alpha q^3 c^* a + \alpha q^3 c^* azz^* \qquad (3.101)$$

Thus, if we choose $\alpha = 1$, then other coefficients can be found as

$$\gamma = \alpha = 1 \qquad (3.102)$$

$$\beta = \delta = \alpha q = q \qquad (3.103)$$

which means

$$(a+qc^*z^*)^{-1}(az-qc^*)=(az-q^2c^*)(a+q^2c^*z^*)^{-1}\ . \qquad (3.104)$$

Therefore, instead of (3.94), the following equality can also be written

$$\tilde{z}^*\tilde{z}=((-c+a^*z^*)(az-q^2c^*))((cz+a^*)(a+q^2c^*z^*))^{-1}\ . \qquad (3.105)$$

Let us look at the (1,2) and (2,1) elements of RHS and LHS matrices in (3.76). They give us the following equalities;

$$(1+\tilde{z}\tilde{z}^*)^{-1/2}\tilde{u}^*=(a+qc^*z^*)(1+zz^*)^{-1/2}u^* \qquad (3.106)$$

$$(1+\tilde{z}^*\tilde{z})^{-1/2}\tilde{u}=(cz+a^*)(1+z^*z)^{-1/2}u\ . \qquad (3.107)$$

Taking inverse of (3.106) and multiplying it by (3.107) gives the following one

$$(1+\tilde{z}^*\tilde{z})^{-1/2}\tilde{u}\tilde{u}(1+\tilde{z}\tilde{z}^*)^{1/2}=(cz+a^*)(1+z^*z)^{-1/2}uu(1+zz^*)^{1/2}(a+qc^*z^*)^{-1}\ . \qquad (3.108)$$

Since u commutes with everything and $(1+zz^*)^{-1/2}=q(1+z^*z)^{-1/2}$, above equality becomes

$$\tilde{u}^2=(cz+a^*)(a+qc^*z^*)^{-1}u^2 \qquad (3.109)$$

which is the linear fractional transformation equation for u^2. Since u is central, transformation law of u can be written as

$$\tilde{u}=\left[(cz+a^*)(a+qc^*z^*)^{-1}\right]^{1/2}u=vu\ . \qquad (3.110)$$

Here as expected, v is also unitary. This can be verified by showing that the following relation holds.

$$(a+qc^*z^*)(cz+a^*)^{-1}=(a^*+qcz)^{-1}(c^*z^*+a)\,. \qquad (3.111)$$

In order to see that this equality holds, let us take the following steps:

$$(a^*+qcz)(a+qc^*z^*)=(c^*z^*+a)(cz+a^*) \qquad (3.112)$$

$$a^*a+qa^*c^*z^*+qcaz+q^2cc^*zz^*=c^*cz^*z+c^*az^*+acz+aa^*\,. \qquad (3.113)$$

By considering the commutation relation of the elements of the unitary quantum matrices, (3.113) becomes

$$(1-cc^*)+q^2cc^*zz^*=c^*cz^*z+(1-q^2c^*c) \qquad (3.114)$$

$$cc^*(q^2zz^*-1)=cc^*(z^*z-q^2) \qquad (3.115)$$

$$q^2zz^*-1=z^*z-q^2 \qquad (3.116)$$

$$q^2zz^*-z^*z=1-q^2 \qquad (3.117)$$

$$zz^*-q^{-2}z^*z=q^{-2}-1\,. \qquad (3.118)$$

Thus, we see that starting from equation (3.111), we attain the equation (3.118) which is the commutation relation satisfied by stereographic coordinates z and z^* which implies that v is unitary.

