INVESTIGATING PRESERVICE TEACHERS' CONTENT KNOWLEDGE ON MULTIPLICATION AND DIVISION

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Dedicated to my mom and my sister

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ABSTRACT

INVESTIGATING PRESERVICE TEACHERS' CONTENT KNOWLEDGE ON MULTIPLICATION AND DIVISION

The aim of the current study was to investigate preservice teachers' content knowledge on multiplication and division. The study was conducted in five universities in Turkey, during the Spring semester of 2021-2022 academic year. A total of 111 senior university students from middle school mathematics, middle school science, secondary school mathematics, chemistry, and physics education programs participated in the study. The instrument including fifteen items developed by the researcher was used as a data collection source. Before the items on the instrument were developed, three indicators of understanding of multiplication and division multiplicatively were determined in light of the literature. Data were analyzed descriptively to examine preservice teachers' performances on each item. Results regarding understanding of a quantity revealed that preservice teachers experienced difficulties in specifying the quantities and determining the referents of the quantities, especially of the intensive quantities in multiplication and division problems. Also, results related figural representation indicated that preservice teachers were not able to multiplicatively explain the pictorial figures they drew in equal groups problem situations especially. Similarly, results regarding understanding of multiplicative problem situations showed that although preservice teachers were able to construct multiplication or division problems, they were not able to provide multiplicative explanations requiring for related problems. Results also suggested that preservice teachers were not successful at explaining multiplicative relationship between the quantities in the problems. It was also concluded that they were inclined to make explanations in a procedural way.

ÖZET

ÖĞRETMEN ADAYLARININ ÇARPMA VE BÖLMEYE İLİŞKİN ALAN BİLGİLERİNİN İNCELENMESİ

Bu çalışmanın amacı öğretmen adaylarının çarpma ve bölmeye ilişkin alan bilgilerini incelemektir. Çalışma, 2021-2022 akademik yılının Bahar döneminde, Türkiye'de yer alan beş farklı üniversitede gerçekleştirilmiştir. Çalışmaya ilköğretim matematik, fen bilgisi, ortaöğretim matematik, kimya ve fizik öğretmenliği programlarında öğrenim gören toplam 111 üniversite son sınıf öğrencisi katılmıştır. Verileri toplamak için araştırmacı tarafından hazırlanan on beş maddelik bir ölçme aracı kullanılmıştır. Ölçme aracındaki maddeler geliştirilmeden önce ilgili literatür ışığında çarpma ve bölmeyi çarpımsal olarak anlamanın bileşeni olan üç gösterge belirlenmiştir. Toplanan veri, öğretmen adaylarının her bir maddedeki performanslarını incelemek için betimsel istatistiklerle analiz edilmiştir. Elde edilen sonuçlar, öğretmen adaylarının çarpma ve bölme problemlerinde nicelikleri belirlemede ve niceliklerin birimlerini saptamada güçlük yaşadıklarını ortaya koymuştur. Sonuçlar ayrıca, öğretmen adaylarının özellikle eşit gruplar içeren problem durumları için çizdikleri model temsillerini çarpımsal olarak açıklayamadıklarını göstermiştir. Çarpımsal problem durumlarının anlaşılmasına ilişkin sonuçlar, öğretmen adaylarının çarpma ve bölme problemleri tasarlayabilmelerine rağmen ilgili problemler için gerekli olan çarpımsal açıklamaları yapamadıklarını göstermiştir. Ek olarak sonuçlar, öğretmen adaylarının problemlerdeki nicelikler arasındaki çarpımsal ilişkiyi açıklamada başarılı olmadıklarını ortaya koymuştur. Ayrıca, öğretmen adaylarının işlemsel şekilde açıklama yapma eğiliminde oldukları belirlenmiştir.

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LIST OF SYMBOLS

C	Continuous
cm	Centimeter
cm^2	Square Centimeter
D	Discrete
E	Extensive Quantity
E'	Extensive Quantity
E''	Extensive Quantity
G	Gibbs Energy
ha	Hectare
hr	Hour
I	Intensive Quantity
kg	Kilogram
m	Mass
m	Meter
m^2	Square Meter
mi	Mile
p	Pressure
ppm	Parts Per Million
Q	Physical Quantity
{Q}	Numerical Value of Physical Quantity
[Q]	Unit of Physical Quantity
T	Temperature
V	Volume
μ	Partial Molar Gibbs Energy
κ	Cohen’s kappa coefficient
p_0	The proportion of units in which the scorers agree
p_e	The proportion of units for which agreement is expected by chance

LIST OF ACRONYMS / ABBREVIATIONS

COVID-19	Coronavirus Disease 2019
CTT	Classical Test Theory
CCK	Common Content Knowledge
GHG	Greenhouse Gas
HCK	Horizon Content Knowledge
KCS	Knowledge of Content and Students
KCT	Knowledge of Content and Teaching
MKT	Mathematics Knowledge for Teaching
NCTM	National Council of Teachers of Mathematics
SCK	Specialized Content Knowledge
SMK	Subject Matter Content Knowledge
PCK	Pedagogical Content Knowledge
PCHT	Preservice Chemistry Teachers
PMMT	Preservice Middle School Mathematics Teachers
PSCT	Preservice Science Teachers
PSMT	Preservice Secondary Mathematics Teachers
PPHT	Preservice Physics Teachers
RIR	Item-rest Correlation

1. INTRODUCTION

Some researchers referred to multiplication as “one of two fundamental operations, along with addition, which can be defined so that it is an appropriate choice for representing and solving problems in many different situations” (Otto, Caldwell, Lubinski, & Hancock, 2011, p. 8). These situations are also related to different disciplines other than mathematics such as science including physics, and chemistry. For instance, the problem situation “Jose drives at an average speed of 50 miles per hour for 6 hours. How many miles does Jose drive?” is an example of a rate problem confronted in science contexts (Otto, et al., 2011, p. 21) as well as mathematics contexts. Otto et al. (2011) also defined division by “its inverse relationship with multiplication” (p. 23).

Multiplication and division are important in the learning of mathematics and science concepts. They underlie many crucial mathematical domains such as measurement, ratio, rate, fractions, proportionality, proportional reasoning, and algebraic thinking (Thompson & Saldanha, 2003; Lamb & Booker, 2004; Otto, et al., 2011; Hino & Kato, 2019; Izsák & Beckmann, 2019). These domains are mostly involved in science including physics and chemistry as well as mathematics.

Understanding multiplicative concepts such as multiplication, division, rational number, etc. are crucial for both children and teachers to obtain mathematical understanding (Behr, Khoury, Harel, Post, & Lesh, 1997). Also, researchers point that understanding of multiplication and division multiplicatively is the backbone of multiplicative reasoning which is necessary for students to conceptualize ratio and proportion (Simon and Placa, 2012). Similarly, National Council of Teachers of Mathematics (NCTM, 2000) asserted that “multiplicative reasoning is more than just doing multiplication and division” (p. 143) as it further involves understanding intensive quantities generated in multiplication and division (Schwartz, 1988). Moreover, multiplication and division concepts “provide a foundation for mathematics far beyond grades 3-5, it is critical that students have opportunities to develop the necessary understanding for future mathematics learning” (Otto et al., 2011, p. 81).

According to Otto et al. (2011), the ability to making sense of the factors in a multiplicative expression is one of the characteristics of the proper understanding of multiplication (also division). Similarly, Smith and Smith (2006) highlight the importance of the understanding of quantity on the foundation for understanding of multiplication. In other words, the ability to correctly explain numerical value and unit of each quantity in multiplicative expression provides a justification for why multiplication or division is used as an operation. In the problem "A farm of 45.8 ha procedures 6850 kg of corn per ha. What will the yield?", multiplication operation for solution is shown as "45.8 (ha) × 6850 (kg per ha) = 313 730 (kg)" by pointing to the units of the quantities (Vergnaud, 1983, p. 129). To exemplify, for someone who has difficulty in conceiving the quantities may write a multiplication expression like "45.8 (ha) × 6850 (ha) = 313 730 (kg)". Even if this person performs a multiplication operation, it cannot be said that she conceptualizes multiplication since the unit of resulting quantity cannot be "kg" in this expression in terms of the nature of multiplication. This is also the case in division.

Furthermore, Smith and Smith (2006) specify that understanding of multiplicative problem situations is also important for understanding of multiplication. For making sense of multiplication and division problems, someone also need to have the ability to write multiplicative word problems. For instance, in the study of Simon (1993), preservice teachers were asked to write a story problem that includes dividing 3/4 by 1/4 to assess their conceptual knowledge of division of fractions. This also gave information about preservice teachers' knowledge of the meanings of division (i.e., partitive and quotative division).

Likewise, to able use multiple representations in describing multiplicative situations and understand the relationship between different representations also provide developing a robust understanding of multiplication (and division) (Otto et al. 2011). Additionally, representations created by students provide opportunity for teachers to have an idea about how students understand quantities and operations (Quintero, 1986). Explanations can also accompany with these representations since someone who conceptualizes multiplication and division is able to multiplicatively describe multiplicative situations (Otto et al., 2011). For example, "There are 5 shelves of books in Dan's room. Dan put 8 books on each shelf. How many books are there in his room?" is a multiplicative problem solved by using multiplication (Fischbein, Deri, Nello, & Marino, 1985). This situation can be multiplicatively explained as “the number of books in the room is 5 times as large as the number of books on each shelf”. This explanation also provides realizing multiplicative relationship between the quantities in the situation. Additionally, as Thompson and Saldanha (2003) states, understanding multiplication multiplicatively includes understanding of reciprocal relationships between the quantities like “the number of books on each shelf is 1/5 times as large as the number of books in the room”.

Albeit the importance in the learning of mathematics and science, many studies examining students’ understanding of multiplication and division revealed that students have difficulties and misconceptions about both operations (Fischbein, Deri, Nello, & Marino, 1985; Smith & Smith, 2006; Simon, Kara, Norton, & Placa, 2018; Wahyu, Kuzu, Subarinah, Ratnasari, & Mahfudy, 2020). Particularly, the intuitive meanings of multiplication and division have some limitations that arise as misconceptions (Fischbein et al., 1985; Schwartz, 1988; Tirosh & Graeber, 1989; Simon & Placa, 2012). In addition, previous research has pointed that preservice and in-service teachers also have deficient understanding and misconceptions related to multiplication and division (Graeber, Tirosh, & Glover, 1989; Ball, 1990; Simon, 1993; Byerley & Thompson, 2014).

As students’ learning and teaching activities in classrooms are highly influenced by what a teacher knows about what she is teaching (Fennema & Franke, 1992), teachers’ understanding of multiplication and division should transcend the content they intend they teach their students (Otto et al., 2011). For instance, the study of Lamb and Booker (2004) indicated that in the class of a teacher who had limited conceptual knowledge, students’ explanations were based on the procedural understanding of the division operation. Results suggest that teachers should also understand multiplication and division multiplicatively beyond the primitive meanings of these operations to realize students’ misconceptions about the related concepts. Although they do not directly teach multiplication and division, this is also the case for middle school science and secondary school mathematics, chemistry, and physics teachers since the contents that they teach include and are based on multiplication and division and those concepts are closely linked to each other specifically in terms of multiplicative reasoning.

It is in this respect that, the aim of this study is to investigate preservice middle school mathematics, middle school science, secondary school mathematics, physics, and chemistry teachers' content knowledge revealing if they understand multiplication and division multiplicatively. In particular, the following research questions will be scrutinized in this study.

1.1. Research Questions

R.Q.1. What is the content knowledge of preservice middle school mathematics, middle school science, secondary school mathematics, chemistry, and physics teachers on multiplication and division?

- R.Q.1.a. How do preservice teachers perform about understanding of a quantity?
- R.Q.1.b. How do preservice teachers perform about figural representation?
- R.Q.1.c. How do preservice teachers perform about understanding of multiplicative problem situations?

1.2. Significance of the Study

This study contributes to education literature in many ways. Firstly, it will provide a different perspective in terms of examining the understanding of multiplication and division through specific components. For instance, the grasp of quantities (especially intensive quantity) has of great importance for conceptual understanding of multiplication and division since these operations have a nature of transforming referents, so producing new quantity (Schwartz, 1988).

Secondly, this study might contribute to determining the current situation of preservice teachers' content knowledge on the concepts of multiplication and division. This may lead to further studies examining science including physics and chemistry teachers' understanding of the other concepts requiring multiplicative meanings of multiplication and division. Similarly, teachers and curriculum developers might use the items in the instrument for teaching multiplication and division.

Finally, it is significant to investigate whether preservice teachers conceptualize multiplication and division operations multiplicatively. This might shed light on whether their understanding transcends the domains they will teach based on multiplication and division. This in turn might have implications to make some regulations in teacher training programs.

2. LITERATURE REVIEW

This chapter first provides operational definition of quantity from both the point of view of mathematics and science as it is important in the learning of multiplication and division. Then, the chapter continues with operational definitions of multiplication and division based on the quantities involved in these operations as well as previous research on both students and teachers' understanding of them.

2.1. Quantity

The notion of quantity has been highly emphasized in the literature and defined variously by researchers from different fields including mathematics (Schwartz, 1988; Thompson, 1990; Steffe, 1991; Kaput, 1995; Smith & Thompson, 2008; Moore, Carlson, & Oehrtman, 2009), physics and chemistry (Cohen, Cvitaš, Frey, Holström, Kuchitsu, Marquardt, Mills, Pavese, Quack, Stohner, Strauss, Takami, & Thor, 2007). Quantity has of paramount importance in the learning of number operations such as addition, subtraction, multiplication, and division (Nunes, Dorneles, Lin, & Rathgeb-Schnierer, 2016), in the learning of algebra (Silverman and Thompson, 2008), and the understanding of the concepts like rate and slope (Johnson, 2013). As a specific example from science, González (in press) asserted that:

> "Understanding the energy budget requires developing meaning for the quantities representing the abundance of GHG in the atmosphere and the intensity of the energy flows between the Sun, the surface, and the atmosphere, as well as the relationships that exist between such quantities".

As the focus of this study is on the operations of multiplication and division, in this section, using examples, definitions of quantity from different fields, the types of quantities, and how quantities are generated in relation to the operations acted upon quantities are provided.

Particularly, in physics, quantities are defined as physical quantities, and "the value of a physical quantity Q can be expressed as the product of a numerical value {Q} and a unit

[Q]" (Cohen et al., 2007, p. 3). For example, "12 kg" refers to the quantity "mass" as the product of the numerical value 12 and the unit "kg". Moreover, physical quantities are divided into the base (fundamental) quantities and derived quantities such that "by convention physical quantities are organized in a dimensional system built upon seven *base quantities*, each of which is regarded as having its own dimension" (Cohen et al., 2007, p. 4). Those seven base quantities are namely length, mass, time, electric current, thermodynamic temperature, amount of substance, and luminous intensity (Cohen et al., 2007). Cohen et al. (2007) also pointed that "all other quantities are called *derived quantities* and are regarded as having dimensions derived algebraically from the seven base quantities by multiplication and division" (p. 4). For instance, area, volume, density, velocity, and force are examples of derived quantities. Similarly,

> "the quantities used in mathematics are derived from the surround by acts of either counting or measuring, depending on whether we are quantifying discrete or continuous properties of the surround. Alternatively, they may be derived from counted and measured quantities by the successive application suitably defined mathematical operations" (Schwartz, 1988, p. 41).

Schwartz (1988) further stated that all quantities that are created by using these methods have referents. For example, in the quantity "5 kg" which represents the weight of flour, "5" shows the numerical value and "kg" shows the referent of the quantity. Mathematical operations stated above in the second way of acquiring quantities are called *referent preserving composition* and *referent transforming composition* (Schwartz, 1988). Referent preserving compositions refers to "composing two like quantities to produce a third like quantity is fundamentally the sort of composition of quantities that the arithmetic acts of addition and subtraction afford" (Schwartz, 1988, p. 41). So, directly countable and measurable quantities are called extensive quantities composition of which also producing an extensive quantity (Schwartz, 1988). Therefore, extensive quantity is additive (Steffe, 1991). For instance, length, mass, area, and volume can be given as examples of this type of quantity. Moreover, as Schwartz (1988) explained, extensive quantities may be the form of either discrete (D) or continuous (C). For instance, while "child, candy, and bag" are discrete, "gram, year, and hour" are continuous extensive quantities (Schwartz, 1988).

On the contrary, “composing two, either like or unlike, quantities to produce a third quantity that is, in general, like neither of the two original quantities is referred to as referent transforming composition. Multiplication and division are referent transforming compositions” (Schwartz, 1988, p. 41). So, quantities that are not directly measurable are called intensive quantity (Schwartz, 1988). Furthermore, it is defined as “a generalization of the notion of an attribute density” (Schwartz, 1988, p. 43); “in other words, the intensity of a trait” such as speed, temperature, and color are intensive quantities (Simon & Placa, 2012, p. 35). Thompson (1990) defined intensive quantities as “quantities whose measures are non-additive as in normal arithmetic” (p. 5) based on the terminology of Cohen and Nagel (1939). “Temperatures, densities, and frequencies are not additive in general” (Thompson, 1990, p. 5). Schwartz (1988) asserted that an intensive quantity can be regarded as a relationship between two extensive quantities and that these three quantities are related to one another via multiplication or division. For instance, while “20 cookies” and “5 boxes” represent extensive (i.e., counted) quantities, “20 cookies/5 boxes” represents an intensive quantity whose referent is neither cookies nor boxes. A division operation is done to find how many cookies are in every box and “4 cookies for every box” or “4 cookies per box” that is (4 cookies/boxes) is obtained. Besides, when (4 cookies/boxes) and (5, boxes) are given, (20, cookies) can be formed by multiplying intensive and extensive quantities. In this instance, “cookies/boxes” (D/D) represents the intensive quantity formed by the relationship between two discrete extensive quantities. Additionally, “liters/bottle” (C/D) or “children/month” (D/C) represents the intensive quantity formed by the relationship between discrete and continuous extensive quantities. Finally, “kilometers/minute” (C/C) represents the intensive quantity formed by the relationship between two continuous extensive quantities. Extensive and intensive quantities are also defined in physics. Cohen et al. (2007) stated that:

> “A quantity that is additive for independent, noninteracting subsystems is called *extensive*; examples are mass m, volume V, Gibbs energy G. A quantity that is independent of the extent of the system is called *intensive*; examples are temperature T, pressure p, chemical potential (partial molar Gibbs energy) μ” (p. 7).

Although Thompson does not disagree with Schwartz’s ideas, Thompson’s notion of quantity as a cognitive entity differs from the aforementioned definitions. That is, Thompson (2011) specifically highlighted “the point that quantities are mental constructions, and that their creation is often effortful, is central to mathematics education” (p. 34). Thompson

(1990) defined quantity as “a quality of something that one has conceived as admitting some measurement process” (p. 4) with explicitly or implicitly conceiving of an appropriate unit. In other words, once an individual considers a phenomenon including objects with measurable qualities, then she has conceived a quantity in her mind. Thus, a quantity is schematic in mind such that there is an object, quality of the object, and an appropriate unit to measure such object, and a process by which one might assign a numerical value to it (Thompson, 1994). This process is called quantification (Thompson, 1990; 1994). Though, for someone to comprehend a quantity, she does not necessarily explicitly need to measure, “rather, the only prerequisite for a conception of a quantity is to have a process in mind” (Thompson, 2011, p. 35). That is to say, it is not required to determine a particular numerical value (amount of measure) of the quality to understand the quantity (Johnson, 2014). So, quantity exists independently of the numbers. Based on the explanations above, for instance, when an individual considers the area as a measurable quality of rectangle, the individual can envision that “the area of a rectangle as the amount of flat surface “covered” by the rectangle without determining the area of the rectangle” (Johnson, 2014, p. 268). Hence, the area can be conceived as a quantity. When quantification is considered in the context of climate change, for instance, the object conceptualized can be “the atmosphere”, measurable quality of this object can be “the relative abundance of atmospheric GHG”, and unit of measure for the quality can be “ppm” González (in press).

Although taking into consideration the definitions provided by Cohen et al. (2007) and Schwartz (1988), in this study, Thompson’s definition of quantity is used. This is especially because, as Schwartz’s (1988) characterization of quantities is based on the ordered pairs of the form (number, unit), Thompson (1994) argued that “to characterize quantities as ordered pairs may be useful formally, but it does not provide insight into what people understand when they reason quantitatively about situations, and it severely confounds notions of number and notions of quantity (Thompson, 1989, in press)” (p. 188). In addition, in this study, it is focused on preservice teachers’ meanings of multiplication and division as referent transforming operations. So, preservice teachers’ being attentive to the notion of quantity especially the intensive quantity as they conceive it would be of paramount importance. In particular, in this study, preservice teachers’ meaning of the two types of intensive quantity is examined specifically. These are called the scale conversion factor (Schwartz, 1988) and unit measure conversion factor (Schwartz, 1988; Simon & Placa, 2012). Firstly, consider the following problem that "The weight of Alex is 45 kg, and the weight of Alex's father is twice as much. What is the weight of Alex's father?". In this example, the intensive quantity represented by "2 kg/kg" is the scale conversion factor. By multiplying the weight of Alex (extensive quantity) and this intensive quantity, an unknown weight (i.e., 90 kg) is obtained. So, as an intensive quantity, "2 kg/kg", "does not change the nature of the referent on which it operates. However, it does change the magnitude of its measure" (Schwartz, 1988, p. 49). Secondly, consider the extensive quantity "3 m" whose attribute (i.e., quality) is "the length of a wardrobe". If the length of the wardrobe is measured in centimeter rather than meter, the same attribute of the same wardrobe is still described. That is, the quantity "100 cm/m" whose attribute is "the length conversion factor" is intensive quantity. As a result, the extensive quantity "300 cm" that has the same attribute (i.e., the length of the wardrobe) is generated. So, when a unit measure conversion factor is used in a multiplication or division, "it does not change the nature of the referent but only the numerical description of its measure" (Schwartz, 1988, p. 49). Thus, as Thompson and Saldanha (2003) stated, "a change of unit does not change the quantity's magnitude" (p. 101). So, in this study, whether preservice teachers are attentive to these (intensive) quantities as well as the extensive quantities while solving problems of the type like the ones shared above or if they could determine their meaning in making sense of an explanation of a problem solution would be important. In the next section, the multiplication and division based on the quantities inherent in these operations are explained.

2.2. Multiplication and Division

In this part, a definition of multiplication and division from a mathematical view is provided. Multiplication as a referent transforming composition in producing quantities (Schwartz, 1988), is one of the fundamental arithmetic operations used to solve real-world problems (Otto, et al., 2011). So, it generates new quantities having referents. For instance, a scalar product of two vectors "force" and "displacement" produces a new quantity "work" that is neither force nor displacement, and it is not also vector (White-Brahmia & Olsho, in press). Thompson (1990) identified multiplication as an arithmetic operation to evaluate (to calculate the numerical value of) the quantity that is "the result of a multiplicative combination of two quantities" (p. 24). Any situation that necessitates a multiplication operation can be written as an equation "$A \times B = C$" to represent the multiplicative relationship between the quantities of the situation. In this equation, "$A \times B$" is called multiplicative expression (Otto et al., 2011), and while A and B are factors, C is the product of multiplication (Van de Walle, Karp, & Bay Williams, 2010).

Similarly, division is another referent transforming composition creating a new quantity (Schwartz, 1988), which also as an arithmetic operation evaluates a quantity as "the result of a multiplicative comparison of two quantities" (Thompson, 1990, p. 24). Besides, Otto et al. (2011), approached division as having an inverse relationship with multiplication. The equation above "$A \times B = C$" equals "$C \times (1/A) = B$" in which $1/A$ represents the multiplicative inverse of A. So, the number "B" is found through dividing C by A (i.e., $C \div A$, for $A \neq 0$). This relationship is explained in detail in the following sections.

2.2.1. Multiplicative Situations Modeled by Multiplication and Division

Many researchers classified the multiplicative situations as those where multiplication and division are used (Vergnaud, 1983; Schwartz, 1988; Nesher, 1988; Greer, 1992; Watanabe, 2003; Van de Walle et al., 2010; Otto et al., 2011). In this study, preservice teachers' meanings of multiplication and division (i.e., primitive meanings and conceptual understanding) through the representations of the multiplicative situations will be examined. This is important because Otto et al. (2011) emphasized that "each multiplicative expression developed in the context of a problem situation has an accompanying explanation, and different representations and ways of reasoning about a situation can lead to different expressions or equations" (p. 12). Hence, examining different multiplicative situations or structures are needed to understand various representations and expressions preservice teachers might utilize while solving the problems. This might allow how preservice teachers make sense of and view multiplication and division from a broader perspective.

Multiplicative situations containing whole numbers or positive integers are mainly classified into four groups: *equal groups or sets, multiplicative comparison, rectangular array/area, and cartesian product (combination)* (Greer, 1992; Watanabe, 2003; Van de Walle et al., 2010).

2.2.1.1. Equal Groups. "Equal groups" is the most given and emphasized situation in introducing multiplication and division both in the literature and the textbooks (Watanabe, 2003; Greer, 1992; Fujii & Iitaka, 2012). This situation involves the number of groups of equal size some examples of which,

> "… are the mathematization of cases of natural replication (for example, n people have 5n fingers), repetition of a sequence of actions (for example, taking three steps four times), and human practices such as giving the same number of objects to a number of people" (Greer, 1992, p. 276, 277).

Consider the following problem "3 children have 4 cookies each. How many cookies do they have altogether?" (Greer, 1992, p. 276). In this problem, there is the number of groups (i.e., 3 children) having the same number of cookies (i.e., 4 cookies) in each group, so the total number of cookies can be obtained by multiplication "3 × 4". Additionally, Watanabe (2003) specified that Japanese textbooks start the multiplication unit by demonstrating the pictures containing both equal and nonequal groups. For instance, at the beginning of the unit of the Japanese textbook called "Tokyo Shoseki's Mathematics International", there is a picture of children on various rides (Fujii & Iitaka, Grade 2, 2012). All rides (groups) are equal sized except for the teacups ride because there are different numbers of children on each teacup. Thus, it is aimed to have students comprehend that when there are equal-sized groups, multiplication can be used whereas multiplication cannot be used for unequal-sized groups. If there are 4 children in each gondola and there are 3 gondolas, the total number of children on gondolas is obtained by "4 × 3", so there are 12 children altogether.

Greer (1992) also proposed that "an alternative way of conceptualizing the equal groups situation is in terms of a rate" (p. 277). For instance, "If there are 4 cookies per child, how many cookies do 3 children have?" (Greer, 1992, p. 277) where an implicit invariant relationship exists between number of children and number of cookies such that the number per group is multiplied by the number of groups to find the total number. Greer (1992) further stated that "… the situation described in the example is the particular instantiation of this relationship when the number of children is 3" (p. 277).

