T.R.
YUZUNCU YIL UNIVERSITY
INSTITUTE OF NATURAL AND APPLIED SCIENCES
DEPARTMENT OF MATHEMATICS

SOME RESULTS ON FRACTIONAL BOUNDARY VALUE PROBLEMS WITH SLIT STRIP AND MULTI STRIP BOUNDARY CONDITIONS

M.Sc. THESIS

PRESENTED BY: Diyar Hashim Malo HAJANI
SUPERVISOR: Asst. Prof. Dr. Zeynep KAYAR

VAN-2017

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ACCEPTANCE and APPROVAL PAGE

This thesis entitled "Some results on fractional boundary value problems with slit-strip and multi strip boundary conditions" presented by Diyar Hashim Malo HAJANI under supervision of Asst. Prof. Zeynep KAYAR in the department of Mathematics has been accepted as a M. Sc. thesis according to Legislations of Graduate Higher Education on 05/05/2017 with unanimity of votes members of jury.

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This thesis has been approved by the committee of The Institute of Natural and Applied Science on ……/……../…….. with decision number ……………..

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THESIS STATEMENT

All information presented in the thesis obtained in the frame of ethical behavior and academic rules. In addition all kinds of information that does not belong to me have been cited appropriately in the thesis prepared by the thesis writing rules.

Signature

Diyar Hashim Malo HAJANI

ÖZET

YARIK ŞERİT ve ÇOKLU ŞERİT SINIR KOŞULLARINA SAHİP KESİRLİ SINIR DEĞER PROBLEMLERİ ÜZERİNE BAZI SONUÇLAR

HAJANI, Diyar Hashim Malo
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Bu tezde yarık şerit ve çoklu şerit sınır koşullarına sahip kesirli sınır değer problemlerinin çözümlerinin varlık ve tekliği için teoremler elde edilmiştir. Kesirli diferansiyel denklemler bir çok fiziksel olayın modellenmesini adi diferansiyel denklemlerden daha doğru, gerçekçi ve pratik yaptıkları için matematikte önemlidirler. Kesirli integral ve türev operatörlerinin avantajı ya da doğalarının dikkat çekici özellikleri yerel olmamalarıdır, yani kesirli türevi içeren dinamik sistemin ya da sürecin gelecek zamandaki durumu hem şu andaki hem de geçmiş zamandaki durumuna bağlıdır. Bu operatörlerin hafıza ya da miras özellikleri bir çok materyalin ve sürecin geçmiş tarihini izlememize izin verir. Kesirli sınır değer problemleri ise matematiksel bakış açısına göre popüler bir araştırma alanıdır ve biyoloji, epidemiyoloji, fizik, mühendislik, kimya, hidroloji, finans, klasik mekanik, kuantum mekaniği, viskoelastisite, elektrik devreleri, nöron modellemsi ve benzeri alanlarda uygulamalara sahiptir.

Bu tez beş bölümden oluşmaktadır. Birinci bölüm giriş niteliğinde olup yarık şerit ve çoklu şerit sınır koşullarına sahip kesirli sınır değer problemleri ile ilgili literatür taramasını içermektedir. Birinci bölümde fonksiyonel analiz ve kesirli diferansiyel denklemlerden gelen gerekli tanım ve teoremleri içeren ön kavramlar verilecektir.

İkinci bölümde yarık şerit tipi sınır koşullarına sahip $\rho \in (n-1, n]$. mertebeden kesirli diferansiyel denklemlerin çözümlerinin varlık ve tekliği için teoremler sunulacaktır. Yarık şerit tipi sınır koşulları şu anlama gelir: Keyfi uzunluktaki kesişmeyen iki alt aralığın (şeridin) toplam etkisi bilinmeyen fonksiyonun alt aralıklar dışındaki ya da yarıktaki noktadaki değeriyle bağlantılıdır. Sonuçlarımızı elde etmek için standart sabit nokta teoremleri (Banach daralma dönüşümü prensibi, Krasnoselski sabit nokta teoremi, Leary-Schauder alternatifi ve Tek değerli dönüşümler için lineer olmayan alternatif) kullanılacaktır ve sonuçlarımızın uygulanabilirliğini doğrulayan bazı örnekler gösterilecektir. Daha sonra bu sonuçlar yarıkdaki keyfi sayıdaki yerel olmayan nokta koşullu, yerel olmayan çoklu alt şerit koşullu ve Riemann-Liouville tipindeki yarık-şerit sınır koşullu kesirli sınır değer problemlerine uygulanacaktır.

Üçüncü bölüm sonlu sayıda, çoklu ve keyfi uzunluktaki kesişmeyen şeritleri içeren Riemann-Liouville tipi sınır koşullu lineer olmayan keyfi mertebeden kesirli diferansiyel denklemlerin çözümlerinin varlık ve tekliği için teoremlere ayrılmıştır. Bu koşulların fiziksel anlamı şudur: Sensörler aralığın ortasında yer aldığı için sınırdaki kontroller enerjiyi yayar ya da emer. Çözümlerin varlık sonuçları Krasnoselski sabit nokta teoremi, Leary-Schauder alternatifi ve Tek değerli dönüşümler için lineer olmayan alternatif uygulanarak elde edilecektir. Çözümlerin teklik sonuçları ise Banach daralma dönüşümü prensibi sayesinde oluşturulacaktır. Sonuçlarımızı göstermek için bir çok örnek verilecektir.

Son bölüm sonuç niteliğinde olup bu tezde yaptıklarımızın özeti şeklindedir.

Anahtar kelimeler: Kesirli sınır değer problemleri, Sabit nokta, Şerit koşulları, Varlık ve teklik.

ABSTRACT

SOME RESULTS ON FRACTIONAL BOUNDARY VALUE PROBLEMS WITH SLIT STRIP AND MULTI STRIP BOUNDARY CONDITIONS

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In this thesis, existence and uniqueness theorems for fractional boundary value problems with slit strip and multi strip boundary conditions are established. Fractional differential equations are essential in mathematics due to the fact that they are more accurate, realistic and practical than ordinary differential equations in modelling several physical phenomena. The main advantage or the remarkable property of fractional integral and differential operators is that they are nonlocal in nature which means that the future state of a dynamical system or process involving fractional derivative depends on its current state as well as its past states. This memory and hereditary properties of these operators allow us to trace the past history of several materials and processes. Moreover, theory of fractional boundary value problems is a very popular research area from mathematics point of view and have applications in biology, epidemiology, physics, engineering, chemistry, hydrology, finance, classical mechanics, quantum mechanics, visco-elasticity, electrical circuits and neuron modelling and so on.

This thesis consists of five chapters. Chapter 1 is introductory and contains literature review for fractional boundary value problems with slit strip and multi strip boundary conditions. In Chapter 2 preliminary concepts including necessary definitions and theorems from functional analysis and fractional differential equations will be given.

In Chapter 3 existence and uniqueness theorems for a fractional differential equation of order $q \in (n-1, n]$ with slit-strips type boundary conditions will be presented. The slit-strips type boundary condition means that the total effect of the two nonintersecting subintervals (strips) of arbitrary lengths is connected to evaluation of the unknown function at the point out of the subintervals or in the aperture (slit). In order to prove our results, standart fixed point theorems (Banach's contraction mapping principle, Krasnoselski's fixed point theorem, Leary-Schauder alternative and Nonlinear

alternative for single valued maps) will be used and some examples will be shown to confirm that our results are theoretically applicable. Then these results will be applied to fractional boundary value problems with arbitrary number of nonlocal points in the slit, the nonlocal multi-substrips conditions and Riemann-Liouville type slit-strips boundary conditions.

Chapter 4 will be devoted to establish the existence and uniqueness theory for nonlinear fractional differential equations of arbitrary order with Riemann-Liouville type boundary conditions involving nonintersecting finite many strips of arbitrary length. Physical meaning of these conditions is that since the sensors situate in the middle of the interval, the controllers at the boundary of the interval disperse or take in energy. The existence results will be obtained by applying Krasnoselski's fixed point theorem, Leary-Schauder alternative and Nonlinear alternative for single valued maps, while the uniqueness of the solutions will be established by means of Banach's contraction mapping principle. Several examples will be given to illustrate our results.

The last chapter serves as a conclusion and is a summary of our findings.

Keywords: Existence and uniqueness, Fixed point, fractional boundary value problem, Strip conditions.

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Diyar Hashim Malo HAJANI

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1. INTRODUCTION

Most of real world phenomena and processes depend on the present, future and past time behaviours during their developments. This remembering effect or memory and hereditary properties of various materials and processes can not be described by integer order models. Therefore the problem of obtaining the models for such processes which contain a memory term insuring the history and its impact to the present and future appears. The solution of this problem is given by fractional order models including fractional derivative, which are non local, and contain more degrees of freedom than the integer order models.

Contrary to the integer-order differential operator, the fractional-order differential operator is nonlocal in nature, that is, the future state of a dynamical system or process involving fractional derivative depends on its current state as well its past states. In fact, this feature of fractional order operators has contributed towards the popularity of fractional-order models, which are recognized as more realistic and practical than the classical integer-order models in which past effects of the processes are neglected. Therefore they have attracted a great deal of attention and the theory of which has developed rapidly due to the increase applications in various fields, such as epidemiology (Demirci, 2017), physics (Glockle and Nonnenmacher, 1995; Sokolov et al., 2002; Goulart, 2017), engineering (Hilfer, 2000), chemistry (Metzler et al., 1995; Yuste et al., 2004), biology (Magin, 2006), hydrology (Benson et al., 2000a; Benson et al., 2000b; Liu et al., 2004), finance (Scalas et al., 2000), classical mechanics (Engheta, 1996; Freed et al.,2002; Debnath; 2004), quantum mechanics (Mehaute, 1983; Mainardi, 1996, Mainardi, 2010) and references therein. Moreover fractional differential equations not only generalize and unify the corresponding theory of ordinary differential equations but also provide more mathematical description for many real world phenomena. Despite the fact that dynamic behaviour of fractional differential equations is more complex than the behaviour of ordinary differential equations, the former is richer and more fruitful in applications than the latter.

The theory of fractional derivatives goes back to Leibniz's note in his letter to L'Hopital, dated 30 September 1695, in which the meaning of a one-half ordered derivative was discussed. Leibniz's note led to the appearance of the theory of derivatives and integrals of arbitrary order, which by the end of nineteenth century took more or less finished form due primarily to Liouville, Grünwald, Letnikov, Riemann and Caputo. Recently, there have been several books on the subject of fractional derivatives and fractional integrals, see Miller and

Ross (1993), Samko et al. (1993), Podlubny (1999), Kilbas et al. (2006), Sabatier et al. (2007).

Boundary value problems for nonlinear differential equations arise in a variety of areas such as applied mathematics, physics, and variational problems of control theory. In recent years, the study of boundary value problems of fractional order has attracted the attention of many scientists and researchers and the subject has been developed in several disciplines. Before 2010, comprehensive exibition of the existence and uniqueness results can be found in the survey Agarwal et al. (2010) and references therein. Significant development of the topic over the past few years clearly indicates its popularity by involving classical, nonlocal, multipoint, periodic/anti-periodic, fractional-order boundary conditions (Zhang, 2006; Benchohra et al., 2008; Ahmad and Nieto, 2009; Benchohra et al., 2009; Rehman and Khan, 2010; Agarwal and Ahmad, 2011; Ahmad et al., 2011; Ahmad and Nieto, 2011; Ahmad and Nieto, 2012; Wang et al., 2013; Yan et al., 2013; Alsaedi et al., 2014; Choudhary and Daftardar-Gejji, 2014; Ahmad and Ntouyas, 2016; Su et al., 2017; Agarwal et al., 2017) and integral boundary conditions (Ahmad and Ntouyas, 2011; Sudsutad and Tariboon, 2012; Ahmad and Alsaedi, 2012; Ahmad et al., 2013; Zhang et al., 2013; Darwish and Ntouyas, 2014; Mahmudov and Unul, 2014; Wang et al., 2014; Ahmad et al., 2015; Liu et al., 2015; Qiao and Zhou, 2017). As a matter of fact, the literature on the topic is now well enriched with a variety of results covering theoretical as well as application aspects of the subject. In consequence, fractional calculus has evolved as an interesting topic of research and its tools have played a key role in improving the mathematical modeling of many physical and engineering phenomena.The nonlocal nature of fractional-order operators is one of the salient features accounting for the practical utility of the subject. With the aid of fractional calculus, it has now become possible to trace the history of many important materials and processes.

1.1. Structure of Thesis

This thesis summarizes the results of Ahmad and Agarwal (2014) and Ahmad et al. (2013).

In this chapter, we will give literature review for existence and uniqueness results of fractional boundary value problems with slit-strip and multistrip boundary conditions. In Chapter 2 we will outline some basic concepts from functional analysis and fractional differential and integral operator which are needed for the construction of our new results and presenting their proofs. In Chapter 3 we will give existence and uniqueness of solutions for a fractional differential equation of order $q \in (n-1, n]$ with slit-strips type boundary conditions. The slit-strips type boundary condition states that the sum of the influences due to finite strips of arbitrary lengths is related to the value of the unknown function at an arbitrary position (nonlocal point) in the slit (a part of the boundary off the two strips). The desired results will be obtained by applying standard tools of the fixed point theory and will be well illustrated with the aid of examples. We will also extend our discussion to the cases of arbitrary number of nonlocal points in the slit, the nonlocal multi-substrips conditions and Riemann-Liouville type slit-strips boundary conditions. Chapter 4 will be devoted to establish the existence theory for nonlinear fractional differential equations of arbitrary order with Riemann-Liouville type boundary conditions involving nonintersecting finite many strips of arbitrary length. Our results will be based on some standard tools of fixed point theory. For the illustration of the results, some examples will be also discussed.

1.2. Literature Review

Nonlocal conditions, introduced by (Bitsadze and Samarskii, 1969) are regarded as more plausible than the classical initial/boundary conditions in view of their ability to describe certain peculiarities of chemical, physical or other processes happening inside the domain. Computational fluid dynamics (CFD) studies of blood flow indicate that it is not always possible to assume circular cross-section of blood arteries. Several approaches have been proposed to resolve this issue. However, the idea of introducing integral boundary conditions (Ahmad et al., 2008) is found to be quite a productive one as integral boundary conditions have various applications in applied fields such as blood flow problems, chemical engineering, thermoelasticity, underground water flow, population dynamics, etc. Regarding the application of the strip conditions of fixed size, we know that such conditions appear in the mathematical modeling of real world problems, for example, see Asghar et al. (1996), Ahmad et al. (2001). Thus, the present idea of nonlocal strip conditions will be quite fruitful in modeling the strip problems as one can choose an arbitrary set of strips of desired size, which can be fixed according to the requirement by fixing the nonlocal parameters involved in the problem. Furthermore, these conditions can be understood in the sense that the controllers at the end-points of the interval dissipate/absorb energy due to the sensors of finite lengths (continuous distribution of intermediate points of arbitrary length: subsegments of the interval) located at the intermediate positions of the interval.

Ahmad and Ntouyas (2012) extended four-point nonlocal boundary conditions $y(0) = \sigma y(\mu), y(1) = \eta y(\upsilon), \sigma, \eta \in \mathbb{R}, 0 < \mu, \upsilon < 1$ to nonlocal strip conditions

$$y(0) = \sigma \int_{\alpha}^{\beta} y(s)ds, \quad y(1) = \eta \int_{\gamma}^{\delta} y(s)ds, \quad 0 < \alpha < \beta < \gamma < \delta < 1, \tag{1.2.1}$$

for the fractional differential equation ${}_0^C D^q y(x) = f(x, y(x)),\ x \in [0,1],\ 1 < q \leq 2$, where ${}_0^C D^q$ denotes the Caputo fractional derivative of order q, $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a given continuous function and σ, η are appropriately chosen real numbers. The boundary conditions in (1.2.1) can be regarded as six-point nonlocal boundary conditions, which reduces to the typical integral boundary conditions in the limit $\alpha, \gamma \to 0$ and $\beta, \delta \to 1$.

Then Ahmad et al. (2015) investigated the existence and uniqueness of solutions for a differential equation of fractional-order $q \in (1,2]$ subject to nonlocal boundary conditions involving Caputo derivative of the form

$$\begin{aligned} {}_0^C D^q y(x) &= f(x, y(x)), \quad x \in [0,1],\ 1 < q \leq 2 \\ y(0) = \delta y(\sigma), \quad a\,{}_0^C D^q y(\rho_1) + b\,{}_0^C D^q y(\rho_2) &= c \int_{\beta_1}^{\beta_2} D^{\mu} y(s)ds, \end{aligned} \tag{1.2.2}$$

where $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a given continuous function, $0 < \rho_1 < \sigma < \beta_1 < \beta_2 < \rho_2 < 1$, $0 < \mu < 1$ and δ, a, b, c are real constants. The integral boundary conditions in the fractional boundary value problem (1.2.2) can be interpreted as the linear combination of the values of Caputo derivative of the unknown function of order $\mu \in (0,1)$ at nonlocal positions ρ_1 and ρ_2 (off the strip) is proportional to the strip contribution of the Caputo derivative of the unknown function, occupying the position (β_1, β_2).

Alsaedi et al. (2015) considered the boundary value problem of fractional integrodifferential equations with nonlocal integral boundary conditions of the form

$$\begin{aligned} {}_0^C D^q y(x) &= Af(x, y(x)) + BI^r g(x, y(x)), \quad x \in [0,1],\ 1 < q \leq 2 \\ y(0) = \beta y(\theta), \quad y(\xi) &= \alpha \int_{\eta}^{1} y(s)ds, \end{aligned} \tag{1.2.3}$$

where $f, g : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a given continuous function, $0 < r < 1$ and α, β, A, B are real constants. The boundary conditions introduced in the problem (1.2.3) are of nonlocal strip type and describe the situation when the receptors at the end points of the boundary are influenced by the nonlocal contributions due to interior points and strips of the domain for the problem. According to these conditions, the value of the unknown function at the left end point $x = \theta$ is proportional to its value at a nonlocal point θ while the value at an arbitrary (local) point ξ is proportional to the contribution due to a substrip of arbitrary length $(1-\eta)$. These conditions appear in the mathematical modelling of physical problems when different parts (nonlocal points and substrips of arbitrary length) of the domain are involved in the input data for the process under consideration. The problem (1.2.3) can also be termed as a five-point nonlocal fractional boundary value problem.

Ahmad and Ntouyas (2015) studied nonlocal boundary value problems of fractional differential equations with slit-strips integral boundary conditions of the form

$$\begin{aligned} {}_0^C D^q y(x) &= f(x,y(x)), \quad x \in [0,1],\ 1 < q \leq 2 \\ y(0) &= h(x), \\ y(\mu) &= a\int_0^{\alpha} y(s)ds + b\int_{\beta}^{1} y(s)ds, \quad 0 < \alpha < \mu < \beta < 1, \end{aligned} \tag{1.2.4}$$

where ${}_0^C D^q$ denotes the Caputo fractional derivative of order q, $f : [0,1]\times\mathbb{R} \to \mathbb{R}$ and $h : C([0,1],\mathbb{R}) \to \mathbb{R}$ are given continuous functions, and a, b are real constants. We emphasize that the integral boundary condition in (1.2.4) can be interpreted as the sum of the influences due to finite strips of arbitrary lengths is proportional to the value of the unknown function at an arbitrary position (nonlocal point) in the slit (a part of the boundary off the two strips), and the nonlocal term $h(x)$ in (1.2.4) may be understood as $h(x) = \sum_{j=1}^{p} k_j y(x_j)$ where k_j are given constants for all $j = 1,\ldots,p$, and $0 < t_1 < \ldots < t_p \leq 1$.

Ahmad et al. (2016) considered a class of boundary value problems of Caputo type fractional differential equations of arbitrary order involving a nonlocal substrip condition given by

$$\begin{aligned} {}_0^C D^q y(x) &= f(x,y(x)), \quad x \in [0,1],\ n-1 < q \leq n,\ n \geq 2 \\ y(0) = \delta y(\sigma), &\quad y'(0) = y''(0) = \ldots = y^{(n-2)}(0) = 0,\ \delta \in \mathbb{R} \\ ay'(\zeta_1) + by'(\zeta_2) &= c\int_{\eta}^{\xi} y'(s)ds, \quad 0 < \sigma < \zeta_1 < \eta < \xi < \zeta_2 < 1, \end{aligned} \tag{1.2.5}$$

where ${}_0^C D^q$ denotes Caputo derivative of order q, $f : [0,1]\times\mathbb{R} \to \mathbb{R}$ is a given continuous function. The integral boundary condition in the problem (1.2.5) implies that the linear combination of the values of the first-order derivative of the unknown function at nonlocal positions ζ_1 and ζ_2 (off the strip) is proportional to its strip contribution occupying the position (η, ξ).

Recently, Ahmad et al. (2017) established sufficient conditions for the existence and uniqueness of solutions for a boundary value problem of fractional differential equations with nonlocal and average type integral boundary conditions in the form

$$\begin{aligned} {}_0^C D^q y(x) &= f(x, y(x), {}_0^C D^\beta y(x)), \quad x \in [0,1],\ n-1 < q \leq n,\ n \geq 2 \\ y''(0) &= y'''(0) = \ldots = y^{(n-2)}(0) = 0, \\ y(0) + y'(0) &= h(x),\ \int_0^\eta y(s)ds = \xi, \quad 0 < \eta < 1 \end{aligned} \tag{1.2.6}$$

where ${}_0^C D^q, {}_0^C D^\beta$ denote the Caputo fractional derivatives of order $q \in (n-1, n), n \geq 2$ and $\beta \in (0,1), f : [0,1] \times \mathbb{R} \to \mathbb{R}, h : C([0,1], \mathbb{R}) \to \mathbb{R}$ are given continuous functions and $\xi \in \mathbb{R}$ is a real constant.

