ÇUKUROVA UNIVERSITY
INSTITUTE OF NATURAL AND APPLIED SCIENCES

MSc THESIS

Andre DUSHIMIRIMANA

FREE LIE ALGEBRAS AND THEIR HILBERT SERIES

DEPARTMENT OF MATHEMATICS

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We certify that the thesis titled above was reviewed and approved for the award of degree of the Master of Science by the board of jury on 15/06/2017.

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ABSTRACT

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In the present work, the canonical bases of finitely generated graded algebras such as the commutative associative polynomial algebra, free associative algebra and free metabelian Lie algebra were investigated, and an elementary proof was proposed in the computation of Hilbert series of free metabelian Lie algebra.

Keywords: Free Lie algebras, Free metabelian Lie algebras, Hall bases, Hilbert Series.

ÖZ

YÜKSEK LİSANS TEZİ

SERBEST LIE CEBİRLERİ VE ONLARIN HİLBERT SERİLERİ

Andre DUSHIMIRIMANA

ÇUKUROVA ÜNİVERSİTESİ
FEN BİLİMLERİ ENSTİTÜSÜ
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: Doç. Dr. Tuncay TUNÇ

Bu çalışmada derecelendirilmiş cebirler olan sonlu üretilmiş değişmeli ve birleşmeli polinomlar cebirinin, serbest birleşmeli cebirin ve serbest metabelyen Lie cebirinin kanonik bazları incelenmiş ve serbest metabelyen Lie cebirinin Hilbert serisinin hesabında elementer bir ispat önerilmiştir.

Anahtar Kelimeler: Serbest Lie cebirleri, Serbest metabelyen Lie cebirleri, Hall bazları, Hilbert Serileri.

EXTENDED ABSTRACT

In this thesis it is shown how to construct a canonical basis for unitary polynomial algebra $K[X_m] = K[x_1, \dots, x_m]$ of finite rank m generated by X_m, unitary associative algebra $K\langle X_m\rangle = K\langle x_1, \dots, x_m\rangle$ of finite rank m generated by X_m (not necessarily commutative), free Lie algebra $L_m = K\langle x_1, \dots, x_m\rangle$ of finite rank m generated by X_m, free Akivis algebra $A_m = K\langle x_1, \dots, x_m\rangle$ of finite rank m generated by X_m, free metabelian Lie algebra

$$F_m = {}^{L_m}\!/_{L_m''} = {}^{L_m}\!\Big/_{\left[[L_m, L_m], [L_m, L_m]\right]} = K\langle y_1, \dots, y_m\rangle$$

of finite rank m generated by $\{y_1, \dots, y_m\}$. All these algebras are graded vector spaces. Recall that V is called a graded K- vector space if it has a direct sum

$$V = \bigoplus_{n\geq 0} V^{(n)} = \sum_{n\geq 0} V^{(n)}$$

where V^n is subspace and $n \geq 1$. Since $V = \bigoplus_{n\geq 0} V^{(n)}$ is a finitely generated graded K-algebra, then we could define the Hilbert function of V as

$$H(V, n) = \dim_K V^{(n)}$$

where $dim_K(V^{(n)})$ is the dimension of the vector space $V^{(n)}$over K. For all $n \geq 0$, we can define formal power series in this form:

$$H(V, t) = Hilb(V, t) = \sum_{n\geq 0} \dim_K V^{(n)}\, t^n$$

is called the Hilbert (or Poincaré) series of V.

We investigate and formulate the dimensions of homogeneous subspaces $\dim_K V^{(n)}$ for $V = K[X_m]$, $V = K\langle X_m\rangle$, $V = F_m$ and find the Hilbert series of these graded algebras. In proving the formula for Hilbert series of free metabelian Lie algebra F_m, which is

$$H(F_m, t) = 1 + mt + \frac{mt - 1}{(1 - t)^m}$$

we give an elementary proof which is main objective of this thesis. The first proof given by Drensky (1994) is relatively theoretical and very nice however it is not easy to follow without a good background, while the proof proposed here is so simple to understand even in undergraduate stage.

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Andre DUSHIMIRIMANA

CONTENTS

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1. INTRODUCTION

The Hilbert Series known also as Poincaré Series was firstly proposed by David Hilbert himself in the years of 1890s when he wanted to study the context of finitely generated commutative algebras. As the years come up the Hilbert Series have been intensively studied, like in 1890 to 1893, Hilbert described the shape of commutative algebra by using the most four famous results which are the basis theorems based on finite generation of invariant, the theorem of zeros also called Nullustellensatz in German, the polynomial nature which is named as the Hilbert function and Syzygy theorem. By definition the Hilbert series is the formal power series in the variables of the degrees ring whose coefficients are the dimensions of the corresponding graded component.

In this thesis we investigate on the notions of free Lie algebras, polynomial algebras, graded algebras and free metabelian algebras. We suppose that $L_m = K\langle x_1, \dots, x_m\rangle$ is the free Lie algebra of finite rank m generated by $x_1, \dots, x_m$ over a field K of characteristic zero and let $F_m = K\langle x_1 + L_m^{''}, \dots, x_m + L_m^{''}\rangle = K\langle y_1, \dots, y_m\rangle$ be the free metabelian Lie algebra of rank m over K. Then the discussion here is based on graded algebras. We introduce the freeness of Lie algebras and their monomials, where these helped us to find the canonical basis of free (metabelian) Lie algebras. We investigate Hilbert series of both polynomial algebra and free metabelian Lie algebra and we come out with an elementary proof of formula of Hilbert series of free metabelian Lie algebra which was our main objective of our thesis.

Our work has two major chapters, one is coupled the basic definitions and theorems here we recalled some definition and we tried to discuss and prove some theorems and propositions as well which will conduct us to the well understandable of our objectives.

Second chapter is computational results. Here we have discussed the graded bases and we investigate the free metabelian Lie algebra thereafter we come out with a simple way to formulate the Hilbert Series of metabelian Lie algebra.

2. BASIC DEFINITIONS AND THEOREMS

Definition 2.1: Let $(K\,,+,\cdot)$ be a field of characteristic 0 and let V be a nonempty set together with the operations of addition $\oplus: V \times V \to V$ and scalar multiplication $\odot: K \times V \to V$. V is called a vector space over K if the following conditions hold for every element $u, v, w \in V$ and scalars $\alpha, \beta \in K$:

(i) Commutativity: $u \oplus v = v \oplus u$

(ii) Associativity: $(u \oplus v) \oplus w = u \oplus (v \oplus w)$ and $(\alpha \cdot \beta) \odot v = \alpha \odot (\beta \odot v)$

(iii) Additive identity: There exists an element $0 \in V$ such that $0 \oplus v = v \oplus 0 = v$ for all $v \in V$;

(iv) Additive inverse: For any $v \in V$, there exists an element w such that $v \oplus w = w \oplus v = 0$;

(v) Multiplicative identity: $1 \odot v = v$;

(vi) Distributivity:

$$\alpha \odot (u \oplus v) = (\alpha \odot u) \oplus (\alpha \odot v)$$
$$(\alpha + \beta) \odot u = (\alpha \odot u) \oplus (\beta \odot u).$$

Usually, a vector space over $\mathbb{R}$ is called a real vector space and a vector space over $\mathbb{C}$ is called a complex vector space. Elements of the vector space V are called vectors.

Example 2.2: K^n is a vector space over K under the operations

$$(v_1, \ldots, v_n) \oplus (w_1, \ldots, w_n) = (v_1 + w_1, \cdots, v_n + w_n)$$
$$\alpha \odot (v_1, \ldots, v_n) = (\alpha \cdot v_1, \ldots, \alpha \cdot v_n),$$

where $\alpha, v_1, \ldots, v_n, w_1, \ldots, w_n \in K$.

Definition 2.3: Let V and W be two vector spaces. A linear transformation f is a function from V to W such that

$$f\big((\alpha \odot v_1) \oplus v_2\big) = \big(\alpha \odot f(v_1)\big) \oplus f(v_2)$$

for all $v_1, v_2 \in V$ and $\alpha \in K$. Sometimes we can say that f is linear just for meaning of linear transformation.

For simplicity, we will shorten the followings from now on: $\alpha \odot v = \alpha v$, $u \oplus v = u + v$.

Definition 2.4: A linear transformation is an isomorphism if it is one-to-one and onto. If there is an isomorphism $f: V \to W$, we say that V is isomorphic to W and write $V \cong W$.

The inverse of an isomorphism is an isomorphism and the composition of two isomorphisms is also an isomorphism, when defined.

Definition 2.5: Let V be a finite dimensional vector space. End (V) is the space of linear homomorphisms from V to V. End (V) is a vector space over K.

Definition 2.6: A vector space V over a field K is called algebra (or K-algebra) if V is equipped with a binary operation $*: V \times V \to V$ called multiplication, such that for any $u, v, w \in V$ and $\alpha \in K$ we have the following identities:

$$(u + v) * w = u * w + v * w,$$
$$u * (v + w) = u * v + u * w,$$
$$\alpha(u * v) = (\alpha u) * v = u * (\alpha v).$$

A basis of V as a vector space is said to be a basis of algebra V. If V is finite dimensional then V has a finite basis over K. The algebra V is a commutative algebra if V is a commutative ring with respect to $*$.

Usually we denote the multiplication of V by $\cdot$ however we are going to use uv instead of $u \cdot v$ and $u \times v$, etc. Clearly, the notion of algebra generalizes both the notion of vector space and of ring. We do not require $1 \in V$ and associativity of V.

Remark 2.7: Let V be an n-dimensional algebra over a field K with the basis $\{ e_1, \dots, e_n\}$. For each pair (e_i, e_j) we can express the product $e_i e_j$ as a linear combination of the basic elements of V as follows:

$$e_i e_j = \sum_{k=1}^{n} \alpha_{ij}^{k} e_k,$$

where $\alpha_{ij}^{k} \epsilon K$.

The bilinear multiplication in V is completely determined by n^3 multiplication constants $\alpha_{ij}^{k} \in K$. To prove this let us take $x = \sum_{i=0}^{n} \gamma_i e_i$ and $y = \sum_{j=0}^{n} \sigma_j e_j$ as two arbitrary elements of V then

$$xy = \left(\sum_{i=0}^{n} \gamma_i e_i\right)\left(\sum_{j=0}^{n} \sigma_j e_j\right) = \sum_{i,j=1}^{n} \gamma_i \sigma_j e_i e_j = \sum_{i,j,k=1}^{n} \gamma_i \sigma_j \alpha_{ij}^{k} e_k.$$

Therefore the multiplication can be completely specified by giving a set of n^3 constants $\alpha_{ij}^{k} \in K$. These constants are called structure constants, because they determine the algebra structure of V. If i and j are fixed then the only finite number of constants α_{ij}^{k} are not zero. If for all $, j, k$ $\alpha_{ij}^{k} = 0$, then the product of two elements of V is 0, otherwise, $\alpha_{ij}^{k} \neq 0$ which is our case where we have a basis for V over K.

Definition 2.8: A subalgebra of algebra V over K is a subset of elements that is closed under addition, multiplication, and scalar multiplication. If a subset S of V is a subalgebra, then for all $s_1, s_2 \in S$ and $\alpha \in K$, we have that $s_1 * s_2$, $s_1 + s_2$, and αs_1 are all in S.

A subset I of V is a left ideal if for all $s_1, s_2 \in I, p \in V$, and $\alpha \in K$ it satisfies the following conditions:

i) $s_1 + s_2 \in I$,

ii) $\alpha s_1 \in I$,

iii) $p * s_1 \in I$.

If condition (iii) is replaced by $s_1 * p \in I$, then I is a right ideal. A two-sided ideal is a subset that is both a left and a right ideal. The term ideal is usually taken to mean a two-sided ideal. When the algebra is commutative, then both of these notions of ideal are equivalent.

Examples 2.9:

(i) Let x be an indeterminate over K. It is known that

$$K[x] = \left\{\sum_{i=1}^{n} \alpha_i x^i \middle| \, n \in \mathbb{N}, \alpha_i \in K\right\}$$

$$K[x_1, \dots, x_n] = \left\{\sum c_{i_1,\dots,i_n} x_1^{i_1}, \dots, x_n^{i_n} \,|(i_1, \dots, i_n) \in \mathbb{N}^n, c_{i_1,\dots,i_n} \in K\right\}$$

are algebras over K.

(ii) Let $M_n(K)$ denote the collection of $n \times n$ matrices. The $M_n(K)$ is of dimension n^2 as a vector space over K in which the addition operation is the usual matrix addition and multiplication is matrix multiplication.

(iii) Let $U_n(K)$ be the subset of $M_n(K)$ consisting of all upper triangular matrices.

$$U_n(K) = \left\{(a_{ij}) : 1 \leq i, j \leq n, a_{ij} = 0 \; for \; i > j\right\}$$

$U_n(K)$ is a subalgebra of $M_n(K)$.

(iv) Let $sl_n(K)$ be the set of $n \times n$ matrices with trace zero and with multiplication

$$[A, B] = AB - BA, \qquad A, B \in sl_n(K)$$

The trace of a $n \times n$ square matrix is defined by

$$Tr(A) = a_{11} + a_{22} + \cdots a_{nn} = \sum_{i=1}^{n} a_{ii}.$$

The trace of product of two square matrices is independent of the order of multiplication using Einstein Summation

$$Tr(AB) = \sum_{i=1}^{m} (AB)_{ii}$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} A_{ij} B_{ji} = \sum_{j=1}^{n} \sum_{i=1}^{m} B_{ji} A_{ij} = \sum_{j=1}^{n} (BA)_{jj} = Tr(BA)$$

Therefore, the trace of the commutator of A and B is given by

$$Tr([A, B]) = Tr(AB) - Tr(BA)$$

This shows that $sl_n(K)$ is a K- algebra.

Definition 2.10: A homomorphism between two algebras U and V over a field K is a map $\phi: U \to V$ such that for all $\alpha \in K$ and $u, v \in U$,

$$\phi(\alpha u) = \alpha\phi(u)$$
$$\phi(u + v) = \phi(u) + \phi(v)$$
$$\phi(u * v) = \phi(u) * \phi(v).$$

A homomorphism ϕ of algebra U into algebra V is called

(i) a monomorphism if ϕ is one-to-one,

(ii) an epimorphism if ϕ is onto V, and

(iii) an isomorphism if ϕ is one-to-one and maps U onto V.

If ϕ is an isomorphism of algebra U onto V, then ϕ^{-1} is an isomorphism of V onto U.

A homomorphism from U to itself is called an endomorphism and an isomorphism of algebra U to U is called an automorphism.

Theorem 2.11: Let ϕ be an algebra homomorphism between algebras U and V, then $\phi(0_U) = 0_V$.

Proof: $\phi(0_U) = \phi(0_U + 0_U) = \phi(0_U) + \phi(0_U)$

Let $y = \phi(0_U) \in V$

$$y = y + y$$
$$y + (-y) = (y + y) + (-y)$$
$$0_V = y + \underbrace{(y + (-y))}_{0_V}$$
$$0_V = y = \phi(0_U).$$

Theorem 2.12: Let ϕ: $U \to V$ be a homomorphism of algebras. Then the kernel $\mathrm{Ker}(\phi)$ of ϕ

$$\mathrm{Ker}(\phi) = \{u \in U \mid \phi(u) = 0\}$$

is a two-sided ideal of U and the factor algebra $U/\mathrm{Ker}(\phi)$ is isomorphic to the image

$$\mathrm{Im}(\phi) = \{\phi(u) \mid u \in U\}$$

of ϕ.

Proof: From theorem 2.11, $0 \in \mathrm{Ker}(\phi)$, so $Ker\phi \neq \emptyset$ and if $u, v \in \mathrm{Ker}\phi$ then $\phi(u) = \phi(v) = 0$, thus $\phi(u) - \phi(v) = \phi(u - v) = 0$. For any $r \in U, \phi(ru) = \phi(r)\phi(u) = \phi(r) \cdot 0 = 0$. Similarly, $\phi(ur) = 0$. Thus $u - v$, ur, ru are also in $\mathrm{Ker}\phi$, hence $\mathrm{Ker}\phi$ is a two-sided ideal of U.

On the other hand, let us consider the map $\varphi : U/\mathrm{Ker}\phi \to \mathrm{Im}\phi \subset V$, defined by $u + \mathrm{Ker}\phi \to \phi(u)$. We have to show that φ is well defined:

Let $u + \mathrm{Ker}\phi = v + \mathrm{Ker}\phi$, then $u - v \in \mathrm{Ker}\phi$ and so $\phi(u - v) = 0 \Rightarrow \phi(u) - \phi(v) = 0 \Rightarrow \phi(u) = \phi(v) \Rightarrow \varphi(u + \mathrm{Ker}\phi) = \varphi(v + \mathrm{Ker}\phi)$. Hence φ is well defined.

φ is a homomorphism: Let $u + \mathrm{Ker}\phi, v + \mathrm{Ker}\phi \in U/\mathrm{Ker}\phi$. Making use of the fact that ϕ is a homomorphism, we have

$$\begin{aligned}\varphi\big((u + \mathrm{Ker}\phi) + (v + \mathrm{Ker}\phi)\big) &= \varphi\big((u + v) + \mathrm{Ker}\phi\big) = \phi(u + v) \\ &= \phi(u) + \phi(v) = \varphi(u + \mathrm{Ker}\phi) + \varphi(v + \mathrm{Ker}\phi)\end{aligned}$$

$$\begin{aligned}\varphi((u + \mathrm{Ker}\phi)(v + \mathrm{Ker}\phi) &= \varphi\big((uv) + \mathrm{Ker}\phi\big) = \phi(uv) = \phi(u)\phi(v) \\ &= \varphi(u + \mathrm{Ker}\phi)\varphi(v + \mathrm{Ker}\phi)\end{aligned}$$

Finally let us prove that φ is bijective. If $u + \mathrm{Ker}\phi \in \mathrm{Ker}\varphi$, then

$$\varphi(u + \mathrm{Ker}\phi) = \phi(u) = 0$$

and so $u \in \mathrm{Ker}\phi$ or equivalently $u + \mathrm{Ker}\phi = \mathrm{Ker}\phi$, hence kernel of φ contains only the zero element $\mathrm{Ker}\phi$, so that φ is injective. Now let $v \in Im\varphi$, then there exist $u \in A$ such that $\phi(u) = v$ or equivalently that $\varphi(u + \mathrm{Ker}\phi) = \phi(u) = v$. Thus $v \in \mathrm{Im}\varphi$ and so φ is surjective. Therefore φ is an isomorphism.

