DIFFUSION CONTROL IN CLOSED-LOOP SUPPLY CHAINS: SUCCESSIVE PRODUCT GENERATIONS WITH REMANUFACTURING POTENTIAL

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By

Büşra Güray

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We certify that we have read this thesis and that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

Emre Nadar(Advisor)

Ayşe Selin Kocaman

Ece Zeliha Demirci

Approved for the Graduate School of Engineering and Science:

Orhan Arıkan
Director of the Graduate School

ABSTRACT

DIFFUSION CONTROL IN CLOSED-LOOP SUPPLY CHAINS: SUCCESSIVE PRODUCT GENERATIONS WITH REMANUFACTURING POTENTIAL

Büşra Güray
M.S. in Industrial Engineering
Advisor: Emre Nadar
June 2023

We consider a durable-good producer who aims to jointly optimize its sales decisions for two successive product generations that are remanufacturable. The customer arrivals are governed by the generalized Norton-Bass diffusion process over a finite selling horizon. The remanufactured-item sales are constrained by the available end-of-use returns in each time period for each product generation. We investigate whether the producer can profit from partially satisfying the second-generation product demand to smooth out the second-generation diffusion curve and increase the total remanufactured-item sales in the long run. We show that the partial-fulfillment policy is optimal for fast-clockspeed products if (i) the profit margin ratio of the remanufactured item to the new item is large enough for the second-generation product, (ii) the profit margin ratio of the first-generation new item to the second-generation new item is high enough, (iii) the fraction of customers who are willing to buy the remanufactured item is only moderately large for each product generation, and (iv) the number of customers who are initially attracted by the first-generation product and willing to buy the remanufactured item is not too large. We also characterize the environmentally critical time period beyond which the optimal initiation of partial demand fulfillment leads to no improvement in the total remanufacturing volume for the second-generation product.

Keywords: sales planning; closed-loop supply chains; remanufacturing; multi-generation product diffusion; partial demand fulfillment.

ÖZET

KAPALI DÖNGÜ TEDARİK ZİNCİRLERİNDE DİFÜZYON KONTROLÜ: YENİDEN ÜRETİM POTANSİYELİ OLAN ARDIŞIK ÜRÜN NESİLLERİ

Büşra Güray
Endüstri Mühendisliği, Yüksek Lisans
Tez Danışmanı: Emre Nadar
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Yeniden üretilebilinir, ardışık iki ürün neslinin satış değerlerini optimize etmeyi amaçlayan bir dayanıklı tüketim malı üreticisini ele alıyoruz. Talep, sınırlı bir satış ufkunda genelleştirilmiş Norton-Bass difüzyon sürecine göre gelir.Yeniden üretim seçeneği, kullanım sonu ürün iadeleri sınırlaması altında gerçekleştirilir. Üreticinin, yeni-nesil yayılma eğrisini yumuşatmak ve uzun vadede toplam yeniden üretilmiş ürün satışlarını artırmak için yeni-nesil ürün talebini kısmen karşılamaktan kazanç sağlayıp sağlayamayacağını araştırıyoruz. Kısmi talep karşılama politikasının, (i) yeniden üretilmiş ürünün yeni-ürüne kâr marjı oranının yeni-nesil ürün için yeterince büyük olması, (ii) erken-nesil yeni ürünün yeni-nesil yeni ürüne kâr marjı oranının yeterince yüksek olması, (iii) yeniden üretilmiş ürün satın almak isteyen müşterilerin oranının olduğu ancak yalnızca orta derecede yüksek ve (iv) başlangıçta erken-nesil tarafından çekilen müşterilerden gelen yeniden üretilmiş ürün talebi çok büyük olmadığı durumlarda optimal olduğunu gösteriyoruz. Ayrıca kısmi talebin karşılanmasının, yeni neslin yeniden üretilmiş toplam hacminde iyileştirmeye yol açmadığı kritik zamanı da karakterize ediyoruz.

Anahtar sözcükler: satış planlama, kapalı-devre tedarik zincirleri, yeniden imalat, çok nesilli ürün yayılımı, kısmi talep karşılama.

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Chapter 1

Introduction

The amount of e-waste generated continues to grow more rapidly than the amount of e-waste disposed of in a responsible manner (United Nations [1]). In 2016, 44.7 million metric tonnes (Mt) of e-waste were generated worldwide, while 8.9 Mt of e-waste were properly disposed of (Baldé et al. [2]). In 2019, however, 53.6 Mt of e-waste were generated whereas 9.3 Mt of e-waste were properly disposed of. In 2030, the amount of e-waste is projected to strike 74.7 Mt (Forti et al. [3]). Improper disposal of such large amounts of e-waste threatens the achievement of the United Nations' 2030 Agenda for Sustainable Development, thereby increasing the significance of product recovery options.

For sustainable consumption and production, many producers of consumer electronics now develop circular economy approaches. Many firms tend to design new products that are easy to reuse, remanufacture, or recycle at the end of their use or life. Many firms also extend their business models by integrating the used-item acquisition and disposition processes into their existing supply chains and by entering the repair and/or refurbishing markets. This movement has led to the rise of closed-loop supply chains (CLSCs).

In the smartphone industry, for example, Apple utilizes several different channels to acquire their used devices for refurbishing practices (the Apple Trade In

Program, the iPhone Upgrade Program, and the third-party trade-in platforms). Apple focuses on improving the product longevity to maintain its value even after years of consumer usage. This not only enables the consumers to use their devices over longer time horizons but also allows the firm to refurbish and remarket the acquired used items more profitably. For example, the iPhone 6s (a 2015 year product) still had a monetary value for trade-in as of January 2022. Apple sells a wide range of refurbished products on their official website with a one-year warranty and Apple Certified Refurbished promise (Apple [4], [5]). Huawei's online "Vmall" program allows its customers to easily and swiftly return their used devices. Huawei inspects the product returns to group them into several different quality conditions. Some used items can be refurbished and resold through official sales channels, while some other used items can be recycled for material recovery. The e-waste is disassembled and disposed of by the certified third parties aiming to reduce environmental pollution (Huawei [6]). Samsung's "Eco-Partner Certification" program audits their suppliers every two years for sustainable sourcing of their raw materials. Samsung has collected 5.07 Mt of e-waste since 2009 for material recovery and reuse in new product manufacturing (Samsung [7]).

The economic value of remanufacturing the acquired used items depends heavily on the supply-loop constraints, which are shaped to a large extent by the diffusion dynamics of successive product generations in the fast-clockspeed industries. Although there is a vast amount of literature on CLSCs (see Atasu et al. [8], Guide and Wassenhove [9], Ferguson and Souza [10], Akçalı and Çetinkaya [11], Hassini et al. [12], Souza [13], and Govindan et al. [14] for comprehensive reviews), only a few papers to our knowledge have modeled their CLSC problems by taking a diffusion-process perspective (Debo et al. [15], Akan et al. [16], Nadar [17], Nadar et al. [18], and Uzunlar [19]). Such a perspective is particularly useful for accurately gauging the demand and return curves of fast-clockspeed durable goods such as consumer electronics. The above diffusion papers (except Uzunlar [19]), however, have restricted their CLSC settings to a single product generation. Successive generations of the same product as well as their new and refurbished versions may exist together in the same marketplace (McKie et al. [20]), leading to interdependent demand and return curves across generations that should be incorporated into the diffusion process (Norton and Bass [21], Michalakelis et al. [22], Jiang and Jain [23]).

In the CLSC literature, Uzunlar [19] has examined the optimal sales decisions by developing a more comprehensive perspective on two successive generations of the same durable good. In their problem setting, the demand arrivals follow the generalized Norton-Bass diffusion process developed by Jiang and Jain [23], and the first-generation product returns become available for material recycling so that the recycled content can be profitably used in the second-generation product manufacturing. Uzunlar [19] shows that slowing down the second-generation diffusion process via partial demand fulfillment is optimal for fast-clockspeed products when the first- and second-generation demand curves largely overlap and the second-generation product launch has only a limited impact on the customer base. Under these conditions, the partial-fulfillment policy enables the producer to integrate a much larger amount of the recycled material into new-item manufacturing. It also improves the total profit, despite some potential lost sales, since the use of virgin raw materials is reasonably assumed to be more expensive than the use of recycled materials. However, in the CLSC settings with successive product generations, it is still unknown whether partial demand fulfillment would ever be optimal if the durable good has the remanufacturing potential rather than the recycling potential. This study aims to fill this gap in the literature.

We study the sales planning problem of a durable-good producer who aims to maximize its total profit from the two successive product generations with remanufacturing potential. Like Uzunlar [19], we employ the generalized Norton-Bass diffusion model developed by Jiang and Jain [23] in our problem setting. This model encompasses the diffusion effect within each product generation as well as the substitution effect among successive product generations. This model also allows us to offer clean mathematical formulations for the leapfrogging behavior (of those customers who purchase the second-generation product by skipping the first-generation product) and the switching behaviour (of those customers who purchase both first- and second-generation products). The end-of-use returns of each product generation from the previous buyers become available for remanufacturing and remarketing. The remanufactured item has a larger profit margin than the new item for each product generation. This assumption is likely to hold in many cases since the remanufacturing process often consumes less material and energy than the new-item manufacturing process (Atasu et al. [24], Guide and Li [25], and Gutowski et al. [26]). In this problem setting, we investigate whether the producer can profit from partially satisfying the second-generation demand to slow down the diffusion process and increase the total remanufactured-item sales in the long run. We note that the partial-fulfillment policy can be implemented in practice by holding in-store inventory less than anticipated demand in certain time periods.

Our main contributions in the CLSC literature are as follows:

- Our study is the first attempt to examine the sales decisions of durable-good producers offering successive product generations with remanufacturing potential. This problem is more complicated than the one described in Uzunlar [19], because we need to keep track of the accumulated number of end-of-use returns for each product generation in our problem formulation. The remanufactured-item sales are bounded by the available end-of-use returns in each time period.

- The partial-fulfillment policy aimed at slowing down the second-generation product diffusion can be more desirable than the myopic-fulfillment policy aimed at selling as many items as possible in each period without taking any forward-looking approach. We outline sufficient conditions for optimality of this partial-fulfillment policy. Although these conditions are mathematically involved, we provide intuitive explanations that may have managerial implications in practice.

- We reveal that the partial-fulfillment policy is optimal if (i) the original diffusion process (free of any sales control) is fast enough, (ii) the profit margin ratio of the first-generation new item to the second-generation new item is high enough, (iii) the fraction of customers willing to buy the remanufactured item is only moderately large for each product generation, and (iv) the number of customers who are initially attracted by the first-generation product and willing to buy the remanufactured item is not too large.

The remainder of this thesis is organized as follows: Chapter 2 reviews the related literature. Chapter 3 formulates the sales-planning problem. Chapter 4 presents the analytical results as well as their illustrations with numerical experiments. Chapter 5 concludes. Unabridged versions of the analytical statements and their proofs are contained in Appendix A.

Chapter 2

Related Literature

Consumer purchasing behavior plays an important role in the theory of product (or technology) adoption and diffusion that has received much attention in the marketing literature. The new-product diffusion model introduced by Bass [27], [28] has by far been the most extensively studied model in the literature. The Bass diffusion model separates the social system into two groups depending on the consumer behaviour with regard to the timing of adoption. The "innovators" are not affected by others during their initial purchases of new products, while the "imitators" are influenced by the social system in their adoption of new products. The diffusion demand at any point in time is given by the number of customers who have not yet attempted to buy the item multiplied with the purchase likelihood of one such customer at that point in time. The purchase likelihood at any point in time increases linearly with the number of previous adopters who potentially influence the current customers. Bass [27], [28] provides strong empirical evidence to validate this diffusion model for durable goods.

Taking a major step toward multi-generation diffusion models, Norton and Bass [21] have extended the Bass diffusion model by incorporating the substitution effect between successive product generations that are simultaneously offered in the market. The Norton-Bass model involves a different customer base unique to each product generation such that the customers of earlier generations can also buy newer generations. The Norton-Bass model has been generalized by Jiang and Jain [23] for an explicit characterization of two types of substitution across generations: leapfrogging behavior vs. switching behavior. Some potential adopters of an early-generation product buy the new-generation product by skipping the early-generation product (leapfroggers), while some others buy the early-generation product now and the new-generation product later (switchers). The leapfrogging behaviour cannibalizes the early-generation demand, whereas the switching behavior induces repeat purchases by the same customer across generations. In this thesis, we employ the generalized Norton-Bass model that has been proposed by Jiang and Jain [23].

For forward supply chains (FSCs), one stream of research has studied the sales planning problem with single-generation diffusion models in the presence of limited production capacity. The myopic-fulfillment policy may induce the diffusion demand to peak in a short amount of time in a large magnitude. When the peak demand exceeds the production capacity, some demand cannot be fulfilled on time and some portion of the unmet demand may be lost. In order to address this issue, previous literature has evaluated the use of two alternative strategies: (i) Some inventory can be built by delaying the new-product launch and this inventory can be used to meet the demand during the peak-demand periods, or (ii) some demand can be deliberately backlogged in initial periods (even if inventory is available) to slow-down the diffusion process and smooth out the demand curve. The latter sales strategy corresponds to our partial-fulfillment policy in this study. While Ho et al. [29], [30] and Shen et al. [31] provide conditions under which the partial-fulfillment policy cannot be optimal, Kumar and Swaminathan [32] and Shen et al. [33] provide conditions under which it can be optimal. Several other papers have developed new-product diffusion models under supply constraints by considering the possibility of social exposure from those customers with backlogged demands. See Jain et al. [34] and Keith et al. [35]. The models described in these papers allow the waitlisted customers to generate a positive or negative word-of-mouth feedback about the product and thus influence the future diffusion demand. Another stream of research has constructed new-product diffusion models by including dynamic pricing decisions. See Bass [36], Li and

Huh [37], Li [38], and Zhang et al. [39]. Unlike these papers, we consider a CLSC setting as well as a multi-generation diffusion model in our study.

In the FSC literature, there are also papers that have focused on the market entry timing problem for new products with multi-generation diffusion models. Wilson and Norton [40] consider a setting in which the profit margin of the early-generation product is larger than that of the new-generation product. Their results suggest that the new-generation product should be launched immediately or should never be introduced (i.e., the "Now or Never" policy). Mahajan and Müller [41] consider the setting of Wilson and Norton [40] in the discounted-profit case. Their results suggest that the new-generation product should be launched immediately or its launch should be delayed until the maturity stage of the early-generation product (i.e., the "Now or Maturity" policy). Krankel et al. [42] show that the launch of the new-generation product should be delayed to benefit from technological advances to a larger extent. Tang et al. [43] show that the launch of the new-generation product should also be delayed in the presence of intense competition between manufacturers. Ke et al. [44] offer the optimal product-launch strategy by taking into account the inventory holding costs. Guo and Chen [45] examine the product-launch timing and pricing problem in response to different types of strategic consumer behavior. Jiang et al. [46] prove that the "Now or Never policy" is optimal for subscribe-to-use products.

Several other papers have examined the product rollover and product-introduction frequency strategies. Building upon the Norton-Bass diffusion model, Druehl et al. [47] consider the time-pacing strategy for new-product development in which newer generations are introduced at regular time intervals. While the fast pace of product introductions may cannibalize the early-generation demand too quickly, the slow pace can exploit the customers' willingness-to-pay for more advanced technology only to a limited extent. They find that the optimal pace is in general correlated with the diffusion rate, the market growth rate, and the margin decay rate. Koca et al. [48] study the product rollover problem by incorporating dynamic pricing and inventory decisions. They find that the single roll strategy is more desirable than the dual roll strategy if the diffusion process is fast, the market is highly responsive to preannouncements, and the new-generation product provides a large performance improvement over the early-generation product. Liaoa and Seifert [49] analytically derive the optimal pace of product introductions under the single roll strategy. They formulate the sales quantity of each product generation as a function of the technical decay and installed base effects. Our study is different from all of these papers with its focus on a CLSC setting.

In the CLSC literature, one stream of research has adopted the single-generation diffusion models in the absence of flexibility to manipulate the diffusion process. See Georgiadis et al. [50], Georgiadis and Athanasiou [51], and Wang et al. [52]. Unlike these papers, we consider a multi-generation diffusion model and provide the flexibility to manipulate the diffusion process via our sales planning. Another stream of research allows for endogenous modeling of the diffusion process in their CLSC settings. Gaur et al. [53] study the CLSC configuration problem by developing an integrated optimization model for new and reconditioned products, employing the Bass diffusion process as an endogenous model input, and presenting a real-world case study from battery manufacturing. They find that the sales-price ratio of the new battery is negatively correlated with the maximum acquisition price of the used battery, and the sales-price ratios of new and reconditioned batteries should be jointly considered to determine the total net profit. Debo et al. [15] endogenize the pricing decisions for new and remanufactured versions of the same product generation over an infinite selling horizon. They extend the diffusion model in Bass [36] by incorporating supply constraints, repeat purchases, variable market sojourn time, and imperfectly substitutable new and remanufactured items. They assume that the coefficient of imitation is a function of the installed base of new products. Robotis et al. [54] consider a producer who leases perfectly substitutable new and remanufactured products to consumers with a constrained production and service capacity. The producer controls the diffusion process via leasing price and leasing duration decisions. Akan et al. [16] look into the sales decisions for imperfectly substitutable new and remanufactured products in a setting in which the producer has an ample production capacity and controls the diffusion process via pricing and sales decisions. The partial-fulfillment policy cannot be optimal for the remanufactured product in their setting. Unlike these papers, we consider a multi-generation diffusion model in our study.

In a more recent paper, Nadar et al. [18] take a new look at the sales planning problem for imperfectly substitutable new and remanufactured products of the same generation that are available over a finite selling horizon. They employ the Bass diffusion model by allowing for partial backlogging for any unmet demand and heterogeneous consumers with respect to their preferences for return timing. They show that the partial-fulfillment policy is optimal for new items when the diffusion process is fast, the remanufactured-item profit margin is high, and the remanufactured-item demand is only moderately large. The partial-fulfillment policy should be initiated earlier if the diffusion process is faster, the remanufactured-item demand is lower, or the number of end-of-use returns is larger. The unmet demand should be larger when the word-of-mouth effect dominates the diffusion process or when the remanufactured-item demand is lower. In this study, we extend the problem setting in Nadar et al. [18] by allowing for two successive generations of the same product as well as their new and remanufactured versions that may exist together in the same marketplace. We therefore employ the generalized Norton-Bass model of Jiang and Jain [23] for our extension.

Finally, as mentioned in Chapter 1, Uzunlar [19] implements the generalized Norton-Bass model into their sales planning problem for two successive product generations with recycling potential. Early-generation product returns are acquired for material recovery and reuse in new-generation product manufacturing. Uzunlar [19] establishes sufficient conditions for optimality of the partial-fulfillment policy in two possible scenarios of the consumer return process: the trade-up programs vs. the recycling programs via convenient mail-back options. The partial-fulfillment policy is more likely to be optimal if most returns are collected from the latter channel. In this study, unlike Uzunlar [19], we consider two successive product generations with remanufacturing potential. Remanufacturing is different from recycling in that it "recovers value from a used product by replacing worn components or reprocessing used parts to restore the used product to like-new condition for resale" (Nadar et al. [55], p. 1).

Chapter 3

Problem Formulation

We consider a durable-good producer who sells two successive generations of a product over a discrete-time horizon of T periods. The first-generation product can be purchased by customers in each period. However, since the second-generation product is launched in period τ, it can be purchased by customers only in periods $t \geq \tau$. The customer arrival process follows the generalized Norton-Bass model proposed by Jiang and Jain [23] with slight modifications. In the generalized Norton-Bass model, a consumer population of size m_1 is initially attracted by the first-generation product. The consumers arriving in periods $t < \tau$ are "switchers." These consumers first purchase the first-generation product and then want to adopt the second-generation product in periods $t \geq \tau$. The consumers arriving in periods $t \geq \tau$ for their initial purchase of the product can be either "leapfroggers" or "switchers." The leapfroggers buy the second-generation product by skipping the first-generation product. The switchers buy the first-generation product in the current period and want to adopt the second-generation product in future periods. Another consumer population of size m_2 is only attracted by the second-generation product and can buy the product only in periods $t \geq \tau$.

The generalized Norton-Bass model assumes that all demand for the two generations is immediately met in each period. This model also has the property that the Bass diffusion dynamics hold separately for the two consumer populations. The Bass diffusion demand for the first-generation product in period $t \geq 1$, if the population of size m_2 were ignored, would be given by

$$\breve{d}_{1t}^{B} = \left(p_1 + \frac{q_1 \breve{D}_{1t}^{B}}{m_1}\right)\left(m_1 - \breve{D}_{1t}^{B}\right), \tag{3.1}$$

where p_1 and q_1 are the coefficients of innovation and imitation for the first-generation product, respectively, and $\breve{D}_{1t}^{B}$ is the total number of consumers who are initially attracted by the first-generation product and have bought the product up to period t (i.e., $\breve{D}_{11}^{B} = 0$ and $\breve{D}_{1t}^{B} = \sum_{i=1}^{t-1} \breve{d}_{1i}^{B}$, $\forall t > 1$). In the Bass diffusion model, the term $\left(p_1 + \frac{q_1 \breve{D}_{1t}^{B}}{m_1}\right)$ often refers to the initial-purchase likelihood of an individual consumer in period t. Therefore, since $\breve{D}_{1t}^{B}$ converges to m_1 for a large t, we assume that $p_1 + q_1 \leq 1$. See Bass ([27], [28]) for details. Likewise, the Bass diffusion demand for the second-generation product in period $t \geq \tau$, if the population of size m_1 were ignored, would be given by

$$\breve{d}_{2t}^{B} = \left(p_2 + \frac{q_2 \breve{D}_{2t}^{B}}{m_2}\right)(m_2 - \breve{D}_{2t}^{B}), \tag{3.2}$$

where p_2 and q_2 are the coefficients of innovation and imitation for the second-generation product, respectively, and $\breve{D}_{2t}^{B}$ is the total number of consumers who are only attracted by the second-generation product and have bought the product up to period t (i.e., $\breve{D}_{2t}^{B} = 0$, $\forall t \leq \tau$, and $\breve{D}_{2t}^{B} = \sum_{i=\tau}^{t-1} \breve{d}_{2i}^{B}$, $\forall t > \tau$). Similar to our assumption in equation (1), we assume that $p_2 + q_2 \leq 1$.