Agarwal et al. (2017) studied nonlinear boundary value problems of Liouville-Caputo type fractional differential equations supplemented with nonlocal multi-point conditions involving lower order fractional derivative as

$$\begin{aligned} {}_0^C D^q y(x) &= f(x, y(x)), \quad x \in [0,1],\ 1 < q \leq 2, \\ y(0) &= \delta y(\sigma), \quad \delta \in \mathbb{R} \\ a\, {}_0^C D^p y(\zeta_1) + b\, {}_0^C D^p y(\zeta_2) &= \sum_{i=1}^{m-2} \alpha_i y(\beta_i), \quad 0 < p < 1, \end{aligned} \tag{1.2.7}$$

where ${}_0^C D^q$ denote Caputo derivative of order q and $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a given continuous function and $\delta, a, b, \alpha_i \in \mathbb{R}$. The multi-point boundary conditions in (1.2.7) implies that the linear combination of the values of the fractional derivative of the unknown function at nonlocal positions ζ_1 and ζ_2 is equal to the linear combination of the values of the unknown function at $\beta_i, i = 1, 2, \ldots, m-2$, while the value of the unknown function at the left end point $(x = 0)$ of the interval $[0,1]$ is proportional to its value at the nonlocal position σ.

In the second problem, Agarwal et al. (2017) discussed the existence of solutions of

$$\begin{aligned} {}_0^C D^q y(x) &= f(x, y(x)), \quad x \in [0,1],\ 1 < q \leq 2, \\ y(0) &= \delta_1 \int_0^\sigma y(s)ds, \quad \delta \in \mathbb{R} \\ a\, {}_0^C D^p y(\zeta_1) + b\, {}_0^C D^p y(\zeta_2) &= \sum_{i=1}^{m-2} \alpha_i y(\beta_i), \quad 0 < p < 1. \end{aligned} \tag{1.2.8}$$

In (1.2.8), the first boundary condition can be interpreted as the value of the unknown function at $x = 0$ is proportional to the continuous distribution of the unknown function over a strip of an arbitrary length σ.

Although the existence and uniqueness theory for fractional boundary conditions subject to nonlocal, integral and strip boundary conditions is very well developed, there are few results obtained for fractional boundary conditions subject to multi strip boundary conditions. Multi-strip boundary conditions may be regarded as the generalization of multi-point boundary conditions. As far we know, the first work has been done by Ahmad and Ntouyas (2012). The authors considered a fractional boundary value problem with a nonlocal strip condition of the form:

$$\begin{aligned} {}_0^C D^q y(x) &= f(x, y(x), {}_0^C D^{\beta} y(x)), \quad x \in [0,1],\ m-1 < q \leq m,\ m \geq 2,\ m \in \mathbb{N} \\ y(0) = y'(0) = y''(0) &= \ldots = y^{(n-2)}(0) = 0, \\ y(1) &= \sum_{i=1}^{n-2} \alpha_i \int_{\zeta_i}^{\eta_i} y(s)ds, \quad 0 < \zeta_i < \eta_i < 1,\ i = 1, 2, \ldots, n-2 \end{aligned} \tag{1.2.9}$$

in which the second boundary condition can be viewed as an extension of a multi-point nonlocal boundary condition:

$$y(1) = \sum_{i=1}^{n-2} \alpha_i y(\eta_i), \tag{1.2.10}$$

where ${}_0^C D^q$ denotes Caputo derivative of order q and $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a given continuous function. The strip boundary condition in problem (1.2.9) can be regarded as a multi-point nonlocal integral boundary condition In fact, the strip condition corresponds to a continuous distribution of the values of the unknown function on arbitrary finite segments (ζ_i, η_i) of the interval $[0,1]$ and the effect of these strips is accumulated at $x = 1$.

Then Tariboon et. al (2014) discussed the existence and uniqueness of solutions for boundary value problems involving multiterm fractional integral boundary conditions of the form

$$\begin{aligned} &{}_0^C D^q y(x) = f(x, y(x), {}_0^C D^{\beta} y(x)), \quad x \in [0,T],\ 1 < q \leq 2, \\ &\sum_{i=1}^{m} \lambda_i I^{\alpha_i} y(\eta_i) = w_1,\ \sum_{j=1}^{n} \mu_j (I^{\beta_j} y(T) - I^{\beta_j} y(\xi_j)) = w_2 \end{aligned} \tag{1.2.11}$$

where ${}_0^C D^q$ denotes the Caputo fractional derivative of order , and $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $\eta_i, \xi_j \in (0,T), \lambda_i, \mu_j \in \mathbb{R}$, for all $i = 1,2,\ldots,m,\ j = 1,2,\ldots,n,\ w_1, w_2 \in \mathbb{R}$ and I^p is the Riemann-Liouville fractional integral of order $p > 0$ $(p = \alpha_i, \beta_j, i = 1,2,\ldots,m,\ j = 1,2,\ldots,n)$.

2. PRELIMINARIES

In this section we give basic definitions and concepts from functional analysis referring to the book of Meise and Vogt (1997) and from the theory of fractional differential equations. Let $\mathbb{R}_+ = [0,\infty)$.

Definition 2.0.1. *(Meise and Vogt, 1997) (Metric space) Let X be a set. A metric on X is a function $d : X \times X \to R_+$ with the following properties:*

M1) $d(x,y) = d(y,x)$ *for all* $x,y \in X$

M2) $d(x,z) = d(x,y) + d(y,z)$ *for all* $x,y,z \in X$

M3) $d(x,y) = 0$ *if and only if* $x = y$*.*

A metric space (X; d) is a nonempty set X on which a metric d is given.

Definition 2.0.2. *(Meise and Vogt, 1997) (Vector space) A linear space Y over $\mathbb{R}$, also called real vector space over $\mathbb{R}$, is a nonempty set Y, in which an addition and a multiplication $+ : Y \times Y \to Y$ and $\cdot : \mathbb{R} \times Y \to Y$ with the following properties are defined, respectively:*

V1) $(Y,+)$ *is an Abelian group with the zero element* 0,

V2) $c(x+y) = cx + cy$ *for all* $c \in \mathbb{R},\ x,y \in Y$,

V3) $(c_1 + c_2)x = c_1 x + c_2 x$ *for all* $c_1, c_2 \in \mathbb{R},\ x \in Y$,

V4) $(c_1 c_2)x = c_1(c_2 x)$ *for all* $c_1, c_2 \in \mathbb{R},\ x \in Y$,

V5) $1x = x$ *for all* $x \in X$.

The elements of Y are called vectors.

Definition 2.0.3. *(Meise and Vogt, 1997) (Convergent sequence) A sequence $\{x_n\}$ in a metric space (X,d) converges to x if and only if $\lim_{n\to\infty} d(x_n,x) = 0$.*

Definition 2.0.4. *(Meise and Vogt, 1997) (Cauchy sequence) A sequence $\{x_n\}$ in a metric space (X,d) is called a Cauchy sequence if to every $\varepsilon > 0$ there corresponds an integer N such that $d(x_m,x_n) < \varepsilon$ whenever $m > N$ and $n > N$.*

Definition 2.0.5. *(Meise and Vogt, 1997) (Complete metric space) A metric space X is said to be complete if every Cauchy sequence in X is convergent.*

Definition 2.0.6. *(Meise and Vogt, 1997) (Normed space) Let Y be a real vector space. A norm on Y is a function $||.|| : Y \to \mathbb{R}$ with the following properties:*

(N1) $||\lambda x|| = |\lambda| \; ||x||$ *for all* $\lambda \in \mathbb{R}, x \in Y$.

(N2) $||x+y|| \leq ||x|| + ||y||$ *for all* $x, y \in Y$.

(N3) $||x|| = 0$ *if only if* $x = 0$.

A normed space $(Y, ||.||)$ is a real vector space Y on which a norm is defined.

Definition 2.0.7. *(Meise and Vogt, 1997) (Canonical metric) A metric d defined on Y by $d(x,y) = ||x-y||$ for all $x, y \in Y$ is called the canonical metric of the normed space Y .*

Definition 2.0.8. *(Meise and Vogt, 1997) (Banach space) A Banach space is a normed space which is complete under its canonical metric.*

Definition 2.0.9. *(Meise and Vogt, 1997) (Closed set) A set S is closed if and only if for every convergent sequence of elements of S its limit is also in S.*

Definition 2.0.10. *(Meise and Vogt, 1997) (Convex set) A subset S of a real vector space Y is said to be convex if $\alpha x + (1-\alpha)y \in S$ for all $\alpha \in [0,1]$ and $x, y \in Y$.*

Definition 2.0.11. *(Meise and Vogt, 1997) (Uniform bounded set) A set S in a metric space X is uniform bounded if $\exists M > 0$ such that $d(x,y) \leq M$ for all $x, y \in X$.*

Definition 2.0.12. *(Meise and Vogt, 1997) (Equicontinuous set) A set S in a metric space X is equicontinuous if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $x, y \in X$ and all $f \in S$, if $d(x,y) < \delta$ then $d(f(x), f(y)) < \varepsilon$.*

Definition 2.0.13. *(Meise and Vogt, 1997) (Compact set in $\mathbb{R}^n$) A set $S \subset \mathbb{R}^n$ is compact if and only if for any sequence $\{x_n\}_{n \in \mathbb{N}} \subset S$, there is a subsequence that converges to a point in S.*

Definition 2.0.14. *(Meise and Vogt, 1997) (Relatively compact set) A set S is said to be relatively compact if its closure $\overline{S}$ is compact.*

Definition 2.0.15. *(Meise and Vogt, 1997) (Compact operator) A compact operator is a linear operator L from a Banach space X to another Banach space Y, such that the image under L of any bounded subset of X is a relatively compact subset (has compact closure) of Y.*

Definition 2.0.16. *(Meise and Vogt, 1997) (Completely continuous operator) Let X,Y be a Banach space. An operator $T : D \subset X \to Y$ is said to be completely continuous if it is continuous and maps any bounded subset of D into a relatively compact subset of Y.*

Theorem 2.0.1. *(Meise and Vogt, 1997) (Banach's contraction mapping principle) Let (X,d) be a complete metric space and $T : X \to X$ satisfies $d(T(x),T(y)) \leq Ld(x,y)$ for all $x,y \in X$ where $L \in [0,1)$. Then F has a unique fixed point $u \in X$.*

Theorem 2.0.2. *(Meise and Vogt, 1997) (Krasnoselski's fixed point theorem) Let M be a closed, convex, non-empty subset of a Banach space Y. Suppose that A and B map M into Y and that*

(i) $Ax + By \in M$ *for all* $x,y \in M$,

(ii) *A is compact and continuous,*

(iii) *B is a contraction mapping.*

Then there exists $y \in M$ such that $Ay + By = y$.

Theorem 2.0.3. *(Meise and Vogt, 1997) (Leary-Schauder alternative) Let X be a Banach space. Assume that $T : X \to X$ is completely continuous operator and the set*

$$V = \{u \in X : u = \sigma T u, 0 < \sigma < 1\} \tag{2.0.1}$$

is bounded. Then T has a unique fixed point $u \in X$.

Theorem 2.0.4. *(Meise and Vogt, 1997) (Arzela-Ascoli Theorem) Let X be a compact metric space and $F \subset C(X)$. Then F is relatively compact if and only if F is uniformly bounded and equicontinuous.*

Theorem 2.0.5. *(Meise and Vogt, 1997) Let X be a Banach space. Assume that Ψ is an open bounded subset of X with $\psi \in \Psi$ and let $T : \overline{\Psi} \to X$ be a completely continuous operator such that $||\Psi v|| \leq ||v||$ for all $v \in$. Then Ψ has a fixed point in $\overline{\Psi}$.*

Theorem 2.0.6. *(Meise and Vogt, 1997) (Nonlinear alternative for single valued maps, Nonlinear alternative of Leray-Schauder type) Let X be a Banach space, C a closed, convex subset of X, U an open subset of C and $0 \in U$. Suppose that $F : \overline{U} \to C$ is a continuous, compact (i.e, $F(\overline{U})$ is a relatively compact subset of C) map. Then either*

(i) F has a fixed point in $\overline{U}$ or

(ii) there is a $u \in \partial U$ (the boundary of U in C) and $\lambda \in (0,1)$ with $u = \lambda F(u)$.

We give the definition of Euler's Gamma function which is the generalization of factorial function, $n!$, to the noninteger real numbers and complex numbers and Beta function which will be used in the sequel.

Definition 2.0.17. *(Miller and Ross, 1993; Samko et al., 1993; Podlubny, 1999, Kilbas et al., 2006; Sabatier et al., 2007) Euler's Gamma function is defined by the integral of the form*

$$\Gamma(\omega) = \int_0^\infty e^{-s} s^{\omega-1} ds, \ \Re(\omega) > 0.$$

Theorem 2.0.7. *(Miller and Ross, 1993; Samko et al., 1993; Podlubny, 1999, Kilbas et al., 2006; Sabatier et al., 2007) Let $\Re(\omega) > 0$. Then $\Gamma(\omega+1) = \omega\Gamma(\omega)$.*

Proof. (Miller and Ross, 1993; Samko et al., 1993; Podlubny, 1999, Kilbas et al., 2006; Sabatier et al., 2007) If we use the definition of Gamma function for $\Gamma(\omega+1)$ and if we use intagration by parts formula, we have

$$\Gamma(\omega+1) = \int_0^\infty e^{-s} s^{\omega} ds = \lim_{T\to\infty} \int_0^T e^{-s} s^{\omega} ds = \lim_{T\to\infty} \left[-e^{-s} s^{\omega}|_0^T + \omega \int_0^T e^{-s} s^{\omega-1} ds \right] = \omega\Gamma(\omega).$$

□

Definition 2.0.18. *(Miller and Ross, 1993; Samko et al., 1993; Podlubny, 1999, Kilbas et al., 2006; Sabatier et al., 2007) The following integral defines the Beta function for $\Re(\alpha) > 0, \Re(\beta) > 0$.*

$$B(\alpha,\beta) = \int_0^1 s^{\alpha-1}(1-s)^{\beta-1} ds.$$

Theorem 2.0.8. *(Miller and Ross, 1993; Samko et al., 1993; Podlubny, 1999, Kilbas et al., 2006; Sabatier et al., 2007) Let $\Re(\alpha) > 0, \Re(\beta) > 0$. Then Beta function has the following properties.*

1. $B(\alpha,\beta) = B(\beta,\alpha)$.

2. $B(\alpha,\beta) = \dfrac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$.

3. $\displaystyle\int_c^d (r-c)^{\alpha-1}(d-r)^{\beta-1}dr = (d-c)^{\alpha+\beta-1}B(\alpha,\beta)$.

Proof. (Miller and Ross, 1993; Samko et al., 1993; Podlubny, 1999, Kilbas et al., 2006; Sabatier et al., 2007)

1. By using the substitution $s = 1-r$, we obtain

$$B(\alpha,\beta) = \int_0^1 s^{\alpha-1}(1-s)^{\beta-1}ds = \int_0^1 (1-r)^{\alpha-1}r^{\beta-1}dr = B(\beta,\alpha).$$

2. The definition of Gamma function implies that

$$\Gamma(\alpha)\Gamma(\beta) = \int_0^\infty e^{-s}s^{\alpha-1}ds\int_0^\infty e^{-r}r^{\beta-1}dr = \int_0^\infty\int_0^\infty e^{-s-r}s^{\alpha-1}r^{\beta-1}dsdr. \qquad (2.0.2)$$

By changing the variables $s = uv, r = u(1-v)$ in the above integral, the Jacobian becomes as $\dfrac{\partial(s,r)}{\partial(u,v)} = \begin{vmatrix} v & u \\ 1-v & -u \end{vmatrix} = -uv-u+uv = -u$. Because $u > 0$, it can be seen that

$$dsdr = \left|\frac{\partial(s,r)}{\partial(u,v)}\right|dudv = ududv.$$

After changing the variables, the domain $0 < s,r < \infty$ is transformed into $0 < u < \infty, 0 < v < 1$. We conclude from equation (2.0.2) that

$$\begin{aligned}\Gamma(\alpha)\Gamma(\beta) &= \int_0^\infty\int_0^\infty e^{-s-r}s^{\alpha-1}r^{\beta-1}dsdr = \int_0^1\int_0^\infty e^{-u}u^{\alpha-1}v^{\alpha-1}u^{\beta-1}(1-v)^{\beta-1}ududv \\ &= \int_0^\infty e^{-u}u^{\alpha+\beta-1}du\int_0^1 v^{\alpha-1}(1-v)^{\beta-1}dv = \Gamma(\alpha+\beta)B(\alpha,\beta).\end{aligned}$$

3. If we use the substitution $s = \dfrac{r-c}{d-c}$ in the definition of Beta function, we obtain

$$\begin{aligned}B(\alpha,\beta) &= \int_0^1 s^{\alpha-1}(1-s)^{\beta-1}ds = \int_c^d (r-c)^{\alpha-1}(d-c)^{-\alpha-\beta+1}(d-r)^{\beta-1}dr \\ &= (d-c)^{-\alpha-\beta+1}\int_c^d (r-c)^{\alpha-1}(d-r)^{\beta-1}dr.\end{aligned}$$

□

We will present the definitions of Riemann-Liouville fractional integral, Riemann-Liouville fractional derivative and the Caputo fractional derivative and their basic properties.

Definition 2.0.19. *(Miller and Ross, 1993; Samko et al., 1993; Podlubny, 1999, Kilbas et al., 2006; Sabatier et al., 2007) Let* $\rho \geq 0$ *and* φ *be a continuous function defined on* $[a,b]$*. The Riemann Lioville fractional integral of order* ρ *is defined by*

$$({}_aI^{\rho}\varphi)(x) = \frac{1}{\Gamma(\rho)}\int_a^x (x-s)^{\rho-1}\varphi(s)ds \;\; for \;\; \rho > 0$$

and ${}_aI^0\varphi(x) = \varphi(x)$ *for* $\rho = 0$.

Definition 2.0.20. *(Miller and Ross, 1993; Samko et al., 1993; Podlubny, 1999, Kilbas et al., 2006; Sabatier et al., 2007) The Riemann Liouville fractional derivative of order* $\rho \geq 0$ *is defined by*

$$({}_aD^{\rho}\varphi)(x) = \begin{cases} ({}_aD^m\,{}_aI^{m-\rho}\varphi)(x) = \dfrac{1}{\Gamma(m-\rho)}\dfrac{d^m}{dx^m}\displaystyle\int_0^x (x-t)^{m-\rho-1}\varphi(t)dt, & \rho > 0 \\ \varphi(x), & \rho = 0 \end{cases}$$

where m is the smallest integer greater or equal than ρ.

Definition 2.0.21. *(Miller and Ross, 1993; Samko et al., 1993; Podlubny, 1999, Kilbas et al., 2006; Sabatier et al., 2007) The Caputo fractional derivative of order* $\rho \geq 0$ *is defined by*

$$({}^C_aD^{\rho}\varphi)(x) = \begin{cases} ({}_aI^{m-\rho}\,{}^C_aD^m\varphi)(x) = \dfrac{1}{\Gamma(m-\rho)}\displaystyle\int_0^x (x-t)^{m-\rho-1}\varphi^{(m)}(t)dt, & \rho > 0 \\ \varphi(x), & \rho = 0 \end{cases}$$

where m is the smallest integer greater or equal than ρ.

Definition 2.0.22. *(Miller and Ross, 1993; Samko et al., 1993; Podlubny, 1999, Kilbas et al., 2006; Sabatier et al., 2007) Let* $\rho \geq 0$*. If* $\varphi(x) \in C[a,b]$ *then* ${}^C_aD^{\rho}({}_aI^{\rho}\varphi)(x) = \varphi(x)$.

Lemma 2.0.1. *(Miller and Ross, 1993; Samko et al., 1993; Podlubny, 1999, Kilbas et al., 2006; Sabatier et al., 2007) If* $\varphi(x) \in AC^m[a,b]$ *or* $\varphi(x) \in C^m[a,b]$*, then for some constants* c_i*,* $i = 1,2,\ldots,m$*, one has*

$${}_aI^{\alpha}\,{}^C_aD^{\rho}\varphi(x) = \varphi(x) + c_1 + c_2(x-a) + c_3(x-a)^2 \ldots + c_m(x-a)^{m-1},$$

where m is the smallest integer greater or equal than α.

3. SOME NEW VERSIONS OF FRACTIONAL BOUNDARY VALUE PROBLEMS WITH SLIT-STRIPS CONDITIONS

This chapter is devoted to the results of Ahmad and Agarwal (2014). We are interested in differential equation with Caputo derivative of order $\rho \in (m-1, m],\ m \geq 2$

$$ {}_0^C D^{\rho} \phi(x) = \varphi(x, \phi(x)),\ x \in [0,1], \tag{3.0.1} $$

subject to slit strips boundary conditions of the form

$$ \begin{gathered} \phi(0) = 0,\ \phi'(0) = 0, \ldots,\ \phi^{(m-2)}(0) = 0, \\ \phi(\beta) = d_1 \int_0^{\alpha} \phi(w)dw + d_2 \int_{\gamma}^{1} \phi(w)dw, \quad 0 < \alpha < \beta < \gamma. \end{gathered} \tag{3.0.2} $$

Without further mention that

1. $\varphi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function with respect to its both arguments,

2. d_1, d_2 are constant numbers.

The following theorem is the fundamental argument to prove the main theorems.