Definition 2.13: Let U, V and W be K-modules where K is a commutative ring with 1. A tensor product of U and V is a K-module $V \otimes W$ along with a bilinear map $\tau: U \times V \to U \otimes V$, such that for every K-module W and every bilinear map $\varphi: U \times V \to W$, there exists a unique linear map $\phi: U \otimes V \to W$ such that the diagram

$$\begin{tikzcd} U \otimes V \arrow[dr, dashed, "\phi"] & \\ U \times V \arrow[u, "\tau"] \arrow[r, "\varphi"'] & W \end{tikzcd}$$

commutes, that is, $\varphi = \phi \circ \tau$.

The existence of the map ϕ satisfying the above conditions is called the universal property of the tensor product. Note that extending the above definition and gives us the definition of tensor product of usual algebras over the field K.

The next proposition came from (Garrett, 2008).

Proposition 2.14: Tensor product $U \otimes V$ exists and is unique up to unique isomorphism.

Theorem 2.15: Let U and V be a vector spaces with the bases $\mathfrak{B}_U$ and $\mathfrak{B}_V$ respectively. Then the set

$$\{u \otimes v : u \in \mathfrak{B}_U \; and \; v \in \mathfrak{B}_V\}$$

is a basis of $U \otimes V$.

Lemma 2.16: $U \otimes V$ is isomorphic to $V \otimes U$.

The next example is in Conrad (2016).

Example 2.17: $K[x] \otimes K[y] \cong K[x, y]$.

Solution: $K[x] \otimes K[y]$ is made up of a finite sum of elements like $f(x) \otimes g(y)$ and since $f(x)$ $g(y)$ are polynomials with coefficients in K, if

$f(x) = \sum_{i=1}^{n} a_i \, x^i$ and $g(y) = \sum_{j=1}^{m} b_j \, y^j$ where $a_i, b_j \in K$, then

$$f(x) \otimes g(y) = \sum_{i=1}^{n} a_i \, x^i \otimes \sum_{j=1}^{m} b_j \, y^j$$

Let's define a map ϕ in order to make isomorphism

$$\phi(f \otimes g) = \sum_{i,j} a_i \, b_j x^i y^j$$

where this sum spans over all possible i, j considered. Using the property of tensor products $K[x] \otimes K[y]$, ϕ is a well-defined in K-module homomorphism.

Define the following map:

$$\psi\left(\sum_{i,j} c_{i,j}\, x^i y^j\right) = \sum_{i,j} c_{i,j}\,(x^i \otimes y^j)$$

It takes an element like the one we get from ϕ and places a tensor in between the x and y terms and pulls out the coefficient. If we get $c_{i,j} = a_i b_j$ then by the properties of tensors:

$$c_{i,j}\left(x^i \otimes y^i\right) = b_j\left(a_i x^i \otimes y^i\right) = a_i x^i \otimes b_j y^j$$

this is linear and a well-defined homomorphism and the compositions of these maps give us the identity. The only thing that is really different is the properties of tensor products so,

$$\sum_{i,j} a_i\, x^i \otimes b_j y^j = \sum_i a_i\, x^i \otimes \sum_j b_j\, y^j = f \otimes g$$

This is what we want and we get the number of these equalities by linearity. Similarly,

$$\sum_{i,j} \phi\left(c_{i,j} x^i \otimes y^j\right) = \sum_{i,j} c_{i,j}\, x^i y^j$$

and so we see that this actually gives an isomorphism. Thus, we have that $[x] \otimes K[y] \cong K[x, y]$.

The following definition came from Szymiczek (1997).

Definition 2.18: Let B be a class of algebras and A be an algebra generated by a set X. The algebra A is called a free algebra in the class B, if it is generated by X and for any algebra $C \in B$, every mapping $\varphi: X \to C$ can be extended to a unique algebra homomorphism $\bar{\varphi}: A \to C$, such that the following diagram commutes, i.e. $\bar{\varphi} i = \varphi$:

$$\begin{tikzcd} X \arrow[rr, "i"] \arrow[dr, "\varphi"'] & & A \arrow[dl, "\bar{\varphi}"] \\ & C & \end{tikzcd}$$

where $i: X \to A$ is the inclusion map. The cardinality $|X|$ of the set X is called the rank of A.

Example 2.19: The unitary polynomial algebra $K[X_m] = K[x_1, \dots, x_m]$ of rank m, where m is finite, is generated by X_m, and the set X_m generates $K[X_m]$ as an algebra in the class of all unitary commutative and associative algebras.

Remark 2.20: A basis $\mathfrak{B}_{K[x]}$ of $K[x]$ with one generator is

$$\mathfrak{B}_{K[x]} = \{1, x, x^2, x^3, x^4, \dots\}.$$

while we have a basis $\mathfrak{B}_{K[X_m]}$ of $K[X_m]$ is

$$\mathfrak{B}_{K[X_m]} = \{x_1^{\alpha_1} \cdots x_m^{\alpha_m} \mid \alpha_1, \dots, \alpha_m \geq 0\}.$$

Example 2.21: $K\langle X_m\rangle = K\langle x_1, \dots, x_m\rangle$ be the polynomial algebra which is associative but not commutative. Then $K\langle X_m\rangle$ is a free K-algebra in the class of all unital associative K-algebras.

Remark 2.22: Subalgebras of a free associative algebra is not necessarily free because the subalgebra of the polynomial ring $K[X]$ generated by x^2 and x^3 is not free.

Definition 2.23: Let V be a vector space over K. V is called a graded K- vector space if it has a direct sum

$$V = \bigoplus_{n\geq 0} V^{(n)} = \sum_{n\geq 0} V^{(n)}$$

where V^n is subspace and $n \geq 0$. An element $v \in V^{(n)}$ is said to be homogeneous of degree n and write $\deg v = n$ or $|v| = n$. At the same time, we define a multigrading on V if

$$V = \sum_{i=1}^{m} \sum_{n_i \geq 0} V^{(n_1,\dots,n_m)},$$

and $V^{(n_1,\dots,n_m)}$ is called homogenous component of degree $(n_1,\dots,n_m)$.

The subspace W of the graded vector space $V = \sum_{n\geq 0} V^{(n)}$ is graded with $W = \sum_{n\geq 0}(W \cap V^{(n)})$. In this case, the factor space V/W can also be naturally graded and we say that V/W inherits the grading of V.

Let us consider a construction of $\mathcal{T}(V)$ and correspond to tensor algebra as follows. A vector space $\mathcal{T}(V)$ is the direct sum:

$$\mathcal{T}(V) = \mathbb{K} \oplus V \oplus V^{\oplus 2} \oplus \cdots \oplus V^{\oplus n} \cdots = \sum_{i=0}^{\infty} V^{\oplus i}$$

Thus the elements of $\mathcal{T}(V)$ are finite sums $v_{i_1} + v_{i_2} + \cdots + v_{i_j}$, $v_{i_j} \in V^{\oplus i_j}$. The multiplication in $\mathcal{T}$ can be defined by extending the multiplication

$$V^{\otimes n} \times V^{\otimes m} \to V^{\otimes (n+m)} \qquad (v, w) \mapsto v \otimes w,$$

bilinearly to all of $\mathcal{T}(V)$. The tensor algebra $\mathcal{T}(V)$ of a vector space V is a graded associative algebra with 1. Note that by construction, the elements of $\mathcal{T}(V)$ are sums of products of elements of V, that is, $\mathcal{T}(V)$ is generated by V.

The following example is in Drensky (2000)

Example 2.24: Polynomial algebras $K[x_1, \cdots, x_m]$ and the free associative algebra $K\langle x_1, \cdots, x_m\rangle$ are graded.

Definition 2.25: If $V = \oplus_{n\geq 0} V^{(n)}$ is a finitely generated graded K-algebra, then we define the Hilbert function of V as

$$H(V, n) = \dim_K(V^{(n)})$$

where $\dim_K(V^{(n)})$ is the dimension of the vector space $V^{(n)}$ over K.

Since V is finitely generated for each non-negative integer j, then there are many finitely monomials of degree j in the generators of V.

Definition 2.26: Suppose that $V = \sum_{n\geq 0} V^{(n)}$ is a graded vector space with $\dim V^{(n)} < \infty$ for all $n \geq 0$, then we can define formal power series

$$H(V,t) = Hilb(V,t) = \sum_{n\geq 0} dim_K(V^{(n)})\, t^n$$

called the Hilbert (or Poincaré) series of V. If the vector space

$$V = \sum_{n\geq 0} V^{(n_1,\cdots,n_m)}, \quad n = (n_1,\cdots,n_m),$$

is multigraded, then the Hilbert series of V is

$$H(V,t_1,\cdots,t_m) = Hilb(V,t_1,\cdots,t_m) = \sum_{n\geq 0} dim_K(V^{(n_1,\cdots,n_m)})\, t_1^{n_1}\cdots t_m^{n_m}.$$

If $V = \oplus V^{(n)}$ and $W = \oplus W^{(n)}$ are graded vector spaces with the same grading, then $V \oplus W$, $V \otimes W$ are also graded with homogeneous components

$$(V \oplus W)^{(n)} = V^{(n)} \oplus W^{(n)}, \qquad (V \otimes W)^{(n)} = \oplus_{n'+n''=n} V^{(n')} \otimes W^{(n'')}.$$

The Hilbert series of $V \oplus W$ and $V \otimes W$, satisfy this relations

$$H(V \oplus W,t) = H(V,t) + H(W,t)$$
$$H(V \otimes W,t) = H(V,t)\cdot H(W,t)$$

Example 2.27: Let us prove that

$$Hilb(K[x],t) = \frac{1}{1-t}$$

Proof: Let $V = K[x]$ and let V have the basis $\{1,x,x^2,x^3,\cdots\}$

$$V^{(0)} = K = Sp\{1\} \rightarrow \dim_K\left(V^{(0)}\right) = 1$$
$$V^{(1)} = Sp\{x\} = K\cdot x = \{c.x \mid c \in K\} \rightarrow \dim_K\left(V^{(1)}\right) = 1$$
$$V^{(2)} = Sp\{x^2\} = K\cdot x^2 = \{c\cdot x^2 \mid c \in K\} \rightarrow \dim_K\left(V^{(2)}\right) = 1$$
$$\vdots$$
$$H(V,t) = \sum_{n\geq 0} dim_K\left(V^{(n)}\right)\cdot t^n$$
$$H(V,t) = \sum_{n\geq 0} 1\cdot t^n = 1 + t + t^2 + \cdots = \frac{1}{1-t}$$

Note that we assume t is a real variable where the series converges to a rational function.

Example 2.28: Let $K[x_1, x_2, \cdots, x_n]$ be the polynomial algebra then

$$H(K[x_1, x_2, \cdots, x_n], t) = \frac{1}{(1-t)^n}$$

Proof: This can be proved by induction on the number of variables n. The case $n = 1$ is trivial from the Example 2.27. We are going to consider the case for $n > 1$, suppose now that it holds in $n - 1$ variables $x_1, x_2, \cdots x_{n-1}$, so

$$H(K[x_1, x_2, \cdots, x_{n-1}], t) = \frac{1}{(1-t)^{n-1}}$$

By Example 2.27 we have inductively that

$$\begin{aligned} K[x_1, x_2, \cdots, x_n] &= (K[x_1] \otimes K[x_2] \otimes \cdots \otimes K[x_{n-1}]) \otimes K[x_n] \\ &= K[x_1, x_2, \cdots, x_{n-1}] \otimes K[x_n] \end{aligned}$$

On the other hand we have

$$H(V \otimes W, t) = H(V, t) \cdot H(W, t)$$

which gives that

$$\begin{aligned} H(K[x_1, x_2, \cdots, x_n], t) &= H(K[x_1, x_2, \cdots, x_{n-1}] \otimes K[x_n], t) \\ &= H(K[x_1, x_2, \cdots, x_{n-1}], t) \cdot H(K[x_n], t) \\ &= \frac{1}{1-t} \cdot \frac{1}{(1-t)^{n-1}} \end{aligned}$$

Therefore

$$H\left(V^{(n)}, t\right) = \frac{1}{(1-t)^n}$$

Example 2.29: Let $K\langle x_1, x_2, \ldots, x_m\rangle$ be the free associative algebra then

$$H(K\langle x_1, x_2, \ldots, x_m\rangle, t) = \frac{1}{1-mt}$$

Proof: In the case $m = 1$, $K\langle x_1\rangle = K[x_1]$ and thus

$$H(K\langle x_1\rangle, t) = \frac{1}{1-t}$$

Let $K\langle x_1, x_2, \ldots, x_m\rangle^{(n)}$ be the n-th homogeneous subspace of $K\langle x_1, x_2, \ldots, x_m\rangle$. For the dimension of $K\langle x_1, x_2, \ldots, x_m\rangle^{(n)}$, we need the number of elements in its canonical basis

$$B_n = \left\{x_1^{l_1}, x_2^{l_2}, \ldots, x_m^{l_m} \,:\, \sum l_j = n\right\}.$$

Number of such elements is equivalent to the possibilities of words written in length n filled by $x_1, x_2, \ldots, x_m$. Because the words are not commutative, for any position, we have m-candidates. Hence $\dim_K K\langle x_1, x_2, \ldots, x_m\rangle^{(n)} = m^n$

The Hilbert Series of free associative algebra becomes

$$H(K\langle x_1, x_2, \ldots, x_m\rangle, t) = \sum_{n=0}^{\infty} \dim_K K\langle x_1, x_2, \ldots, x_m\rangle^{(n)} . t^n$$

$$= \sum_{n=0}^{\infty} m^n t^n = \sum_{n=0}^{\infty} (mt)^n = \frac{1}{1-mt}$$

Definition 2.30: A Lie algebra L is a vector space over K endowed with a bilinear map $[.,.] : L \times L \to L, (x, y) \to [x, y]$, such that the following conditions hold:

$$L(1): [x, x] = 0$$

$$L(2): \big[[x, y], z\big] + \big[[y, z], x\big] + \big[[z, x], y\big] = 0$$

where $x, y, z \in L$. The Lie bracket $[x, y]$ is often referred to as the commutator of x and y. Condition $(L2)$ is known as the Jacobi identity. As the Lie bracket $[.,.]$ is bilinear, we have

$$0 = [x+y, x+y] = [x, x] + [x, y] + [y, x] + [y, y] = [x, y] + [y, x].$$

Hence condition $(L1)$ implies

$$[x, y] = -[y, x] \text{ for all } x, y \in L.$$

One can rewrite The Jacobi identity as

$$\big[[x, y], z\big] = \big[x, [y, z]\big] + \big[[x, z], y\big].$$

Definition 2.31: For every element $x \in L$ we define the map ad_x, called adjoint action, as follows:

$$ad_x: L \longrightarrow L, y \longmapsto [x, y]$$

Here the linear map

$$ad: x \longmapsto ad_x,$$

is called the adjoint representation which is one of the well known examples of Lie homomorphisms. Let us show that ad is a homomorphism. For all $z \in L$ we have that

$$\begin{aligned}
[ad(x), ad(y)](z) &= [ad_x, ad_y](z) = (ad_x ad_y - ad_y ad_x)(z) \\
&= (ad_x ad_y)(z) - (ad_y ad_x)(z) \\
&= ad_x\left(ad_y(z)\right) - ad_y\left(ad_x(z)\right) \\
&= ad_x([y,z]) - ad_y([x,z]) = [x,[y,z]] - [y,[x,z]] \\
&= [[z,y],x] + [[x,z],y] = -[[y,x],z] = [[x,y],z] \\
&= ad_{[x,y]}(z) = ad([x,y])(z)
\end{aligned}$$

Thus

$$ad([x,y]) = [ad(x), ad(y)]$$

and this implies that ad is a homomorphism.

Definition 2.32: The center of Lie algebra L is defined by

$$Z(L) = \{x \in L;\ [x,y] = 0, \text{for all } y \in L\}.$$

Proposition 2.33: The center of Lie algebra L is an ideal in L and is the kernel of the adjoint representation ad.

Definition 2.34: The derived subalgebra $L^{'}$ of Lie algebra L is defined as follows:

$$L^{'} = [L, L] = \{[x,y]: x, y \in L\}$$

Proposition 2.35: The derived subalgebra $L^{'}$ is an ideal of L.

Proof: From definition 2.34, let $x, y \in L$, so we have

$$[\underbrace{x}_{\in L}, \underbrace{y}_{\in L}] \in L^{'}, \quad [\underbrace{y}_{\in L}, \underbrace{x]}_{\in L} \in L^{'}$$

therefore the subalgebra $L^{'}$ is an ideal of L.

Definition 2.36: For all non-negative integer m, the ideal $L^{(m)}$ of L is defined as

$$L^{(0)} = L,$$
$$L^{(1)} = [L, L] = L',$$
$$L^{(2)} = [L^{(1)}, L^{(1)}],$$
$$\vdots$$
$$L^{(m+1)} = [L^{(m)}, L^{(m)}]$$

Hence, we can iterate to define the derived sequence of L as follows:

$$L = L^{(0)} \supseteq L^{(1)} \supseteq \cdots \supseteq L^{(m)} \supseteq \cdots$$

The first derived $L^{(1)}$ and second derived $L^{(2)}$ can be denoted by L' and L'', respectively.