The numbers in the multiplication expression of the equal groups situation have distinct roles. While the number of groups is called the *multiplier*, the number of objects in each group is called the *multiplicand* (Greer, 1992; Watanabe, 2003; Fujii & Iitaka, 2012). The distinction is mostly highlighted in the Japanese textbook series (Watanabe, 2003). Students are encouraged to write the quantities in math sentences to realize the distinction between multipliers and multiplicands. This distinction influences the implementation of multiplication (Watanabe, 2003). For instance, when the two examples above about cookie and gondola are examined, it is realized that there is a difference between them in terms of the order of the multiplicand and multiplier. In the multiplicative expression of the former example, the multiplier (i.e., 3) is written before the multiplicand (i.e., 4). Watanabe (2003) reported that this is the case in the traditional U.S. system. On the contrary, the multiplicand (i.e., 4) is written before the multiplier (i.e., 3) in the expression of the latter example. Watanabe (2003) stated that the reason for this difference is language difference,

> “When one says "5 times 3" in English, it seems to be more natural to interpret this as "5 times" of a group of 3. Similarly, in Japanese, "5 ka-ke-ru 3" more naturally implies the set of 5 three times” (p. 118).

That is to say, the expression of the example of the cookie (3×4) can be interpreted as “3 times of a group of 4” and the expression of the example of the gondola (4×3) can be interpreted as “4 ka-ke-ru 3” (i.e., the set of 4 three times). Additionally, Anghileri (1989) explained the expression in which the multiplier is written before the multiplicand as “times” and the expression in which the multiplicand is written before the multiplier as “multiplied by”. Then, 3×4 read as “3 times 4” specifies “$4 + 4 + 4$” and 3×4 read as “3 multiplied by 4” specifies “$3 + 3 + 3 + 3$” (Watanabe, 2003). Although both multiplication sentences (i.e., 3×4) are the same, their interpretations are different. Besides, “the phrase "multiplied by" is considered to be mathematically ‘correct’ because it conforms with the conventions for addition, subtraction and division. Each word or phrase implies that the first number in an expression is operated upon by the second” (Anghileri, 1989, p. 368). This view corresponds to the terms which are operand (i.e., the size of each group) and operator (i.e., the number of equal-sized groups) used by Fischbein et al. (1985). It can be summarized that *the operand* is *the multiplicand*, and *the operator* is *the multiplier*.

Division can be also utilized for equal groups situations depending on the context of the problem. For example, like in multiplication, both equal and non-equal sharing situations are demonstrated in the picture in the beginning of the division unit of the aforementioned Japanese textbook. In this example, students are expected to "discuss the difference between how the fried noodles and the juice are divided" (Fujii & Iitaka, Grade 3, 2012, p. A24). In the first picture, noodles are unequally shared among four people, and in the second one, the juice is equally shared among four people. Hence, it is highlighted that division includes equal sharing of quantities.

2.2.1.2. Multiplicative Comparison. "Multiplicative comparison" includes the situation that "one quantity is so many times more than another quantity" (Watanabe, 2003, p. 113). For example, "Jill picked 6 apples. Mark picked 4 times as many apples as Jill. How many apples did Mark pick?" (Van de Walle et al., 2010, p. 155). Here, "how many times" (i.e., multiplier) and "the base quantity" (i.e., multiplicand) are given and "the quantity compared" is expected to be found. The number of apples Mark picked is acquired by 4×6. Greer (1992) called the multiplier (i.e., 4) in this expression as a multiplicative factor. In the Japanese textbook named "Tokyo Shoseki's Mathematics International", the association between "times as much" and multiplication is presented (Fujii & Iitaka, Grade 2, 2012). Firstly, students are asked to color in the tape 3 times as long as the length of the given tape. Then, the textbook gives the length of the given tape (e.g., 2 cm), and the students are asked to write math sentence to find the length of the tape they color. The length of this tape is found by "$2 \times 3 = 6$ cm". Thus, it is provided that students can conceive the multiplicative comparison in the multiplication operation. If "the quantity compared" is given and "how many times" or "the base quantity" is asked, then division operation is used in this type of situations.

2.2.1.3. Rectangular Array/Area. "Rectangular array/area" refers to situations that:

> "the total number of items arranged in an array can be obtained by multiplying the number of rows and the number of columns. Equivalently, the area of a rectangle can be obtained by multiplying the length and the width of the rectangle" (Watanabe, 2003, p. 113).

If the situation gives the area of the rectangle and the length of one of the sides and asks the other side, it is solved by division.

2.2.1.4. Cartesian Product. “Cartesian product” problem situations also referred to combination, combinatorics, and counting problems (Van de Walle et al., 2010; Otto et al., 2011). It comprises the situation that “given two sets, the total number of pairs one can make by selecting one item from one set and one from the other can be obtained by multiplying the numbers of items in the sets” (Watanabe, 2003, p. 113). If there are five pants and three blouses, the number of different combinations of pant and blouse outfits can be obtained by 5×3. If the number of the combination and the one set are given, the other set is found by division.

Although this classification works well with whole numbers, Greer (1992) further classified the multiplicative situations involving fractions and decimals in addition to the multiplicative situations including positive integers. This classification is demonstrated below in Table 2.1.

Table 2.1. Multiplicative situations classified by Greer (1992).

Positive Integers	**Fractions**	**Decimals**
Equal Groups	Rational Rate	Rate Measure Conversion
Multiplicative Comparison	Multiplicative Comparison Part/Whole	Multiplicative Comparison Part/Whole Multiplicative Change
Rectangular Array/Area	Rectangular Area	Rectangular Area
Cartesian Product		Product of Measures

Particularly, Greer (1992) classified *equal groups situations* into “rational rate” (involving fractions) and “rate” and “measure conversion” (involving decimals); *multiplicative comparison situations* into “multiplicative comparison” and “part/whole” (involving fractions) and “multiplicative comparison”, “part/whole”, and “multiplicative change” (involving decimals); *rectangular array/area situations* into a rectangular area (involving fractions and decimals); *cartesian product situations* into “product of measures” (involving decimals). In this study, this classification of situations is adopted to model multiplicative problems including whole numbers or positive integers, fractions, and decimals. Greer (1992) also presented the correspondence of the categories identified by Vergnaud (1983, 1988) and Schwartz (1988) with his classification.

Isomorphism of measures which is one of three main multiplicative problem structures defined by Vergnaud (1983) corresponds to the Greer’s (1992) *equal groups situations* and *multiplicative comparison situations* containing *positive integers* or *whole numbers*, *fractions*, and *decimals*. “Isomorphism of measures covers all situations where there is a direct proportion between two measure spaces M_1 and M_2” (Greer, 1992, p. 282). In this structure illustrated in Figure 2.1 drawn by the researcher of this study, the quantities (a, b, and x) can be positive integers, fractions, or decimals (Greer, 1992). Depending on which one of the three quantities is unknown, multiplication and division are used to solve the problem.

M_1	M_2
1	a
b	x

Figure 2.1. Isomorphism of measures.

It is already told above that Schwartz (1988) stressed both extensive quantity (E) and intensive quantity (I) in the multiplicative expressions. Greer (1992) declared that the problems having the structures “$I \times E = E'$” (multiplication), “$E' / E = I$” (division) and “E'

/ $I = E$" (division) by Schwartz (1988) corresponds to Vergnaud's (1983) *isomorphism of measures* and Greer's (1992) *equal groups situations* and *multiplicative comparison situations* containing *positive integers* or *whole numbers*, *fractions*, and *decimals*. This (I E E') semantic triad asserted by Schwartz are explained later.

Additionally, the *product of measures* which is one of three main multiplicative problem structures defined by Vergnaud (1983) corresponds to Greer's (1992) *rectangular array/area situations* and *cartesian product situations* containing *positive integers* or *whole numbers* and *decimals*. This structure "consists of the cartesian composition of two measure-spaces, M_1 and M_2, into a third, M_3" (Vergnaud, 1983, p. 134) as demonstrated in Figure 2.2 drawn by the researcher of the study.

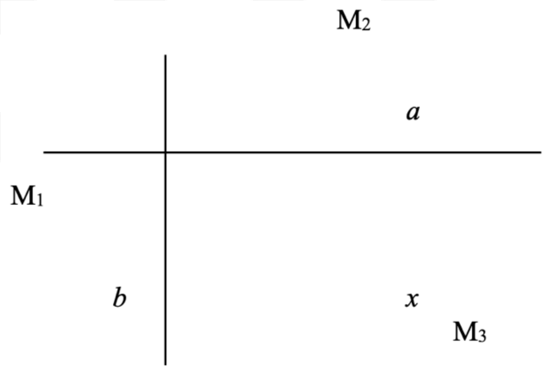

Figure 2.2. Product of measures.

Furthermore, Schwartz's (1988) multiplicative structures "$E \times E' = E''$" (multiplication), "$E'' / E' = E$" (division), and "$E'' / E = E'$" (division) corresponds to *product of measures* by Vergnaud (1983). This (I E' E'') semantic triad also correspond to Greer's (1992) *rectangular array/area situations* and *cartesian product situations* containing *positive integers* or *whole numbers* and *decimals*. These correspondences are elaborated in the following section. He emphasized the property of this triad as "the multiplication operation maps a quantity in one space onto a quantity in another space" (p. 51). Schwartz (1988) also specified that the extensive quantity (E'') having new referent is the representation of "an expansion of the semantic space of the problem" (p. 51). Greer (1992) also remarked that:

> “the distinctions between classes of situations are important pedagogically and provide an analytical framework useful for guiding research. However, it is important to realize that the way in which a situation is interpreted is not inherent in the situation but depends on the student's construal of it” (p. 279).

For example, rectangular array/area and cartesian product situations including whole numbers can be also conceived by equal groups. It depends on how to interpret the meaning of the multiplication or division problem situation. This is further explained by providing examples. So, in the following section, the meanings of multiplication and division (i.e., primitive meanings and conceptual understanding) through the representations of these multiplicative structures/situations are discussed. Additionally, the relationship between multiplication and division is demonstrated by the problem situations.

2.3. Meanings of Multiplication and Division

2.3.1. Meanings of Multiplication

<u>2.3.1.1. Repeated Addition.</u> Fischbein et al. (1985) hypothesized that “each fundamental operation of arithmetic generally remains linked to an implicit, unconscious, and primitive intuitive model” (p. 4). The primitive model they hypothesized for multiplication is “repeated addition, in which a number of collections of the same size are put together” (Fischbein et al., 1985, p. 6).

Regarding the multiplicative situations mentioned above, repeated addition meaning can be explained in the equal groups situation (Anghileri, 1989). Consider the following problem that “Mark has 4 bags of apples. There are 6 apples in each bag. How many apples does Mark have altogether?” (Van de Walle et al., 2010, p. 155). For the problem, the answer of the multiplicative expression “4×6” read as “4 times 6” can be found by *adding six four times*, so calculating “$6 + 6 + 6 + 6$” (i.e., *repeated addition*).

It is emphasized above that Greer (1992) categorized equal groups situations containing fractions as *rational rate*:

"The characterization of equal groups as a rate, as in 4 pizzas per child, generalizes to what might be termed a rational rate, such as 14/3 pizzas per child, which can be expressed in terms of integers as the equivalent rate of 14 pizzas for every 3 children" (p. 278).

This situation can be also expressed by repeated addition such that "14/3 + 14/3 + 14/3" because the multiplier (i.e., 3) is a positive integer.

Besides, Greer (1992) declared that "it is also possible to view the situation in terms of a many-one correspondence" (p. 277). Namely, the multiplicative comparison situation above "Jill picked 6 apples. Mark picked 4 times as many apples as Jill. How many apples did Mark pick?" (Van de Walle et al., 2010, p. 155) is understood as equal groups situation in terms of rate (i.e., 4 apples of Mark's for every 1 apple of Jill's), and it can be solved by repeated addition (i.e., 4 apples + 4 apples + 4 apples + 4 apples+ 4 apples + 4 apples = 24 apples, Mark picked). Alternatively, it can be solved by adding 6 apples four times (i.e., 6 apples + 6 apples + 6 apples + 6 apples = 24 apples). However, this is valid for only problems comprising positive integer/whole number multipliers.

For instance, the multiplication problem involving positive integers says that "Connie wants to buy 4 plastic cars. They cost 5 dollars each. How much does she have to pay?" (Vergnaud, 1988, p. 144). In this situation, M_1 is "numbers of cars", M_2 is "costs", a is "5", b is "4", and "x" is sought in terms of Vergnaud's (1983) isomorphism of measures in Figure 2.1 [11]. This situation can be also solved by repeated addition (i.e., 5 dollars + 5 dollars + 5 dollars + 5 dollars + 5 dollars = 20 dollars).

Moreover, rectangular array/area situation involving whole numbers or positive integers can be expressed by repeated addition meaning of multiplication. To exemplify, "A classroom has a rectangular arrangement of desks with 5 rows of 4 desks. How many desks are in the classroom?" (Otto et al., 2011, p. 14). This situation can be represented by using symbols like in Figure 2.3.

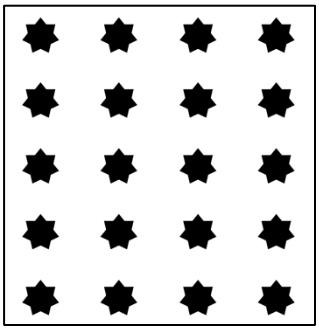

Figure 2.3. Representation of rectangular array situation.

When it is looked at horizontally, there are 5 rows and 4 desks in each row (i.e., 5 rows of 4 desks / 5 groups of 4), so it can be reasoned that "4 desks + 4 desks + 4 desks + 4 desks + 4 desks = 20 desks" or "5 × 4". When it is looked at vertically, there are 4 columns and 5 desks in each row (i.e., 4 columns of 5 desks / 4 groups of 5), so it can be reasoned that "5 desks + 5 desks + 5 desks + 5 desks = 20 desks" or "4 × 5". As seen in the example, factors (i.e., 4 and 5) play both roles (i.e., multiplier and multiplicand). Greer (1992) also explained rectangular area such that

"… where the sides of the rectangle are integral, say 4 cm by 3 cm. In this case, the rectangle can be partitioned into squares of side 1 cm so that the area may be found by *counting* these squares-it is *literally* 12 square cms" (p. 277).

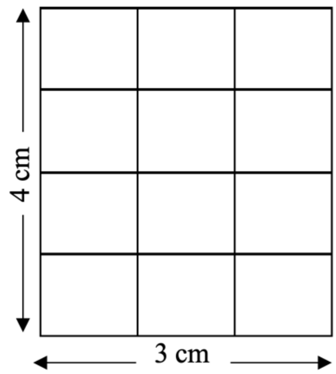

Figure 2.4. Representation of rectangular area situation.

This thinking is based on copying the region with square of side 1 cm (i.e., a unit of area) repeatedly to cover another region and then, counting the number of the regions that form a measure of area (i.e., 12 square cms) (Thompson, 2011). However, it includes additive thinking. When it is looked horizontally, there are 3 square cms in each row, so it can be written "3 square cms + 3 square cms + 3 square cms + 3 square cms = 12 square cms". When it is looked vertically, there are 4 square cms in each column, so it can be written "4 square cms + 4 square cms + 4 square cms = 12 square cms". Like in the rectangular array situation above, factors (i.e., 3 and 4) play both roles (i.e., multiplier and multiplicand) which show a symmetrical structure.

Cartesian product problem situations containing whole numbers or positive integers can be also interpreted as the equal groups situation (Greer, 1992). Hence, it can be thought by the repeated addition. If there are five pants and three blouses, the number of different combinations of pant and blouse outfits can be expressed as "5 different pants (with blouse 1) + 5 different pants (with blouse 2) + 5 different pants (with blouse 3)" or "3 different blouses (with pant 1) + 3 different blouses (with pant 2) + 3 different blouses (with pant 3) + 3 different blouses (with pant 4) + 3 different blouses (with pant 5)".

<u>2.3.1.2. Limitations of Repeated Addition.</u> Researchers point that this intuitive model of multiplication has crucial limitations and flaws (Fischbein et al., 1985; Schwartz, 1988; Tirosh & Graeber, 1989; Greer, 1992; Thompson & Saldanha, 2003).

Firstly, repeated addition meaning does not see multiplication as commutative (Fischbein et al., 1985; Tirosh & Graeber, 1989). According to the commutative property of multiplication, when the order of the factors is replaced, the result is still the same (e.g., $3 \times 5 = 5 \times 3$). It is important to note that although the result of these two different multiplication operations is the same, their interpretations are different (Otto et al., 2011). While "3" is the multiplier and "5" is the multiplicand in the first one, "5" is the multiplier and "3" is the multiplicand in the second one. The commutative property of multiplication over addition supposes that "$3 \times 0.65 = 0.65 \times 3$". However, 3×0.65 can be conceived by repeated addition as $0.65 + 0.65 + 0.65$ whereas 0.65×3 cannot be conceived by repeated addition because 3 cannot be added 0.65 times. That is, the fact that the results of these two operations are the same cannot be grasped under the repeated addition meaning.

Additionally, in the situations understood with this model, although there is no restriction for multiplicand and it can be any positive number, multiplier must be a whole number (Fischbein et al., 1985; Tirosh & Graeber, 1989; Greer, 1992). In the apple example (an equal groups situation) above by Van de Walle et al. (2010), both multiplier and multiplicand are whole numbers, so it can be understood with repeated addition (i.e., adding 6 four times). Besides, Greer (1992) specifically classified the situation in which the multiplier is a positive integer as "equal measures" and it can be also interpreted with repeated addition. For instance, the situation tells that "3 children each have 4.2 liters of orange juice. How much orange juice do they have altogether" (Greer, 1992, p. 280). The expression for the situation can be also written as "$4.2 + 4.2 + 4.2$" (adding 4.2 three times) in the repeated addition meaning. However, "it is difficult to say what "$5\ 2/3 \times 4$" means. It cannot mean, "add 4 five and two-thirds times" (Thompson & Saldanha, 2003, p. 104). Likewise, "5.2 times 3" also cannot be intuitively conceived by repeated addition because 3 cannot be added 5.2 times. It can be deduced that repeated addition works well with whole number multipliers, but it does not work with fractional and decimal multipliers that are non-integer numbers. Moreover, because the multiplier must always be a whole number in repeated addition, the result that is larger than multiplicand is obtained, and this situation might lead to the misconception that multiplication necessarily and always "makes bigger" (Fischbein et al, 1985; Schwartz, 1988; Tirosh & Graeber, 1989; Greer, 1992). On the contrary, the result of the multiplication "0.2 times 6" which cannot be conceived by repeated addition is smaller than the multiplicand. This example violates the constraint of this implicit model (i.e., repeated addition). Schwartz (1988) considered this misconception above as "a procedural flaw of repeated addition".

Schwartz (1988) also emphasized the importance of the relationship between numbers and their referents, and the need for the intensive quantity for a proper understanding of multiplication. The general structure of the multiplication described by Schwartz (1988) can be demonstrated as "(number of items per group) × (number of groups) = (total number of items)". In this demonstration, referents accompany the numbers, and the first quantity (multiplicand) is called intensive quantity. “(5.0, candies/bag, kind of party favor) × (6, bags)” can be an example expression (Schwartz, 1988, p. 46). In this multiplication, “the product of a quantity with the referent *candies/bag* and a quantity with the referent *bags* give rise to a quantity whose referent is *neither* candies/bag or bags” (Schwartz, 1988, p. 46). That is to say, the resulting quantity (product) is (30, candies) having a new referent because multiplication is referent transforming composition. However, addition is referent preserving compositions. So, the repeated addition model sees the quantities separately and accepts two of them as extensive (i.e., the first quantity as having the referent “candies”). Then, the model makes correspondences between quantities (e.g., “5 candies” corresponds to “1 bag”, “10 candies” corresponds to “2 bags”, and so on). When the first quantity is iterated six times, the resulting quantity having the same referent (i.e., candies) with one of the quantities (i.e., multiplicand) is generated. This is also referred to as *additive thinking* and addition is referent preserving composition. However, the nature of referent transforming compositions (i.e., multiplication and division) includes producing a third quantity with a referent different from the original quantities. That is why Schwartz (1988) called this situation “a conceptual flaw of repeated addition”. Therefore, in this study, how preservice teachers give meaning to the multiplication operation embedded in the situations with an emphasis on the nature of quantities inherent to the situations will be important. This is especially as Thompson and Saldanha (2003) stated, “multiplication is not the same as repeated addition. To be fair, we should say conceptualized multiplication is not the same as repeated addition.” (p. 103). This is because “envisioning the result of having multiplied is to anticipate a multiplicity. One may engage in repeated addition to evaluate the result of multiplying, but envisioning adding some amount repeatedly cannot support conceptualizations of multiplication” (Thompson & Saldanha, 2003, p. 103).

In the light of all limitations explained above, beyond the repeated addition, another meaning (i.e., the operational and conceptual understanding) of multiplication should be investigated to conceptualize multiplication including whole numbers, fractional and decimal multiplier multiplicatively and to realize the referents of the quantities in multiplication.

2.3.1.3. Multiplicative Meaning. Thompson and Saldanha (2003) proposed that conceiving 3×4 as "three fours" encourages to conceive that the result of the multiplication is "three times as large as four". This understanding is different from seeing 3×4 as "adding four three times" (i.e., repeated addition meaning). Otto et al. (2011) asserted a parallel idea on multiplication such that multiplication is scaling [1]. Consider the following equal groups situation that "A recipe calls for 2 cups of flour for a cake. How many cups of flour will need if we are going to make 5 cakes?". In this problem, the number of cups (i.e., quantity) of flour needed for five cakes is five times as much as the number of cups (i.e., quantity) of flour needed for one cake. That is, five times as much flour as the two cups is needed. The situation can be written as the expression "5×2". The first factor "5" is a *scaling factor* (also called a *scalar operator* defined by Vergnaud) and scales the number of cups of flour needed (i.e., second quantity) (Otto et al. 2011). Besides, the aforementioned Japanese textbook also introduced this understanding of multiplication (i.e., times as many/much) starting from Grade 2 (Karagöz Akar, Watanabe, & Turan, in press). For example, in the Tokyo Shoseki's book, it is asked to find "the length of two 3cm strips of paper put together" (Fujii & Iitaka, Grade 2, 2012, p. B10). For this situation, it is explained that:

> "If a piece of tape is as long as two 3cm strips of paper put together, we can say the tape is 2 times as long as the 3cm tape. You can use the multiplication math sentence 3×2 to find the length that is two times as long as 3cm" (Fujii & Iitaka, Grade 2, 2012, p. B10).

"That is the result of the product, i.e., 6, is emphasized relative to the size of one of the quantities, i.e., 3" (Karagöz Akar, Watanabe, & Turan, in press). Furthermore, the *rational rate* situation (i.e., 14/3 pizzaz per child) can be also expressed in terms of the multiplicative meaning can be that "the number of pizzaz (i.e., 14 pizzas) 3 children eat is three times as much as the number of pizzaz (i.e., 14/3 pizzaz) one child eats.

As already stated above, Greer (1992) categorized equal groups situations containing decimals as *rate* such as "A boat moves at a steady speed of 4.2 meters per second. How far does it move in 3.3 seconds?" (p. 280). Greer (1992) further wrote the structure of this problem situation as "x [measure$_1$] $\times$ y [measure$_2$ per measure$_1$] = xy [measure$_2$]" (p. 278). That is, the problem is demonstrated by this structure as "3.3 seconds $\times$ 4.2 meters per second = 13.9 meters". Quantities (and their referents) are emphasized in this multiplication expression. It is said that in 3.3 seconds, a boat moves 3.3 times as much far as in a second (i.e., 3.3 times as much as 4.2 meters).

Equal groups situations containing decimals are also categorized as *measure conversion* such as “A yard is about 3.0 feet. About how long is 4.5 yards in feet?”. In this multiplication situation, the result (i.e., 13.5) is 4.5 times as much as 3.0 feet.

Besides, multiplicative comparison problem structure directly refers to the multiplicative meaning of multiplication (i.e., conceptual understanding of multiplication) because it emphasizes the relative size comparison. For instance, “Dan has 5 marbles. Ruth has 4 times as many marbles as Dan. How many marbles does Ruth have?” (Nesher, 1988). In this multiplicative situation, “how many times” (i.e., 4) and “the base quantity” (i.e., 5 marbles) are given, and “the quantity compared” (i.e., the number of marbles Ruth has) is expected to be found.

Greer (1992) categorized multiplicative comparison situations containing fraction multipliers (i.e., multiplicative factor) as *multiplicative comparison* and *part/whole* [39]. To exemplify, multiplicative comparison situation “John has 3 times as many apples as Mary” can be inverted the problem including fraction multiplier that “Mary has $\frac{1}{3}$ as many apples as John” (Greer, 1992, p. 278). It can be continued such as “If John 36 apples, how many apples Mary has?”. Furthermore, “There are 36 children in a class, of whom $\frac{2}{3}$ are girls. How many girls are there in the class?” is an example of a part-whole relationship (Greer, 1992, p. 278). This situation is also interpreted as “There are 36 children in a class. The number of girls is $\frac{2}{3}$ times as many as the number of children. How many girls are there in the class?”.

Greer (1992) classified multiplicative comparison situations containing decimal multiplier as *multiplicative comparison*, *part/whole*, and *multiplicative change*. While “Iron weighs 0.88 times as much as copper. A piece of copper weighs 4.2 gms. How much would a piece of iron the same size weigh?” is an example of the first class (Greer, 1992, p. 278), “A piece of elastic can be stretched to 3.3 times its original length. What is the length of a piece 4.2 meters long when fully stretched?" is an example of the third one (Greer, 1992, p. 280).

In the example "A farm of 45.8 ha procedures 6850 kg of corn per ha. What will the yield?" (Vergnaud, 1983, p. 129), M_1 is "areas", M_2 is "weights of corn", a is "6850", and b is "45.8" in terms of the isomorphism of measures in Figure 2.1 above. However, this situation cannot be conceived by repeated addition due to the decimal multiplier. Therefore, different thinking is needed to comprehend and solve this multiplication problem. Vergnaud (1983) mentioned two different ways which are using the *scalar operator* and using the *function operator*. Scalar operator ($\times b$) illustrated in Figure 2.5 drawn by the researcher of the study includes "transposing in M_2, from a to x, the operator that links 1 to b in M_1" (Vergnaud, 1983, p. 130).

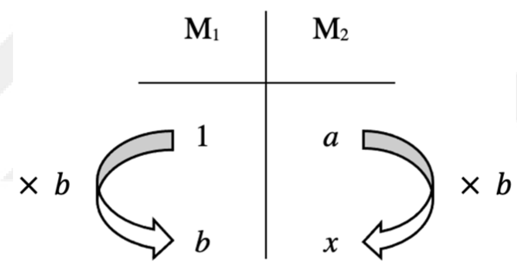

Figure 2.5. The use of scalar operator "$\times b$".

"$\times b$" is a scalar operator because it has no dimension, being a ratio of two magnitudes of the same kind" (Vergnaud, 1988, p. 130). For this example, that cannot be conceptualized in repeated addition due to the decimal multiplier, with the scalar operator it is understood that the area of the farm (i.e., 45.8 ha) is 45.8 times as much as 1 ha, and the weights of the corn produced in this farm is also 45.8 times as much as the weight of the corn produced in 1 ha (i.e., 6850 kg). Furthermore, function operator ($\times a$) illustrated in Figure 2.6 drawn by the researcher of the study includes "transposing on the lower line, from b to x, the operator that links 1 to a on the upper line" (Vergnaud, 1988, p. 130).

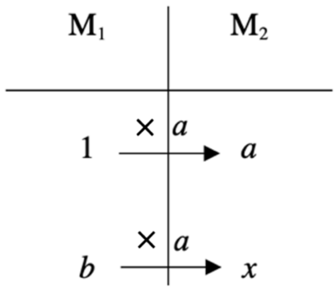

Figure 2.6. The use of function operator "× *a*".