Theorem 3.0.1. *(Ahmad and Agarwal, 2014) Let $g \in C[0,1]$. Then $\phi(x) \in AC^m[0,1]$ is the unique solution of*

$$ {}_0^C D^{\rho} \phi(x) = g(x),\ x \in [0,1],\ \rho \in (m-1, m] \tag{3.0.3} $$

$$ \phi(0) = 0,\ \phi'(0) = 0, \ldots,\ \phi^{m-2}(0) = 0, \tag{3.0.4} $$

$$ \phi(\beta) = d_1 \int_0^{\alpha} \phi(w)dw + d_2 \int_{\gamma}^{1} \phi(w)dw, \quad 0 < \alpha < \beta < \gamma \tag{3.0.5} $$

provided that ϕ satisfies the following:

$$ \begin{aligned} \phi(x) = \frac{x^{m-1}}{C} \Bigg[& \frac{d_1}{\Gamma(\rho+1)} \int_0^{\alpha} (\alpha - s)^{\rho} g(s) ds + \frac{d_2}{\Gamma(\rho+1)} \int_0^{1} (1-s)^{\rho} g(s) ds \\ & - \frac{d_2}{\Gamma(\rho+1)} \int_0^{\gamma} (\gamma - s)^{\rho} g(s) ds - \frac{1}{\Gamma(\rho)} \int_0^{\beta} (\beta - s)^{\rho-1} g(s) ds \Bigg] + \frac{1}{\Gamma(\rho)} \int_0^{x} (x-s)^{\rho-1} g(s) ds \end{aligned} \tag{3.0.6} $$

where

$$ C = \beta^{m-1} - \frac{1}{m} (d_1 \alpha^m + d_2 - d_2 \gamma^m) \neq 0. $$

Proof. By employing Definition 2.0.1, for arbitrary constants $c_0, c_1, \ldots, c_{m-1}$, the general solution of equation (3.0.3) may be presented as

$$\phi(x) = \frac{1}{\Gamma(\rho)}\int_0^x (x-w)^{\rho-1} g(w)dw + c_0 + c_1 x + \ldots + c_{m-1}x^{m-1}. \tag{3.0.7}$$

Using the boundary conditions (3.0.4) in equation (3.0.7) implies that $c_0 = c_1 = \ldots = c_{m-2} = 0$.

If we apply boundary condition (3.0.5) to the both sides of equation (3.0.7), we obtain

$$\begin{aligned}\phi(\beta) &= c_{m-1}\beta^{m-1} + \frac{1}{\Gamma(\rho)}\int_0^\beta (\beta-w)^{\rho-1} g(w)dw \\ &= \frac{d_1}{\Gamma(\rho)}\int_0^\alpha \left(c_{m-1}w^{m-1} + \int_0^w (w-s)^{\rho-1} g(s)ds\right) dw \\ &+ \frac{d_2}{\Gamma(\rho)}\int_\gamma^1 \left(c_{m-1}w^{m-1} + \int_0^w (w-s)^{\rho-1} g(s)ds\right) dw.\end{aligned}$$

By evaluating the first integrals on the second and third lines of the above equation and then changing the orders of the second integrals on the same lines, we have

$$\begin{aligned}c_{m-1} = \frac{1}{C}\Bigg[&\frac{d_1}{\Gamma(\rho+1)}\int_0^\alpha (\alpha-s)^\rho g(s)ds + \frac{d_2}{\Gamma(\rho+1)}\int_0^1 (1-s)^\rho g(s)ds \\ &- \frac{d_2}{\Gamma(\rho+1)}\int_0^\gamma (\gamma-s)^\rho g(s)ds - \frac{1}{\Gamma(\rho)}\int_0^\beta (\beta-s)^{\rho-1} g(s)ds\Bigg],\end{aligned}$$

where $C = \beta^{m-1} - \frac{1}{m}(d_1\alpha^m + d_2 - d_2\gamma^m)$.

Rewriting equation (3.0.7) with the found value of c_{m-1}, the expression of the unique solution (3.0.6) can be obtained. □

3.1. Main Results

Consider a Banach space $H = C([0,1],\mathbb{R})$ containing continuous functions which are defined on the domain $[0,1]\times\mathbb{R}$ on which a uniform convergence topology is defined via the sup norm, $||\phi|| = \sup_{x\in[0,1]} |\phi(x)|$.

Theorem 3.0.1 defines the following operator $\mathcal{P} : H \to H$ as

$$\begin{aligned}(\mathcal{P}\phi)(x) = \frac{x^{m-1}}{C}\Bigg[&\frac{d_1}{\Gamma(\rho+1)}\int_0^\alpha (\alpha-s)^\rho \varphi(s,\phi(s))ds + \frac{d_2}{\Gamma(\rho+1)}\int_0^1 (1-s)^\rho \varphi(s,\phi(s))ds \\ &- \frac{d_2}{\Gamma(\rho+1)}\int_0^\gamma (\gamma-s)^\rho \varphi(s,\phi(s))ds - \frac{1}{\Gamma(\rho)}\int_0^\beta (\beta-s)^{\rho-1}\varphi(s,\phi(s))ds\Bigg] \\ &+ \frac{1}{\Gamma(\rho)}\int_0^x (x-s)^{\rho-1}\varphi(s,\phi(s))ds.\end{aligned} \tag{3.1.1}$$

It is clear that the problem (3.0.1)-(3.0.2) may have a solution provided that the related fixed point equation $\mathcal{P}\phi = \phi$ possesses a solution; that means admitted a fixed point.

In short,

$$Z = \frac{1}{|C|\Gamma(\rho+2)}\left[|d_1|\alpha^{\rho+1} + |d_2|(1-\gamma^{\rho+1}) + (\rho+1)(\beta^{\rho} + |C|)\right]. \tag{3.1.2}$$

Theorem 3.1.1. *(Ahmad and Agarwal, 2014) Assume that the continuous real valued function* $\varphi : [0,1]\times\mathbb{R}\to\mathbb{R}$ *satisfies the following assumption, which is Lipschitz condition:*

$(A_1) : |\varphi(x,\phi_1) - \varphi(x,\phi_2)| \leq L|\phi_1 - \phi_2|$ *for all* $x \in [0,1]$, $L > 0$, $\phi_1, \phi_2 \in \mathbb{R}$. *In that case boundary value problem (3.0.1)-(3.0.2) does have a unique solution provided that*

$$L < \frac{1}{Z}. \tag{3.1.3}$$

Proof. Proof will be done by employing Theorem 2.0.1. Let us define a closed and bounded set

$$B_r = \{\phi \in H : ||\phi|| \leq r\},$$

where $r \geq \dfrac{MZ}{1-LZ}$, $M = \sup\limits_{x\in[0,1]} |\varphi(x,0)| < \infty$.

Step 1. $\mathcal{P}B_r \subset B_r$: The triangular inequality implies that

$$|\varphi(x,\phi(x))| \leq |\varphi(x,\phi(x)) - \varphi(x,0)| + |\varphi(x,0)| \leq L||\phi|| + M \leq Lr + M,\ \phi \in B_r.$$

On the other hand

$$\begin{aligned} ||\mathcal{P}\phi|| \leq \sup_{x\in[0,1]} \Bigg\{ \frac{x^{m-1}}{|C|} \Bigg[&\frac{|d_1|}{\Gamma(\rho+1)} \int_0^{\alpha} (\alpha - z)^{\rho} |\varphi(z,\phi(z))| dz + \frac{1}{\Gamma(\rho)} \int_0^{\beta} (\beta - z)^{\rho-1} |\varphi(z,\phi(z))| dz \\ &+ \frac{|d_2|}{\Gamma(\rho+1)} \left| \int_0^{\gamma} (1-z)^{\rho} \varphi(z,\phi(z)) dz + \int_{\gamma}^1 (1-v)^{\rho} \varphi(z,\phi(z)) dz - \int_0^{\gamma} (\gamma - z)^{\rho} \varphi(z,\phi(z)) dz \right| \Bigg] \\ &+ \frac{1}{\Gamma(\rho)} \int_0^x (x-z)^{\rho-1} |\varphi(z,\phi(z))| dz \Bigg\}. \end{aligned} \tag{3.1.4}$$

If we manipulate second line of the above inequality, one can get

$$\begin{aligned} &\left| \int_0^{\gamma} (1-r)^{\rho} \varphi(z,\phi(z)) dz + \int_{\gamma}^1 (1-z)^{\rho} \varphi(z,\phi(z)) dz - \int_0^{\gamma} (\gamma - z)^{\rho} \varphi(z,\phi(z)) dz \right| \\ &= \left| \int_0^{\gamma} \left[(1-z)^{\rho} - (\gamma - z)^{\rho}\right] \varphi(z,\phi(z)) dz + \int_{\gamma}^1 (1-z)^{\rho} \varphi(z,\phi(z)) dz \right| \\ &\leq \int_0^{\gamma} \left[(1-z)^{\rho} - (\gamma - z)^{\rho}\right] |\varphi(z,\phi(z))| dz + \int_{\gamma}^1 (1-z)^{\rho} |\varphi(z,\phi(z))| dz = (Lr+M)\frac{1-\gamma^{\rho+1}}{\rho+1}. \end{aligned} \tag{3.1.5}$$

After using triangular inequality, if we take the integrals in inequality (3.1.4) and by using inequality (3.1.5), we have

$$||\mathcal{P}\phi|| \leq (Lr+M)\sup_{x\in[0,1]}\left\{\frac{x^{m-1}}{|C|}\left[\frac{|d_1|\alpha^{\rho+1}}{\Gamma(\rho+2)}+\frac{|d_2|(1-\gamma^{\rho+1})}{\Gamma(\rho+2)}+\frac{\beta^{\rho}}{\Gamma(\rho+1)}\right]+\frac{x^{\rho}}{\Gamma(\rho+1)}\right\}$$
$$\leq (Lr+M)Z.$$

By using the condition $r \geq \dfrac{MZ}{1-LZ}$, we get

$$||\mathcal{P}\phi|| \leq (Lr+M)Z = LZr+MZ \leq -Mz+r+Mz = r.$$

Step 2. $\mathcal{P}$ is a contraction: For every $x\in[0,1]$ and for all $\phi_1,\phi_2\in H$, the following holds:

$$\begin{aligned}||\mathcal{P}\phi_1-\mathcal{P}\phi_2|| \leq \sup_{x\in[0,1]}&\left\{\frac{x^{m-1}}{|C|}\left[\frac{|d_1|}{\Gamma(\rho+1)}\int_0^{\alpha}(\alpha-z)^{\rho}|\varphi(z,\phi_1(z))-\varphi(z,\phi_2(z))|dv\right.\right.\\&+\frac{1}{\Gamma(\rho)}\int_0^{\beta}(\beta-z)^{\rho-1}|\varphi(z,\phi_1(z))-\varphi(z,\phi_2(z))|dz\\&+\frac{|d_2|}{\Gamma(\rho+1)}\int_0^{\gamma}\left[(1-z)^{\rho}-(\gamma-z)^{\rho}\right]|\varphi(z,\phi_1(z))-\varphi(z,\phi_2(z))|dz\\&\left.+\frac{|d_2|}{\Gamma(\rho+1)}\int_{\gamma}^{1}(1-z)^{\rho}|\varphi(z,\phi_1(z))-\varphi(z,\phi_2(z))|dz\right]\\&\left.+\frac{1}{\Gamma(\rho)}\int_0^{x}(x-z)^{\rho-1}|\varphi(z,\phi_1(z))-\varphi(z,\phi_2(z))|dz.\right\}\end{aligned}$$

The Lipschitz condition (A_1) leads to

$$\begin{aligned}||\mathcal{P}\phi_1-\mathcal{P}\phi_2|| \leq \sup_{x\in[0,1]}&\left\{\frac{Lx^{m-1}}{|C|}\left[\frac{|d_1|}{\Gamma(\rho+1)}\int_0^{\alpha}(\alpha-z)^{\rho}|\phi_1(z)-\phi_2(z)|dz\right.\right.\\&+\frac{1}{\Gamma(\rho)}\int_0^{\beta}(\beta-z)^{\rho-1}|\phi_1(z)-\phi_2(z)|dz\\&+\frac{|d_2|}{\Gamma(\rho+1)}\int_0^{\gamma}\left[(1-z)^{\rho}-(\gamma-z)^{\rho}\right]|\phi_1(z)-\phi_2(z)|dz\\&\left.+\frac{|d_2|}{\Gamma(\rho+1)}\int_{\gamma}^{1}(1-z)^{\rho}|\phi_1(z)-\phi_2(z)|dz\right]\\&\left.+\frac{L}{\Gamma(\rho)}\int_0^{x}(x-z)^{\rho-1}|\phi_1(z)-\phi_2(z)|dz\right\}\end{aligned}$$

Applying the same procedure followed in Step 1 results in $||\mathcal{P}\phi_1-\mathcal{P}\phi_2|| \leq LZ||\phi_1-\phi_2||$. From condition (3.1.3), we can conclude that $\mathcal{P}$ is contraction. Therefore Banach's contraction mapping principle provides the existence of unique solution of boundary value problem (3.0.1)-(3.0.2). □

Example 3.1.1. *(Ahmad and Agarwal, 2014) Examine boundary value problem with fractional derivative*

$$\begin{aligned} {}_0^C D^{9/2}\phi(x) &= e^x \arctan\phi + \frac{\sin\phi}{(x+4)^{1/2}} + (x+1)^{3/2},\ x\in[0,1], \\ \phi(0)=0,\ \phi'(0)&=0,\ldots,\ \phi''(0)=0,\ldots,\ \phi'''(0)=0, \\ \phi(1/2) &= d_1\int_0^{1/3}\phi(w)dw + d_2\int_{2/3}^1 \phi(w)dw, \end{aligned} \tag{3.1.6}$$

where $m=5, \rho=9/2, d_1=1, d_2=1, \alpha=1/3, \beta=1/2, \gamma=2/3$ *and*

$$\varphi(x,\phi(x)) = e^x\arctan\phi + \frac{\sin\phi}{(x+4)^{1/2}} + (x+1)^{3/2}.$$

On the other hand $|C| = |\beta^{m-1} - \frac{1}{m}(d_1\alpha^m + d_2 - d_2\gamma^m)| = 0.111986$ *and*

$$Z = \frac{1}{|C|\Gamma(\rho+2)}\left[|d_1|\alpha^{\rho+1} + |d_2|(1-\gamma^{\rho+1}) + (\rho+1)(\beta^\rho + |C|)\right] = 0.0544011$$

By applying Mean Value Theorem for $\phi_1 < \xi < \phi_2, x\in[0,1]$, *we obtain*

$$\begin{aligned} |\varphi(x,\phi_1) - \varphi(x,\phi_2)| &= e^x|\arctan\phi_1 - \arctan\phi_2| + \frac{|\sin\phi_1 - \sin\phi_2|}{(x+4)^{1/2}} = |\varphi_\phi(x,\phi)(\xi)||\phi_1-\phi_2| \\ &= \left|\frac{e^x}{1+\xi^2} + \frac{\cos\xi}{(x+4)^{1/2}}\right||\phi_1-\phi_2| \le \left(e+\frac{1}{2}\right)|\phi_1-\phi_2|, \end{aligned}$$

which implies $L = e+1/2$ *and* $L < 1/Z$ *is satisfied. Since the hypotheses of Theorem 3.1.1 are satisfied, (3.1.6) does have a unique solution.*

Theorem 3.1.2. *(Ahmad and Agarwal, 2014) Let* $\varphi : [0,1]\times\mathbb{R}\to\mathbb{R}$ *be a continuous function and* (A_1) *and*

$$(A_2):\ |\varphi(x,\phi)| \le f(x) \text{ for all } (x,\phi)\in[0,1]\times\mathbb{R} \text{ and } f\in([0,1],\mathbb{R}^+)$$

hold. If

$$\frac{L}{|C|\Gamma(\rho+2)}\left[|d_1|\alpha^{\rho+1} + |d_2|(1-\gamma^{\rho+1}) + (\rho+1)\beta^\rho\right] < 1, \tag{3.1.7}$$

then the boundary value problem (3.0.1)-(3.0.2) possesses a solution.

Proof. Theorem 2.0.2 will be employed to show the existence of solution of the boundary value problem (3.0.1)-(3.0.2). For this purpose, we define the set $B_r = \{\phi\in H : ||\phi|| \le r\} \subset H$ for which $Z||f|| \le r$. We also divide the operator $\mathcal{P}$ into two parts as

$$(\mathcal{P}_1\phi)(x) = \frac{1}{\Gamma(\rho)}\int_0^x (x-v)^{\rho-1}\varphi(z,\phi(z))dz$$

and

$$(\mathcal{P}_2\phi)(x) = \frac{x^{m-1}}{C}\left[\frac{d_1}{\Gamma(\rho+1)}\int_0^{\alpha}(\alpha-z)^{\rho}\varphi(z,\phi(z))dz + \frac{d_2}{\Gamma(\rho+1)}\int_0^{1}(1-z)^{\rho}\varphi(z,\phi(z))dz\right.$$
$$\left. -\frac{d_2}{\Gamma(\rho+1)}\int_0^{\gamma}(\gamma-z)^{\rho}\varphi(z,\phi(z))dz - \frac{1}{\Gamma(\rho)}\int_0^{\beta}(\beta-z)^{\rho-1}\varphi(z,\phi(z))dz\right]$$

Step 1. $\mathcal{P}_1\phi_1+\mathcal{P}_2\phi_2 \in B_r$ for all $\phi_1,\phi_2 \in B_r$: Let us consider

$$||\mathcal{P}_1\phi_1+\mathcal{P}_2\phi_2|| \leq \sup_{x\in[0,1]}\left\{\frac{1}{\Gamma(\rho)}\int_0^{x}(x-z)^{\rho-1}|\varphi(z,\phi_1(z))|dz\right.$$
$$+\frac{x^{m-1}}{|C|}\left[\frac{|d_1|}{\Gamma(\rho+1)}\int_0^{\alpha}(\alpha-z)^{\rho}|\varphi(z,\phi_2(z))|dz\right.$$
$$+\frac{1}{\Gamma(\rho)}\int_0^{\beta}(\beta-z)^{\rho-1}|\varphi(z,\phi_2(z))|dz$$
$$+\frac{|d_2|}{\Gamma(\rho+1)}\int_0^{\gamma}[(1-z)^{\rho}-(\gamma-z)^{\rho}]\,|\varphi(z,\phi_2(z))|dz$$
$$\left.\left.+\int_{\gamma}^{1}(1-z)^{\rho}|\varphi(z,\phi_2(z))|dz\right]\right\}.$$

By using condition (A_2), we obtain

$$||\mathcal{P}_1\phi_1+\mathcal{P}_2\phi_2|| \leq \sup_{x\in[0,1]}\left\{\frac{1}{\Gamma(\rho)}\int_0^{x}(x-z)^{\rho-1}f(z)dz\right.$$
$$+\frac{x^{m-1}}{|C|}\left[\frac{|d_1|}{\Gamma(\rho+1)}\int_0^{\alpha}(\alpha-z)^{\rho}f(z)dz+\frac{1}{\Gamma(\rho)}\int_0^{\beta}(\beta-z)^{\rho-1}f(z)dz\right.$$
$$\left.\left.+\frac{|d_2|}{\Gamma(\rho+1)}\int_0^{\gamma}[(1-z)^{\rho}-(\gamma-z)^{\rho}]f(z)dz+\int_{\gamma}^{1}(1-z)^{\rho}f(z)dz\right]\right\}$$

which yields

$$||\mathcal{P}_1\phi_1+\mathcal{P}_2\phi_2|| \leq ||f||\left\{\frac{x^{m-1}}{|C|}\left[\frac{|d_1|}{\Gamma(\rho+1)}\int_0^{\alpha}(\alpha-z)^{\rho}dz+\frac{1}{\Gamma(\rho)}\int_0^{\beta}(\beta-z)^{\rho-1}dz\right.\right.$$
$$\left.+\frac{|d_2|}{\Gamma(\rho+1)}\int_0^{\gamma}[(1-z)^{\rho}-(\gamma-z)^{\rho}]dz+\int_{\gamma}^{1}(1-z)^{\rho}dz\right]$$
$$\left.+\frac{1}{\Gamma(\rho)}\int_0^{x}(x-z)^{\rho-1}dz\right\} \leq ||f||Z \leq r.$$

Step 2. $\mathcal{P}_2$ is contraction: For all $\phi_1, \phi_2 \in B_r$ and for each $x \in [0,1]$, we have

$$\begin{aligned}||\mathcal{P}_2\phi_1 - \mathcal{P}_2\phi_2|| \leq \sup_{x\in[0,1]} \Bigg\{ \frac{x^{m-1}}{|C|} \Bigg[& \frac{|d_1|}{\Gamma(\rho+1)} \int_0^{\alpha} (\alpha - z)^{\rho} |\varphi(z,\phi_1(z)) - \varphi(z,\phi_2(z))| dz \\ & + \frac{1}{\Gamma(\rho)} \int_0^{\beta} (\beta - z)^{\rho-1} |\varphi(z,\phi_1(z)) - \varphi(z,\phi_2(z))| dz \\ & + \frac{|d_2|}{\Gamma(\rho+1)} \int_0^{\gamma} [(1-z)^{\rho} - (\gamma - z)^{\rho}] |\varphi(z,\phi_1(z)) - \varphi(z,\phi_2(z))| dz \\ & + \frac{|d_2|}{\Gamma(\rho+1)} \int_{\gamma}^{1} (1-z)^{\rho} |\varphi(z,\phi_1(z)) - \varphi(z,\phi_2(z))| dz \Bigg] \Bigg\}\end{aligned}$$

Lipschitz condition (A_1) leads to

$$\begin{aligned}||\mathcal{P}_2\phi_1 - \mathcal{P}_2\phi_2|| \leq \sup_{x\in[0,1]} \Bigg\{ \frac{Lx^{m-1}}{|C|} \Bigg[& \frac{|d_1|}{\Gamma(\rho+1)} \int_0^{\alpha} (\alpha - z)^{\rho} |\phi_1(z) - \phi_2(z)| dz \\ & + \frac{1}{\Gamma(\rho)} \int_0^{\beta} (\beta - z)^{\rho-1} |\phi_1(z) - \phi_2(z)| dz \\ & + \frac{|d_2|}{\Gamma(\rho+1)} \int_0^{\gamma} [(1-z)^{\rho} - (\gamma - z)^{\rho}] |\phi_1(z) - \phi_2(z)| dz \\ & + \frac{|d_2|}{\Gamma(\rho+1)} \int_{\gamma}^{1} (1-z)^{\rho} |\phi_1(z) - \phi_2(z)| dz \Bigg] \Bigg\}\end{aligned}$$

Taking supremum of the both sides for $x \in [0,1]$ implies

$$||\mathcal{P}_2\phi_1 - \mathcal{P}_2\phi_2|| \leq ||\phi_1 - \phi_2|| \frac{L}{|C|} \left[\frac{|d_1|}{\Gamma(\rho+2)} \alpha^{\rho+1} + \frac{1}{\Gamma(\rho+1)} \beta^{\rho} + \frac{|d_2|}{\Gamma(\rho+2)} (1 - \gamma^{\rho+1}) \right].$$

From condition (3.1.7), we can conclude that $\mathcal{P}_2$ is contraction.