The following lemma is stated in Erdmann and Wildon (2006)

Lemma 2.37: Let I be an ideal of Lie algebra L, if $L/_I$ is an abelian then $L' = [L, L] \subset I$.

Definition 2.38: L is solvable if for some $m \geq 1$, $L^{(m)} = 0$.

Definition 2.39: The lower center of a Lie algebra L is the series with terms

$$L^1 = L' = [L, L] \quad \text{and } L^k = [L^{k-1}, L^1] \quad \text{for } k \geq 2.$$

Thus, $L \supseteq L^1 \supseteq L^2 \supseteq L^3 \supseteq \cdots$ is called central series and $L^k/_{L^{k+1}}$ is in the center of $L/_{L^{k+1}}$.

Definition 2.40: The Lie algebra L is said to be nilpotent if for some positive integer m, $L^m = 0$.

Lemma 2.41: Any nilpotent Lie algebra L is solvable.

Proof: Let's use induction of k to show that $L^{(k)} \subseteq L^k$

For $k = 1$, $L^{(1)} = L' = [L, L] = L^1$

For $k - 1$, let's consider $L^{(k-1)} \subseteq L^{k-1}$.

$L^k = [L^{(k-1)}, L^{(k-1)}] \subseteq [L^{k-1}, L] = L^k$.

For $m \geq 1$, we have $L^m = \{0\}$, then $L^{(m)} \subseteq L^m = \{0\}$. Hence $L^{(m)} = \{0\}$. Therefore L is solvable.

Theorem 2.42: Let X be a set. There exist a Lie algebra generated by X, satisfying no relations other than $(L1)$ and $(L2)$.

Proof: Let $M(X)$ be the set called free magma on X then by inductive on the integer $n \geq 1$, we define the sets X_n by writing

$$X_1 = X$$

$$X_n = \bigcup_{p=1}^{n-1} X_p \times X_{n-p}$$

where $p = 1,2, \cdots, n-1$ if X is finite, so each X_n. The sum set of the family $(X_n)_{n\geq 1}$ is denoted by $M(X)$; each of sets X_n is identified with a subset of $M(X)$ hence we can also write

$$M(X) = \bigcup_{n=1}^{\infty} X_p.$$

Let $a, b \in M(X)$ and let p and q denote the integers such that $a \in X_p$ and $b \in X_q$ and let $n = p + q$; the image of the ordered pair (a, b) under the canonical injection of $X_p \times X_{n-p}$ into X_n is denoted by $a \cdot b$ and called the product of a and b. Every mapping of X into magma M can be extended in a unique way to a magma homomorphism of $M(X)$ into M.

Since $a \in M(X)$; the unique integer n such that $a \in X_n$ is called the length of a and denoted by $l(a)$, then $l(a \cdot b) = l(a) + l(b)$ for $a, b \in M(X)$. The set X is the subset of $M(X)$ consisting of the elements of length 1. Every elements a of length ≥ 2 can be written uniquely in the form $a = b \cdot c$.

Let K be a field and let $A(X)$ be the vector space over K spanned by $M(X)$. If we extend the binary operation on $M(X)$ bilinearly to $A(X)$, then $A(X)$ becomes a (non-associative) algebra; it is called the free algebra over K on X. Let $f \in A(X)$; if also $f \in M(X)$, then f is said to be a monomial. For $f \in A(X)$ we define $\deg(f)$ to be the maximum of the degrees of a, where a runs over all $a \in M(X)$ that occur in f with non-zero coefficient.

Let I_o be the ideal of $A(X)$ generated by all elements

$$(a, a) \quad \text{for } a \in M(X)$$

$$(a, b) + (b, a) \quad \text{for } a, b \in M(X)$$

$$\big(a, (b, c)\big) + \big(b, (c, a)\big) + (c, (a, b)) \quad \text{for } a, b, c \in M(X).$$

Set $L(X) = A(X)/I_o$. Let $\mathfrak{B}$ be a basis of $L(X)$ consisting of (images of) elements of $M(X)$. Then it is immediate that we have $(x, x) = 0$, $(x, y) + (y, x) = 0$ and $\big(x, (y, z)\big) + \big(y, (z, x)\big) + \big(z, (x, y)\big) = 0$ for all $x, y, z \in \mathfrak{B}$. It follows that the relations $(L1)$ and $(L2)$ are holds for all elements of $L(X)$ so that $L(X)$ is Lie algebra. Therefore we will use the bracket to denote the product in $L(X)$.

Definition 2.43: The Lie algebra $L(X)$ is called the free Lie algebra on X.

Definition 2.44: If R is an associative algebra and the Lie algebra L is isomorphic to subalgebra of $R^{(-)}$, we say that R is an enveloping algebra of L. The associative algebra $U = U(L)$ is the universal enveloping algebra of the lie algebra L, if L is a subalgebra of $U^{(-)}$ and U has the following universal property: For any associative algebra R and any homomorphism of Lie algebras $\phi : L \to R^{(-)}$ there exist a unique homomorphism of associative algebras $\psi : U \to R$ which extends ϕ, i.e. ψ is equal to ϕ on L.

Theorem 2.45 (The Poincaré-Birkhoff –Witt Theorem): Every Lie algebra L has a unique (up to an isomorphism) universal enveloping algebra $U(L)$. If L has a basis $\{e_i \mid i \in I\}$, and the set of indices I is ordered, then $U(L)$ has a basis 1 and

$$e_{i_1} \cdots e_{i_p}, \; i_1 \leq \cdots \leq i_p, \; i_k \in I, \qquad p = 1,2, \cdots$$

3. COMPUTATIONAL RESULTS

3.1. Hall Basis

A set M with binary operations is called magma, and the condition of being associative is not necessarily. A basis for a free Lie algebra is the Hall set or the Hall basis, which is a particular kind of subset of the free magma on an alphabet.

Serre (1962) defined the magma M as free magma on X denoted as M_X, for the set X which is defined inductively by the family of sets X_n for $n \geq 1$ and satisfy the following conditions:

a. $X_1 = X$

b. $X_n = \bigcup_{p+q=n} X_p \times X_q \;\; n \geq 2$ (disjoint union) where we put $M_X = \bigcup_{n=1}^{\infty} X_n$ and define $M_X \times M_X \to M_X$ for $w, w^{'}$ of M_X, we note that the natural p and q such that $w \in X_p$ and $w^{'} \in X_q$, suppose that $n = p + q$ is image of couple of $(w, w^{'})$ of canonical injection of $X_p \times X_q \to X_{p+q} \subset M_X$ this is denoted by $w \cdot w^{'}$ and is called the product of w and $w^{'}$ where the arrow is the canonical inclusion resulting from b.

An element w of M_X is called a non- associative word on X and the unique of natural n such that $w \in X_n$ is called length and is denoted by $l(w)$. We have $l(w \cdot w^{'}) = l(w) + l(w^{'})$ for $w, w^{'}$ in M_X. The set of X is a party of M_X formed by the element of length 1 and M_X is defined as a finite and nonempty set where it can be identified with a set of binary, complete, rooted trees with leaves labeled by elements in X. All element w of length ≥ 2 is written in unique way with this form

$$w = w^{'} \cdot w^{''}$$

where $w^{'}$ and $w^{''}$ is its immediate left and right subtree respectively. The binary operation of M_X is the mapping $M_X \times M_X \to M_X$, $(w^{'}, w^{''})$. We define the degree $|w|$ of a tree w to be the number of its leaves, i.e $|w| = 1$ if $w \in X$ and $|(w^{'}, w^{''})| = |w^{'}| + |w^{''}|$.

There is a canonical map from M_X onto X^* (free monoid over X) defined by $f(a) = a$ if $a \in X$ and $f(w) = f(w^{'})f(w^{''})$ is of degree greater than 1. Degree (length) of a tree (word) is usually defined to be the number of its leaves (letters). We note that $|w| = |f(w)|$, therefore $f(w)$ is called the foliage of w.

Definition 3.1.1: A subset H of M_X is called a Hall set if its holds the following conditions

i) H has total order $\leq_M$;

ii) $X \subseteq H$;

iii) For any tree $h = (h^{'}, h^{''})$ in H, we have $h^{''} \in H$ and $h > h^{''}$.

iv) For any tree $h = (h^{'}, h^{''})$ in H, we have $h \in H$ iff $h^{'}, h^{''} \in H$ and $h^{'} > h^{''}$ and either $h^{'} \in X$ and $(h^{'})^{''} \leq_M h^{''}$.

where X is called generating set and the elements in X are called generators. Here $h^{'}$, $h^{''}$ are denoted as left and right subtree of h respectively and $(h^{'})^{''}$ is denoted the right subtree of the left subtree of h.

Definition 3.1.2: For a fixed Hall set, Hall tree is called an element of Hall set and every subtree of Hall tree is also Hall tree.

Definition 3.1.3: We call the Hall word the foliage of the Hall tree.

Corollary 3.1.4: Reutenauer (1993) show how every Hall word is the foliage of a unique Hall tee.

Here we may identify clearly Hall trees and Hall words, where Hall set in X^* is called the image under f of Hall set in M_X, for a given corresponding total order on Hall set.

Reutenauer (1993) and Hu (2009) identify on how each node in a Hall tree can be interpreted as a lie bracket. In the example given there a Hall tree $h = (h^{'}, h^{''})$ of order greater than or equal to 2 can be written as

$$h = [h^{'}, h^{''}]$$

where $h^{'}, h^{''}$ is left and right subtree respectively and $[*,*]$ is denoted as the lie bracket.

Actually, Hall set are not unique that why a specific Hall set depends on how you define the total order. In this thesis we define the order of Hall set of $2, 3, 4$ generators using the idea generated by Hall (1950) where he said that in the free ring, every element may be written as linear combination of the "Left normed" elements $\left[[[x_1, x_2], x_3], \cdots, x_n\right] = [x_1, x_2, x_3, \cdots, x_n]$ but these are not independent. Let $X = \{x, y\}$ and define a total order on X by setting $x >_X y$. Let M_X denote the free magma on X; so M_X is the set of all nonassociative words in $\{x, y\}$. Any $h \in M_X$ can be written uniquely as $h = [h^{'}, h^{''}]$ where $h^{'}, h^{''} \in M_X$. Now we can define the total order $>_{M_X}$ or simply $>_M$ on M_X by making $>_M$ agree with $>_X$ on X, then for $h, t \in M_X/X$, with $t = [t^{'}, t^{''}]$, we define $h >_M t$ iff:

a) $\deg(h) > \deg(t)$, or
b) $\deg(h) = \deg(t)$, but $h^{'} >_M t^{'}$, or
c) $\deg(h) = \deg(t)$ and $h^{'} = t^{'}$, but $h^{''} >_M t^{''}$.

Example 3.1.5: Hall Words of 2 Generators:
The construction of the recursive computation to generate Hall words (Hall trees) of degree $n \leq 8$ denotes $L_2 = K\langle x, y\rangle$ for $x > y$ are illustrate accordingly by total order $>_M$:

$n = 1$ x, y $x > y$

$n = 2$ $[x, y]$

$n = 3$ $\left[[x, y], y\right], [[x, y], x]$

$n = 4$ $\left[\left[[x, y], y\right], y\right], \left[\left[[x, y], y\right], x\right], \left[\left[[x, y], x\right], x\right]$

$n = 5$ $\left[\left[\left[[x, y], y\right], y\right], y\right], \left[\left[\left[[x, y], y\right], y\right], x\right], \left[\left[\left[[x, y], y\right], x\right], x\right],$

$\left[\left[\left[[x, y], x\right], x\right], x\right], \left[\left[[x, y], y\right], [x, y]\right], \left[\left[[x, y], x\right], [x, y]\right]$

$$n = 6 \qquad [[[[[x,y],y],y],y],y], [[[[[x,y],y],y],y],x], [[[[[x,y],y],y],x],x],$$

$$[[[[[x,y],y],x],x],x], [[[[[x,y],x],x],x],x], [[[[x,y],y],y],[x,y]],$$

$$[[[[x,y],y],x],[x,y]], [[[[x,y],x],x],[x,y]], [[[x,y],x],[[x,y],y]]$$

$$n = 7 \qquad [[[[[[x,y],y],y],y],y],y], [[[[[[x,y],y],y],y],y],x],$$

$$[[[[[[x,y],y],y],y],x],x], [[[[[[x,y],y],y],x],x],x],$$

$$[[[[[[x,y],y],x],x],x],x], [[[[[[x,y],x],x],x],x],x],$$

$$[[[[[x,y],y],y],y],[x,y]], [[[[[x,y],y],y],x],[x,y]],$$

$$[[[[[x,y],y],x],x],[x,y]], [[[[[x,y],x],x],x],[x,y]],$$

$$[[[x,y],y],[x,y],[x,y]], [[[x,y],x],[x,y],[x,y]],$$

$$[[[[x,y],y],y],[[x,y],y]], [[[[x,y],y],y],[[x,y],x]],$$

$$[[[[x,y],y],x],[[x,y],y]], [[[[x,y],y],x],[[x,y],x]],$$

$$[[[[x,y],x],x],[[x,y],y]], [[[[x,y],x],x],[[x,y],x]]$$

$$n = 8 \qquad [[[[[[[x,y],y],y],y],y],y],y], [[[[[[[x,y],y],y],y],y],y],x],$$

$$[[[[[[[x,y],y],y],y],y],x],x], [[[[[[[x,y],y],y],y],x],x],x],$$

$$[[[[[[[x,y],y],y],x],x],x],x], [[[[[[[x,y],y],x],x],x],x],x],$$

$$[[[[[[[x,y],x],x],x],x],x],x], [[[[[[x,y],y],y],y],y],[x,y]],$$

$$[[[[[[x,y],y],y],y],x],[x,y]], [[[[[[x,y],y],y],x],x],[x,y]],$$

$$[[[[[[x,y],y],x],x],x],[x,y]], [[[[[[x,y],x],x],x],x],[x,y]],$$

$$[[[[x,y],y],y],[x,y],[x,y]], [[[[x,y],y],x],[x,y],[x,y]],$$

$$[[[[x,y],x],x],[x,y],[x,y]], [[[[[x,y],y],y],y],[[x,y],y]],$$

$$[[[[[x,y],y],y],y],[[x,y],x]], [[[[[x,y],y],y],x],[[x,y],y]],$$

$$[[[[[x,y],y],y],x],[[x,y],x]], [[[[[x,y],y],x],x],[[x,y],y]],$$

$$[[[[[x,y],y],x],x],[[x,y],x]], [[[[[x,y],x],x],x],[[x,y],y]],$$

$$[[[[[x,y],x],x],x],[[x,y],x]], [[[x,y],y],[x,y],[[x,y],y]],$$

$$[[[x,y],y],[x,y],[[x,y],x]], [[[x,y],x],[x,y],[[x,y],y]],$$

$$[[[x,y],x],[x,y],[[x,y],x]], [[[[x,y],x],x],[[[x,y],y],x]],$$

$$[[[[x,y],x],x],[[[x,y],y],y]], [[[[x,y],y],x],[[[x,y],y],y]].$$

Example 3.1.6: Hall Words of 3 Generators:

Let $X = \{x, y, z\}$ and define a total order on X by setting $x >_X y >_X z$. Let M_X denote the free magma on X; so M_X is the set of all nonassociative words in $\{x, y, z\}$. All the properties used in Example 3.1.5 are equally applied to Hall words of 3 generators case.

The construction of the recursive computation to generate Hall words (Hall trees) of degree $n \leq 4$ denotes $L_3 = K\langle x, y, z\rangle$ for $x > y > z$ are illustrate accordingly by total order $>_M$:

$n = 1$ $\quad x, y, z \qquad x > y > z$

$n = 2$ $\quad [x, y], [x, z], [y, z]$

$n = 3$ $\quad [[x, y], y], [[x, y], x], [[x, z], z], [[x, z], y], [[x, z], x], [[y, z], z],$
$[[y, z], y], [[y, z], x]$

$n = 4$ $\quad \big[[[x, y], y], y\big], \big[[[x, y], y], x\big], \big[[[x, y], x], x\big], \big[[[x, z], z], z\big],$
$\big[[[x, z], z], y\big], \big[[[x, z], z], x\big], \big[[[x, z], y], y\big], \big[[[x, z], y], x\big],$
$\big[[[x, z], x], x\big], \big[[[y, z], z], z\big], \big[[[y, z], z], y\big], \big[[[y, z], z], x\big], \big[[[y, z], y], y\big],$
$\big[[[y, z], y], x\big], \big[[[y, z], x], x\big], \big[[x, y], [x, z]\big], \big[[x, y], [y, z]\big], \big[[x, z], [y, z]\big].$

Example 3.1.7: Hall Words of 4 Generators:

Let $X = \{x, y, z, t\}$ and define a total order on X by setting $x >_X y >_X z >_X t$. Let M_X denote the free magma on X; so M_X is the set of all nonassociative words in $\{x, y, z, t\}$. All the properties used in the Hall word of Example 3.1.5 and Example 3.1.6 are equally applied to Hall words of 4 generators case.

The construction of the recursive computation to generate Hall words (Hall trees) of degree $n \leq 3$ denotes $L_4 = K\langle x, y, z, t\rangle$ for $x > y > z > t$ are illustrate accordingly by total order $>_M$:

$n = 1$ $\quad x, y, z, t \qquad x > y > z > t$

$n = 2$ $\quad [x, y], [x, z], [x, t], [y, z], [y, t], [z, t]$

$n = 3$ $\quad [[x, y], y], [[x, y], x], [[x, z], z], [[x, z], y], [[x, z], x], [[x, t], t],$
$[[x, t], z], [[x, t], y], [[x, t], x], [[y, z], z], [[y, z], y], [[y, z], x], [[y, t], t],$

$$[[y,t],z],[[y,t],y],[[y,t],x],[[z,t],t],[[z,t],z],[[z,t],y],[[z,t],x].$$

The Hall word of 2,3 and 4 generators can also be calculated using Witt Dimension Formula and Möbius Function so, we are going to look on this as following:

Definition 3.1.8: Suppose that n is a positive integer, if n is divisible by the square of a prime number, we can define $\mu(n)$ as follow:

$$\mu(n)=\begin{cases} 1, \text{if } n \text{ is a squarefree positive integer with even number} \\ -1, if\ n \text{ a squarefree positive integer with odd number} \\ 0, \text{if } n \text{ is not squarefree} \end{cases}$$

The function $\mu: \mathbb{N} \to \{-1,0,1\}$ defined above is called Möbius Function.