"x *a* is a function operator because it represents the coefficient of the linear function from M_1 to M_2. Its dimension is the quotient of two other dimensions (e.g., cents per cake, kg per ha)" (Vergnaud, 1988, p. 130). So, this problem can be interpreted by function operator such as "45.8 ha × 6850 kg per ha = 313730 kg". Quantities (and their referents) are emphasized in this multiplication, and the intensive quantity 6850 kg/ha "corresponds to the functional relationship between the two measures" (Simon & Placa, 2012, p. 36).

Schwartz's (1988) (*I E E*') semantic triad also explains the multiplicative meaning of multiplication and corresponds to isomorphism of measures by emphasizing the intensive quantity and the referents of the quantities. To exemplify, "A boat moves at a steady speed of 4.2 meters per second. How far does it go in 3.3 seconds?" (Greer, 1992, p. 284). This *rate* problem (i.e., equal groups situation involving decimal multiplier) fits into the structure by Schwartz as follows 4.2 meters per second (*I*) × 3.3 seconds (*E*) = 13.9 meters (*E*')

The rectangular array/area example represented in Figure 2.3 and explained with repeated addition above can be also thought multiplicatively. Firstly, understanding of this situation multiplicatively involves that there are 4 desks for each row and because there are 5 rows, there are 5 times as many desks as 4 desks. Similarly, there are 5 desks for each column and because there are 4 columns, there are 4 times as many desks as 5 desks.

Understanding of the rectangular area situation multiplicatively in Figure 2.4 above can be expressed as "there are 3 squares of side 1 cm in each row, so the area of each row 3 square cm. Because there are 4 rows, there are 4 times as many square cm as 3 square cm".

That is, the area of entire representations is 4 times as much as the area of each row. Another thinking of this situation can be explained as "there are 4 squares of side 1 cm in each column, so the area of each column 4 square cm. Because there are 3 columns, there are 3 times as many square cm as 4 square cm". That is, the area of the rectangle is 3 times as much as the area of each column. Additionally, based on the explanations of Thompson and Saldanha (2003) about the measurement and quantities, 3 centimeters is a length that is 3 times as long as one centimeter and 4 centimeters is a length that is 4 times as long as one centimeter. So, the area of the rectangle is 12 cm^2 that is 12 times the area of a rectangle of dimension 1 cm by 1 cm.

Greer (1992) classified rectangular array/area situations containing fractions and decimals into *rectangular area* such as "What is the area of a rectangle 3.3 meters long by 4.2 meters wide?" (p. 280). Like in the above, 3.3 meters is a length that is 3.3 times as long as one meter and 4.2 meters is a length that is 4.2 times as long as one meter. Then, the area of rectangle (i.e., 13.9 m^2) is explained as "13.9 times the area of rectangle of dimension 1 m by 1 m.

Consider the cartesian product problem that "How many different outfits can Sue make if she has 3 shirts (white, red, and blue) and 2 skirts (tan and navy)" (Otto et al., 2011, p. 16). The number of different outfits can be found by 3×2. This situation can be modeled by a tree diagram like in Figure 2.7 drawn by the researcher of the study. In this model, it is realized that there are three different options for a shirt which are white shirt, red shirt, and blue shirt, and each shirt option leads to two different options for a skirt which are tan skirt and navy skirt. As seen in the model, six different outfits are obtained.

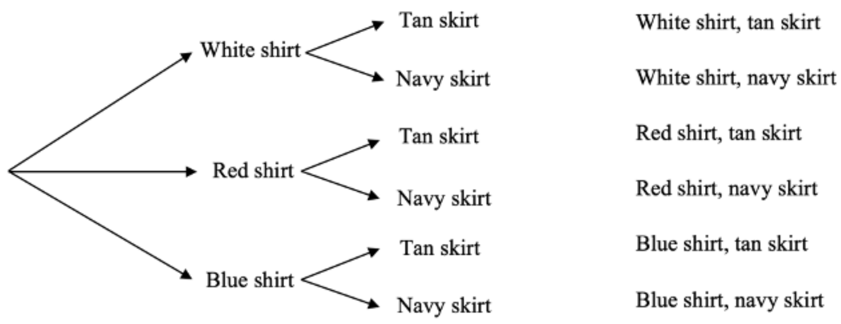

Figure 2.7. A tree diagram modeled for different shirt and skirt outfits.

For the description of this situation by multiplicative meaning, Otto et al. (2011) clarified that "for each shirt, Sue has two ways to complete the outfit. Because she has three choices for the color of the shirt, she has three times as many different outfits as the number of skirts" (p. 17). In this interpretation, the number of skirts (i.e., 2) is the base quantity, the number of shirts (i.e., 3) is how many times, and the number of different outfits asked to be found (i.e., 6) is the quantity compared. Thus, multiplication makes sense. The number of different outfits can be also found by 2×3 as modeled in Figure 2.8. There are two different options for a skirt and each skirt option leads to three different options for a shirt. Similarly, six different outfits are obtained.

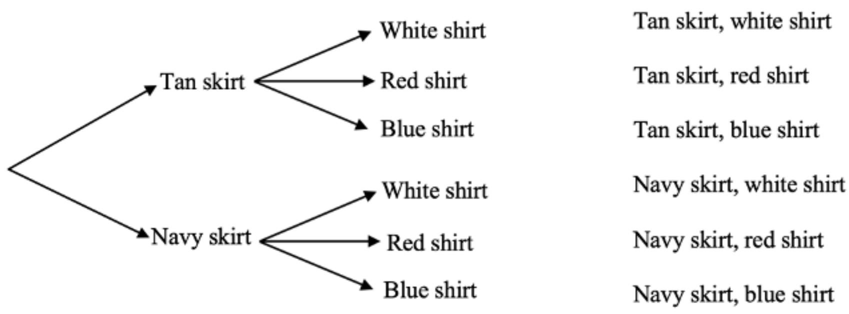

Figure 2.8. Another tree diagram modeled for different shirt and skirt outfits.

In this model, for each skirt, Sue has three routes to make a combination for the outfit. Because she has two options for the color of the skirt, she has two times as many different outfits as the number of shirts. In this interpretation, the number of shirts (i.e., 3) is the base quantity, the number of skirts (i.e., 2) is how many times, and the number of different outfits asked to be found (i.e., 6) is the quantity compared.

Greer (1992) classified cartesian product situations containing decimals into *product of measures*. For instance, "If an appliance uses 3.3 kilowatts for 4.2 hours, how many kilowatt-bours of electricity does it consume?" (Greer, 1992, p. 278). Greer (1992) also specified that "each part of the 3.3 kilowatts is combined with each part of the 4.2 hours" (p. 278). When this situation is also thought with the explanations of Thompson and Saldanha (2003) about the measures of quantities, it can be said that 3.3 kilowatts is 3.3 times as large as one kilowatt and 4.2 hours is 4.2 times as large as one hour, so the measure of electricity (i.e., 13.9 kilowatt-hours) is 13.9 times the measure of electricity of 1 kilowatt by 1 hour.

In the example "What is the area of a rectangular room that is 7 m long and 4.4 m wide?" (Vergnaud, 1983, p. 135), M_1 is "widths", M_2 is "lengths", M_3 is "areas", *a* is "7", and *b* is "4.4" in terms of the product of measures in Figure 2.2 above. However, this situation cannot be conceived by repeated addition due to the decimal factor. So, this is explained by the multiplicative meaning of multiplication such that "the area of rectangle (30.8 m^2) is 30.8 times the measure of the area of rectangle of dimension 1 m by 1 m.

Schwartz's (1988) (E E' E'') semantic triad also explains the multiplicative meaning of multiplication and corresponds to product of measures by emphasizing the referents of the quantities. For example, "(3, blouses) × (5, skirts) = (15, outfits)" and "(3.2, cm width of rectangle) × (6.4, cm, length of rectangle) = (20.48, cm^2, area of rectangle)" (Schwartz, 1988, p. 51).

2.3.2. Meanings of Division

As already stated above, the equal groups situation can be considered as an asymmetrical structure because the roles of multiplier and multiplicand are different, and there is an inverse relationship between multiplication and division. That is, the problem context is a division situation when either the multiplier or multiplicand is unknown (Van de Walle et al., 2010). Therefore, two types of division emerge (Greer, 1992). These are division by multiplier (i.e., multiplicand is unknown) and division by multiplicand (i.e., multiplier is unknown). The primitive intuitive meaning related to division (by multiplier) is partitive division and the primitive intuitive meaning related to division (by multiplicand) is quotative (Fischbein et al., 1985; Greer, 1992; Thompson and Saldanha, 2003; Van de Walle et al., 2010; Otto et al., 2011). Besides, the relative size meaning of the division by the multiplier and the division by the multiplicand is explained in the following parts.

2.3.2.1. Partitive Division. This type of division is also referred to as *(fair-)sharing* (Fischbein et al., 1985; Thompson and Saldanha, 2003; Van de Walle et al., 2010; Otto et al., 2011) or *partition(ing)* (Greer, 1992; Thompson and Saldanha, 2003; Van de Walle et al., 2010). In the partitive division, "an object or collection of objects is divided into a number of equal fragments or subcollections" (Fischbein et al., 1985, p. 7). Thompson and Saldanha (2003) also defined partitive division as "the action of distributing an amount of something among a number of recipients so that each recipient receives the same amount" (p. 106). The size of the object or the number of the objects is called *dividend*, the number of equal fragments or subcollections is called *divisor*, and the size of each fragment or subcollection is called *quotient* (Harel, Behr, Post, & Lesh, 1994). This type of division can be exemplified by equal groups situation such as "Mark has 24 apples. He wants to share them equally among his 4 friends. How many apples will each friend receive" (Van de Walle et al., 2010, p. 155). This problem requires partitioning the apples into a particular number of groups. So, the result is obtained when 24 is divided by 4 (i.e., $24 \div 4$). When this division situation is transformed into a multiplication situation, the quantity "4 friends" is called the multiplier. In this division problem, the number of groups (i.e., multiplier) is known whereas the number of apples each friend receives (i.e., multiplicand) is unknown.

The partitive division problem can be given in the multiplicative comparison situation including the whole number such as "Mark picked 24 apples. He picked 4 times as many apples as Jill. How many apples did Jill pick?" (Van de Walle et al., 2010, p. 155). When it is thought in terms of a many-one correspondence, it can be deduced that there are 4 apples of Mark for 1 apple of Jill. So, Mark's apples (i.e., 24 apples) are divided into 4 groups. Here, "the quantity compared" and "how many times" (i.e., multiplier) are given, and "the base quantity" (i.e., multiplicand) is expected to be found.

For example, the partitive division problem involving positive integers says that "Connie wants to share her sweets with Jane and Susan. Her mother gave her 12 sweets. How many sweets will each receive?" (Vergnaud, 1983, p. 131). In this situation, M_1 is "numbers of children", M_2 is "numbers of sweets", b is "3", x is "12", and "a" is sought in terms of Vergnaud's (1983) isomorphism of measures. He asserted that scalar operator (/ b) to the quantity x (illustrated in Figure 2.9 drawn by the researcher of the study) can be used to solve this problem.

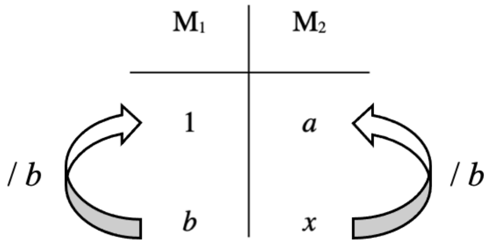

Figure 2.9. The use of scalar operator "/ b".

It can be thought that when 12 sweets are delivered one by one to three children, each child receives 4 sweets. This is also the case for rectangular array situation in Figure 2.3. When it is looked at horizontally, it can be asked, "If there are 20 desks in 5 rows, how many desks are there in each row?". It can be thought that when 20 desks are delivered one by one to 5 rows, there are 4 desk in each row. When it is looked at vertically, it can be asked, "If there are 20 desks in 4 columns, how many desks are there in each column?". It can be thought that when 20 desks are delivered one by one to 4 columns, there are 5 objects in each column.

<u>2.3.2.2. Quotative Division.</u> This type of division is also referred as *measurement* (Fischbein et al., 1985; Van de Walle et al., 2010; Otto et al., 2011), *segmenting* (Thompson and Saldanha, 2003) or *repeated subtraction* (Van de Walle et al., 2010; Otto et al., 2011). In the quotative division, "one seeks to determine how many times a given quantity is contained in a larger quantity" (Fischbein et al., 1985, p. 7). It is also defined as "the action of putting an amount into parts of a given size" (Thompson & Saldanha, 2003, p. 106). The equal groups situation problem "Mark has 24 apples that he wants to give to his friends. Each friend receives 6 apples. How many friends does Mark have?" is an example of a quotative division

situation because it is needed to put the apples into particular group sizes. So, the result is obtained through dividing 24 by 6 (i.e., 24 ÷ 6). It can be also solved by repeated subtraction until finishing the apples such as "24 – 6 =18, 18 – 6 = 12, 12 – 6 = 6, 6 – 6 = 0". The number of times subtraction operation is done gives the answer (i.e., 4) of the problem. Again, when it is transformed into a multiplication situation, the quantity "6 apples for each friend" is called the multiplicand. In this situation, the number of apples in each bag (i.e., multiplicand) is known whereas the number of bags (i.e., multiplier) is unknown.

Furthermore, the quotative division including the whole number can be also used in the multiplicative comparison situation such as "Mark picked 24 apples, and Jill picked only 6. How many times as many apples did Mark pick as Jill did?" (Van de Walle et al., 2010, p. 155). It can be also done by using repeated subtraction. Here, "the quantity compared" and "the base quantity" (i.e., multiplicand) are given, and "how many times" (i.e., multiplier) is expected to be found.

Additionally, the quotative division situation involving positive integers is exemplified by the problem that "Peter has $15 to spend and he would like to buy miniature cars. They cost $3 each. How many cars can he buy?" (Vergnaud, 1983, p. 132). M_1 is "numbers of cars", M_2 is "costs", a is "3", x is "15", and "b" is sought in terms of Vergnaud's (1983) isomorphism of measures in Figure 2.1. He also stated that functional operator (/ a) to the quantity x (illustrated in Figure 2.10 drawn by the researcher of the study) can be used to solve this problem.

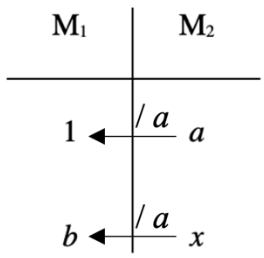

Figure 2.10. The use of function operator "/ *a*".

In this situation, repeated subtraction (until getting to zero) can be used as "$15 - 3 = 12, 12 - 3 = 9, 9 - 3 = 6, 6 - 3 = 3, 3 - 3 = 0$". In this case, the number of times subtraction operation is done gives the answer (i.e., 5). It can be also thought additively as "add $3 + 3 + 3 + 3 + 3$ until getting to 15". In this case, the number of times a is added gives b.

This is again the case for rectangular array situation. When it is looked at horizontally, it can be asked, "There are 20 desks and 4 desks in each row. How many rows are there?". It can be thought by repeated subtraction such that the number of times subtraction operation (i.e., subtracting 4 from 20 until getting zero) is done gives the answer (i.e., 5). When it is looked at vertically, it can be asked, "There are 20 desks and 4 desks in each column. How many columns are there?". It can be thought again by repeated subtraction such that the number of times subtraction operation (i.e., subtracting 5 from 20 until getting zero) is done gives the answer (i.e., 4)

Additionally, as seen in the two interpretations above for the outfit example in Figure 2.7 and Figure 2.8, the factors in the cartesian product may have two roles (i.e., base quantity or how many times). That is, they have symmetrical roles (i.e., equivalent roles) like in the rectangular area situation, so there is one type of division problem that can be created in the cartesian product situation (Greer, 1992). For the example above, if the number of shirts (i.e., 3) or the number of skirts (i.e., 2) is unknown, the same division operation is used.

2.3.2.3. Limitations of Partitive and Quotative Division. Researchers point that these two types of division involving whole number (i.e., the intuitive meaning of division) have some limitations that arise as misconceptions (Fischbein et al., 1985; Schwartz, 1988; Tirosh & Graeber, 1989; Simon & Placa, 2012). They are proposed as the primitive model associated with division by Fischbein et al. (1985). Particularly, the partitive division has some constraints such that "the dividend must be larger than the divisor; the divisor (operator) must be a whole number; and the quotient must be smaller than the dividend (operand)" (Fischbein et al., 1985, p. 7). Schwartz (1988) explained the situation in which the divisor is not a whole number in the following way,

> "…one wishes to divide the quantity (46. 4, mi, distance traveled) by the quantity (3.2, hr, time trip takes). It is clear that the mental image of sharing or partitioning one quantity evenly does not work easily here.

The related division problem, dividing the quantity (46.4, mi, distance traveled) by the quantity (14.5, mi/hr, average speed on trip), does not allow for a partitioning interpretation at all" (p. 47).

Moreover, the aforementioned last two constraints above cause the misbelief on the part of students such that "division always makes smaller" (Fischbein et al., 1985; Graeber & Tirosh, 1990). Schwartz (1988) considered it as "a procedural flaw of partitive division" by asserting that this model of division leads to expect that a quotient that is smaller than the dividend is acquired in the division operation.

Karagöz Akar, Watanabe and Turan (in press) also asserted that "the idea behind partitive division seems to be multiplication as repeated addition whereas the idea behind quotitive division seems to be repeated subtraction". Therefore, the flaw in the referents of the quantities in multiplication is also seen in the partitive division. This type of division has a profound "conceptual flaw" because it might lead to thinking that the generated quantity (i.e., quotient) has the same referent with one of the quantities (i.e., dividend) in the operation (Schwartz, 1988). To exemplify, suppose that one wants to equitably share (30, candies) among (6, bags). The result of (30, candies) / (6, bags) that is (5, candies/bags) will represent the generated intensive quantity (i.e., 5 candies per bag) with a new referent. However, distributing candies one by one to the bags may be another procedure for the partitive division. In this procedure, the correspondence between the quantities (e.g., if "30 candies" corresponds to "6 bags" and then, "5 candies" corresponds to "1 bag") ignore the intensive quantity with new referent (candies/bag) and accept this quantity as having the referent "candies" (i.e., 5, candies). This shows additive thinking in the division, not multiplicative thinking. Schwartz (1988) stressed that the sharing model of division fails due to these procedural and conceptual flaws above.

Besides, the quotative division has the constraint that "the dividend must be larger than the divisor" (Fischbein et al., 1985, p. 7). Hence, in the situations where the divisor is larger than the dividend, this division type does not work. Simon and Placa (2012) also highlighted that the quotative (measurement) type of division does not enhance the grasp of intensive quantities because whole-number division includes reasoning with only extensive quantities. When the quotative division problem, "How many 20 cm pieces of ribbon can be made from a 140 cm length of ribbon?" is examined, it is understood that 140 cm is measured with a 20 cm piece of ribbon (Simon & Placa, 2012, p. 38). The additive procedure that is adding 20 cm + 20 cm + 20 cm, … until 140 cm is reached may be utilized to solve this problem. Although 20 cm/piece is an intensive quantity, all quantities in the problem are regarded as extensive (Simon & Placa, 2012).

Based on the constraints of partitive and quotative division, operational and conceptual understanding of division (by multiplier and by multiplicand) should be addressed to multiplicatively conceive division problems in which dividend is smaller than divisor or quotient, and in which divisor is the whole number and non-integer numbers such as fraction and decimal. Thompson and Saldanha (2003) specified that "operational understanding of division entails a conceptual isomorphism between them" (p. 106). Moreover, conceptual understanding of division is also essential to realize the referents of the quantities in the division.

2.3.2.4. Relative Size Meaning. As specified above, understanding of multiplication multiplicatively (i.e., times as many/much meaning) includes that the product is relative to the size of one of the quantities. Moreover, this understanding also emphasizes that the product is in reciprocal relationships to the factors (Thompson & Saldanha, 2003). To exemplify, thinking "3×4" multiplicatively is expressed as (3×4) is 3 times as large as 4, (3×4) is 4 times as large as 3, 3 is 1/4 as large as (3×4), and 4 is 1/3 as large as (3×4). Such understanding of multiplication is also important to understand division multiplicatively because conceptualization of division also includes relative size comparison (Thompson & Saldanha, 2003). To give an example, the quotient 24/6 tells that 24 is some number of times as large as 6. Additionally, the quotient is considered as the measure of relative size of the quantities. Conceiving the quotient as this meaning "helps students make sense of situations where division is used" (Byerley & Thompson, 2014, p. 217).

By regarding primitive and intuitive models of division (i.e., the understanding of whole-number division, partitive and quotative division), Byerley, Hatfield, and Thompson (2012) remarked:

> "These two meanings for division do not require multiplicative reasoning. A third model for division, relative size, requires students to reason multiplicatively; the relative size model for division calls upon a

comparison between the size of one quantity with respect to another quantity (Thompson & Saldanha, 2003). Division as relative size allows students to be able to reason about non-integer divisors. If division is viewed partitively, it only makes sense to divide a number into n equal parts if n is an [*sic*] whole number" (p. 359).

To exemplify, suppose that it is asked to divide the quantity 79.2 mi (distance traveled) by the quantity 4.8 hr (time elapsed). The relative size model works easily here because "constant speed measured in miles per hour tells us that the number of miles traveled is so many times as large as the number of hours elapsed" (Byerley & Thompson, 2014, p. 218). Considering Thompson and Saldanha's (2003) illustration on the conceptualization of division as relative size, equal groups division situations (i.e., the apple examples) above are reasoned multiplicatively as such:

In the division by multiplier, when 24 apples (4 × 6) are shared among 4 friends, each friend receives the same number of apples (6 apples per friend). In terms of the relative size meaning of division, 6 is considered as 1/4 as large as the total number of apples (24). That is, each friend receives the number of apples that is 1/4 of the total number of apples. In division by multiplicand, when 24 apples (6 × 4) are shared so that each friend receives 4 apples, the number of friends is 6. In terms of the relative size meaning, 6 is again considered as 1/4 as large as the total number of apples (24). That is, the number of parts made by putting all apples into 4-apple sized parts is 1/4 as large as the number of all apples. With this understanding, it is also made sense why the numerical operation evaluating these two types of situations is the same. To sum up, comparing the equal groups situations (i.e., the apple example) in the partitive and the quotative division, while each child receives 4 apples in the first division, 6 children can receive the apples in the second division respectively.

As emphasized above, Greer (1992) categorized equal groups situations containing fractions as *rational rate*. This situation is exemplified as the division (by multiplier) problem such that "if 14 pizzas are shared among 3 children, how many pizzas will each child get?". Greer (1992) explained this situation as follows:

"14 pizzas cannot be equally divided among 3 children, as long as a pizza is considered an indivisible whole; each child can be given 4 pizzas, with 2 left over. By a shift of perspective, whereby a pizza is considered as something that can be cut into fractions (fractured), a solution becomes possible" (p. 277).

Based on his description of how this division situation is conceived, if 14 pizzas reconceptualized as 14.3 (1/3 pizzas) are divided among 3 children, so 14 (1/3 pizzas) are given per child, and then it is converted back to 14/3 pizzas per person. That is to say, the number of pizzas one child gets (i.e., 14/3 pizzas) is 1/3 as large as the number of pizzas 3 children get (14 pizzas).

It is already declared that Greer (1992) also classified equal groups situations including decimals as *rate*. Division by multiplier exemplifies the situation such as “A boat moves 13.9 meters in 3.3 seconds. What is its average speed in meters per second?” (Greer, 1992, p. 280). This can be explained that in a second, a boat moves 1/3.3 as large as in 3.3 seconds (i.e., 1/3.3 as large as 13.9 meters). Average speed measured in meters per second expresses that the number of meters moved is so many times as large as the number of seconds elapsed. That is, the quotient (i.e., 4.2 meters/second) is a measure of relative size of two quantities. Division by multiplicand exemplifies the situation such as “How long does it take a boat to move 13.9 meters at a speed of 4.2 meters per second?” (Greer, 1992, p. 280). The number of meters moved (i.e., 13.9) is 4.2 times as large as the number of seconds elapsed.

Greer (1992) also classified equal groups situations including decimals as *measure conversion*. For the division by multiplier, this situation can be expressed as “4.5 yards is about 13.5 feet. About how many feet are there in a yard?”. The resulting quantity (3.0 ft/yd) is an *intensive quantity* and the measure of the size of the length in feet relative to the size of the length in yard. For the division by multiplicand, the situation can be written as “A yard is about 3.0 feet. About how long in yards is 13.5 feet?”. In this situation, 13.5 feet is measured with 3.0 feet/yards.

Besides, multiplicative comparison problem structure directly refers to the relative size meaning of division (i.e., conceptual understanding of division) because of the focus on the relative size comparison. When example by Nesher (1988) above can be written as a division by multiplier situation such as “Ruth has 20 marbles He has 4 times as many marbles as Ruth. How many marbles does Dan have?”. Additionally, the problem “Ruth has 20 marbles, and Dan 5 marbles. How many times as many marbles does Ruth have as Dan does?” is a division by multiplicand situation.

Multiplicative comparison and *part/whole* are the classes of the multiplicative comparison situations containing fractions (Greer, 1992). The problem says that "Mary has 1/3 as many apples as John. If Mary has 12 apples, how many apples John has?" shows a multiplicative comparison situation in division (by multiplier). It can be inverted such as "John has 3 times as many apples as Mary" and can be solved by multiplication. Moreover, it can be solved by dividing 12 by 1/3. The problem says that "Mary has 12 apples and John has 36 apples. How many apples have Mary relative to John?" is multiplicative comparison situation in division (by multiplicand). While the situation "$\frac{2}{3}$ of the children in a class are girls. If there are 24 girls, how many children are there in the class?" is a part/whole situation in division (by multiplier), the situation "24 out of 36 children in a class are girls. What fraction of the students are girls?" is a part/whole situation in division (by multiplicand).

Additionally, *multiplicative comparison*, *part/whole*, and *multiplicative change* are the classes of the multiplicative comparison situations including decimals. "Iron is 0.88 times as heavy as copper. If a piece of iron weighs 3.7 kg. How much does a piece of copper the same size weigh?" (Greer, 1992, p. 280) is the first class in division (by multiplier). "If equally sized pieces of iron and copper weigh 3.7 kg and 4.2 kg respectively. How heavy is iron relative to copper?" (Greer, 1922, p. 280) is the first class in division (by multiplicand). "A piece of elastic can be stretched to 3.3 times its original length. When fully stretched it is 13.9 meters long. What was its original length?" (Greer, 1922, p. 280) is the third class in division (by multiplier). "A piece of elastic 4.2 meters long can be stretched to 13.9 meters. By what factor is it lengthened?" (Greer, 1922, p. 280) is the third class in division (by multiplicand). Consequently, in all these multiplicative situations, if "base quantity" is asked, the division by multiplier is utilized whereas if "how many times" is asked, the division by multiplicand is utilized.