Step 3. $\mathcal{P}_1$ is relatively compact:

(i) $\mathcal{P}_1$ is continuous: Since φ is continuous, $\mathcal{P}_1$ is also continuous.

(ii) $\mathcal{P}_1$ is uniformly bounded: Let $\mathcal{B}_r \subset H$ be a bounded set. In order to find uniform boundedness of $\mathcal{P}_1$, we will analyze the following:

$$||(\mathcal{P}_1\phi)(x)|| \leq \sup_{x\in[0,1]} \frac{1}{\Gamma(\rho)} \int_0^x (x-s)^{\rho-1} |\varphi(s,\phi(s))| ds.$$

Using condition (A_2) results in $||(\mathcal{P}_1\phi)(x)|| \leq \dfrac{||f||}{\Gamma(\rho+1)}$.

(iii) $\mathcal{P}_1$ is equicontinuous: We will show that $(\mathcal{P}_1\phi)(x_2) \to (\mathcal{P}_1\phi)(x_1)$ provided $x_1 \to x_2$ for all $x_1, x_2 \in [0,1], x_1 < x_2$. For this purpose let $\sup\limits_{(x,\phi)\in[0,1]\times B_r} |\varphi(x,\phi(x))| = \varphi_n$. Then

$$\begin{aligned}|(\mathcal{P}_1\phi)(x_2) - (\mathcal{P}_1\phi)(x_1)| &\leq \int_0^{x_2} \frac{(x_2-v)^{\rho-1}|\varphi(v,\phi(v))|}{\Gamma(\rho)}dv - \int_0^{x_1} \frac{(x_1-v)^{\rho-1}|\varphi(v,\phi(v))|}{\Gamma(\rho)} \\ &\leq \int_0^{x_1} \frac{\left[(x_2-v)^{\rho-1} - (x_1-v)^{\rho-1}\right]|\varphi(v,\phi(v))|}{\Gamma(\rho)} \\ &+ \int_{x_1}^{x_2} \frac{(x_2-v)^{\rho-1}|\varphi(v,\phi(v))|}{\Gamma(\rho)}dv.\end{aligned}$$

By taking supremum of the both sides for $(x,\phi) \in [0,1] \times B_r$, we obtain

$$\begin{aligned}||(\mathcal{P}_1\phi)(x_2) - (\mathcal{P}_1\phi)(x_1)|| \leq \frac{\varphi_n}{\Gamma(\rho)} \Bigg(&\int_0^{x_1} \left[(x_2-v)^{\rho-1} - (x_1-v)^{\rho-1}\right] dv \\ &+ \int_{x_1}^{x_2} (x_2-v)^{\rho-1} dv \Bigg) = \frac{\varphi_n}{\Gamma(\rho+1)}(x_2^{\rho} - x_1^{\rho})\end{aligned}$$

approaching zero independent of x as $x_1 \to x_2$.

Since the parts (ii)-(iii) satisfy the hypotheses of Arzela Ascoli Theorem 2.0.4, the operator $\mathcal{P}_1 : B_r \to B_r$ is relatively compact and so, from Theorem 2.0.2, it is completely continuous. Thus the boundary value problem (3.0.1)-(3.0.2) does have a solution.

□

Theorem 3.1.3. *(Ahmad and Agarwal, 2014) Suppose that for $x \in [0,1], \phi \in H$, the function φ is bounded, i.e $|\varphi(x,\phi)| \leq L_1$, where $L_1 > 0$. In that case the boundary value problem (3.0.1)-(3.0.2) does have a solution.*

Proof. So as to show the presence of solution, Theorem 2.0.3 will be used. First our aim is to prove that $\mathcal{P}$ is completely continuous:

(i) $\mathcal{P}$ is continuous: Since φ is continuous, $\mathcal{P}_1$ is also continuous.

(ii) $\mathcal{P}$ is uniformly bounded: Suppose that $\mathcal{B} \subset H$ is a bounded set and $\phi \in \mathcal{B}$. In order to show uniform boundedness of $\mathcal{P}$, we will analyze the following:

$$|\mathcal{P}\phi| \leq \frac{x^{m-1}}{|C|}\left[\frac{|d_1|}{\Gamma(\rho+1)}\int_0^{\alpha}(\alpha-z)^{\rho}|\varphi(z,\phi(z))|dz + \frac{1}{\Gamma(\rho)}\int_0^{\beta}(\beta-z)^{\rho-1}|\varphi(z,\phi(z))|dz\right.$$
$$+\frac{|d_2|}{\Gamma(\rho+1)}\left|\int_0^{\gamma}(1-z)^{\rho}\varphi(z,\phi(z))dz + \int_{\gamma}^{1}(1-z)^{\rho}\varphi(z,\phi(z))dz\right.$$
$$\left.\left.-\int_0^{\gamma}(\gamma-z)^{\rho}\varphi(z,\phi(z))dz\right|\right]$$
$$+\frac{1}{\Gamma(\rho)}\int_0^{x}(x-z)^{\rho-1}|\varphi(z,\phi(z))|dz.$$

Using condition $|\varphi(x,\phi)| \leq L_1$, and applying the same procedure followed in Step 1 of the proof of Theorem 3.1.1 lead to

$$|\mathcal{P}\phi| \leq L_1 Z = L_2.$$

(iii) $\mathcal{P}$ is equicontinuous: We will show that f$(\mathcal{P}_1\phi)(x_2) \to (\mathcal{P}_1\phi)(x_1)$ provided $x_1 \to x_2$ for all $x_1, x_2 \in [0,1], x_1 < x_2$. For this purpose let us consider

$$|(\mathcal{P}\phi)'(x)| \leq \frac{(m-1)x^{m-2}}{|C|}\left[\frac{|d_1|}{\Gamma(\rho+1)}\int_0^{\alpha}(\alpha-z)^{\rho}|\varphi(z,\phi(z))|dz\right.$$
$$+\frac{1}{\Gamma(\rho)}\int_0^{\beta}(\beta-z)^{\rho-1}|\varphi(z,\phi(z))|dz$$
$$+\frac{|d_2|}{\Gamma(\rho+1)}\left|\int_0^{\gamma}(1-z)^{\rho}\varphi(z,\phi(z))dz + \int_{\gamma}^{1}(1-z)^{\rho}\varphi(z,\phi(z))dz\right.$$
$$\left.\left.-\int_0^{\gamma}(\gamma-z)^{\rho}\varphi(z,\phi(z))dz\right|\right]$$
$$+\frac{1}{\Gamma(\rho-1)}\int_0^{x}(x-z)^{\rho-2}|\varphi(z,\phi(z))|dz.$$

Using condition $|\varphi(x,\phi)| \leq L_1$, and applying the same procedure followed in Step 1 of the proof of Theorem 3.1.1 lead to

$$|(\mathcal{P}\phi)'(x)| \leq L_1 Z = L_3.$$

This follows that for all $x_1, x_2 \in [0,1], x_1 < x_2$, one can show that

$$|(\mathcal{P}\phi)(x_2) - (\mathcal{P}\phi)(x_1)| \leq \int_{x_1}^{x_2}|(\mathcal{P}\phi)'(x)|dx \leq L_3(x_2 - x_1),$$

implying $\mathcal{P}$ is equicontinuous.

Since the parts (ii)-(iii) satisfy the hypotheses of Arzela Ascoli Theorem 2.0.4, the operator $\mathcal{P} : B \to B$ is relatively compact and so, from Part (i) and Definition 2.0.16 it is completely continuous.

To use Leray-Schauder alternative 2.0.3, let use define the set

$$W = \{\phi \in H : \phi = \sigma \mathcal{P}\phi, 0 < \sigma < 1\}$$

and show the boundedness of it. Let $\phi \in W$. Then one may obtain that

$$\begin{aligned}|\phi(x)| = \sigma|(\mathcal{P}\phi)(x)| \leq \frac{x^{m-1}}{|C|}\Bigg[&\frac{|d_1|}{\Gamma(\rho+1)}\int_0^{\alpha}(\alpha-z)^{\rho}|\varphi(z,\phi(z))|dz \\ &+\frac{1}{\Gamma(\rho)}\int_0^{\beta}(\beta-z)^{\rho-1}|\varphi(z,\phi(z))|dz \\ &+\frac{|d_2|}{\Gamma(\rho+1)}\left|\int_0^{\gamma}(1-z)^{\rho}\varphi(z,\phi(z))dz+\int_{\gamma}^{1}(1-z)^{\rho}\varphi(z,\phi(z))dz-\int_0^{\gamma}(\gamma-z)^{\rho}\varphi(z,\phi(z))dz\right|\Bigg] \\ &+\frac{1}{\Gamma(\rho)}\int_0^{x}(x-z)^{\rho-1}|\varphi(z,\phi(z))|dz \leq L_1 Z = M_1, \quad x\in[0,1].\end{aligned}$$

For any $0 \leq x \leq 1$, since $|\phi(x)| \leq M_1$, the set W is bounded. Since the hypotheses of Theorem 2.0.3 holds, the boundary value problem (3.0.1)-(3.0.2) does have a solution. □

Theorem 3.1.4. *(Ahmad and Agarwal, 2014) Consider the following assumptions:*

A3) For the functions $v \in C([0,1],\mathbb{R}^+)$ and $q : \mathbb{R}^+ \to \mathbb{R}^+, q'(x) \geq 0$, the following holds

$$|\varphi(x,\phi(x))| \leq v(x)q(||\phi||)$$

for all $(x,\phi) \in [0,1]\times\mathbb{R}$;

A4) For the constant $D > 0$, we have $\frac{D}{Zq(D)||v||} > 1$.

In that case the boundary value problem (3.0.1)-(3.0.2) does have a solution.

Proof. So as to prove the existence of solution, Theorem 2.0.6 will be used. For convenience we can define the bounded set $\mathcal{B}_\eta = \{\phi \in C([0,1],\mathbb{R}) : ||\phi|| \leq \eta, \eta > 0\}$. First we will prove that $\mathcal{P} : H \to H$ is completely continuous:

(i) $\mathcal{P}$ is continuous: Since φ is continuous, $\mathcal{P}_1$ is also continuous.

(ii) The image of bounded sets of $C([0,1],\mathbb{R})$ are bounded sets in $C([0,1],\mathbb{R})$:

$$\begin{aligned}||(\mathcal{P}\phi)(x)|| \leq \sup_{x\in[0,1]}\Bigg\{\frac{x^{m-1}}{|C|}\Bigg[&\frac{|d_1|}{\Gamma(\rho+1)}\int_0^{\alpha}(\alpha-z)^{\rho}|\varphi(z,\phi(z))|dz \\ &+\frac{1}{\Gamma(\rho)}\int_0^{\beta}(\beta-z)^{\rho-1}|\varphi(z,\phi(z))|dz \\ &+\frac{|d_2|}{\Gamma(\rho+1)}\left\{\int_0^{\gamma}[(1-z)^{\rho}-(\gamma-z)^{\rho}]\,|\varphi(z,\phi(z))|\,dz+\int_{\gamma}^{1}(1-v)^{\rho}\,|\varphi(v,\phi(v))|\,dv\right\} \\ &+\frac{1}{\Gamma(\rho)}\int_0^{x}(x-z)^{\rho-1}|\varphi(z,\phi(z))|dz\Bigg\}.\end{aligned}$$

Using condition $|\varphi(x,\phi(x))| \leq v(x)q(||\phi||)$ and applying the same procedure followed in Step 1 of the proof of Theorem 3.1.1 result in

$$\begin{aligned}||(\mathcal{P}\phi)(x)|| \leq \sup_{x\in[0,1]} \Bigg\{ \frac{x^{m-1}}{|C|} \Bigg[\frac{|d_1|}{\Gamma(\rho+1)} \int_0^{\alpha} (\alpha-z)^{\rho} v(x)q(||\phi||)dz \\ + \frac{1}{\Gamma(\rho)} \int_0^{\beta} (\beta-z)^{\rho-1} v(x)q(||\phi||)dz \\ + \frac{|d_2|}{\Gamma(\rho+1)} \Bigg\{ \int_0^{\gamma} [(1-z)^{\rho} - (\gamma-z)^{\rho}] v(x)q(||\phi||)dz + \int_{\gamma}^{1} (1-z)^{\rho} v(x)q(||\phi||)dz \Bigg\} \\ + \frac{1}{\Gamma(\rho)} \int_0^{x} (x-z)^{\rho-1} v(x)q(||\phi||)dz \Bigg\}.\end{aligned}$$

Condition (A_4) imples that

$$\begin{aligned}||(\mathcal{P}\phi)(x)|| \leq ||v||q(D) \Bigg\{ \frac{x^{m-1}}{|C|} \Bigg[\frac{|d_1|}{\Gamma(\rho+1)} \int_0^{\alpha} (\alpha-z)^{\rho} dz \\ + \frac{1}{\Gamma(\rho)} \int_0^{\beta} (\beta-z)^{\rho-1} dz + \frac{|d_2|}{\Gamma(\rho+1)} (1-\gamma^{\rho+1}) \Bigg] \\ + \frac{1}{\Gamma(\rho)} \int_0^{x} (x-z)^{\rho-1} dz \Bigg\} \leq ||v||q(D)Z < D.\end{aligned}$$

Therefore $||(\mathcal{P}\phi)(x)|| \leq D$.

(iii) The image of bounded sets of $C([0,1],\mathbb{R})$ are equicontinuous sets of $C([0,1],\mathbb{R})$: Choosing $0 \leq x_1 < x_2 \leq 1$ implies that

$$\begin{aligned}|(\mathcal{P}\phi)(x_1) - (\mathcal{P}\phi)(x_2)| \leq \frac{x_2^{m-1} - x_1^{m-1}}{|C|} \Bigg\{ \frac{|d_1|}{\Gamma(\rho+1)} \int_0^{\alpha} (\alpha-v)^{\rho} |\varphi(v,\phi(v))| dv \\ + \frac{1}{\Gamma(\rho)} \int_0^{\beta} (\beta-v)^{\rho-1} |\varphi(v,\phi(v))| dv \\ + \frac{|d_2|}{\Gamma(\rho+1)} \Bigg[\int_0^{\gamma} [(1-v)^{\rho} - (\gamma-v)^{\rho}] |\varphi(v,\phi(v))| dv \\ + \int_{\gamma}^{1} (1-v)^{\rho} |\varphi(v,\phi(v))| dv \Bigg] \Bigg\} \\ + \frac{1}{\Gamma(\rho)} \Bigg[\int_0^{x_2} (x_2-v)^{\rho-1} |\varphi(v,\phi(v))| dv - \int_0^{x_1} (x_1-v)^{\rho-1} |\varphi(v,\phi(v))| dv \Bigg]\end{aligned}$$

Using condition $|\varphi(x,\phi(x))| \leq v(x)q(||\phi||)$ and assumption (A_1) and applying the same procedure followed in Step 1 of the proof of Theorem 3.1.1 result in

$$|(\mathcal{P}\phi)(x_1) - (\mathcal{P}\phi)(x_2)| \leq ||v||q(\eta) \left[Z \frac{x_2^{m-1} - x_1^{m-1}}{|C|} + \frac{x_2^{\rho} - x_1^{\rho}}{\Gamma(\rho+1)} \right].$$

When $x_2 \to x_1$,

$$|(\mathcal{P}\phi)(x_1) - (\mathcal{P}\phi)(x_2)| \to 0$$

not dependently of $\phi \in B_\eta$.

Since the parts (ii)-(iii) satisfy the hypotheses of Arzela Ascoli Theorem 2.0.4, the operator $\mathcal{P} : H \to H$ is relatively compact and so, from Part (i) and Definition 2.0.16 it is completely continuous.

Now we will show that the second conclusion of Theorem 2.0.6 can not be satisfied. For this purpose let us take the set $\phi \in W$ as in the proof of Theorem 3.1.3. Then employing the similar arguments in the proof that $\mathcal{P}$ is bounded, for any $x \in [0,1]$ it can be obtained that

$$\begin{aligned}||\phi(x)|| = \sigma||(\mathcal{P}\phi)(x)|| \leq \sup_{x\in[0,1]} \Bigg\{ \frac{x^{m-1}}{|C|} \Bigg[\frac{|d_1|}{\Gamma(\rho+1)} \int_0^\alpha (\alpha - v)^\rho |\varphi(v,\phi(v))| dv \\ + \frac{1}{\Gamma(\rho)} \int_0^\beta (\beta - v)^{\rho-1} |\varphi(v,\phi(v))| dv \\ + \frac{|d_2|}{\Gamma(\rho+1)} \Bigg[\int_0^\gamma [(1-v)^\rho - (\gamma - v)^\rho] |\varphi(v,\phi(v))| dv \\ + \int_\gamma^1 (1-v)^\rho |\varphi(v,\phi(v))| dv \Bigg] \Bigg\} \\ + \frac{1}{\Gamma(\rho)} \int_0^x (x-v)^{\rho-1} |\varphi(v,\phi(v))| dv \Bigg\}.\end{aligned}$$

Using condition $|\varphi(x,\phi(x))| \leq v(x)q(||\phi||)$ and applying the same procedure followed in Step 1 of the proof of Theorem 3.1.1 result in

$$\begin{aligned}||\phi(x)|| \leq \sup_{x\in[0,1]} \Bigg\{ \frac{x^{m-1}}{|C|} \Bigg[\frac{|d_1|}{\Gamma(\rho+1)} \int_0^\alpha (\alpha - z)^\rho v(x) q(||\phi||) dz \\ + \frac{1}{\Gamma(\rho)} \int_0^\beta (\beta - z)^{\rho-1} v(x) q(||\phi||) dz \\ + \frac{|d_2|}{\Gamma(\rho+1)} \Bigg[\int_0^\gamma [(1-z)^\rho - (\gamma - z)^\rho] v(x) q(||\phi||) dz \\ + \int_\gamma^1 (1-z)^\rho v(x) q(||\phi||) dz \Bigg] \Bigg\} \\ + \frac{1}{\Gamma(\rho)} \int_0^x (x-z)^{\rho-1} v(x) q(||\phi||) dz \Bigg\}.\end{aligned}$$

Besides, condition (A_4) yields that

$$||\phi(x)|| \leq ||v||q(||\phi||)\left\{\frac{x^{m-1}}{|C|}\left[\frac{|d_1|}{\Gamma(\rho+1)}\int_0^{\alpha}(\alpha-z)^{\rho}dz + \frac{1}{\Gamma(\rho)}\int_0^{\beta}(\beta-z)^{\rho-1}dz + \frac{|d_2|}{\Gamma(\rho+1)}(1-\gamma^{\rho+1})dz\right] + \frac{1}{\Gamma(\rho)}\int_0^{x}(x-z)^{\rho-1}dz\right\} \leq ||v||q(||\phi||)Z$$

or

$$\frac{||\phi||}{||v||q(||\phi||)Z} \leq 1.$$

Therefore from assumption (A_4), there is a constant E for which $||\phi|| \neq E$. Now we set the set $U = \{\phi \in C([0,1],\mathbb{R}) : ||\phi|| < E+1\}$. It can be shown as above that $\mathcal{P} : \overline{U} \to C([0,1],\mathbb{R})$ is continuous and completely continuous. Choosing U in this form yields that there does not exist $\phi \in \partial U$ satisfying $\phi = \sigma\mathcal{P}(\phi)$ for $0 < \sigma < 1$. Hence we can conclude from Theorem 2.0.6 that there exist a fixed point $\phi \in \overline{U}$ of $\mathcal{P}$. Hence the boundary value problem (3.0.1)-(3.0.2) does have a solution $\phi \in \overline{U}$. □

Example 3.1.2. *(Ahmad and Agarwal, 2014) Examine the boundary value problem with fractional derivative*

$$\begin{aligned} {}_0^CD^{9/2}\phi(x) &= \frac{(x+2)(\phi+2)^2}{1+(\phi+2)^2} + (x+1)\sin\phi,\ x \in [0,1], \\ \phi(0) = 0,\ \phi'(0) &= 0, \dots,\ \phi''(0) = 0, \dots,\ \phi'''(0) = 0, \\ \phi(1/2) &= d_1\int_0^{1/3}\phi(w)dw + d_2\int_{2/3}^{1}\phi(w)dw, \end{aligned} \tag{3.1.8}$$

where $m = 5, \rho = 9/2, d_1 = 1, d_2 = 1, \alpha = 1/3, \beta = 1/2, \gamma = 2/3$ *and*

$$\varphi(x,\phi(x)) = \frac{(x+2)(\phi+2)^2}{1+(\phi+2)^2} + (x+1)\sin\phi.$$

Moreover

$$\begin{aligned} |\varphi(x,\phi(x))| &= \left|\frac{(x+2)(\phi+2)^2}{1+(\phi+2)^2} + (x+1)\sin\phi\right| \leq (x+2)\left|\frac{(\phi+2)^2}{1+(\phi+2)^2}\right| + (x+1)\,|\sin\phi| \\ &\leq (x+2)\left|\frac{(\phi+2)^2}{1+(\phi+2)^2}\right| + (x+1) \leq (x+2)\left[\left|\frac{(\phi+2)^2}{1+(\phi+2)^2}\right| + 1\right] \leq v(x)q(||\phi||), \end{aligned}$$

where $v(x) = x+2,\ q(||\phi||) = 1 + ||\phi||,\ ||\phi|| = \left|\frac{(\phi+2)^2}{1+(\phi+2)^2}\right|$

Since

$$Z = \frac{1}{|C|\Gamma(\rho+2)}\left[|d_1|\alpha^{\rho+1} + |d_2|(1-\gamma^{\rho+1}) + (\rho+1)(\beta^{\rho} + |C|)\right] = 0.0544011.$$

From assumption (A_4) *we have* $\frac{D}{Zq(D)||v||} > 1$ *which implies* $D > 0.195033$. *Since all the hypotheses of Theorem 3.1.4 hold, (3.1.8) does have a solution.*