For more use Bourbaki (2006).

Table 1: The values of Möbius function $\boldsymbol{\mu}$ for $\boldsymbol{n \geq 10}$

n	1	2	3	4	5	6	7	8	9	10
$\mu(n)$	1	-1	-1	0	-1	1	-1	0	0	1

Recall that given two integers $n_1 \geq 1, n_2 \geq 2$, we can write $n_1 | n_2$ if n_1 divides n_2.

From Bahturin (1987),

Theorem 3.1.9: If L_d is the free Lie algebra with generators d, then the dimension of the space of homogeneous expression of degree n or the number of Hall words in Hall basis is:

$$\dim L_d^{(n)} \sum_{d|n} \mu(d) r^{n|d}$$

where the μ is Möbius function.

For $L_2 = K\langle x,y\rangle \quad x > y \quad$ we get

a) $n = 1 \quad r = 2$

$$\dim L_2^{(1)} = \frac{1}{1}\sum_{d|1} \mu(d) 2^{1|d} = 1\mu(1)\cdot 2 = 2$$

b) $n = 2 \;\; r = 2$

$$\dim L_2^{(2)} = \frac{1}{2}\sum_{d|2} \mu(d) 2^{2|d} = \frac{1}{2}\left[\mu(1)2^2 + \mu(2)2^{\frac{2}{2}}\right] = \frac{1}{2}(4-2) = 1$$

c) $n = 3 \quad r = 2$

$$\dim L_2^{(3)} = \frac{1}{3}\sum_{d|3} \mu(d) 2^{3|d} = \frac{1}{3}\left[\mu(1)2^{\frac{3}{1}} + \mu(3)2^{\frac{3}{3}}\right] = \frac{1}{3}(2^3-2) = \frac{6}{3} = 2$$

d) $n = 4 \quad r = 2$

$$\dim L_2^{(4)} = \frac{1}{4}\sum_{d|4} \mu(d) 2^{4|d} = \frac{1}{4}\left[\mu(1)2^{\frac{4}{1}} + \mu(2)2^{\frac{4}{2}} + \mu(4)2^{\frac{4}{4}}\right] = \frac{1}{4}(2^4-2^2) = \frac{12}{4} = 3$$

e) $n = 5 \quad r = 2$

$$\dim L_2^{(5)} = \frac{1}{5}\sum_{d|5} \mu(d) 2^{5|d} = \frac{1}{5}\left[\mu(1)2^{\frac{5}{1}} + \mu(5)2^{\frac{5}{5}}\right] = \frac{1}{5}(2^5-2) = \frac{30}{5} = 6$$

f) $n = 6 \qquad r = 2$

$$\dim L_2^{(6)} = \frac{1}{6}\sum_{d|6} \mu(d) 2^{6|d} = \frac{1}{6}\left[\mu(1)2^{\frac{6}{1}} + \mu(2)2^{\frac{6}{2}} + \mu(3)2^{\frac{6}{3}} + \mu(6)2^{\frac{6}{6}}\right]$$

$$= \frac{1}{6}(2^6-2^3-2^2+2) = \frac{54}{6} = 9$$

g) $n = 7 \quad r = 2$

$$\dim L_2^{(7)} = \frac{1}{7}\sum_{d|7} \mu(d) 2^{7|d} = \frac{1}{7}\left[\mu(1)2^{\frac{7}{1}} + \mu(7)2^{\frac{7}{7}}\right] = \frac{1}{7}(2^7-2) = \frac{126}{7} = 18$$

h) $n = 8 \quad r = 2$

$$\dim L_2^{(8)} = \frac{1}{8}\sum_{d|8} \mu(d)2^{8|d} = \frac{1}{8}\left[\mu(1)2^{\frac{8}{1}} + \mu(2)2^{\frac{8}{2}} + \mu(4)2^{\frac{8}{4}} + \mu(8)2^{\frac{8}{8}}\right]$$

$$= \frac{1}{8}(2^8 - 2^4) = \frac{240}{8} = 30$$

i) $n = 9 \quad r = 2$

$$\dim L_2^{(9)} = \frac{1}{9}\sum_{d|9} \mu(d)2^{9|d} = \frac{1}{9}\left[\mu(1)2^{\frac{9}{1}} + \mu(3)2^{\frac{9}{3}} + \mu(9)2^{\frac{9}{9}}\right] = \frac{1}{9}(2^9 - 2^3)$$

$$= \frac{504}{9} = 56$$

j) $n = 10 \quad r = 2$

$$\dim L_2^{(10)} = \frac{1}{10}\sum_{d|10} \mu(d)2^{10|d} = \frac{1}{10}\left[\mu(1)2^{\frac{10}{1}} + \mu(2)2^{\frac{10}{2}} + \mu(5)2^{\frac{10}{5}} + \mu(10)2^{\frac{10}{10}}\right]$$

$$= \frac{1}{10}(2^{10} - 2^5 - 2^2 + 2) = \frac{990}{10} = 99$$

For $L_3 = K\langle x, y, z\rangle \quad x > y > z$ we get

a) $n = 1 \quad r = 3$

$$\dim L_3^{(1)} = \frac{1}{1}\sum_{d|1} \mu(d)3^{1|d} = 1\mu(1) \cdot 3 = 3$$

b) $n = 2 \quad r = 3$

$$\dim L_3^{(2)} = \frac{1}{2}\sum_{d|2} \mu(d)3^{2|d} = \frac{1}{2}\left[\mu(1)3^2 + \mu(2)3^{\frac{2}{2}}\right] = \frac{1}{2}(3^2 - 3) = \frac{6}{2} = 3$$

c) $n = 3 \quad r = 3$

$$\dim L_3^{(3)} = \frac{1}{3}\sum_{d|3} \mu(d)3^{3|d} = \frac{1}{3}\left[\mu(1)3^{\frac{3}{1}} + \mu(3)3^{\frac{3}{3}}\right] = \frac{1}{3}(3^3 - 3) = \frac{24}{3} = 8$$

d) $n = 4 \quad r = 3$

$$\dim L_3^{(4)} = \frac{1}{4}\sum_{d|4} \square(d)3^{4|d} = \frac{1}{4}\left[\mu(1)3^{\frac{4}{1}} + \mu(2)3^{\frac{4}{2}} + \mu(4)3^{\frac{4}{4}}\right] = \frac{1}{4}(3^4 - 3^2) = \frac{72}{4}$$

$$= 18$$

e) $n = 5 \quad r = 3$

$$\dim L_3^{(5)} = \frac{1}{5}\sum_{d|5} \square(d)3^{5|d} = \frac{1}{5}\left[\mu(1)3^{\frac{5}{1}} + \mu(5)3^{\frac{5}{5}}\right] = \frac{1}{5}(3^5 - 3) = \frac{240}{5} = 48$$

f) $n = 6 \quad r = 3$

$$\dim L_3^{(6)} = \frac{1}{6}\sum_{d|6} \square(d)3^{6|d} = \frac{1}{6}\left[\mu(1)3^{\frac{6}{1}} + \mu(2)3^{\frac{6}{2}} + \mu(3)3^{\frac{6}{3}} + \mu(6)3^{\frac{6}{6}}\right]$$

$$= \frac{1}{6}(3^6 - 3^3 - 3^2 + 3) = \frac{696}{6} = 116$$

g) $n = 7 \quad r = 3$

$$\dim L_3^{(7)} = \frac{1}{7}\sum_{d|7} \square(d)3^{7|d} = \frac{1}{7}\left[\mu(1)3^{\frac{7}{1}} + \mu(7)3^{\frac{7}{7}}\right] = \frac{1}{7}(3^7 - 3) = \frac{2184}{7} = 312$$

h) $n = 8 \quad r = 3$

$$\dim L_3^{(8)} = \frac{1}{8}\sum_{d|8} \square(d)3^{8|d} = \frac{1}{8}\left[\mu(1)3^{\frac{8}{1}} + \mu(2)3^{\frac{8}{2}} + \mu(4)3^{\frac{8}{4}} + \mu(8)3^{\frac{8}{8}}\right]$$

$$= \frac{1}{8}(3^8 - 3^4) = \frac{6480}{8} = 810$$

i) $n = 9 \quad r = 3$

$$\dim L_3^{(9)} = \frac{1}{9}\sum_{d|9} \square(d)3^{9|d} = \frac{1}{9}\left[\mu(1)3^{\frac{9}{1}} + \mu(3)3^{\frac{9}{3}} + \mu(9)3^{\frac{9}{9}}\right] = \frac{1}{9}(3^9 - 3^3)$$

$$= \frac{19656}{9} = 2184$$

j) $n = 10 \quad r = 3$

$$\dim L_3^{(10)} = \frac{1}{10}\sum_{d|10} \mu(d)3^{10|d} = \frac{1}{10}\left[\mu(1)3^{\frac{10}{1}} + \mu(2)3^{\frac{10}{2}} + \mu(5)3^{\frac{10}{5}} + \mu(10)3^{\frac{10}{10}}\right]$$

$$= \frac{1}{10}(3^{10} - 3^5 - 3^2 + 3) = \frac{58800}{10} = 5880$$

For $L_4 = K\langle x, y, z, t\rangle \quad x > y > z > t$ we get

a) $n = 1 \quad r = 4$

$$\dim L_4^{(1)} = \frac{1}{1}\sum_{d|1} \mu(d)4^{1|d} = 1\mu(1)\cdot 4 = 4$$

b) $n = 2 \ \ r = 4$

$$\dim L_4^{(2)} = \frac{1}{2}\sum_{d|2} \mu(d)4^{2|d} = \frac{1}{2}\left[\mu(1)4^2 + \mu(2)4^{\frac{2}{2}}\right] = \frac{1}{2}(4^2 - 4) = \frac{12}{2} = 6$$

c) $n = 3 \ \ r = 4$

$$\dim L_4^{(3)} = \frac{1}{3}\sum_{d|3} \mu(d)4^{3|d} = \frac{1}{3}\left[\mu(1)4^{\frac{3}{1}} + \mu(3)4^{\frac{3}{3}}\right] = \frac{1}{3}(4^3 - 4) = \frac{60}{3} = 20$$

d) $n = 4 \ \ r = 4$

$$\dim L_4^{(4)} = \frac{1}{4}\sum_{d|4} \mu(d)4^{4|d} = \frac{1}{4}\left[\mu(1)4^{\frac{4}{1}} + \mu(2)4^{\frac{4}{2}} + \mu(4)4^{\frac{4}{4}}\right] = \frac{1}{4}(4^4 - 4^2)$$

$$= \frac{240}{4} = 60$$

e) $n = 5 \ \ r = 4$

$$\dim L_4^{(5)} = \frac{1}{5}\sum_{d|5} \mu(d)4^{5|d} = \frac{1}{5}\left[\mu(1)4^{\frac{5}{1}} + \mu(5)4^{\frac{5}{5}}\right] = \frac{1}{5}(4^5 - 4) = \frac{1020}{5} = 204$$

f) $n = 6 \quad r = 4$

$$\dim L_4^{(6)} = \frac{1}{6}\sum_{d|6} \mu(d)4^{6|d} = \frac{1}{6}\left[\mu(1)4^{\frac{6}{1}} + \mu(2)4^{\frac{6}{2}} + \mu(3)4^{\frac{6}{3}} + \mu(6)4^{\frac{6}{6}}\right]$$

$$= \frac{1}{6}(4^6 - 4^3 - 4^2 + 4) = \frac{4020}{6} = 670$$

g) $n = 7 \quad r = 4$

$$\dim L_4^{(7)} = \frac{1}{7}\sum_{d|7} \mu(d)4^{7|d} = \frac{1}{7}\left[\mu(1)4^{\frac{7}{1}} + \mu(7)4^{\frac{7}{7}}\right] = \frac{1}{7}(4^7 - 4) = \frac{16380}{7}$$

$$= 2340$$

h) $n = 8 \quad r = 4$

$$\dim L_4^{(8)} = \frac{1}{8}\sum_{d|8} \mu(d)4^{8|d} = \frac{1}{8}\left[\mu(1)4^{\frac{8}{1}} + \mu(2)4^{\frac{8}{2}} + \mu(4)4^{\frac{8}{4}} + \mu(8)4^{\frac{8}{8}}\right]$$

$$= \frac{1}{8}(4^8 - 4^4) = \frac{65280}{8} = 8160$$

i) $n = 9 \quad r = 4$

$$\dim L_4^{(9)} = \frac{1}{9}\sum_{d|9} \mu(d)4^{9|d} = \frac{1}{9}\left[\mu(1)4^{\frac{9}{1}} + \mu(3)4^{\frac{9}{3}} + \mu(9)4^{\frac{9}{9}}\right] = \frac{1}{9}(4^9 - 4^3)$$

$$= \frac{262080}{9} = 29120$$

j) $n = 10 \quad r = 4$

$$\dim L_4^{(10)} = \frac{1}{10}\sum_{d|10} \mu(d)4^{10|d} = \frac{1}{10}\left[\mu(1)4^{\frac{10}{1}} + \mu(2)4^{\frac{10}{2}} + \mu(5)4^{\frac{10}{5}} + \mu(10)4^{\frac{10}{10}}\right]$$

$$= \frac{1}{10}(4^{10} - 4^5 - 4^2 + 4) = \frac{1047540}{10} = 104754$$

Table 2: Summary of Dimension

$\dim L_d^{(n)} = \frac{1}{n}\sum_{d\mid n} \square(d) r^{n\mid d}$	$L_2 = K\langle x,y\rangle$ $x > y$	$L_3 = K\langle x,y,z\rangle$ $x > y > z$	$L_4 = K\langle x,y,z,t\rangle$ $x > y > z > t$
$n = 1$	2	3	4
$n = 2$	1	3	6
$n = 3$	2	8	20
$n = 4$	3	18	60
$n = 5$	6	48	204
$n = 6$	9	116	670
$n = 7$	18	312	2340
$n = 8$	30	810	8160
$n = 9$	56	2184	29120
$n = 10$	99	5880	104754

Definition 3.1.10: A vector space A over the field K with bilinear and trilinear operations $[x,y]$ and $[x,y,z]$ is said to be an Akivis algebra if the bilinear operation is anticommutative and the two operations are verifying the Akivis identity

$$[[x,y],z] + [[y,z],x] + [[z,x],y] =$$
$$[x,y,z] + [y,z,x] + [z,x,y] - [x,z,y] - [y,x,z] - [z,y,x].$$

For more look on Akivis (1976).

Example 3.1.11: A nonassociative algebra A with commutator $[x,y]=xy-yx$ and associator $[x,y,z]=(xy)z-x(yz)$ becomes an Akivis Algebra.

Here we present few homogenous Akivis elements on one generator:

$$x,[x,x,x],\big[[x,x,],x\big],$$

$$\Big[\big[[x,x,x],x\big],x\Big],\big[[x,x,x],x,x\big],[x,[x,x,x],x],\big[x,x,[x,x,x]\big]$$

The following proposition is in Bremner, Hentzel and Peresi (2005)

Proposition 3.1.12: If $X=\{x\}$, then we have the dimensions

$$\dim A^{(1)}=1,\dim A^{(2)}=0,\dim A^{(3)}=1,\dim A^{(4)}=1,\dim A^{(5)}=4,$$

$$\dim A^{(6)}=7,\dim A^{(7)}=23,\dim A^{(8)}=53,\dim A^{(9)}=157,$$

where $A^{(n)}$ is the homogeneous subspace of degree n.