For example, consider the following division (by multiplier) situation that "Mrs. Johnson bought some large peaches. Nine peaches weigh 2 kg. How much does one peach weigh, on the average?" (Vergnaud, 1983, p. 131). M_1 is "numbers of peaches", M_2 is "weights", b is "9", x is "2", and a is sought in terms of the isomorphism of measures. However, this one cannot be conceived by the primitive meaning of division (i.e., partitive division) because the dividend is smaller than the divisor. Therefore, Vergnaud's scalar operator (/ b) to the quantity *a* illustrated in Figure 2.9 can be used. This can be explained as "The weight of one peach is 1/9 as large as the weight of 9 peaches (i.e., 2 kg).

Additionally, the division (by multiplicand) situation is exemplified by the problem that "Dad drives 55 miles per hour on the freeway. How long will it take him to get to his mother's house, which is 410 miles away?" (Vergnaud, 1983, p. 132). M_1 is "durations", M_2 is "distances", *a* is "55", *x* is "410", and "*b*" is sought in terms of the isomorphism of measures. Vergnaud's functional operator (/ a) to the quantity *b* illustrated in Figure 2.10 can be used. The number of miles (i.e., 410) is 55 times as large as the number of hours elapsed.

Schwartz's (1988) (*I E E*') semantic triad also explains the relative size meaning of division and corresponds to isomorphism of measures of Vergnaud by emphasizing the intensive quantity and the referents of the quantities. A boat example above for both types of division fits into the structure by Schwartz as follows: "13.9 meters (*E*') / 3.3 seconds (*E*) = 4.2 meters per second (*I*) (division by multiplier)" and "13.9 meters (*E*') / 4.2 meters per second (*I*) = 3.3 seconds (*E*) (division by multiplicand)"

In this rectangular array division problem "A classroom has a rectangular arrangement of desks with 5 rows. If there are 20 desks in the classroom, how many desks are in each row?", the number of the desks in each row is 1/5 as many as 20. Besides, in the rectangular area division problem "If the area of a rectangle is 12 cm^2 and the length is 4 cm, what is the width?", the width measured in centimeters tells that the number of square centimeters is so many times as large as the number of centimeters. Greer (1992) classified rectangular array/area situations containing fractions and decimals into *rectangular area* such as "If the area of a rectangle is 13.9 m^2 and the length is 3.3 m what is the width" (p. 280). This situation is conceived by relative size like in the area problem above.

Consider the following cartesian product problem "If there are 12 different routes from A to C via B, and 3 routes from A to B, how many routes are there from B to C?" (Greer, 1992, p. 280). This division problem is explained by relative size meaning such as "the number of routes from B to C is 1/3 as many as the number of the routes from A to C (i.e., 12). Also, *product of measures* defined by Greer (1992) for the cartesian product situations involving decimals is exemplified by the division situation "A heater uses 3.3 kilowatts per hours. For how long can it be used on 13.9 kilowatt-hour of electricity?" (p. 280). The relative size meaning of division explains this situation such that the time measured in hour tells that the number of kilowatt-hours is so many times as large as the number of kilowatts.

Schwartz's (1988) (E E' E'') semantic triad also explains the relative size meaning of division and corresponds to product of measures by emphasizing the referents of the quantities [9]. For example, "(15, outfits) (E'') / (3, blouses) (E') = (5, skirts) (E)" or "(15, outfits) (E'') / (5, skirts) (E) = (3, blouses) (E')" (cartesian product) and "20.48 cm^2 (E'') / 3.2 cm (E') = 6.4 cm (E)" or "20.48 cm^2 (E'') / 6.4 cm (E) = 3.2 cm (E')" (rectangular area).

2.4. Mathematical Knowledge for Teaching

Many studies have been conducted on the issue of teacher knowledge to define it and to describe its components (Shulman, 1986; Grossman, 1990). Among these studies, Shulman's (1986) study is one of the most important research on teacher knowledge because the model he introduced provided a basis for other studies. Shulman (1986) explained the teachers' content knowledge in three categories which are subject matter content knowledge (also referred to as content knowledge) (SMK), pedagogical content knowledge (PCK), and curricular knowledge. He defined content knowledge as "the amount and organization of knowledge per se in the mind of the teacher" (p. 9). On the other hand, pedagogical content knowledge is described as a "particular form of content knowledge that embodies the aspects of content most germane to its teachability" (Shulman, 1986, p. 9). The last category of teachers' content knowledge is curricular knowledge expressed as "understandings about the curricular alternatives available for instruction" (Shulman, 1986, p. 10). Based on the term pedagogical content knowledge described by Shulman (1986), some researchers developed their own models for describing the components of teacher knowledge (Tamir, 1998; Cochran, DeRuiter, & King, 1993; Magnusson, Krajcik, & Borko, 1999; Tatto, Peck, Schwille, Bankov, Senk, Rodriguez, Ingvarson, Reckase, & Rowley, 2012).

In the field of mathematics education, Mathematics Knowledge for Teaching (MKT) developed by Ball and her colleagues (2008) contributes to examining mathematical knowledge and effective mathematics teaching. Besides, they develop an instrument to measure such knowledge (Hill, Schilling, & Ball, 2004). In the MKT model, there are six categories that are settled under subject matter knowledge and pedagogical content knowledge as seen in Figure 2.11 drawn by the researcher of the study. SMK is subdivided into common content knowledge (CCK), specialized content knowledge (SCK), and horizon content knowledge (HCK). PCK is subdivided into knowledge of content and students (KCS), knowledge of content and teaching (KCT), and knowledge of content and curriculum.

<table>
<tr><th colspan="2">SUBJECT MATTER KNOWLEDGE</th><th colspan="2">PEDAGOGICAL CONTENT KNOWLEDGE</th></tr>
<tr><td>Common content knowledge (CCK)</td><td rowspan="2">Specialized content knowledge (SCK)</td><td>Knowledge of content and students (KCS)</td><td rowspan="2">Knowledge of content and curriculum</td></tr>
<tr><td>Horizon content knowledge (HCK)</td><td>Knowledge of content and teaching (KCT)</td></tr>
</table>

Figure 2.11. Mathematical knowledge for teaching model.

Common content knowledge refers to “mathematical knowledge and skill used in settings other than teaching” (Ball et al., 2008, p. 399). This knowledge includes being able to correctly solve the problems, to correctly use mathematical terms, to identify students’ incorrect answers, and to realize wrong definitions in the textbooks. This knowledge is not unique to teaching since it can be possessed by other professions such as physics teaching or engineering (Delaney, Ball, Hill, Schilling, & Zopf, 2008).

Hill, Sleep, Lewis, and Ball (2007) defined specialized content knowledge as “mathematical knowledge that is used in teaching, but not directly taught to students” (p.

132). This knowledge is unique to teachers and to teaching. For instance, teachers need to know how numbers and operations are represented using diagrams and how mathematical concepts and algorithms are explained (Hill & Lubienski, 2007).

Horizon content knowledge is described as "an awareness of how mathematical topics are related over the span of mathematics included in the curriculum" (Ball et al, 2008, p. 403). It includes knowing how the same topic is taught in past and later grades. This knowledge provides teachers to holistically see mathematical connections.

Moreover, the first category of pedagogical content knowledge, the knowledge of content and students is considered "amalgamated knowledge that teachers possess about how students learn content" (Hill et al., 2007, p. 133). It includes being aware of which examples students are might find motivating, which type of task they might do, and which type of problem they might find confusing.

Knowledge of content and teaching is considered a combination of both knowledge of teaching and knowledge of content (Ball et al., 2008). This knowledge includes being able to design appropriate instruction and to determine proper examples for effective students' discussions related to mathematical ideas.

Lastly, the knowledge of content and curriculum is parallel with curricular knowledge in Shulman's model. It includes knowing which mathematical contents in the curriculum are related to each other.

2.5. Research on Teacher Knowledge on Multiplication and Division

Although a variety of researchers have conducted studies to investigate students' knowledge and understanding of multiplication and division (Fischbein et al., 1985; Anghileri, 1989; Byerley et al., 2012), a restricted number of studies focus on preservice teachers' understanding. In the literature, several researchers investigated the preservice teachers' misconceptions related to multiplication and division (Graeber, et al. 1989; Tirosh & Graeber, 1989; Tirosh & Graeber, 1990). Additionally, some others focused on preservice (Ball, 1990; Simon, 1993; Tekin-Sitrava, Özel, & Işık, 2020) and in-service teachers' conceptual knowledge of division (Lamb & Booker, 2004; Byerley & Thompson, 2014).

Particularly, Graeber and colleagues (1989) constructed a study to examine if preservice elementary teachers had the same misconceptions held by the students in the study of Fischbein et al. (1985). In order to accomplish this aim, a test with small changes including multiplication and division problems created by Fischbein et al. (1985) were presented to 129 female preservice teachers. They were asked to determine and write an expression with the correct operation that could be used to solve these problems. Then, 33 of them were interviewed to gain deeper information about their conceptions and reasoning related to the problems. The findings of the study demonstrated that more than 25% of the preservice teachers wrote a division expression for the multiplication problems where the operator (i.e., multiplier) was decimal less than 1 and the contexts directed the answer less than the given operand (i.e., multiplicand). Additionally, half of the interviewees who wrote a multiplication expression for the division problem including a decimal divisor reasoned that they determined the multiplication because the answer of the problem would be bigger than the given whole number (i.e., dividend) in the problem. These two situations showed that preservice teachers also had misconceptions such that “division always makes smaller” and “multiplication always makes bigger”. In the problems where the dividend was decimal and divisor was a whole number, 22 of 33 the interviewees changed the role of dividend and divisor by claiming that dividend should be greater than the divisor. However, one of the interviewees who reversed the role of decimal dividend and whole number divisor had to rechange the roles and answered the problem correctly due to the decimal divisor. Therefore, the researchers concluded that the misconception “divisor must be a whole number” had more impact than “dividend must be larger than divisor”.

Similarly, Tirosh and Graeber (1989) aimed to explicitly reveal preservice elementary teachers’ beliefs about multiplication and division. Participants consisting of 136 preservice teachers were asked to evaluate a set of six items related to the misconceptions/misbeliefs about multiplication and division mentioned above as true or false and to explain their justifications. Participants were also asked to solve some multiplication and division word problems by writing expressions and computing. Then, about one-half of the participants were interviewed. The results of this study were parallel to those obtained by Graeber et al. (1989). Although the percentage of the preservice teachers who correctly labeled the items about multiplication was high, the data obtained from the expressions of the participants demonstrated that about half of them wrote a division expression for the multiplication problem where the operator (i.e., multiplier) was decimal less than 1. This was also evidenced by the findings of the interviews. Furthermore, more than half of the preservice teachers explicitly believed that the *quotient must be smaller than the dividend*. This was supported by the fact that in the division problems including decimal divisor less than 1, about half of the participants wrote multiplication expressions because they expected the answer to be larger than the given whole number (i.e., dividend). All these findings indicated that the preservice *elementary* teachers implicitly held the misbeliefs, “multiplication always makes bigger” and “division always makes smaller”.

To analyze preservice teachers’ understanding of division, Ball (1990) interviewed 19 preservice *elementary* and *secondary* teachers about three division problems including fractions, zero decimal, and algebraic equations, respectively. They were asked to solve and create a representation for each problem. The results of the analysis of their responses showed that although most of the participants solved division by fraction problem correctly, only 5 preservice teachers created appropriate representation for the problem. The researcher reported that most preservice teachers thought division only in terms of partitive meaning. Besides, preservice teachers had significant difficulty in explaining what division by zero and division by algebraic expression meant. In the light of these all results, the researcher also concluded that preservice teachers were not able to make sense of the division problems and to think multiplicatively.

Simon (1993) administered open-ended problems to preservice elementary teachers and then interviewed some of them to examine their knowledge of division. According to the data from the written instrument, the division problems created by the preservice teachers were mostly partitive division (74%) and 17% of them were quotative division. While thirty percent of the participants were able to write a problem for division by a fraction, the rest were not. Moreover, none of the preservice teachers could develop a method to find the remainder of the given division by using a calculator. This pointed that they did not have knowledge of the units of the quantities in the division.

One of the studies focused on the impact of Professional Development on the students’ knowledge of division. Lamb and Booker (2004) investigated the relationship between the teachers’ conceptual knowledge after professional development and their students’ knowledge of division. The study was conducted in two phases one year apart because of the professional development sessions. While there were 24 seventh-grade students and one teacher in Phase 1, there were 23 seventh-grade students and one teacher in Phase 2. In both phases, students and their teachers were asked to solve division questions without any time limitation. Then, six students and their teachers from each phase were interviewed. According to the test results of the two questions from the six questions, in Phase 2, more students correctly solved the division algorithm than in Phase 1. Additionally, the findings of the interviews demonstrated that the explanations of the students in Phase 1 were based on the procedural understanding of the division operation. The researchers of the study asserted that these students' explanations resulted from their teacher who asked her students to apply the same procedures she showed to solve the problems. On the other hand, the Phase 2 teacher who experienced Professional Development integrated concrete materials and games into the lessons. Thus, her students had a more conceptual understanding of division because they comprehended the sharing by considering place value.

The relative size meaning of division is an almost untouched area in the literature. In the study designed by Byerley and Thompson (2014), secondary in-service teachers were asked to select the appropriate response for the two items to reveal their meanings related to relative size [21]. The results of the responses given for the first item indicated that most of the teachers did not think multiplicatively in the comparison of the relative size of the distance with respect to the time. This showed that they did not have the meaning of quotient (i.e., speed for this item) as a measure of relative size.

In addition to the international studies, preservice or in-service teachers' understanding of multiplication or division has not been emphasized sufficiently in Turkey. For example, Tekin-Sitrava et al. (2020) conducted a qualitative study to investigate 25 preservice elementary (primary) school teachers' knowledge on the meaning and modeling of division. For this purpose, preservice teachers were asked two open-ended problems related to sharing (i.e., partitive) and grouping (i.e., quotative) meanings of division, and then they were interviewed. The findings of the study indicated that about one-third of the preservice teachers had sufficient knowledge about the meanings of division, so they knew both sharing and grouping meanings of the division operation. More than half of them had *insufficient* knowledge because they knew one meaning of division. Only one teacher confused these two meanings. The results also revealed that about half of the preservice teachers had sufficient knowledge about the modeling of division because they correctly modeled the problems related to both sharing and grouping meanings. About one-third of them correctly modeled only one problem, so they did not have sufficient modeling knowledge. Only five preservice teachers had no modeling knowledge. Overall, it was concluded that the number of preservice teachers who knew sharing meaning was higher than the number of those who knew grouping meaning. On the contrary, the number of preservice teachers who knew the modeling of sharing division was less than the number of those who knew the modeling of grouping division. While preservice primary school teachers made sense of division as sharing, they modeled division problems by thinking of grouping meaning. The researchers explained the reason for this situation such that grouping the total number of objects by the number of objects in a group may be easier modeling for students, preservice teachers, or in-service teachers than distributing the total number of objects by a certain number (e.g., one by one).

3. METHODOLOGY

3.1. Research Design

This study aims to investigate preservice senior middle school mathematics, middle school science, secondary mathematics, chemistry, and physics teachers' content knowledge on the concepts of multiplication and division. Data were collected from five different universities in Turkey. Since the data were gathered from the participants at a single point in time, a cross-sectional survey design was used (Creswell, 2015).

3.2. Participants

The participants of this study were selected from preservice senior university students studying middle school mathematics, middle school science, secondary school mathematics, chemistry, and physics education programs whose language of education is English in Turkey because the items in the instrument were developed in English. In Turkey, there are five universities having middle school mathematics education program whose language of education is English. In only two of them have all these five education programs whose language of education is English. Therefore, preservice senior university students studying currently on middle school mathematics, secondary mathematics, science, chemistry, and physics education programs in these five universities were selected conveniently as the sample of the study. The participants were asked to voluntarily participate in the study. In this study, the abbreviations were used for preservice teachers (i.e., PMMT for preservice middle school mathematics teachers, PSMT for preservice secondary mathematics teachers, PSCT for preservice science teachers, PPHT for preservice physics teachers, PCHT for preservice chemistry teachers). Totally, 111 preservice teachers participated in the study. The number of senior preservice teachers who were expected to be involved in the study and the number of senior preservice teachers who participated in the study are shown below in Table 3.1. While 252 participants were expected to participate the study, 111 of them participated in the study.

Table 3.1. The expected and participated numbers of preservice teachers.

University	A	B	C	D	E	Total
PMMT (Expected)	40	36	10	13	20	119
PMMT (Participated)	8	27	8	12	17	72
PSMT (Expected)	13	12	-	-	-	25
PSMT (Participated)	6	6	-	-	-	12
PSCT (Expected)	32	22	-	-	-	54
PSCT (Participated)	7	10	-	-	-	17
PCHT (Expected)	14	9	-	-	-	23
PCHT (Participated)	8	1	-	-	-	9
PPHT (Expected)	15	16	-	-	-	31
PPHT (Participated)	1	0	-	-	-	1

3.3. Instrument

The aim of the study is to characterize preservice teachers' understanding on the concepts of multiplication and division. During the development of the items in the instrument, what was wanted to measure (i.e., preservice teachers' understanding of multiplication and division multiplicatively) was firstly clearly determined and components of it were specified in the light of the related literature.

Regarding the related literature, Smith and Smith (2006) emphasized the importance of understanding of four concepts which are "quantity", "multiplicative problem situations", "equal groups", and "units relevant to multiplication" for understanding of multiplication. Firstly, understanding quantity includes interpreting a numerical value of quantity and a unit of quantity. Understanding multiplicative problem situations includes making sense of word problems having multiplicative situations and an ability to discern the relationship between multiplication and division problems. Also, understanding equal groups is related to understanding of function of equal groups in multiplicative situations. Finally, understanding units relevant to multiplication refers to being attentive to the fact that the quantities in multiplication have different units. It is about considering multiplication as a referent transforming composition defined by Schwartz (1988).

In addition, Otto et al. (2011) identified six characteristics of problem solvers who have robust understanding of multiplication (i.e., understanding of multiplication multiplicatively) as follows: (1) "Describe many varied situations in a multiplicative manner" (2) "Use multiple representations to describe multiplicative situations" (3) "Provide justifications for using multiplication as an operation" (4) "Make connections among different representations" (5) "Trace the "answer," or product, of a multiplicative expression to the interpretation of a representation" (6) "Give meanings to the factors in a multiplicative expression".

Based on these descriptions, three comprehensive components indicating proper understanding of multiplication and division multiplicatively were determined. These components were labeled as "indicators" in the current study. These three indicators are:

- Understanding of a quantity (understanding units relevant to multiplication and division)
- Figural representation (pointing to the underlying reasoning)
- Understanding of multiplicative problem situations (the meaning of multiplicative situations)

After that, an item pool was generated and formats for the items were decided. For this step, the problems and tasks included in the related literature were comprehensively reviewed by the researcher. All of them were organized in the table by grouping in terms of the name of the study, research questions of the study, problems/tasks, page number, and rationale behind the problem. After all the problems in this table were examined, the researcher and the advisor wrote the items and created the item pool. The item pool firstly included 20 items consisting of multiplication and division problems written based on the indicators.

While some multiplication and division problems in the items were modified from the previous studies (e.g., Vergnaud, 1983; Greer, 1992; Simon, 1993; Van de Walle et al., 2010; Otto et al., 2011; Thompson, 2011) and the textbook (e.g., Fujii & Iitaka, 2012), others were written by the researcher and the advisor. In addition to the problems, the items themselves were created by the researcher and the advisor after many discussions.

Additionally, the items in the instrument were developed in the context of multiplicative situations explained in the previous chapter. Multiplicative situations containing whole numbers or positive integers are mainly classified into four groups which are equal groups or sets, multiplicative comparison, rectangular array/area, and cartesian product (combination) (Greer, 1992; Watanabe, 2003; Van de Walle et al., 2010).

Then, a document was designed by the researcher for the expert review of the item pool. This document included the information about the research topic, the purpose of the study, the sample of the study, research questions, how multiplication and division are conceptualized by the researcher, and how the items in the instrument were developed. The document was sent to four academics who are experts in mathematics education and who has done research on multiplication and division and measurement and evaluation. Based on the information included in the document, experts were requested to provide their expert opinions on the followings:

- How do the items and the indicators align? (i.e., Are the items appropriate to measure the indicators?)
- Rubric including possible responses and possible scoring for each item is also provided. How do the items align with the rubric? (i.e., Is the rubric appropriate for the assigned items?)
- Is it possible to check the language of the items?

They were requested to state their opinions, explanations, or recommendations in the table being after each item rubric. This table in the expert review document is demonstrated in Figure 3.1.

	APPROPRIATE	**PARTIALLY APPROPRIATE**	**NOT APPROPRIATE**
Is the item appropriate to measure the indicator?			
Explanation:			
Is the rubric appropriate for the item?			
Explanation:			
Is the language appropriate?			
Explanation:			

Figure 3.1. The table in the expert review document.

In the light of experts' comments and suggestions related to each item, 20 items in the instrument were edited and some of the items were excluded. Besides, the language of some problems were improved, and the scoring of the items were decided. For instance, in the initial items asking to draw more than one pictorial figure, all experts had a common comment about that asking for two or more was problematic. They asserted that there was no consensus on what constituted "different" among these pictorial figures. Therefore, these items were revised and then, participants were asked to draw one pictorial figure in the finalized instrument. Moreover, experts recommended the use of standardized scoring for the items as 0, 1, 2 and scoring of the responses were determined in this way. Some numbers and words included in the initial items were changed by regarding experts' comments. For example, in the initial version of item 3, participants were asked to write "multiplicative" word problem. However, two of the experts specified that this terminology might be taken as "multiplication" by preservice teachers, or this might not be understood. So, it was changed, and participants were asked to write "multiplication or division" word problem. Finally, 15 items were remained in the instrument according to the indicators. Indicators and task characteristics of each item in the instrument were shown in Table 3.2 below. Besides, in this step, one preservice teacher was asked to solve the items in the instrument for deciding the duration of the implementation of the instrument and for receiving feedback about the clarity and comprehensiveness of the items.

Table 3.2. Indicators and task characteristics of the items in the instrument.

Item	**Indicator**	**Task Characteristics**
Item 1	Understanding of a Quantity	Equal Groups Positive Integer
Item 2	Figural Representation	Equal Groups Positive Integer
Item 3	Understanding of Multiplicative Problem Situations	Rectangular Array Positive Integer
Item 4	Understanding of a Quantity	Multiplicative Comparison Decimal

Table 3.2. Indicators and task characteristics of the items in the instrument. (cont.)

Item	Indicator	Task Characteristics
Item 5	Figural Representation	Multiplicative Comparison Decimal
Item 6	Understanding of Multiplicative Problem Situations	Any Multiplicative Situation Positive Integer
Item 7	Understanding of Multiplicative Problem Situations	Rectangular Area Any Number Structure
Item 8	Understanding of Multiplicative Problem Situations	Rectangular Area Decimal
Item 9	Understanding of a Quantity	Equal Groups Decimal
Item 10	Figural Representation	Equal Groups Decimal
Item 11	Understanding of a Quantity	Equal Groups Fraction
Item 12	Understanding of Multiplicative Problem Situations	Any Multiplicative Situation Positive Integer
Item 13	Understanding of Multiplicative Problem Situations	Any Multiplicative Situation Positive Integer
Item 14	Understanding of a Quantity	Multiplicative Comparison Positive Integer
Item 15	Figural Representation	Multiplicative Comparison Positive Integer

According to Table 3.2 above, five items (i.e., item 1, item 4, item 9, item 11, and item 14) were created for the indicator “Understanding of a Quantity”. Multiplication and division are arithmetic operations that produce new quantity having a referent (i.e., unit). Thus, the quantity (especially intensive quantity) should be properly understood to conceive multiplication and division (Schwartz, 1988; Smith & Smith, 2006). These items were designed with the aim of eliciting preservice teachers’ ability to explain the referents of three elements in their solutions for the multiplicative problems. Additionally, while some of the items (i.e., item 1, item 9, and item 11) included equal groups situation, two of them (i.e., item 4 and item 14) included multiplicative comparison situation. Since equal groups situation is the first emphasized situation in introducing multiplication and division (Watanabe, 2003; Greer, 1992), it was considered that most of the participants correctly solved the problem. Hence, their explanations about the quantities were more emphasized rather than their solutions. Some resources call equal group problems as "repeated addition" problems (e.g., Van de Walle, et al. 2010). The reason of it can be that repeated addition is a primitive model for whole number multiplication. Also, because multiplicative comparison situation directly assigns the understanding of multiplication and division multiplicatively, the explanations of preservice teachers in these three items were very important. Furthermore, since the problems in two items (i.e., item 1 and item 14) included a whole number, it can be easily detected whether preservice teachers used additive method to solve the problems. Non-integer numbers (i.e., decimal or fraction) were used in the other three items to investigate participants' understanding by eliminating the use of additive methods. Particularly, the problem in item 11 was modified from one of the problems in the study conducted by Simon (1993). This problem required further thinking on the quantities and their units since it did not ask the quantity (i.e., number of cookies) that was a result of division operation.

Also, as given in Table 3.2, four items (i.e., item 2, item 5, item 10, and item 15) were developed under the indicator "Figural Representation". The ability to use multiple representations in identifying multiplicative situations is one of the hallmarks for a proper understanding of multiplication (Otto et al., 2011). The representation can be pictorial, so these items expected participants to draw pictorial figures representing the problems in previous items. Showing their own representations is also important to reveal preservice teachers' understanding of the quantities in the problems and of the operations used to solve them. In these items, accompanying explanation was also asked from the participants because it is "always necessary to clarify the reasoning behind an expression or equation" (Otto, et al., 2011, p. 13).

Finally, six items (i.e., item 3, item 6, item 7, item 8, item 12, and item 13) were written under the indicator "Understanding of Multiplicative Problem Situations". These items were directed to write multiplicative problems and to explain multiplicative relationship between the quantities in the problems. For example, in item 3, a rectangular array model was presented for participants to write a multiplication or division problem related to this model, so it gives participants an opportunity to use different multiplicative situations in their own problems. They were also expected to explain why they chose which operation for the problem. Being able to justify their use of an operation provides evidence of the understanding the meaning of multiplication or division (Otto et al., 2011). Especially, item 7 was modified from one of the problems in the study of Thompson (2011). For understanding of why the multiplication of n and m is meaningful to get the area of the rectangle having n inches wide and m centimeters high, one needs to conceptualize that unit of area is multiplicatively derived (i.e., inch-cm) (Thompson, 1995).

Since the items were devised in a constructed response format, the participants' responses were polytomously scored as 0, 1, and 2. The scores for each item were explained in detail below (Table 3.3) because the nature of scoring in each item was different. However, labeling 0 points as incorrect, 1 point as partial, and 2 points as correct were common for each item. In addition to the scores for the participants' responses, numerical response codes also were assigned for categorizing preservice teachers' responses. These response codes were uniquely determined for each item (see in Appendix B). Although there were already possible responses in the initial rubric before implementing the instrument, response categories were developed and shaped with reading the data as a whole and item by item. The rubric including preservice teachers' responses and the scoring related to these responses in the current study were described in Table 3.3 below. There were also response categories in the rubric where no participants' response was included in that category. These were given in Appendix B.

Table 3.3. Rubric for the participants' responses.

	0	1	2
Item 1 Item 4 Item 9 Item 14	-no answer/solution -incorrect solutions -no explanation or incorrect explanations for a correct solution	-insufficient explanations (i.e., writing one or two referents) for a correct solution	-a sufficient explanation (i.e., writing three referents) for a correct solution
Item 11	-no answer/solution -incorrect solutions	-partial solution -correct solutions with insufficient explanations	-correct solutions with correct explanations

Table 3.3. Rubric for the participants' responses. (cont.)