3.2. Nonlocal multi-point case on the aperture

Let us consider differential equation with Caputo derivative of order $\rho \in (m-1, m]$, $m \geq 2$

$$ {}_0^C D^{\rho}\phi(x) = \varphi(x, \phi(x)),\ x \in [0,1], \tag{3.2.1}$$

subject to the following boundary conditions

$$\begin{aligned} &\phi(0) = 0,\ \phi'(0) = 0, \ldots,\ \phi^{(m-2)}(0) = 0, \\ &\sum_{j=1}^{n} \lambda_j \phi(\beta_j) = d_1 \int_0^{\alpha} \phi(w)dw + d_2 \int_{\gamma}^{1} \phi(w)dw, \quad d_1, d_2, \lambda_j \in \mathbb{R}, \end{aligned} \tag{3.2.2}$$

where $0 < \alpha < \beta_1 < \beta_2 < \ldots < \beta_n < \gamma < 1$. By using Theorem 3.0.1, the general solution of (3.2.1)-(3.2.2) can be obtained as

$$\begin{aligned} \phi(x) = \frac{x^{m-1}}{C_n}\Bigg[&\frac{d_1}{\Gamma(\rho+1)}\int_0^{\alpha}(\alpha-s)^{\rho}\varphi(s,\phi(s))ds + \frac{d_2}{\Gamma(\rho+1)}\int_0^{1}(1-s)^{\rho}\varphi(s,\phi(s))ds \\ &-\frac{d_2}{\Gamma(\rho+1)}\int_0^{\gamma}(\gamma-s)^{\rho}\varphi(s,\phi(s))ds - \frac{1}{\Gamma(\rho)}\sum_{j=1}^{n}\int_0^{\beta_j}\lambda_j(\beta_j-s)^{\rho-1}\varphi(s,\phi(s))ds\Bigg] \\ &+\frac{1}{\Gamma(\rho)}\int_0^{x}(x-s)^{\rho-1}\varphi(s,\phi(s))ds, \end{aligned}$$

where $C_n = \sum_{j=1}^{n}\lambda_j\beta_j^{m-1} - \frac{1}{m}(d_1\alpha^m + d_2 - d_2\gamma^m) \neq 0$. Then definition of Z presented in (3.1.2) turns into

$$Z_n = \frac{1}{|C_n|\Gamma(\rho+2)}\left[|d_1|\alpha^{\rho+1} + |d_2|(1-\gamma^{\rho+1}) + (\rho+1)\left(\sum_{j=1}^{n}\lambda_j\beta_j^{\rho} + |C_n|\right)\right].$$

All the results established in Section 3.1 can be obtained for the fractional boundary value problem (3.2.1)-(3.2.2) by defining the following operator which is the counterpart of operator defined in (3.1.1) as

$$(\mathcal{P}_n\phi)(x) = \frac{x^{m-1}}{C_n}\left[\frac{d_1}{\Gamma(\rho+1)}\int_0^{\alpha}(\alpha-s)^{\rho}\varphi(s,\phi(s))ds + \frac{d_2}{\Gamma(\rho+1)}\int_0^{1}(1-s)^{\rho}\varphi(s,\phi(s))ds \right.$$
$$\left. - \frac{d_2}{\Gamma(\rho+1)}\int_0^{\gamma}(\gamma-s)^{\rho}\varphi(s,\phi(s))ds - \frac{1}{\Gamma(\rho)}\sum_{j=1}^{n}\int_0^{\beta_j}\lambda_j(\beta_j-s)^{\rho-1}\varphi(s,\phi(s))ds\right]$$
$$+ \frac{1}{\Gamma(\rho)}\int_0^{x}(x-s)^{\rho-1}\varphi(s,\phi(s))ds.$$

3.3. Nonlocal multi-substrips case

Let us consider differential equation with Caputo derivative of order $\rho \in (m-1,m]$, $m \geq 2$

$$ {}_0^C D^{\rho}\phi(x) = \varphi(x,\phi(x)),\ x \in [0,1], \tag{3.3.1}$$

subject to the following boundary conditions

$$\begin{aligned} &\phi(0) = 0,\ \phi'(0) = 0, \dots,\ \phi^{(m-2)}(0) = 0, \\ &\phi(\beta) = \sum_{j=1}^{n} a_j \int_{\alpha_j}^{\delta_j}\phi(w)dw + \sum_{j=1}^{r} b_j \int_{\gamma_j}^{\theta_j}\phi(w)dw, \quad a_j, b_j \in \mathbb{R}, \end{aligned} \tag{3.3.2}$$

where $0 = \alpha_1 < \delta_1 < \alpha_2 < \delta_2 < \ldots < \alpha_n < \delta_n = \alpha < \beta < \gamma_1 = \gamma < \theta_1 < \gamma_2 < \theta_2 < \ldots < \gamma_r < \theta_r = 1$. By using Theorem 3.0.1, one can obtain the general solution of (3.3.1)-(3.3.2) as

$$\phi(x) = \frac{x^{m-1}}{C_{nr}}\left[\sum_{j=1}^{n}\frac{a_j}{\Gamma(\rho)}\int_{\alpha_j}^{\delta_j}\int_0^{w}(w-s)^{\rho-1}\varphi(s,\phi(s))dsdw \right.$$
$$+ \sum_{j=1}^{r}\frac{b_j}{\Gamma(\rho)}\int_{\gamma_j}^{\theta_j}\int_0^{w}(w-s)^{\rho-1}\varphi(s,\phi(s))dsdw$$
$$\left. - \frac{1}{\Gamma(\rho)}\int_0^{\beta}(\beta-s)^{\rho-1}\varphi(s,\phi(s))ds\right] + \frac{1}{\Gamma(\rho)}\int_0^{x}(x-s)^{\rho-1}\varphi(s,\phi(s))ds,$$

where $C_{nr} = \beta^{m-1} - \sum_{j=1}^{n}\frac{a_j}{m}\left(\delta_j^m - \alpha_j^m\right) - \sum_{j=1}^{r}\frac{b_j}{m}\left(\theta_j^m - \gamma_j^m\right) \neq 0$. Then definition of Z presented in (3.1.2) turns into

$$Z_{nr} = \frac{1}{|C_{nr}|\Gamma(\rho+2)}\left[\sum_{j=1}^{n}|a_j|\left(\delta_j^{\rho+1} - \alpha_j^{\rho+1}\right) + \sum_{j=1}^{r}|b_j|\left(\theta_j^{\rho+1} - \gamma_j^{\rho+1}\right) + \left(\beta^{\rho} + |C_{nr}|\right)(\rho+1)\right].$$

All the results established in Section 3.1 can be obtained for the fractional boundary value problem (3.3.1)-(3.3.2) by defining the following operator which is the counterpart of operator defined in (3.1.1) as

$$(\mathcal{P}_{nr}\phi)(x) = \frac{x^{m-1}}{C_{nr}}\left[\sum_{j=1}^{n}\frac{a_j}{\Gamma(\rho)}\int_{\alpha_j}^{\delta_j}\int_0^w (w-s)^{\rho-1}\varphi(s,\phi(s))dsdw\right.$$
$$+\sum_{j=1}^{r}\frac{b_j}{\Gamma(\rho)}\int_{\gamma_j}^{\theta_j}\int_0^w (w-s)^{\rho-1}\varphi(s,\phi(s))dsdw$$
$$\left.-\frac{1}{\Gamma(\rho)}\int_0^{\beta}(\beta-s)^{\rho-1}\varphi(s,\phi(s))ds\right]+\frac{1}{\Gamma(\rho)}\int_0^x (x-s)^{\rho-1}\varphi(s,\phi(s))ds.$$

3.4. Riemann-Liouville slit-strips boundary conditions

Let us consider differential equation with Caputo derivative of order $\rho \in (m-1,m]$, $m \geq 2$

$$ {}_0^C D^{\rho}\phi(x) = \varphi(x,\phi(x)),\ x \in [0,1], \tag{3.4.1}$$

subject to the following boundary conditions

$$\phi(0)=0,\ \phi'(0)=0,\ldots,\ \phi^{(m-2)}(0)=0,$$
$$\phi(\beta) = d_1\int_0^{\alpha}\frac{(\alpha-w)^{\mu-1}}{\Gamma(\mu)}\phi(w)dw + d_2\int_{\gamma}^{1}\frac{(1-w)^{\mu-1}}{\Gamma(\mu)}\phi(w)dw, \quad 0<\alpha<\beta<\gamma<1,\ \mu>0. \tag{3.4.2}$$

By using Theorem 3.0.1, one can obtain the general solution of (3.4.1)-(3.4.2) as

$$\phi(x) = \frac{x^{m-1}}{C_{RL}}\left[\frac{d_1\Gamma(\rho)}{\Gamma(\rho+\mu)}\int_0^{\alpha}(\alpha-s)^{\rho+\mu+1}\varphi(s,\phi(s))ds\right.$$
$$\left.+\frac{d_2}{\Gamma(\mu)}\int_{\gamma}^{1}\int_0^w (1-w)^{\mu-1}(w-s)^{\rho-1}\varphi(s,\phi(s))dsdw - \frac{1}{\Gamma(\rho)}\int_0^{\beta}(\beta-s)^{\rho-1}\varphi(s,\phi(s))ds\right]$$
$$+\frac{1}{\Gamma(\rho)}\int_0^x (x-s)^{\rho-1}\varphi(s,\phi(s))ds,$$

where $C_{RL} = \beta^{m-1} - \dfrac{d_1\alpha^{\mu+m-1}\Gamma(m)}{\Gamma(\mu+m)} - \dfrac{d_2}{\Gamma(\mu)}\displaystyle\int_{\gamma}^{1}(1-w)^{\mu-1}w^{m-1}dw$. Then definition of Z presented in (3.1.2) turns into

$$Z_{RL} = \frac{1}{|C_{RL}|}\left[\frac{|d_1|\Gamma(\rho)\alpha^{\mu+\rho}}{\Gamma(\rho+\mu+1)} + \frac{|d_2|}{\rho\Gamma(\mu)}\int_{\gamma}^{1}(1-w)^{\mu-1}w^{\rho}dw + \frac{\beta^{\rho}}{\Gamma(\rho+1)} + \frac{|C_{RL}|}{\Gamma(\rho+1)}\right].$$

All the results established in Section 3.1 can be shown for the fractional boundary value problem (3.3.1)-(3.3.2) by defining the following operator which is the counterpart of operator defined in (3.1.1) as

$$
\begin{aligned}
(\mathcal{P}_{RL}\phi)(x) = \frac{x^{m-1}}{C_{RL}}\Bigg[&\frac{d_1\Gamma(\rho)}{\Gamma(\rho+\mu)}\int_0^{\alpha}(\alpha-s)^{\rho+\mu+1}\varphi(s,\phi(s))ds \\
&+\frac{d_2}{\Gamma(\mu)}\int_{\gamma}^{1}\int_0^{w}(1-w)^{\mu-1}(w-s)^{\rho-1}\varphi(s,\phi(s))dsdw-\frac{1}{\Gamma(\rho)}\int_0^{\beta}(\beta-s)^{\rho-1}\varphi(s,\phi(s))ds\Bigg] \\
&+\frac{1}{\Gamma(\rho)}\int_0^{x}(x-s)^{\rho-1}\varphi(s,\phi(s))ds.
\end{aligned}
$$

4. A STUDY OF NONLINEAR FRACTIONAL DIFFERENTIAL EQUATIONS OF ARBITRARY ORDER WITH RIEMANN-LIOUVILLE TYPE MULTISTRIP BOUNDARY CONDITIONS

This chapter is devoted to the results of Ahmad et al. (2013). We are interested in differential equation with Caputo derivative of order $\rho \in (m-1, m]$, $m \geq 2$

$$ {}_{0}^{C}D^{\rho}\phi(x) = \varphi(x, \phi(x)),\ x \in [0, K], \tag{4.0.1}$$

subject to finitely many multistrip integral boundary conditions of the form

$$\begin{aligned} &\phi(0) = 0,\ \phi'(0) = 0, \ldots,\ \phi^{(m-2)}(0) = 0, \\ &\phi(K) = \sum_{j=1}^{k} \gamma_j [I^{\lambda_j}\phi(\delta_j) - I^{\lambda_j}\phi(\theta_j)]. \end{aligned} \tag{4.0.2}$$

Without further mention that

1. $\varphi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function with respect to its both arguments,

2. $\lambda_j > 0$ for $j = 1, 2, \ldots, k$ and I^{λ_j} is the λ_j-th order Riemann Liouville fractional integral,

3. $0 < \theta_1 < \delta_1 < \theta_2 < \ldots < \theta_k < \delta_k < K$ and γ_j are constant numbers for $j = 1, 2, \ldots, k$.

In order to prove the main theorems, the next theorem is the fundamental argument.

Theorem 4.0.1. *(Ahmad et al., 2013) Let $g \in C[0, K]$. Then $\phi(x) \in AC^m[0, K]$ is the unique solution of*

$$ {}_{0}^{C}D^{\rho}\phi(x) = g(x),\ x \in [0, K],\ \rho \in (m-1, m] \tag{4.0.3}$$

$$\phi(0) = 0,\ \phi'(0) = 0, \ldots,\ \phi^{m-2}(0) = 0, \tag{4.0.4}$$

$$\phi(K) = \sum_{j=1}^{k} \gamma_j [I^{\lambda_j}\phi(\delta_j) - I^{\lambda_j}\phi(\theta_j)] \tag{4.0.5}$$

if and only if ϕ satisfies the following:

$$\begin{aligned} \phi(x) = &\frac{1}{\Gamma(\rho)} \int_0^x (x-w)^{\rho-1} g(w)dw - \frac{x^{m-1}}{\Lambda\Gamma(\rho)} \int_0^K (K-w)^{\rho-1} g(w)dw \\ &+ \frac{x^{m-1}}{\Lambda} \sum_{j=1}^{k} \frac{\gamma_j}{\Gamma(\lambda_j+\rho)} \left[\int_0^{\delta_j} (\delta_j - w)^{\lambda_j+\rho-1} g(w)dw - \int_0^{\theta_j} (\theta_j - w)^{\lambda_j+\rho-1} g(w)dw \right] \end{aligned} \tag{4.0.6}$$

where

$$\Lambda = K^{m-1} - \sum_{j=1}^{k} \gamma_j \frac{\left(\delta_j^{\lambda_j+m-1} - \theta_j^{\lambda_j+m-1}\right)\Gamma(m)}{\Gamma(\lambda_j+m)} \neq 0.$$

Proof. By employing Definition 2.0.1, for arbitrary constants $c_0, c_1, \ldots, c_{m-1}$, one can write the general solution of equation (4.0.3) as

$$\phi(x) = \frac{1}{\Gamma(\rho)} \int_0^x (x-w)^{\rho-1} g(w)dw - c_0 - c_1 x - \ldots - c_{m-1}x^{m-1}. \tag{4.0.7}$$

Using the boundary conditions (4.0.4) in equation (4.0.7) implies that $c_0 = c_1 = \ldots = c_{m-2} = 0$. If we apply Riemann Liouville integral operator I^{λ_j} to the both sides of equation (4.0.7), we obtain

$$\begin{aligned} I^{\lambda_j}\phi(x) &= \frac{1}{\Gamma(\lambda_j)} \int_0^x (x-w)^{\lambda_j-1} \left(\frac{1}{\Gamma(\rho)} \int_0^w (w-s)^{\rho-1} g(s)ds - c_{m-1}w^{m-1} \right) dw \\ &= \frac{1}{\Gamma(\rho)} \frac{1}{\Gamma(\lambda_j)} \int_0^x \int_0^w (x-w)^{\lambda_j-1}(w-s)^{\rho-1} g(s)dsdw - \frac{c_{m-1}}{\Gamma(\lambda_j)} \int_0^x (x-w)^{\lambda_j-1} w^{m-1}dw. \end{aligned} \tag{4.0.8}$$

By changing the orders of the integral and by the third property of Beta function represented in Theorem 2.0.8, the double integral turns into

$$\begin{aligned} \int_0^x \int_0^w (x-w)^{\lambda_j-1}(w-s)^{\rho-1} g(s)dsdw &= \int_0^x g(s) \left(\int_s^x (x-w)^{\lambda_j-1}(w-s)^{\rho-1}dw \right) ds \\ &= \int_0^x g(s)(x-s)^{\lambda_j+\rho-1} B(\lambda_j, \rho)ds \\ &= \frac{\Gamma(\rho)\Gamma(\lambda_j)}{\Gamma(\rho+\lambda_j)} \int_0^x g(s)(x-s)^{\lambda_j+\rho-1}ds. \end{aligned}$$

The third property of Beta function represented in Theorem 2.0.8 yields that

$$\int_0^x (x-w)^{\lambda_j-1} w^{m-1}dw = x^{\lambda_j+m-1} B(\lambda_j, m) = x^{\lambda_j+m-1} \frac{\Gamma(m)\Gamma(\lambda_j)}{\Gamma(m+\lambda_j)}.$$

Therefore equation (4.0.8) becomes as

$$I^{\lambda_j}\phi(x) = \frac{1}{\Gamma(\rho+\lambda_j)} \int_0^x g(s)(x-s)^{\lambda_j+\rho-1}ds - \frac{c_{m-1}\Gamma(m)}{\Gamma(\lambda_j+m)} x^{\lambda_j+m-1}. \tag{4.0.9}$$

After evaluating equation (4.0.9) at the points $x = \delta_j$ and $x = \theta_j$ and applying the boundary condition (4.0.5) to equation (4.0.7) lead to

$$\begin{aligned} \phi(K) = \frac{1}{\Gamma(\rho)} \int_0^K (K-w)^{\rho-1} g(w)dw - c_{m-1}K^{m-1} &= c_{m-1} \sum_{j=1}^{k} \frac{\Gamma(m)\gamma_j}{\Gamma(m+\lambda_j)} \left[\theta_j^{m+\lambda_j-1} - \delta_j^{m+\lambda_j-1} \right] \\ &+ \sum_{j=1}^{k} \frac{\gamma_j}{\Gamma(\rho+\lambda_j)} \left[\int_0^{\delta_j} g(s)(\delta_j - s)^{\lambda_j+\rho-1}ds - \int_0^{\theta_j} g(s)(\theta_j - s)^{\lambda_j+\rho-1}ds \right]. \end{aligned}$$

Hence we obtain that

$$c_{m-1} = \frac{1}{\Lambda\Gamma(\rho)} \int_0^K (K-w)^{\rho-1} g(w)dw$$
$$- \sum_{j=1}^{k} \frac{\gamma_j}{\Lambda\Gamma(\rho+\lambda_j)} \left[\int_0^{\delta_j} g(s)(\delta_j - s)^{\lambda_j+\rho-1} ds - \int_0^{\theta_j} g(s)(\theta_j - s)^{\lambda_j+\rho-1} ds \right].$$

Rewriting equation (4.0.7) with the found value of c_{m-1}, the expression of the unique solution (4.0.6) can be obtained. □

4.1. Main Results

Consider a Banach space $H = C([0,K],\mathbb{R})$ containing continuous functions which are defined on the domain $[0,K]\times\mathbb{R}$ on which a uniform convergence topology is defined via the sup norm, $||\phi|| = \sup_{x\in[0,K]} |\phi(x)|$.

Theorem 4.0.1 defines the following operator $\mathcal{P} : H \to H$ as

$$(\mathcal{P}\phi)(x) = \frac{1}{\Gamma(\rho)} \int_0^x (x-z)^{\rho-1} \varphi(z,\phi(z))dz - \frac{x^{m-1}}{\Lambda\Gamma(\rho)} \int_0^K (K-z)^{\rho-1} \varphi(z,\phi(z))dz$$
$$+ \sum_{j=1}^{k} \frac{x^{m-1}\gamma_j}{\Lambda\Gamma(\lambda_j+\rho)} \left[\int_0^{\delta_j} (\delta_j - z)^{\lambda_j+\rho-1} \varphi(z,\phi(z))dz - \int_0^{\theta_j} (\theta_j - z)^{\lambda_j+\rho-1} \varphi(z,\phi(z))dz \right]$$

It is clear that the problem (4.0.1)-(4.0.2) may have a solution provided that the related fixed point equation $\mathcal{P}\phi = \phi$ possesses a solution; that means admitted a fixed point.

Let us define

$$Z = \frac{K^\rho}{\Gamma(\rho+1)} + \frac{K^{\rho+m-1}}{|\Lambda|\Gamma(\rho+1)} + \sum_{j=1}^{k} \frac{K^{m-1}}{\Gamma(\lambda_j+\rho+1)} \left|\frac{\gamma_j}{\Lambda}\right| (\delta_j^{\lambda_j+\rho} - \theta_j^{\lambda_j+\rho}). \tag{4.1.1}$$

Theorem 4.1.1. *(Ahmad et al., 2013) Assume that the continuous real valued function* φ : $[0,K]\times\mathbb{R}\to\mathbb{R}$ *satisfies the following assumption, which is Lipschitz condition:*

$(A_1) : |\varphi(x,\phi_1) - \varphi(x,\phi_2)| \leq L|\phi_1 - \phi_2|$ *for all* $x \in [0,k]$, $L > 0$, $\phi_1,\phi_2 \in \mathbb{R}$.

In that case boundary value problem (4.0.1)-(4.0.2) does have a unique solution provided that

$$L < \frac{1}{Z}. \tag{4.1.2}$$

Proof. We will use Theorem 2.0.1. We first define the set

$$B_r = \{\phi \in H : ||\phi|| \leq r\},$$

where $r \geq \dfrac{MZ}{1-LZ}$, $M = \sup_{k\in[0,K]} |\varphi(x,0)| < \infty$.