We illustrate the bases elements until length 4:

Basis elements of $A^{(1)}$: x

Basis elements of $A^{(2)}$: none

Basis elements of $A^{(3)}$: $[x,x,x]$

Basis elements of $A^{(4)}$: $\big[[x,x,x],x\big]$

For the Basis elements of $A^{(n)}$ for $n\geq 5$, we have to generate all possible monomials of degree n, containing anticommutative binary and ternary operations. These monomials will span the subspace of Akivis elements. In the case of degree 5, a basis of subspace of Akivis elements is spanned by 4 elements below:

$$\Big[\big[[x,x,x],x\big],x\Big],\qquad \big[[x,x,x],x,x\big],\qquad [x,[x,x,x],x],\qquad \big[x,x,[x,x,x]\big]$$

From the above elements, there are 14 nonassociative monomials from 14, so those monomials of degree 5 are:

1. $\Big(\big(((xx)x)x\big)x\Big)$ 2. $\Big(\big((x(xx))x\big)x\Big)$ 3. $\big(((xx)x)(xx)\big)$ 4. $\big(((xx)(xx))x\big)$

5. $\Big(\big(x((xx)x)\big)x\Big)$ 6. $\Big(\big(x(x(xx))\big)x\Big)$ 7. $\big((x(xx))(xx)\big)$ 8. $\big((xx)((xx)x)\big)$

9. $\big((xx)(x(xx))\big)$ 10. $\Big(x\Big(\big((xx)x\big)x\Big)\Big)$ 11. $\Big(x\Big(\big(x(xx)\big)x\Big)\Big)$ 12. $\Big(x\big((xx)(xx)\big)\Big)$

13. $\Big(x\Big(x\big((xx)x\big)\Big)\Big)$ 14. $\Big(x\Big(x\big(x(xx)\big)\Big)\Big)$

Let's write the bases of subspace of Akivis elements in times of monomials in other to find our matrix.

$$\big[[[x,x,x],x],x\big] = [[\{(xx)x - x(xx)\},x],x]$$
$$= [(\{(xx)x - x(xx)\}x - x\{(xx)x - x(xx)\}),x]$$
$$= (\{(xx)x - x(xx)\}x - x\{(xx)x - x(xx)\})x$$
$$-x(\{(xx)x - x(xx)\}x - x\{(xx)x - x(xx)\})$$
$$= \Big(\Big(\big((xx)x\big)x\Big)x\Big) - \Big(\Big(\big(x(xx)\big)x\Big)x\Big) - \Big(\Big(x\big((xx)x\big)\Big)x\Big)$$
$$+ \Big(\Big(x\big(x(xx)\big)\Big)x\Big) - \Big(x\Big(\big((xx)x\big)x\Big)\Big) + \Big(x\Big(\big(x(xx)\big)x\Big)\Big)$$
$$+ \Big(x\Big(x\big((xx)x\big)\Big)\Big) - \Big(x\Big(x\big(x(xx)\big)\Big)\Big)$$

$$\big[[x,x,x],x,x\big] = [(xx)x - x(xx),x,x] = (\{(xx)x - x(xx)\}x)x - x(\{(xx)x -$$
$$xxxx\ = xxxxx - xxxxx$$

$$-\Big(\big((xx)x\big)(xx)\Big)\Big(\big(x(xx)\big)(xx)\Big)$$

$$[x,[x,x,x],x] = [x,(xx)x - x(xx),x] = (x\{(xx)x - x(xx)\})x - x(\{(xx)x -$$
$$xxxx = xxxxx - xxxxx - xxxxx + xxxxx$$

$$\big[x,x,[x,x,x]\big] = [x,x,(xx)x - x(xx)] = (xx)\{(xx)x - x(xx)\} - x(x\{(xx)x -$$
$$xxx = xxxxx - xxxxx - xxxxx + xxxxx$$

So we form a matrix 4×14:

$$\begin{pmatrix} 1 & -1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & -1 & 1 & 0 & 1 & -1 \\ 1 & -1 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & -1 & 1 \end{pmatrix}$$

The rank of this matrix is 4, therefore the space of Akivis element with degree 5 has dimension 4.

For the degree 6, a basis of subspace of Akivis elements is spanned by 7 elements below:

$$\left[\left[[[x,x,x],x],x\right],x\right], \left[[[x,x,x],x],x,x\right], \left[[[x,x,x],x,x],x\right], \; \left[x,\left[[x,x,x],x\right],x\right],$$

$$\left[[x,[x,x,x],x],x\right], \; \left[x,x,\left[[x,x,x],x\right]\right], \; \left[[x,x,[x,x,x]],x\right]$$

There are 42 nonassociative monomials, and those monomials of degree 6 are:

1. $(((((xx)x)x)x)x)$
2. $((((x(xx))x)x)x)$
3. $((((xx)x)x)(xx))$
4. $((((xx)x)(xx))x)$
5. $(((x((xx)x))x)x)$
6. $(((x(x(xx)))x)x)$
7. $(((x(xx))x)(xx))$
8. $(((x(xx))(xx))x)$
9. $(((xx)((xx)x))x)$
10. $(((xx)(x(xx)))x)$
11. $((x(((xx)x)x))x)$
12. $((x((x(xx))x))x)$
13. $((x(((xx)x)x))x)$
14. $((x((x(xx))x))x)$
15. $((x(x((xx)x)))x)$
16. $((x(x(x(xx))))x)$
17. $((x((xx)x))(xx))$
18. $((x(x(xx)))(xx))$
19. $((xx)(((xx)x)x))$
20. $((xx)((x(xx))x))$
21. $((xx)(x((xx)x)))$
22. $((xx)(x(x(xx))))$
23. $(x((((xx)x)x)x))$
24. $(x(((x(xx))x)x))$

25. $\left(x\left(\left(x\left((xx)x\right)\right)x\right)\right)$ 26. $\left(x\left(\left(x\left(x(xx)\right)\right)x\right)\right)$ 27. $\left(x\left(\left((xx)x\right)(xx)\right)\right)$

28. $\left(x\left(\left(x(xx)\right)(xx)\right)\right)$ 29. $\left(x\left(x\left(\left((xx)x\right)x\right)\right)\right)$ 30. $\left(x\left(x\left(\left(x(xx)\right)x\right)\right)\right)$

31. $\left(x\left(x\left(x\left((xx)x\right)\right)\right)\right)$ 32. $\left(x\left((xx)\left((xx)x\right)\right)\right)$ 33. $\left(x\left((xx)\left(x(xx)\right)\right)\right)$

34. $\left(x\left(x\left(x\left((xx)x\right)\right)\right)\right)$ 35. $\left(\left(\left((xx)(xx)\right)x\right)x\right)$ 36. $\left(\left(x\left((xx)(xx)\right)\right)x\right)$

37. $(xx)\left((xx)(xx)\right)$ 38. $\left(\left((xx)x\right)\left(x(xx)\right)\right)$ 39. $\left(\left(x(xx)\right)\left((xx)x\right)\right)$

40. $\left(\left((xx)(xx)\right)(xx)\right)$ 41. $\left(\left((xx)x\right)\left((xx)x\right)\right)$ 42. $\left(x\left(x\left(x\left(x(xx)\right)\right)\right)\right)$.

Let's write the bases of subspace of Akivis elements in times of monomials in other to find our matrix.

$$\left[\left[[[x,x,x],x],x\right],x\right] = \left[[[\{(xx)x - x(xx)\},x],x],x\right]$$
$$= \left[[(\{(xx)x - x(xx)\}x - \{(xx)x - x(xx)\}),x],x\right]$$
$$= [(\{(xx)x - x(xx)\}x - x\{(xx)x - x(xx)\})x$$
$$-x(\{(xx)x - x(xx)\}x - x\{(xx)x - x(xx)\}),x]$$
$$= \{(\{(xx)x - x(xx)\}x - x\{(xx)x - x(xx)\})x$$
$$- x(\{(xx)x - x(xx)\}x - x\{(xx)x - x(xx)\})\}x$$
$$-x\{(\{(xx)x - x(xx)\}x - x\{(xx)x - x(xx)\})x$$
$$- x(\{(xx)x - x(xx)\}x - x\{(xx)x - x(xx)\})\}$$
$$= \left(\left(\left(\left((xx)x\right)x\right)x\right)x\right) - \left(\left(\left(\left(x(xx)\right)x\right)x\right)x\right)$$
$$- \left(\left(\left(x\left((xx)x\right)\right)x\right)x\right) + \left(\left(\left(x\left(x(xx)\right)\right)x\right)x\right)$$
$$- \left(\left(x\left(\left((xx)x\right)x\right)\right)x\right) + \left(\left(x\left(\left(x(xx)\right)x\right)\right)x\right)$$

$$+\left(\left(x\left(x((xx)x)\right)\right)x\right)-\left(\left(x\left(x(x(xx))\right)\right)x\right)$$

$$-\left(x\left(\left(((xx)x)x\right)x\right)\right)+\left(x\left(\left((x(xx))x\right)x\right)\right)$$

$$+\left(x\left(\left(x((xx)x)\right)x\right)\right)-\left(x\left(\left(x(x(xx))\right)x\right)\right)$$

$$+\left(x\left(x\left(((xx)x)x\right)\right)\right)-\left(x\left(x\left((x(xx))x\right)\right)\right)$$

$$-\left(x\left(x\left(x((xx)x)\right)\right)\right)+\left(x\left(x\left(x(x(xx))\right)\right)\right)$$

$$\left[[[x,x,x],x],x,x\right]=[[\{(xx)x-x(xx)\},x],x,x]$$

$$=[(\{(xx)x-x(xx)\}x-x\{(xx)x-x(xx)\}),x,x]$$

$$=\{(\{(xx)x-x(xx)\}x-x\{(xx)x-x(xx)\})x\}x$$

$$-\{((xx)x-x(xx)\}x-x\{(xx)x-x(xx)\})(xx)$$

$$=\left(\left(\left(((xx)x)x\right)x\right)x\right)-\left(\left(\left((x(xx))x\right)x\right)x\right)$$

$$-\left(\left(\left(x((xx)x)\right)x\right)x\right)+\left(\left(\left(x(x(xx))\right)x\right)x\right)$$

$$-\left(\left(((xx)x)x\right)(xx)\right)+\left(\left((x(xx))x\right)(xx)\right)$$

$$+\left(\left(x((xx)x)\right)(xx)\right)-\left(\left(x(x(xx))\right)(xx)\right)$$

$$\left[[[x,x,x],x,x],x\right]=[[\{(xx)x-x(xx)\},x,x],x]$$

$$=[\{(\{(xx)x-x(xx)\}x)x-\{(xx)x-x(xx)\}(xx)\}(xx)\},x]$$

$$=\{(\{(xx)x-x(xx)\}x)x-\{(xx)x-x(xx)\}(xx)\}x$$

$$-x\{(\{(xx)x-x(xx)\}x)x-\{(xx)x-x(xx)\}(xx)\}$$

$$=\left(\left(\left(((xx)x)x\right)x\right)x\right)-\left(\left(\left((x(xx))x\right)x\right)x\right)$$

$$-\left(\left(\left((xx)x\}(xx)\right)x\right)+\left(\left(\left(x(xx)\right)(xx)\right)x\right)$$

$$-\left(x\left(\left(\left((xx)x\right)x\right)x\right)\right)+\left(x\left(\left(\left(x(xx)\right)x\right)x\right)\right)$$

$$+\left(x\left(\left((xx)x\right)(xx)\right)\right)-\left(x\left(\left(x(xx)\right)(xx)\right)\right)$$

$$\begin{aligned}
\left[x,\left[[x,x,x],x\right],x\right] &= [x,[\{(xx)x-x(xx)\},x],x] \\
&= [x,(\{(xx)x-x(xx)\}x-x\{(xx)x-x(xx)\}),x] \\
&= \{x(\{(xx)x-x(xx)\}x-x\{(xx)x-x(xx)\})\}x \\
&\quad -x\{(\{(xx)x-x(xx)\}x-x\{(xx)x-x(xx)\})x\} \\
&= \left(\left(x\left(\left((xx)x\right)x\right)\right)x\right)-\left(\left(x\left(\left(x(xx)\right)x\right)\right)x\right) \\
&\quad -\left(\left(x\left(x\left((xx)x\right)\right)\right)x\right)+\left(\left(x\left(x\left(x(xx)\right)\right)\right)x\right) \\
&\quad -\left(x\left(\left(\left((xx)x\right)x\right)x\right)\right)+\left(x\left(\left(\left(x(xx)\right)x\right)x\right)\right) \\
&\quad +\left(x\left(\left(x\left((xx)x\right)\right)x\right)\right)-\left(x\left(\left(x\left(x(xx)\right)\right)x\right)\right)
\end{aligned}$$

$$\begin{aligned}
\left[[x,[x,x,x],x],x\right] &= \left[[x,\{(xx)x-x(xx)\},x],x\right] \\
&= [\{(x\{(xx)x-x(xx)\})x-x(\{(xx)x-x(xx)\}x)\},x] \\
&= \{(x\{(xx)x-x(xx)\})x-x(\{(xx)x-x(xx)\}x)\}x \\
&\quad -x\{(x\{(xx)x-x(xx)\})x-x(\{(xx)x-x(xx)\}x)\} \\
&= \left(\left(\left(x\left((xx)x\right)\right)x\right)x\right)-\left(\left(\left(x\left(x(xx)\right)\right)x\right)x\right) \\
&\quad -\left(\left(x\left(\left((xx)x\right)x\right)\right)x\right)+\left(\left(x\left(\left(x(xx)\right)x\right)\right)x\right) \\
&\quad -\left(x\left(\left(x\left((xx)x\right)\right)x\right)\right)+\left(x\left(\left(x\left(x(xx)\right)\right)x\right)\right)
\end{aligned}$$

$$+\left(x\left(x\left(\left((xx)x\right)x\right)\right)\right)-\left(x\left(x\left(\left(x(xx)\right)x\right)\right)\right)$$

$$\begin{aligned}
\big[x,x,\big[[x,x,x],x\big]\big] &= [x,x,[\{(xx)x-x(xx)\},x]] \\
&= [x,x,(\{(xx)x-x(xx)\}x-x\{(xx)x-x(xx)\})] \\
&= (xx)(\{(xx)x-x(xx)\}x-x\{(xx)x-x(xx)\}) \\
&\quad -x\{x(\{(xx)x-x(xx)\}x-x\{(xx)x-x(xx)\})\} \\
&= \left((xx)\left(\left((xx)x\right)x\right)\right)-\left((xx)\left(\left(x(xx)\right)x\right)\right) \\
&\quad -\left((xx)\left(x\left((xx)x\right)\right)\right)+\left((xx)\left(x\left(x(xx)\right)\right)\right) \\
&\quad -\left(x\left(x\left(\left((xx)x\right)x\right)\right)\right)+\left(x\left(x\left(\left(x(xx)\right)x\right)\right)\right) \\
&\quad +\left(x\left(x\left(x\left((xx)x\right)\right)\right)\right)-\left(x\left(x\left(x\left(x(xx)\right)\right)\right)\right)
\end{aligned}$$

$$\begin{aligned}
\big[\big[x,x,[x,x,x]\big],x\big] &= [[x,x,\{(xx)x-x(xx)\}],x] \\
&= [\{(xx)\{(xx)x-x(xx)\}-x(x\{(xx)x-x(xx)\})\},x] \\
&= \{(xx)\{(xx)x-x(xx)\}-x(x\{(xx)x-x(xx)\})\}x \\
&\quad -x\{(xx)\{(xx)x-x(xx)\}-x(x\{(xx)x-x(xx)\})\} \\
&= \left(\left((xx)\left((xx)x\right)\right)x\right)-\left(\left((xx)\left(x(xx)\right)\right)x\right) \\
&\quad -\left(\left(x\left(x\left((xx)x\right)\right)\right)x\right)+\left(\left(x\left(x\left(x(xx)\right)\right)\right)x\right) \\
&\quad -\left(x\left((xx)\left((xx)x\right)\right)\right)+\left(x\left((xx)\left(x(xx)\right)\right)\right) \\
&\quad +\left(x\left(x\left(x\left((xx)x\right)\right)\right)\right)-\left(x\left(x\left(x\left(x(xx)\right)\right)\right)\right)
\end{aligned}$$

So, from here the matrix of 7×42 is as follows:

$$
\begin{bmatrix}
1 & * & 0 & 0 & * & 1 & 0 & 0 & 0 & 0 & * & 1 & 0 & 0 & 1 & * & 0 & 0 & 0 & 0 & 0 & 0 & * & 1 & 1 & * & 0 & 0 & 1 & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & * & * & 0 & * & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & * & 0 & * & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & 1 & 0 & 0 & 1 & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & 0 & 0 & * & 1 & 0 & 0 & 0 & 0 & 0 & 0 & * & 1 & 1 & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & * & 0 & 0 & 0 & 0 & 0 & 0 & * & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & 1 & 0 & 0 & 1 & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & * & 1 & 0 & 0 & 0 & 0 & 0 & 0 & * & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & 0 & 0 & 0 & 0 & * & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & *
\end{bmatrix}
$$

In the above matrix -1 is replaced by $*$.

This matrix has rank 7: Here the columns $42, 3, 4, 11, 13, 19$ and 10 are chosen respectively to formulate the following matrix 7×7

$$\begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

The determinant becomes:

$$\begin{vmatrix} 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & -1 \end{vmatrix} = (-1) \begin{vmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \end{vmatrix}$$

$$= (-1)(1)\begin{vmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{vmatrix} = (-1)(1)(-1)\begin{vmatrix} 1 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

$$= (-1)(1)(-1)(1)\begin{vmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{vmatrix}$$

$$= (-1)(1)(-1)(1)(-1)\begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} = (-1)(1)(-1)(1)(-1)(-1)$$

$$= 1 \neq 0$$

Therefore the space of Akivis element with degree 6 has dimension 7.

For the degree 7, a basis of subspace of Akivis elements is spanned by 23 elements below:

$$\Big[\Big[\big[[[x,x,x],x],x\big],x\Big],x\Big], \quad \Big[\big[[[x,x,x],x],x\big],x,x\Big], \Big[\big[[[x,x,x],x],x,x\big],x\Big],$$

$$\Big[\big[[[x,x,x],x,x],x\big],x\Big], \quad \big[[[x,x,x],x,x],x,x\big], \quad \big[[[x,x,x],x],[x,x,x]\big],$$

$$\big[[[x,[x,x,x],x],x],x\big], \quad \Big[\big[[x,x,[x,x,x]],x\big],x\Big], \quad [[x,x,x],x,[x,x,x]],$$

$$[[x,x,x],[x,x,x],x], \quad \big[[x,[[x,x,x],x],x],x\big], \quad \Big[\big[x,x,[[x,x,x],x]\big],x\Big],$$

$$[[x,[x,x,x],x],x,x], \quad \big[[x,x,[x,x,x]],x,x\big], \quad \Big[x,\big[[[x,x,x],x],x\big],x\Big],$$

$$\Big[x,x,\big[[[x,x,x],x],x\big]\Big], \quad [x,[[x,x,x],x,x],x], \quad \big[x,x,[[x,x,x],x,x]\big],$$

$$[x,[x,[x,x,x],x],x], \quad [x,x,[x,[x,x,x],x]], \quad \big[x,[x,x,[x,x,x]],x\big],$$

$$\Big[x,x,\big[x,x,[x,x,x]\big]\Big], \quad [x,[x,x,x],[x,x,x]].$$

Table 3: In the same way, the computation is founded in this way up to degree **9**

n	matrix size	$\dim A^{(n)}$
7	23×132	23
8	54×429	53
9	162×1430	157

3.2. Hilbert Series of Free Metabelian Lie Algebras

Let L_m be the free Lie algebra over the field K freely generated by a finite set $X = \{x_1, \cdots, x_m\}$ with $m \geq 2$. We assume that the elements of X are Lie monomials of length 1. If x and y are Lie monomials of any length, then length of $[x, y]$ is the sum of lengths x and y.