	0	1	2
Item 2 Item 5 Item 10 Item 15	-no answer/pictorial figure -incorrect or additive pictorial figures	- no explanation, incorrect explanations, insufficient explanations for correct pictorial figures -only calculations with multiplication/division or/and repeated addition for correct pictorial figures	-multiplicative explanations for correct pictorial figures
Item 3 Item 6	-no answer -incorrect problems	- no explanation, incorrect explanations, insufficient explanations for correct multiplication/division problems -only calculations with multiplication/division or/and repeated addition for correct multiplication/division problems	-multiplicative (or sufficient) explanations for correct multiplication/division problems
Item 7	-no answer -"not meaningful" answer -incorrect explanations for "meaningful" answer -only calculations with multiplication or repeated addition for "meaningful" answer	-insufficient explanations for "meaningful" answer	-multiplicative explanations for "meaningful" answer
Item 8	-no answer -incorrect explanations -only drawing -only calculations with multiplication or/and additive explanations	-insufficient explanations	-multiplicative explanation
Item 12	-no answer -incorrect problems	-no explanation -incorrect explanations -insufficient explanations (i.e., writing one or two referents) for their correct division problems	-sufficient explanation (i.e., writing three referents) for their correct division problems
Item 13	-no answer/explanation -incorrect explanations -additive relationship -only calculations with multiplication/division or/and repeated addition	-insufficient explanations	-multiplicative relationships

3.4. Data Collection

Finalized items (i.e., 15 items) in the instrument were administered to preservice teachers at a single point in time. The data were collected from five universities in Turkey in the second semester of 2021-2022 academic year. Firstly, the instructors from relevant departments in these universities were reached to give an information about the study and to ask permission to collect data in their classes.

In University A, the instructors did not give permission to implement the items to the preservice teachers in their classes. However, they allowed to give brief information about the study and to reach volunteer preservice teachers who wanted to participate in the study. Due to the COVID-19 pandemic conditions, some courses were conducted face-to-face on campuses and some of them were conducted online. Therefore, face-to-face courses were attended to inform preservice teachers about the study and then, consent form and the instrument were distributed for each volunteer participant. In addition, their contact information (i.e., a telephone number or an e-mail address) was also taken to reach them later. Online courses were also attended to inform them and then again, consent form and the instrument were given in pdf format to volunteer participants. Their contact information was also obtained by them. The volunteer participants were given a week to solve the items. Some of these participants delivered the instrument as a hard copy and some of them sent to their responses on the instrument in pdf format by e-mail.

In University B, one of the research assistants in the department had been contacted before to ask for help in data collection process. Consent forms and instruments (as many as the number of students in the classes) as well as the document of ethics committee approval for each class were sent to the research assistant by cargo to deliver them to the instructors. The instructors were requested to administer the instrument to the preservice teachers under their supervision during classroom hours. However, the instructor of PPHT (i.e., preservice physics teachers) could not be reached, so the data from PPHT could not be collected. Some of the instructors implemented the instrument to preservice teachers during classroom hours. Some instructors distributed the instruments to preservice teachers to solve the items out of classroom hours and they wanted them to bring the instruments in the next class. After the data collection process was finished in University B, the research assistant collected all documents from the instructors and sent them back again by cargo.

In University C, consent forms and instruments (as many as the number of students in the classes) as well as the document of ethics committee approval for each class were sent to the instructor of PMMT (i.e., preservice middle school mathematics teachers) by cargo. Under the supervision of the instructor, preservice teachers solved the items during a classroom hour. After that, all documents were received from the instructor by the researcher.

In University D, the instructor of PMMT gave permission the researcher to administer the items to the preservice teachers in the class. Finally, in University E, consent forms and instruments (as many as the number of students in the classes) as well as the document of ethics committee approval for each class were sent to the instructor of PMMT by e-mail. After printing all documents, the instructor implemented the instrument to preservice teachers during a classroom hour. After that, the instructor sent all documents back again by cargo.

3.5. Data Analysis

In this section, the procedures for scoring and coding data were clarified. Then, descriptive statistics related to the responses of participants for each item, statistical analyses consisting of reliability, validity, and item analyses (i.e., item difficulty and item discrimination) on the instrument were explained.

3.5.1. Descriptive Statistics

Firstly, data were read by the researcher one by one according to initial rubric including scoring for possible responses. However, scores were not given to the responses in the items at this point. While reading the data, participants' responses were categorized by assigning numerical response codes unique to each item. The response categories that were not included in the initial rubric were added to the finalized rubric with their scores and codes. After that, participants' responses were read item by item according to the finalized rubric including scoring and response codes for each response category. At this point, both scores (i.e., 0, 1, 2) and response codes were given to participants' responses in each item. Then, descriptive statistics (i.e., frequencies) based on the percentages of the scores and response codes in each item were calculated to analyze and characterize preservice teachers' performances for each item on the indicators.

3.5.2. Reliability

Reliability refers to "the degree to which a test consistently measures whatever it is measuring" (Gay, Mills, & Airasian, 2012, p. 165). It is the consistency (i.e., reproducibility) of the scores. There are five types of reliability which are stability (test-retest reliability), equivalence (equivalent-forms reliability), equivalence and stability, internal consistency reliability, and scorer/rater reliability (Gay, et al., 2012). Since in the current study, participants' responses were scored item by item, the reliability of scoring is significant. Thus, inter-rater reliability was concerned in the study. Inter-rater (i.e., interjudge) reliability refers to "the consistency of two or more independent scorers, raters, or observers" (Gay et al., 2012, p. 168). For inter-rater reliability, Cohen's kappa coefficient (κ) that measures the agreement between two scorers was used (Cohen, 1960). The coefficient is calculated by the formula of

$$\kappa = \frac{(p_0 - p_e)}{1 - p_e}, \tag{3.1}$$

where p_0 is the proportion of units in which the scorers agree and p_e is "the proportion of units for which agreement is expected by chance" (Cohen, 1960, p. 39).

In this study, the other scorer (i.e., independent from the researcher of the study) was voluntarily selected as an in-service primary school teacher knowing English. Firstly, she was informed about the study and the items. Then, the nature of scoring of the items (i.e., 0, 1, and 2) was explained to her. After copies of the data collected (without participants' personal information) were given the other scorer, she gave a score for participants' responses for each item.

According to Cohen's kappa coefficients as seen in Table 3.4, there was perfect agreement between two scorers in item 3, item 7, item 8, and item 13 since the coefficient of these items was 1. Also, there was almost perfect agreement between two scorers in item 1, item 2, item 4, item 5, item 6, item 9, item 10, item 12, item 14, and item 15 since their coefficients were higher than 0.81 (Landis & Koch, 1977). The coefficient of item 11 (i.e., 0.80) indicated substantial agreement between the scorers. So, it can be concluded that the inter-rater reliability/agreement was provided.

Table 3.4. Cohen's kappa statistic for the items.

Item	**Cohen's kappa coefficient (κ)**
Item 1	0.98
Item 2	0.91
Item 3	1.00
Item 4	0.96
Item 5	0.90
Item 6	0.94
Item 7	1.00
Item 8	1.00
Item 9	0.98
Item 10	0.90
Item 11	0.80
Item 12	0.96
Item 13	1.00
Item 14	0.97
Item 15	0.81

3.5.3. Validity

Validity is "the degree to which a test measures what it is supposed to measure and, consequently, permits appropriate interpretation of scores" (Gay, et al., 2012, p. 160). There are fundamentally three types of validity which are content validity, criterion-related validity, and construct validity (DeVellis, 2017). In this study, content validity that is "the degree to which a test measures an intended content area" was included (Gay, et al. 2012). It was determined by taking experts' opinions on the appropriateness of the items for measuring the indicators (i.e., the intended content area), the appropriateness of the rubric for the items, and the appropriateness of the language of the items.

3.5.4. Item Difficulty and Item Discrimination

Classical Test Theory (CTT) was used as a framework for item analyses. This framework assumes that a person's score on a test is the sum of her true score and measurement error (Crocker & Algina, 2008). In this regard, item difficulty and item discrimination were computed. Item difficulty is the proportion of participants who correctly answer the item (Crocker & Algina, 2008). Since the items were developed in a constructed response format (i.e., polytomous items coded as 0, 1, 2), the mean of the participant scores in each item was calculated to show the difficulty levels of the items. Although participants' total scores were not considered in this study, item discrimination was also evaluated to have more information about the items and to provide a support results related to item difficulty. Item discrimination provides information about how effectively an item in discriminating participants who have high scores and have low scores (Crocker & Algina, 2008). For this, item-rest correlation (RIR) which is the correlation between the item and the total score excluding that item was calculated.

4. RESULTS

In this section, results of the current study will be presented in two parts. Firstly, item analyses of the items (i.e., item difficulty and item discrimination) in the instrument will be interpreted. Additionally, preservice teachers' performances on the instrument will be examined regarding three indicators which are "understanding of a quantity", "figural representation", and "understanding of multiplicative problem situations".

4.1. Item Difficulty of the Items on the Instrument

Considering item difficulty in Table 4.1, it can be stated that item 7 which is about evaluating the meaningfulness of the multiplication to find the area of a rectangle with different units of the side lengths, item 8 which is about explaining the relationship between two rectangles, item 10 which is about drawing a pictorial figure with an explanation for a multiplication problem including non-integer multiplier, and item 13 which is about constructing the multiplicative relationship in a division problem were very difficult items ($p = 0.02$, $p = 0.06$, $p = 0.10$, and $p = 0.02$, respectively). Additionally, item 5 which is about drawing a pictorial figure with an explanation for multiplicative comparison problem, item 11 which is about solving a multiplicative word problem including part-whole relationship between the quantities, and item 15 which is about providing a representation for the multiplicative comparison problem were difficult items ($p = 0.48$, $p = 0.53$, and $p = 0.56$, respectively). The items in the instrument excluding these seven (i.e., item 1, item 2, item 3, item 4, item 6, item 9, item 12, and item 14) were moderately difficult (i.e., having p values 1.36; 0.84; 0.70; 0.86; 1.19; 1.04; 1.24; 0.86 respectively). It was concluded that the items in the instrument were difficult ($p_{mean} = 0.66$).

According to the item-rest correlation (RIR) demonstrated in Table 4.1, item 9, item 12, and item 14 were good discriminators since the discrimination indices of these items were higher than 0.30. The items (i.e., item 1, item 4, item 5, item 6, item 11, item 13, and item 15) whose discrimination indices ranged between 0.2 and 0.3 were moderately discriminative. Since their indices are lower than 0.30, these seven items might be problematic, so they can be revised to be used in the instrument. The discrimination indices of the other five items (i.e., item 2, item 3, item 7, item 8, and item 10) were very low, so these items were problematic. Regarding the item difficulties for the item 7, item 8, and item 10 (having *p* values 0.02, 0.06, 0.10, respectively), it can be deduced that these three items were very difficult for all preservice teachers in the study. This might be the reason why the discrimination between the participants having high scores and participants having low scores on the instrument were low (i.e., having r values -0.041, 0.074, and -0.015, respectively).

Table 4.1. Item difficulty and item discrimination values.

Item	Item Difficulty	Item Discrimination
Item 1	1.36	0.253
Item 2	0.84	0.084
Item 3	0.70	0.110
Item 4	0.86	0.261
Item 5	0.48	0.270
Item 6	1.19	0.227
Item 7	0.02	-0.041
Item 8	0.06	0.074
Item 9	1.04	0.303
Item 10	0.10	-0.015
Item 11	0.53	0.219
Item 12	1.24	0.373
Item 13	0.02	0.250
Item 14	0.86	0.305
Item 15	0.56	0.222

4.2. Analysis of Preservice Teachers' Performances for Each Item on the Indicators

A distribution of preservice teachers' responses for each item were shown in Figure 4.1. Responses of the participants who got 0 points were labelled as incorrect, responses of those who got 1 point were labelled as partial, and responses of those who got 2 points were labeled as correct.

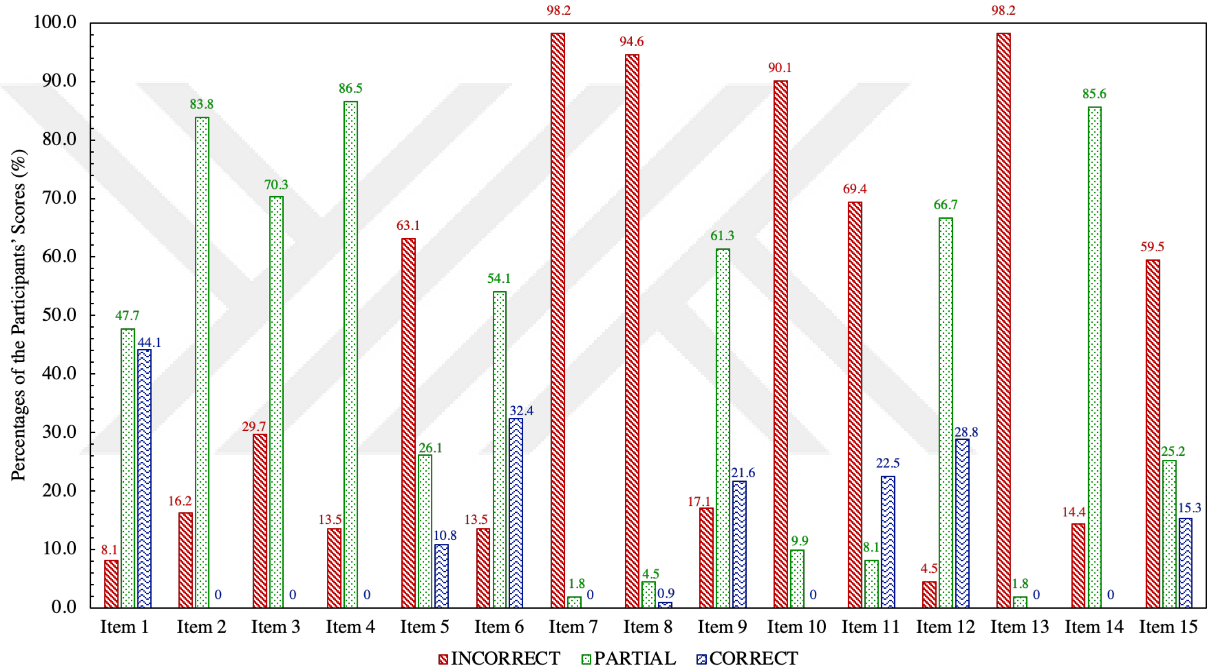

Figure 4.1. Preservice teachers' responses for each item.

According to Figure 4.1, no preservice teachers got full points from the seven items (i.e., item 2, item 3, item 4, item 7, item 10, item 13, and item 14). Particularly, in both item 4 and item 14 about the indicator of understanding of a quantity, preservice teachers were not able to write the referents of the three elements in their correct solutions. Besides, in both item 2 and item 10 about the indicator of figural representation, they were not able to provide multiplicative explanations for the correct pictorial figures they drew. Finally, in item 3, item 7, and item 13 about the indicator of understanding of multiplicative problem situations, none of the participants were able to provide explanations in terms of multiplicatively understanding. All these suggested that under each indicator, there were the items in which the participants were not able to provide a correct response. These results might be partly due to the item difficulty levels since the difficulty level of the items 4 and 14 were 0.86, the difficulty level of the items 2 and 10 were 0.84 and 0.10, respectively, and the difficulty level of the items 3, 7, and 13 were 0.70, 0.02, and 0.02, respectively as the data earlier indicated.

4.2.1. Preservice Teachers' Performances Regarding Their Understanding of a Quantity

For the indicator "Understanding of a Quantity", there were five items (i.e., item 1, item 4, item 9, item 11, and item14) in the instrument. They were related to understanding referents (i.e., units) relevant to multiplication or division. Therefore, these items were directed to explain the referents of the elements in the participants' solutions for kinds of the problems in the items. Among these items in this indicator, while item 11 was more difficult than the others for participants, item 1 was easier than the rest as illustrated in Figure 4.1 above. Hence, in this section, preservice teachers' responses were clarified item by item.

Item 1 included multiplication problem stating that "Marry has 438 bags of oranges. There are 6 oranges in each bag. How many oranges does Marry have altogether?". This item expected to explain the referent of each element in the participants' solution for the problem having equal groups situation. Participants were expected to correctly describe the quantities and their referents in this problem as 438 bags (i.e., extensive quantity), 6 oranges in each bag or 6 oranges/bag (i.e., intensive quantity), and 2628 oranges (i.e., extensive quantity).

Particularly, according to Figure 4.1, most of the preservice teachers (91.8% of all participants) were able to address at least one referent in their correct solutions. However, only 48% of them (44.1% of all participants) were able to write the referents of three quantities. Also, 52% of them (47.7% of all participants) specified the referent of one or two quantities (extensive or intensive) in their correct solutions. In addition, among these participants who correctly wrote at least one referent, 53% of them (48.6% of all participants) were able to explain the referent of the intensive quantity as 6 oranges/bag in the problem. Other participants only emphasized extensive quantities. These results revealed that only almost half of the preservice teachers were aware of the intensive quantity and know the referent of it. Relatively high percentage of the participants who got full points (i.e., correctly solved the item) on this item might have stemmed from several reasons. Firstly, the problem had a familiar context for participants since they may have encountered such problems even in their daily lives and experienced finding the total number of objects when the number of objects in each group and number of groups were known. Secondly, "equal groups" including whole number is the first given situation in introducing multiplication at schools. So, participants may have been more familiar with this situation.

In addition, as seen in Figure 4.1, 8.1% of the participants (i.e., only 9 participants) did not correctly solve the item (i.e., they did not correctly write any referents) and some of them did not answer the item. Among them, there were also incorrect solutions such as dividing 438 oranges in total by 6 oranges in each bag to find 73 bags. Furthermore, while the responses that did not include any explanation for the referents were considered incorrect response, incorrect explanations for the referents were also seen as incorrect response. For instance, some participants provided multiplication of 438 (oranges) and 6 (bags) as incorrect explanations for the referents. These results suggested that there were even 9 preservice mathematics and science teachers who cannot correctly determine the quantities in whole number multiplication problem having equal groups situation.

Item 4 consisted of multiplication problem having a multiplicative comparison such that "Gold is 1.8 times as heavy as silver. If a piece of silver weighs 5 kilograms, how much does a piece of gold the same size weigh?". The item also necessitated explaining the quantities and their referents in preservice teachers' solutions. The quantities with their referents for this problem were specified as 5 kg (i.e., extensive quantity), 1.8 kg/kg (i.e., intensive quantity) , and 9 kg (i.e., extensive quantity).

As demonstrated in Figure 4.1., 86.5% of the participants were able to indicate one or two referents from two extensive quantities, but not the intensive quantity. Besides, none of the participants were able to correctly specify the referents of three elements. These suggested that none of the preservice teachers were able to write the referent of the intensive quantity as 1.8 kg/kg in the problem. These also pointed that preservice teachers were not able to comprehend the nature of the scale conversion factor that is a kind of an intensive quantity. This might partly be since the value of the intensive quantity was non-integer. Additionally, the participants describing “1.8” as the ratio of the weight of a piece of gold to the weight of a piece of silver may have thought that the ratio had no unit or referent.

Also, Figure 4.1 showed that 13.5% of the preservice teachers were not able to correctly define any referents. Among these participants, there were also those who provided incorrect solutions as well as those who did not answer the item. Incorrect solutions were based on miscalculation with multiplication like “5 × 1.8 = 5.40” or “5 × 18/10 = 5.40”. For the referents, also the responses of the participants who provided no explanation or provided incorrect explanation were considered incorrect for the item.

Item 9, under the indicator of understanding of a quantity, required writing the referent of each element for the equal groups multiplication problem asked as “A farm of 46.4 ha produces 6750 kg of corn per ha. What will be the yield?”. For this problem, the quantities were defined as 46.4 ha (i.e., extensive quantity), 6750 kg/ha (i.e., intensive quantity), and 313 200 kg (i.e., extensive quantity).

As Figure 4.1 showed, most of the participants (82.9% of all participants) were able to specify at least one referent in their correct solutions. Particularly, only 26% of them (21.6% of all participants) were able to write the referent of three quantities. In addition, more than half of the preservice teachers (61.3% of all participants) specified the referent of one or two quantities (extensive or intensive) in their correct solutions. Among these preservice teachers addressing at least one referent, only 41.3% of them (34.2% of all participants) were able to explain the intensive quantity as 6750 kg/ha. However, others only specified extensive quantities. Although both comprised equal groups situation, the participants were more successful in emphasizing the intensive quantity in item 1 than in item 9. There may be various underlying reasons for this result. To exemplify, preservice teachers may have not conceived the unit “ha” in item 9 since it is not a commonly used unit in area word problems. Also, while the problem in item 1 involved whole number, item 9 involved a decimal number (i.e., non-integer).

Moreover, 17.1% of the preservice teachers did not correctly point any referents of the quantities. Among them, apart from the participants who did not answer the item, some participants were not able to correctly solve the problem because they divided 6750 by 46.4 or divided 46.4 by 6750. In addition to those, who did not write any explanation for the referents, the responses of the participants who incorrectly explained the referents like “6750 ha” were regarded as incorrect.

Item 11 involved a multiplicative word problem including part-whole and part-part relationship between the quantities such that “Jason has 32 cups of flour. He makes cookies that require 3/8 of a cup each. If he makes as many such cookies as he has flour for, how much flour will be left over?”. Preservice teachers were expected to solve this problem and then, to explain their solutions by pointing to each element and the unit of each element in their solutions.

As indicated in Figure 4.1, only 22.5% of the participants were able to correctly solve and explain the problem. Some of them investigated how many cookies were made from 32 cups of flour if one cookie was made from 3/8 cups of flour. They claimed that the number of cups per cookies was 3/8 times as much as the number of cookies made. So, they divided 32 (cups) by 3/8 (cups per cookie) and found 85 1/3 (cookies). Then, they decided that 85 cookies were made in total, and 1/3 of the cookies (i.e., $1/3 \times 3/8$ cups $= 1/8$ cups of flour) were left over.

On the other hand, most participants (69.4% of them) were not able to correctly solve the problem. So, this item was the most difficult item on the indicator of understanding of a quantity. Although the context was familiar for the participants in both daily lives and word problems, they may have had difficulty in comprehending and solving the problem since it included division by fraction. In particular, some of them who incorrectly solved the problem multiplied 32 by 3/8. They thought that 3/8 of 32 cups (i.e., 12 cups) were needed to make one cookie, so 24 cups were needed for two cookies. Finally, 24 cups of flour were used, and 8 cups were left over. This showed that these preservice teachers were not able to comprehend the problem because the problem stated that 3/8 cups were needed for one cookie, not 12 cups. In addition, some participants also multiplied 32 by 3/8. They thought that because 3/8 of each cup were used, 32 × 3/8 (i.e., 12) cups of flour were used in total. So, 32 – 12 = 20 cups were left over. Likewise, some of them multiplied 32 by 5/8. They thought that since 3/8 of each cup were used, 5/8 of each cup were left over. So, 32 × 5/8 (i.e., 20 cups) were left over in total. These preservice teachers whose answer were "20 cups" did not mention the amount of cookies made and they were not able to realize that the cookies could still be made from 20 cups of flour. In addition, some of the participants who gave an incorrect solution divided 32 by 3/8 (answer is 85 1/3). They asserted that 85 cookies were made, and 1/3 cups of flour were left over. All these responses suggested that most of the preservice teachers were not able to conceive the relationship between the quantities (i.e., the number of cups and the number of cookies made). In other words, they were not aware of the fact that the number of cups of flour per cookie was 3/8 times as much as the number of cookies.

Finally, Figure 4.1 further indicated that 8.1% of the participants were not able to provide a correct explanation despite correctly solving the problem. Their responses included partial solution or insufficient explanation that did not include any solutions or comments about the amount of remaining cups of flour.

Item 14 contained a division problem with a multiplicative comparison situation such that "An adult lion weighs 6 times as much as a lion kitten. If the adult lion weighs 72 kg, how much does the kitten weigh in kg?". It also asked to explain the quantities with their referents as 72 kg (i.e., extensive quantity), 6 kg/kg (i.e., intensive quantity), and 12 kg (i.e., extensive quantity).

As Figure 4.1 illustrated, 85.6% of the preservice teachers were able to express one or two referents from two extensive quantities, but not the intensive quantity. Also, any participants were not able to correctly write the referents of three quantities. These revealed that none of the preservice teachers were able to indicate the referent of the intensive quantity as 6 kg/kg in the problem. The participants describing "6" as the ratio of the weight of an adult lion to the weight of a lion kitten may have thought that the ratio had no unit or referent. Comparing these results with the aforementioned, even though item 4 included non-integer number in a multiplicative comparison situation, distribution of the participants' responses for these two items (i.e., item 4 and item 14) were very similar. This showed that number structure may not have influenced participants' responses related to quantities since the situations were the same. On the contrary, results seem to point that preservice teachers might have difficulties in determining the referents of especially the intensive quantities in problem situations involving multiplication and division. This was further indicated by the fact that 14.4% of the preservice teachers did not correctly address any of the referents. Among them, there were also those who provided incorrect solutions as well as those who did not answer the item. Incorrect solutions were based on the multiplication like 72 kg × 6 = 432 kg" and on miscalculation with division like "72 kg ÷ 6 = 9 kg". Likewise, for the referents, the responses of the participants who made no explanation or made incorrect explanation were considered as incorrect response for the item.

To sum up, regarding the participants' responses to these five items related to writing the units relevant to multiplication or division problems, it can be deduced that preservice teachers had difficulty in specifying the referents of the quantities, especially of the intensive quantities. Moreover, their performances in writing the referents of three quantities of the problems with equal groups situation (i.e., item 1 and item 9) were better than of the those with multiplicative comparison situation (i.e., item 4 and item 14). In multiplicative comparison situations, the unit of the intensive quantity (i.e., kg/kg) were not determined by any participants although some of them emphasized that this quantity was the ratio of the weights. This may be since "equal groups" which is the most emphasized situation in textbooks and literature may give participants more opportunity to examine each quantity with their referents, separately like "number of objects in each group", "number of groups", and "total number of objects". On the other hand, in multiplicative comparison situation, participants may have tended to ignore the referent of ratio (e.g., the scale conversion factor).

4.2.2. Preservice Teachers' Performances Regarding Figural Representation

For the indicator "Figural Representation", there were four items (i.e., item 2, item 5, item 10, and item 15) in the instrument. They were related to the ability to draw a pictorial figure for the problem situations in the previous items. They also required to point the underlying reasoning of these figures drawn. Among these items in this indicator, while item

10 was more difficult than the others for preservice teachers, item 2 was easier than the rest as Figure 4.1 demonstrated above. In what follows, preservice teachers' responses were examined item by item in this part.

Item 2 asked participants to draw a pictorial figure representing the problem in item 1, along with their explanations for the figure. Also, a possible expected multiplicative explanation was "The number of oranges is 438 *times as many as* the number of oranges in one bag". Figure 4.1 showed that 83.8% of the participants were able to provide a correct pictorial figure. All these representations consisted of 438 bags or groups including six oranges in each as seen in Figure 4.2 below. Also, as shown in Figure 4.1, 18 preservice teachers (16.2% of all participants) were not able to draw a correct pictorial figure for the multiplication problem having equal groups situation in item 1.