In addition to the Bass diffusion dynamics described above, the substitution effect among the two generations is also incorporated by the generalized Norton-Bass model: The first-generation demand in period $t \geq 1$ is defined as

$$\breve{d}_{1t}^{NB} = \breve{d}_{1t}^{B} - \frac{\breve{d}_{1t}^{B} \breve{D}_{2(t+1)}^{B}}{m_2} \tag{3.3}$$

and the second-generation demand in period $t \geq \tau$ is defined as

$$\breve{d}_{2t}^{NB} = \breve{d}_{2t}^{B} + \frac{\breve{d}_{1t}^{B} \breve{D}_{2(t+1)}^{B}}{m_2} + \frac{\breve{D}_{1t}^{B} \breve{d}_{2t}^{B}}{m_2}. \tag{3.4}$$

The terms $\frac{\breve{d}_{1t}^{B} \breve{D}_{2(t+1)}^{B}}{m_2}$ and $\frac{\breve{D}_{1t}^{B} \breve{d}_{2t}^{B}}{m_2}$ above represent the numbers of leapfrogging and switching adopters from the customer population initially attracted by the first-generation product in period t, respectively. The word-of-mouth effect can be generated in this customer population only by the adopters of this customer population, even if these adopters have purchased the second-generation product. We refer the reader to Jiang and Jain [23] for further explanations. Note that $\breve{d}_{1t}^{NB} = \breve{d}_{1t}^{B}$ and $\breve{d}_{2t}^{NB} = 0$ if $t < \tau$.

Unlike the generalized Norton-Bass model, our sales planning model provides the flexibility that the producer can partially satisfy the demand that arises from the customer population only attracted by the second-generation product in any period. Such a sales plan can be implemented in practice under the condition that the customers from the population of size m_1 arrive earlier than the customers from the population of size m_2 during each period $t \geq \tau$. Under this condition, if the on-hand inventory of second-generation items in period t is larger than the total number of leapfrogging and switching adopters in period t but smaller than the total demand in period t, some customers only from the population of size m_2 cannot buy the item in period t. Therefore, for our sales planning model, we define s_{2t}^{B} as the sales volume for the customer population of size m_2 in period $t \geq \tau$ and S_{2t}^{B} as the cumulative sales volume for this customer population up to period t (i.e., $S_{2t}^{B} = 0$, $\forall t \leq \tau$, and $S_{2t}^{B} = \sum_{i=\tau}^{t-1} s_{2i}^{B}$, $\forall t > \tau$).

We now incorporate the sales decisions into the Bass diffusion demand for the second-generation product that we previously defined for the generalized Norton-Bass model. With a slight change of notation, we reformulate the Bass diffusion demands in period t for our sales planning model as follows:

$$d_{1t}^{B} = \left(p_1 + \frac{q_1 D_{1t}^{B}}{m_1}\right)\left(m_1 - D_{1t}^{B}\right) \tag{3.5}$$

and

$$d_{2t}^{B} = \left(p_2 + \frac{q_2 S_{2t}^{B}}{m_2}\right)\left(m_2 - D_{2t}^{B}\right), \tag{3.6}$$

where $D_{11}^{B} = 0$, $D_{1t}^{B} = \sum_{i=1}^{t-1} d_{1i}^{B}$, $\forall t > 1$, $D_{2t}^{B} = 0$, $\forall t \leq \tau$, and $D_{2t}^{B} = \sum_{i=\tau}^{t-1} d_{2i}^{B}$, $\forall t > \tau$. Although the Bass diffusion demand for the first-generation product is not affected by the sales decisions, the observed demands for both generations are linked to the sales decisions through the terms d_{2t}^{B} and D_{2t}^{B}. Specifically, for our sales planning model, the first-generation demand in period $t \geq 1$ is given by

$$d_{1t}^{NB} = d_{1t}^{B} - \frac{d_{1t}^{B} D_{2(t+1)}^{B}}{m_2} \tag{3.7}$$

and the second-generation demand in period $t \geq \tau$ is given by

$$d_{2t}^{NB} = d_{2t}^{B} + \frac{d_{1t}^{B} D_{2(t+1)}^{B}}{m_2} + \frac{D_{1t}^{B} d_{2t}^{B}}{m_2}. \tag{3.8}$$

Note that $d_{1t}^{NB} = \breve{d}_{1t}^{NB} = \breve{d}_{1t}^{B}$ and $\breve{d}_{2t}^{NB} = 0$, $\forall t < \tau$. Our diffusion model includes the generalized Norton-Bass model as a special case when the producer chooses to meet all demand in each period. See Figure 3.1 for an illustration of two different sales plans in our diffusion model. We note from Figure 3.1 that partially satisfying the demand from the population of size m_2 slows down the second-generation product diffusion. This curbs the leapfrogging behavior and increases the first-generation product sales.

We might also allow the producer to partially satisfy the demand that arises from the customer population initially attracted by the first-generation product in any period. However, partial fulfillment of the first-generation demand slows down the first-generation product diffusion and escalates the leapfrogging behavior. This reduces the total number of repeat purchases over the entire selling horizon. Such a sales plan also slows down the consumer return process. This limits the producer's benefit from recycling in second-generation product manufacturing (if recycling is possible). Similar issues also arise when some demand for the second-generation product from leapfrogging adopters is rejected. Finally, the total number of repeat purchases again declines when some demand for the second-generation product from switching adopters is rejected. Therefore, intuitively, rejecting some demand from the customer population initially attracted by the first-generation product is not advisable.

We denote by s_{2t} the sales volume of the second-generation product in period $t \geq \tau$. Since the customer population initially attracted by the first-generation product can always buy the product according to our sales planning model, $s_{2t} = s_{2t}^{B} + \frac{d_{1t}^{B} D_{2(t+1)}^{B}}{m_2} + \frac{D_{1t}^{B} d_{2t}^{B}}{m_2}$, $\forall t \geq \tau$. We assume that the fraction α of the unmet demand in any period is backlogged while the remaining fraction is lost. We also assume that the customers whose demands were rejected in any previous period keep no memory of how many periods they have waited till their demands are satisfied. These assumptions have been widely adopted in the related literature;

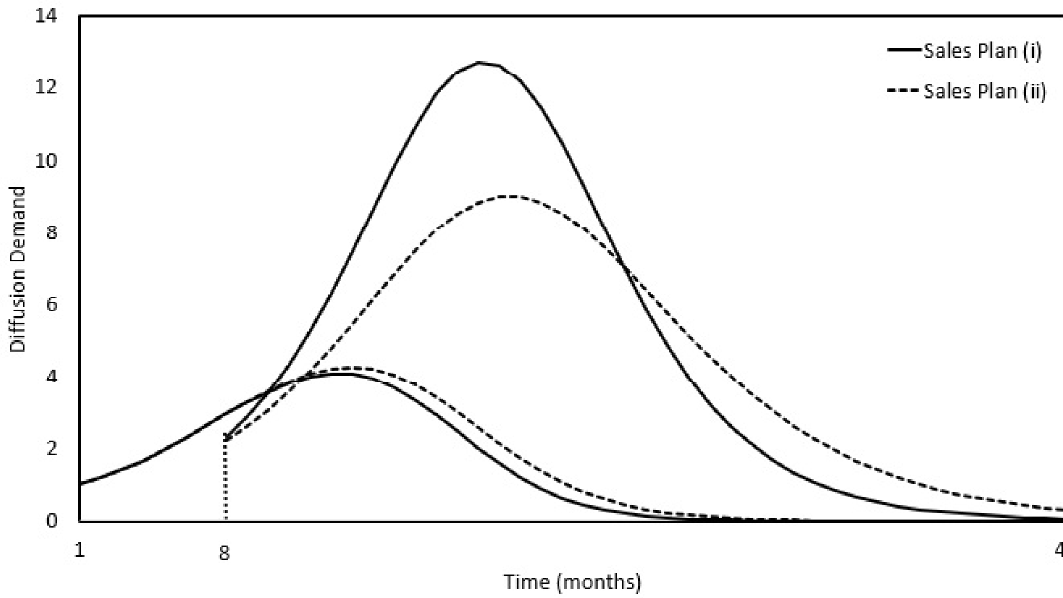


Figure 3.1: Multi-generation product diffusion when $\tau = 8$, $T = 48$, $p_1 = 0.01$, $p_2 = 0.02$, $q_1 = q_2 = 0.20$, and $m_1 = m_2 = 100$. The first-generation product is introduced in period 1 and the second-generation product is released in period 8. Sales plan (i) meets all demand for both generations in each period and corresponds to the generalized Norton-Bass model. Sales plan (ii) meets all demand from the population of size m_1 and 70% of the diffusion demand from the population of size m_2 in each period.

see, Ho et al. [29], [30], Kumar and Swaminathan [32], Shen et al. [33], [31], and Nadar et al. [18].

In the CLSC literature, Nadar et al. [18] have studied the sales planning problem for a single generation of a remanufacturable durable good. We extend their sales planning model to successive product generations by allowing the first-generation product to be remanufacturable. Such an extension (denoted as Model 1 below) involves the Norton-Bass diffusion dynamics for the two generations, while the setting in Nadar et al. [18] relies only on the Bass diffusion dynamics for the single generation. In addition, we consider a further extension (denoted as Model 2 below) that allows both the first- and second-generation products to be remanufacturable.

- Model 1: The first-generation product returns can be remanufactured and remarketed. This model takes into account the evolution of the end-of-use return inventory for the first-generation product. Our analysis of Model 1 facilitates that of Model 2.

- Model 2: This model extends Model 1 by allowing the consumer returns of both first- and second-generation products to be remanufacturable and remarketable. This model takes into account the evolution of the end-of-use return inventories for both product generations.

3.1 Model 1

In this model, the producer is able to remanufacture and remarket the first-generation product returns. Also, recall that the producer can partially satisfy the demand from the population of size m_2 in any period. We denote by b_t the accumulated number of backorders in period t from the population of size m_2. The second-generation sales volume in period t must be no smaller than the total number of leapfrogging and switching adopters in period t, and must be no larger than the total second-generation demand observed in period t:

$$\frac{d_{1t}^B D_{2(t+1)}^B}{m_2} + \frac{D_{1t}^B d_{2t}^B}{m_2} \leq s_{2t} \leq d_{2t}^{NB} + b_t. \tag{3.9}$$

Note that $0 \leq s_{2t}^B \leq d_{2t}^B + b_t$ by definitions of s_{2t} and d_{2t}^{NB}. Taking $b_\tau = 0$, we can calculate $b_t,\ \forall t > \tau$, with the following recursion:

$$b_{t+1} = \alpha \left(d_{2t}^{NB} + b_t - s_{2t} \right). \tag{3.10}$$

The fraction β_i of the first-generation products sold in period t are returned to the producer by consumers at the end of their use within the next i periods. These end-of-use returns can be used for remanufacturing and resale in period $t + i$ or later. Letting $\beta \triangleq \sum_i \beta_i \leq 1$ where $i \in \{1, 2, ..., T\}$, we note that the fraction $(1 - \beta)$ of the first-generation products sold in any period never become available in the future. The fraction γ of the population of size m_1 prefers to buy the remanufactured item if there are remanufactured items in inventory and buys the new item otherwise. This fraction of the population is price sensitive and represents the functionality-oriented segment in Nadar et al. [18]. The remaining fraction buys only new items and represents the newness-conscious segment in Nadar et al. [18]. The consumer preferences for new versus remanufactured items remain the same throughout the selling horizon.

The demand for first-generation remanufactured items is given by γd_{1t}^{NB} in period t. We define r_t as the sales volume for first-generation remanufactured items in period t. We also define e_t as the end-of-use return volume available at the beginning of period t. The producer sells as many remanufactured items as possible in each period. Thus:

$$r_t = \min\left\{\gamma d_{1t}^{NB}, e_t\right\}. \tag{3.11}$$

Taking $e_1 = 0$, we can calculate $e_{t+1},\ \forall t \geq 1$, with the following recursion:

$$e_{t+1} = e_t - r_t + \sum_{i=1}^{t} \beta_i d_{1(t+1-i)}^{NB}. \tag{3.12}$$

Recall that the producer rejects no demand from the population of size m_1. If any customer is unable to purchase the first-generation remanufactured item due to limited return availability, she wants to buy the first-generation new item and the producer meets all such demand in each period. Therefore, the total sales volume for first-generation items (new or remanufactured) equals the diffusion demand d_{1t}^{NB} in each period t.

We define c_{1n} as the unit manufacturing cost of the first-generation new item and p_{1n} as the unit selling price of this item. We also define c_{1r} as the unit remanufacturing cost of the first-generation product return and p_{1r} as the unit selling price of the first-generation remanufactured item. Finally, we define c_{2n} as the unit manufacturing costs of the second-generation item and p_{2n} as the unit selling price of this item. We assume that $p_{1n} > c_{1n}$, $p_{1r} > c_{1r}$, $p_{2n} > c_{2n}$, and $(p_{1r} - c_{1r}) > (p_{1n} - c_{1n})$. The producer aims to maximize its total profit from the two generations over the entire selling horizon:

$$\max_{s_{2\tau},..,s_{2T}} \left\{ (p_{1n} - c_{1n}) \sum_{t=1}^{T} \left(d_{1t}^{NB} - r_t\right) + (p_{1r} - c_{1r}) \sum_{t=1}^{T} r_t + (p_{2n} - c_{2n}) \sum_{t=\tau}^{T} s_{2t} \right\}$$

subject to (3.5)-(3.12). The sales decisions for the population of size m_2 start to influence the first two summations above (i.e., the first-generation profits) in period $\tau+1$ with the formulations in (3.6) and (3.7). See Figure 3.2 for an illustration

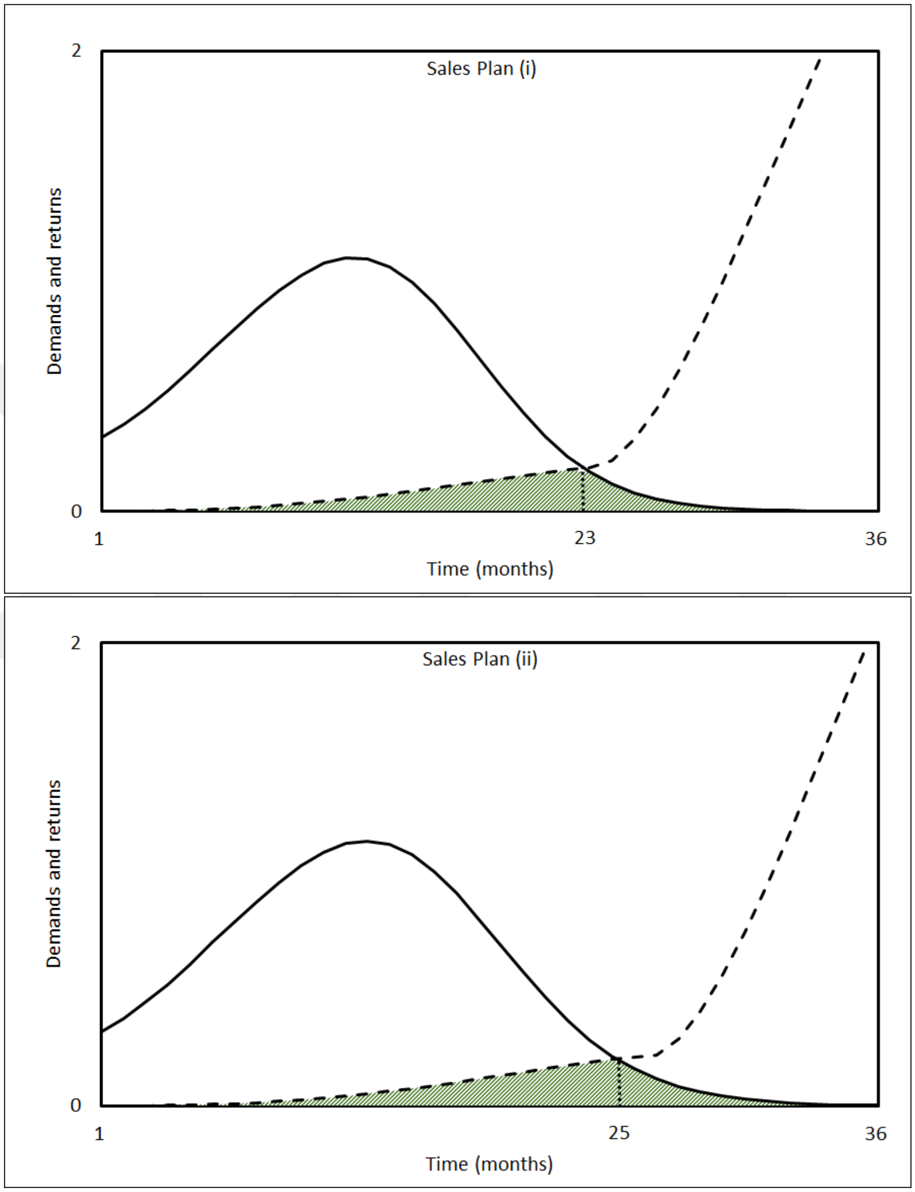


Figure 3.2: First-generation remanufactured-item demands (solid lines) and accumulated product returns (dashed lines) for two sales plans (i)-(ii). Sales plan (i) meets all demand for both generations, selling as many remanufactured items as possible. Sales plan (ii) meets all demand from the population of size m_1 and 70% of the diffusion demand from the population of size m_2, selling as many remanufactured items as possible. The shaded areas represent the total remanufactured-item sales volumes. The area in (ii) is larger by 12% than in (i). $\tau = 8$, $p_1 = 0.01$, $p_2 = 0.02$, $q_1 = q_2 = 0.20$, $m_1 = m_2 = 100$, $\gamma = 0.27$, and $\beta_i = (12\%) \times \mathbb{P}\{i - 0.5 \leq X \leq i + 0.5\}$, $\forall i \geq 1$, where $X \sim \text{Weibull}(25, 2)$.

of how different sales plans may influence the first-generation remanufactured-item sales. We note from Figure 3.2 that partially satisfying the demand from the population of size m_2 has the potential to improve the total sales volume for first-generation remanufactured items.

3.2 Model 2

In this model, the producer is able to remanufacture and remarket the consumer returns of both first- and second-generation products. We define n_{1t} as the first-generation new-item sales volume in period t and r_{1t} as the first-generation remanufactured-item sales volume in period t. Note that $n_{1t} + r_{1t} = d_{1t}^{NB}$. We also define n_{2t} as the second-generation new-item sales volume in period t and r_{2t} as the second-generation remanufactured-item sales volume in period t. The fraction β_i of the first-generation (or second-generation) products sold in period t are returned to the producer by consumers at the end of their use within the next i periods. These end-of-use returns can be used for remanufacturing and resale in period $t + i$ or later. The fraction γ_1 of the population of size m_1 prefers to buy the remanufactured item if there are remanufactured items in inventory and buys the new item otherwise. Likewise, the fraction γ_2 of the population of size m_2 prefers to buy the remanufactured item if there are remanufactured items in inventory and buys the new item otherwise.

We assume that, during each period, the newness-conscious customers attempt to purchase the new item earlier than the functionality-oriented customers who switch to buying the new item since the remanufactured item is unavailable. Also, during each period $t \geq \tau$, recall that the customers from the population of size m_1 arrive earlier than the customers from the population of size m_2. Our sales planning model in this section allows the producer to only reject the new-item demand from the functionality-oriented segment of the population of size m_2. We denote by b_{2t}^r the accumulated number of backorders in period t from this specific customer segment. Therefore, the second-generation sales volume in period t is constrained as follows:

$$\frac{d_{1t}^{B} D_{2(t+1)}^{B}}{m_2} + \frac{D_{1t}^{B} d_{2t}^{B}}{m_2} + (1-\gamma_2) d_{2t}^{B} \leq s_{2t} = n_{2t} + r_{2t} \leq d_{2t}^{NB} + b_{2t}^{r}. \tag{3.13}$$

Also, the second-generation new-item sales volume is constrained as follows:

$$\frac{(1-\gamma_1) d_{1t}^{B} D_{2(t+1)}^{B}}{m_2} + \frac{(1-\gamma_1) D_{1t}^{B} d_{2t}^{B}}{m_2} + (1-\gamma_2) d_{2t}^{B} \leq n_{2t} \leq d_{2t}^{NB} + b_{2t}^{r}. \tag{3.14}$$

Taking $b_{2\tau}^{r} = 0$, we can calculate b_{2t}^{r}, $\forall t > \tau$, with the following recursion:

$$b_{2(t+1)}^{r} = \alpha \left(d_{2t}^{NB} + b_{2t}^{r} - s_{2t} \right). \tag{3.15}$$

The demand for first-generation remanufactured items is given by $\gamma_1 d_{1t}^{NB}$ in period t. We define e_{1t} as the first-generation end-of-use return volume available at the beginning of period t. Thus:

$$r_{1t} = \min \left\{ \gamma_1 d_{1t}^{NB}, e_{1t} \right\}. \tag{3.16}$$

Taking $e_{11} = 0$, we can calculate $e_{1(t+1)}$, $\forall t \geq 1$, with the following recursion:

$$e_{1(t+1)} = e_{1t} - r_{1t} + \sum_{i=1}^{t} \beta_i d_{1(t+1-i)}^{NB}. \tag{3.17}$$

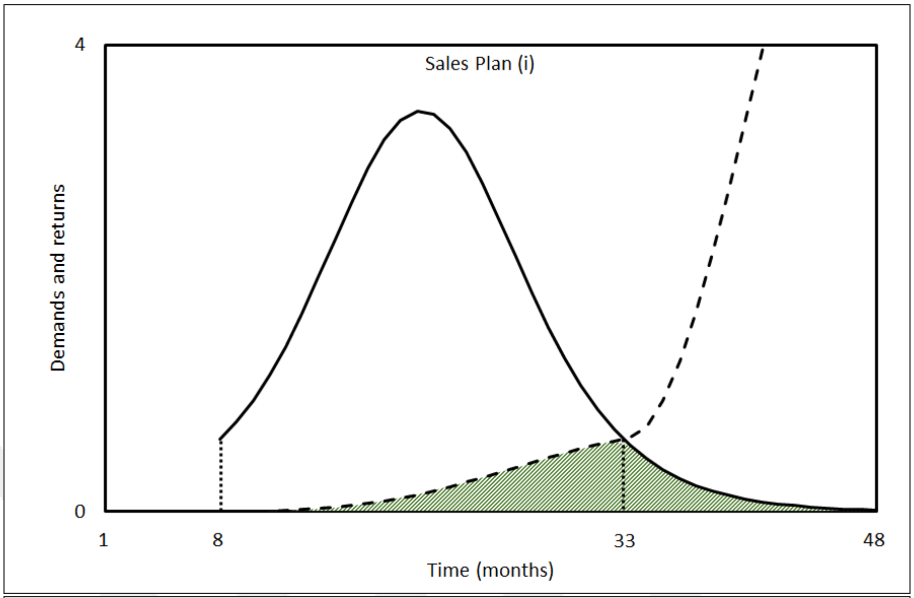
The demand for second-generation remanufactured items is given by $\gamma_2 d_{2t}^{B} + \frac{\gamma_1 d_{1t}^{B} D_{2(t+1)}^{B}}{m_2} + \frac{\gamma_1 D_{1t}^{B} d_{2t}^{B}}{m_2} + b_{2t}^{r}$ in period $t \geq \tau$. We define e_{2t} as the second-generation end-of-use return volume available at the beginning of period t. Thus:

$$r_{2t} = \min \left\{ \gamma_2 d_{2t}^{B} + \frac{\gamma_1 d_{1t}^{B} D_{2(t+1)}^{B}}{m_2} + \frac{\gamma_1 D_{1t}^{B} d_{2t}^{B}}{m_2} + b_{2t}^{r}, e_{2t} \right\}. \tag{3.18}$$

Taking $e_{2\tau} = 0$, we can calculate $e_{2(t+1)}$, $\forall t \geq \tau$, with the following recursion:

$$e_{2(t+1)} = e_{2t} - r_{2t} + \sum_{i=1}^{t} \beta_i s_{2(t+1-i)}. \tag{3.19}$$

Note that $s_{2t} = 0$, $\forall t < \tau$. The above formulations imply that the producer sells as many remanufactured items as possible in each period.