We will show the monomials of the form $\left[[z_1, z_2], z_3\right]$ by $[z_1, z_2, z_3]$ for the sake of simplicity. Hence we can extent this inductively as

$$[z_1, z_2, z_3, \cdots, z_k] = \left[\cdots \left[[z_1, z_2], z_3\right], \cdots, z_k\right], k \geq 3,$$

for all $z_1, z_2, \cdots, z_k \in L_m$. In this way every element of L_m can be written as a linear combination of the left normed monomials, which means that, the set of all left normed Lie monomials spans the whole free Lie algebra L_m on X.

Definition 3.2.1: The quotient algebra

$$F_m = {L_m}/{L_m''} = {L_m}/{\left[[L_m, L_m], [L_m, L_m]\right]}$$

is called the free metabelian Lie algebra of rank m defined by metabelian identity $\left[[x, y], [z, t]\right] = 0$. The second derived ideal of F_m is equal to zero. F_m is generated by free generators $y_1 = x_1 + L_m'', \cdots, y_m = x_m + L_m''$ and this algebra is of a basis consisting of $y_1, \cdots, y_m$ together with all left normed monomials of the form:

$$[y_{i_1}, y_{i_2}, \cdots, y_{i_k}], i_1 > i_2 \leq i_3 \leq \cdots \leq i_k \leq m.$$

Let $w \in F_m^{'}$ and let $x, y \in F_m$ be arbitrary elements. Then as a consequence of Jacobi identity we have

$$[w, x, y] + [x, y, w] + [y, w, x] = 0.$$

Since $0 = [x, y, w] \in F_m^{''} = \{0\}$ we get

$$[w, x, y] = -[y, w, x] = [w, y, x].$$

Thus, $F_m^{'}$ is furnished with a natural structure of module of $K[y_1, \cdots, y_m]$.

Example 3.2.2: Basis elements in 2 Generators:

Let $F_2 = \langle x, y \rangle$ be a free metabelian Lie algebra with generators x, y over a field K of characteristic 0.

The alternating of generators with the condition of $x > y$ is used to construct F_2 which is a closed formula and generate the numbers of basis. For $m \leq 9$, the number of elements in basis are illustrate manually in the following way by total order $>$;

$n = 1$ $\quad x, y \quad x > y$

$n = 2$ $\quad [x, y]$

$n = 3$ $\quad [[x, y], y],\ [[x, y], x]$

$n = 4$ $\quad [[[x, y], y], y],\ [[[x, y], y], x],\ [[[x, y], x], x]$

$n = 5$ $\quad [[[[x, y], y], y], y],\ [[[[x, y], y], y], x],$

$[[[[x, y], y], x], x],\ [[[[x, y], x], x], x]$

$n = 6$ $\quad [[[[[x, y], y], y], y], y],\ [[[[[x, y], y], y], y], x],$

$[[[[[x, y], y], y], x], x],\ [[[[[x, y], y], x], x], x],$

$[[[[[x, y], x], x], x], x]$

$n = 7$ $\quad [[[[[[x,y],y],y],y],y],y],\ [[[[[[x,y],y],y],y],y],x],$

$[[[[[[x,y],y],y],y],x],x],\quad [[[[[[x,y],y],y],x],x],x],$

$[[[[[[x,y],y],x],x],x],x],\quad [[[[[[x,y],x],x],x],x],x]$

$n = 8$ $\quad [[[[[[[x,y],y],y],y],y],y],y],\ [[[[[[[x,y],y],y],y],y],y],x],$

$[[[[[[[x,y],y],y],y],y],x],x],\quad [[[[[[[x,y],y],y],y],x],x],x],$

$[[[[[[[x,y],y],y],x],x],x],x],\quad [[[[[[[x,y],y],x],x],x],x],x],$

$[[[[[[[x,y],x],x],x],x],x],x]$

$n = 9$ $\quad [[[[[[[[x,y],y],y],y],y],y],y],y],$

$$\left[\left[\left[\left[\left[[[x,y],y],y\right],y\right],y\right],y\right],x\right],\left[\left[\left[\left[\left[[[x,y],y],y\right],y\right],y\right],y\right],x\right],x\right],$$

$$\left[\left[\left[\left[\left[[[x,y],y],y\right],y\right],y\right],x\right],x\right],x\right],\left[\left[\left[\left[\left[[[x,y],y],y\right],y\right],x\right],x\right],x\right],x\right],$$

$$\left[\left[\left[\left[\left[[[x,y],y],y\right],x\right],x\right],x\right],x\right],x\right],\left[\left[\left[\left[\left[[[x,y],y],x\right],x\right],x\right],x\right],x\right],x\right],$$

$$\left[\left[\left[\left[\left[[[x,y],x],x\right],x\right],x\right],x\right],x\right],x\right].$$

Example 3.2.3: Basis elements in 3 Generators:

Let $F_2 = \langle x, y, z\rangle$ be a free metabelian Lie algebra with generators x, y, z over a field K of characteristic 0. The alternating of generators with conditions $x > y > z$, $[x, y] > z$ and $y \leq z$ are used to construct F_3 which is a closed formula and generate the numbers of basis. For $m \leq 6$, the number of basis are illustrate manually in the following way by total order $>$;

$n = 1$ $\quad x, y, z \qquad x > y > z$

$n = 2$ $\quad [x, y],\ [x, z],\ [y, z]$

$n = 3$ $\quad [[x, y], y],\ [[x, y], x],\ [[x, z], z],\ [[x, z], y],\ [[x, z], x],$
$[[y, z], z], [[y, z], y], [[y, z], x]$

$n = 4$ $\quad \left[[[x, y], y], y\right],\ \left[[[x, y], y], x\right],\ \left[[[x, y], x], x\right],\ \left[[[x, z], z], z\right],$
$\left[[[x, z], z], y\right], \left[[[x, z], z], x\right], \left[[[x, z], y], y\right], \left[[[x, z], y], x\right],$
$\left[[[x, z], x], x\right], \left[[[y, z], z], z\right], \left[[[y, z], z], y\right], \left[[[y, z], z], x\right],$
$\left[[[y, z], y], y\right], \left[[[y, z], y], x\right], \left[[[y, z], x], x\right]$

$n = 5$ $\Big[\big[[[x,y],y],y\big],y\Big]$, $\Big[\big[[[x,y],y],y\big],x\Big]$, $\Big[\big[[[x,y],y],x\big],x\Big]$,

$\Big[\big[[[x,y],x],x\big],x\Big]$, $\Big[\big[[[x,z],z],z\big],z\Big]$, $\Big[\big[[[x,z],z],z\big],y\Big]$,

$\Big[\big[[[x,z],z],z\big],x\Big]$, $\Big[\big[[[x,z],z],y\big],y\Big]$, $\Big[\big[[[x,z],z],y\big],x\Big]$,

$\Big[\big[[[x,z],z],x\big],x\Big]$, $\Big[\big[[[x,z],y],y\big],y\Big]$, $\Big[\big[[[x,z],y],y\big],x\Big]$,

$\Big[\big[[[x,z],y],x\big],x\Big]$, $\Big[\big[[[x,z],x],x\big],x\Big]$, $\Big[\big[[[y,z],z],z\big],z\Big]$,

$\Big[\big[[[y,z],z],z\big],y\Big]$, $\Big[\big[[[y,z],z],z\big],x\Big]$, $\Big[\big[[[y,z],z],y\big],y\Big]$,

$\Big[\big[[[y,z],z],y\big],x\Big]$, $\Big[\big[[[y,z],z],x\big],x\Big]$, $\Big[\big[[[y,z],y],y\big],y\Big]$,

$\Big[\big[[[y,z],y],y\big],x\Big]$, $\Big[\big[[[y,z],y],x\big],x\Big]$, $\Big[\big[[[y,z],x],x\big],x\Big]$

$n = 6$ $\Bigg[\Big[\big[[[x,y],y],y\big],y\Big],y\Bigg]$, $\Bigg[\Big[\big[[[x,y],y],y\big],y\Big],x\Bigg]$,

$$\left[\left[\left[[[x,y],y],y\right],x\right],x\right],\left[\left[\left[[[x,y],y],x\right],x\right],x\right],\left[\left[\left[[[x,y],x],x\right],x\right],x\right],$$

$$\left[\left[\left[[[x,z],z],z\right],z\right],z\right],\left[\left[\left[[[x,z],z],z\right],z\right],y\right],\left[\left[\left[[[x,z],z],z\right],z\right],x\right],$$

$$\left[\left[\left[[[x,z],z],z\right],y\right],y\right],\left[\left[\left[[[x,z],z],z\right],y\right],x\right],\left[\left[\left[[[x,z],z],z\right],x\right],x\right],$$

$$\left[\left[\left[[[x,z],z],y\right],y\right],y\right],\left[\left[\left[[[x,z],z],y\right],y\right],x\right],\left[\left[\left[[[x,z],z],y\right],x\right],x\right],$$

$$\left[\left[\left[[[x,z],z],x\right],x\right],x\right],\left[\left[\left[[[x,z],y],y\right],y\right],y\right],\left[\left[\left[[[x,z],y],y\right],y\right],x\right],$$

$$\left[\left[\left[[[x,z],y],y\right],x\right],x\right],\left[\left[\left[[[x,z],y],x\right],x\right],x\right],\left[\left[\left[[[x,z],x],x\right],x\right],x\right],$$

$$\left[\left[\left[[[y,z],z],z\right],z\right],z\right],\left[\left[\left[[[y,z],z],z\right],z\right],y\right],\left[\left[\left[[[y,z],z],z\right],z\right],x\right],$$

$$\left[\left[\left[[[y,z],z],z\right],y\right],y\right],\left[\left[\left[[[y,z],z],z\right],y\right],x\right],\left[\left[\left[[[y,z],z],z\right],x\right],x\right],$$

$$\left[\left[\left[[[y,z],z],y\right],y\right],y\right],\left[\left[\left[[[y,z],z],y\right],y\right],x\right],\left[\left[\left[[[y,z],z],y\right],x\right],x\right],$$

$$\left[\left[\left[[[y,z],z],x\right],x\right],x\right],\left[\left[\left[[[y,z],y],y\right],y\right],y\right],\left[\left[\left[[[y,z],y],y\right],y\right],x\right],$$

$$\left[\left[\left[[[y,z],y],y\right],x\right],x\right],\left[\left[\left[[[y,z],y],x\right],x\right],x\right],\left[\left[\left[[[y,z],x],x\right],x\right],x\right].$$

Example 3.2.4: Basis elements in 4 Generators:

Let $F_4 = \langle x,y,z,t\rangle$ be a free metabelian Lie algebra with generators x,y,z,t over a field K of characteristic 0. The alternating of generators with conditions $x > y > z$, $[x,y] > z$ and $y \leq z$ are used to construct F_4 which is a closed formula and generate the numbers of basis. For $m \leq 5$, the number of basis are illustrate manually in the following way by total order $>$;

$n = 1$ $\quad x,y,z,t \qquad x > y > z > t$

$n = 2$ $[x,y],\ [x,z],\ [x,t],\ [y,z],\ [y,t],\ [z,t]$

$n = 3$ $[[x,y],y],\ [[x,y],x], [[x,z],z],\ [[x,z],y],\ [[x,z],x],$

$[[x,t],t], [[x,t],z], [[x,t],y], [[x,t],x], [[y,z],z], [[y,z],y],$

$[[y,z],x], [[y,t],t], [[y,t],z], [[y,t],y], [[y,t],x], [[z,t],t],$

$[[z,t],z], [[z,t],y], [[z,t],x]$

$n = 4$ $\left[[[x,y],y],y\right], \left[[[x,y],y],x\right], \left[[[x,y],x],x\right], \left[[[x,z],z],z\right],$

$\left[[[x,z],z],y\right], \left[[[x,z],z],x\right], \left[[[x,z],y],y\right], \left[[[x,z],y],x\right],$

$\left[[[x,z],x],x\right], \left[[[x,t],t],t\right], \left[[[x,t],t],z\right], \left[[[x,t],t],y\right],$

$\left[[[x,t],t],x\right], \left[[[x,t],z],z\right], \left[[[x,t],z],y\right], \left[[[x,t],z],x\right],$

$\left[[[x,t],y],y\right], \left[[[x,t],y],x\right], \left[[[x,t],x],x\right], \left[[[y,z],z],z\right],$

$\left[[[y,z],z],y\right], \left[[[y,z],z],x\right], \left[[[y,z],y],y\right], \left[[[y,z],y],x\right],$

$\left[[[y,z],x],x\right], \left[[[y,t],t],t\right], \left[[[y,t],t],z\right], \left[[[y,t],t],y\right],$

$\left[[[y,t],t],x\right], \left[[[y,t],z],z\right], \left[[[y,t],z],y\right], \left[[[y,t],z],x\right],$

$\left[[[y,t],y],y\right], \left[[[y,t],y],x\right], \left[[[y,t],x],x\right], \left[[[z,t],t],t\right], \left[[[z,t],t],z\right],$

$\left[[[z,t],t],y\right], \left[[[z,t],t],x\right], \left[[[z,t],z],z\right], \left[[[z,t],z],y\right], \left[[[z,t],z],x\right],$

$\left[[[z,t],y],y\right], \left[[[z,t],y],x\right], \left[[[z,t],x],x\right]$

$n = 5$ $\left[\left[[[x,y],y],y\right],y\right], \left[\left[[[x,y],y],y\right],x\right], \left[\left[[[x,y],y],x\right],x\right],$

$\left[\left[[[x,y],x],x\right],x\right], \left[\left[[[x,z],z],z\right],z\right], \left[\left[[[x,z],z],z\right],y\right],$

$\left[\left[[[x,z],z],z\right],x\right], \left[\left[[[x,z],z],y\right],y\right], \left[\left[[[x,z],z],y\right],x\right],$

$\left[\left[[[x,z],z],x\right],x\right], \left[\left[[[x,z],y],y\right],y\right], \left[\left[[[x,z],y],y\right],x\right],$

$$\Big[\big[[[x,z],y],x\big],x\Big],\Big[\big[[[x,z],x],x\big],x\Big],\Big[\big[[[x,t],t],t\big],t\Big],$$

$$\Big[\big[[[x,t],t],t\big],z\Big],\Big[\big[[[x,t],t],t\big],y\Big],\Big[\big[[[x,t],t],t\big],x\Big],$$

$$\Big[\big[[[x,t],t],z\big],z\Big],\Big[\big[[[x,t],t],z\big],y\Big],\Big[\big[[[x,t],t],z\big],x\Big],$$

$$\Big[\big[[[x,t],t],y\big],y\Big],\Big[\big[[[x,t],t],y\big],x\Big],\Big[\big[[[x,t],t],x\big],x\Big],$$

$$\Big[\big[[[x,t],z],z\big],z\Big],\Big[\big[[[x,t],z],z\big],y\Big],\Big[\big[[[x,t],z],z\big],x\Big],$$

$$\Big[\big[[[x,t],z],y\big],y\Big],\Big[\big[[[x,t],z],y\big],x\Big],\Big[\big[[[x,t],z],x\big],x\Big],$$

$$\Big[\big[[[x,t],y],y\big],y\Big],\Big[\big[[[x,t],y],y\big],x\Big],\Big[\big[[[x,t],y],x\big],x\Big],$$

$$\Big[\big[[[x,t],x],x\big],x\Big],\Big[\big[[[y,z],z],z\big],z\Big],\Big[\big[[[y,z],z],z\big],y\Big],$$

$$\Big[\big[[[y,z],z],z\big],x\Big],\Big[\big[[[y,z],z],y\big],y\Big],\Big[\big[[[y,z],z],y\big],x\Big],$$

$$\Big[\big[[[y,z],z],x\big],x\Big],\Big[\big[[[y,z],y],y\big],y\Big],\Big[\big[[[y,z],y],y\big],x\Big],$$

$$\Big[\big[[[y,z],y],x\big],x\Big],\Big[\big[[[y,z],x],x\big],x\Big],\Big[\big[[[y,t],t],t\big],t\Big],$$

$$\Big[\big[[[y,t],t],t\big],z\Big],\Big[\big[[[y,t],t],t\big],y\Big],\Big[\big[[[y,t],t],t\big],x\Big],$$

$$\Big[\big[[[y,t],t],z\big],z\Big],\Big[\big[[[y,t],t],z\big],y\Big],\Big[\big[[[y,t],t],z\big],x\Big],$$

$$\Big[\big[[[y,t],t],y\big],y\Big],\Big[\big[[[y,t],t],y\big],x\Big],\Big[\big[[[y,t],t],x\big],x\Big],$$

$$\Big[\big[[[y,t],z],z\big],z\Big],\Big[\big[[[y,t],z],z\big],y\Big],\Big[\big[[[y,t],z],z\big],x\Big],$$

$$\Big[\big[[[y,t],z],y\big],y\Big],\Big[\big[[[y,t],z],y\big],x\Big],\Big[\big[[[y,t],z],x\big],x\Big],$$

$$\Big[\big[[[y,t],y],y\big],y\Big],\Big[\big[[[y,t],y],y\big],x\Big],\Big[\big[[[y,t],y],x\big],x\Big],$$

$$\Big[\big[[[y,t],x],x\big],x\Big], \Big[\big[[[z,t],t],t\big],t\Big], \Big[\big[[[z,t],t],t\big],z\Big],$$

$$\Big[\big[[[z,t],t],t\big],y\Big], \Big[\big[[[z,t],t],t\big],x\Big], \Big[\big[[[z,t],t],z\big],z\Big],$$

$$\Big[\big[[[z,t],t],z\big],y\Big], \Big[\big[[[z,t],t],z\big],x\Big], \Big[\big[[[z,t],t],y\big],y\Big],$$

$$\Big[\big[[[z,t],t],y\big],x\Big], \Big[\big[[[z,t],t],x\big],x\Big], \Big[\big[[[z,t],z],z\big],z\Big],$$

$$\Big[\big[[[z,t],z],z\big],y\Big], \Big[\big[[[z,t],z],z\big],x\Big], \Big[\big[[[z,t],z],y\big],y\Big],$$

$$\Big[\big[[[z,t],z],y\big],x\Big], \Big[\big[[[z,t],z],x\big],x\Big], \Big[\big[[[z,t],y],y\big],y\Big],$$

$$\Big[\big[[[z,t],y],y\big],x\Big], \Big[\big[[[z,t],y],x\big],x\Big], \Big[\big[[[z,t],x],x\big],x\Big].$$

Theorem 3.2.5: Consider equation $n_1 + n_2 + \cdots + n_r = n$, for $r \in \mathbb{Z}^+$, $n, n_i \geq 0$ and $1 \leq i \leq r$. Then number of distinct nonnegative integer solutions satisfying the equation is

$$\binom{n+r-1}{r-1}.$$

For more on combinatorial use Lipschutz (1966).