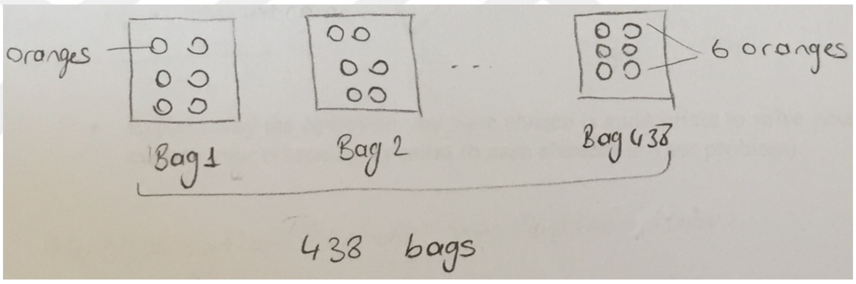

Figure 4.2. An example of a participant's answer for correct pictorial figure in item 2.

However, among the ones who were able to provide a correct pictorial figure, no one were able to provide a sufficient (i.e., multiplicative) explanation although item 2 had less difficulty than the other items in the same indicator. In particular, instead of multiplicative explanation, more than half of them (54.8% of them, 45.9% of all participants) provided insufficient explanations for their correct pictorial figures. These explanations contained only referring the problem situation itself such that "There are 2628 oranges in total". Besides, some of the participants who drew a correct figural representation (38.7% of them, 32.4% of all participants) only performed calculation with repeated addition or multiplication. This result revealed that their understanding was based on procedural knowledge in terms of multiplication. Additionally, while there were responses that did not include any explanations for correct representations, there was also response including an incorrect explanation pointing to incorrect problem like that "Each bag has 6 oranges and how many oranges are there in 438th bag?". While the problem asked the number of oranges in total 438 bags, it investigated the number of the oranges in 438th bag.

Among the ones who were not able to draw a correct pictorial figure, in addition to one preservice teacher who did not answer the item, there were also those who provided an additive pictorial figure that included adding 6 oranges in each bag 438 times as well as those who drew an incorrect pictorial figure. Incorrect ones were based on drawing 73 bags and 6 oranges in each bag or drawing 6 bags and 438 oranges in each bag. These results were parallel to the some of the responses for item 1 shared earlier in terms of the referents of the quantities. These results indicated that preservice teachers did not comprehend the quantities in the problem. That is, all these results pointed that the preservice teachers were not able to reason multiplicative structure of the quantities in the problem.

Item 5 also required providing a correct figural representation for multiplicative comparison multiplication problem in item 4 by pointing to the explanation for the representation. As Figure 4.1 indicated, only 36.9% of the preservice teachers were able to draw a correct pictorial figure. This type of pictorial figure included the representation of the relationship between the quantities whose sizes are the same (i.e., between 1 kg silver and 1.8 kg gold or 5 kg silver and 9 kg gold) as exemplified in Figure 4.3 below.

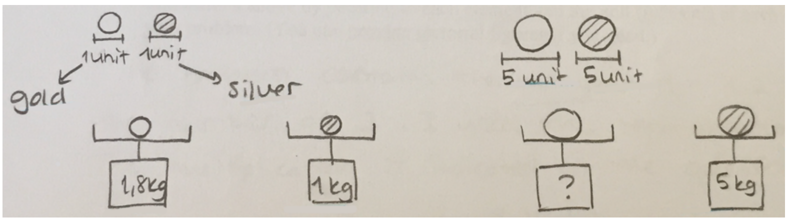

Figure 4.3. An example of a participant's answer for correct pictorial figure in item 5.

Among them, only 12 participants (29.3% of them, 10.8% of all participants) were able to explain the multiplicative relationship between the quantities such that "The weight of a piece of gold is 1.8 times as much as the weight of a piece of silver the same size". However, among the participants drawing a correct figural representation, 70.7% of them

(26.1% of all participants) were not able provide a possible multiplicative explanation for their figures. In addition to the participants who did not provide any explanations for the representations they drew, there were those who demonstrated only calculation with repeated addition or multiplication, as well as those who gave an insufficient explanation. The insufficient explanations were based on only expressing the problem situation like “A piece of gold weighs 9 kilograms” or “When a piece of silver is 5 kg, a piece of gold the same size is 9 kg”.

On the other hand, as illustrated in Figure 4.1, most of the preservice teachers (63.1% of all participants) were not able to draw a correct pictorial figure for the problem. Among these participants, except for the participants who did not draw any pictorial figures, 48.6% of them (30.6% of all participants) drew incorrect pictorial representations. Some of this type of responses were based on the representations in which two objects that are not the same size (two objects represented gold and silver themselves, not their weights). This revealed that these participants were not able to properly understand the problem. However, the sizes of gold and silver should be the same for the relationship between their weights to be constant. Also, some of these responses contained the representation of 1.8 kg silver and of 9 kg gold formed by 5 of 1.8 kg silver. This showed that some of the preservice teachers did not conceive the quantities and their referents in the problem. Moreover, among those who were not able to provide a correct representation, 35.7% of them (22.5% of all participants) demonstrated additive representation for the problem. These representations can be divided into three categories. First one consisted of adding the representation of 5 kg silver and the 8/10 (or 4/5) of 5 kg silver as exemplified in Figure 4.4 below. Second one is adding the 1 kg silver 5 times and adding the 1.8 kg gold 5 times. Third one included the addition of the representations of 5 kg silver and of the half of the 5 kg silver and of the 3/10 of the 5 kg silver. These results suggested that although the problem is a multiplicative comparison problem that directly reflected the multiplicative meaning of multiplication, some preservice teachers revealed their additive reasoning on multiplication.

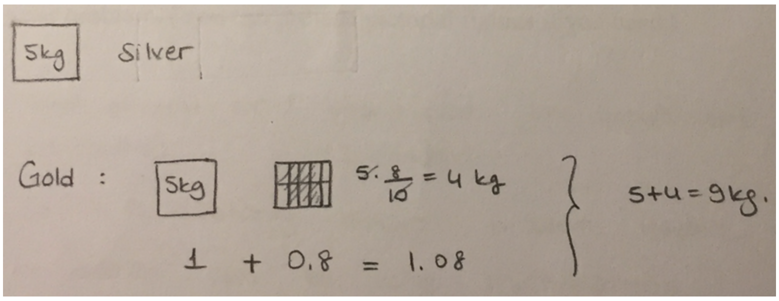

Figure 4.4. An example of a participant's answer for additive pictorial figure in item 5.

Item 10 also expected participants to draw a pictorial figure along with the explanation to represent the multiplication problem including equal groups situation in item 9. As indicated in Figure 4.1, only 11 participants (9.9% of all participants) were able to provide a correct pictorial figure. This type of pictorial figure included the representation of the relationship between the quantities 6750 kg corn for 1 ha and 313 200 kg corn for 46.4 ha as demonstrated in Figure 4.5. However, among them, no one were able to provide a possible sufficient (i.e., multiplicative) explanation such that "The weight of a corn produced in 46.4 ha (i.e., the yield) is 46.4 times as much as the weight of a corn produced in one ha (i.e., 6750 kg)".

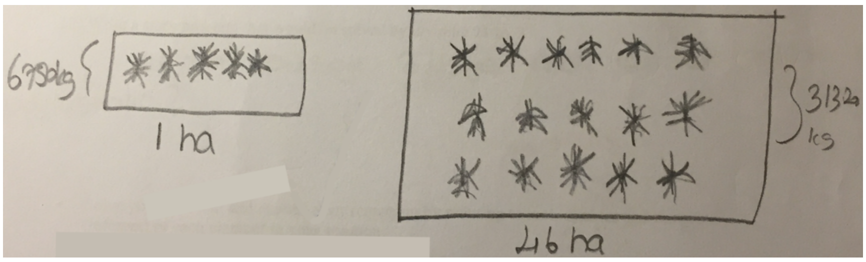

Figure 4.5. An example of a participant's answer for correct pictorial figure in item 10.

In fact, rather than making a multiplicative explanation, some of them provided insufficient explanations for their correct pictorial figures. These explanations were based on only expressing the problem situation such as "The yield will be 313 200 kg". Additionally, some of those explained the figure they draw by only calculation with multiplication like "46.4 × 6750 = 313 200 kg".

Figure 4.1 further showed that most of the preservice teaches (90.1% of all participants) were not able to provide a correct representation for the problem, so this further indicated the difficulty of item 10. Among these participants, apart from the participants who did not answer the item, 70% of them (63.1% of all participants) drew incorrect figural representation as demonstrated in Figure 4.6 below. Some of these preservice teachers drew a representation as if an area of 46.4 ha was divided equally into the areas of 1 ha, so they thought that 46.4 pieces of 6750 kg corn (since there are 6750 kg for one ha). However, this representation is not correct and meaningful since 6750 kg cannot be added 46.4 times. Likewise, some of the incorrect representations included drawing 6750 kg corn for 46.4 ha instead of for 1 ha. Particularly, additive pictorial figures were drawn by 17% of them (15.3% of all participants). These type of the representations can be analyzed in two categories. First one consisted of adding the representations of 6750 kg of corn 46 times and of 4/10 of the 6750 kg of corn. Second one is adding 675 kg of corn 464 times. These responses pointed that preservice teachers' representational skills were based on the additive reasoning.

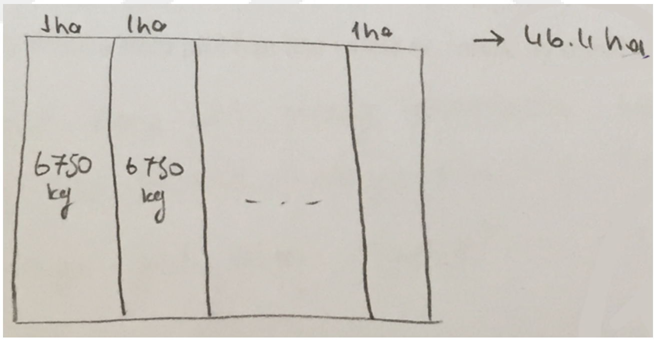

Figure 4.6. An example of a participant's answer for incorrect pictorial figure in item 10.

Item 15 which was the last item under the indicator of figural representation necessitated providing figural representation for multiplicative comparison division problem in item 14. As Figure 4.1 illustrated that only 40.5% of the preservice teachers were able to draw a correct representation. Some of the participants who provided a correct representation drew a tape diagram representing the weight of the lion kitten and another tape diagram that is 6 times as large as the previous one. The example of this representation was given in the Figure 4.7 below. Among them, only 17 participants (37.8% of them, 15.3% of all participants) were able to explain the multiplicative relationship between the quantities such that “The weight of the adult lion is 6 *times as much as* the weight of the lion kitten”. For example, the participant in Figure 4.7 specified that “lion kitten is 6 times lighter than adult lion”. Also, more than half of the preservice teachers (59.5% of all participants) were not able to draw a correct pictorial figure for the problem.

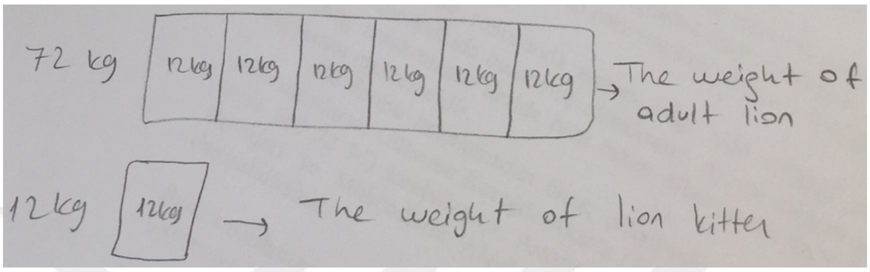

Figure 4.7. An example of a participant’s answer for correct pictorial figure in item 15.

Moreover, among the preservice teachers providing a correct figural representation, 62.2% of them (25.2% of all participants) were not able to multiplicatively explain the figures they drew. Other than the participants who did not provide any explanations, there were also those who performed only calculation with repeated addition or division as well as those who gave an insufficient explanation. These explanations were based on only expressing the problem situation like “A lion kitten weighs 12 kilograms”.

Conversely, among the ones (59.5% of all participants) were not able to draw a correct pictorial figure for the problem, except for the preservice teachers who did not draw any pictorial figures, most of them (72.7% of them, 43.2% of all participants) provided additive representation. These representations can be divided into three categories. First one consisted of adding the representation of six of 1 kg lion kitten (to reach 6 kg adult lion) or adding the representation of six of 12 kg lion kitten (to reach 72 kg adult lion). Second category is adding the representation of 12 of 1 kg lion kitten and adding the representation of 12 of 6 kg adult lion. Third one is a fraction bar (i.e., tape diagram) including six equal parts (one out of the six parts represents the 6 kg lion kitten and whole parts represent the 72 kg adult lion). An example for additive representation was given in Figure 4.8 below. It highlights the inclusion meaning. These responses gave an information about the preservice teachers' additive thinking on the figural representation. Likewise, while some incorrect pictorial figures were based on the representation of the lion kitten whose weight is 72 kg, some of them contained that the representation of the weight of the lion kitten is 6 times as much as the weight of the adult lion rather than the weight of the adult lion is 6 times as much as the weight of the lion kitten. These responses demonstrated that these preservice teachers were not able to either understand the problem or they did not internalize the referents of the quantities in the problem.

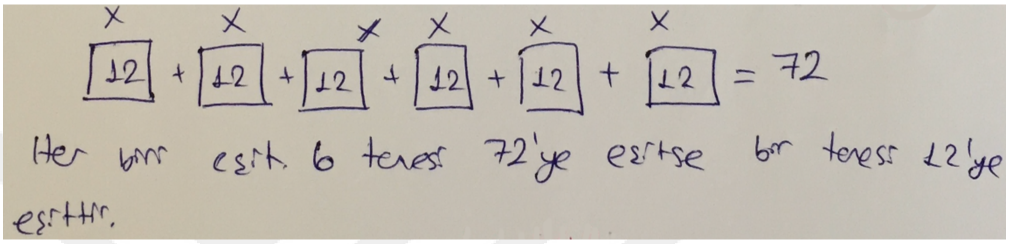

Figure 4.8. An example of a participant's answer for additive pictorial figure in item 15.

Finally, concerning the participants' responses to these four items related to drawing a figural representation aligning with their explanations, it can be concluded that preservice teachers had difficulty in explaining the representations multiplicatively. Especially, for the problems with equal groups situations in item 2 and item 10, multiplicative explanations were not provided by any participants. Since item 5 and item 15 were asked for the problems involving multiplicative comparison situation, preservice teachers may have presented multiplicative explanations for them. Yet, the number of the preservice teachers who provided multiplicative explanations for their correct pictorial figures were low (i.e., 10.8% of all participants in item 5 and 15.3% of all participants in item 15). Finally, some of the participants used repeated addition to explain the figural representations they drew in item 2 and item 15. A possible reason of this result can be that while item 2 and item 15 included whole numbers, item 5 and item 10 included non-integers.

4.2.3. Preservice Teachers' Performances Regarding Their Understanding of Multiplicative Problem Situations

For the indicator "Understanding of Multiplicative Problem Situations", there were six items (i.e., item 3, item 6, item 7, item 8, item 12, and item 13) in the instrument. They were related to the ability to write a multiplication or division word problem and the ability to construct a multiplicative relationship between the quantities in the problems. Among these items in this indicator, while item 7 and item 13 were more difficult than the others for preservice teachers, item 12 was easier than the rest as Figure 4.1 showed above. Preservice teachers' responses were interpreted item by item in this section.

Item 3 asked to write a multiplication or division word problem for the rectangular array model (4×5) and to justify the reason of their operation choice for the problem by pointing to the referents of the quantities in the problem context. Figure 4.1 expressed that 70.3% of the participants were able to write multiplication, partitive division, or quotative division problems modeled by the figure. Also, as indicated in Figure 4.1, 29.7% of the preservice teachers were not able to write a multiplicative word problem modeled by given figure.

Particularly, among 28 participants providing division problems, while 21 of them wrote partitive division problems, 6 of them wrote quotative division problems. The reason why partitive division was more preferred by the participants may be that the action of partitioning more evokes the meaning of dividing. Moreover, although the figure given in the item represented the rectangular array situation model, preservice teachers also used other multiplicative situations. Half of the participants writing multiplicative problems (35.1% of all participants) used equal groups situation and 9 participants (11.5% of them, 8.1% of all participants) used rectangular area situation. That is, 38.5% of the participants writing multiplicative problems (27% of all participants) used rectangular array situation. This revealed that many preservice teachers were able to make connections among different multiplicative situations and representations. In addition, the reason why half of the participants included equal groups situation in their problems may be that it is the first given and the most underlined multiplicative situation to introduce multiplication and division both in the literature and in textbooks. However, among the preservice teachers who wrote a multiplicative problem, no one were able to explain why the operation they have chosen was appropriate to solve their problems as they were required to provide their reasoning in terms of multiplicative explanation of the relationship between the quantities in their problems. This may demonstrate that preservice teachers were not able to internalize multiplication and division in terms of the multiplicatively pointing to the multiplicative relationship between the quantities in the problem context.

In fact, while explaining their choice of operation for their problems, most of the participants (54.1% of all participants) provided an insufficient explanation by stating the problem itself such that “There are 20 object in total”. Furthermore, some of the participants who wrote a correct word problem (17.9% of them, 12.6% of all participants) only did calculations by repeated addition or multiplication procedurally to find the answer of the problem. This again pointed that their explanations had a procedural basis. Besides, there were four responses including incorrect or unrelated explanations for correct problems. These responses were related to teaching to the students, but they did not focus on the appropriateness of the chosen operation to solve the problem.

Furthermore, as indicated in Figure 4.1, among the 29.7% of the preservice teachers who were not able to write a multiplicative word problem modeled by given figure, in addition to those who did not answer the item, there were also participants who wrote incorrect problems. Some of them wrote other types of the problems like “How many objects/stars are there in the figure?”. Although this problem can be solved by using multiplication, it is not a multiplication word problem. Therefore, it was not considered correct problem. Besides, some of them wrote multiplication problems that were not modeled by the figure (e.g., writing problems asking for 5×5 or 4×4 instead of 4×5). For instance, one problem asked “We have 4 pizzas for dinner, and each pizza has 4 slices. So, how many slices do we have?” was an example of participant’s answer for incorrect word problem.

Item 6 necessitated writing a word problem for one of the given mathematical expressions (i.e., 3×1, 3×2, 3×3, 3×4, and 3×5) in the item and to relate it to the other four mathematical expressions by pointing to the referents of the quantities in the problem they wrote. As shown in Figure 4.1, 86.5% of the preservice teachers were able to create multiplication, partitive division, or quotative division problems related to one of the expressions. Because there was no restriction to address a multiplicative situation, participants used three type situations which were equal groups, multiplicative comparison, and rectangular area in writing their problems. As Figure 4.1 stated, also 13.5% of the preservice teachers were not able to write a correct problem related to one of the expressions. In addition to those who did not answer the item, some of them created incorrect problems such as additive problems including these expressions or problems asking for writing a multiplication table of 3.

Particularly, 37.5% of the participants writing multiplicative problems (32.4% of all participants) were able to provide a sufficient and full explanation for the correct problem they wrote. A full explanation included making a correct relationship between the word problem with the other four mathematical expressions given by pointing to each element and the unit of each element in the problem. However, among the participants writing a correct problem, 62.5% of them (54.1% of all participants) were not able to sufficiently explain the relationship between the word problem with the other four mathematical expressions. While some of them provided insufficient explanation such as explanations in which the referents of the quantities were not included, some of them only solved their problems by using repeated addition or multiplication instead of forming a relationship. There were also responses including incorrect or unrelated explanations focusing on teaching aspect as well as those including no explanations.

Item 7 expected preservice teachers to justify whether the multiplication of the sides (i.e., n and m) were meaningful to find the area of a rectangle with different units of the side lengths (i.e., n inches wide and m centimeters high). The correct response for this item including a "meaningful" answer supported by a solid reasoning with a multiplicative explanation was expected to be as follows: "n inches is a length that is n times as long as one inch and m centimeters is as a length that is m times as long as one centimeter. So, the area of a rectangle is nm in-cm that is nm times the area of a rectangle of dimension 1 inch by 1 centimeter".

However, no one were able to provide such a sufficient (i.e., multiplicative) explanation in this way. Furthermore, as Figure 4.1 demonstrated, only two preservice teachers justified their "meaningful" answer with insufficient explanation that included only calculation with multiplication (including the units of the sides). That is, their explanations only involved finding the area that was nm (inch-cm) by multiplying n (inches) and m (centimeters). Although their explanations were insufficient, this showed that only these two participants among all preservice teachers (i.e., 1.8% of all participants) might have internalized the multiplication as a referent transforming composition that generates a quantity having a new referent. That is, they may have comprehended that the unit of area having new referent is multiplicatively derived (i.e., inch-cm).

In addition, among 98.2% of the preservice teachers who were not able to provide a sufficient explanation stated above, about half of them (45.9% of them, 45% of all participants) found the situation "meaningful" and 49.5% of all participants found the situation "not meaningful".

Particularly, regarding the ones (45% of all participants) who found the situation as meaningful, they were not able to correctly highlight the referents of the quantities in the problem. Among them, 32% of them (14.4% of all participants) did not correctly explain their "meaningful" responses. For example, some participants multiplied the sides whose unit of length was the same such as "n (centimeters) × m (centimeters) = nm (cm^2)" or "n (inch) × m (inch) = nm ($inch^2$)". This indicated that they did not either realize the referents of the quantities in the situation or even if they have realized they might have ignored the referents of the quantities. Besides, some participants made an additive explanation by ignoring the units of the quantities. They asserted that the area of a rectangle could be found by adding m (n times) like m + m + . . . + m. Since they did not specify the units of the side lengths, it cannot be concluded that their thinking was based on repeatedly copying the region of sides (1 inch by 1 cm) or copying the region of sides (1 cm by 1 cm). However, it is apparent that these preservice teachers approached the area concept in terms of additive thinking. Additionally, some of the participants supported their "meaningful" responses with only multiplying the sides (not including units of the sides) such as "n × m" or "m × n".

Regarding the ones (49.5% of all participants) who found the situation "not meaningful", while some of them did not support their ideas with an explanation, most of these participants asserted that the units of the side lengths should not be different to find the area by multiplying. Besides, some of them highlighted that the sides of the rectangle can be multiplied after converting inches to centimeters such that $2.54 \text{ cm} \times 1 \text{ cm} = 2.54 \text{ cm}^2$.

Item 8, under the indicator of understanding of multiplicative problem situations, required explaining the relationship between the area of a rectangle with 4.2 meters long and 3.3 meters wide and the area of a rectangle with 1 meter long and 1 meter wide by pointing to the referents of the elements. The correct response for this item consisted of an explanation such that "4.2 meters is a length that is 4.2 times as long as one meter and 3.3 meters is as a length that is 3.3 times as long as one meter. So, the area of a rectangle is 13.86 square meters that is 13.86 times the area of a rectangle of dimension 1 meter by 1 meter". However, only one preservice teacher was able to provide such a sufficient (i.e., multiplicative) explanation as shown in Figure 4.1.

Furthermore, only 5 participants provided insufficient explanations including the idea such that "The area of a rectangle is 13.86 square meters that is 13.86 times the area of a rectangle of dimension 1 meter by 1 meter". This pointed that these five preservice teachers formed a multiplicative relationship between the areas of two rectangles, but in their explanations, they did not multiplicatively relate the sides of the rectangles.

Figure 4.1 also indicated that most of the participants (94.6% of them) were not able to construct any relationships between these two rectangles. Though, among them, 35.2% of them (33.3% of all participants) only calculated the areas of the rectangles by multiplying the sides such that only writing the expression "$4.2 \text{ m} \times 3.3 \text{ m} = 13.86 \text{ m}^2$" or writing both the expressions "$4.2 \text{ m} \times 3.3 \text{ m} = 13.86 \text{ m}^2$" and "$1 \text{ m} \times 1 \text{ m} = 1 \text{ m}^2$". This showed that many preservice teachers' thinking on the rectangular area problems was procedural. Also, the responses of 19% of them (18% of all participants) were based on additive explanations with or without representation. For instance, as exemplified in Figure 4.9 below, some of them divided the rectangle into 12 squares dimension of 1 meter by 1 meter. They showed the remaining areas as 4 rectangles (of dimension 0.3 meters by 1 meter), 3 rectangles (of dimension 0.2 meters by 1 meter), and 1 rectangle (of dimension 0.2 meters by 0.3 meters). Additive explanations also included the idea such that the rectangle (4.2 m by 3.3 m) consisted of 1386 of the squares (0.1 m by 0.1 m).

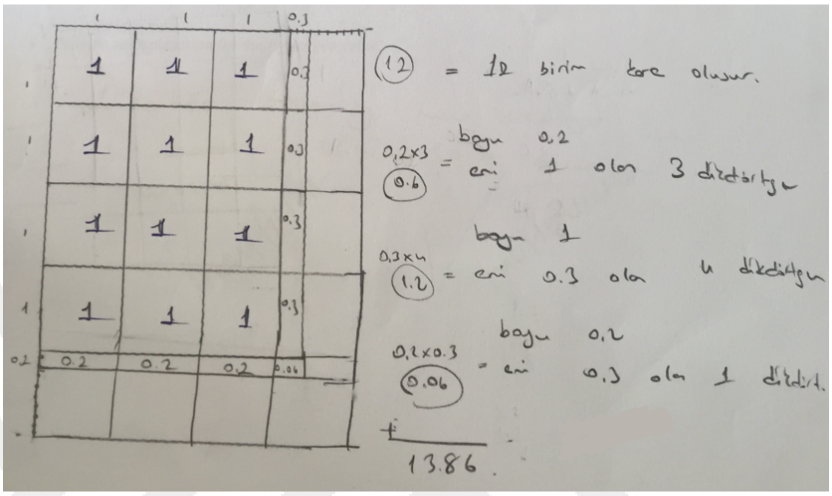

Figure 4.9. An example of a participant’s answer for additive explanation in item 8.

Finally, among the participants who did not form any relationships between the rectangles, other than those who did not answer the item, there were also answers including incorrect explanation or incorrect pictorial figures for the relationship between the areas of two rectangles. For example, some of them specified that the rectangles whose area was 13.86 m^2 was consisted of 13.86 of the squares (1 m by 1 m). However, this thinking is neither correct nor meaningful since the squares (1 m^2) cannot be added 13.86 times. Also, some of the preservice teachers only provided drawings without any explanation. In particular, they drew only the pictorial representations of these two rectangles, but they did not explain the relationship between the areas.

Item 12 asked preservice teachers to write a story problem that would be solved by dividing 21 by 3 and to explain each element and the referent of each element in their solutions. As Figure 4.1 expressed, 95.5% of the participants were able to correctly write a division problem. Also, 4.5% of the participants did not answer the item. There was also only one participant writing an incorrect problem.

Among the ones who were able to correctly write a division problem, while 77.4% of them wrote a partitive division problem, 22.6% of them wrote a quotative division problem. The reason why partitive division was more preferred again may be that the action of distributing an amount of something among the number of recipients more pushes more the thinking of division (like in item 3). Besides, all preservice teachers who wrote a division problem included equal groups as a situation in their problems. The reason why equal groups situation was more preferred by the participants may be that it is a situation on which it is easy to construct a problem.