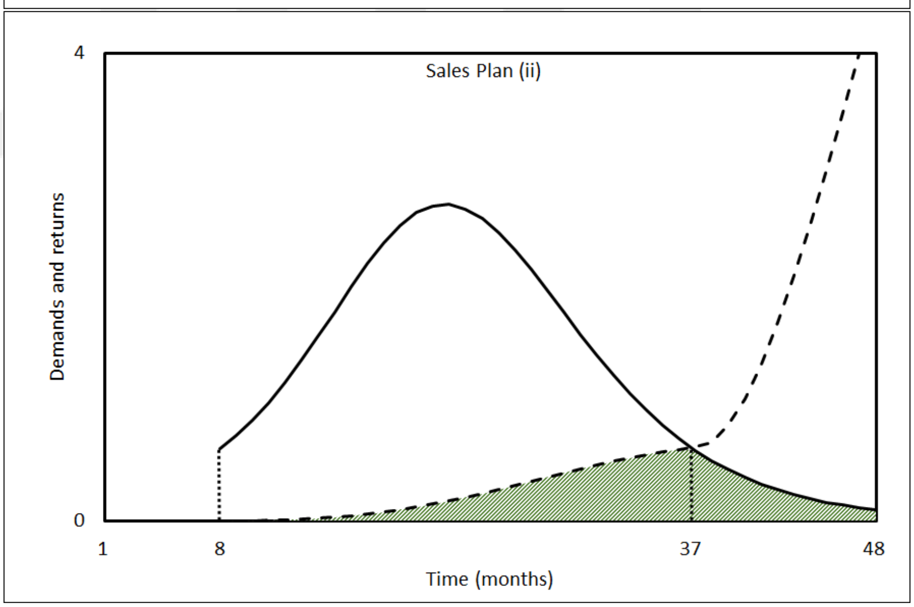


Figure 3.3: Second-generation remanufactured-item demands (solid lines) and accumulated product returns (dashed lines) for two sales plans (i)-(ii). Sales plan (i) meets all demand for both generations, selling as many remanufactured items as possible. Sales plan (ii) meets all demand from the population of size m_1 and 70% of the diffusion demand from the population of size m_2, selling as many remanufactured items as possible. The shaded areas represent the total remanufactured-item sales volumes. The area in (ii) is larger by 26% than in (i). $\tau = 8$, $p_1 = 0.01$, $p_2 = 0.02$, $q_1 = q_2 = 0.20$, $m_1 = m_2 = 100$, $\gamma_1 = \gamma_2 = 0.27$, and $\beta_i = (12\%) \times \mathbb{P}\{i - 0.5 \leq X \leq i + 0.5\}$, $\forall i \geq 1$, where $X \sim$ Weibull$(25, 2)$.

We define c_{2r} as the unit remanufacturing cost of the second-generation product return and p_{2r} as the unit selling price of the second-generation remanufactured item. We assume that $p_{2r} > c_{2r}$ and $(p_{2r} - c_{2r}) > (p_{2n} - c_{2n})$. The producer aims to maximize its total profit from the two generations over the entire selling horizon:

$$\max_{\substack{n_{21},..,n_{2T},\\ r_{21},..,r_{2T}}} \left\{ (p_{1n} - c_{1n}) \sum_{t=1}^{T} n_{1t} + (p_{1r} - c_{1r}) \sum_{t=1}^{T} r_{1t} + (p_{2n} - c_{2n}) \sum_{t=\tau}^{T} n_{2t} + (p_{2r} - c_{2r}) \sum_{t=\tau}^{T} r_{2t} \right\}$$

subject to (3.5)-(3.8) and (3.13)-(3.19). See Figure 3.3 for an illustration of how different sales plans may influence the second-generation remanufactured-item sales. We note from Figure 3.3 that partially satisfying the demand from the population of size m_2 has the potential to improve the total sales volume for second-generation remanufactured items.

Chapter 4

Analysis of Sales Plans

In this section, we investigate whether and when the producer might be better off by slowing down the product diffusion by partially satisfying the demand from the customer population only attracted by the second-generation product. For our investigation, we partition the producer's feasible sales plans into two different classes:

(i) The myopic-fulfillment policy: This policy meets all demand for the first- and second-generation products in each period, selling as many remanufactured items as possible. These fulfillment decisions are myopically optimal in each period. The diffusion process implied by this policy is equivalent to the generalized Norton-Bass model. We include the breve ($\breve{}$) in the problem variables under the myopic-fulfillment policy. Note that $\breve{s}^{B}_{2t} = \breve{d}^{B}_{2t}$, $\breve{S}^{B}_{2t} = \breve{D}^{B}_{2t}$, and $\breve{s}_{2t} = \breve{d}^{NB}_{2t}, \forall t \geq \tau$.

(ii) The partial-fulfillment policy: This policy meets all demand from the population of size m_1 but rejects some demand from the population of size m_2, again selling as many remanufactured items as possible in each period. We include the hat ($\hat{}$) in the problem variables under the partial-fulfillment policy.

The following proposition highlights two important implications of the partial-fulfillment policy for the demand structure of both product generations:

Proposition 1 *For all* $t > \tau$, $\widehat{d}_{1t}^{NB} \geq \breve{d}_{1t}^{NB}$. *When* T *is sufficiently large,* $\sum_{t=\tau}^{T} \widehat{D}_{1t}^{B} \widehat{d}_{2t}^{B} > \sum_{t=\tau}^{T} \breve{D}_{1t}^{B} \breve{d}_{2t}^{B}$.

Proof. Our proof steps are similar to those used by Uzunlar [19] to prove their Proposition 1. In each period t, we take the term $\frac{d_{1t}^{B} D_{2(t+1)}^{B}}{m_2}$ as the number of leapfrogging adopters, while Uzunlar [19] takes the term $\frac{d_{1t}^{B} D_{2t}^{B}}{m_2}$ as the number of leapfrogging adopters. The proof of Proposition 1 in Uzunlar [19] applies with this modification. □

Proposition 1 states that the partial-fulfillment policy helps the producer improve the first-generation demand after the second-generation product is launched and the partial-fulfillment policy is initiated. See Figure 3.1 for an example. Now suppose that the selling horizon is long enough so that all customers from both populations can arrive to buy the second-generation product. In this case, Proposition 1 states that the partial-fulfillment policy also helps the producer observe more switching adopters (but fewer leapfrogging adopters), in comparison with the myopic-fulfillment policy. Thus, the partial-fulfillment policy improves the number of repeat purchases across generations if the selling horizon is long enough. But this advantage comes at the expense of lost sales due to some unmet demand in the population of size m_2.

In Sections 4.1 and 4.2, we establish the conditions that provide the optimality of the partial-fulfillment policy in Models 1 and 2, respectively. See Theorems 1 and 2 below. Since our original theorems include lengthy mathematical expressions, we present below the simplified but intuitive versions of our theorems, relegating the unabridged versions to the Appendix A. Despite the complex nonlinear nature of the sales planning problem, all of the conditions in Theorems 1 and 2 rely on the generalized Norton-Bass model (free of any sales control) and the resulting closed-loop dynamics, and thus can be easily checked without solving any optimization problem.

4.1 Sufficient Conditions for Partial-Fulfillment Optimality: Model 1

Theorem 1. *The partial-fulfillment policy is optimal in Model 1 if, under the myopic-fulfillment policy, there exists a period $\psi < T$ such that*

(i) the accumulated first-generation return volume exceeds the first-generation remanufactured-item demand in each period $t \geq \psi$, and

(ii) the rejection of a unit of the second-generation product demand in period ψ induces a loss of diffusion demand in period $\psi + 1$ that is below a certain threshold (see Appendix A for the detailed expression) and a backlogged demand for the second-generation product in period $\psi + 1$ that is above a certain threshold (again see Appendix A for the detailed expression).

Proof. See Appendix A.

The above theorem shows that the producer can improve its total profit by rejecting some demand from the population of size m_2 in some period (including period ψ at least) under two conditions that must hold together: The partial-fulfillment policy curbs the leapfrogging behaviour and increases the first-generation product sales. Condition (i) in Theorem 1 implies that the first-generation product returns in future periods will still suffice to meet all of the future remanufactured-item demand that is to be raised by partial demand fulfillment. The partial-fulfillment policy thus also improves the remanufactured-item sales. Condition (ii) in Theorem 1 ensures that the revenue gain from improved first-generation sales outweighs the revenue loss associated with lower second-generation sales (that are reduced via the partial-fulfillment policy).

Considering a remanufacturable durable good of a single generation, Nadar et al. [18] show that the optimal initiation of the partial-fulfillment policy occurs in early stages of the product life cycle when the product returns are scarce: The total sales volume is reduced via the partial-fulfillment policy in Nadar et al. [18]. Their policy can only be desirable if it yields a greater sales volume for the remanufactured item. In their study, the remanufactured-item sales can only be increased via the partial-fulfillment policy initiated early enough. In this thesis, however, the existence of successive product generations nullifies this result: The partial-fulfillment policy in our study can be optimally initiated even when the accumulated number of product returns is greater than the demand for remanufactured items in future periods. This result can be explained by the additional benefit of the partial-fulfillment policy that can only arise in the multi-generation setting: It can potentially improve the repeat purchases across generations (and thus the total sales volume) even if initiated in future periods.

Using Theorem 1, we conduct numerical experiments to investigate the optimality of the partial-fulfillment policy in different environments. We construct a base scenario by choosing the parameter values calibrated for the smartphone industry: $T = 48$ months, $\tau = 12$, $p = p_1 = p_2 = 0.05$, $q = q_1 = q_2 = 0.35$, $m_1 = m_2$, $(p_{1n} - c_{1n}) = (p_{2n} - c_{2n})$, $(p_{1n} - c_{1n})/(p_{1r} - c_{1r}) = 0.5$, $\alpha = 0.88$, $\gamma = 0.27$, $\beta = 0.12$, and $\beta_i = \beta \times \mathbb{P}\{i - 0.5 \leq X \leq i + 0.5\}$, $\forall i \geq 1$, where X has a Weibull distribution with scale parameter 25 and shape parameter 2. These parameter values are consistent with those used by Jiang et al. [46] and Nadar et al. [18] in their numerical experiments for consumer electronics products. We refer the reader to Stremersch et al. [56] for an empirical justification for the constant values of the diffusion parameters (p and q) across the two generations in the consumer electronics industry. We generate many other instances by varying the parameter values of the base scenario. Specifically, we consider various values for p and q, various values for τ and m_1/m_2, various values for τ and γ, various values for τ and $(p_{2n} - c_{2n})/(p_{1n} - c_{1n})$, and various values for τ and $(p_{1r} - c_{1r})/(p_{1n} - c_{1n})$, respectively.

The partial-fulfillment optimality conditions in Theorem 1 fail to hold in each of our instances: The revenue gain from improved first-generation product sales (thanks to the partial-fulfillment policy) does not suffice to compensate the revenue loss from reduced second-generation product sales (due to the partial-fulfillment policy). Conversely, on a similar experimental test bed, the partial-fulfillment policy in Nadar et al. [18] is optimal for single-generation remanufacturable products in many instances. All these results together suggest that the benefit of circularity cannot be exploited fruitfully in our study by allowing only the first-generation product to be remanufacturable. Extending circularity to the second-generation product may amplify the value of the partial-fulfillment policy, motivating us to examine Model 2 in Section 4.2.

4.2 Sufficient Conditions for Partial-Fulfillment Optimality: Model 2

Theorem 2. *The partial-fulfillment policy is optimal if, under the myopic-fulfillment policy, there exists a period $\kappa < T$ such that*

(i) the accumulated first-generation return volume exceeds the first-generation remanufactured-item demand in each period $t > \kappa$,

(ii) the second-generation remanufactured-item demand exceeds the accumulated second-generation return volume in each period $t \leq \kappa$ whereas the reverse is true in each period $t > \kappa$, and

(iii) the rejection of a unit of the second-generation product demand in period κ induces a loss of diffusion demand in period $\kappa + 1$ that is below a certain threshold (see Appendix A for the detailed expression) and a backlogged demand for the second-generation product in period $\kappa + 1$ that is above a certain threshold (again see Appendix A for the detailed expression).

Suppose that the above conditions are met and T is sufficiently large. Then, if the partial-fulfillment policy is initiated after period κ, it provides no improvement in the total remanufacturing volume for the second-generation product.

Proof. See Appendix A.

The above theorem shows that the producer can improve its total profit by rejecting some demand from the population of size m_2 in some period (i.e., period κ) under three conditions that must hold together: Condition (i) in Theorem 2 is analogous to condition (i) in Theorem 1. The partial-fulfillment policy increases the first-generation remanufactured-item sales under condition (i). While the unmet demand reduces the future returns from the second-generation product sales, the delayed demand has the potential to increase the future demand for the second-generation remanufactured items. Condition (ii) in Theorem 2 implies that the second-generation product returns in future periods will still suffice to meet all of the future remanufactured-item demand that is possibly larger via the partial-fulfillment policy. The second-generation remanufactured-item sales can thus be potentially improved thanks to the partial-fulfillment policy. Condition (iii) in Theorem 2 ensures that the revenue gain from improved first-generation product sales and improved second-generation remanufactured-item sales outweigh the revenue loss associated with lower second-generation new-item sales (that are reduced via the partial-fulfillment policy).

Using Theorem 2, we conduct numerical experiments to investigate the optimality of the partial-fulfillment policy as well as the environmentally critical time period κ for the optimal initiation of the partial-fulfillment policy in different environments. For our base scenario in Section 4.1, we choose the values of the additional parameters unique to Model 2 as follows: $\gamma_1 = \gamma_2 = 0.27$ and $(p_{2n} - c_{2n})/(p_{2r} - c_{2r}) = 0.5$. We generate many other instances by varying the parameter values of the base scenario. Specifically, we consider various values for p and q, various values for τ and m_1/m_2, various values for τ and γ_1, various values for τ and γ_2, various values for τ and $(p_{2n} - c_{2n})/(p_{1n} - c_{1n})$, and various values for τ and $(p_{2n} - c_{2n})/(p_{2r} - c_{2r})$, respectively. Figures 4.1–4.3 exhibit our numerical results. We observe from Figures 4.1–4.3 that, unlike Model 1, the partial-fulfillment optimality conditions in Theorem 2 hold in a very large number of our instances for Model 2: The additional revenue gain from improved second-generation remanufactured-item sales (thanks to the partial-fulfillment policy) can indeed offset the revenue loss from reduced second-generation new-item sales (due to the partial-fulfillment policy).

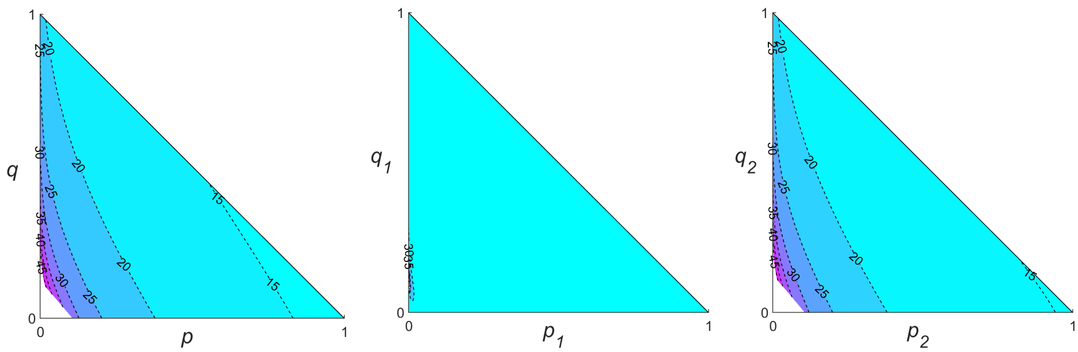


Figure 4.1: The diffusion parameters vs. the environmentally critical period κ. Colored regions indicate the partial-fulfillment optimality. Note that $p + q \leq 1$.

Figure 4.1 shows that the partial-fulfillment policy is optimal when the diffusion parameters p and q are large: Increasing p and q accelerates the diffusion process so that the delayed demand via the partial-fulfillment policy may still be observed in large amounts before the selling horizon ends. Figure 4.1 also shows that the diffusion parameters of the population of size m_2 (p_2 and q_2) have a larger impact on the critical time period κ (described in Theorem 2) than those of the population of size m_1 (p_1 and q_1): The critical time period κ is determined by the closed-loop dynamics of the second-generation product and the population of size m_2 plays a greater role on these dynamics than the population of size m_1. We also observe that the critical time period κ decreases as the diffusion parameters p_2 and q_2 increase: The accumulated return volume rises above the remanufactured-item demand earlier in the selling horizon when the diffusion process is faster. Finally, we note that the critical time period κ is larger when p_1 is too small and q_1 is moderately small: The diffusion process is too slow for the population of size m_1 in this case. This escalates the leapfrogging behavior and scales up the second-generation product demand. The accumulated return volume thus rises above the remanufactured-item demand later in the selling horizon.

Figure 4.2 shows that the critical time period κ may increase as m_1/m_2 increases when $\gamma_1 \geq \gamma_2$, while it may decrease as m_1/m_2 increases when $\gamma_1 < \gamma_2$. Notice that an increment in the size of either population increases the second-generation product sales and returns. Also, recall that the demand for second-generation remanufactured items equals $\gamma_2 d_{2t}^B + \frac{\gamma_1 d_{1t}^B D_{2(t+1)}^B}{m_2} + \frac{\gamma_1 D_{1t}^B d_{2t}^B}{m_2} + b_{2t}^r$ in period

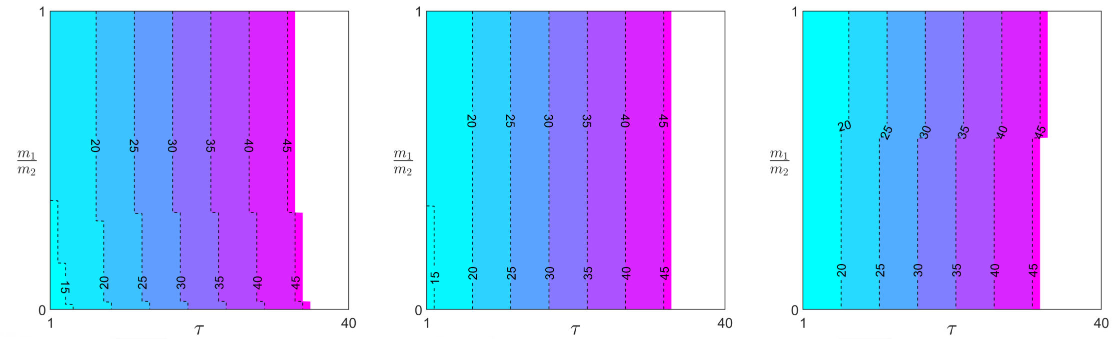


Figure 4.2: The population parameters vs. the environmentally critical period κ. Colored regions indicate the partial-fulfillment optimality. In the left plot $\gamma_1 = 0.5$ and $\gamma_2 = 0.1$. In the middle plot $\gamma_1 = \gamma_2 = 0.27$. In the right plot $\gamma_1 = 0.1$ and $\gamma_2 = 0.5$.

$t \geq \tau$. While the first term of this demand comes from the population of size m_2, the second and third terms come from the population of size m_1. When $\gamma_1 \geq \gamma_2$ (or $\gamma_1 < \gamma_2$), an increment in m_1 improves the second-generation remanufactured-item demands more (or less) significantly than the second-generation product returns, and thus it takes longer (or shorter) for the accumulated return volume to rise above the remanufactured-item demand in the selling horizon.