From above theorem we can obtain the number of basis of free metabelian Lie algebra in different generators. Let $\mathfrak{B}_{m,n}$ indicate the canonical basis for the homogeneous subspace of degree n in the free metabelian Lie algebra F_m, and let $\eta_{m,n}$ denote the number of elements in $\mathfrak{B}_{m,n}$ which coincides with $\dim F_m^{(n)}$.

Example 3.2.6: Number of Elements of Basis in 2 Generators

Consider the free metabelian Lie algebra $F_2 = K\langle x, y\rangle$ with 2 generators and assuming that $x > y$. Here we shall compute $\eta_{2,n}$.

$n = 1$, $\mathfrak{B}_{2,1} = \{x, y\}$ and $\eta_{2,1} = 2$

$n = 2$, $\mathfrak{B}_{2,2} = \{[x, y]\}$ and $\eta_{2,2} = 1$

$n = 3$, $\mathfrak{B}_{2,3} = \big\{[[x, y], y], [[x, y], x]\big\}$ and $\eta_{2,3} = 2$

$n = 4$, $\mathfrak{B}_{2,4} = \left\{\left[[[x,y],y],y\right], \left[[[x,y],y],x\right], \left[[[x,y],x],x\right]\right\}$ and $\eta_{2,4} = 3$

$\vdots$

In general

$$\mathfrak{B}_{2,n} = \{[x, y, \underbrace{p_1, \cdots, p_{n-2}}_{n-2 \text{ position}}] : y \leq p_1 \leq \cdots \leq p_{n-2} \leq x\}$$

where $p_j \in \{x,y\}, j = 1, \ldots, n-2$ and $n \geq 2$. Thus considering $n-2$ positions filled by p_j s, must start by several ys followed by xs. That is because, the number $\eta_{2,n}$ is the number of nonnegative solutions of equation

$$n_1 + n_2 = n - 2$$

where n_1 indicates the number of ys and n_2 stands for the number of xs used in those $n-2$ positions. Hence by Theorem 3.2.5 we have

$$\eta_{2,n} = \binom{(n-2)+2-1}{2-1} = \binom{n-1}{1} = n-1$$

Example 3.2.7: Number of Elements of Basis in 3 Generators

Now let $F_3 = K\langle x, y, z\rangle$ be the free metabelian Lie algebra with condition of $x > y > z$. Similar steps as in the previous computations give

$n = 1$, $\mathfrak{B}_{3,1} = \{x, y, z\}$ and $\eta_{3,1} = 3$

$n = 2$, $\mathfrak{B}_{3,2} = \{[x,y],[x,z],[y,z]\}$ and $\eta_{3,2} = 3$

$$n = 3,\quad \mathfrak{B}_{3,3} = \left\{\begin{matrix}\left[[x,y],y\right], \left[[x,y],x\right], \left[[x,z],z\right], \left[[x,z],y\right], \left[[x,z],x\right], \\ \left[[y,z],z\right], \left[[y,z],y\right], \left[[y,z],x\right]\end{matrix}\right\},\ \eta_{3,3} = 8.$$

In general

$$\mathfrak{B}_{3,n} = B_1 \cup B_2 \cup B_3$$

where

$$B_1 = \{[x, y, \underbrace{p_1, \cdots, p_{n-2}}_{n-2 \text{ position}}] : y \leq p_1 \leq \cdots \leq p_{n-2} \leq x\}$$

$$B_2 = \{[x, z, \underbrace{p_1, \cdots, p_{n-2}}_{n-2 \text{ position}}] : z \leq p_1 \leq \cdots \leq p_{n-2} \leq x\}$$

$$B_3 = \{[y, z, \underbrace{p_1, \cdots, p_{n-2}}_{n-2 \text{ position}}] : z \leq p_1 \leq \cdots \leq p_{n-2} \leq x\}$$

Therefore,

$$\eta_{3,n}=|B_1|+|B_2|+|B_3|=|B_1|+2|B_2|=\eta_{2,n}+2|B_2|$$

The key point is to find number of elements in B_1. Again, considering $n-2$ positions filled by p_j s, must start by several zs followed by ys and then by xs. That is because, the number $\eta_{2,n}$ is the number of nonnegative solutions of equation

$$n_1+n_2+n_3=n-2$$

where n_1 indicates the number of zs, n_2 indicates the number of ys, and n_3 stands for the number of xs used in those $n-2$ positions. Theorem … gives

$$|B_2|=\binom{(n-2)+3-1}{3-1}=\binom{n}{2}$$

Consequently

$$\eta_{3,n}=\binom{n-1}{1}+2\binom{n}{2}=n^2-1$$

Example 3.2.8: Number of Elements of Basis in 4 Generators

Now let $F_3=K\langle x,y,z\rangle$ be the free metabelian Lie algebra with condition of $x>y>z$. Similar steps as in the previous computations give

$n=1$, $\mathfrak{B}_{3,1}=\{x,y,z\}$ and $\eta_{3,1}=3$

$n=2$, $\mathfrak{B}_{3,2}=\{[x,y],[x,z],[y,z]\}$ and $\eta_{3,2}=3$

$n=3$, $\mathfrak{B}_{3,3}=\left\{\begin{matrix}[[x,y],y],[[x,y],x],[[x,z],z],[[x,z],y],[[x,z],x], \\ [[y,z],z],[[y,z],y],[[y,z],x]\end{matrix}\right\}$, $\eta_{3,3}=8$.

In general

$$\mathfrak{B}_{3,n}=B_1\cup B_2\cup B_3$$

where

$$B_1=\{[x,y,\underbrace{p_1,\cdots,p_{n-2}}_{n-2\text{ position}}]:y\le p_1\le\cdots\le p_{n-2}\le x\}$$

$$B_2=\{[x,z,\underbrace{p_1,\cdots,p_{n-2}}_{n-2\text{ position}}]:z\le p_1\le\cdots\le p_{n-2}\le x\}$$

$$B_3=\{[y,z,\underbrace{p_1,\cdots,p_{n-2}}_{n-2\text{ position}}]:z\le p_1\le\cdots\le p_{n-2}\le x\}$$

Therefore,

$$\eta_{3,n}=|B_1|+|B_2|+|B_3|=|B_1|+2|B_2|=\eta_{2,n}+2|B_2|$$

The key point is to find number of elements in B_2. Again, considering $n-2$ positions filled by p_j s, must start by several zs followed by ys and then by xs. That is because, the number $\eta_{2,n}$ is the number of nonnegative solutions of equation

$$n_1+n_2+n_3=n-2$$

where n_1 indicates the number of zs, n_2 indicates the number of ys, and n_3 stands for the number of xs used in those $n-2$ positions. Theorem … gives

$$|B_2|=\binom{(n-2)+3-1}{3-1}=\binom{n}{2}$$

Consequently

$$\eta_{3,n}=\binom{n-1}{1}+2\binom{n}{2}=n^2-1$$

$$\eta_{4,n}=\binom{n-1}{1}+2\binom{n}{2}+3\binom{n+1}{3}=\frac{n-1}{2}(n+1)(n+2).$$

Proposition 3.2.9: Generalization for Number of Elements of Basis in m Generators

To generalize a formula for the number of homogeneous basis of degree n in the free metabelian Lie algebra of rank m with generators $y_1,\cdots,y_m$. We have to prove by induction what we obtained in the cases 2, 3, and 4 generators respectively.

Theorem 3.2.10: Dimension $\eta_{m,n}$ of $F_m^{(n)}$ is

$$\eta_{m,n}=\binom{n-1}{1}+2\binom{n}{2}+\cdots+(m-1)\binom{n+m-3}{m-1}$$

Proof: We prove the statement by induction on number of generators $m\geq 2$. Let us assume $y_1>\cdots>y_m$. As illustrated before, it is straightforward to check the formula for $m=2,3,4$. Now let

$$\eta_{m-1,n} = \binom{n-1}{1} + \cdots + (m-2)\binom{n+m-4}{m-2}$$

And let us explicitly write down the elements of $\mathfrak{B}_{m,n}$. Firstly it is a direct result that

$$\mathfrak{B}_{m,n} = B_2 \cup B_3 \cup \cdots \cup B_{m-1} \cup B_m$$

where the subset B_j is of elements of the form

$$[\_, y_j, \underbrace{p_1, \cdots, p_{n-2}}_{n-2 \text{ position}}]$$

Thus p_is can be chosen from the set $\{y_1, \cdots, y_j\}$. The first place allows $y_1, \cdots, y_{j-1}$ and number of elements in B_j is $(j-1)$ times number of nonnegative solutions of equation

$$n_1 + \cdots + n_j = n-2$$

such that n_k indicates number of y_ks used for those $n-2$ places. One must observe from this point that

$$|\mathfrak{B}_{m-1,n}| = \eta_{m-1,n} = |B_2 \cup B_3 \cup \cdots \cup B_{m-1}|$$

Hence it is sufficient to show that

$$|B_m| = (m-1)\binom{n+m-3}{m-1}$$

The elements of B_m are of the form

$$[\_, y_m, \underbrace{p_1, \cdots, p_{n-2}}_{n-2 \text{ position}}]$$

Thus p_is can be chosen from the set $\{y_1, \cdots, y_m\}$. The first place allows $y_1, \cdots, y_{m-1}$ and number of elements in B_m is $(m-1)$ times number of nonnegative solutions of equation

$$n_1 + \cdots + n_m = n-2$$

such that n_k indicates number of y_ks used for those $n-2$ places. By Theorem 3.2.5 this number is

$$|B_m| = (m-1)\binom{(n-2)+m-1}{m-1}$$

which completes the proof.

Example 3.2.11: Table of Dimension of Free Metabelian Lie Algebra

From the general formula of dimension of free metabelian Lie algebra we can make table of free metabelian Lie algebra of 2, 3 and 4 generator up to $n \leq 10$.

General formula is

$$\dim F_m^{(n)} = \frac{n-1}{(m-2)!}(n+1)(n+2)\cdots(n+m-3)$$

For $m = 2$, we have $\dim F_2^{(n)} = n - 1$.

So, we obtain,

$n = 1,\quad n - 1 = 1 - 1 = 0$

$n = 2,\quad n - 1 = 2 - 1 = 1$

$n = 3,\quad n - 1 = 3 - 1 = 2,$

$n = 4,\quad n - 1 = 4 - 1 = 3,$

$n = 5,\quad n - 1 = 5 - 1 = 4,$

$n = 6,\quad n - 1 = 6 - 1 = 5,$

$n = 7,\quad n - 1 = 7 - 1 = 6,$

$n = 8,\quad n - 1 = 8 - 1 = 7,$

$n = 9,\quad n - 1 = 9 - 1 = 8,$

$n = 10,\quad n - 1 = 10 - 1 = 9.$

For $m = 3$, we have $\dim F_3^{(n)} = n^2 - 1$.

So, we obtain,

$n = 1,\quad n^2 - 1 = 0,$

$n = 2,\quad n^2 - 1 = 4 - 1 = 3,$

$n = 3,\quad n^2 - 1 = 9 - 1 = 8,$

$n = 4,\quad n^2 - 1 = 16 - 1 = 15,$

$n = 5, \qquad n^2 - 1 = 25 - 1 = 24,$

$n = 6, \qquad n^2 - 1 = 36 - 1 = 35,$

$n = 7, \qquad n^2 - 1 = 49 - 1 = 48,$

$n = 8, \qquad n^2 - 1 = 64 - 1 = 63,$

$n = 9, \qquad n^2 - 1 = 81 - 1 = 80,$

$n = 10, \qquad n^2 - 1 = 100 - 1 = 99.$

For $m = 4$, the dimension is

$$\dim F_4^{(n)} = \frac{1}{2}(n-1)(n+1)(n+2) = \frac{1}{2}(n^3 + 2n^2 - n - 2).$$

So, we obtain,

$n = 1,\ \frac{1}{2}(n^3 + 2n^2 - n - 2) = \frac{1}{2}(1 + 2 - 1 - 2) = 0,$

$n = 2,\ \frac{1}{2}(n^3 + 2n^2 - n - 2) = \frac{1}{2}(2^3 + 2 \cdot 2^2 - 2 - 2) = \frac{1}{2}(8 + 8 - 4) = \frac{12}{2}$
$= 6,$

$n = 3,\ \frac{1}{2}(n^3 + 2n^2 - n - 2) = \frac{1}{2}(3^3 + 2 \cdot 3^2 - 3 - 2) = \frac{1}{2}(27 + 18 - 5) = \frac{40}{2}$
$= 20,$

$n = 4,\ \frac{1}{2}(n^3 + 2n^2 - n - 2) = \frac{1}{2}(4^3 + 2 \cdot 4^2 - 4 - 2) = \frac{1}{2}(64 + 32 - 6) = \frac{90}{2}$
$= 45,$

$n = 5,\ \frac{1}{2}(n^3 + 2n^2 - n - 2) = \frac{1}{2}(5^3 + 2 \cdot 5^2 - 5 - 2) = \frac{1}{2}(125 + 50 - 7)$
$= \frac{168}{2} = 84,$

$n = 6,\ \frac{1}{2}(n^3 + 2n^2 - n - 2) = \frac{1}{2}(6^3 + 2 \cdot 6^2 - 6 - 2) = \frac{1}{2}(216 + 72 - 8)$
$= \frac{280}{2} = 140,$

$n = 7,\ \frac{1}{2}(n^3 + 2n^2 - n - 2) = \frac{1}{2}(7^3 + 2 \cdot 7^2 - 7 - 2) = \frac{1}{2}(343 + 98 - 9)$
$= \frac{432}{2} = 216,$

$n=8,\ \frac{1}{2}(n^3+2n^2-n-2)=\frac{1}{2}(8^3+2\cdot 8^2-8-2)=\frac{1}{2}(512+128-10)$

$$=\frac{630}{2}=315,$$

$n=9,\ \frac{1}{2}(n^3+2n^2-n-2)=\frac{1}{2}(9^3+2\cdot 9^2-9-2)=\frac{1}{2}(729+162-11)$

$$=\frac{880}{2}=440,$$

$n=10,\ \frac{1}{2}(n^3+2n^2-n-2)=\frac{1}{2}(10^3+2\cdot 10^2-10-2)$

$$=\frac{1}{2}(1000+200-12)\ =\frac{1188}{2}=594.$$

Table 4: Summary of Dimension

	$F_2=K\langle x,y\rangle$	$F_3=K\langle x,y,z\rangle$	$F_4=K\langle x,y,z,t\rangle$
1	2	3	4
2	1	3	6
3	2	8	20
4	3	15	45
5	4	24	84
6	5	35	140
7	6	48	216
8	7	63	315
9	8	80	440
10	9	99	594

Example 3.2.12: Hilbert Series of $F_2=K\langle x,y\rangle$

Let $F_2=K\langle x,y\rangle$ be the free metabelian Lie algebra with generators x and y. We know that number of elements in homogeneous subspace of degree n is equal to $\eta_{2,n}=n-1$ for $n\geq 2$. The Hilbert Series of F_2 comes out from this formula:

$$H(F_2,t)=\sum_{n=0}^{\infty}(\dim F_2^{(n)})\cdot t^n=0+2\cdot t^1+\sum_{n=2}^{\infty}\eta_{2,n}\cdot t^n$$

since $\dim F_2^{(0)} = 0$ and $\dim F_2^{(1)} = 2$. Hence we have

$$H(F_2, t) = 2t + \sum_{n=2}^{\infty} (n-1)t^n$$

$$= 2t + \sum_{n=2}^{\infty} n\,t^n - \sum_{n=2}^{\infty} t^n = 2t + B - A$$

where A and B are the power series where

$$A = \sum_{n=2}^{\infty} t^n \quad \text{and} \quad B = \sum_{n=2}^{\infty} n\,t^n$$

We are going to solve this accordingly:

For $A = \sum_{n=2}^{\infty} t^n$, we know this from the Example 2.27 for polynomial algebra where we have

$$H(K[x], t) = \sum_{n=0}^{\infty} \dim K[x]^{(n)}\, t^n$$

$$= \sum_{n=0}^{\infty} t^n = 1 + t + t^2 + t^3 + \cdots = \frac{1}{1-t}.$$

Recall that in writing the rational formula on right side of equality, we assume that t is a variable lying in the set of real numbers such that the series converges. Thus we can have derivative in computations.