3.6 Differential Calculus on Quantum 2-sphere in terms of z and $\bar{z}$

In this section, we will review the transformation laws for dz and operator ∂[38]. In order to see how these transformation laws can be written, first consider the following equations

$$zdz=q^{-2}dzz \qquad (3.119)$$

$$\bar{z}dz=q^{2}dz\bar{z} \qquad (3.120)$$

$$zd\bar{z}=q^{-2}d\bar{z}z \qquad (3.121)$$

$$\bar{z}d\bar{z} = q^2 d\bar{z}\bar{z} \tag{3.122}$$

$$(dz)^2 = (d\bar{z})^2 = 0 \tag{3.123}$$

$$dzd\bar{z} = -q^{-2} d\bar{z}dz\,. \tag{3.124}$$

Note that in these relations $\bar{z}$ is equal to z^* and these relations can be derived by using the relations for unitary quantum group $SU_q(2)$ and its R- matrix.

Now, define the operator d and then write the commutation relations for operator ∂.

$$d = dz\partial + d\bar{z}\bar{\partial} \tag{3.125}$$

$$\partial z = 1 + q^{-2} z\partial \tag{3.126}$$

$$\partial\bar{z} = q^2 \bar{z}\partial \tag{3.127}$$

$$\bar{\partial} z = q^{-2} z\bar{\partial} \tag{3.128}$$

$$\bar{\partial}\bar{z} = 1 + q^2 \bar{z}\bar{\partial} \tag{3.129}$$

$$\partial\bar{\partial} = q^{-2}\bar{\partial}\partial\,. \tag{3.130}$$

Above relations can be derived by using the relations (3.119-123) with the consideration undeformed Leibniz rule for d and d^2 is equal to zero. Up to now, we wrote the equations which we will need in order to derive the transformation law for dz. Now let us start derivation by considering the transformation law for z

$$z = (az + b)(cz + d)^{-1} \tag{3.131}$$

which implies

$$dz = d\left[(a + b)(cz + d)^{-1}\right] \tag{3.132}$$

By considering (3.125), the above equality can be rewritten as

$$dz = dz\partial\left[(az+b)(cz+d)^{-1}\right] + d\bar{z}\bar{\partial}\left[(az+b)(cz+d)^{-1}\right]. \quad (3.133)$$

It is clear that second part of the above summation is equal to zero. For the first part, let us take the following steps:

$$dz = dz\partial\left[(az+b)(cz+d)^{-1}\right]. \quad (3.134)$$

By considering (3.126), equation (3.134) becomes

$$dz = dz\left[a(cz+d)^{-1} + (q^{-2}az+b)\partial\ (cz+d)^{-1}\right]. \quad (3.135)$$

Now, our problem is finding $\partial\ (cz+d)^{-1}$. Since derivative of any constant is equal to zero, following equality holds.

$$\partial\left[(cz+d)(cz+d)^{-1}\right] = 0 \quad (3.136)$$

which implies

$$c(cz+d)^{-1} + (q^{-2}cz+d)\partial\ (cz+d)^{-1} = 0 \quad (3.137)$$

$$\partial\ (cz+d)^{-1} = -(q^{-2}cz+d)^{-1}c(cz+d)^{-1}. \quad (3.138)$$

If we put RHS of the above equality instead of $\partial\ (cz+d)^{-1}$ in the equation (3.135), it becomes

$$dz = dz(a-(q^{-2}az+b)(q^{-2}cz+d)^{-1}c)(cz+d)^{-1}. \quad (3.139)$$

This equality is very complicated. In order to simplify it, let us take the following steps;

$$(q^{-2}az+b)(q^{-2}cz+d)^{-1}=(\alpha cz+\beta d)^{-1}(\gamma\ az+\delta b)\ . \qquad (3.140)$$

As it is known, here $\alpha,\beta,\gamma,\delta$ are the coefficients which we shall determine.

$$(\alpha cz+\beta d)(q^{-2}az+b)=(\gamma az+\delta b)(q^{-2}cz+d) \qquad (3.141)$$

$$\alpha\, q^{-2}caz^2+\alpha\, cbz+\beta\, q^{-2}daz+\beta\, db=\gamma\, q^{-2}acz^2+\gamma\, adz+\delta\, q^{-2}bcz+\delta\, bd\ . \qquad (3.142)$$