Figure 4.3 shows that the critical time period κ tends to increase as γ_1 or γ_2 grows. Since the demand for remanufactured items is higher when γ_1 or γ_2 is larger, it takes longer for the accumulated return volume to exceed the remanufactured-item demand in this case. We note that the accumulated return volume never reaches the remanufactured-item demand when γ_1 or γ_2 is too high and τ is large enough. Figure 4.3 also shows that the partial-fulfillment policy is optimal when the profit margin of the second-generation remanufactured item is significantly larger than that of the second-generation new item; condition (iii) in Theorem 2 is violated otherwise. Likewise, the partial-fulfillment policy is optimal when the profit margin of the first-generation new item is sufficiently large relative to (but not necessarily greater than) that of the second-generation new item; condition (iii) in Theorem 2 is violated otherwise. Finally, we note from Figures 4.2 and 4.3 that the partial-fulfillment policy is optimal when τ is not too large. The second-generation product sales start earlier in this case so that a large number of returns can arrive before the selling horizon ends. When τ is large,

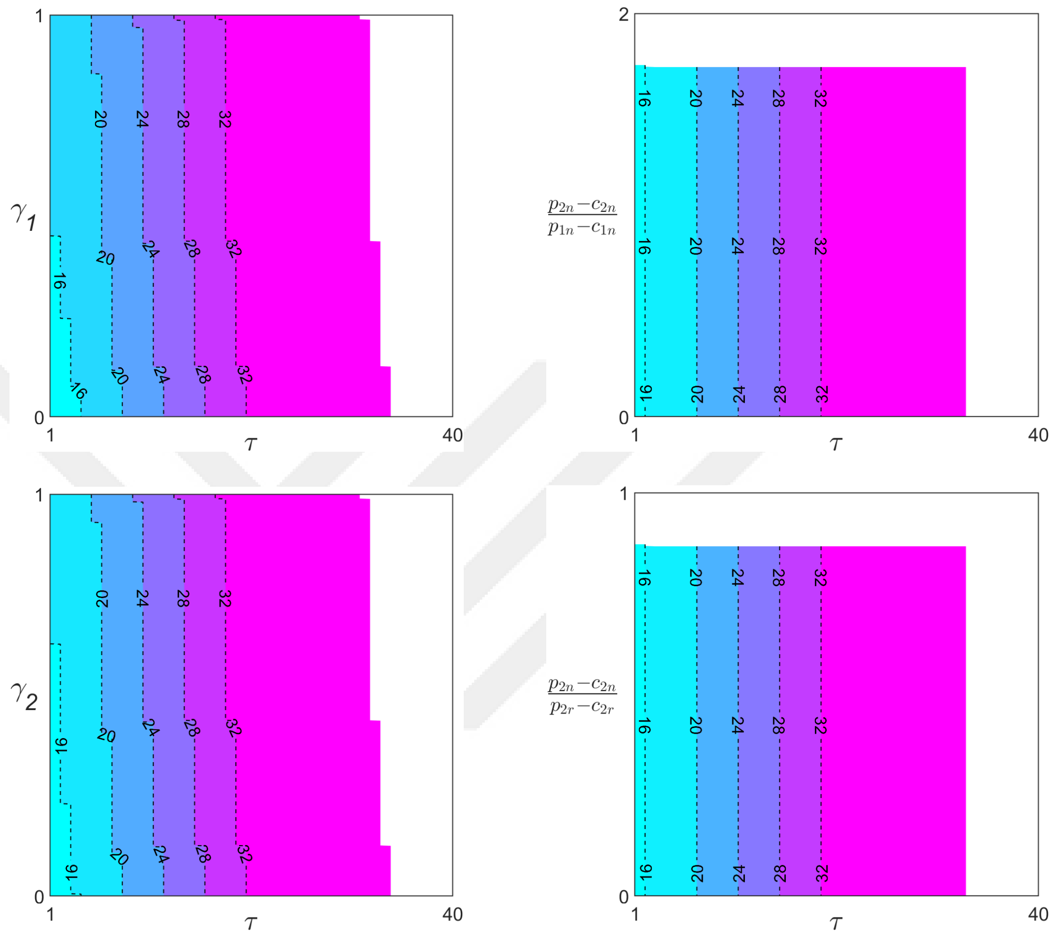


Figure 4.3: The remanufacturing parameters and profit margins vs. the environmentally critical period κ. Colored regions indicate the partial-fulfillment optimality.

on the other hand, the selling horizon is not long enough to observe the entire second-generation demand. The partial-fulfillment policy in this case slows down the second-generation product diffusion by inducing a loss of diffusion demand at the end of the selling horizon.

Chapter 5

Concluding Remarks

In this study, we have implemented the generalized Norton-Bass diffusion model into the sales planning problem for two successive product generations along with their new and remanufactured versions over a finite selling horizon. We have incorporated the supply-loop constraints related to the end-of-use items required for remanufacturing into our problem setting. To our knowledge, our study is the first attempt to model the multi-generation diffusion process for durable goods with remanufacturing potential. The partial-fulfillment policy has the potential to delay the second-generation diffusion demand (due to lowered word-of-mouth effect) and to reduce the second-generation new-item sales (due to partial backlogging). Our results, however, show that the partial-fulfillment policy can still help improve the total profit in the long run under certain conditions.

Specifically, the partial-fulfillment policy is desirable when the diffusion process is fast, the second-generation product is released early, the profit margin ratio of the remanufactured item to the new item is large for the second-generation product, the profit margin ratio of the first-generation new item to the second-generation new item is high, and the remanufactured items are attractive only for a limited number of customers. These conditions enable the partial-fulfillment policy to profitably raise the remanufactured-item sales. We have also conducted numerical experiments with data-calibrated instances for smartphones (compiled from the related literature) to investigate the optimality of partial-fulfillment policy in realistic scenarios. The partial-fulfillment policy is more likely to be desirable when the producer enters the refurbished market for both product generations than when it restricts its refurbishing operations to the first-generation product.

This study thus widens our knowledge of diffusion control for remanufacturable durable goods: Based on similar experimental test beds, the partial-fulfillment policy can be desirable for a single product generation that is remanufacturable (see Nadar et al. [18]), whereas it can be undesirable for successive product generations only with the first-generation product being remanufacturable. However, again based on a similar test bed, the partial-fulfillment policy can be desirable if the producer expands its remanufacturing capacity to the second-generation product for a further benefit from the higher profit margin of the remanufactured item. Furthermore, the benefit of partial-fulfillment policy is likely to be larger in the multi-generation case (with both generations being remanufacturable) than in the single-generation case. This is because a slower second-generation diffusion process, in the multi-generation case, induces more switchers and thus more repeat purchases across generations.

Future extensions of this study may consider the options of recycling and remanufacturing together for the acquired used items of successive product generations. The producer might classify the used items into two distinct categories: the end-of-use returns available for remanufacturing vs. the end-of-life returns available for recycling. In the literature, Uzunlar [19] has revealed the potential benefits of the partial-fulfillment policy in a multi-generation setting in which the recycled content is obtained from the first-generation product returns and used in the second-generation product manufacturing. Combining their results with ours, one may intuitively expect the partial-fulfillment policy to be still desirable in the multi-generation setting with both recycling and remanufacturing options (again under certain conditions like a fast diffusion process).

Future research may also consider alternative used-item acquisition strategies rather than the one assumed in this study. For example, the trade-up programs have recently received considerable attention in the consumer electronics industry. These programs encourage the adopters of first-generation products to bring their old devices when they decide to buy newer-generation products. Therefore, in the presence of trade-up programs, any manipulation of the diffusion process has also the potential to influence the consumers' timing of returns. Another direction for future research is to extend our problem setting by incorporating the used-item acquisition costs and decisions that are possibly dependent on the used-item condition. In the literature, Nadar et al. [55] identify the role of different quality levels of used items in the optimal procurement strategies. Future work may also extend our problem setting by allowing for nonstationarity and/or randomness in the consumers' willingness-to-pay for remanufactured items. In the literature, Abbey et al. [57] present empirical evidence indicating the significant variability in consumer's discount factors for remanufactured items.

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Appendix A

Unabridged Versions of the Analytical Statements and Their Proofs

We provide below the unabridged versions of Theorems 1 and 2 in Chapter 4 (Theorems A.1 and A.2 below) and their proofs.

Theorem A.1.

(a) Suppose that $p_{1n} - c_{1n} > p_{2n} - c_{2n}$. For optimality of partial demand fulfillment in some period over the T-period selling horizon, it is sufficient that $\exists \psi \in \{\tau, ..., T-1\}$ s.t. $\breve{e}_t > \gamma \breve{d}_{1t}^{NB}$ for $t > \psi$, $\alpha > \frac{(1+A)(1-2B)}{1-B}$, and

$$\alpha > 1 + AE - \left(\frac{p_{1n} - c_{1n}}{p_{2n} - c_{2n}}\right) \frac{(1-\gamma) A\breve{d}_{1(\psi+1)}^{B}}{m_2} - \left(\frac{p_{1r} - c_{1r}}{p_{2n} - c_{2n}}\right) \frac{\gamma A\breve{d}_{1(\psi+1)}^{B}}{m_2},$$

where $A = q_2 - \frac{q_2 \breve{D}_{2(\psi+1)}^{B}}{m_2}$, $B = p_2 + \frac{q_2 \breve{D}_{2(\psi+1)}^{B}}{m_2}$, and $E = 1 + \frac{\breve{D}_{1(\psi+2)}^{B}}{m_2}$.

(b) Suppose that $p_{1n} - c_{1n} < p_{2n} - c_{2n}$. For optimality of partial demand fulfillment in some period over the T-period selling horizon, it is sufficient

that $\exists \psi \in \{\tau, ..., T-1\}$ *s.t.* $\breve{e}_t > \gamma \breve{d}_{1t}^{NB}$ *for* $t > \psi$, $\alpha > \frac{(1+A)(1-2B)}{1-B}$, *and*

$$\alpha > 1 + AE - \left(\frac{p_{1n} - c_{1n}}{p_{2n} - c_{2n}}\right) \frac{(1-\gamma) A\breve{d}_{1(\psi+1)}^B}{m_2} - \left(\frac{p_{1r} - c_{1r}}{p_{2n} - c_{2n}}\right) \frac{\gamma A\breve{d}_{1(\psi+1)}^B}{m_2}$$
$$- \left(\frac{p_{1n} - c_{1n}}{p_{2n} - c_{2n}} - 1\right) \sum_{t=\psi+2}^{T} \frac{A\breve{d}_{1t}^B}{m_2},$$

where $A = q_2 - \frac{q_2 \breve{D}_{2(\psi+1)}^B}{m_2}$, $B = p_2 + \frac{q_2 \breve{D}_{2(\psi+1)}^B}{m_2}$, *and* $E = 1 + \frac{\breve{D}_{1(\psi+2)}^B}{m_2}$.

Proof of Theorem A.1. Suppose that the conditions stated in Theorem A.1 hold. We will show that rejecting a demand of size $\epsilon > 0$ from the customers unique to the second-generation product in period $\psi \geq \tau$ while meeting all the remaining demand for both generations in period ψ and all demand for both generations in each period $t \neq \psi$ (sales plan i) is more profitable than meeting all demand for both generations in each period (sales plan ii). We use the hat ($\widehat{\ }$) and the breve ($\breve{}$) to denote the variables of sales plans (i) and (ii), respectively. Let $\widehat{D}_{1t}^{NB} = \sum_{i=1}^{t-1} \widehat{d}_{1i}^{NB}$ for $t > 1$ and $\widehat{D}_{11}^{NB} = 0$, $\breve{D}_{1t}^{NB} = \sum_{i=1}^{t-1} \breve{d}_{1i}^{NB}$ for $t > 1$ and $\breve{D}_{11}^{NB} = 0$, $\widehat{D}_{2t}^{NB} = \sum_{i=\tau}^{t-1} \widehat{d}_{2i}^{NB}$ for $t > \tau$ and $\widehat{D}_{2\tau}^{NB} = 0$, and $\breve{D}_{2t}^{NB} = \sum_{i=\tau}^{t-1} \breve{d}_{2i}^{NB}$ for $t > \tau$ and $\breve{D}_{2\tau}^{NB} = 0$. We make the following observations:

(1) Periods t $< \psi$: $\widehat{d}_{1t}^{NB} = \breve{d}_{1t}^{NB}$ and $\widehat{s}_{2t} = \breve{s}_{2t} = \widehat{d}_{2t}^{NB} = \breve{d}_{2t}^{NB}$.

(2) Period ψ: $\widehat{d}_{1\psi}^{NB} = \breve{d}_{1\psi}^{NB}$ and $\widehat{s}_{2\psi} = \breve{s}_{2\psi} - \epsilon$.

(3) Period $\psi+1$: Note that $\widehat{d}_{1t}^{B} = \breve{d}_{1t}^{B}, \forall t$. Rejecting a demand of size ϵ from the customers unique to the second-generation product in period ψ leads to a backlogged demand of size $\alpha\epsilon$ in period $\psi + 1$. Note that:

$$
\begin{aligned}
\widehat{d}^{NB}_{1(\psi+1)} &= \widehat{d}^{B}_{1(\psi+1)}\left(1-\frac{\widehat{D}^{B}_{2(\psi+2)}}{m_2}\right)\\
&= \widehat{d}^{B}_{1(\psi+1)}\left(1-\frac{\sum_{i=\tau}^{\psi+1}\widehat{d}^{B}_{2i}}{m_2}\right)\\
&= \widehat{d}^{B}_{1(\psi+1)}\left(1-\frac{\sum_{i=\tau}^{\psi}\widehat{d}^{B}_{2i}}{m_2}-\frac{\widehat{d}^{B}_{2(\psi+1)}}{m_2}\right)\\
&= \breve{d}^{B}_{1(\psi+1)}\left(1-\frac{\sum_{i=\tau}^{\psi}\breve{d}^{B}_{2i}}{m_2}-\left(p_2+\frac{q_2\widehat{S}^{B}_{2(\psi+1)}}{m_2}\right)\left(\frac{m_2-\widehat{D}^{B}_{2(\psi+1)}}{m_2}\right)\right)\\
&= \breve{d}^{B}_{1(\psi+1)}\left(1-\frac{\sum_{i=\tau}^{\psi}\breve{d}^{B}_{2i}}{m_2}-\left(p_2+\frac{q_2\breve{D}^{B}_{2(\psi+1)}}{m_2}-\frac{q_2\epsilon}{m_2}\right)\left(\frac{m_2-\breve{D}^{B}_{2(\psi+1)}}{m_2}\right)\right)\\
&= \breve{d}^{B}_{1(\psi+1)}\left(1-\frac{\sum_{i=\tau}^{\psi}\breve{d}^{B}_{2i}}{m_2}-\left(p_2+\frac{q_2\breve{D}^{B}_{2(\psi+1)}}{m_2}\right)\left(\frac{m_2-\breve{D}^{B}_{2(\psi+1)}}{m_2}\right)\right)+\frac{A\epsilon\breve{d}^{B}_{1(\psi+1)}}{m_2}\\
&= \breve{d}^{B}_{1(\psi+1)}\left(1-\frac{\sum_{i=\tau}^{\psi}\breve{d}^{B}_{2i}}{m_2}-\frac{\breve{d}^{B}_{2(\psi+1)}}{m_2}\right)+\frac{A\epsilon\breve{d}^{B}_{1(\psi+1)}}{m_2}\\
&= \breve{d}^{B}_{1(\psi+1)}\left(1-\frac{\sum_{i=\tau}^{\psi+1}\breve{d}^{B}_{2i}}{m_2}\right)+\frac{A\epsilon\breve{d}^{B}_{1(\psi+1)}}{m_2}\\
&= \breve{d}^{B}_{1(\psi+1)}\left(1-\frac{\breve{D}^{B}_{2(\psi+2)}}{m_2}\right)+\frac{A\epsilon\breve{d}^{B}_{1(\psi+1)}}{m_2}\\
&= \breve{d}^{NB}_{1(\psi+1)}+\frac{A\epsilon\breve{d}^{B}_{1(\psi+1)}}{m_2}.
\end{aligned}
$$

Also, note that:

$$\begin{aligned}
\widehat{d}^{NB}_{2(\psi+1)} &= \widehat{d}^{B}_{2(\psi+1)} + \frac{\widehat{D}^{B}_{1(\psi+1)}\widehat{d}^{B}_{2(\psi+1)}}{m_2} + \frac{\widehat{d}^{B}_{1(\psi+1)}\widehat{D}^{B}_{2(\psi+2)}}{m_2} \\
&= \widehat{d}^{B}_{2(\psi+1)} + \frac{\breve{D}^{B}_{1(\psi+1)}\widehat{d}^{B}_{2(\psi+1)}}{m_2} + \frac{\breve{d}^{B}_{1(\psi+1)}\sum_{i=\tau}^{\psi+1}\widehat{d}^{B}_{2i}}{m_2} \\
&= \widehat{d}^{B}_{2(\psi+1)}\left(1+\frac{\breve{D}^{B}_{1(\psi+1)}}{m_2}\right) + \frac{\breve{d}^{B}_{1(\psi+1)}\sum_{i=\tau}^{\psi+1}\widehat{d}^{B}_{2i}}{m_2} \\
&= \widehat{d}^{B}_{2(\psi+1)}\left(1+\frac{\breve{D}^{B}_{1(\psi+1)}}{m_2}\right) + \frac{\breve{d}^{B}_{1(\psi+1)}\sum_{i=\tau}^{\psi}\breve{d}^{B}_{2i}}{m_2} + \frac{\breve{d}^{B}_{1(\psi+1)}\widehat{d}^{B}_{2(\psi+1)}}{m_2} \\
&= \left(p_2+\frac{q_2\widehat{S}^{B}_{2(\psi+1)}}{m_2}\right)\left(m_2-\widehat{D}^{B}_{2(\psi+1)}\right)\left(1+\frac{\breve{D}^{B}_{1(\psi+1)}}{m_2}\right) + \frac{\breve{d}^{B}_{1(\psi+1)}\sum_{i=\tau}^{\psi}\breve{d}^{B}_{2i}}{m_2} \\
&\quad + \frac{\breve{d}^{B}_{1(\psi+1)}\widehat{d}^{B}_{2(\psi+1)}}{m_2} \\
&= \left(p_2+\frac{q_2\left(\widehat{D}^{B}_{2(\psi+1)}-\epsilon\right)}{m_2}\right)\left(m_2-\widehat{D}^{B}_{2(\psi+1)}\right)\left(1+\frac{\breve{D}^{B}_{1(\psi+1)}}{m_2}\right) \\
&\quad + \frac{\breve{d}^{B}_{1(\psi+1)}\sum_{i=\tau}^{\psi}\breve{d}^{B}_{2i}}{m_2} + \frac{\breve{d}^{B}_{1(\psi+1)}\widehat{d}^{B}_{2(\psi+1)}}{m_2} \\
&= \left(p_2+\frac{q_2\breve{D}^{B}_{2(\psi+1)}}{m_2}\right)\left(m_2-\breve{D}^{B}_{2(\psi+1)}\right)\left(1+\frac{\breve{D}^{B}_{1(\psi+1)}}{m_2}\right) \\
&\quad - \frac{q_2\epsilon}{m_2}\left(m_2-\breve{D}^{B}_{2(\psi+1)}\right)\left(1+\frac{\breve{D}^{B}_{1(\psi+1)}}{m_2}\right) + \frac{\breve{d}^{B}_{1(\psi+1)}\sum_{i=\tau}^{\psi}\breve{d}^{B}_{2i}}{m_2} + \frac{\breve{d}^{B}_{1(\psi+1)}\widehat{d}^{B}_{2(\psi+1)}}{m_2} \\
&= \breve{d}^{B}_{2(\psi+1)}\left(1+\frac{\breve{D}^{B}_{1(\psi+1)}}{m_2}\right) - \frac{q_2\epsilon}{m_2}\left(m_2-\breve{D}^{B}_{2(\psi+1)}\right)\left(1+\frac{\breve{D}^{B}_{1(\psi+1)}}{m_2}\right) \\
&\quad + \frac{\breve{d}^{B}_{1(\psi+1)}\breve{D}^{B}_{2(\psi+1)}}{m_2} + \frac{\breve{d}^{B}_{1(\psi+1)}\widehat{d}^{B}_{2(\psi+1)}}{m_2}
\end{aligned}$$

$$
\begin{aligned}
&= \breve{d}^B_{2(\psi+1)}\left(1+\frac{\breve{D}^B_{1(\psi+1)}}{m_2}\right) - \frac{q_2\epsilon}{m_2}\left(m_2 - \breve{D}^B_{2(\psi+1)}\right)\left(1+\frac{\breve{D}^B_{1(\psi+1)}}{m_2}\right) \\
&\quad + \frac{\breve{d}^B_{1(\psi+1)}\breve{D}^B_{2(\psi+2)}}{m_2} + \frac{\breve{d}^B_{1(\psi+1)}\widehat{d}^B_{2(\psi+1)}}{m_2} - \frac{\breve{d}^B_{1(\psi+1)}\breve{d}^B_{2(\psi+1)}}{m_2} \\
&= \breve{d}^{NB}_{2(\psi+1)} - \frac{q_2\epsilon}{m_2}\left(m_2 - \breve{D}^B_{2(\psi+1)}\right)\left(1+\frac{\breve{D}^B_{1(\psi+1)}}{m_2}\right) + \frac{\breve{d}^B_{1(\psi+1)}\left(\widehat{d}^B_{2(\psi+1)} - \breve{d}^B_{2(\psi+1)}\right)}{m_2} \\
&= \breve{d}^{NB}_{2(\psi+1)} - \frac{q_2\epsilon}{m_2}\left(m_2 - \breve{D}^B_{2(\psi+1)}\right)\left(1+\frac{\breve{D}^B_{1(\psi+1)}}{m_2}\right) \\
&\quad + \frac{\breve{d}^B_{1(\psi+1)}\left(m_2 - \breve{D}^B_{2(\psi+1)}\right)(-q_2\epsilon)}{(m_2)^2} \\
&= \breve{d}^{NB}_{2(\psi+1)} - \frac{q_2\epsilon}{m_2}\left(m_2 - \breve{D}^B_{2(\psi+1)}\right)\left(\left(1+\frac{\breve{D}^B_{1(\psi+1)}}{m_2}\right) + \frac{\breve{d}^B_{1(\psi+1)}}{m_2}\right) \\
&= \breve{d}^{NB}_{2(\psi+1)} - \epsilon A\left(1+\frac{\breve{D}^B_{1(\psi+2)}}{m_2}\right) \\
&= \breve{d}^{NB}_{2(\psi+1)} - \epsilon AE.
\end{aligned}
$$

We thus obtain $\widehat{s}_{2(\psi+1)} = \widehat{d}^{NB}_{2(\psi+1)} + \alpha\epsilon = \breve{d}^{NB}_{2(\psi+1)} - \epsilon AE + \alpha\epsilon = \breve{s}_{2(\psi+1)} - \epsilon AE + \alpha\epsilon$. Also, $\widehat{S}^B_{2(\psi+2)} = \sum_{i=\tau}^{\psi+1}\widehat{s}^B_{2i} = \sum_{i=\tau}^{\psi+1}\widehat{d}^B_{2i} - \epsilon + \alpha\epsilon = \sum_{i=\tau}^{\psi+1}\breve{d}^B_{2i} - \epsilon - A\epsilon + \alpha\epsilon = \breve{D}^B_{2(\psi+1)} + \breve{d}^B_{2(\psi+1)} - \epsilon - A\epsilon + \alpha\epsilon$ and $\widehat{D}^B_{2(\psi+1)} + \widehat{d}^B_{2(\psi+1)} + A\epsilon = \breve{D}^B_{2(\psi+1)} + \breve{d}^B_{2(\psi+1)}$.