Step 1. $\mathcal{P}B_r \subset B_r$: The triangular inequality implies that

$$|\varphi(x,\phi(x))| \leq |\varphi(x,\phi(x)) - \varphi(x,0)| + |\varphi(x,0)| \leq L||\phi|| + M \leq Lr + M,\ \phi \in B_r.$$

Moreover,

$$\begin{aligned}
|\mathcal{P}\phi| \leq & \frac{1}{\Gamma(\rho)}\int_0^x (x-z)^{\rho-1}|\varphi(z,\phi(z))|dz + \frac{K^{m-1}}{|\Lambda|\Gamma(\rho)}\int_0^K (K-z)^{\rho-1}|\varphi(z,\phi(z))|dz \\
& + \sum_{j=1}^{k} \frac{K^{m-1}}{\Gamma(\lambda_j+\rho)}\left|\frac{\gamma_j}{\Lambda}\right| \left|\int_0^{\delta_j}(\delta_j-z)^{\lambda_j+\rho-1}\varphi(z,\phi(z))dz - \int_0^{\theta_j}(\theta_j-z)^{\lambda_j+\rho-1}\varphi(z,\phi(z))dz\right| \\
\leq & \frac{M}{\Gamma(\rho)}\int_0^x (x-z)^{\rho-1}dz + \frac{MK^{m-1}}{|\Lambda|\Gamma(\rho)}\int_0^K (K-z)^{\rho-1}dz \\
& + \sum_{j=1}^{k} \frac{K^{m-1}}{\Gamma(\lambda_j+\rho)}\left|\frac{\gamma_j}{\Lambda}\right| \left|\int_0^{\delta_j}(\delta_j-z)^{\lambda_j+\rho-1}\varphi(z,\phi(z))dz - \int_0^{\theta_j}(\theta_j-z)^{\lambda_j+\rho-1}\varphi(z,\phi(z))dz\right|.
\end{aligned} \tag{4.1.3}$$

Let

$$\begin{aligned}
S &= \sum_{j=1}^{k} \frac{K^{m-1}}{\Gamma(\lambda_j+\rho)}\left|\frac{\gamma_j}{\Lambda}\right| \left|\int_0^{\delta_j}(\delta_j-z)^{\lambda_j+\rho-1}\varphi(z,\phi(z))dz - \int_0^{\theta_j}(\theta_j-z)^{\lambda_j+\rho-1}\varphi(z,\phi(z))dz\right| \\
&= \sum_{j=1}^{k} S_1 S_2,
\end{aligned}$$

where

$$S_1 = \frac{K^{m-1}}{\Gamma(\lambda_j+\rho)}\left|\frac{\gamma_j}{\Lambda}\right|,$$

$$S_2 = \left|\int_0^{\delta_j}(\delta_j-z)^{\lambda_j+\rho-1}\varphi(z,\phi(z))dz - \int_0^{\theta_j}(\theta_j-z)^{\lambda_j+\rho-1}\varphi(z,\phi(z))dz\right|.$$

Since $\delta_j > \theta_j$ for all $j = 1,2,\ldots,k$, seperating the first integral from 0 to θ_j and from θ_j to δ_j yields,

$$\begin{aligned}
S &= \sum_{j=1}^{k} S_1 S_2 \\
&= \sum_{j=1}^{k} S_1 \left|\int_0^{\theta_j}\left[(\delta_j-z)^{\lambda_j+\rho-1} - (\theta_j-z)^{\lambda_j+\rho-1}\right]\varphi(z,\phi)dz + \int_{\theta_j}^{\delta_j}(\delta_j-z)^{\lambda_j+\rho-1}\varphi(z,\phi)dz\right| \\
&\leq \sum_{j=1}^{k} S_1 \left[\int_0^{\theta_j}\left|(\delta_j-z)^{\lambda_j+\rho-1} - (\theta_j-z)^{\lambda_j+\rho-1}\right| |\varphi(z,\phi)|\,dz + \int_{\theta_j}^{\delta_j}(\delta_j-z)^{\lambda_j+\rho-1}\,|\varphi(z,\phi)|\,dz\right].
\end{aligned}$$

$m-1 \leq \rho \leq m, m \geq 2$, implies that $0 \leq \lambda_j + \rho - 1$ and $(\delta_j - v)^{\lambda_j+\rho-1} - (\theta_j - v)^{\lambda_j+\rho-1} \geq 0$. Therefore S becomes as

$$S \leq \sum_{j=1}^{k} S_1 (Lr+M) \left\{ \int_0^{\theta_j} \left[(\delta_j - z)^{\lambda_j+\rho-1} - (\theta_j - z)^{\lambda_j+\rho-1} dz \right] + \int_{\theta_j}^{\delta_j} (\delta_j - z)^{\lambda_j+\rho-1} dz \right\}.$$

If we take the integrals of the above inequality, one can obtain that

$$S \leq \sum_{j=1}^{k} \frac{(Lr+M)K^{m-1}}{\Gamma(\lambda_j+\rho+1)} \left| \frac{\gamma_j}{\Lambda} \right| (\delta_j^{\lambda_j+\rho} - \theta_j^{\lambda_j+\rho}). \tag{4.1.4}$$

By taking integrals in (4.1.3) and using (4.1.4), we have

$$\begin{aligned} |\mathcal{P}\phi| &\leq (Lr+M) \left[\frac{K^{\rho}}{\Gamma(\rho+1)} + \frac{K^{\rho+m-1}}{|\Lambda|\Gamma(\rho+1)} + \sum_{j=1}^{k} \frac{K^{m-1}}{\Gamma(\lambda_j+\rho+1)} \left| \frac{\gamma_j}{\Lambda} \right| (\delta_j^{\lambda_j+\rho} - \theta_j^{\lambda_j+\rho}) \right] \\ &= (Lr+M)Z. \end{aligned}$$

By using the condition $r \geq \dfrac{MZ}{1-LZ}$, we get

$$|\mathcal{P}\phi| \leq (Lr+M)Z \leq LZr + MZ \leq -Mz + r + Mz = r.$$

The desired result can be obtained by taking supremum of the both sides of the above inequality.

Step 2. $\mathcal{P}$ is a contraction: For every $x \in [0,1]$ and for all $\phi_1, \phi_2 \in H$, the following holds:

$$\begin{aligned} ||\mathcal{P}\phi_1 - \mathcal{P}\phi_2|| \leq \sup_{x\in[0,K]} \Bigg\{ & \frac{1}{\Gamma(\rho)} \int_0^x (x-z)^{\rho-1} |\varphi(z,\phi_1(z)) - \varphi(z,\phi_2(z))| dz \\ & + \frac{x^{m-1}}{|\Lambda|\Gamma(\rho)} \int_0^K (K-z)^{\rho-1} |\varphi(z,\phi_1(z)) - \varphi(z,\phi_2(z))| dz \\ & + \sum_{j=1}^{k} \frac{x^{m-1}}{\Gamma(\lambda_j+\rho)} \left| \frac{\gamma_j}{\Lambda} \right| \Bigg[\int_0^{\delta_j} (\delta_j - z)^{\lambda_j+\rho-1} |\varphi(z,\phi_1(z)) - \varphi(z,\phi_2(z))| dz \\ & - \int_0^{\theta_j} (\theta_j - z)^{\lambda_j+\rho-1} |\varphi(z,\phi_1(z)) - \varphi(z,\phi_2(z))| dz \Bigg] \Bigg\}. \end{aligned}$$

The condition (4.1.2) leads to

$$\begin{aligned} ||\mathcal{P}\phi_1 - \mathcal{P}\phi_2|| \leq \sup_{x\in[0,K]} \Bigg\{ & \frac{L}{\Gamma(\rho)} \int_0^x (x-z)^{\rho-1} |\phi_1(z) - \phi_2(z)| dz \\ & + \frac{Lx^{m-1}}{|\Lambda|\Gamma(\rho)} \int_0^K (K-z)^{\rho-1} |\phi_1(z) - \phi_2(z)| dz \\ & + \sum_{j=1}^{k} \frac{Lx^{m-1}}{\Gamma(\lambda_j+\rho)} \left| \frac{\gamma_j}{\Lambda} \right| \Bigg[\int_0^{\delta_j} (\delta_j - z)^{\lambda_j+\rho-1} |\phi_1(z) - \phi_2(z)| dz \\ & - \int_0^{\theta_j} (\theta_j - z)^{\lambda_j+\rho-1} |\phi_1(z) - \phi_2(z)| dz \Bigg] \Bigg\}. \end{aligned}$$

Applying the same procedure followed in Step 1 results in $||\mathcal{P}\phi_1 - \mathcal{P}\phi_2|| \leq LZ||\phi_1 - \phi_2||$. From condition (4.1.2), we can conclude that $\mathcal{P}$ is contraction. Therefore Banach's contraction mapping principle provides the presence of unique solution of boundary value problem (4.0.1)-(4.0.2). □

Example 4.1.1. *(Ahmad et al., 2013) Examine the four-strip nonlocal boundary value problem*

$$\begin{aligned} {}_0^C D^{9/2}\phi(x) &= \varphi(x,\phi(x)),\ x \in [0,2], \\ \phi(0) = 0,\ \phi'(0) = 0, \ldots,\ \phi''(0) &= 0, \ldots,\ \phi'''(0) = 0, \\ \phi(2) &= \sum_{j=1}^{4} \gamma_j [I^{\lambda_j}\phi(\delta_j) - I^{\lambda_j}\phi(\theta_j)], \end{aligned} \tag{4.1.5}$$

where $\rho = 9/2, m = 5, \theta_1 = 1/4, \delta_1 = 1/2, \theta_2 = 2/3, \delta_2 = 1, \theta_3 = 5/4, \delta_3 = 4/3, \theta_4 = 3/2, \delta_4 = 7/4, \gamma_1 = 5, \gamma_2 = 10, \gamma_3 = 15, \gamma_4 = 25, \lambda_1 = 5/4, \lambda_2 = 7/4, \lambda_3 = 9/4, \lambda_4 = 11/4.$

On the other hand $\Lambda = K^{m-1} - \sum_{j=1}^{k} \gamma_j \frac{\left(\delta_j^{\lambda_j+m-1} - \theta_j^{\lambda_j+m-1}\right)\Gamma(m)}{\Gamma(\lambda_j+m)} \simeq 9,334784$ *and*

$$Z = \frac{K^{\rho}}{\Gamma(\rho+1)} + \frac{K^{\rho+m-1}}{|\Lambda|\Gamma(\rho+1)} + \sum_{j=1}^{k} \frac{K^{m-1}}{\Gamma(\lambda_j+\rho+1)} \left|\frac{\gamma_j}{\Lambda}\right| (\delta_j^{\lambda_j+\rho} - \theta_j^{\lambda_j+\rho}) \simeq 1.406972.$$

- *Let* $\varphi(x,\phi(x)) = \dfrac{\arctan\phi}{(x+8)^{1/3}} + (4+3\sin 2x)^{1/2}$. *By applying Mean Value Theorem we obtain*

$$\begin{aligned} |\varphi(x,\phi_1) - \varphi(x,\phi_2)| &= \frac{|\arctan\phi_1 - \arctan\phi_2|}{(x+8)^{1/3}} = \varphi_\phi(x,\phi)(\xi)|\phi_1-\phi_2| = \frac{|\phi_1-\phi_2|}{(1+\xi^2)(x+8)^{1/3}} \\ &\leq \frac{|\phi_1-\phi_2|}{(x+8)^{1/3}} \leq \frac{1}{2}|\phi_1-\phi_2|, \end{aligned}$$

  which implies $L = 1/2$ *and* $L < 1/Z$ *is satisfied. Since the assumptions of Theorem 4.1.1 are fulfilled, the unique solution exist for (4.1.5).*

- *Let* $\varphi(x,\phi(x)) = \dfrac{\phi}{7} + \dfrac{\arctan\phi}{(x+8)^{1/3}} + (4+3\sin 2x)^{1/2}$ *be an unbounded nonlinear function. Then*

$$\begin{aligned} |\varphi(x,\phi_1) - \varphi(x,\phi_2)| &= \frac{|\phi_1-\phi_2|}{7} + \frac{|\arctan\phi_1 - \arctan\phi_2|}{(x+8)^{1/3}} \leq |\phi_1-\phi_2|\left(\frac{1}{7}+\frac{1}{2}\right) \\ &= \frac{9}{14}|\phi_1-\phi_2|, \end{aligned}$$

  which yields that $L = \dfrac{9}{14} < \dfrac{1}{Z}$, *where* $Z = 1.406972$. *Since all the hypotheses of Theorem 4.1.1 hold, the unique solution exist for (4.1.5).*

Theorem 4.1.2. *(Ahmad et al., 2013) Suppose* $\phi \in \mathbb{R}$ *is bounded for* $x \in [0,K]$*, i.e* $|\varphi(x,\phi)| \leq L_1$*, where* $L_1 > 0$*. In that case the boundary value problem (4.0.1)-(4.0.2) does have a solution.*

Proof. To prove that (4.0.1)-(4.0.2) has a solution, Theorem 4.1.2 will be used. First we will prove that $\mathcal{P}$ is completely continuous:

(i) $\mathcal{P}$ is continuous: Since φ is continuous, $\mathcal{P}$ is also continuous.

(ii) $\mathcal{P}$ is uniformly bounded: Let $\mathcal{B} \subset H$ be a bounded set. In order to find uniform boundedness of $\mathcal{P}$, we will analyze the following:

$$|(\mathcal{P}\phi)(x)| \leq \frac{1}{\Gamma(\rho)}\int_0^x (x-z)^{\rho-1}|\varphi(z,\phi(z))|dz + \frac{x^{m-1}}{|\Lambda|\Gamma(\rho)}\int_0^K (K-z)^{\rho-1}|\varphi(z,\phi(z))|dz$$
$$+\sum_{j=1}^{k}\frac{x^{m-1}\gamma_j}{|\Lambda|\Gamma(\lambda_j+\rho)}\left|\int_0^{\delta_j}(\delta_j-z)^{\lambda_j+\rho-1}\varphi(z,\phi(z))dz\right.$$
$$\left.-\int_0^{\theta_j}(\theta_j-z)^{\lambda_j+\rho-1}\varphi(z,\phi(z))dz\right|.$$

Using condition $|\varphi(x,\phi)| \leq L_1$, and applying the same procedure followed in Step 1 of the proof of Theorem 3.1.1 result in

$$|(\mathcal{P}\phi)(x)| \leq L_1\left[\frac{K^{\rho}}{\Gamma(\rho+1)} + \frac{K^{\rho+m-1}}{|\Lambda|\Gamma(\rho+1)} + \sum_{j=1}^{k}\frac{K^{m-1}}{\Gamma(\lambda_j+\rho+1)}\left|\frac{\gamma_j}{\Lambda}\right|(\delta_j^{\lambda_j+\rho}-\theta_j^{\lambda_j+\rho})\right] = L_2.$$

Therefore $||(\mathcal{P}\phi)(x)|| \leq L_2$.

(iii) $\mathcal{P}$ is equicontinuous: We will show that $|(\mathcal{P}\phi)(x_2)-(\mathcal{P}\phi)(x_1)| \to 0$ provided $x_1 \to x_2$ for all $0 \leq x_1, x_2 \leq K$. Consider

$$|(\mathcal{P}\phi)'(x)| \leq \frac{1}{\Gamma(\rho-1)}\int_0^x (x-z)^{\rho-2}|\varphi(z,\phi(z))|dz$$
$$+\frac{(m-1)x^{m-2}}{|\Lambda|\Gamma(\rho)}\int_0^K (K-z)^{\rho-1}|\varphi(z,\phi(z))|dz$$
$$+\sum_{j=1}^{k}\frac{(m-1)x^{m-2}\gamma_j}{|\Lambda|\Gamma(\lambda_j+\rho)}\left|\int_0^{\delta_j}(\delta_j-z)^{\lambda_j+\rho-1}\varphi(z,\phi(z))dz\right.$$
$$\left.-\int_0^{\theta_j}(\theta_j-z)^{\lambda_j+\rho-1}\varphi(z,\phi(z))dz\right|.$$

Using condition $|\varphi(x,\phi)| \leq L_1$, and applying the same procedure followed in Step 1 of the proof of Theorem 3.1.1 lead to

$$|(\mathcal{P}\phi)'(x)| \leq L_1\left[\frac{K^{\rho-1}}{\Gamma(\rho)} + \frac{(m-1)K^{\rho+m-2}}{|\Lambda|\Gamma(\rho+1)} + \sum_{j=1}^{k}\frac{(m-1)K^{m-2}}{\Gamma(\lambda_j+\rho+1)}\left|\frac{\gamma_j}{\Lambda}\right|(\delta_j^{\lambda_j+\rho}-\theta_j^{\lambda_j+\rho})\right] = L_3.$$

This follows that for all $x_1, x_2 \in [0,K], x_1 < x_2$, one can show that

$$|(\mathcal{P}\phi)(x_2) - (\mathcal{P}\phi)(x_1)| \leq \int_{x_1}^{x_2} |(\mathcal{P}\phi)'(x)|dx \leq L_3(x_2 - x_1),$$

implying $\mathcal{P}$ is equicontinuous.

Since the parts (ii)-(iii) satisfy the hypotheses of Arzela Ascoli Theorem 2.0.4, the operator $\mathcal{P} : H \to H$ is relatively compact and so, from Part (i) and Definition 2.0.16 it is completely continuous.

To use Leray-Schauder alternative 2.0.3, let use define the set $W = \{\phi \in H : \phi = \sigma\mathcal{P}\phi, 0 < \sigma < 1\}$ and show the boundedness of it. Let $\phi \in W$. Then for any $x \in [0,K]$ it can be obtained

$$\begin{aligned}
|\phi(x)| = \sigma|(\mathcal{P}\phi)(x)| \leq{}& \frac{1}{\Gamma(\rho)}\int_0^x (x-w)^{\rho-1}|\varphi(w,\phi(w))|dw + \frac{x^{m-1}}{|\Lambda|\Gamma(\rho)}\int_0^K (K-w)^{\rho-1}|\varphi(w,\phi(w))|dw \\
&+ \sum_{j=1}^{k} \frac{x^{m-1}\gamma_j}{|\Lambda|\Gamma(\lambda_j+\rho)}\bigg|\int_0^{\delta_j}(\delta_j - w)^{\lambda_j+\rho-1}\varphi(w,\phi(w))dw \\
&\qquad - \int_0^{\theta_j}(\theta_j - w)^{\lambda_j+\rho-1}\varphi(w,\phi(w))dw\bigg| \\
\leq{}& L_1\left[\frac{K^\rho}{\Gamma(\rho+1)} + \frac{K^{\rho+m-1}}{|\Lambda|\Gamma(\rho+1)} + \sum_{j=1}^{k}\frac{K^{m-1}}{\Gamma(\lambda_j+\rho+1)}\left|\frac{\gamma_j}{\Lambda}\right|(\delta_j^{\lambda_j+\rho} - \theta_j^{\lambda_j+\rho})\right] = M_1.
\end{aligned}$$

Since $|\phi(x)| \leq M_1$, for any $x \in [0,K]$, the set W is bounded. □

Example 4.1.2. *(Ahmad et al., 2013) In the boundary value problem (4.1.5) let us take*

$$\varphi(x,\phi(x)) = \frac{3e^{(2-|\phi(x)|)^{3/2}}\left[\cos 4x + 2\ln(1+4\sin^2\phi(x))\right]}{(10+\cos\phi(x))^{1/2}}. \tag{4.1.6}$$

Note that $|\varphi(x,\phi(x))| \leq e^{2\sqrt{2}}(1+\ln 25) = L_1$. *Since the hypotheses of Theorem 4.1.2 are satisfied, boundary value problem (4.1.5) does have a solution for* $\varphi(x,\phi(x))$ *presented in (4.1.6).*

Theorem 4.1.3. *(Ahmad et al., 2013) Suppose that for sufficiently small* $\eta > 0, 0 < |x| < \eta$ *and for* $0 < \nu \leq 1/Z$, *the function* φ *satisfies* $|\varphi(x,\phi)| \leq \nu|\phi|$. *In that case the boundary value problem (4.0.1)-(4.0.2) does have a solution.*

Proof. So as to present the presence of solution, Theorem 2.0.5 will be used. For convenience we can define the bounded set $\mathcal{B}_\eta = \{\phi \in H : ||\phi|| < \eta\}$. Let us choose $\phi \in H$ such that $||\phi|| = \eta$, i.e $\phi \in \partial\mathcal{B}_\eta$. First we will prove that $\mathcal{P}$ is completely continuous:

(i) $\mathcal{P}$ is continuous: Since φ is continuous, $\mathcal{P}$ is also continuous.

(ii) $\mathcal{P}$ is uniformly bounded: In order to show uniform boundedness of $\mathcal{P}$, we will analyze the following:

$$\begin{aligned}||(\mathcal{P}\phi)(x)|| \leq \sup_{x\in[0,K]} \Bigg\{ &\frac{1}{\Gamma(\rho)}\int_0^x (x-z)^{\rho-1}|\varphi(z,\phi(z))|dz \\ &+ \frac{x^{m-1}}{|\Lambda|\Gamma(\rho)}\int_0^K (K-z)^{\rho-1}|\varphi(z,\phi(z))|dz \\ &+ \sum_{j=1}^{k} \frac{x^{m-1}\gamma_j}{|\Lambda|\Gamma(\lambda_j+\rho)}\Bigg|\int_0^{\delta_j}(\delta_j-z)^{\lambda_j+\rho-1}\varphi(z,\phi(z))dz \\ &\qquad - \int_0^{\theta_j}(\theta_j-z)^{\lambda_j+\rho-1}\varphi(z,\phi(z))dz\Bigg|\Bigg\}.\end{aligned}$$

Using condition $|\varphi(x,\phi)| \leq \nu|\phi|$, and applying the same procedure followed in Step 1 in the proof of Theorem 3.1.1 result in

$$\begin{aligned}||(\mathcal{P}\phi)(x)|| &\leq \sup_{x\in[0,K]} \Bigg\{ \frac{1}{\Gamma(\rho)}\int_0^x (x-z)^{\rho-1}\nu|\phi|dz + \frac{x^{m-1}}{|\Lambda|\Gamma(\rho)}\int_0^K (K-z)^{\rho-1}\nu|\phi|dz \\ &+ \sum_{j=1}^{k} \frac{x^{m-1}\gamma_j}{|\Lambda|\Gamma(\lambda_j+\rho)}\left[\int_0^{\theta_j}\left|(\delta_j-z)^{\lambda_j+\rho-1}-(\theta_j-z)^{\lambda_j+\rho-1}\right||\phi|\,dz \right.\\ &\qquad \left. + \int_{\theta_j}^{\delta_j}(\delta_j-z)^{\lambda_j+\rho-1}\,|\phi|\,dz\right]\Bigg\} \\ &= \nu||\phi||\Bigg\{\frac{1}{\Gamma(\rho)}\int_0^x (x-z)^{\rho-1}dz + \frac{x^{m-1}}{|\Lambda|\Gamma(\rho)}\int_0^K (K-z)^{\rho-1}dz \\ &+ \sum_{j=1}^{k} \frac{x^{m-1}\gamma_j}{|\Lambda|\Gamma(\lambda_j+\rho)}\left[\int_0^{\theta_j}\left|(\delta_j-z)^{\lambda_j+\rho-1}-(\theta_j-z)^{\lambda_j+\rho-1}\right|dz \right.\\ &\qquad \left. + \int_{\theta_j}^{\delta_j}(\delta_j-z)^{\lambda_j+\rho-1}dz\right]\Bigg\} \leq \nu||\phi||Z \leq ||\phi|| = \eta.\end{aligned}$$

Therefore $||(\mathcal{P}\phi)(x)|| \leq \eta$.