This equality implies that following equalities also hold

$$\alpha q^{-2}caz^2=\gamma q^{-2}acz^2 \qquad (3.143)$$

$$\beta db=\delta bd \qquad (3.144)$$

$$\alpha cbz+\beta q^{-2}daz=\gamma adz+\delta q^{-2}bcz\ . \qquad (3.145)$$

By using (3.2-7), coefficients can be found as

$$\alpha=\delta=1 \qquad (3.146)$$

$$\beta=q \qquad (3.147)$$

$$\gamma=q^{-1}\ . \qquad (3.148)$$

Thus, equation (3.140) can be rewritten as

$$(q^{-2}az+b)(q^{-2}cz+d)^{-1}=(cz+qd)^{-1}(q^{-1}az+b)\ . \qquad (3.149)$$

If we use this equality in the equation (3.139), then it becomes

$$dz=dz(a-(cz+qd)^{-1}(q^{-1}az+b)c)(cz+d)^{-1}\ . \qquad (3.150)$$

This equality is still complicated, therefore we should continue to simplification by taking the following steps

$$dz = dz(a-(q^{-1}cz+d)^{-1}q^{-1}(q^{-1}az+b)c)(cz+d)^{-1} \tag{3.151}$$

$$dz = dz(q^{-1}cz+d)^{-1}((q^{-1}cz+d)a-q^{-1}(q^{-1}az+b)c)(cz+d)^{-1} . \tag{3.152}$$

Let us say x instead of $(q^{-1}cz+d)a-q^{-1}(q^{-1}az+b)c$ and try to find what x is

$$x = (q^{-1}cz+d)a-q^{-1}(q^{-1}az+b)c \tag{3.153}$$

$$x = q^{-1}caz+d-q^{-2}acz-q^{-1}bc . \tag{3.154}$$

Since $ac = qca$ and $ad = da-(q-q^{-1})bc$ equalities are hold, equation (3.154) becomes

$$x = ad-qbc = 1 \tag{3.155}$$

which means that equation (3.152) can be rewritten as

$$dz = dz(q^{-1}cz+d)^{-1}(cz+d)^{-1} . \tag{3.156}$$

Since d is invariant and it is defined with the equation (3.125), transformation law of the operator ∂ can be observed easily in the following form

$$\partial \rightarrow (cz+d)(q^{-1}cz+d)\partial . \tag{3.157}$$

4. CONCLUSION

In this study, we reviewed the special unitary quantum group $SU_q(2)$ in the matrix representation and discuss the parametrization of this quantum group. We saw that the representation of $SU_q(2)$ by the operator a and c obeying commutation relations (3.12-18) and (3.34) can be regarded as a q-oscillator ($q<1$) and $c=u(1-a^*a)^{1/2}$ where u is central. It was also seen that all the commutation relations which are satisfied by the elements of $SU_q(2)$ matrices can be derived by only using (3.11) and (3.19).

The main object of this thesis was to discuss the parametrization of $SU_q(2)$ in terms of the stereographic quantum coordinates of the quantum sphere $SU_q(2)/U(1)$ z, z^* and its central unitary phase u. We observed that since z obeys the commutation relation (3.56), this representation gives a q^{-1}-oscillator ($q^{-1}>1$) representation of $SU_q(2)$. Hence, it is also observed that from the point of view of $SU_q(2)$, the q-oscillator ($q<1$) and q^{-1}-oscillator ($q^{-1}>1$) are just different "coordinates" of the same algebraic structure. For both cases, if q goes to one, then commutation relations (3.34) and (3.56) reduce to ordinary commuting number relations.

In section 3, we also derived the transformation laws for z, z^* and u respectively such that the transformations given by (3.92) and (3.110) completely define the action of the quantum group $SU_q(2)$ on the "coordinates" z and u of $SU_q(2)$. For $q\to 1$, these transformation laws reduce to their classical form.

Finally, we superficially studied transformation law for differential structures on S_q^2. Studying this part in detail will be worthwhile for future.

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