(4) Periods $t \geq \psi + 2$: We know from Proposition 1 that $\widehat{d}^{NB}_{1t} \geq \breve{d}^{NB}_{1t}$, $\forall t \geq \tau$. We next consider the second-generation products:

$$
\begin{aligned}
\widehat{s}_{2(\psi+2)} &= \widehat{d}^{NB}_{2(\psi+2)} \\
&= \widehat{d}^B_{2(\psi+2)} + \frac{\widehat{D}^B_{1(\psi+2)}\widehat{d}^B_{2(\psi+2)}}{m_2} + \frac{\widehat{d}^B_{1(\psi+2)}\widehat{D}^B_{2(\psi+3)}}{m_2} \\
&= \widehat{d}^B_{2(\psi+2)}\left(1+\frac{\widehat{D}^B_{1(\psi+2)}}{m_2}\right) + \frac{\widehat{d}^B_{1(\psi+2)}\widehat{D}^B_{2(\psi+3)}}{m_2} \\
&= \widehat{d}^B_{2(\psi+2)}\left(1+\frac{\breve{D}^B_{1(\psi+2)}}{m_2}\right) + \frac{\breve{d}^B_{1(\psi+2)}\widehat{D}^B_{2(\psi+3)}}{m_2} \\
&= \widehat{d}^B_{2(\psi+2)}\left(1+\frac{\breve{D}^B_{1(\psi+2)}}{m_2}\right) + \frac{\breve{d}^B_{1(\psi+2)}\breve{D}^B_{2(\psi+3)}}{m_2} - \left(\widehat{d}^{NB}_{1(\psi+2)} - \breve{d}^{NB}_{1(\psi+2)}\right) \\
&\geq \breve{d}^B_{2(\psi+2)}\left(1+\frac{\breve{D}^B_{1(\psi+2)}}{m_2}\right) + \frac{\breve{d}^B_{1(\psi+2)}\breve{D}^B_{2(\psi+3)}}{m_2} - \left(\widehat{d}^{NB}_{1(\psi+2)} - \breve{d}^{NB}_{1(\psi+2)}\right) \\
&= \breve{d}^{NB}_{2(\psi+2)} - \left(\widehat{d}^{NB}_{1(\psi+2)} - \breve{d}^{NB}_{1(\psi+2)}\right).
\end{aligned}
$$

The above inequality holds as we know from the proof of Theorem EC. 1. in Nadar et al.[18] that $\widehat{d}^B_{2(\psi+2)} \geq \breve{d}^B_{2(\psi+2)}$ when $B(1+A)+(1-B)(\alpha-1-A) > 0$. Thus $\widehat{s}_{2(\psi+2)} = \widehat{d}^{NB}_{2(\psi+2)} \geq \breve{d}^{NB}_{2(\psi+2)} - \left(\widehat{d}^{NB}_{1(\psi+2)} - \breve{d}^{NB}_{1(\psi+2)}\right) = \breve{s}_{2(\psi+2)} - \left(\widehat{d}^{NB}_{1(\psi+2)} - \breve{d}^{NB}_{1(\psi+2)}\right)$. Proceeding similarly, it can be shown that $\widehat{d}^B_{2t} \geq \breve{d}^B_{2t}$ and $\widehat{s}_{2t} = \widehat{d}^{NB}_{2t} \geq \breve{d}^{NB}_{2t} - \left(\widehat{d}^{NB}_{1t} - \breve{d}^{NB}_{1t}\right) = \breve{s}_{2t} - \left(\widehat{d}^{NB}_{1t} - \breve{d}^{NB}_{1t}\right)$, $\forall t \geq \psi+2$. Since $\breve{D}^B_{2(\psi+2)} - \widehat{D}^B_{2(\psi+2)} = A\epsilon$ and $\widehat{d}^B_{2t} \geq \breve{d}^B_{2t}$, $\forall t \geq \psi+2$, we obtain $\breve{D}^B_{2t} - \widehat{D}^B_{2t} \leq A\epsilon$ and $\widehat{d}^{NB}_{1t} - \breve{d}^{NB}_{1t} = \frac{\breve{d}^B_{1t}\left(\breve{D}^B_{2(t+1)} - \widehat{D}^B_{2(t+1)}\right)}{m_2} \leq \frac{A\epsilon\breve{d}^B_{1t}}{m_2}$, $\forall t \geq \psi+2$. As we also know from Lemma 2 of Nadar et al.[18] that $\breve{D}^B_{2t} > \widehat{D}^B_{2t}$, $\forall t \geq \psi+2$, we obtain $\breve{d}^B_{2t} + A\epsilon \geq \widehat{d}^B_{2t}$, $\forall t \geq \psi+2$.

We now consider the first-generation remanufactured items and returns:

(1) Periods $t \leq \psi$: $\breve{e}_t = \widehat{e}_t$, $\widehat{n}_t = \breve{n}_t$, and $\widehat{r}_t = \breve{r}_t$.

(2) Period $\psi+1$: Since $\breve{e}_{\psi+1} > \gamma\breve{d}^{NB}_{1(\psi+1)}$, $\breve{n}_{\psi+1} = (1-\gamma)\breve{d}^{NB}_{1(\psi+1)}$ and $\breve{r}_{\psi+1} = \gamma\breve{d}^{NB}_{1(\psi+1)}$. Since $\breve{e}_{\psi+1} = \breve{e}_\psi - \breve{r}_\psi + \sum_{i=1}^{\psi}\beta_i\breve{d}^{NB}_{1(\psi+1-i)} > \gamma\breve{d}^{NB}_{1(\psi+1)}$ and $\widehat{d}^{NB}_{1\psi} = \breve{d}^{NB}_{1\psi}$, $\exists\epsilon$ s.t. $\widehat{e}_{\psi+1} = \widehat{e}_\psi - \widehat{r}_\psi + \sum_{i=1}^{\psi}\beta_i\widehat{d}^{NB}_{1(\psi+1-i)} = \breve{e}_\psi - \breve{r}_\psi + \sum_{i=1}^{\psi}\beta_i\breve{d}^{NB}_{1(\psi+1-i)} = \breve{e}_{\psi+1} > \gamma\breve{d}^{NB}_{1(\psi+1)} + \frac{\gamma A\epsilon\breve{d}^B_{1(\psi+1)}}{m_2} = \gamma\widehat{d}^{NB}_{1(\psi+1)}$. Thus, $\widehat{e}_{\psi+1} = \breve{e}_{\psi+1}$, $\widehat{r}_{\psi+1} = \breve{r}_{\psi+1} + \frac{\gamma A\epsilon\breve{d}^B_{1(\psi+1)}}{m_2}$, and $\widehat{n}_{\psi+1} = \breve{n}_{\psi+1} + (1-\gamma)\frac{A\epsilon\breve{d}^B_{1(\psi+1)}}{m_2}$.

(3) Periods $t \geq \psi+2$: Since $\breve{e}_{\psi+2} > \gamma\breve{d}^{NB}_{1(\psi+2)}$, $\breve{n}_{\psi+2} = (1-\gamma)\breve{d}^{NB}_{1(\psi+2)}$ and $\breve{r}_{\psi+2} = \gamma\breve{d}^{NB}_{1(\psi+2)}$. Also, recall that $\widehat{d}^{NB}_{1(\psi+1)} = \breve{d}^{NB}_{1(\psi+1)} + \frac{A\epsilon\breve{d}^B_{1(\psi+1)}}{m_2}$ and $0 \leq \widehat{d}^{NB}_{1t} - \breve{d}^{NB}_{1t} \leq \frac{A\epsilon\breve{d}^B_{1t}}{m_2}$, $\forall t \geq \psi+2$. Since $\breve{e}_{\psi+2} = \breve{e}_{\psi+1} - \breve{r}_{\psi+1} + \sum_{i=1}^{\psi+1}\beta_i\breve{d}^{NB}_{1(\psi+2-i)} > \gamma\breve{d}^{NB}_{1(\psi+2)}$, $\exists\epsilon$ s.t.

$$\begin{aligned}
\widehat{e}_{\psi+2} &= \widehat{e}_{\psi+1} - \widehat{r}_{\psi+1} + \sum_{i=1}^{\psi+1}\beta_i\widehat{d}^{NB}_{1(\psi+2-i)} \\
&= \widehat{e}_{\psi+1} - \widehat{r}_{\psi+1} + \beta_1\widehat{d}^{NB}_{1(\psi+1)} + \sum_{i=2}^{\psi+1}\beta_i\widehat{d}^{NB}_{1(\psi+2-i)} \\
&= \breve{e}_{\psi+1} - \breve{r}_{\psi+1} - \frac{\gamma A\epsilon\breve{d}^B_{1(\psi+1)}}{m_2} + \beta_1\breve{d}^{NB}_{1(\psi+1)} + \frac{\beta_1 A\epsilon\breve{d}^B_{1(\psi+1)}}{m_2} + \sum_{i=2}^{\psi+1}\beta_i\breve{d}^{NB}_{1(\psi+2-i)} \\
&= \breve{e}_{\psi+1} - \breve{r}_{\psi+1} + \sum_{i=1}^{\psi+1}\beta_i\breve{d}^{NB}_{1(\psi+2-i)} - \frac{\gamma A\epsilon\breve{d}^B_{1(\psi+1)}}{m_2} + \frac{\beta_1 A\epsilon\breve{d}^B_{1(\psi+1)}}{m_2} \\
&= \breve{e}_{\psi+2} - \frac{\gamma A\epsilon\breve{d}^B_{1(\psi+1)}}{m_2} + \frac{\beta_1 A\epsilon\breve{d}^B_{1(\psi+1)}}{m_2} > \gamma\breve{d}^{NB}_{1(\psi+2)} + \frac{\gamma A\epsilon\breve{d}^B_{1(\psi+2)}}{m_2} \geq \gamma\widehat{d}^{NB}_{1(\psi+2)}.
\end{aligned}$$

Thus, $\widehat{r}_{\psi+2} = \gamma\widehat{d}^{NB}_{1(\psi+2)} > \breve{r}_{\psi+2}$ and $\widehat{n}_{\psi+2} = (1-\gamma)\widehat{d}^{NB}_{1(\psi+2)} > \breve{n}_{\psi+2}$. Proceeding similarly, it can be shown that $\widehat{r}_t \geq \breve{r}_t$ and $\widehat{n}_t \geq \breve{n}_t$, $\forall t \geq \psi+2$.

In order to prove part (a) of Theorem A.1, suppose that $p_{1n} - c_{1n} > p_{2n} - c_{2n}$. We will show that sales plan (i) is more profitable than sales plan (ii). Combining all of the above observations:

$$
\begin{aligned}
&(p_{1n}-c_{1n})\sum_{t=1}^{T}\left(\widehat{d}^{NB}_{1t}-\widehat{r}_t\right)+(p_{1r}-c_{1r})\sum_{t=1}^{T}\widehat{r}_t+(p_{2n}-c_{2n})\sum_{t=\tau}^{T}\widehat{s}_{2t}\\
&=(p_{1n}-c_{1n})\left[\sum_{t=1}^{\psi-1}\left(\widehat{d}^{NB}_{1t}-\widehat{r}_t\right)+\left(\widehat{d}^{NB}_{1\psi}-\widehat{r}_{1\psi}\right)+\left(\widehat{d}^{NB}_{1(\psi+1)}-\widehat{r}_{\psi+1}\right)+\sum_{t=\psi+2}^{T}\left(\widehat{d}^{NB}_{1t}-\widehat{r}_t\right)\right]\\
&\quad+(p_{1r}-c_{1r})\left[\sum_{t=1}^{\psi-1}\widehat{r}_t+\widehat{r}_\psi+\widehat{r}_{\psi+1}+\sum_{t=\psi+2}^{T}\widehat{r}_t\right]\\
&\quad+(p_{2n}-c_{2n})\left[\sum_{t=\tau}^{\psi-1}\widehat{s}_{2t}+\widehat{s}_{2\psi}+\widehat{s}_{2(\psi+1)}+\sum_{t=\psi+2}^{T}\widehat{s}_{2t}\right]\\
&=(p_{1n}-c_{1n})\left[\sum_{t=1}^{\psi-1}\left(\breve{d}^{NB}_{1t}-\widehat{r}_t\right)+\left(\breve{d}^{NB}_{1\psi}-\widehat{r}_\psi\right)+\left(\breve{d}^{NB}_{1(\psi+1)}+\frac{A\epsilon\breve{d}^{B}_{1(\psi+1)}}{m_2}-\widehat{r}_{\psi+1}\right)\right.\\
&\quad\left.+\sum_{t=\psi+2}^{T}\breve{d}^{B}_{1t}-\sum_{t=\psi+2}^{T}\frac{\breve{d}^{B}_{1t}\widehat{D}^{B}_{2(t+1)}}{m_2}-\sum_{t=\psi+2}^{T}\widehat{r}_t\right]+(p_{1r}-c_{1r})\left[\sum_{t=1}^{\psi-1}\widehat{r}_t+\widehat{r}_\psi+\widehat{r}_{\psi+1}+\sum_{t=\psi+2}^{T}\widehat{r}_t\right]\\
&\quad+(p_{2n}-c_{2n})\left[\sum_{t=\tau}^{\psi-1}\breve{s}_{2t}+\widehat{s}_{2\psi}+\widehat{s}_{2(\psi+1)}+\sum_{t=\psi+2}^{T}\widehat{d}^{B}_{2t}\left(1+\frac{\breve{D}^{B}_{1t}}{m_2}\right)+\sum_{t=\psi+2}^{T}\frac{\breve{d}^{B}_{1t}\widehat{D}^{B}_{2(t+1)}}{m_2}\right]\\
&=(p_{1n}-c_{1n})\left[\sum_{t=1}^{\psi-1}\left(\breve{d}^{NB}_{1t}-\widehat{r}_t\right)+\left(\breve{d}^{NB}_{1\psi}-\widehat{r}_\psi\right)+\left(\breve{d}^{NB}_{1(\psi+1)}+\frac{A\epsilon\breve{d}^{B}_{1(\psi+1)}}{m_2}-\widehat{r}_{\psi+1}\right)\right.\\
&\quad\left.+\sum_{t=\psi+2}^{T}\breve{d}^{B}_{1t}-\sum_{t=\psi+2}^{T}\widehat{r}_t\right]-\left[(p_{1n}-c_{1n})-(p_{2n}-c_{2n})\right]\sum_{t=\psi+2}^{T}\frac{\breve{d}^{B}_{1t}\widehat{D}^{B}_{2(t+1)}}{m_2}\\
&\quad+(p_{1r}-c_{1r})\left[\sum_{t=1}^{\psi-1}\widehat{r}_t+\widehat{r}_\psi+\widehat{r}_{\psi+1}+\sum_{t=\psi+2}^{T}\widehat{r}_t\right]\\
&\quad+(p_{2n}-c_{2n})\left[\sum_{t=\tau}^{\psi-1}\breve{s}_{2t}+\widehat{s}_{2\psi}+\widehat{s}_{2(\psi+1)}+\sum_{t=\psi+2}^{T}\widehat{d}^{B}_{2t}\left(1+\frac{\breve{D}^{B}_{1t}}{m_2}\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
&\geq (p_{1n}-c_{1n})\Bigg[\sum_{t=1}^{\psi-1}\left(\breve{d}_{1t}^{NB}-\widehat{r}_t\right)+\left(\breve{d}_{1\psi}^{NB}-\widehat{r}_\psi\right)+\left(\breve{d}_{1(\psi+1)}^{NB}+\frac{A\epsilon\breve{d}_{1(\psi+1)}^{B}}{m_2}-\widehat{r}_{\psi+1}\right)\\
&\quad+\sum_{t=\psi+2}^{T}\breve{d}_{1t}^{B}-\sum_{t=\psi+2}^{T}\widehat{r}_t\Bigg]-[(p_{1n}-c_{1n})-(p_{2n}-c_{2n})]\sum_{t=\psi+2}^{T}\frac{\breve{d}_{1t}^{B}\breve{D}_{2(t+1)}^{B}}{m_2}\\
&\quad+(p_{1r}-c_{1r})\left[\sum_{t=1}^{\psi-1}\widehat{r}_t+\widehat{r}_\psi+\widehat{r}_{\psi+1}+\sum_{t=\psi+2}^{T}\widehat{r}_t\right]\\
&\quad+(p_{2n}-c_{2n})\left[\sum_{t=\tau}^{\psi-1}\breve{s}_{2t}+\widehat{s}_{2\psi}+\widehat{s}_{2(\psi+1)}+\sum_{t=\psi+2}^{T}\breve{d}_{2t}^{B}\left(1+\frac{\breve{D}_{1t}^{B}}{m_2}\right)\right]\\
&=(p_{1n}-c_{1n})\Bigg[\sum_{t=1}^{\psi-1}\left(\breve{d}_{1t}^{NB}-\widehat{r}_t\right)+\left(\breve{d}_{1\psi}^{NB}-\widehat{r}_\psi\right)+\left(\breve{d}_{1(\psi+1)}^{NB}+\frac{A\epsilon\breve{d}_{1(\psi+1)}^{B}}{m_2}-\widehat{r}_{\psi+1}\right)\\
&\quad+\sum_{t=\psi+2}^{T}\breve{d}_{1t}^{B}-\sum_{t=\psi+2}^{T}\widehat{r}_t\Bigg]-[(p_{1n}-c_{1n})-(p_{2n}-c_{2n})]\sum_{t=\psi+2}^{T}\frac{\breve{d}_{1t}^{B}\breve{D}_{2(t+1)}^{B}}{m_2}\\
&\quad+(p_{1r}-c_{1r})\left[\sum_{t=1}^{\psi-1}\widehat{r}_t+\widehat{r}_\psi+\widehat{r}_{\psi+1}+\sum_{t=\psi+2}^{T}\widehat{r}_t\right]\\
&\quad+(p_{2n}-c_{2n})\left[\sum_{t=\tau}^{\psi-1}\breve{s}_{2t}+\breve{s}_{2\psi}-\epsilon+\breve{s}_{2(\psi+1)}-\epsilon AE+\alpha\epsilon+\sum_{t=\psi+2}^{T}\breve{d}_{2t}^{B}\left(1+\frac{\breve{D}_{1t}^{B}}{m_2}\right)\right]\\
&\geq (p_{1n}-c_{1n})\Bigg[\sum_{t=1}^{\psi-1}\left(\breve{d}_{1t}^{NB}-\breve{r}_t\right)+\left(\breve{d}_{1\psi}^{NB}-\breve{r}_\psi\right)+\Bigg(\breve{d}_{1(\psi+1)}^{NB}+\frac{A\epsilon\breve{d}_{1(\psi+1)}^{B}}{m_2}\\
&\quad-\left(\breve{r}_{(\psi+1)}+\frac{\gamma A\epsilon\breve{d}_{1(\psi+1)}^{B}}{m_2}\right)\Bigg)+\sum_{t=\psi+2}^{T}\breve{d}_{1t}^{B}-\sum_{t=\psi+2}^{T}\breve{r}_t\Bigg]\\
&\quad+(p_{1r}-c_{1r})\left[\sum_{t=1}^{\psi-1}\breve{r}_t+\breve{r}_\psi+\left(\breve{r}_{\psi+1}+\frac{\gamma A\epsilon\breve{d}_{1(\psi+1)}^{B}}{m_2}\right)+\sum_{t=\psi+2}^{T}\breve{r}_t\right]\\
&\quad-[(p_{1n}-c_{1n})-(p_{2n}-c_{2n})]\sum_{t=\psi+2}^{T}\frac{\breve{d}_{1t}^{B}\breve{D}_{2(t+1)}^{B}}{m_2}\\
&\quad+(p_{2n}-c_{2n})\left[\sum_{t=\tau}^{\psi-1}\breve{s}_{2t}+\breve{s}_{2\psi}-\epsilon+\breve{s}_{2(\psi+1)}-\epsilon AE+\alpha\epsilon+\sum_{t=\psi+2}^{T}\breve{d}_{2t}^{B}\left(1+\frac{\breve{D}_{1t}^{B}}{m_2}\right)\right]
\end{aligned}
$$

$$= (p_{1n} - c_{1n}) \left[\sum_{t=1}^{T} \left(\breve{d}_{1t}^{NB} - \breve{r}_t \right) + \left(\frac{A\epsilon \breve{d}_{1(\psi+1)}^{B}}{m_2} - \frac{\gamma A\epsilon \breve{d}_{1(\psi+1)}^{B}}{m_2} \right) \right]$$
$$+(p_{1r} - c_{1r}) \left(\sum_{t=1}^{T} \breve{r}_t + \frac{\gamma A\epsilon \breve{d}_{1(\psi+1)}^{B}}{m_2} \right) + (p_{2n} - c_{2n}) \left(\sum_{t=\tau}^{T} \breve{s}_{2t} - \epsilon AE + \alpha\epsilon - \epsilon \right)$$
$$= (p_{1n} - c_{1n}) \sum_{t=1}^{T} \left(\breve{d}_{1t}^{NB} - \breve{r}_t \right) + (p_{1r} - c_{1r}) \sum_{t=1}^{T} \breve{r}_t + (p_{2n} - c_{2n}) \sum_{t=\tau}^{T} \breve{s}_{2t}$$
$$+(p_{1n} - c_{1n}) \left(\frac{A\epsilon \breve{d}_{1(\psi+1)}^{B}}{m_2} - \frac{\gamma A\epsilon \breve{d}_{1(\psi+1)}^{B}}{m_2} \right) + (p_{1r} - c_{1r}) \left(\frac{\gamma A\epsilon \breve{d}_{1(\psi+1)}^{B}}{m_2} \right)$$
$$+(p_{2n} - c_{2n}) (-\epsilon AE + \alpha\epsilon - \epsilon).$$

In order to show that

$$(p_{1n} - c_{1n}) \sum_{t=1}^{T} \left(\widehat{d}_{1t}^{NB} - \widehat{r}_t \right) + (p_{1r} - c_{1r}) \sum_{t=1}^{T} \widehat{r}_t + (p_{2n} - c_{2n}) \sum_{t=\tau}^{T} \widehat{s}_{2t} \geq$$
$$(p_{1n} - c_{1n}) \sum_{t=1}^{T} \left(\breve{d}_{1t}^{NB} - \breve{r}_t \right) + (p_{1r} - c_{1r}) \sum_{t=1}^{T} \breve{r}_t + (p_{2n} - c_{2n}) \sum_{t=\tau}^{T} \breve{s}_{2t},$$

it suffices to show that

$$(p_{1n} - c_{1n}) \left(\frac{A\breve{d}_{1(\psi+1)}^{B}}{m_2} - \frac{\gamma A\breve{d}_{1(\psi+1)}^{B}}{m_2} \right) + (p_{1r} - c_{1r}) \frac{\gamma A\breve{d}_{1(\psi+1)}^{B}}{m_2} - (p_{2n} - c_{2n}) (1 + AE - \alpha) > 0.$$