(iii) $\mathcal{P}$ is equicontinuous: We will show that $|(\mathcal{P}\phi)(x_2)-(\mathcal{P}\phi)(x_1)| \to 0$ provided $x_1 \to x_2$ for all $0 \le x_1, x_2 \le K$. For this purpose we analyze that

$$
\begin{aligned}
|(\mathcal{P}\phi)'(x)| \le & \frac{1}{\Gamma(\rho-1)} \int_0^x (x-z)^{\rho-2} |\varphi(z,\phi(z))| dz \\
& + \frac{(m-1)x^{m-2}}{|\Lambda|\Gamma(\rho)} \int_0^K (K-z)^{\rho-1} |\varphi(z,\phi(z))| dz \\
& + \sum_{j=1}^{k} \frac{(m-1)x^{m-2}\gamma_j}{|\Lambda|\Gamma(\lambda_j+\rho)} \left| \int_0^{\delta_j} (\delta_j - z)^{\lambda_j+\rho-1} \varphi(z,\phi(z)) dz \right. \\
& \left. \qquad - \int_0^{\theta_j} (\theta_j - z)^{\lambda_j+\rho-1} \varphi(z,\phi(z)) dz \right|.
\end{aligned}
$$

Using condition $|\varphi(x,\phi)| \le L_1$, and applying the same procedure followed in Step 1 of the proof of Theorem 3.1.1 lead to

$$
|(\mathcal{P}\phi)'(x)| \le L_1 \left[\frac{K^{\rho-1}}{\Gamma(\rho)} + \frac{(m-1)K^{\rho+m-2}}{|\Lambda|\Gamma(\rho+1)} + \sum_{j=1}^{k} \frac{(m-1)K^{m-2}}{\Gamma(\lambda_j+\rho+1)} \left| \frac{\gamma_j}{\Lambda} \right| (\delta_j^{\lambda_j+\rho} - \theta_j^{\lambda_j+\rho}) \right] = L_3.
$$

This follows that for all $x_1, x_2 \in [0,K], x_1 < x_2$, one can show that

$$
|(\mathcal{P}\phi)(x_2) - (\mathcal{P}\phi)(x_1)| \le \int_{x_1}^{x_2} |(\mathcal{P}\phi)'(x)| dx \le L_3(x_2 - x_1),
$$

implying $\mathcal{P}$ is equicontinuous.

Since the parts (ii)-(iii) satisfy the hypotheses of Arzela Ascoli Theorem 2.0.4, the operator $\mathcal{P} : H \to H$ is relatively compact and so, from Part (i) and Definition 2.0.16 it is completely continuous. Hence we can conclude from Theorem 2.0.5 that boundary value problem (4.0.1)-(4.0.2) does have a solution.

□

Example 4.1.3. *(Ahmad et al., 2013) In the boundary value problem (4.1.5) let us take*

$$
\varphi(x,\phi(x)) = \phi(a^5+\phi^4)^{1/5} + 2(1+\cos(x^4+3))^5(1-\cos\phi), \phi \ne 0, a > 0. \qquad (4.1.7)
$$

If ϕ is small enough and if all its power bigger than 1 are neglected then

$$
|\phi(a^5+\phi^4)^{1/5} + 2(1+\cos(x^4+3))^5(1-\cos\phi)| \le a|\phi|.
$$

If we choose $a \le 1/Z$, then the assumptions of Theorem 4.1.3 are fulfilled and for ϕ presented in (4.1.7), boundary value problem (4.1.5) does have a solution.

Theorem 4.1.4. *(Ahmad et al., 2013) Consider the following assumptions:*

A2) For the functions $v \in C([0,1],\mathbb{R}^+)$ and $q : \mathbb{R}^+ \to \mathbb{R}^+, q'(x) \geq 0$, the following holds

$$|\varphi(x,\phi(x))| \leq v(x)q(||\phi||)$$

for all $(x,\phi) \in [0,K] \times \mathbb{R}$;

A3) For the constant $D > 0$, we have $\frac{D}{Zq(D)||v||} > 1$.

In that case the boundary value problem (4.0.1)-(4.0.2) does have a solution.

Proof. Theorem 2.0.6 will be employed to show the presence of solution. For convenience we can define the bounded set $\mathcal{B}_\eta = \{\phi \in C([0,K],\mathbb{R}) : ||\phi|| \leq \eta, \eta > 0\}$.

(i) $\mathcal{P}$ is continuous: Since φ is continuous, $\mathcal{P}$ is also continuous.

(ii) The image of bounded sets in $C([0,K],\mathbb{R})$ are bounded sets in $C([0,K],\mathbb{R})$: Consider

$$\begin{aligned}||(\mathcal{P}\phi)(x)|| \leq \sup_{x\in[0,K]} \Bigg\{ &\frac{1}{\Gamma(\rho)} \int_0^x (x-w)^{\rho-1} |\varphi(w,\phi(w))| dw \\ &+ \frac{x^{m-1}}{|\Lambda|\Gamma(\rho)} \int_0^K (K-w)^{\rho-1} |\varphi(w,\phi(w))| dw \\ &+ \sum_{j=1}^{k} \frac{x^{m-1}\gamma_j}{|\Lambda|\Gamma(\lambda_j+\rho)} \left| \int_0^{\delta_j} (\delta_j - w)^{\lambda_j+\rho-1} \varphi(w,\phi(w)) dw \right. \\ &\left. - \int_0^{\theta_j} (\theta_j - w)^{\lambda_j+\rho-1} \varphi(w,\phi(w)) dw \right| \Bigg\}.\end{aligned}$$

Using condition $|\varphi(x,\phi(x))| \leq v(x)q(||\phi||)$ and applying the same procedure followed in Step 1 of the proof of Theorem 3.1.1 result in

$$\begin{aligned}||(\mathcal{P}\phi)(x)|| \leq \sup_{x\in[0,K]} \Bigg\{ &\frac{1}{\Gamma(\rho)} \int_0^x (x-w)^{\rho-1} v(w)q(||\phi||) dw \\ &+ \frac{x^{m-1}}{|\Lambda|\Gamma(\rho)} \int_0^K (K-w)^{\rho-1} v(w)q(||\phi||) dw \\ &+ \sum_{j=1}^{k} \frac{x^{m-1}\gamma_j}{|\Lambda|\Gamma(\lambda_j+\rho)} \Bigg[\int_0^{\theta_j} \left| (\delta_j - w)^{\lambda_j+\rho-1} - (\theta_j - w)^{\lambda_j+\rho-1} \right| v(w)q(||\phi||) dw \\ &+ \int_{\theta_j}^{\delta_j} (\delta_j - w)^{\lambda_j+\rho-1} v(w)q(||\phi||) dw \Bigg] \Bigg\}.\end{aligned}$$

Besides,

$$||(\mathcal{P}\phi)(x)|| \leq ||v||q(\eta)\left\{\frac{1}{\Gamma(\rho)}\int_0^x (x-w)^{\rho-1}dw + \frac{x^{m-1}}{|\Lambda|\Gamma(\rho)}\int_0^K (K-w)^{\rho-1}dw\right.$$
$$+\sum_{j=1}^{k}\frac{x^{m-1}\gamma_j}{|\Lambda|\Gamma(\lambda_j+\rho)}\left[\int_0^{\theta_j}\left|(\delta_j-w)^{\lambda_j+\rho-1}-(\theta_j-w)^{\lambda_j+\rho-1}\right|dw\right.$$
$$\left.\left.+\int_{\theta_j}^{\delta_j}(\delta_j-w)^{\lambda_j+\rho-1}dw\right]\right\} \leq ||v||q(\eta)Z = Z_1.$$

Therefore $||(\mathcal{P}\phi)(x)|| \leq Z_1$.

(iii) The image of bounded sets in $C([0,K],\mathbb{R})$ are equicontinuous sets of $C([0,K],\mathbb{R})$: Choosing $0 \leq x_1 < x_2 \leq K$ implies that

$$|(\mathcal{P}\phi)(x_2)-(\mathcal{P}\phi)(x_1)| = \left|\frac{1}{\Gamma(\rho)}\int_0^{x_2}(x_2-z)^{\rho-1}\varphi(z,\phi)dz\right.$$
$$-\frac{1}{\Gamma(\rho)}\int_0^{x_1}(x_1-z)^{\rho-1}\varphi(z,\phi)dz$$
$$+\frac{x_2^{m-1}-x_1^{m-1}}{|\Lambda|\Gamma(\rho)}\int_0^K (K-z)^{\rho-1}\varphi(z,\phi)dz$$
$$+\sum_{j=1}^{k}\frac{[x_2^{m-1}-x_1^{m-1}]\gamma_j}{|\Lambda|\Gamma(\lambda_j+\rho)}\left[\int_0^{\theta_j}\left|(\delta_j-z)^{\lambda_j+\rho-1}-(\theta_j-z)^{\lambda_j+\rho-1}\right|\varphi(z,\phi)dz\right.$$
$$\left.\left.+\int_{\theta_j}^{\delta_j}(\delta_j-z)^{\lambda_j+\rho-1}\varphi(z,\phi)dz\right]\right|.$$

Using condition $|\varphi(x,\phi(x))| \leq v(x)q(||\phi||)$ and applying the same procedure followed in Step 1 of the proof of Theorem 3.1.1 result in

$$|(\mathcal{P}\phi)(x_2)-(\mathcal{P}\phi)(x_1)| \leq \frac{1}{\Gamma(\rho)}\int_0^{x_1}\left|(x_2-z)^{\rho-1}-(x_1-z)^{\rho-1}\right||\varphi(z,\phi)|\,dz$$
$$+\frac{1}{\Gamma(\rho)}\int_{x_1}^{x_2}(x_2-z)^{\rho-1}|\varphi(z,\phi)|\,dz$$
$$+\frac{x_2^{m-1}-x_1^{m-1}}{|\Lambda|\Gamma(\rho)}\int_0^K (K-z)^{\rho-1}|\varphi(z,\phi)|\,dz$$
$$+\sum_{j=1}^{k}\frac{[x_2^{m-1}-x_1^{m-1}]|\gamma_j|}{|\Lambda|\Gamma(\lambda_j+\rho)}\left[\int_{\theta_j}^{\delta_j}(\delta_j-z)^{\lambda_j+\rho-1}|\varphi(z,\phi)|\,dz\right.$$
$$\left.+\int_0^{\theta_j}\left|(\delta_j-z)^{\lambda_j+\rho-1}-(\theta_j-z)^{\lambda_j+\rho-1}\right||\varphi(z,\phi)|\,dz\right].$$

Assumption (A_2) implies that

$$\begin{aligned}|(\mathcal{P}\phi)(x_2)-(\mathcal{P}\phi)(x_1)| \leq & \frac{1}{\Gamma(\rho)}\int_0^{x_1}\left|(x_2-z)^{\rho-1}-(x_1-z)^{\rho-1}\right|v(z)q(\eta)dz \\ &+\frac{1}{\Gamma(\rho)}\int_{x_1}^{x_2}(x_2-z)^{\rho-1}v(z)q(\eta)dz \\ &+\frac{x_2^{m-1}-x_1^{m-1}}{|\Lambda|\Gamma(\rho)}\int_0^K(K-z)^{\rho-1}v(z)q(\eta)dz \\ &+\sum_{j=1}^k\frac{[x_2^{m-1}-x_1^{m-1}]\,|\gamma_j|}{|\Lambda|\Gamma(\lambda_j+\rho)}\left[\int_{\theta_j}^{\delta_j}(\delta_j-z)^{\lambda_j+\rho-1}v(z)q(\eta)dz\right. \\ &\left.+\int_0^{\theta_j}\left|(\delta_j-z)^{\lambda_j+\rho-1}-(\theta_j-z)^{\lambda_j+\rho-1}\right|v(z)q(\eta)dz\right].\end{aligned}$$

When $x_2\to x_1$,

$$|(\mathcal{P}\phi)(x_2)-(\mathcal{P}\phi)(x_1)|\to 0$$

not depending on $\phi\in B_\eta$.

Since the parts (ii)-(iii) satisfy the hypotheses of Arzela Ascoli Theorem 2.0.4, the operator $\mathcal{P}:H\to H$ is relatively compact and so, from Part (i) and Definition 2.0.16 it is completely continuous.

Now we will show that the second conclusion of Theorem 2.0.6 can not be satisfied. For this purpose let us take the set $\phi\in W$ as in the proof of Theorem 4.1.2. Then employing the similar arguments in the proof that $\mathcal{P}$ is bounded, for any $x\in[0,K]$ one may find that

$$\begin{aligned}|\phi(x)|=\sigma|(\mathcal{P}\phi)(x)| \leq & \frac{1}{\Gamma(\rho)}\int_0^x(x-w)^{\rho-1}|\varphi(w,\phi(w))|dw \\ &+\frac{x^{m-1}}{|\Lambda|\Gamma(\rho)}\int_0^K(K-w)^{\rho-1}|\varphi(w,\phi(w))|dw \\ &+\sum_{j=1}^k\frac{x^{m-1}\gamma_j}{|\Lambda|\Gamma(\lambda_j+\rho)}\left|\int_0^{\delta_j}(\delta_j-w)^{\lambda_j+\rho-1}\varphi(w,\phi(w))dw\right. \\ &\left.-\int_0^{\theta_j}(\theta_j-w)^{\lambda_j+\rho-1}\varphi(w,\phi(w))dw\right| \\ \leq & ||v||q(||\phi||)\left[\frac{K^\rho}{\Gamma(\rho+1)}+\frac{K^{\rho+m-1}}{|\Lambda|\Gamma(\rho+1)}+\sum_{j=1}^k\frac{K^{m-1}}{\Gamma(\lambda_j+\rho+1)}\left|\frac{\gamma_j}{\Lambda}\right|(\delta_j^{\lambda_j+\rho}-\theta_j^{\lambda_j+\rho})\right].\end{aligned}$$

If we take supremum of the both sides of the above inequality for $x\in[0,K]$, we have

$$||\phi||\leq||v||q(||\phi||)Z$$

or

$$\frac{||\phi||}{||v||q(||\phi||)Z} \leq 1.$$

Therefore from assumption (A_3), there is a constant E for which $||\phi|| \neq E$. Now we set the set $U = \{\phi \in C([0,K],\mathbb{R}) : ||\phi|| < E+1\}$. It can be shown as above that $\mathcal{P} : \overline{U} \to C([0,K],\mathbb{R})$ is continuous and completely continuous. Chhosing U in this form yields that there does not exist $\phi \in \partial U$ satisfying $\phi = \sigma\mathcal{P}(\phi)$ for $0 < \sigma < 1$. Hence we can conclude from Theorem 2.0.6 that $\mathcal{P}$ possesses a fixed point $\phi \in \overline{U}$. Hence the boundary value problem (4.0.1)-(4.0.2) does have a solution $\phi \in \overline{U}$. □

Example 4.1.4. *(Ahmad et al., 2013)*

- *In the boundary value problem (4.1.5) let us take*

$$\varphi(x,\phi(x)) = \frac{1}{\sqrt{x+4}}\left(1+\frac{\phi}{1+\phi}\right). \tag{4.1.8}$$

  Then $|\varphi(x,\phi(x))| \leq \frac{1}{\sqrt{x+4}}\left(1+\frac{|\phi|}{1+|\phi|}\right) \leq v(x)q(||\phi||)$, *where* $v(x) = \frac{1}{\sqrt{x+4}}, q(||\phi||) = 2$. *Therefore* $||v|| = 1/2, Z = 1.406972$ *and assumption* (A_3) *imply that* $E > Z$. *Hence for* $\varphi(x,\phi(x))$ *presented in (4.1.8), the assumptions of Theorem 4.1.4 are fulfilled. Thus (4.1.5) does have a solution.*

- *In (4.1.5) let us take*

$$\varphi(x,\phi(x)) = \frac{1}{\sqrt{x+4}}\left(1+\frac{\phi}{1+\phi}+\frac{\phi}{2}\right). \tag{4.1.9}$$

  Then $|\varphi(x,\phi(x))| \leq \frac{1}{\sqrt{x+4}}\left(1+\frac{|\phi|}{1+|\phi|}+\frac{|\phi|}{2}\right) \leq v(x)q(||\phi||)$, *where*

$$v(x) = \frac{1}{\sqrt{x+4}}, q(||\phi||) = 2+||\phi||/2.$$

  Therefore $||v|| = 1/2, Z = 1.406972$ *and assumption* (A_3) *imply that* $E > 2.170392$. *Hence for* ϕ *presented in (4.1.9), the assumptions of Theorem 4.1.4 are fulfilled. Thus (4.1.5) does have a solution.*

5. CONCLUSION

In this thesis, existence and uniqueness theorems for fractional boundary value problems with slit strip and multi strip boundary conditions have been established. Fractional differential equations are essential in mathematics due to the fact that they are more accurate, realistic and practical than ordinary differential equations in modelling several physical phenomena. The main advantage or the remarkable property of fractional integral and differential operators is that they are nonlocal in nature which means that the future state of a dynamical system or process involving fractional derivative depends on its current state as well its past states. This memory and hereditary properties of these operators allow us to trace the past history of several materials and processes. Moreover, theory of fractional boundary value problems is a very popular research area from mathematics point of view and have applications in biology, epidemiology, physics, engineering, chemistry, hydrology, finance, classical mechanics, quantum mechanics, visco-elasticity, electrical circuits and neuron modelling and so on.

This thesis summarizes the results of (Ahmad et al., 2013; Ahmad and Agarwal, 2014).

In Chapter 3 existence and uniqueness theorems for a differential equation with fractional derivative of order $q \in (n-1, n]$ with slit-strips type boundary conditions have been presented. The slit-strips type boundary condition means that the total effect of the two nonintersecting subintervals (strips) of arbitrary lengths is connected to evaluation of the unknown function at the point out of the subintervals or in the aperture (slit). In order to prove our results, standart fixed point theorems (Banach's contraction mapping principle, Krasnoselski's fixed point theorem, Leary-Schauder alternative and Nonlinear alternative for single valued maps) will be used and some examples will be shown to confirm that our results are theoretically applicable. Then these results will be applied to fractional boundary value problems with arbitrary number of nonlocal points in the slit, the nonlocal multi-substrips conditions and Riemann-Liouville type slit-strips boundary conditions.

Chapter 4 have been devoted to establish the existence and uniqueness theory for nonlinear fractional differential equations of arbitrary order with Riemann-Liouville type boundary conditions involving nonintersecting finite many strips of arbitrary length. Physical meaning of these conditions is that since the sensors situate in the middle of the interval, the controllers at the boundary of the interval disperse or take in energy. The existence results have been established via Krasnoselski's fixed point theorem, Leary-Schauder alternative and Nonlinear alternative for single valued maps, the uniqueness of the solutions have been obtained via Banach's contraction mapping principle. Several examples have been given to illustrate our results.

REFERENCES

Agarwal, R. P., Ahmad, B., 2011. Existence theory for anti-periodic boundary value problems of fractional differential equations and inclusions. ***Computers & Mathematics with Applications***, **62** (3): 1200–1214.

Agarwal, R. P., Alsaedi, A., Alsharif, A., Ahmad, B., 2017. On nonlinear fractional order boundary value problems with nonlocal multipoint conditions involving Liouville Caputo derivative. ***Differential Equations & Applications***, **9** (2): 147-160.

Agarwal, R. P., Benchohra, M., Hamani, S. 2010. A survey on existence results for boundary value problems of nonlinear fractional differential equations and ınclusions, ***Acta Applicandae Mathematicae***, **109**: 973–1033.

Ahmad, B., Alsaedi, A., 2012. Nonlinear fractional differential equations with nonlocal fractional integro-differential boundary conditions. ***Boundary Value Problems***, **2012**: 124, 10 pp.

Ahmad, B., Alsaedi, A., Alghamdi, B. S., 2008. Analytic approximation of solutions of the forced Duffing equation with integral boundary conditions. ***Nonlinear Analysis. Real World Applications***, **9**: 1727-1740.

Ahmad, B., Alsaedi, A., Alsharif, A., 2015. Existence results for fractional-order differential equations with nonlocal multi-point-strip conditions involving Caputo derivative. ***Advances in Difference Equations***, **2015**: 348.

Ahmad, B., Alsaedi, A., Alsharif, A., 2016. On mild solutions to Caputo type differential equations of arbitrary order with nonlocal multi-point-strip conditions. ***Journal of Fractional Calculus and Applications,*** **7** (2): 109-120.

Ahmad, B., Alsaedi, A., Assolami, A., Agarwal, R. P., 2013. A new class of fractional boundary value problems, ***Advances in Difference Equations***, **2013**: 373.

Ahmad, B., Hayat, T., Asghar, S., 2001. Diffraction of a plane wave by an elastic knife-edge adjacent to a strip. ***The Canadian Applied Mathematics Quarterly,*** **9**: 303–316.

Ahmad, B., Matar, M. M., Agarwal, R. P., 2015. Existence results for fractional differential equations of arbitrary order with nonlocal integral boundary conditions. ***Boundary Value Problems***, **2015**: 220, 13 pp.

Ahmad, B., Nieto, J. J., 2009. Existence of solutions for nonlocal boundary value problems of higher-order nonlinear fractional differential equations. ***Abstract and Applied Analysis***, **2009**, Art. ID 494720, 9 pp.