However, among the participants who created a division problem, only 30.2% of them (28.8% of all participants) were able to explain the referents of three quantities including intensive quantity in their problems. Also, almost half of the participants (49.5% of all participants) specified the referent of one or two quantities (extensive or intensive) in their problems. Among these participants who correctly wrote at least one referent, 37.9% of them (29.7% of all participants) were able to explain the referent of the intensive quantity and others only emphasized extensive quantities. These results further clarified that most of the preservice teachers were not aware of the intensive quantity in division problems. Also, some of them did not provide any explanation for the referents, and some of them incorrectly explain the referents.

Item 13 directed preservice teachers to explain the multiplicative relationship between the elements in the problems they have written in item 12. To exemplify, for a provided partitive division problem "If 21 pencils are equally distributed to 3 students, how many pencils does each student have?", multiplicative relationship can be described as follows: "the number of the pencils each student has is 1/3 times as large as the total number of pencils" or "the total number of pencils is 3 times as large as each student has". Additionally, for a provided quotative division problem "If groups of 3 students are formed in a class having 21 students, how many groups are formed in total?", multiplicative relationship can be described as "the total number of groups is 1/3 times as large as the total number of students" or "the total number of students is 3 times as large as the total number of groups".

However, among the preservice teachers who wrote a division problem, no one were able to explain the multiplicative relationship between the quantities providing such explanations. In fact, most of them (80.6% of them, 48.6% of all participants) used multiplication operation. There could be multiple reasons why most of them justified multiplicative relationship as a multiplication. For instance, preservice teachers may have regarded "multiplicative relationship" through only multiplication. Alternatively, they may have driven to focus on multiplication due to language. However, there is also a multiplicative relationship in division. Also, only 2 participants provided an insufficient explanation. For example, one of these participants had written a partitive division "A girl starts to run 3 km every day. If she has run 21 km total, how many days been since she has started?" in item 12. This participant stated that "the total km she runs is increasing as multiples of 3" and demonstrated the calculations for the total km she runs by days as "3×1 for first day, 3×2 for second day, …, 3×7 for seventh day". This suggested that this participant was aware of the multiplicative relationship between the total km run and the number of days. So, instead of providing multiplicative explanations, most of the participants (60.4% of all participants) performed only calculation with division or multiplication. Among them, only 6 participants used division operation. In addition, apart from the participants who did not answer the problem, some participants provided incorrect or unrelated explanations like only drawing a pictorial figure, some presented additive explanations for the item by using repeated addition or repeated subtraction. Shortly, these responses revealed that preservice teachers had a difficulty in constructing multiplicative relationship between the quantities.

These results pointing to the indicator "Understanding of Multiplicative Problem Situations", the participants' responses to three items (i.e., item 3, item 6, and item 12) related to writing a multiplication or division problem point to some important conclusions. Firstly, many preservice teachers had ability to create multiplicative problems. In terms of the operation choice of the participants in writing these problems, it can be expressed that partitive division was more used than the quotative division. When the multiplicative situations included in the problems were examined, it can be asserted that "equal groups" was the most preferable situation by the preservice teachers. Especially, in item 12 where participants only wrote division problems, all of them emphasized the equal groups situation. However, in responding to these three items, cartesian product was not included as a problem situation by any of the participants.

Secondly, in the other three items (i.e., item 7, item 8, and item 13), it was revealed that most of the preservice teachers had difficulty in describing multiplicative relationship between the quantities and they had again difficulty in pointing to the referents of intensive quantities in the problem situations. These results actually align with the results pointing to item difficulty, where these were very difficult items as interpreted above in Table 4.1. Particularly, these results showed that participants had an inclination to share their explanations in a procedural way. This was also revealed in their responses to rectangular area problems in which they showed more of a procedural understanding. That is, many of them explained the relationship between the quantities by only making a calculation such as multiplying the sides. For example, in item 13 where directly asked the multiplicative relationship between the quantities in the problems they wrote, explanations with only multiplication were the most common response by the preservice teachers.

5. DISCUSSION

In this section, firstly, the results related to the study will be discussed in the light of the research questions and relevant literature. Then, limitations of the study will be explained, and recommendations considering further research will be discussed.

5.1. Discussion of the Results Considering Indicators

The purpose of the current study was to investigate preservice teachers' content knowledge on multiplication and division. The items were written as pointing to three indicators created by syncretizing the characteristics of the problem solvers who properly conceptualize multiplication and division (Otto et al., 2011; Smith & Smith, 2006). The three indicators were "understanding of a quantity", "figural representation", and "understanding of multiplicative problem situations".

Specifically, results regarding the indicator, understanding of a quantity, showed that except for item 11, although the percentage of the number of preservice teachers who correctly found the answer of multiplication and division problems was high in the four items (i.e., item 1, item 4, item 9, and item 14), they were not able to determine the quantities' referents in the problem situations. Their high performance in solving these four items might have been because of these preservice teachers' familiarity with the contexts of the problems. However, the main purpose of these items was to investigate if the participants correctly wrote the referents of the quantities provided in the problem situations. In addition, results regarding the indicator, figural representation, indicated that while none of the participants were able to provide multiplicative explanations for the problems having equal groups situations in item 2 and item 10, the number of the preservice teachers who multiplicatively explained multiplicative comparison problems was also low (12 participants for item 5 and 17 participants for item 15). Finally, results under the indicator, understanding of multiplicative problem situations, showed that although preservice teachers were successful at creating multiplication and division problems, they had difficulty in constructing and explaining multiplicative relationship between the quantities in the problems. So, the results related to three indicators in this study suggested that preservice middle school mathematics, middle school science, secondary school mathematics, physics, and chemistry teachers did not have content knowledge on understanding of multiplication and division multiplicatively.

Particularly, in the first indicator "understanding of a quantity" related to writing the units (i.e., referents) of the elements of multiplication and division problems (i.e., item 1, item 4, item 9, item 11, and item14), preservice teachers experienced difficulties in identifying the referents of the quantities, especially intensive quantities. To understand multiplication multiplicatively, intensive quantities should be conceived since multiplication is an operation having new referents (Schwartz, 1988; Simon & Placa, 2012). However, results showed that while many participants were able to specify the intensive quantities and their referents in equal groups problems (i.e., 48.6% of all participants for item 1 and 34.2% of all participants for item 9), none of the participants were able to address intensive quantities (i.e., 1.8 kg/kg and 12 kg/kg) in multiplicative comparison multiplication and division problems (i.e., item 4 and item 14). Though, some of the participants only expressed that "1.8" or "12" is the ratio of the weights (i.e., the extensive quantities in the problems) without pointing to their referents. These results point that preservice teachers seem not be aware of the scale conversion factor, that is a kind of intensive quantity such that "it does not change the nature of the referent on which it operates. However, it does change the magnitude of its measure" (Schwartz, 1988, p. 49). In addition, in item 11 including division by a fraction, investigating preservice teachers' understanding of quantities, the percentage of the number of preservice teachers correctly solving the problem was low. The reason of this might be that the problem in item 11 cannot be correctly solved if the quantities in the problem are not properly conceived. Consistent with the study of Simon (1993), results revealed that some of the participants chose multiplication operation to solve the problem instead of division. The reason might be that they did not choose division operation as follows "division by a fraction less than one violates primitive models of division" (Simon, 1993, p. 241). The incorrect operation choice also shows consistency with the results of other studies (Fischbein et al., 1985; Graeber et al., 1989). Also, although some of the preservice teachers used division for solving the problem, they did not correctly specify the amount of the cups of flour left over as they were not able to keep tracking of the referents of the quantities (i.e., the total number of cups of flour, the number of cups of flour used for one cookie, and the number of cookies made). As Simon (1993) highlighted, “whereas the most common algorithm for division by a fraction would yield a fraction or mixed number that would refer to the number of cookies, the problem asks for leftover flour” (p. 241). That is, because the total amount of flour and the amount of flour used for one cookie were given in the problem, participants may have tended to focus on only finding the amount of cookies made. So, they may have ignored the amount of flour left over. On the other hand, the reason why preservice teachers were more aware of intensive quantities in equal groups situation may be that equal groups situation is the first introduced multiplicative situation at schools. Hence, participants may have been more familiar with this situation and since they had been more exposed to this kind of situation in also daily lives, they may have had more opportunity to comprehend the elements of it such that “the number of objects in each groups” implying to intensive quantity and “the number of groups” and “the total number of object” implying extensive quantities.

In the second indicator “figural representation”, drawing a pictorial figure with providing appropriate multiplicative explanations pointing to multiplicative relationship between the quantities in drawings were expected of the participants on the items (item 2, item 5, item 10, and item 15). These items were especially placed after the previous items (whose indicators were understanding of a quantity) to make preservice teachers understood that they needed to show their reasoning about quantities. As Otto et al. (2011) emphasized, pictorial representations together with provided explanations are important as they point to learner’s robust understanding of the problems and also “always necessary to clarify the reasoning” (p. 13). However, results showed that preservice had difficulty in drawing a pictorial representation that reflected their multiplicative reasoning on the problems. In addition, participants had lack of ability to provide explanations for their correct figural representations in terms of multiplicative reasoning.

Regarding the results showing that preservice had difficulty in drawing a pictorial representation that reflected their multiplicative reasoning on the problems, for example, in item 5, expecting a drawing for multiplicative comparison problem including a decimal, 22.5% of the preservice teachers provided additive pictorial figures. As an example, some of them represented “9 kg gold” as adding “5 kg and 8/10 of the 5 kg”. This representation might lay behind the idea of that “9 kg” is consisted of 5 kg and 4 kg that is eight out of ten parts of “5 kg”. Likewise, in item 15 expecting a drawing for multiplicative comparison problem including a whole number, some preservice teachers gave responses based on the idea such that “12 kg is one out of six parts of 72 kg”. All these results seem to point that these preservice teachers might be thinking additively while reasoning in multiplication and division problems involving fractions and decimals. That is, these ways of thinking on multiplicative comparison problems contradict to the multiplicative ideas such that “9 kg is 1.8 (or 18/10) times as large as 5 kg” or “12 kg is 1/6 times as large as 72 kg” (Thompson & Saldanha, 2003). Besides, results in item 15 showed similarity with the results pointing to the preservice teachers’ insufficient knowledge of modeling of division problems (Tekin-Sitrava et al., 2020). They concluded that about half of the preservice primary school teachers did not have sufficient modeling knowledge related to division operation.

Regarding the results related with explanations, for instance, some participants performed only calculations for the problems rather than providing their reasoning even if they provided correct figural representation. This pointed that some preservice teachers had procedural understanding. However, it does not mean that these participants were not able to make higher reasoning. Still, the fact that the intensive quantities in the problem cannot be identified and the pictorial figures related to the same problem cannot be multiplicatively explained seem to suggest that preservice teachers did not understand multiplication and division multiplicatively. Moreover, although any preservice teacher providing correct representations did not point to the multiplicative relationship between the quantities in equal groups situations, there were preservice teachers who multiplicatively explained their correct pictorial figures in multiplicative comparison situations. The reason might be that the latter situation already has multiplicative structure in itself.

In the last indicator “understanding of multiplicative problem situations”, results from the items 3, 6, 7, 8, 12, and 13 showed preservice teachers were proficient at writing multiplication and division problems. Results further showed that among the participants who created division problems, partitive division was more preferred than quotative division. These results are consistent with the preservice teachers’ tendency to think about division in partitive terms in other studies (Graeber et al., 1986; Ball, 1990; Tekin-Sitrava et al., 2020). In addition, although the model given in item 3 was an array representation, some preservice teachers also wrote multiplication and division problems including other multiplicative situations also wrote multiplication problems including other multiplicative situations such as equal groups and rectangular area situations. This result supported participants' ability to make connections between multiplicative situations. However, preservice teachers had difficulty in justifying why the operation they chose was appropriate to solve their problems in item 3. These justifications give an idea about preservice teachers' understanding on the meanings of multiplication and division as highlighted by Otto et al. (2011). According to Otto et al. (2011), the ability to justify using multiplication as an operation is also component of robust understanding of multiplication. Similarly, in item 12, all preservice teachers writing a division problem included only equal groups situation in their problems. Any of them did not write division problems having situations like multiplicative comparison, rectangular array/area, or cartesian product. Their reliance on equal groups situation may demonstrate that participants might have limited knowledge of situations representing division. Moreover, cartesian product situation was not included in multiplication and division problems by any of the participants. The reason of it may be that this multiplicative situation is less mentioned in introducing multiplication and division in most curricula (Van de Walle et al., 2010). Therefore, preservice teachers may have not been much familiar with this situation to create these types of problems.

Results also showed that contrary to preservice teachers' ability to write multiplication and division problems, they had lack of ability to interpret multiplicative relationships between the quantities involved in their problems. For example, item 7 asked participants to explain the meaningfulness of finding the area of a rectangle once the sides, given n inches and m centimeters, were multiplied. Half of the participants found multiplication of the sides having different units for this situation "not meaningful" and some of them thought that after converting inches to centimeters, the sides of the rectangle can be multiplied (i.e., $2.54 \text{ cm} \times 1 \text{ cm} = 2.54 \text{ cm}^2$). Similarly, although approximately half of them found the situation "meaningful", they were not able to justify why sides were multiplied to find the area. Most of these participants computed the area by multiplying the sides such as "$n \times m$" or "$m \times n$" (i.e., by not including units of the sides). This might be because these preservice teachers do not realize that "n inches is a length that is n times as long as one inch and m centimeters is as a length that is m times as long as one centimeter. So, the area of a rectangle is nm in-cm that is nm times the area of a rectangle of dimension 1 inch by 1 cm" (Thompson &

Saldanha, 2003; Thompson, 2011). That is, these results further point that preservice teachers might have lack of understanding multiplicative problem situations.

Additionally, in item 8 that expected preservice teachers to explain the relationship between the areas of the rectangles (one of them has 4.2 meters long and 3.3 meters wide, the other has 1 meter long and 1 meter wide), the responses such that $(4 \times 3) + (4 \times 0.3) + (3 \times 0.2) + (0.3 \times 0.2)$ were considered as additive explanation. This actually shows distributive property of multiplication (i.e., 4.2×3.3) over addition (Otto et al., 2011). As Otto et al. (2011) emphasized, this property is important in terms of understanding computational processes. However, it does not support multiplicative reasoning since the multiplicative relationship between these two areas can be explained such that 4.2 meters is a length that is 4.2 times as long as one meter and 3.3 meters is as a length that is 3.3 times as long as one meter (Thompson & Saldanha, 2003). So, the area of a rectangle is 13.86 square meters which can be considered as 13.86 times the area of a rectangle of dimension 1 meter by 1 meter.

5.2. Limitations of the Study and Recommendations for Further Research

The current study has limitations, and the results of the study may have been influenced by these limitations. Firstly, this study was limited to five universities whose language of education is English in Turkey. After translating the language of the items to Turkish, implementing the items to preservice teachers from the universities whose language of education is Turkish, implementing the items to preservice teachers from different universities whose language of education is Turkish may provide further insight about how preservice secondary and middle school science and mathematics teachers in Turkey reason on multiplication and division. Additionally, participants may be selected from in-service middle school and secondary school mathematics and science teachers to be able to see the relationship between their content knowledge and teaching the concepts related to multiplication and division.

Secondly, content knowledge of preservice teachers was investigated in this study. A further study focusing on pedagogical content knowledge on multiplication and division may be conducted with preservice and in-service primary school teachers who directly teach these two concepts to their students. Thus, their knowledge of content and students and knowledge of content and teaching can be examined. Also, the results of the studies about both preservice and in-service teachers' subject matter knowledge and pedagogical content knowledge on understanding of multiplication and division multiplicatively might provide important knowledge for making some changes in teacher education.

Another limitation is that the data from some participants could not collected under the supervision of the instructor of the course or under the supervision of the researcher in the classroom hours. However, it can be concluded that the results of these data showed consistency with the results of other studies.

Fourth limitation of this study is related to the source of data collection. Preservice teachers were asked to answer the items on the instrument. Interviews with the participants could give more detailed information about their understanding of multiplication and division. For instance, participants who provided multiplicative explanations in multiplicative comparison problems could be probed by questions directed to multiplicative reasoning in the interview and thus, whether their explanations were really based on multiplicative reasoning could be determined more explicitly detailing their reasoning. Furthermore, preservice teachers who did not provide any explanations for their correct figural representations could be driven to explain verbally in the interview.

Finally, since the participants' responses were analyzed item by item and their total scores were not evaluated in this study, judgments about overall instrument cannot be made. After the items in this study were revised, the study aimed at developing an instrument for measuring preservice teachers' understanding on multiplication and division can be conducted with more participants. For that, more items can be written and then, a factorial structure can be established by eliminating the items where participants' responses support that factorial structure.

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APPENDIX A: INSTRUMENT

Dear Teacher Candidate,

This instrument has been improved to determine your subject matter knowledge on the concepts of multiplication and division. It is very important to write your e-mail address so that you can share your views with us about the study, and we can get your opinions about the instrument.

During solving the questions in the instrument,

- Please, read each material carefully.
- You can answer the questions in English or Turkish.

Student Number:

Male ☐ Female ☐ Others ☐

University:

Department/Program:

Term:

GPA (Grade Point Average):

E-mail:

Figure A.1. Instrument 1.

1) Marry has 438 bags of oranges. There are 6 oranges in each bag. How many oranges does Marry have altogether?

- Solve this problem by writing mathematical expression.

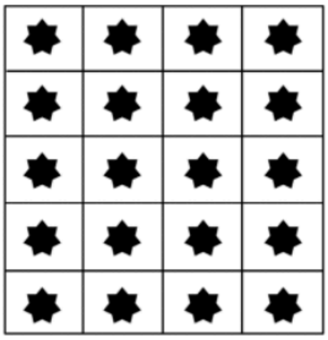
- Explain each element and the unit (referent) of each element in your solution.

2) Draw a pictorial figure to represent the situation in the problem above (in the 1st question).

- Provide an explanation for the figure you draw.

Figure A.2. Instrument 2.

3) Given the figure below,



- Write a multiplication or a division word problem modeled by the figure.

- Explain why the operation you have chosen is appropriate to solve your problem? (Please explain your reasoning by point to each element in your problem).

Figure A.3. Instrument 3.

4) Gold is 1.8 times as heavy as silver. If a piece of silver weighs 5 kilograms, how much does a piece of gold the same size weigh?

- Solve this problem by writing mathematical expression.

- Explain each element and the unit (referent) of each element in your solution.

5) Draw a pictorial figure to represent the situation in the problem above (in the 4th question).

- Provide an explanation for the figure you draw.

Figure A.4. Instrument 4.

6) $3 \times 1 = \square$

$3 \times 2 = \square$

$3 \times 3 = \square$

$3 \times 4 = \square$

$3 \times 5 = \square$

- Write a word problem for one of the given mathematical expressions above.

- Explain how your word problem might be related with the other four mathematical expressions above by pointing to each element and the unit (referent) of each element in your problem. (You can provide pictorial figures if you need.)

Figure A.5. Instrument 5.

7) We have a rectangle that is *n* inches long and *m* centimeters wide. We multiplied *n* by *m* to get the area of the rectangle.

- Do you think this is meaningful? Why?

- Explain your reasoning by pointing to each element and the unit (referent) of each element in the problem. (You can provide drawings if you need.)

8) The area of a rectangle that is 4.2 meters long and 3.3 meters wide is 13.86 square meters.

- Explain the relationship between the area of this rectangle and the area of a square of dimension 1 meter by 1 meter. (Please point to each element and the unit (referent) of each element in the problem. You can provide drawings if you need.)

Figure A.6. Instrument 6.

9) A farm of 46.4 ha produces 6750 kg of corn per ha. What will be the yield?

- Solve this problem by writing mathematical expression.

- Explain each element and the unit (referent) of each element in your solution.

10) Draw a pictorial figure to represent the situation in the problem above (in the 9^{th} question).

- Provide an explanation for the figure you draw.

Figure A.7. Instrument 7.

11) Jason has 32 cups of flour. He makes cookies that require 3/8 of a cup each. If he makes as many such cookies as he has flour for, how much flour will be left over?

- Solve this problem by writing mathematical expression.

- Explain your reasoning. (Please explain your reasoning by pointing to each element and the unit (referent) of each element in your problem.)

12) Write a story problem that would be solved by dividing 21 by 3.

- Solve your problem and explain your reasoning by pointing to each element and the unit (referent) of each element in your solution.

Figure A.8. Instrument 8.

13) Explain the **multiplicative relationship** between the elements in **your problem.** (You can provide drawings if you need.) (in the 12^{th} question).

14) An adult lion weighs 6 times as much as a lion kitten. If the adult lion weighs 72 kg, how much does the kitten weigh in kg?

- Solve this problem by writing mathematical expression.

- Explain each element and the unit (referent) of each element in your solution.

15) Draw a pictorial figure to represent the situation in the problem above (in the 14^{th} question).

- Provide an explanation for the figure you draw.

Figure A.9. Instrument 9.

APPENDIX B: RUBRIC

ITEM 1		
Responses	**Score**	**Response Code**
No answer/solution	0	99
Incorrect solution with/without explanation Example: $438 \div 6 = 73$	0	70
Correct solution with repeated addition and no explanation for referents Example: $6 + 6 + 6 + \ldots + 6 = 2628$	0	10
Correct solution with repeated addition and incorrect explanation for referents of elements Example: $6 + 6 + 6 + \ldots + 6 = 2628$	0	11
Correct solution with multiplication (*calculation mistake or calculation not completed is accepted) and no explanation for referents of elements Example: $438 \times 6 = 24$ or $6 \times 438 = 2628$	0	12
Correct solution with multiplication and incorrect explanation for written referents	0	13
Correct solution with multiplication and correct explanation for 1 element Example: $438 \times 6 = 2628$ or $6 \times 438 = 2628$ and 6 (bags or groups; number of bags or number of groups)- This is incorrect. 438 (oranges or objects; number of oranges or number of objects)- This is incorrect. 2628 (oranges or objects; number of oranges or number of objects)	1	14
Correct solution with multiplication and correct explanation for 2 elements	1	15

Figure B.1. Rubric 1.

Example: 438 × 6 = 24 or 6 × 438 = 2628 and 438 (bags or groups; number of bags or number of groups) 6 (oranges or objects; number of oranges or number of objects)- This is incorrect. 2628 (oranges or objects; number of oranges or number of objects)		
Correct solution with multiplication and correct explanation for 3 elements Example: 438 × 6 = 24 or 6 × 438 = 2628 and 438 (bags or groups; number of bags or number of groups) 6 (oranges/bag or objects/group; number of oranges in each bag or number of objects in each group) 2628 (oranges or objects; number of oranges or number of objects)	2	20
ITEM 2		
Responses	**Score**	**Response Code**
No answer	0	99
No pictorial figure	0	99
Incorrect pictorial figure with/without explanation Example: Drawing 73 bags or groups including six oranges or objects in each or Drawing 6 bags including 438 oranges in each	0	70
Additive pictorial figure with/without explanation Example:	0	71

Figure B.2. Rubric 2.

Adding 6 oranges in each bag 438 times		
Correct pictorial figure and no explanation Example: Drawing 438 bags or groups including six oranges or objects in each. or Drawing an arrow from each six orange or object to show that 438 orange or objects are produced from each orange. or Number line representation (Two parallel number lines) one in which 1 unit shows "the number of oranges in each bag") and one in which 438 units show "the number of oranges in total")	1	14
Correct pictorial figure and incorrect/unrelated explanation Example: One of the pictorial figures exemplified above. and Each bag has 6 oranges and how many oranges are there in 438th bag?	1	18
Correct pictorial figure and insufficient explanation Example: One of the pictorial figures exemplified above. and "There are 2628 oranges in total." or "There are 438 bags/groups, six oranges/objects in each bag/group, and 2628 oranges in total/in 438 bags/in 438 groups	1	15
Correct pictorial figure and explanation with multiplication Example: $438 \times 6 = 2628$	1	12

Figure B.3. Rubric 3.

Correct pictorial figure and repeated addition explanation Example: One of the pictorial figures exemplified above. and "There are 6 oranges + 6 oranges + . . . + 6 oranges = 2628 oranges in total."	1	16
Correct pictorial figure with both multiplication and repeated addition	1	17
Correct pictorial figure and sufficient (i.e., multiplicative) explanation Example: One of the pictorial figures exemplified above. and "The number of oranges is 438 times as many as the number of oranges in one bag."	2	20
ITEM 3		
Responses	**Score**	**Response Code**
No answer	0	99
Other (Incorrect) with or without explanation Example: How many objects are there in the figure? or The problem asking for 5×5 or 4×4 or Calculate the total number of the stars in the given figure	0	71
Multiplication or division word problem and no explanation Examples: • If there are 4 stars/desks/objects in each row/group, how many stars/desks/objects in 5 rows/groups? (multiplication)		

Figure B.4. Rubric 4.

- If there are 5 stars/desks/objects in each column/group, how many stars/desks/objects in 4 columns/groups? (multiplication) - If there are 20 stars/desks/objects in 5 rows/groups, how many stars/desks/objects are there in each row/group? (division) - If there are 20 stars/desks/objects in 4 columns/groups, how many stars/desks/objects are there in each column/group? (division) - There are 20 stars/desks/objects and 4 stars/desks/objects in each row/group. How many rows/groups are there? (division) - There are 20 stars/desks/objects and 5 stars/desks/objects in each column/group. How many column/groups are there? (division)	1	14
Multiplication or division word problem and incorrect/unrelated explanation	1	18
Multiplication or division word problem and only calculation with multiplication or division Example: One of the multiplicative word problems exemplified above. and $5 \times 4 = 20, 4 \times 5 = 20, 20 \div 5 = 4, 20 \div 4 = 5$	1	12
Multiplication or division word problem and insufficient explanation Example: One of the multiplicative word problems exemplified above. and "There are 20 objects in total." or Only explanation of the referents of the elements or If there are 5 rows and 5 columns, we need to multiply.	1	15

Figure B.5. Rubric 5.

Multiplication or division word problem and repeated addition or subtraction explanation Example: One of the multiplicative word problems exemplified above. and • 4 (objects) + 4 (objects) + 4 (objects) + 4 (objects) + 4 (objects) = 20 (objects) (repeated addition) • 5 (objects) + 5 (objects) + 5 (objects) + 5 (objects) = 20 (objects) (repeated addition) • It can be thought that when 20 objects are delivered one by one to 5 groups, there are 4 objects in each group (partitive division) • It can be thought that when 20 objects are delivered one by one to 4 groups, there are 5 objects in each group (partitive division) • $20 - 4 = 16, 16 - 4 = 12, 12 - 4 = 8, 8 - 4 = 4, 4 - 4 = 0$, the number of times subtraction operation is done gives the answer (i.e., 5) (repeated subtraction) (quotative division) • $20 - 5 = 15, 15 - 5 = 10, 10 - 5 = 5, 5 - 5 = 0$ the number of times subtraction operation is done gives the answer (i.e., 4) (repeated subtraction) (quotative division)	1	16
Multiplication or division word problem and (i.e., multiplicative) explanation Example: One of the multiplicative word problems exemplified above. and Sufficient (i.e., multiplicative) explanations for multiplication operation: • The number of stars/desks/objects in 5 rows/groups is 5 times as many as the number of stars/desks/objects in each row/group. 5 (the number of groups) × 4 (the number of objects in each group) = 20 (the number of objects)		

Figure B.6. Rubric 6.