The above inequality holds as we assume

$$\alpha > 1 + AE - \left(\frac{p_{1n} - c_{1n}}{p_{2n} - c_{2n}} \right) \frac{(1-\gamma) A\breve{d}_{1(\psi+1)}^{B}}{m_2} - \left(\frac{p_{1r} - c_{1r}}{p_{2n} - c_{2n}} \right) \frac{\gamma A\breve{d}_{1(\psi+1)}^{B}}{m_2}.$$

In order to prove part (b) of Theorem A.1, suppose that $p_{1n} - c_{1n} < p_{2n} - c_{2n}$. We will show that sales plan (i) is more profitable than sales plan (ii). With similar arguments to those used in the proof of part (a):

$$(p_{1n} - c_{1n}) \sum_{t=1}^{T} \left(\widehat{d}_{1t}^{NB} - \widehat{r}_t \right) + (p_{1r} - c_{1r}) \sum_{t=1}^{T} \widehat{r}_t + (p_{2n} - c_{2n}) \sum_{t=\tau}^{T} \widehat{s}_{2t}$$
$$= (p_{1n} - c_{1n}) \left[\sum_{t=1}^{\psi-1} \left(\widehat{d}_{1t}^{NB} - \widehat{r}_t \right) + \left(\widehat{d}_{1\psi}^{NB} - \widehat{r}_\psi \right) + \left(\widehat{d}_{1(\psi+1)}^{NB} - \widehat{r}_{(\psi+1)} \right) + \sum_{t=\psi+2}^{T} \left(\widehat{d}_{1t}^{NB} - \widehat{r}_t \right) \right]$$
$$+(p_{1r} - c_{1r}) \sum_{t=1}^{T} \widehat{r}_t + (p_{2n} - c_{2n}) \left[\sum_{t=\tau}^{\psi-1} \widehat{s}_{2t} + \widehat{s}_{2\psi} + \widehat{s}_{2(\psi+1)} + \sum_{t=\psi+2}^{T} \widehat{s}_{2t} \right]$$

$$\geq (p_{1n}-c_{1n})\Bigg[\sum_{t=1}^{\psi}\left(\breve{d}_{1t}^{NB}-\breve{r}_t\right)+\breve{d}_{1(\psi+1)}^{NB}+\frac{A\epsilon\breve{d}_{1(\psi+1)}^{B}}{m_2}-\breve{r}_{(\psi+1)}-\frac{\gamma A\epsilon\breve{d}_{1(\psi+1)}^{B}}{m_2}$$

$$+\sum_{t=\psi+2}^{T}\left(\breve{d}_{1t}^{NB}+\left(\widehat{d}_{1t}^{NB}-\breve{d}_{1t}^{NB}\right)-\breve{r}_t\right)\Bigg]$$

$$+(p_{1r}-c_{1r})\left[\sum_{t=1}^{\psi}\breve{r}_t+\breve{r}_{(\psi+1)}+\frac{\gamma A\epsilon\breve{d}_{1(\psi+1)}^{B}}{m_2}+\sum_{t=\psi+2}^{T}\breve{r}_t\right]$$

$$+(p_{2n}-c_{2n})\left[\sum_{t=\tau}^{\psi-1}\widehat{s}_{2t}+\widehat{s}_{2\psi}+\widehat{s}_{2(\psi+1)}+\sum_{t=\psi+2}^{T}\widehat{s}_{2t}\right]$$

$$\geq (p_{1n}-c_{1n})\Bigg[\sum_{t=1}^{\psi}\left(\breve{d}_{1t}^{NB}-\breve{r}_t\right)+\breve{d}_{1(\psi+1)}^{NB}+\frac{A\epsilon\breve{d}_{1(\psi+1)}^{B}}{m_2}-\breve{r}_{(\psi+1)}-\frac{\gamma A\epsilon\breve{d}_{1(\psi+1)}^{B}}{m_2}$$

$$+\sum_{t=\psi+2}^{T}\left(\breve{d}_{1t}^{NB}+\left(\widehat{d}_{1t}^{NB}-\breve{d}_{1t}^{NB}\right)-\breve{r}_t\right)\Bigg]$$

$$+(p_{1r}-c_{1r})\left[\sum_{t=1}^{\psi}\breve{r}_t+\breve{r}_{(\psi+1)}+\frac{\gamma A\epsilon\breve{d}_{1(\psi+1)}^{B}}{m_2}+\sum_{t=\psi+2}^{T}\breve{r}_t\right]$$

$$+(p_{2n}-c_{2n})\left[\sum_{t=\tau}^{\psi-1}\breve{s}_{2t}+\breve{s}_{2\psi}-\epsilon+\breve{s}_{2(\psi+1)}-\epsilon AE+\alpha\epsilon+\sum_{t=\psi+2}^{T}\left(\breve{s}_{2t}-\left(\widehat{d}_{1t}^{NB}-\breve{d}_{1t}^{NB}\right)\right)\right]$$

$$= (p_{1n}-c_{1n})\Bigg[\sum_{t=1}^{\psi}\left(\breve{d}_{1t}^{NB}-\breve{r}_t\right)+\breve{d}_{1(\psi+1)}^{NB}+\frac{A\epsilon\breve{d}_{1(\psi+1)}^{B}}{m_2}-\breve{r}_{(\psi+1)}-\frac{\gamma A\epsilon\breve{d}_{1(\psi+1)}^{B}}{m_2}$$

$$+\sum_{t=\psi+2}^{T}\left(\breve{d}_{1t}^{NB}-\breve{r}_t\right)\Bigg]+(p_{1r}-c_{1r})\left[\sum_{t=1}^{\psi}\breve{r}_t+\breve{r}_{(\psi+1)}+\frac{\gamma A\epsilon\breve{d}_{1(\psi+1)}^{B}}{m_2}+\sum_{t=\psi+2}^{T}\breve{r}_t\right]$$

$$+[(p_{1n}-c_{1n})-(p_{2n}-c_{2n})]\sum_{t=\psi+2}^{T}\left(\widehat{d}_{1t}^{NB}-\breve{d}_{1t}^{NB}\right)$$

$$+(p_{2n}-c_{2n})\left[\sum_{t=\tau}^{\psi-1}\breve{s}_{2t}+\breve{s}_{2\psi}-\epsilon+\breve{s}_{2(\psi+1)}-\epsilon AE+\alpha\epsilon+\sum_{t=\psi+2}^{T}\breve{s}_{2t}\right]$$

$$\geq (p_{1n}-c_{1n})\Bigg[\sum_{t=1}^{\psi}\left(\breve{d}_{1t}^{NB}-\breve{r}_t\right)+\breve{d}_{1(\psi+1)}^{NB}+\frac{A\epsilon\breve{d}_{1(\psi+1)}^{B}}{m_2}-\breve{r}_{(\psi+1)}-\frac{\gamma A\epsilon\breve{d}_{1(\psi+1)}^{B}}{m_2}$$

$$+\sum_{t=\psi+2}^{T}\left(\breve{d}_{1t}^{NB}-\breve{r}_t\right)\Bigg]+[(p_{1n}-c_{1n})-(p_{2n}-c_{2n})]\sum_{t=\psi+2}^{T}\frac{A\epsilon\breve{d}_{1t}^{B}}{m_2}$$

$$+(p_{1r}-c_{1r})\left[\sum_{t=1}^{\psi}\breve{r}_t+\breve{r}_{(\psi+1)}+\frac{\gamma A\epsilon\breve{d}_{1(\psi+1)}^{B}}{m_2}+\sum_{t=\psi+2}^{T}\breve{r}_t\right]$$

$$+(p_{2n}-c_{2n})\left[\sum_{t=\tau}^{\psi-1}\breve{s}_{2t}+\breve{s}_{2\psi}-\epsilon+\breve{s}_{2(\psi+1)}-\epsilon AE+\alpha\epsilon+\sum_{t=\psi+2}^{T}\breve{s}_{2t}\right]$$

$$= (p_{1n} - c_{1n})\Bigg[\sum_{t=1}^{\psi}\left(\breve{d}_{1t}^{NB} - \breve{r}_t\right) + \breve{d}_{1(\psi+1)}^{NB} + \frac{A\epsilon \breve{d}_{1(\psi+1)}^{B}}{m_2} - \breve{r}_{(\psi+1)} - \frac{\gamma A\epsilon \breve{d}_{1(\psi+1)}^{B}}{m_2}$$

$$+ \sum_{t=\psi+2}^{T}\left(\breve{d}_{1t}^{NB} - \breve{r}_t\right)\Bigg] + \left[(p_{1n} - c_{1n}) - (p_{2n} - c_{2n})\right]\frac{A\epsilon\left(\breve{D}_{1(T+1)}^{B} - \breve{D}_{1(\psi+2)}^{B}\right)}{m_2}$$

$$+(p_{1r} - c_{1r})\Bigg[\sum_{t=1}^{\psi}\breve{r}_t + \breve{r}_{(\psi+1)} + \frac{\gamma A\epsilon \breve{d}_{1(\psi+1)}^{B}}{m_2} + \sum_{t=\psi+2}^{T}\breve{r}_t\Bigg]$$

$$+(p_{2n} - c_{2n})\left[\sum_{t=\tau}^{\psi-1}\breve{s}_{2t} + \breve{s}_{2\psi} - \epsilon + \breve{s}_{2(\psi+1)} - \epsilon AE + \alpha\epsilon + \sum_{t=\psi+2}^{T}\breve{s}_{2t}\right]$$

$$= (p_{1n} - c_{1n})\sum_{t=1}^{T}\left(\breve{d}_{1t}^{NB} - \breve{r}_t\right) + (p_{1r} - c_{1r})\sum_{t=1}^{T}\breve{r}_t + (p_{2n} - c_{2n})\sum_{t=\tau}^{T}\breve{s}_{2t}$$

$$+(p_{1n} - c_{1n})\left(\frac{A\epsilon \breve{d}_{1(\psi+1)}^{B}}{m_2} - \frac{\gamma A\epsilon \breve{d}_{1(\psi+1)}^{B}}{m_2}\right)$$

$$+\left[(p_{1n} - c_{1n}) - (p_{2n} - c_{2n})\right]\frac{A\epsilon\left(\breve{D}_{1(T+1)}^{B} - \breve{D}_{1(\psi+2)}^{B}\right)}{m_2}$$

$$+(p_{1r} - c_{1r})\frac{\gamma A\epsilon \breve{d}_{1(\psi+1)}^{B}}{m_2} - (p_{2n} - c_{2n})\left(\epsilon + \epsilon AE - \alpha\epsilon\right).$$

In order to show that

$$(p_{1n} - c_{1n})\sum_{t=1}^{T}\left(\widehat{d}_{1t}^{NB} - \widehat{r}_t\right) + (p_{1r} - c_{1r})\sum_{t=1}^{T}\widehat{r}_t + (p_{2n} - c_{2n})\sum_{t=\tau}^{T}\widehat{s}_{2t} \geq$$

$$(p_{1n} - c_{1n})\sum_{t=1}^{T}\left(\breve{d}_{1t}^{NB} - \breve{r}_t\right) + (p_{1r} - c_{1r})\sum_{t=1}^{T}\breve{r}_t + (p_{2n} - c_{2n})\sum_{t=\tau}^{T}\breve{s}_{2t},$$

it suffices to show that

$$\left((p_{1n} - c_{1n}) - (p_{2n} - c_{2n})\right)\frac{A\left(\breve{D}_{1(T+1)}^{B} - \breve{D}_{1(\psi+2)}^{B}\right)}{m_2} + (p_{1n} - c_{1n})\left(\frac{A\breve{d}_{1(\psi+1)}^{B}}{m_2} - \frac{\gamma A\breve{d}_{1(\psi+1)}^{B}}{m_2}\right)$$

$$+(p_{1r} - c_{1r})\frac{\gamma A\breve{d}_{1(\psi+1)}^{B}}{m_2} - (p_{2n} - c_{2n})\left(1 + AE - \alpha\right) > 0.$$

The above inequality holds as we assume

$$\alpha > 1 + AE - \left(\frac{p_{1n} - c_{1n}}{p_{2n} - c_{2n}}\right)\frac{(1-\gamma)\,A\breve{d}_{1(\psi+1)}^{B}}{m_2} - \left(\frac{p_{1r} - c_{1r}}{p_{2n} - c_{2n}}\right)\frac{\gamma A\breve{d}_{1(\psi+1)}^{B}}{m_2}$$

$$-\left(\frac{p_{1n} - c_{1n}}{p_{2n} - c_{2n}} - 1\right)\sum_{t=\psi+2}^{T}\frac{A\breve{d}_{1t}^{B}}{m_2}. \qquad \square$$

Theorem A.2.

(a) *Suppose that* $p_{1n} - c_{1n} > p_{2n} - c_{2n}$. *For optimality of partial demand fulfillment in some period over the T-period selling horizon, it is sufficient that* $\exists \kappa \in \{\tau, ..., T-1\}$ *s.t.* $\breve{e}_{2t} < \gamma_2 \breve{d}_{2t}^B + \frac{\gamma_1 \breve{d}_{1t}^B \breve{D}_{2(t+1)}^B}{m_2} + \frac{\gamma_1 \breve{D}_{1t}^B \breve{d}_{2t}^B}{m_2}$ *for* $\tau \leq t \leq \kappa$, $\breve{e}_{2t} > \gamma_2 \breve{d}_{2t}^B + \frac{\gamma_1 \breve{d}_{1t}^B \breve{D}_{2(t+1)}^B}{m_2} + \frac{\gamma_1 \breve{D}_{1t}^B \breve{d}_{2t}^B}{m_2}$ *and* $\breve{e}_{1t} > \gamma_1 \breve{d}_{1t}^{NB}$ *for* $t > \kappa$, $\alpha > \frac{(1+A)(1-2B)}{1-B}$, *and*

$$\alpha > \left(\frac{p_{2n} - c_{2n}}{p_{2r} - c_{2r}}\right)(1 + AE) - \left(\frac{p_{1n} - c_{1n}}{p_{2r} - c_{2r}}\right) \frac{(1-\gamma_1) A \breve{d}_{1(\kappa+1)}^B}{m_2} - \left(\frac{p_{1r} - c_{1r}}{p_{2r} - c_{2r}}\right) \frac{\gamma_1 A \breve{d}_{1(\kappa+1)}^B}{m_2} + \left(1 - \frac{p_{2n} - c_{2n}}{p_{2r} - c_{2r}}\right)\left(A\left(\gamma_2 + \frac{\gamma_1 \breve{D}_{1(\kappa+2)}^B}{m_2}\right) + \sum_{t=\kappa+2}^{T} \frac{\gamma_1 A \breve{d}_{1t}^B}{m_2}\right),$$

where $A = q_2 - \frac{q_2 \breve{D}_{2(\kappa+1)}^B}{m_2}$, $B = p_2 + \frac{q_2 \breve{D}_{2(\kappa+1)}^B}{m_2}$, *and* $E = 1 + \frac{\breve{D}_{1(\kappa+2)}^B}{m_2}$.

(b) *Suppose that* $p_{2n} - c_{2n} > p_{1n} - c_{1n}$. *For optimality of partial demand fulfillment in some period over the T-period selling horizon, it is sufficient that* $\exists \kappa \in \{\tau, ..., T-1\}$ *s.t.* $\breve{e}_{2t} < \gamma_2 \breve{d}_{2t}^B + \frac{\gamma_1 \breve{d}_{1t}^B \breve{D}_{2(t+1)}^B}{m_2} + \frac{\gamma_1 \breve{D}_{1t}^B \breve{d}_{2t}^B}{m_2}$ *for* $\tau \leq t \leq \kappa$, $\breve{e}_{2t} > \gamma_2 \breve{d}_{2t}^B + \frac{\gamma_1 \breve{d}_{1t}^B \breve{D}_{2(t+1)}^B}{m_2} + \frac{\gamma_1 \breve{D}_{1t}^B \breve{d}_{2t}^B}{m_2}$ *and* $\breve{e}_{1t} > \gamma_1 \breve{d}_{1t}^{NB}$ *for* $t > \kappa$, $\alpha > \frac{(1+A)(1-2B)}{(1-B)}$, *and*

$$\alpha > \left(\frac{p_{2n} - c_{2n}}{p_{2r} - c_{2r}}\right)(1 + AE) - \left(\frac{p_{1n} - c_{1n}}{p_{2r} - c_{2r}}\right) \frac{(1-\gamma_1) A \breve{d}_{1(\kappa+1)}^B}{m_2} - \left(\frac{p_{1r} - c_{1r}}{p_{2r} - c_{2r}}\right) \frac{\gamma_1 A \breve{d}_{1(\kappa+1)}^B}{m_2} - \left(\frac{p_{1n} - c_{1n} - p_{2n} + c_{2n}}{p_{2r} - c_{2r}}\right) \sum_{t=\kappa+2}^{T} \frac{A \breve{d}_{1t}^B}{m_2} + \left(1 - \frac{p_{2n} - c_{2n}}{p_{2r} - c_{2r}}\right)\left(A\left(\gamma_2 + \frac{\gamma_1 \breve{D}_{1(\kappa+2)}^B}{m_2}\right) + \sum_{t=\kappa+2}^{T} \frac{\gamma_1 A \breve{d}_{1t}^B}{m_2}\right),$$

where $A = q_2 - \frac{q_2 \breve{D}_{2(\kappa+1)}^B}{m_2}$, $B = p_2 + \frac{q_2 \breve{D}_{2(\kappa+1)}^B}{m_2}$, *and* $E = 1 + \frac{\breve{D}_{1(\kappa+2)}^B}{m_2}$.

(c) *Suppose that the conditions in part (a) or (b) hold and T is sufficiently large. Then the partial-fulfillment policy, if initiated after period* κ, *leads to no improvement in the total remanufacturing volume for the second-generation product.*

Proof of Theorem A.2. We will show that rejecting a demand of size $\epsilon > 0$ from the functionality-oriented customers unique to the second-generation product in period κ while meeting all the remaining demand in period κ and all demand in each period $t \neq \kappa$ (sales plan i) is more profitable than meeting all demand in each period (sales plan ii). We make the following observations:

(1) Periods t $< \kappa$: We know from the proof of Theorem A.1 that $\widehat{d}_{1t}^{NB} = \breve{d}_{1t}^{NB}$, $\widehat{n}_{1t} = \breve{n}_{1t}$, $\widehat{r}_{1t} = \breve{r}_{1t}$, and $\widehat{s}_{2t} = \breve{s}_{2t} = \widehat{d}_{2t}^{NB} = \breve{d}_{2t}^{NB}$, $\forall t < \kappa$. Also, note that $\widehat{n}_{2t} = \breve{n}_{2t}$ and $\widehat{r}_{2t} = \breve{r}_{2t}$, $\forall t < \kappa$.

(2) Period κ: We know from the proof of Theorem A.1 that $\widehat{d}_{1\kappa}^{NB} = \breve{d}_{1\kappa}^{NB}$, $\widehat{n}_{1\kappa} = \breve{n}_{1\kappa}$, and $\widehat{r}_{1\kappa} = \breve{r}_{1\kappa}$. Since $\breve{e}_{2\kappa} < \gamma_2 \breve{d}_{2\kappa}^{B} + \frac{\gamma_1 \breve{d}_{1\kappa}^{B} \breve{D}_{2(\kappa+1)}^{B}}{m_2} + \frac{\gamma_1 \breve{D}_{1\kappa}^{B} \breve{d}_{2\kappa}^{B}}{m_2}$, $\exists \epsilon > 0$ s.t. a demand of size ϵ from the functionality-oriented customers unique to the second-generation product in period κ cannot be met with remanufactured items in period κ. Rejecting a demand of size ϵ from these customers in period κ reduces the new item sales in period κ by ϵ. Thus $\widehat{n}_{2\kappa} = \breve{n}_{2\kappa} - \epsilon$ and $\widehat{r}_{2\kappa} = \breve{r}_{2\kappa}$.