Ahmad, B., Nieto, J. J., 2011. Anti-periodic fractional boundary value problems. ***Computers & Mathematics with Applications,*** **62** (3): 1150–1156.

Ahmad, B., Nieto, J. J., 2012. Anti-periodic fractional boundary value problems with nonlinear term depending on lower order derivative, ***Journal of Fractional Calculus and Applications,*** **15** (3): 451–462.

Ahmad, B., Nieto, J. J., Pimentel, J., 2011. Some boundary value problems of fractional differential equations and inclusions. ***Computers & Mathematics with Applications***, **62** (3): 1238–1250.

Ahmad, B., Ntouyas, S. K., 2011. A four-point nonlocal integral boundary value problem for fractional differential equations of arbitrary order. ***Electronic Journal of Qualitative Theory of Differential Equations***, **2011** (22): 1-15.

Ahmad, B., Ntouyas, S. K., 2012. Existence results for nonlocal boundary value problems of fractional differential equations and inclusions with strip conditions. ***Boundary Value Problems***, **2012**: 55

Ahmad, B., Ntouyas, S. K., 2012. Nonlinear fractional differential equations and inclusions of arbitrary order and multi-strip boundary conditions. ***Electronic Journal of Qualitative Theory of Differential Equations***, **2012** (98): 1–22.

Ahmad, B., Ntouyas, S. K., 2015. Nonlocal fractional boundary value problems with slit-strips boundary conditions. ***Journal of Fractional Calculus and Applications***, **18** (1): 261-280.

Ahmad, B., Ntouyas, S. K., 2016. Boundary value problems for fractional differential equations with integral and ordinary-fractional flux boundary conditions. ***Journal of Nonlinear Science and its Applications***, **9**: 3622–3637.

Ahmad, B., Ntouyas, S. K., Alsaedi, A., 2017. Existence of solutions for fractional differential equations with nonlocal and average type integral boundary conditions. ***Journal of Applied Mathematics and Computing***, **53**: 129–145.

Alsaedi, A., Ahmad, B., Mohamad, N., Ntouyas, S. K., 2014. Uniqueness results for fully anti-periodic fractional boundary value problems with nonlinearity depending on lower-order derivatives. ***Advances in Difference Equations*** , **2014**: 136, 9 pp.

Alsaedi, A., Ntouyas, S. K., Ahmad, B., 2015. New existence results for fractional integrodifferential equations with nonlocal integral boundary conditions. ***Abstract and Applied Analysis,*** **2015**, Article ID 205452, 10 pp.

Asghar, S., Ahmad, B., Ayub, M., 1996. Diffraction from an absorbing half plane due to a finite cylindrical source. ***Acta Acustica***, **82**: 365–367.

Benchohra, M., Hamani, S., Ntouyas, S. K., 2008. Boundary value problems for differentıal equations with fractional order. ***Surveys in Mathematics and its Applications***, **3**: 1-12.

Benchohra, M., Hamani, S., Ntouyas, S. K., 2009. Boundary value problems for differential equations with fractional order and nonlocal conditions. ***Nonlinear Analysis. Theory, Methods & Applications.***, **71**: 2391–2396.

Benson, D. A., Wheatcraft, S. W., Meerschaert, M. M., 2000. Application of a fractional advection dispersion equation. ***Water Resources Research,*** **36**: 1403-1412.

Benson, D. A., Wheatcraft, S. W., Meerschaert, M. M., 2000. The fractional order governing equation of Lvy motion. ***Water Resources Research***, **36**: 1413-1423.

Bitsadze, A., Samarskii, A., 1969. On some simple generalizations of linear elliptic boundary problems. ***Doklady Mathematics***, **10**: 398-400.

Choudhary, S., Daftardar-Gejji, V., 2014. Nonlinear multi-order fractional differential equations with periodic/anti-periodic boundary conditions. ***Fractional Calculus and Applied Analysis***, **17** (2): 333–347.

Darwish, M. A., Ntouyas, S. K., 2014. Existence results for first order boundary value problems for fractional differential equations with four-point integral boundary conditions. ***Miskolc Mathematical Notes***, **15** (1): 51–61.

Debnath, L., Bhatta, D., D., 2004. Solutions to few linear fractional inhomogenous partial differential equations in fluid mechanics. ***Fractional Calculus and Applied Analysis***, **7**: 21-36.

Demirci, E., 2017. A fractional order model for obesity epidemic in a non-constant population. ***Advances in Difference Equations***, **2017**: 2017: 79.

Engheta, N., 1996. On fractional calculus and fractional multipoles in electromagnetism. IEE Trans. ***IEEE Transactions on Antennas and Propagation***, **44**: 554-566.

Freed, A., Diethelm, K., Luchko, Y., 2002. ***Fractional Order Viscoelasticity (FOV): Constitutive Development Using The Fractional Calculus: First annual Report***, NASA Technical Reports Server (NTRS), USA.

Glöckle, W. G., Nonnenmacher, T. F., 1995. A fractional calculus approach to self similar protein Dynamics. ***Biophysical Journal***, **68** (1): 46-53.

Goulart, A. G. O., Lazo, M. J., Suarez, J. M. S., Moreira, D. M., 2017. Fractional derivative models for atmospheric dispersion of pollutants, ***Physica A: Statistical Mechanics and its Applications***, 477: 9-19.

Hilfer, R., 2000. ***Applications of Fractional Calculus in Physics. World Scientific***, River Edge, USA.

Kilbas, A. A., Srivastava, H. M., Trujillo, J. J., 2006. ***Theory and Applications of Fractional Differential Equations.*** North Holand Mathematics Studies, 146, Elsevier Science B. V., The Netherlands.

Liu, F., Anh, V., Turner, I., 2004. Numerical solution of the space fractional Fokker Planck equation. ***Journal of Computational and Applied Mathematics***, **166**: 209-219.

Liu, S., Li, H., Dai, O., 2015. Nonlinear fractional differential equations with nonlocal integral boundary conditions. ***Advances in Difference Equations***, **2015**:187.

Magin, R. L., 2006. ***Fractional Calculus in Bioengineering***. Begel House Publisher., Inc., Connecticut.

Mahmudov, N. I.,. Unul, S., 2014. Existence of solutions of $\alpha \in (2,3]$ order fractional three-point boundary value problems with integral conditions. ***Abstract and Applied Analysis,*** **2014**, Art. ID 198632, 12 pp

Mainardi, F. 1997. Fractional Calculus: Some Basic Problems in Continuum and Statistical Mechanics, Chap 7. ***Fractals and Fractional Calculus in Continuum Mechanics.*** CISM Courses and Lectures, Springer, Vienna.

Mainardi, F. 2010. ***Fractional Calculus And Waves in Linear Viscoelasticity, An Introduction to Mathematical Models***. Imperial College Press, London.

Mechaute, A. L., Crepy, G., 1983. Introduction to transfer and motion in fractal media: the geometry of kinetics. ***Solid State Ionics***, **9**: 17-30.

Meise, R., Vogt, D., 1997. ***Introduction to Functional Analysis***. The Clarendon Press, Oxford University Press, New York.

Metzler, R., Schick, W., Kilian, H. G., Nonnenmacher, T. F., 1995. Relaxation in filled poşymers: a fractional calculus approach. ***The Journal of Chemical Physics***, **103** (16): 7180-7186.

Miller, K. S., Ross, B., 1993. ***Introduction to Fractional Calculus and Differential Equations***. John Wiley & Sons, New York, USA.

Podlubny, I, 1999. ***Fractional Differential Equations***. Mathematics in Science and Engineering, Academic Press, 198, San Diego, USA.

Qiao, Y., Zhou, Z., 2017. Existence of solutions for a class of fractional differential equations with integral and anti-periodic boundary conditions. ***Boundary Value Problems***, **2017**:11, 9 pp.

Rehman, M., Khan, R. A., 2010. Existence and uniqueness of solutions for multi-point boundary value problems for fractional differential equations. ***Applied Mathematics Letters,*** **23** (9): 1038–1044.

Sabatier, J., Agrawal, O. P., Machado, J. A. T., 2007. ***Advances in Factional Calculus. Theoretical Developments and Applications in Physics and Engineering***. Springer-Verlag, Dordrecht.

Samko, S. G., Kilbas, A. A., Marichev, O. I., 1993. ***Fractional Integrals and Derivatives.*** Gordon and Breach Science Publishers, Switzerland.

Scalas, E., Gorenflo, R., Mainardi, F., 2000. Fractional calculus and continuous time finance. ***Physica A: Statistical Mechanics and its Applications***, **284**: 376-384.

Sokolov, I. M., Klafter, J., Blumen, A., 2002. Fractional kinetics. ***Physics Today***, **55** (11): 48-53.

Su, C.M., Sun, J. M., Zhao, Y. H. 2017. Existence and uniqueness of solutions for bvp of nonlinear fractional differential equation. ***International Journal of Differential Equations***, **2017**, Art. ID 4683581, 7 pp.

Sudsutad, W., Tariboon, J., 2012. Boundary value problems for fractional differential equations with three-point fractional integral boundary conditions. ***Advances in Difference Equations***, **2012**: 93, 10 pp.

Tariboon, J., Ntouyas, S. K., Singubol, A., 2014. Boundary value problems for fractional differential equations with fractional multiterm integral conditions. ***Journal of Applied Mathematics***, **2014** Article ID 806156: 10.

Wang, G., Cabada, A., Zhang, L., 2014. Integral boundary value problem for nonlinear differential equations of fractional order on an unbounded domain. ***Journal of Integral Equations***, **26** (1): 117-129.

Wang, X., Guo, X., Tang, G., 2013. Anti-periodic fractional boundary value problems for nonlinear differential equations of fractional order. ***Journal of Applied Mathematics & Computing***, **41**: 367–375.

Yan, R., Sun, S., Sun, Y., Han, Z., 2013. Boundary value problems for fractional differential equations with nonlocal boundary conditions. ***Advances in Difference Equations***, **2013**:176

Yuste, S. B., Acedo, L., Linderberg, K., 2004. Reaction front in an A+B→C reaction-subdiffusion process. ***Physical Review E***, **69**: 036126.

Zhang, S., 2006. Existence of solution for a boundary value problem of fractional order. ***Acta Mathematica Scientia. Series B. English Edition***, **26** (2): 220–228.

Zhang, L., Ahmad, B., Wang, G., Agarwal, R. P., Al-Yami, M., Shammakhi, W., 2013. Nonlocal integrodifferential boundary value problem for nonlinear fractional differential equations on an unbounded domain. ***Abstract and Applied Analysis***, **2013**, Article ID 813903, 5 pp.

APPENDIX

GENİŞLETİLMİŞ TÜRKÇE ÖZET
EXTENDED TURKISH SUMMARY

Bu tezde yarık şerit ve çoklu şerit sınır koşullarına sahip kesirli sınır değer problemlerinin çözümlerinin varlık ve tekliği için teoremler elde edilmiştir. Kesirli diferansiyel denklemler içlerinde daha fazla değişken barındırdıkları için bir çok fiziksel olayın modellemesini adi diferansiyel denklemlerden daha doğru, gerçekçi ve pratik yaparlar. Kesirli integral ve türev operatörlerinin avantajı ya da doğalarının dikkat çekici özellikleri yerel olmamalarıdır, yani kesirli türevi içeren dinamik sistemlerin ya da süreçlerin gelecek zamandaki durumu hem şu andaki hem de geçmiş zamandaki durumuna bağlıdır. Bu operatörlerin hafıza ya da miras özellikleri bir çok materyalin ve sürecin geçmiş tarihini izlememize izin verir. Kesirli sınır değer problemleri ise matematiksel bakış aşısına göre popüler bir araştırma alanıdır ve biyoloji, epidemiyoloji, fizik, mühendislik, kimya, hidroloji, finans, klasik mekanik, kuantum mekaniği, viskoelastisite, elektrik devreleri, nöron modellemesi ve benzeri alanlarda uygulamalara sahiptir.

Bu tez beş bölümden oluşmaktadır. Birinci bölüm giriş niteliğinde olup yarık şerit ve çoklu şerit sınır koşullarına sahip kesirli sınır değer problemleri ile ilgili literatür taramasını içermektedir. 2. bölümde fonksiyonel analiz ve kesirli diferansiyel denklemlerden gelen gerekli tanım ve teoremleri içeren ön kavramlar verilmiştir.

Tezin 3. bölümünde Ahmad and Agarwal (2014) makalesi detaylı incelenmiş, elde edilen teoremlerin ispatları açıkca yapılmıştır. Bu bölümde mertebesi $\rho \in (m-1, m],\ m \geq 2$ olan lineer olmayan Caputo tipi kesirli diferansiyel denklem

$$ {}_{0}^{C}D^{\rho}\phi(x) = \varphi(x, \phi(x)),\ x \in [0,1], \tag{5.0.1} $$

aşağıdaki yarık şerit tipi sınır koşullarıyla beraber ele alınmıştır:

$$ \begin{gathered} \phi(0) = 0,\ \phi'(0) = 0, \ldots,\ \phi^{(m-2)}(0) = 0, \\ \phi(\beta) = d_1 \int_0^{\alpha} \phi(w)dw + d_2 \int_{\gamma}^{1} \phi(w)dw, \quad 0 < \alpha < \beta < \gamma. \end{gathered} \tag{5.0.2} $$

Yarık şerit tipi sınır koşulları şu anlama gelir: Keyfi uzunluktaki kesişmeyen iki alt aralığın (şeridin) toplam etkisi bilinmeyen fonksiyonun alt aralıklar dışındaki ya da yarıktaki noktadaki değeriyle bağlantılıdır.

(5.0.1)-(5.0.2) kesirli sınır değer probleminin çözümlerinin varlık ve tekliği standart sabit nokta teoremleri (Banach daralma dönüşümü prensibi, Krasnoselski sabit nokta teoremi, Leary-Schauder alternatifi ve Tek değerli dönüşümler için lineer olmayan alternatif) kullanılarak elde edilmiştir. Bu teoremleri kullanmak amacıyla önce (5.0.1)-(5.0.2) kesirli sınır değer problemi $C = \beta^{m-1} - \frac{1}{m}(d_1\alpha^m + d_2 - d_2\gamma^n) \neq 0$ olmak üzere aşağıdaki integral denkleme dönüştürülmüştür.

$$\begin{aligned} \phi(x) = \frac{x^{m-1}}{C}&\left[\frac{d_1}{\Gamma(\rho+1)}\int_0^\alpha (\alpha-s)^\rho \varphi(s,\phi(s))ds + \frac{d_2}{\Gamma(\rho+1)}\int_0^1 (1-s)^\rho \varphi(s,\phi(s))ds\right. \\ &\left. - \frac{d_2}{\Gamma(\rho+1)}\int_0^\gamma (\gamma-s)^\rho \varphi(s,\phi(s))ds - \frac{1}{\Gamma(\rho)}\int_0^\beta (\beta-s)^{\rho-1}\varphi(s,\phi(s))\right] \\ &+ \frac{1}{\Gamma(\rho)}\int_0^x (x-s)^{\rho-1}\varphi(s,\phi(s))ds \end{aligned} \tag{5.0.3}$$

Daha sonra bu integral denklem kullanılarak

$$\begin{aligned} (\mathcal{P}\phi)(x) = \frac{x^{m-1}}{C}&\left[\frac{d_1}{\Gamma(\rho+1)}\int_0^\alpha (\alpha-s)^\rho \varphi(s,\phi(s))ds + \frac{d_2}{\Gamma(\rho+1)}\int_0^1 (1-s)^\rho \varphi(s,\phi(s))ds\right. \\ &\left. - \frac{d_2}{\Gamma(\rho+1)}\int_0^\gamma (\gamma-s)^\rho \varphi(s,\phi(s))ds - \frac{1}{\Gamma(\rho)}\int_0^\beta (\beta-s)^{\rho-1}\varphi(s,\phi(s))ds\right] \\ &+ \frac{1}{\Gamma(\rho)}\int_0^x (x-s)^{\rho-1}\varphi(s,\phi(s))ds. \end{aligned} \tag{5.0.4}$$

operatörü tanımlanmıştır. (5.0.4) operatörünün sabit noktası, yani $\mathcal{P}\phi = \phi$ eşitliğini sağlayan ϕ değerleri (5.0.1)-(5.0.2) kesirli sınır değer probleminin çözümü olduğundan (5.0.1)-(5.0.2) kesirli sınır değer probleminin çözümlerinin varlık ve teklik problemi (5.0.4) operatörünün sabit noktasının varlık ve teklik problemine indirgenmiştir. Ardından sonuçlarımızın uygulanabilirliğini doğrulayan bazı örnekler gösterilmiştir. Daha sonra bu sonuçlar yarıkdaki keyfi sayıdaki yerel olmayan nokta koşullu, yerel olmayan çoklu alt şerit koşullu ve Riemann-Liouville tipindeki yarık-şerit sınır koşullu kesirli sınır değer problemlerine uygulanmıştır.

Tezin 4. bölümünde Ahmad et al. (2013) makalesi detaylı incelenmiş, elde edilen teoremlerin ispatları açıkca yapılmıştır. Bu bölüm sonlu sayıda, çoklu ve keyfi uzunluktaki kesişmeyen şeritleri içeren Riemann-Liouville tipi sınır koşullu lineer olmayan keyfi mertebeden kesirli diferansiyel denklemlerin çözümlerinin varlık ve tekliği için teoremlere ayrılmıştır. Bu bölümde mertebesi $\rho \in (m-1,m]$, $m \geq 2$ olan lineer olmayan Caputo tipi kesirli diferansiyel denklem

$${}_0^C D^\rho \phi(x) = \varphi(x,\phi(x)),\ x \in [0,K], \tag{5.0.5}$$

aşağıdaki sonlu sayıda, çoklu şeritli integral sınır koşullarıyla beraber ele alınmıştır:

$$\begin{aligned} &\phi(0)=0,\ \phi'(0)=0,\ldots,\ \phi^{(m-2)}(0)=0, \\ &\phi(K)=\sum_{j=1}^{k}\gamma_j[I^{\lambda_j}\phi(\delta_j)-I^{\lambda_j}\phi(\theta_j)]. \end{aligned} \tag{5.0.6}$$

Burada her $j=1,2,\ldots,k$ için $\lambda_j>0$ ve I^{λ_j} mertebesi λ_j olan Riemann Liouville tipi kesirli integraldir. Ayrıca $0<\theta_1<\delta_1<\theta_2<\ldots<\theta_k<\delta_k<K$ ve her $j=1,2,\ldots,k$ için γ_j sabit sayılardır. Ele alınan sınır koşullarının fiziksel anlamı şudur: Sensörler aralığın ortasında yer aldığı için sınırdaki kontroller enerjiyi yayar ya da emer. Çözümlerin varlık sonuçları Krasnoselski sabit nokta teoremi, Leary-Schauder alternatifi ve Tek değerli dönüşümler için lineer olmayan alternatif uygulanarak elde edilecektir. Çözümlerin teklik sonuçları ise Banach daralma dönüşümü prensibi sayesinde oluşturulacaktır. Bu teoremleri kullanmak amacıyla önce (5.0.5)-(5.0.6) kesirli sınır değer problemi $\Lambda=K^{m-1}-\sum_{j=1}^{k}\gamma_j\frac{\left(\delta_j^{\lambda_j+m-1}-\theta_j^{\lambda_j+m-1}\right)\Gamma(m)}{\Gamma(\lambda_j+m)}\neq 0$ olmak üzere aşağıdaki integral denkleme dönüştürülmüştür.

$$\begin{aligned} \phi(x)=&\frac{1}{\Gamma(\rho)}\int_0^x(x-w)^{\rho-1}\varphi(w,\phi(w))dw-\frac{x^{m-1}}{\Lambda\Gamma(\rho)}\int_0^K(K-w)^{\rho-1}\varphi(w,\phi(w))dw \\ &+\sum_{j=1}^{k}\frac{x^{m-1}\gamma_j}{\Lambda\Gamma(\lambda_j+\rho)}\left[\int_0^{\delta_j}(\delta_j-w)^{\lambda_j+\rho-1}\varphi(w,\phi(w))dw-\int_0^{\theta_j}(\theta_j-w)^{\lambda_j+\rho-1}\varphi(w,\phi(w))dw\right] \end{aligned} \tag{5.0.7}$$

Daha sonra bu integral denklem kullanılarak

$$\begin{aligned} (\mathcal{P}\phi)(x)=&\int_0^x\frac{(x-w)^{\rho-1}\varphi(w,\phi(w))dw}{\Gamma(\rho)}-\frac{x^{m-1}}{\Lambda\Gamma(\rho)}\int_0^K(K-w)^{\rho-1}\varphi(w,\phi(w))dw \\ &+\sum_{j=1}^{k}\frac{x^{m-1}\gamma_j}{\Lambda\Gamma(\lambda_j+\rho)}\left[\int_0^{\delta_j}(\delta_j-w)^{\lambda_j+\rho-1}\varphi(w,\phi(w))dw-\int_0^{\theta_j}(\theta_j-w)^{\lambda_j+\rho-1}\varphi(w,\phi(w))dw\right] \end{aligned} \tag{5.0.8}$$

operatörü tanımlanmıştır. (5.0.8) operatörünün sabit noktası, yani $\mathcal{P}\phi=\phi$ eşitliğini sağlayan ϕ değerleri (5.0.5)-(5.0.6) kesirli sınır değer probleminin çözümü olduğundan (5.0.5)-(5.0.6) kesirli sınır değer probleminin çözümlerinin varlık ve teklik problemi (5.0.8) operatörünün sabit noktasının varlık ve teklik problemine indirgenmiştir. Daha sonra sonuçlarımızın uygulanabilirliğini doğrulayan bazı örnekler gösterilmiştir.

Son bölüm sonuç niteliğinde olup bu tezde yaptıklarımızın özeti şeklindedir.

CURRICULUM VITAE

The preparer of this research; Diyar Hashim Malo HAJANI, was born in December 1, 1986 in Duhok. He finished his primary school in Bzav in 1998 and the secondary school in Brayati in 2008. In 2012, got his B.Sc. in Mathematics from the Department of Mathematics, College of Education, University of Zakho.

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