Now, for A we have $n \geq 2$ therefore our equation becomes

$$1 + t + \sum_{n=2}^{\infty} t^n = \frac{1}{1-t}$$

$$1 + t + A = \frac{1}{1-t}$$

and hence

$$A = \sum_{n=2}^{\infty} t^n = \frac{1}{1-t} - (1+t).$$

Now, for B we have,

$$B = \sum_{n=2}^{\infty} n\,t^{n} = 2t^{2} + 3t^{3} + \cdots = t(2t + 3t^{2} + \cdots)$$

$$= t\left(\sum_{n=2}^{\infty} n\,t^{n-1}\right).$$

It is seen that the parenthesis is derivative of A, so we obtain,

$$B = t \cdot A' = t \cdot \left(\sum_{n=2}^{\infty} t^{n}\right)' = t \cdot \left[\frac{1}{1-t} - (1+t)\right]'$$

$$= t \cdot \left[\frac{1'(1-t) - 1(1-t)'}{(1-t)^{2}} - 1\right] = t \cdot \left(\frac{1}{(1-t)^{2}} - 1\right)$$

Hence we have

$$B = \sum_{n=2}^{\infty} n\,t^{n} = \frac{t}{(1-t)^{2}} - t.$$

Let come back on our Hilbert Series

$$H(F_2, t) = 2t + B - A = 2t + \frac{t}{(1-t)^{2}} - t - \left[\frac{1}{1-t} - (1+t)\right]$$

$$= 2t + \frac{t}{(1-t)^{2}} - t - \frac{1}{1-t} + 1 + t = 1 + 2t + \frac{t}{(1-t)^{2}} - \frac{1}{1-t}$$

$$= 1 + 2t + \frac{t - (1-t)}{(1-t)^{2}} = 1 + 2t + \frac{2t-1}{(1-t)^{2}}.$$

Therefore the Hilbert Series of free metabelian Lie algebra F_2 of two generators is equal to:

$$H(F_2, t) = 1 + 2t + \frac{2t-1}{(1-t)^{2}}.$$

Example 3.2.13: Hilbert Series of $F_3 = K\langle x, y, z\rangle$

Let $F_3 = K\langle x,y,z\rangle$ be the free metabelian Lie algebra with generators x,y,z. We know that number of elements in homogeneous subspace of degree n is equal to $\eta_{3,n} = n^2 - 1$ for $n \geq 2$. The Hilbert Series of F_3 comes out from this formula:

$$H(F_3,t) = \sum_{n=0}^{\infty}(\dim F_3^{(n)}) \cdot t^n = 0 + 3\cdot t^1 + \sum_{n=2}^{\infty}\eta_{3,n}\cdot t^n$$

since $\dim F_3^{(0)} = 0$ and $\dim F_3^{(1)} = 3$. Hence we have

$$H(F_3,t) = 3t + \sum_{n=2}^{\infty}(n^2-1)t^n = 3t + \sum_{n=2}^{\infty}n^2\, t^n - \sum_{n=2}^{\infty}t^n = 3t + C - A$$

where A and C are the power series

$$A = \sum_{n=2}^{\infty}t^n \quad \text{and} \quad C = \sum_{n=1}^{\infty}n^2\, t^n$$

It is known from the previous computations that

$$A = \frac{1}{1-t} - (1+t)$$

Now, we are going to find a nice rational expression for C.

$$C = \sum_{n=2}^{\infty}n^2\, t^n = 4t^2 + 9t^3 + \cdots = t(4t + 9t^2 + \cdots)$$

$$= t\left(\sum_{n=2}^{\infty}n^2\, t^{n-1}\right).$$

It is seen that the parenthesis is derivative of B, so we obtain,

$$C = t\cdot B' = t\cdot\left(\sum_{n=2}^{\infty}n\, t^n\right)' = t\cdot\left[\frac{t}{(1-t)^2} - t\right]' = t\cdot\left[\left(\frac{t}{(1-t)^2}\right)' - t'\right]$$

$$= t\cdot\left[\frac{t'(1-t)^2 - t((1-t)^2)'}{((1-t)^2)^2} - t'\right] = t\cdot\left[\frac{1(1-t)^2 - 2t(1-t)(-1)}{(1-t)^4} - 1\right]$$

$$= t\cdot\left[\frac{(1-t)^2+2t(1-t)}{(1-t)^4}-1\right] = t\cdot\left[\frac{(1-t)(1-t+2t)}{(1-t)^4}-1\right]$$

$$= t\cdot\left[\frac{(1+t)}{(1-t)^3}-1\right] = \frac{t^2+t}{(1-t)^3}-t.$$

Hence we have

$$C = \sum_{n=1}^{\infty} n^2\, t^n = \frac{t^2+t}{(1-t)^3}-t.$$

Therefore

$$3t + C - A = 3t + \frac{t^2+t}{(1-t)^3} - t - \left[\frac{1}{1-t}-(1+t)\right]$$

$$= 3t + \frac{t^2+t}{(1-t)^3} - t - \frac{1}{1-t} + 1 + t = 1 + 3t + \frac{t^2+t}{(1-t)^3} - \frac{1}{1-t}$$

$$= 1 + 3t + \frac{t^2+t-(1-t)^2}{(1-t)^3} = 1 + 3t + \frac{t^2+t-1+2t-t^2}{(1-t)^3}$$

$$= 1 + 3t + \frac{3t-1}{(1-t)^3}.$$

Thus the Hilbert Series of free metabelian Lie algebra F_3 of 3 generators is equal to:

$$H(F_3,t) = 1 + 3t + \frac{3t-1}{(1-t)^3}.$$

Example 3.2.14: Hilbert Series of $F_4 = K\langle x,y,z,t\rangle$

Let $F_4 = K\langle x,y,z,t\rangle$ be the free metabelian Lie algebra with generators x,y,z,t. We know that number of elements in homogeneous subspace of degree n is equal to

$$\eta_{4,n} = \frac{1}{2}(n-1)(n+1)(n+2) = \frac{1}{2}(n^3+2n^2-n-2)$$

for $n \geq 2$. The Hilbert Series of F_4 comes out from this formula:

$$H(F_3,t) = \sum_{n=0}^{\infty}(\dim F_4^{(n)})\cdot t^n = 0 + 4\cdot t^1 + \sum_{n=2}^{\infty}\eta_{3,n}\cdot t^n$$

since $\dim F_4^{(0)} = 0$ and $\dim F_4^{(1)} = 4$. Hence we have

$$H(F_4,t) = 0 \cdot 1 + 4 \cdot t + \sum_{n=2}^{\infty} \eta_{4,n} \cdot t^n = 4t + \sum_{n=2}^{\infty} \frac{1}{2}(n^3 + 2n^2 - n - 2) \cdot t^n$$

$$= 4t + \frac{1}{2}\sum_{n\geq 2}^{\infty} n^3\, t^n + \sum_{n\geq 2}^{\infty} n^2\, t^n - \frac{1}{2}\sum_{n=2}^{\infty} n\, t^n - \sum_{n=2}^{\infty} t^n$$

$$=4t + \tfrac{1}{2}D + C - \tfrac{1}{2}B - A$$

such that

$$D = \sum_{n=2}^{\infty} n^3\, t^n, \qquad C = \sum_{n=2}^{\infty} n^2\, t^n, \qquad B = \sum_{n=2}^{\infty} n\, t^n \quad \text{and} \quad A = \sum_{n=2}^{\infty} t^n$$

Now, we are going to find a nice rational expression for D since all of expressions for A, B, C have just been computed.

$$D = \sum_{n=2}^{\infty} n^3\, t^n = 8t^2 + 27t^3 + \cdots = t(8t + 27t^2 + \cdots) = t\left(\sum_{n=2}^{\infty} n^3\, t^{n-1}\right).$$

It is seen that the parenthesis is derivative of C, so we have,

$$D = t \cdot C' = t \cdot \left(\sum_{n=2}^{\infty} n^2\, t^n\right)' = t \cdot \left[\frac{t^2+t}{(1-t)^3} - t\right]' = t \cdot \left[\left(\frac{t^2+t}{(1-t)^3} - t\right)' - t'\right]$$

$$= t \cdot \left[\frac{(t^2+t)'(1-t)^3 - (t^2+t)((1-t)^3)'}{((1-t)^3)^2} - t'\right]$$

$$= t \cdot \left[\frac{(2t+1)(1-t)^3 - 3(t^2+t)(1-t)^2(-1)}{(1-t)^6} - 1\right]$$

$$= t \cdot \left[\frac{(2t+1)(1-t)^3 + 3(t^2+t)(1-t)^2}{(1-t)^6} - 1\right]$$

$$= t \cdot \left[\frac{(1-t)^2[(2t+1)(1-t) + 3(t^2+t)]}{(1-t)^6} - 1\right]$$

$$= t \cdot \left[\frac{2t - 2t^2 + 1 - t + 3t^2 + 3t}{(1-t)^4} - 1\right]$$

$$= t \cdot \left[\frac{t^2 + 4t + 1}{(1-t)^4} - 1\right] = \frac{t^3 + 4t^2 + t}{(1-t)^4} - t.$$

Hence we have

$$D = \sum_{n=2}^{\infty} n^3\, t^n = \frac{t^3 + 4t^2 + t}{(1-t)^4} - t.$$

Therefore the Hilbert Series becomes

$$H(F_4, t) = 4t + \frac{1}{2}D + C - \frac{1}{2}B - A$$

$$= 4t + \frac{1}{2}\left(\frac{t^3 + 4t^2 + t}{(1-t)^4} - t\right) + \frac{t^2 + t}{(1-t)^3} - t - \frac{1}{2}\left(\frac{t}{(1-t)^2} - t\right)$$

$$- \left[\frac{1}{1-t} - (1+t)\right]$$

$$= 4t + \frac{1}{2}\frac{t^3 + 4t^2 + t}{(1-t)^4} - \frac{t}{2} + \frac{t^2 + t}{(1-t)^3} - t - \frac{1}{2}\frac{t}{(1-t)^2} + \frac{t}{2} - \frac{1}{1-t} + 1 + t$$

$$= 1 + 4t + \frac{1}{2}\frac{t^3 + 4t^2 + t}{(1-t)^4} + \frac{t^2 + t}{(1-t)^3} - \frac{1}{2}\frac{t}{(1-t)^2} - \frac{1}{1-t}$$

$$= 1 + 4t + \frac{t^3 + 4t^2 + t + 2(1-t)(t^2 + t) - t(1-t)^2 - 2(1-t)^3}{2(1-t)^4}$$

$$= 1 + 4t + \frac{t^3 + 4t^2 + t + 2t - 2t^3 - t + 2t^2 - t^3 - 2 + 6t - 6t^2 + 2t^3}{2(1-t)^4}$$

$$= 1 + 4t + \frac{8t - 2}{2(1-t)^4} = 1 + 4t + \frac{4t - 1}{(1-t)^4}$$

Therefore the Hilbert Series of free metabelian Lie algebra F_4 of 4 generators is equal to:

$$H(F_4, t) = 1 + 4t + \frac{4t - 1}{(1-t)^4}.$$

To generalize this formula in m generators, we have to prove by induction from what we have in Hilbert series of 2, 3 and 4 generators.

Theorem 3.2.15: For $m \geq 3$, We have

$$H(F_m, t) = H(F_{m-1}, t) + t + \frac{(m-1)t^2}{(1-t)^m}$$

Proof: Direct computations from Examples 3.2.12, 3.2.13, 3.2.14 give

$$H(F_3, t) - H(F_2, t) = 1 + 3t + \frac{3t-1}{(1-t)^3} - \left[1 + 2t + \frac{2t-1}{(1-t)^2}\right]$$

$$= 1 + 3t - 1 - 2t + \frac{3t-1}{(1-t)^3} - \frac{2t-1}{(1-t)^2} = t + \frac{3t - 1 - (2t-1)(1-t)}{(1-t)^3}$$

$$= t + \frac{3t - 1 - (2t - 2t^2 - 1 + t)}{(1-t)^3} = t + \frac{2t^2}{(1-t)^3}.$$

and,

$$H(F_4, t) - H(F_3, t) = 1 + 4t + \frac{4t-1}{(1-t)^4} - \left[1 + 3t + \frac{3t-1}{(1-t)^3}\right]$$

$$= 1 + 4t - 1 - 3t + \frac{4t-1}{(1-t)^4} - \frac{3t-1}{(1-t)^3} = t + \frac{4t - 1 - (3t-1)(1-t)}{(1-t)^4}$$

$$= t + \frac{4t - 1 - (3t - 3t^2 - 1 + t)}{(1-t)^4} = t + \frac{3t^2}{(1-t)^4}.$$

Thus the formula holds for $m = 3,4$.

On the other hand, we have that

$$H(K[Y_m], t) = \sum_{r \geq 0} \dim K[Y_m]^{(r)}\, t^r = \frac{1}{(1-t)^m}$$

For the dimension of $K[Y_m]^{(r)}$ we need to know the canonical forms of basis elements in $\mathfrak{B}_{K[Y_m]^{(r)}}$. They are of the form

$$y_1^{n_1} y_2^{n_2} \cdots y_m^{n_m}$$

such that $n_1, n_2, \ldots, n_m \geq 0$, $n_1 + \cdots + n_m = r$. By the Theorem 3.2.5, the number of such elements is

$$\binom{r+m-1}{m-1}.$$

Hence we have the new expression for $H(K[Y_m], t)$ as follows:

$$H(K[Y_m],t) = \frac{1}{(1-t)^m} = \sum_{r=0}^{\infty} \dim K[Y_m]^{(r)}\, t^r = \sum_{r=0}^{\infty} \binom{r+m-1}{m-1} t^r$$

Let us multiply the equality

$$\sum_{r=0}^{\infty} \binom{r+m-1}{m-1} t^r = \frac{1}{(1-t)^m}$$

by t^2 to get

$$\sum_{r=0}^{\infty} \binom{r+m-1}{m-1} t^{r+2} = \frac{t^2}{(1-t)^m}.$$

Let $n = r+2$, and rewrite the above formula:

$$\sum_{n=2}^{\infty} \binom{(n-2)+m-1}{m-1} t^n = \frac{t^2}{(1-t)^m}$$

$$\Rightarrow \sum_{n=2}^{\infty} \binom{n+m-3}{m-1} t^n = \frac{t^2}{(1-t)^m}$$

By multiplying $(m-1)$ and adding t on both sides we obtain

$$t + \sum_{n=2}^{\infty} (m-1)\binom{n+m-3}{m-1} t^n = t + \frac{(m-1)t^2}{(1-t)^m}$$

Recall that, by Theorem 3.2.10,

$$\eta_{m,n} = \binom{n-1}{1} + 2\binom{n}{2} + \cdots + (m-1)\binom{n+m-3}{m-1}$$

giving the numbers of element in $\mathfrak{B}_{m,n}$, hence

$$\eta_{m,n} - \eta_{m-1,n} = (m-1)\binom{n+m-3}{m-1}.$$

Rewriting the previous equation, we have

$$mt - (m-1)t + \sum_{n=2}^{\infty} (\eta_{m,n} - \eta_{m-1,n})\, t^n = t + \frac{(m-1)t^2}{(1-t)^m}$$

$$\Rightarrow \left(mt + \sum_{n=2}^{\infty} \eta_{m,n}\, t^n\right) - \left((m-1)t + \sum_{n=2}^{\infty} \eta_{m-1,n}\, t^n\right) = t + \frac{(m-1)t^2}{(1-t)^m}$$

$$H(F_m, t) - H(F_{m-1}, t) = t + \frac{(m-1)t^2}{(1-t)^m}$$

Therefore we have

$$H(F_m, t) = H(F_{m-1}, t) + t + \frac{(m-1)t^2}{(1-t)^m}$$

which completes the proof.

The following formula can be found in paper by Drensky (1994).

Corollary 3.2.16: The Hilbert series of the free metabelian Lie algebra F_m is

$$H(F_m, t) = 1 + mt + \frac{mt-1}{(1-t)^m}$$

Proof: Cleary the formula holds for $m = 2{,}3$ and 4 by Examples 3.2.12, 3.2.13, and 3.2.14. Let assume that the formula is true for $m-1$ as the induction hypothesis:

$$H(F_{m-1}, t) = 1 + (m-1)t + \frac{(m-1)t-1}{(1-t)^{m-1}}.$$

By Theorem 3.2.15,

$$H(F_m, t) = H(F_{m-1}, t) + t + \frac{(m-1)t^2}{(1-t)^m}$$

$$= 1 + (m-1)t + \frac{(m-1)t-1}{(1-t)^{m-1}} + t + \frac{(m-1)t^2}{(1-t)^m}$$

$$= 1 + (m-1)t + t + \frac{(m-1)t-1}{(1-t)^{m-1}} + \frac{(m-1)t^2}{(1-t)^m}$$

$$= 1 + t(m-1+1) + \frac{[(m-1)t-1](1-t) + (m-1)t^2}{(1-t)^m}$$

$$= 1 + mt + \frac{(m-1)t - (m-1)t^2 - 1 + t + (m-1)t^2}{(1-t)^m}$$

$$= 1 + mt + \frac{(m-1)t + t - 1}{(1-t)^m} = 1 + mt + \frac{t(m-1+1) - 1}{(1-t)^m}$$

Therefore, for all $m \geq 2$

$$H(F_m, t) = 1 + mt + \frac{mt - 1}{(1-t)^m}$$

This completes the proof.

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BIOGRAPHY

I am Rwandan, born in 1989, first born in the family of mother and 4 children two boys and two sisters. I studied Mathematics and Physics in high school where I passed and successful national exams in 2008 which qualified me to start my undergraduate studies in 2010 at former Kigali Institute of Science and Technology now called University of Rwanda, College of Science and Technology located at Kigali-Rwanda. In September 2013 I got my bachelors degree from faculty of Science in Department of Applied Mathematics and Pure Mathematics as option. Before awarded a scholarship of the Scientific and Technological Research Council of Turkey (TÜBİTAK) for the 2235 Graduate Scholarship Program for Least Developed Countries in September 2014, I worked as a teacher of Mathematics in high school called Ecole Techniques Appliques de Kimisange in Kigali – Rwanda. Now day I am completing my Masters degree in Mathematics at Çukurova University.