(3) Period $\kappa+1$: We know from the proof of Theorem A.1 that $\widehat{d}_{1(\kappa+1)}^{NB} = \breve{d}_{1(\kappa+1)}^{NB} + \frac{A\epsilon \breve{d}_{1(\kappa+1)}^{B}}{m_2}$, $\widehat{r}_{1(\kappa+1)} = \breve{r}_{1(\kappa+1)} + \frac{\gamma_1 A\epsilon \breve{d}_{1(\kappa+1)}^{B}}{m_2}$, $\widehat{n}_{1(\kappa+1)} = \breve{n}_{1(\kappa+1)} + \frac{(1-\gamma_1) A\epsilon \breve{d}_{1(\kappa+1)}^{B}}{m_2}$ and $\widehat{s}_{2(\kappa+1)} = \widehat{d}_{2(\kappa+1)}^{NB} + \alpha\epsilon = \breve{d}_{2(\kappa+1)}^{NB} - \epsilon AE + \alpha\epsilon = \breve{s}_{2(\kappa+1)} - \epsilon AE + \alpha\epsilon$. Since $\breve{e}_{2(\kappa+1)} > \gamma_2 \breve{d}_{2(\kappa+1)}^{B} + \frac{\gamma_1 \breve{d}_{1(\kappa+1)}^{B} \breve{D}_{2(\kappa+2)}^{B}}{m_2} + \frac{\gamma_1 \breve{D}_{1(\kappa+1)}^{B} \breve{d}_{2(\kappa+1)}^{B}}{m_2}$, $\breve{n}_{2(\kappa+1)} = (1-\gamma_2)\breve{d}_{2(\kappa+1)}^{B} + \frac{(1-\gamma_1)\breve{d}_{1(\kappa+1)}^{B} \breve{D}_{2(\kappa+2)}^{B}}{m_2} + \frac{(1-\gamma_1)\breve{D}_{1(\kappa+1)}^{B} \breve{d}_{2(\kappa+1)}^{B}}{m_2}$ and $\breve{r}_{2(\kappa+1)} = \gamma_2 \breve{d}_{2(\kappa+1)}^{B} + \frac{\gamma_1 \breve{d}_{1(\kappa+1)}^{B} \breve{D}_{2(\kappa+2)}^{B}}{m_2} + \frac{\gamma_1 \breve{D}_{1(\kappa+1)}^{B} \breve{d}_{2(\kappa+1)}^{B}}{m_2}$. Demand for the second-generation remanufactured items in period $\kappa+1$ is given by:

$$\begin{aligned}
&\gamma_2 \widehat{d}_{2(\kappa+1)}^{B} + \frac{\gamma_1 \widehat{D}_{1(\kappa+1)}^{B} \widehat{d}_{2(\kappa+1)}^{B}}{m_2} + \frac{\gamma_1 \widehat{d}_{1(\kappa+1)}^{B} \widehat{D}_{2(\kappa+2)}^{B}}{m_2} + \alpha\epsilon \\
&= \gamma_2 \widehat{d}_{2(\kappa+1)}^{B} + \frac{\gamma_1 \breve{D}_{1(\kappa+1)}^{B} \widehat{d}_{2(\kappa+1)}^{B}}{m_2} + \frac{\gamma_1 \breve{d}_{1(\kappa+1)}^{B} \sum_{i=\tau}^{\kappa+1} \widehat{d}_{2i}^{B}}{m_2} + \alpha\epsilon \\
&= \widehat{d}_{2(\kappa+1)}^{B} \left(\gamma_2 + \frac{\gamma_1 \breve{D}_{1(\kappa+1)}^{B}}{m_2} \right) + \frac{\gamma_1 \breve{d}_{1(\kappa+1)}^{B} \sum_{i=\tau}^{\kappa+1} \widehat{d}_{2i}^{B}}{m_2} + \alpha\epsilon
\end{aligned}$$

$$= \widehat{d}^B_{2(\kappa+1)} \left(\gamma_2 + \frac{\gamma_1 \breve{D}^B_{1(\kappa+1)}}{m_2}\right) + \frac{\gamma_1 \breve{d}^B_{1(\kappa+1)} \sum_{i=\tau}^{\kappa} \breve{d}^B_{2i}}{m_2} + \frac{\gamma_1 \breve{d}^B_{1(\kappa+1)} \widehat{d}^B_{2(\kappa+1)}}{m_2} + \alpha\epsilon$$

$$= \left(p_2 + \frac{q_2 \widehat{S}^B_{2(\kappa+1)}}{m_2}\right)\left(m_2 - \widehat{D}^B_{2(\kappa+1)}\right)\left(\gamma_2 + \frac{\gamma_1 \breve{D}^B_{1(\kappa+1)}}{m_2}\right) + \frac{\gamma_1 \breve{d}^B_{1(\kappa+1)} \sum_{i=\tau}^{\kappa} \breve{d}^B_{2i}}{m_2}$$

$$+ \frac{\gamma_1 \breve{d}^B_{1(\kappa+1)} \widehat{d}^B_{2(\kappa+1)}}{m_2} + \alpha\epsilon$$

$$= \left(p_2 + \frac{q_2 \left(\widehat{D}^B_{2(\kappa+1)} - \epsilon\right)}{m_2}\right)\left(m_2 - \widehat{D}^B_{2(\kappa+1)}\right)\left(\gamma_2 + \frac{\gamma_1 \breve{D}^B_{1(\kappa+1)}}{m_2}\right)$$

$$+ \frac{\gamma_1 \breve{d}^B_{1(\kappa+1)} \sum_{i=\tau}^{\kappa} \breve{d}^B_{2i}}{m_2} + \frac{\gamma_1 \breve{d}^B_{1(\kappa+1)} \widehat{d}^B_{2(\kappa+1)}}{m_2} + \alpha\epsilon$$

$$= \left(p_2 + \frac{q_2 \breve{D}^B_{2(\kappa+1)}}{m_2}\right)\left(m_2 - \breve{D}^B_{2(\kappa+1)}\right)\left(\gamma_2 + \frac{\gamma_1 \breve{D}^B_{1(\kappa+1)}}{m_2}\right)$$

$$- \frac{q_2\epsilon}{m_2}\left(m_2 - \breve{D}^B_{2(\kappa+1)}\right)\left(\gamma_2 + \frac{\gamma_1 \breve{D}^B_{1(\kappa+1)}}{m_2}\right) + \frac{\gamma_1 \breve{d}^B_{1(\kappa+1)} \sum_{i=\tau}^{\kappa} \breve{d}^B_{2i}}{m_2}$$

$$+ \frac{\gamma_1 \breve{d}^B_{1(\kappa+1)} \widehat{d}^B_{2(\kappa+1)}}{m_2} + \alpha\epsilon$$

$$= \breve{d}^B_{2(\kappa+1)} \left(\gamma_2 + \frac{\gamma_1 \breve{D}^B_{1(\kappa+1)}}{m_2}\right) - \frac{q_2\epsilon}{m_2}\left(m_2 - \breve{D}^B_{2(\kappa+1)}\right)\left(\gamma_2 + \frac{\gamma_1 \breve{D}^B_{1(\kappa+1)}}{m_2}\right)$$

$$+ \frac{\gamma_1 \breve{d}^B_{1(\kappa+1)} \breve{D}^B_{2(\kappa+1)}}{m_2} + \frac{\gamma_1 \breve{d}^B_{1(\kappa+1)} \breve{d}^B_{2(\kappa+1)}}{m_2} - \frac{\gamma_1 A\epsilon \breve{d}^B_{1(\kappa+1)}}{m_2} + \alpha\epsilon$$

$$= \gamma_2 \breve{d}^B_{2(\kappa+1)} + \frac{\gamma_1 \breve{d}^B_{2(\kappa+1)} \breve{D}^B_{1(\kappa+1)}}{m_2} + \frac{\gamma_1 \breve{d}^B_{1(\kappa+1)} \breve{D}^B_{2(\kappa+2)}}{m_2} - \epsilon\left(A\left(\gamma_2 + \frac{\gamma_1 \breve{D}^B_{1(\kappa+2)}}{m_2}\right) - \alpha\right).$$

Since $\breve{e}_{2(\kappa+1)} = \sum_{i=1}^{\kappa} \beta_i \breve{s}_{2(\kappa+1-i)} > \gamma_2 \breve{d}^B_{2(\kappa+1)} + \frac{\gamma_1 \breve{d}^B_{1(\kappa+1)} \breve{D}^B_{2(\kappa+2)}}{m_2} + \frac{\gamma_1 \breve{D}^B_{1(\kappa+1)} \breve{d}^B_{2(\kappa+1)}}{m_2}$, $\exists \epsilon > 0$ s.t. $\widehat{e}_{2(\kappa+1)} = \sum_{i=1}^{\kappa} \beta_i \widehat{s}_{2(\kappa+1-i)} = \beta_1 \widehat{s}_{2\kappa} + \sum_{i=2}^{\kappa} \beta_i \breve{s}_{2(\kappa+1-i)} = \sum_{i=1}^{\kappa} \beta_i \breve{s}_{2(\kappa+1-i)} - \beta_1\epsilon = \breve{e}_{2(\kappa+1)} - \beta_1\epsilon > \gamma_2 \breve{d}^B_{2(\kappa+1)} + \frac{\gamma_1 \breve{d}^B_{1(\kappa+1)} \breve{D}^B_{2(\kappa+2)}}{m_2} + \frac{\gamma_1 \breve{D}^B_{1(\kappa+1)} \breve{d}^B_{2(\kappa+1)}}{m_2} - \epsilon\left(A\left(\gamma_2 + \frac{\gamma_1 \breve{D}^B_{1(\kappa+2)}}{m_2}\right) - \alpha\right) = \gamma_2 \widehat{d}^B_{2(\kappa+1)} + \frac{\gamma_1 \widehat{D}^B_{1(\kappa+1)} \widehat{d}^B_{2(\kappa+1)}}{m_2} + \frac{\gamma_1 \widehat{d}^B_{1(\kappa+1)} \widehat{D}^B_{2(\kappa+2)}}{m_2} + \alpha\epsilon$. This implies that $\widehat{r}_{2(\kappa+1)} = \gamma_2 \widehat{d}^B_{2(\kappa+1)} + \frac{\gamma_1 \widehat{D}^B_{1(\kappa+1)} \widehat{d}^B_{2(\kappa+1)}}{m_2} + \frac{\gamma_1 \widehat{d}^B_{1(\kappa+1)} \widehat{D}^B_{2(\kappa+2)}}{m_2} + \alpha\epsilon$. Thus, $\widehat{r}_{2(\kappa+1)} = \breve{r}_{2(\kappa+1)} - \epsilon\left(A\left(\gamma_2 + \frac{\gamma_1 \breve{D}^B_{1(\kappa+2)}}{m_2}\right) - \alpha\right)$ and $\widehat{n}_{2(\kappa+1)} = \breve{n}_{2(\kappa+1)} - \epsilon A\left(1 - \gamma_2 + \frac{(1-\gamma_1)\breve{D}^B_{1(\kappa+2)}}{m_2}\right)$.

(4) Periods $t \geq \kappa + 2$: We know from the proof of Theorem A.1 that $\widehat{d}^{NB}_{1t} \geq \breve{d}^{NB}_{1t}$, $\widehat{r}_{1t} \geq \breve{r}_{1t}$, $\widehat{n}_{1t} \geq \breve{n}_{1t}$, and $\widehat{s}_{2t} = \widehat{d}^{NB}_{2t} \geq \breve{d}^{NB}_{2t} - \left(\widehat{d}^{NB}_{1t} - \breve{d}^{NB}_{1t}\right) =$

$\breve{s}_{2t} - \left(\widehat{d}_{1t}^{NB} - \breve{d}_{1t}^{NB}\right) \geq \breve{s}_{2t} - \frac{A\epsilon\breve{d}_{1t}^{B}}{m_2}$, $\forall t \geq \kappa+2$. Since $\breve{e}_{2(\kappa+2)} > \gamma_2\breve{d}_{2(\kappa+2)}^{B} + \frac{\gamma_1\breve{D}_{1(\kappa+2)}^{B}\breve{d}_{2(\kappa+2)}^{B}}{m_2} + \frac{\gamma_1\breve{d}_{1(\kappa+2)}^{B}\breve{D}_{2(\kappa+3)}^{B}}{m_2}$, $\breve{n}_{2(\kappa+2)} = (1-\gamma_2)\breve{d}_{2(\kappa+2)}^{B} + \frac{(1-\gamma_1)\breve{d}_{1(\kappa+2)}^{B}\breve{D}_{2(\kappa+3)}^{B}}{m_2} + \frac{(1-\gamma_1)\breve{D}_{1(\kappa+2)}^{B}\breve{d}_{2(\kappa+2)}^{B}}{m_2}$ and $\breve{r}_{2(\kappa+2)} = \gamma_2\breve{d}_{2(\kappa+2)}^{B} + \frac{\gamma_1\breve{d}_{1(\kappa+2)}^{B}\breve{D}_{2(\kappa+3)}^{B}}{m_2} + \frac{\gamma_1\breve{D}_{1(\kappa+2)}^{B}\breve{d}_{2(\kappa+2)}^{B}}{m_2}$. We also know from the proof of Theorem A.1 that $\breve{d}_{2t}^{B} \leq \widehat{d}_{2t}^{B} \leq \breve{d}_{2t}^{B} + A\epsilon$ and $\widehat{D}_{2t}^{B} \leq \breve{D}_{2t}^{B} \leq A\epsilon + \widehat{D}_{2t}^{B}$, $\forall t \geq \kappa+2$. This implies that $\gamma_2\breve{d}_{2(\kappa+2)}^{B} + \frac{\gamma_1\breve{d}_{1(\kappa+2)}^{B}\breve{D}_{2(\kappa+3)}^{B}}{m_2} + \frac{\gamma_1\breve{D}_{1(\kappa+2)}^{B}\breve{d}_{2(\kappa+2)}^{B}}{m_2} + \epsilon A\left(\gamma_2 + \frac{\gamma_1\breve{D}_{1(\kappa+2)}^{B}}{m_2}\right) \geq \gamma_2\widehat{d}_{2(\kappa+2)}^{B} + \frac{\gamma_1\widehat{d}_{1(\kappa+2)}^{B}\widehat{D}_{2(\kappa+3)}^{B}}{m_2} + \frac{\gamma_1\widehat{D}_{1(\kappa+2)}^{B}\widehat{d}_{2(\kappa+2)}^{B}}{m_2}$. Since $\breve{e}_{2(\kappa+2)} > \gamma_2\breve{d}_{2(\kappa+2)}^{B} + \frac{\gamma_1\breve{d}_{1(\kappa+2)}^{B}\breve{D}_{2(\kappa+3)}^{B}}{m_2} + \frac{\gamma_1\breve{D}_{1(\kappa+2)}^{B}\breve{d}_{2(\kappa+2)}^{B}}{m_2}$, $\exists\epsilon > 0$ s.t.

$$\begin{aligned}
\widehat{e}_{2(\kappa+2)} &= \widehat{e}_{2(\kappa+1)} - \widehat{r}_{2(\kappa+1)} + \sum_{i=1}^{\kappa+1}\beta_i\widehat{s}_{2(\kappa+2-i)} \\
&= \widehat{e}_{2(\kappa+1)} - \widehat{r}_{2(\kappa+1)} + \sum_{i=3}^{\kappa+1}\beta_i\breve{s}_{2(\kappa+2-i)} + \beta_1\widehat{s}_{2(\kappa+1)} + \beta_2\widehat{s}_{2\kappa} \\
&= \breve{e}_{2(\kappa+1)} - \beta_1\epsilon - \widehat{r}_{2(\kappa+1)} + \sum_{i=3}^{\kappa+1}\beta_i\breve{s}_{2(\kappa+2-i)} + \beta_1\widehat{s}_{2(\kappa+1)} + \beta_2\widehat{s}_{2\kappa} \\
&= \breve{e}_{2(\kappa+2)} + \breve{r}_{2(\kappa+1)} - \beta_1\breve{s}_{2(\kappa+1)} - \beta_2\breve{s}_{2\kappa} - \beta_1\epsilon - \widehat{r}_{2(\kappa+1)} + \beta_1\widehat{s}_{2(\kappa+1)} + \beta_2\widehat{s}_{2\kappa} \\
&= \breve{e}_{2(\kappa+2)} - \beta_1\left(\breve{s}_{2(\kappa+1)} - \widehat{s}_{2(\kappa+1)}\right) - \beta_1\epsilon - \beta_2\left(\breve{s}_{2\kappa} - \widehat{s}_{2\kappa}\right) + \breve{r}_{2(\kappa+1)} - \widehat{r}_{2(\kappa+1)} \\
&= \breve{e}_{2(\kappa+2)} - \beta_1\epsilon - \beta_2\epsilon - \beta_1(\epsilon AE - \alpha\epsilon) + \epsilon\left(A\left(\gamma_2 + \frac{\gamma_1\breve{D}_{1(\kappa+2)}^{B}}{m_2}\right) - \alpha\right) \\
&> \gamma_2\breve{d}_{2(\kappa+2)}^{B} + \frac{\gamma_1\breve{d}_{1(\kappa+2)}^{B}\breve{D}_{2(\kappa+3)}^{B}}{m_2} + \frac{\gamma_1\breve{D}_{1(\kappa+2)}^{B}\breve{d}_{2(\kappa+2)}^{B}}{m_2} + \epsilon A\left(\gamma_2 + \frac{\gamma_1\breve{D}_{1(\kappa+2)}^{B}}{m_2}\right) \\
&\geq \gamma_2\widehat{d}_{2(\kappa+2)}^{B} + \frac{\gamma_1\widehat{d}_{1(\kappa+2)}^{B}\widehat{D}_{2(\kappa+3)}^{B}}{m_2} + \frac{\gamma_1\widehat{D}_{1(\kappa+2)}^{B}\widehat{d}_{2(\kappa+2)}^{B}}{m_2}.
\end{aligned}$$

Since $\widehat{e}_{2(\kappa+2)} > \gamma_2\widehat{d}_{2(\kappa+2)}^{B} + \frac{\gamma_1\widehat{d}_{1(\kappa+2)}^{B}\widehat{D}_{2(\kappa+3)}^{B}}{m_2} + \frac{\gamma_1\widehat{D}_{1(\kappa+2)}^{B}\widehat{d}_{2(\kappa+2)}^{B}}{m_2}$, $\widehat{r}_{2(\kappa+2)} = \gamma_2\widehat{d}_{2(\kappa+2)}^{B} + \frac{\gamma_1\widehat{d}_{1(\kappa+2)}^{B}\widehat{D}_{2(\kappa+3)}^{B}}{m_2} + \frac{\gamma_1\widehat{D}_{1(\kappa+2)}^{B}\widehat{d}_{2(\kappa+2)}^{B}}{m_2}$ and $\widehat{n}_{2(\kappa+2)} = (1-\gamma_2)\widehat{d}_{2(\kappa+2)}^{B} + \frac{(1-\gamma_1)\widehat{d}_{1(\kappa+2)}^{B}\widehat{D}_{2(\kappa+3)}^{B}}{m_2} + \frac{(1-\gamma_1)\widehat{D}_{1(\kappa+2)}^{B}\widehat{d}_{2(\kappa+2)}^{B}}{m_2}$. Also note that $\gamma_2\widehat{d}_{2(\kappa+2)}^{B} + \frac{\gamma_1\widehat{D}_{1(\kappa+2)}^{B}\widehat{d}_{2(\kappa+2)}^{B}}{m_2} + \frac{\gamma_1\widehat{d}_{1(\kappa+2)}^{B}\widehat{D}_{2(\kappa+3)}^{B}}{m_2} \geq \gamma_2\breve{d}_{2(\kappa+2)}^{B} + \frac{\gamma_1\breve{D}_{1(\kappa+2)}^{B}\breve{d}_{2(\kappa+2)}^{B}}{m_2} + \frac{\gamma_1\breve{d}_{1(\kappa+2)}^{B}\breve{D}_{2(\kappa+3)}^{B}}{m_2} - \frac{\gamma_1A\epsilon\breve{d}_{1(\kappa+2)}^{B}}{m_2}$. Proceeding similarly, it can be shown that $\widehat{e}_{2t} > \gamma_2\widehat{d}_{2t}^{B} + \frac{\gamma_1\widehat{d}_{1t}^{B}\widehat{D}_{2(t+1)}^{B}}{m_2} + \frac{\gamma_1\widehat{D}_{1t}^{B}\widehat{d}_{2t}^{B}}{m_2} = \widehat{r}_{2t} \geq \breve{r}_{2t} - \frac{\gamma_1A\epsilon\breve{d}_{1t}^{B}}{m_2}$, $\forall t \geq \kappa+2$.

Now we will show that sales plan (i) is more profitable than sales plan (ii).

In order to prove part (a) of Theorem A.2, suppose that $p_{1n} - c_{1n} > p_{2n} - c_{2n}$. Combining all of the above observations:

$$
\begin{aligned}
&(p_{1n} - c_{1n}) \sum_{t=1}^{T} \left(\widehat{d}_{1t}^{NB} - \widehat{r}_{1t} \right) + (p_{1r} - c_{1r}) \sum_{t=1}^{T} \widehat{r}_{1t} + (p_{2n} - c_{2n}) \sum_{t=\tau}^{T} \widehat{n}_{2t} + (p_{2r} - c_{2r}) \sum_{t=\tau}^{T} \widehat{r}_{2t} \\
&= (p_{1n} - c_{1n}) \left[\sum_{t=1}^{\kappa-1} \left(\widehat{d}_{1t}^{NB} - \widehat{r}_{1t} \right) + \left(\widehat{d}_{1\kappa}^{NB} - \widehat{r}_{1\kappa} \right) + \left(\widehat{d}_{1(\kappa+1)}^{NB} - \widehat{r}_{1(\kappa+1)} \right) + \sum_{t=\kappa+2}^{T} \left(\widehat{d}_{1t}^{NB} - \widehat{r}_{1t} \right) \right] \\
&\quad + (p_{1r} - c_{1r}) \sum_{t=1}^{T} \widehat{r}_{1t} + (p_{2n} - c_{2n}) \left[(\sum_{t=\tau}^{\kappa-1} \widehat{n}_{2t} + \widehat{n}_{2\kappa} + \widehat{n}_{2(\kappa+1)} + \sum_{t=\kappa+2}^{T} \widehat{d}_{2t}^{NB} - \sum_{t=\kappa+2}^{T} \widehat{r}_{2t} \right] \\
&\quad + (p_{2r} - c_{2r}) \left[\sum_{t=\tau}^{\kappa-1} \widehat{r}_{2t} + \widehat{r}_{2\kappa} + \widehat{r}_{2(\kappa+1)} + \sum_{t=\kappa+2}^{T} \widehat{r}_{2t} \right] \\
&= (p_{1n} - c_{1n}) \left[\sum_{t=1}^{\kappa-1} \left(\widehat{d}_{1t}^{NB} - \widehat{r}_{1t} \right) + \left(\widehat{d}_{1\kappa}^{NB} - \widehat{r}_{1\kappa} \right) + \left(\widehat{d}_{1(\kappa+1)}^{NB} - \widehat{r}_{1(\kappa+1)} \right) + \sum_{t=\kappa+2}^{T} \left(\widehat{d}_{1t}^{NB} - \widehat{r}_{1t} \right) \right] \\
&\quad + (p_{1r} - c_{1r}) \sum_{t=1}^{T} \widehat{r}_{1t} + (p_{2n} - c_{2n}) \left[\sum_{t=\tau}^{\kappa-1} \widehat{n}_{2t} + \widehat{n}_{2\kappa} + \widehat{n}_{2(\kappa+1)} + \sum_{t=\kappa+2}^{T} \widehat{d}_{2t}^{NB} \right] \\
&\quad + (p_{2r} - c_{2r}) \left[\sum_{t=\tau}^{\kappa-1} \widehat{r}_{2t} + \widehat{r}_{2\kappa} + \widehat{r}_{2(\kappa+1)} \right] + (p_{2r} - c_{2r} - p_{2n} + c_{2n}) \sum_{t=\kappa+2}^{T} \widehat{r}_{2t} \\
&\geq (p_{1n} - c_{1n}) \left[\sum_{t=1}^{\kappa-1} \left(\widehat{d}_{1t}^{NB} - \widehat{r}_{1t} \right) + \left(\widehat{d}_{1\kappa}^{NB} - \widehat{r}_{1\kappa} \right) + \left(\widehat{d}_{1(\kappa+1)}^{NB} - \widehat{r}_{1(\kappa+1)} \right) + \sum_{t=\kappa+2}^{T} \left(\widehat{d}_{1t}^{NB} - \widehat{r}_{1t} \right) \right] \\
&\quad + (p_{1r} - c_{1r}) \sum_{t=1}^{T} \widehat{r}_{1t} + (p_{2n} - c_{2n}) \left[\sum_{t=\tau}^{\kappa-1} \widehat{n}_{2t} + \widehat{n}_{2\kappa} + \widehat{n}_{2(\kappa+1)} + \sum_{t=\kappa+2}^{T} \widehat{d}_{2t}^{NB} \right] \\
&\quad + (p_{2r} - c_{2r}) \left[\sum_{t=\tau}^{\kappa-1} \widehat{r}_{2t} + \widehat{r}_{2\kappa} + \widehat{r}_{2(\kappa+1)} \right] + (p_{2r} - c_{2r} - p_{2n} + c_{2n}) \sum_{t=\kappa+2}^{T} \left[\breve{r}_{2t} - \frac{\gamma_1 A \epsilon \breve{d}_{1t}^{B}}{m_2} \right] \\
&= (p_{1n} - c_{1n}) \left[\sum_{t=1}^{\kappa-1} \left(\widehat{d}_{1t}^{NB} - \widehat{r}_{1t} \right) + \left(\widehat{d}_{1\kappa}^{NB} - \widehat{r}_{1\kappa} \right) + \left(\widehat{d}_{1(\kappa+1)}^{NB} - \widehat{r}_{1(\kappa+1)} \right) + \sum_{t=\kappa+2}^{T} \left(\widehat{d}_{1t}^{NB} - \widehat{r}_{1t} \right) \right] \\
&\quad + (p_{1r} - c_{1r}) \sum_{t=1}^{T} \widehat{r}_{1t} + (p_{2n} - c_{2n}) \left[\sum_{t=\tau}^{\kappa-1} \widehat{s}_{2t} + \widehat{s}_{2\kappa} + \widehat{s}_{2(\kappa+1)} + \sum_{t=\kappa+2}^{T} \widehat{d}_{2t}^{NB} \right] \\
&\quad + (p_{2r} - c_{2r} - p_{2n} + c_{2n}) \left[\sum_{t=\tau}^{\kappa-1} \widehat{r}_{2t} + \widehat{r}_{2\kappa} + \widehat{r}_{2(\kappa+1)} \right] \\
&\quad + (p_{2r} - c_{2r} - p_{2n} + c_{2n}) \sum_{t=\kappa+2}^{T} \left(\breve{r}_{2t} - \frac{\gamma_1 A \epsilon \breve{d}_{1t}^{B}}{m_2} \right)
\end{aligned}
$$