• The number of objects/stars/desks in 4 columns/groups is 4 times as many as the number of objects/stars/desks in each column/group. 4 (the number of groups) × 5 (the number of objects in each group) = 20 (the number of objects) Sufficient explanations for division operation: • The number of the objects in each group is 1/5 times as many as the number of objects. 20 (the number of objects) ÷ 5 (the number of groups) = 4 (the number of objects in each group • The number of the objects in each group is 1/4 times as many as the number of objects. 20 (the number of objects) ÷ 4 (the number of groups) = 5 (the number of objects in each group • The number of the groups is 1/4 as many as 20. 20 (the number of objects) ÷ 4 (the number of objects in each group) = 5 (the number of groups) • The number of the groups is 1/5 as many as 20. 20 (the number of objects) ÷ 5 (the number of objects in each group) = 4 (the number of groups)	2	20
ITEM 4		
Responses	**Score**	**Response Code**
No answer/solution	0	99
Repeated addition with/without explanation Example: 1.8 + 1.8 + 1.8 + 1.8 + 1.8 = 9	0	70
Incorrect solution with/without explanation Example: 5 × 18/10 = 50/18 or 5 × 1.8 = 5.40	0	71

Figure B.7. Rubric 7.

Correct solution with multiplication and no explanation for referents of elements Example: $5 \times 1.8 = 9$ or $1.8 \times 5 = 9$	0	12
Correct solution and incorrect explanation for written referents	0	13
Correct solution with multiplication and correct explanation for 1 element	1	14
Correct solution with multiplication and correct explanation for 2 elements Example: $5 \times 1.8 = 9$ or $1.8 \times 5 = 9$ and 5 (kg, weight of a piece of silver), 1.8 (no referent)- This is incorrect. 9 (kg, weight of a piece of gold)	1	15
Correct solution with multiplication and correct explanation for 3 elements Example: $5 \times 1.8 = 9$ or $1.8 \times 5 = 9$ and 5 (kg, weight of a piece of silver) 1.8 (kg/kg, ratio [gold/silver] weight) 9 (kg, weight of a piece of gold)	2	20
ITEM 5		
Responses	**Score**	**Response Code**
No answer	0	99
No pictorial figure	0	99
Incorrect pictorial figure with/without explanation Example:	0	70

Figure B.8. Rubric 8.

Two objects that are not the same size (representing for gold and silver, not the weights) or Representation of the 1.8 kg silver and of the 9 kg gold that is formed by 5 of 1.8 kg silver		
Additive pictorial figure with/without explanation Example: Adding the representations of the 5 kg silver and of the 8/10 (or 4/5) of the 5 kg silver or Drawing (adding) the 1 kg silver 5 times and (adding) the 1.8 kg gold 5 times or Adding the representations of the 5 kg silver and of the half of the 5 kg silver and of the 3/10 of the 5 kg silver	0	71
Correct pictorial figure and no explanation Example: Number line representation (Two parallel number lines) one for silver (in which 1 unit shows "the weight of a piece of silver") and one for gold (in which 1.8 units show "the weight of a piece of gold") or 1 kg silver corresponding to 1.8 kg gold (their sizes are the same) 5 kg silver corresponding to 5×1.8 kg $= 9$ kg gold (their sizes are the same)	1	14
Correct pictorial figure and incorrect/unrelated explanation Example: One of the pictorial figures exemplified above. and "I solved the problem by the pattern."	1	18

Figure B.9. Rubric 9.

Correct pictorial figure and insufficient explanation Example: One of the pictorial figures exemplified above. and "A piece of gold weighs 9 kilograms." or "When a piece of silver is 5 kg, a piece of gold the same size is 9 kg."	1	15
Correct pictorial figure and explanation with only multiplication Example: $1.8 \times 5 = 9$ kg or $5 \times 1.8 = 9$ kg	1	12
Correct pictorial figure and additive explanation Example: One of the pictorial figures exemplified above. and 1.8 kg + 1.8 kg + 1.8 kg + 1.8 kg + 1.8 kg = 9 kg or The weight of the gold is 18/10 of the weight of the silver (i.e., inclusion meaning) or The weight of the gold equals to 5 kg + 8/10 (or 4/5) of the 5 kg or $1.8 \times$ (the weight of the silver), so the weight of the gold equals to 1.8×5 kg = (1×5 kg) + ($8/10 \times 5$ kg) = 5 kg + 4 kg = 9 kg	1	16
Correct pictorial figure with both multiplication and repeated addition	1	17
Correct pictorial figure and sufficient (i.e., multiplicative) explanation Example:		

Figure B.10. Rubric 10.

One of the pictorial figures exemplified above. and "The weight of a piece of gold is 1.8 times as much as the weight of a piece of silver." or "The weight of a piece of gold is 18/10 times as large as of the weight of a piece of silver." or "The weight of a piece of gold is 18 times as large as 1/10 of the weight of a piece of silver."	2	20
ITEM 6		
Responses	**Score**	**Response Code**
No answer	0	99
Additive word problem with or without explanation (Incorrect)	0	70
Other (Incorrect) with or without explanation Example: "Write a multiplication table of 3" or Addition problem	0	71
Multiplicative word problem and no explanation Example: "If one skirt has 3 buttons, then how many buttons are on 5 skirts?" (3 is a multiplicand, 5 is a multiplier) or "There are 3 shelves of books in Tom's room. Tom puts 5 books on each shelf. How many books are there in his room?" (3 is a multiplier, 5 is a multiplicand)	1	14
Multiplicative word problem and incorrect/unrelated explanation	1	18

Figure B.11. Rubric 11.

Multiplicative word problem and only calculation with multiplication or division Example: One of the multiplicative word problems exemplified above. and "There are $5 \times 3 = 15, 3 \times 5 = 15$ buttons in total." or "There are $3 \times 5 = 15, 5 \times 3 = 15$ books in total." or Forming a relationship with others in terms of only calculation like 3×1 or 1×3	1	15
Multiplicative word problem and additive explanation (including only calculation with repeated addition/subtraction) Example: One of the multiplicative word problems exemplified above. and 3 buttons + 3 buttons + 3 buttons + 3 buttons + 3 buttons = 15 buttons or 5 books + 5 books + 5 books = 15 books	1	16
Multiplicative word problem and calculation/explanation with both multiplication/division and repeated addition/subtraction	1	17
Multiplicative word problem and insufficient explanation Example: Explanations in which referents are not included	1	19
Multiplicative word problem and sufficient (i.e., full) explanation Example: Correct relation and all referents are written	2	20

Figure B.12. Rubric 12.

<table>
<tr><th colspan="3">ITEM 7</th></tr>
<tr><th>Responses</th><th>Score</th><th>Response Code</th></tr>
<tr><td>No answer (for meaningfulness)</td><td>0</td><td>99</td></tr>
<tr><td>“Not meaningful” answer without explanation (Incorrect)</td><td>0</td><td>70</td></tr>
<tr><td>“Not meaningful” answer with explanation (Incorrect)

Example:

The units are different, and they cannot be different

or

After converting inches to centimeters, it would me meaningful as
$2.54\ \text{cm} \times 1\ \text{cm} = 2.54\ \text{cm}^2$</td><td>0</td><td>71</td></tr>
<tr><td>“Meaningful” answer without explanation</td><td>0</td><td>12</td></tr>
<tr><td>“Meaningful” answer with incorrect/unrelated explanation

Example:
n (centimeters) × m (centimeters) = nm (cm^2)

or

n (inch) × m (inch) = nm (inch^2)

or

n × m = nm (br^2)

or

2.54 × n × m centimeters

or

3 m + 3 m + 3 m + 3 m = 12 m or 4 m + 4 m + 4 m = 12 m</td><td>0</td><td>18</td></tr>
<tr><td>“Meaningful” answer with repeated addition explanation (also including explanation without units)

Example: (by giving a number value to n and m)</td><td></td><td></td></tr>
</table>

Figure B.13. Rubric 13.

When it is looked horizontally, there are 3 in-cms in each row, so it can be written "3 in-cms + 3 in-cms + 3 in-cms + 3 in-cms = 12 in-cms" or When it is looked vertically, there are 4 in-cms in each column, so it can be written "4 in-cms + 4 in- cms + 4 in-cms = 12 in-cms" or Adding m + m + m … + m (n times)	0	13
"Meaningful" answer with multiplication (not including units) Example: $n \times m = nm$ or $m \times n = mn$ or $4 \times 3 = 12$ or $3 \times 4 = 12$	0	14
"Meaningful" answer with insufficient explanation Example: n (inches) × m (centimeters) = nm (in-cm) or by giving a number value to n and m; 4 (in) × 3 (cm) = 12 (in-cm)	1	15
"Meaningful" answer with multiplicative explanation(s) Example: • ""n inches" is a length that is n times as long as one inch and "m centimeters" is as a length that is m times as long as one centimeter. So, the area of a rectangle is nm in-cm that is nm times the area of a rectangle of dimension 1 inch by 1 centimeter." (It can be also reasoned in this way by giving a number value to n and m.) or (by giving a number value to n and m) • (Horizontally) The area of a rectangle of dimension 1 inch by 1 cm is 1 in-cm. In this rectangle, there are 3 rectangles of dimension 1 inch by 1 cm in each row. So, the area of each row is 3 times as much as the area of a rectangle of dimension 1 inch by 1 cm (i.e.,	2	20

Figure B.14. Rubric 14.

the area of each row is 3 in-cm). Because there are 4 rows in this rectangle, the area of a rectangle is 4 times as much as the area of each row (i.e., the area of a rectangle is 12 in-cm). or • (Vertically) The area of a rectangle of dimension 1 inch by 1 cm is 1 in-cm. In this rectangle, there are 4 rectangles of dimension 1 inch by 1 cm in each column. So, the area of each column is 4 times as much as the area of a rectangle of dimension 1 inch by 1 cm (i.e., the area of each column is 4 in-cm). Because there are 3 columns in this rectangle, the area of a rectangle is 3 times as much as the area of each column (i.e., the area of a rectangle is 12 in-cm).		
ITEM 8		
Responses	**Score**	**Response Code**
No answer	0	99
Incorrect/Unrelated explanation/pictorial figure Example: Adding 3.3 meters 4.2 times (not meaningful) or Explanation that does not include the relationship or The rectangle whose area is 13.86 m^2 is consisted of adding the squares (1 m^2) 13.86 times (not meaningful) or The area of the rectangle (4.2 m by 3.3 m) is consisted of the squares (1 br^2) or The pictorial figure of the rectangle (4.2 m by 3.3 m) that is consisted of 12 the squares (1 m by 1 m)	0	70
Only calculation Example: 4.2 m × 3.3 m = 13.86 m^2		

Figure B.15. Rubric 15.

or 4.2 m × 3.3 m = 13.86 m^2 and "The area of a rectangle is 13.86 square meters" or 4.2 m × 3.3 m = 13.86 m^2 and 1 m × 1 m = 1 m^2	0	12
Only drawing without explanation Example: The rectangle of dimension 4.2 meters by 3.3 meters or The rectangle of dimension 4.2 meters by 3.3 meters and The rectangle of dimension 1 meter by 1 meter	0	13
Additive explanation Example: 12 (the number of square of dimension 1 meter by 1 meter) + 4 × 0.3 + 3 × 0.2 + 0.3 × 0.2 = 13.86 square meters or (3.3 m × 3.3 m) + (3.3 m × 0.9 m) or This rectangle (4.2 m by 3.3 m) is consisted of 1386 of the squares (0.1 m by 0.1 m)	0	10
Calculation with multiplication and additive explanation	0	11
Insufficient explanation Example:	1	14

Figure B.16. Rubric 16.

The area of a rectangle is 13.86 square meters that is 13.86 times the area of a rectangle of dimension 1 meter by 1 meter.		
Sufficient (i.e., multiplicative) explanation Example: 4.2 m × 3.3 m = 13.86 m^2 and "4.2 meters" is a length that is 4.2 times as long as one meter and "3.3 meters" is as a length that is 3.3 times as long as one meter. So, the area of a rectangle is 13.86 square meters that is 13.86 times the area of a rectangle of dimension 1 meter by 1 meter.	2	20
ITEM 9		
Responses	**Score**	**Response Code**
No answer/solution	0	99
Incorrect solution with/without explanation Example: 6750 ÷ 46.4 = 145.5 kg or 46.4 ÷ 6750 = 0.0068	0	70
Correct solution and no explanation (*calculation mistake is accepted) for referents of elements Example: 46.4 × 6750 = 313 200 kg or 6750 × 46.4 = 313 200 kg	0	12
Correct solution and incorrect explanation for written referents	0	13
Correct solution and correct explanation for 1 element Example: 46.4 (kg) - This is incorrect. 6750 (ha) - This is incorrect. 313 200 (kg)	1	14

Figure B.17. Rubric 17.

Correct solution and correct explanation for 2 elements Example: 46.4 (ha) 6750 (kg)- This is incorrect. 313 200 (kg)	1	15
Correct solution and correct explanation for 3 elements Example: 46.4 (ha) 6750 (kg per ha or kg/ha) 313 200 (kg)	2	20
ITEM 10		
Responses	**Score**	**Response Code**
No answer	0	99
No pictorial figure	0	99
Incorrect pictorial figure with/without explanation Example: Number line representation in which 1 unit shows "46.4 kg corn" and 6750 units show "the yield." or Drawing 6750 kg of corn 46.4 times since there are 6750 kg for one ha (not meaningful) or Representation of the 6750 kg corn for 46.4 ha	0	70
Additive pictorial figure with/without explanation Example: Adding the representations of 6750 kg of corn 46 times and of the 4/10 of the 6750 kg of corn	0	71

Figure B.18. Rubric 18.

or Adding 675 kg of corn 464 times		
Correct pictorial figure and no explanation Example: Number line representation (Two parallel number lines) one in which 1 unit shows "6750 kg corn") and one in which 46.4 units show "the yield") or 1 ha corresponding to 6759 kg of corn, 46.4 ha corresponding to 46.4 × 6750 kg = 313 200 kg of corn	1	14
Correct pictorial figure and incorrect/unrelated explanation	1	18
Correct pictorial figure and insufficient explanation Example: Correct pictorial figure(s) exemplified above. and "The yield will be 313 200 kg."	1	15
Correct pictorial figure and additive explanation Example: Correct pictorial figure(s) exemplified above. and Adding 6750 kg of corn 46.4 times (not meaningful) or There are 1 ha 46.4 times, so there are 6750 kg of corn 46.4 times (not meaningful) or 6750 kg of corn 46 times and of the 4/10 of the 6750 kg of corn or Adding 675 kg of corn 464 times	1	16

Figure B.19. Rubric 19.

Correct pictorial figure and explanation with only multiplication Example: $46.4 \times 6750 = 313\ 200$	1	12
Correct pictorial figure with both multiplication and repeated addition	1	17
Correct pictorial figure and sufficient (i.e., multiplicative) explanation Example: Correct pictorial figure(s) exemplified above. and "The weight of a corn produced in 46.4 ha (i.e., the yield) is 46.4 times as much as the weight of a corn produced in one ha (i.e., 6750 kg)."	2	20
ITEM 11		
Responses	**Score**	**Response Code**
No answer/solution	0	99
Incorrect solution with/without explanation Example: $32 \times 3/8$ 3/8 of 32 cups (i.e., 12 cups) are needed to make one cookie, so 24 cups are needed for two cookies. Finally, 24 cups of flour are used, and 8 cups are left over. or $32 \times 3/8$ Since 3/8 of each cup is used, $32 \times 3/8$ (i.e., 12) cups of flour are used. So, $32 - 12 = 20$ cups are left over. or $32 \times 5/8$ Since 3/8 of each cup is used, 5/8 of each cup is left over. So, $32 \times 5/8 =$ 20 cups are left over. or $32 \div 3/8 = 85\ 1/3^{\circ}$	0	70

Figure B.20. Rubric 20.

85 cookies are made, and 1/3 cups of flour are left over.		
Partial solution Example: 32 ÷ 3/8 = 85 1/3 cookies If 3/8 cups --- 1 cookie, 6/8 cups --- 2 cookies--- For 85 cookies, 255/8 cups are needed.	1	16
Correct solution and no explanation 32 ÷ 3/8 = 85 1/3 cookies 1/3 × 3/8 = 1/8 cups of flour or There are 32 cups of flour (equals to 256/8 cups of flour) 32 ÷ 3/8 = 85 1/3 cookies (85 cookies can be made) For 85 cookies, 85 × 3/8 = 255/8 cups are needed 256/8 − 255/8 =1/8 cups of flour are left over or 3/8 + 3/8 = 6/8 cups (2 cookies can be made from each cup) So, 2 × 32 = 64 cookies can be made 2/8 cups from each cup are left over, so in total 2/8 × 32 = 8 cups are left over. 8 ÷ 3/8 = 64/3 = 21 1/3 cookies can be made from 8 cups 64 + 21 = 85 cookies can be made 1/3 × 3/8 = 1/8 cups of flour are left over	1	14
Correct solution and incorrect/unrelated explanation	1	18
Correct solution and insufficient explanation	1	15
Correct solution and correct explanation 32 ÷ 3/8 = 85 1/3 cookies 1/3 × 3/8 = 1/8 cups of flour or There are 32 cups of flour (equals to 256/8 cups of flour) 32 ÷ 3/8 = 85 1/3 cookies (85 cookies can be made)		

Figure B.21. Rubric 21.

For 85 cookies, 85 × 3/8 = 255/8 cups are needed 256/8 − 255/8 = 1/8 cups of flour are left over and explanation • "1/3 cookies" is a remainder, so 1/3 of the 3/8 cups (i.e., 1/8 cups of flour) left over. or • If one cookie is made from 3/8 cup of flour, how many cookies are made from 32 cups of flour? The number of cups was 3/8 times as much as the number of cookies made. So, 32 (cups) ÷ 3/8 (cup per cookie) = 85 1/3 cookies. That is, 85 cookies are made and (1/3 × 3/8 cups = 1/8 cups of flour) are left over.	2	20
ITEM 12		
Responses	**Score**	**Response Code**
No answer	0	99
Other (Incorrect) with or without explanation	0	71
Division story problem and no solution/explanation for referent of elements Examples: • "Alice walked 21 miles in 3 hours. How many miles per hour (how fast) did she walk?" • "Alex bought apples at 3 cents apiece. The total cost of his apples was 21 cents. How many apples did Alex buy?" • "This month Paul saved 3 times as much money as he did last month. If he saved $21 this month, how much did he save last month?" • "Clara picked 21 apples, and Lisa picked only 3. How many times as many apples did Clara pick as Lisa did?" • "If there are 21 different routes from A to C via B, and 3 routes from A to B, how many routes are there from B to C?" • "A classroom has a rectangular arrangement of desks with 3 rows. If there are 21 desks in the classroom, how many desks are in each row?"	1	17

Figure B.22. Rubric 22.

Responses	Score	Response Code
• "A classroom has a rectangular arrangement of desks with 3 desks in each row. If there are 21 desks in the classroom, how many rows?" • "If the area of a rectangle is 21 m^2 and the length is 3 m, what is the width?"		
Division story problem and incorrect/unrelated explanation	1	18
Division story problem and correct explanation for 1 element	1	14
Division story problem and correct explanations for 2 elements	1	15
Division story problem and correct explanations for 3 elements Example: $21 \div 3 = 7$ and 21 (cent, the total cost of apples) 3 (cent/apple, the cost apiece) 7 (apple, the number of apples)	2	20
ITEM 13		
Responses	**Score**	**Response Code**
No explanation	0	99
Incorrect/Unrelated explanation Example: There is no multiplicative relationship	0	70
Additive relationship (e.g., repeated addition or subtraction) (Incorrect)	0	71
Explanation through division	0	11
Explanation through multiplication Example: $3 \times 7 = 21$ or $7 \times 3 = 21$ or $21 \times 1/3 = 7$	0	12

Figure B.23. Rubric 23.

Repeated addition or subtraction and multiplication or division	0	13
Insufficient explanation Example: Two quantities in the problem changes in the ratio of the answer of the division.	1	14
Multiplicative relationship Examples: • “The miles Alice walks per hour is 1/3 times as much as the miles she walks in 3 hours.” or • “The miles Alice walks in 3 hours is 3 times as much as how far she walks in 1 hour.” • “The number of apples Alex bought 1/3 as large as the total cost of his apples (i.e., 1/3 as large as 21).” • “The amount of money Paul saved last month is 1/3 times as much as the amount of money he saved this month.” • “Clara picked 7 times as many apples as Lisa picked (i.e., 7 is considered as 1/3 as large as 21).” • “The number of routes from B to C is 1/3 times as much as the number of the routes from A to C (i.e., 12).” • “The number of the desks in each row is 1/3 times as much as the number of the desk in the classroom.” • “The number of rows is 1/3 as large as the number of desks in the classroom (1/3 as many as 21).” • “The measure of the width is 1/3 as large as the measure of the area.” • “Multiplicative explanation through multiplication is also accepted.”	2	20
ITEM 14		
Responses	**Score**	**Response Code**

Figure B.24. Rubric 24.

No answer/solution	0	99
Incorrect solution with/without explanation	0	70
Correct solution with division and no explanation for referents of elements Example: 72 ÷ 6 = 12	0	12
Correct solution with division and incorrect explanation	0	13
Correct solution with division and correct explanation for 1 element	1	14
Correct solution with division and correct explanation for 2 elements Example: 72 ÷ 6 = 12 and 72 (kg, weight of the adult lion) 6 (no referent)- This is incorrect. 12 (kg, weight of the lion kitten)	1	15
Correct solution with division and correct explanation for 3 elements Example: 72 ÷ 6 = 12 and 72 (kg, weight of the adult lion) 6 (kg/kg, ratio [adult lion/lion kitten] weight) 12 (kg, weight of the lion kitten)	2	20
ITEM 15		
Responses	**Score**	**Response Code**
No answer	0	99
No pictorial figure	0	99
Incorrect pictorial figure with/without explanation Example: The representation of the lion kitten whose weight is 72 kg or	0	70

Figure B.25. Rubric 25.

Groups drawn for the weight of the lion kitten and adult lion in the representation are not equal		
Additive pictorial figure with/without explanation Example: The representation of adding the 1 kg lion kitten 6 times (to reach 6 kg adult lion) and adding the 12 kg lion kitten 6 times (to reach 72 kg adult lion) or The representation of adding the 1 kg lion kitten 12 times and adding the 6 kg adult lion 12 times or A fraction bar (i.e., tape diagram) including six equal parts (one out of the six parts represents the 6 kg lion kitten and whole parts represent the 72 kg adult lion) (Thompson & Saldanha, 2003) or The representation of that the weight of the adult lion equals to the weight of 6 lion kittens (12 kg + 12 kg + 12 kg + 12 kg + 12 kg + 12 kg) or The representation of the adult lion that is equal to 6 lion kittens (1 lion kitten + 1 lion kitten + 1 lion kitten + 1 lion kitten + 1 lion kitten + 1 lion kitten = 1 adult lion) (incorrect quantities for the problem)	0	71
Correct pictorial figure and no explanation Example: Drawing an object that symbolizes “the weight of the lion kitten” and drawing six of the same objects that symbolize “the weight of the adult lion” or 72 objects of 1 kg (for adult lion) and 12 objects of 1 kg (for lion kitten) or Drawing an object that symbolizes 6 kg adult lion” and drawing 12 of the same objects that symbolize “72 kg adult lion”		

Figure B.26. Rubric 26.

or A fraction bar (i.e., tape diagram) representing the weight of the lion kitten and other fraction bar that is 6 times as large as the previous fraction bar. or Number line representation (Two parallel number lines) one for lion kitten (in which 1 unit shows "the weight of the lion kitten") and one for adult lion (in which 6 units show "the weight of the adult lion") or 1 kg lion kitten corresponding to 6 kg adult lion 12 g lion kitten corresponding to 12×6 kg $= 72$ kg adult lion or Graphical representation that shows the linear relationship the weight of the lion kitten and the weight of the adult lion	1	14
Correct pictorial figure and incorrect/unrelated explanation	1	18
Correct pictorial figure and insufficient explanation Example: One of the pictorial figures exemplified above. and "A lion kitten weighs 12 kilograms." or "12 kg is 1/6 of 72 kg"	1	15
Correct pictorial figure and explanation with multiplication or division Example: $6 \times 12 = 72$ or $12 \times 6 = 72$ or $72 \div 6 = 12$	1	12
Correct pictorial figure and repeated subtraction/additive explanation Example: One of the pictorial figures exemplified above.		

Figure B.27. Rubric 27.

and "Subtract 6's from 72 until getting to zero. The number of times subtraction operation is done gives the answer (i.e., 12). A lion kitten weighs 12 kilograms." or The weight of the adult lion equals to the weight of 6 lion kittens (12 kg + 12 kg + 12 kg + 12 kg + 12 kg + 12 kg) or The weight of the adult lion is addition of the weight of the lion kitten 6 times or "12 kg + 12 kg +. . . + 12 kg (6 times) = 72 kg" or "6 kg + 6 kg + . . . + 6 kg (12 times) = 72 kg" or "12 kg is one out of 6 parts representing 72 kg"	1	16
Correct pictorial figure with both multiplication or division and repeated subtraction/additive explanation	1	17
Correct pictorial figure(s) and sufficient (i.e., multiplicative) explanation Example: One of the pictorial figures exemplified above. and "The weight of the adult lion is 6 times as much as the weight of the lion kitten." or "The weight of the lion kitten is 1/6 times as much as the weight of the adult lion."	2	20

Figure B.28. Rubric 28.

APPENDIX C: ETHICS COMMITTEE APPROVAL

Evrak Tarih ve Sayısı: 14.12.2021-42474

T.C.
BOĞAZİÇİ ÜNİVERSİTESİ REKTÖRLÜĞÜ
Fen Bilimleri ve Mühendislik Alanları İnsan Araştırmaları Etik Kurulu
(FMİNAREK)

Sayı : E-84391427-050.01.04-42474 14.12.2021
Konu : 2021/25 Kayıt no'lu başvurunuz hakkında

Sayın Doç. Dr. Gülseren KARAGÖZ AKAR
Matematik ve Fen Bilimleri Eğitimi Bölüm Başkanlığı - Öğretim Üyesi

"Developing an Instrument for Investigating Preservice Teachers' Mathematical Meanings on Multiplication and Division" başlıklı projeniz ile Boğaziçi Üniversitesi Fen Bilimleri ve Mühendislik Alanları İnsan Araştırmaları Etik Kurulu (FMİNAREK)'e yaptığınız 2021/25 kayıt numaralı başvuru 06.12.2021 tarihli ve 2021/10 No.lu kurul toplantısında incelenerek etik onay verilmesi uygun bulunmuştur. Bu karar tüm üyelerin toplantıya on-line olarak katılımıyla ve oybirliği ile alınmıştır.

COVID-19 önlemleri nedeniyle üyelerden ıslak imza alınamadığından bu onam mektubu tüm üyeler adına Komisyon Başkanı tarafından e-imzalanmıştır.

Saygılarımızla bilginize sunarız.

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