$$= (p_{1n} - c_{1n}) \left[\sum_{t=1}^{\kappa-1} \left(\widehat{d}_{1t}^{NB} - \widehat{r}_{1t} \right) + \left(\widehat{d}_{1\kappa}^{NB} - \widehat{r}_{1\kappa} \right) + \left(\widehat{d}_{1(\kappa+1)}^{NB} - \widehat{r}_{1(\kappa+1)} \right) + \sum_{t=\kappa+2}^{T} \left(\widehat{d}_{1t}^{NB} - \widehat{r}_{1t} \right) \right]$$

$$+ (p_{1r} - c_{1r}) \sum_{t=1}^{T} \widehat{r}_{1t}$$

$$+ (p_{2n} - c_{2n}) \left[\sum_{t=\tau}^{\kappa-1} \breve{s}_{2t} + \breve{s}_{2\kappa} - \epsilon + \breve{s}_{2(\kappa+1)} - \epsilon AE + \epsilon\alpha + \sum_{t=\kappa+2}^{T} \widehat{d}_{2t}^{B} \left(1 + \frac{\breve{D}_{1t}^{B}}{m_2} \right) \right.$$

$$\left. + \sum_{t=\kappa+2}^{T} \frac{\breve{d}_{1t}^{B} \widehat{D}_{2(t+1)}^{B}}{m_2} \right]$$

$$+ (p_{2r} - c_{2r} - p_{2n} + c_{2n}) \left[\sum_{t=\tau}^{\kappa-1} \breve{r}_{2t} + \breve{r}_{2\kappa} + \breve{r}_{2(\kappa+1)} - \epsilon \left(A \left(\gamma_2 + \frac{\gamma_1 \breve{D}_{1(\kappa+2)}^{B}}{m_2} \right) - \alpha \right) \right]$$

$$+ (p_{2r} - c_{2r} - p_{2n} + c_{2n}) \sum_{t=\kappa+2}^{T} \left(\breve{r}_{2t} - \frac{\gamma_1 A \epsilon \breve{d}_{1t}^{B}}{m_2} \right)$$

$$\geq (p_{1n} - c_{1n}) \left[\sum_{t=1}^{\kappa} \left(\breve{d}_{1t}^{NB} - \breve{r}_{1t} \right) + \left(\breve{d}_{1(\kappa+1)}^{NB} + \frac{A\epsilon \breve{d}_{1(\kappa+1)}^{B}}{m_2} - \breve{r}_{1(\kappa+1)} - \frac{\gamma_1 A \epsilon \breve{d}_{1(\kappa+1)}^{B}}{m_2} \right) \right.$$

$$\left. + \sum_{t=\kappa+2}^{T} \breve{d}_{1t}^{B} - \sum_{t=\kappa+2}^{T} \breve{r}_{1t} - \sum_{t=\kappa+2}^{T} \frac{\breve{d}_{1t}^{B} \widehat{D}_{2(t+1)}^{B}}{m_2} \right]$$

$$+ (p_{1r} - c_{1r}) \left[\sum_{t=1}^{\kappa} \breve{r}_{1t} + \breve{r}_{1(k+1)} + \frac{\gamma_1 A \epsilon \breve{d}_{1(\kappa+1)}^{B}}{m_2} + \sum_{t=\kappa+2}^{T} \breve{r}_{1t} \right]$$

$$+ (p_{2n} - c_{2n}) \left[\sum_{t=\tau}^{\kappa-1} \breve{s}_{2t} + \breve{s}_{2\kappa} - \epsilon + \breve{s}_{2(\kappa+1)} - \epsilon AE + \epsilon\alpha + \sum_{t=\kappa+2}^{T} \widehat{d}_{2t}^{B} \left(1 + \frac{\breve{D}_{1t}^{B}}{m_2} \right) \right.$$

$$\left. + \sum_{t=\kappa+2}^{T} \frac{\breve{d}_{1t}^{B} \widehat{D}_{2(t+1)}^{B}}{m_2} \right]$$

$$+ (p_{2r} - c_{2r} - p_{2n} + c_{2n}) \left[\sum_{t=\tau}^{\kappa-1} \breve{r}_{2t} + \breve{r}_{2\kappa} + \breve{r}_{2(\kappa+1)} - \epsilon \left(A \left(\gamma_2 + \frac{\gamma_1 \breve{D}_{1(\kappa+2)}^{B}}{m_2} \right) - \alpha \right) \right]$$

$$+ (p_{2r} - c_{2r} - p_{2n} + c_{2n}) \sum_{t=\kappa+2}^{T} \left(\breve{r}_{2t} - \frac{\gamma_1 A \epsilon \breve{d}_{1t}^{B}}{m_2} \right)$$

$$\geq (p_{1n} - c_{1n}) \left[\sum_{t=1}^{\kappa} \left(\breve{d}_{1t}^{NB} - \breve{r}_{1t} \right) + \left(\breve{d}_{1(\kappa+1)}^{NB} + \frac{A\epsilon \breve{d}_{1(\kappa+1)}^{B}}{m_2} - \breve{r}_{1(\kappa+1)} - \frac{\gamma_1 A \epsilon \breve{d}_{1(\kappa+1)}^{B}}{m_2} \right) \right.$$

$$\left. + \sum_{t=\kappa+2}^{T} \breve{d}_{1t}^{B} - \sum_{t=\kappa+2}^{T} \breve{r}_{1t} \right] + (p_{1r} - c_{1r}) \left[\sum_{t=1}^{\kappa} \breve{r}_{1t} + \breve{r}_{1(k+1)} + \frac{\gamma_1 A \epsilon \breve{d}_{1(\kappa+1)}^{B}}{m_2} + \sum_{t=\kappa+2}^{T} \breve{r}_{1t} \right]$$

$$
\begin{aligned}
&- (p_{1n} - c_{1n} - p_{2n} + c_{2n}) \sum_{t=\kappa+2}^{T} \frac{\breve{d}^B_{1t} \breve{D}^B_{2(t+1)}}{m_2} \\
&+(p_{2n} - c_{2n}) \left[\sum_{t=\tau}^{\kappa-1} \breve{s}_{2t} + \breve{s}_{2\kappa} - \epsilon + \breve{s}_{2(\kappa+1)} - \epsilon AE + \epsilon\alpha + \sum_{t=\kappa+2}^{T} \breve{d}^B_{2t} \left(1 + \frac{\breve{D}^B_{1t}}{m_2}\right) \right] \\
&+(p_{2r} - c_{2r} - p_{2n} + c_{2n}) \left[\sum_{t=\tau}^{\kappa-1} \breve{r}_{2t} + \breve{r}_{2\kappa} + \breve{r}_{2(\kappa+1)} - \epsilon \left(A \left(\gamma_2 + \frac{\gamma_1 \breve{D}^B_{1(\kappa+2)}}{m_2} \right) - \alpha \right) \right] \\
&+(p_{2r} - c_{2r} - p_{2n} + c_{2n}) \sum_{t=\kappa+2}^{T} \left(\breve{r}_{2t} - \frac{\gamma_1 A \epsilon \breve{d}^B_{1t}}{m_2} \right) \\
= &\, (p_{1n} - c_{1n}) \sum_{t=1}^{T} \left(\breve{d}^{NB}_{1t} - \breve{r}_{1t} \right) + (p_{1r} - c_{1r}) \sum_{t=1}^{T} \breve{r}_{1t} + (p_{2n} - c_{2n}) \sum_{t=\tau}^{T} \breve{s}_{2t} \\
&+(p_{2r} - c_{2r} - p_{2n} + c_{2n}) \sum_{t=\tau}^{T} \breve{r}_{2t} \\
&+(p_{1n} - c_{1n}) \frac{(1-\gamma_1) A \epsilon \breve{d}^B_{1(\kappa+1)}}{m_2} + (p_{1r} - c_{1r}) \frac{\gamma_1 A \epsilon \breve{d}^B_{1(\kappa+1)}}{m_2} - (p_{2n} - c_{2n}) (\epsilon + \epsilon AE) \\
&+(p_{2r} - c_{2r}) \epsilon\alpha - (p_{2r} - c_{2r} - p_{2n} + c_{2n}) \left(\epsilon A \left(\gamma_2 + \frac{\gamma_1 \breve{D}^B_{1(\kappa+2)}}{m_2} \right) + \sum_{t=\kappa+2}^{T} \frac{\gamma_1 A \epsilon \breve{d}^B_{1t}}{m_2} \right) \\
= &\, (p_{1n} - c_{1n}) \sum_{t=1}^{T} \left(\breve{d}^{NB}_{1t} - \breve{r}_{1t} \right) + (p_{1r} - c_{1r}) \sum_{t=1}^{T} \breve{r}_{1t} + (p_{2n} - c_{2n}) \sum_{t=\tau}^{T} \breve{n}_{2t} + (p_{2r} - c_{2r}) \sum_{t=\tau}^{T} \breve{r}_{2t} \\
&+(p_{1n} - c_{1n}) \frac{(1-\gamma_1) A \epsilon \breve{d}^B_{1(\kappa+1)}}{m_2} + (p_{1r} - c_{1r}) \frac{\gamma_1 A \epsilon \breve{d}^B_{1(\kappa+1)}}{m_2} - (p_{2n} - c_{2n}) (\epsilon + \epsilon AE) \\
&+(p_{2r} - c_{2r}) \epsilon\alpha - (p_{2r} - c_{2r} - p_{2n} + c_{2n}) \left(\epsilon A \left(\gamma_2 + \frac{\gamma_1 \breve{D}^B_{1(\kappa+2)}}{m_2} \right) + \sum_{t=\kappa+2}^{T} \frac{\gamma_1 A \epsilon \breve{d}^B_{1t}}{m_2} \right).
\end{aligned}
$$

In order to show that

$$
\begin{aligned}
&(p_{1n} - c_{1n}) \sum_{t=1}^{T} (\widehat{d}^{NB}_{1t} - \widehat{r}_{1t}) + (p_{1r} - c_{1r}) \sum_{t=1}^{T} \widehat{r}_{1t} + (p_{2n} - c_{2n}) \sum_{t=\tau}^{T} \widehat{n}_{2t} + (p_{2r} - c_{2r}) \sum_{t=\tau}^{T} \widehat{r}_{2t} \\
&\geq (p_{1n} - c_{1n}) \sum_{t=1}^{T} \left(\breve{d}^{NB}_{1t} - \breve{r}_{1t} \right) + (p_{1r} - c_{1r}) \sum_{t=1}^{T} \breve{r}_{1t} + (p_{2n} - c_{2n}) \sum_{t=\tau}^{T} \breve{n}_{2t} + (p_{2r} - c_{2r}) \sum_{t=\tau}^{T} \breve{r}_{2t}
\end{aligned}
$$

it suffices to show that

$$
\begin{aligned}
&(p_{1n} - c_{1n}) \frac{(1-\gamma_1) A \breve{d}^B_{1(\kappa+1)}}{m_2} + (p_{1r} - c_{1r}) \frac{\gamma_1 A \breve{d}^B_{1(\kappa+1)}}{m_2} - (p_{2n} - c_{2n}) (1 + AE) \\
&+(p_{2r} - c_{2r}) \alpha - (p_{2r} - c_{2r} - p_{2n} + c_{2n}) \left(A \left(\gamma_2 + \frac{\gamma_1 \breve{D}^B_{1(\kappa+2)}}{m_2} \right) + \sum_{t=\kappa+2}^{T} \frac{\gamma_1 A \breve{d}^B_{1t}}{m_2} \right) > 0.
\end{aligned}
$$

The above inequality holds as we assume

$$\alpha > \left(\frac{p_{2n}-c_{2n}}{p_{2r}-c_{2r}}\right)(1+AE) - \left(\frac{p_{1n}-c_{1n}}{p_{2r}-c_{2r}}\right)\frac{(1-\gamma_1)A\breve{d}^B_{1(\kappa+1)}}{m_2} - \left(\frac{p_{1r}-c_{1r}}{p_{2r}-c_{2r}}\right)\frac{\gamma_1 A\breve{d}^B_{1(\kappa+1)}}{m_2}$$
$$+\left(1-\frac{p_{2n}-c_{2n}}{p_{2r}-c_{2r}}\right)\left(A\left(\gamma_2+\frac{\gamma_1\breve{D}^B_{1(\kappa+2)}}{m_2}\right)+\sum_{t=\kappa+2}^{T}\frac{\gamma_1 A\breve{d}^B_{1t}}{m_2}\right).$$

In order to prove part (b) of Theorem A.2, suppose that $p_{2n}-c_{2n} > p_{1n}-c_{1n}$. We will show that sales plan (i) is more profitable than sales plan (ii). With similar arguments to those used in the proof of part (a):

$$(p_{1n}-c_{1n})\sum_{t=1}^{T}\left(\widehat{d}^{NB}_{1t}-\widehat{r}_{1t}\right)+(p_{1r}-c_{1r})\sum_{t=1}^{T}\widehat{r}_{1t}+(p_{2n}-c_{2n})\sum_{t=\tau}^{T}\widehat{n}_{2t}+(p_{2r}-c_{2r})\sum_{t=\tau}^{T}\widehat{r}_{2t}$$
$$=(p_{1n}-c_{1n})\left[\sum_{t=1}^{\kappa-1}\left(\widehat{d}^{NB}_{1t}-\widehat{r}_{1t}\right)+\left(\widehat{d}^{NB}_{1\kappa}-\widehat{r}_{1\kappa}\right)+\left(\widehat{d}^{NB}_{1(\kappa+1)}-\widehat{r}_{1(\kappa+1)}\right)+\sum_{t=\kappa+2}^{T}\left(\widehat{d}^{NB}_{1t}-\widehat{r}_{1t}\right)\right]$$
$$+(p_{1r}-c_{1r})\sum_{t=1}^{T}\widehat{r}_{1t}+(p_{2n}-c_{2n})\left[\sum_{t=\tau}^{\kappa-1}\widehat{n}_{2t}+\widehat{n}_{2\kappa}+\widehat{n}_{2(\kappa+1)}+\sum_{t=\kappa+2}^{T}\widehat{s}_{2t}-\sum_{t=\kappa+2}^{T}\widehat{r}_{2t}\right]$$
$$+(p_{2r}-c_{2r})\left[\sum_{t=\tau}^{\kappa-1}\widehat{r}_{2t}+\widehat{r}_{2\kappa}+\widehat{r}_{2(\kappa+1)}+\sum_{t=\kappa+2}^{T}\widehat{r}_{2t}\right]$$
$$=(p_{1n}-c_{1n})\left[\sum_{t=1}^{\kappa-1}\left(\widehat{d}^{NB}_{1t}-\widehat{r}_{1t}\right)+\left(\widehat{d}^{NB}_{1\kappa}-\widehat{r}_{1\kappa}\right)+\left(\widehat{d}^{NB}_{1(\kappa+1)}-\widehat{r}_{1(\kappa+1)}\right)+\sum_{t=\kappa+2}^{T}\left(\widehat{d}^{NB}_{1t}-\widehat{r}_{1t}\right)\right]$$
$$+(p_{1r}-c_{1r})\sum_{t=1}^{T}\widehat{r}_{1t}+(p_{2n}-c_{2n})\left[\sum_{t=\tau}^{\kappa-1}\widehat{n}_{2t}+\widehat{n}_{2\kappa}+\widehat{n}_{2(\kappa+1)}+\sum_{t=\kappa+2}^{T}\widehat{s}_{2t}\right]$$
$$+(p_{2r}-c_{2r})\left[\sum_{t=\tau}^{\kappa-1}\widehat{r}_{2t}+\widehat{r}_{2\kappa}+\widehat{r}_{2(\kappa+1)}\right]+(p_{2r}-c_{2r}-p_{2n}+c_{2n})\sum_{t=\kappa+2}^{T}\widehat{r}_{2t}$$
$$\geq(p_{1n}-c_{1n})\left[\sum_{t=1}^{\kappa-1}\left(\widehat{d}^{NB}_{1t}-\widehat{r}_{1t}\right)+\left(\widehat{d}^{NB}_{1\kappa}-\widehat{r}_{1\kappa}\right)+\left(\widehat{d}^{NB}_{1(\kappa+1)}-\widehat{r}_{1(\kappa+1)}\right)+\sum_{t=\kappa+2}^{T}\left(\widehat{d}^{NB}_{1t}-\widehat{r}_{1t}\right)\right]$$
$$+(p_{1r}-c_{1r})\sum_{t=1}^{T}\widehat{r}_{1t}+(p_{2n}-c_{2n})\left[\sum_{t=\tau}^{\kappa-1}\widehat{n}_{2t}+\widehat{n}_{2\kappa}+\widehat{n}_{2(\kappa+1)}+\sum_{t=\kappa+2}^{T}\widehat{s}_{2t}\right]$$
$$+(p_{2r}-c_{2r})\left[\sum_{t=\tau}^{\kappa-1}\widehat{r}_{2t}+\widehat{r}_{2\kappa}+\widehat{r}_{2(\kappa+1)}\right]+(p_{2r}-c_{2r}-p_{2n}+c_{2n})\sum_{t=\kappa+2}^{T}\left(\breve{r}_{2t}-\frac{\gamma_1 A\epsilon\breve{d}^B_{1t}}{m_2}\right)$$

$$
\begin{aligned}
&= (p_{1n}-c_{1n})\left[\sum_{t=1}^{\kappa-1}\left(\widehat{d}_{1t}^{NB}-\widehat{r}_{1t}\right)+\left(\widehat{d}_{1\kappa}^{NB}-\widehat{r}_{1\kappa}\right)+\left(\widehat{d}_{1(\kappa+1)}^{NB}-\widehat{r}_{1(\kappa+1)}\right)+\sum_{t=\kappa+2}^{T}\left(\widehat{d}_{1t}^{NB}-\widehat{r}_{1t}\right)\right]\\
&\quad+(p_{1r}-c_{1r})\sum_{t=1}^{T}\widehat{r}_{1t}+(p_{2n}-c_{2n})\left[\sum_{t=\tau}^{\kappa-1}\widehat{s}_{2t}+\widehat{s}_{2\kappa}+\widehat{s}_{2(\kappa+1)}+\sum_{t=\kappa+2}^{T}\widehat{s}_{2t}\right]\\
&\quad+(p_{2r}-c_{2r}-p_{2n}+c_{2n})\left(\sum_{t=\tau}^{\kappa-1}\widehat{r}_{2t}+\widehat{r}_{2\kappa}+\widehat{r}_{2(\kappa+1)}\right)\\
&\quad+(p_{2r}-c_{2r}-p_{2n}+c_{2n})\sum_{t=\kappa+2}^{T}\left(\breve{r}_{2t}-\frac{\gamma_1 A\epsilon\breve{d}_{1t}^{B}}{m_2}\right)\\
&\geq (p_{1n}-c_{1n})\left[\sum_{t=1}^{\kappa-1}\left(\widehat{d}_{1t}^{NB}-\widehat{r}_{1t}\right)+\left(\widehat{d}_{1\kappa}^{NB}-\widehat{r}_{1\kappa}\right)+\left(\widehat{d}_{1(\kappa+1)}^{NB}-\widehat{r}_{1(\kappa+1)}\right)+\sum_{t=\kappa+2}^{T}\left(\widehat{d}_{1t}^{NB}-\widehat{r}_{1t}\right)\right]\\
&\quad+(p_{1r}-c_{1r})\sum_{t=1}^{T}\widehat{r}_{1t}\\
&\quad+(p_{2n}-c_{2n})\left[\sum_{t=\tau}^{\kappa-1}\breve{s}_{2t}+\breve{s}_{2\kappa}-\epsilon+\breve{s}_{2(\kappa+1)}-\epsilon AE+\epsilon\alpha+\sum_{t=\kappa+2}^{T}\left(\breve{s}_{2t}-\left(\widehat{d}_{1t}^{NB}-\breve{d}_{1t}^{NB}\right)\right)\right]\\
&\quad+(p_{2r}-c_{2r}-p_{2n}+c_{2n})\left[\sum_{t=\tau}^{\kappa-1}\breve{r}_{2t}+\breve{r}_{2\kappa}+\breve{r}_{2(\kappa+1)}-\epsilon\left(A\left(\gamma_2+\frac{\gamma_1\breve{D}_{1(\kappa+2)}^{B}}{m_2}\right)-\alpha\right)\right.\\
&\quad\left.+\sum_{t=\kappa+2}^{T}\left(\breve{r}_{2t}-\frac{\gamma_1 A\epsilon\breve{d}_{1t}^{B}}{m_2}\right)\right]\\
&\geq (p_{1n}-c_{1n})\left[\sum_{t=1}^{\kappa}\left(\breve{d}_{1t}^{NB}-\breve{r}_{1t}\right)+\left(\breve{d}_{1(\kappa+1)}^{NB}+\frac{A\epsilon\breve{d}_{1(\kappa+1)}^{B}}{m_2}-\breve{r}_{1(\kappa+1)}-\frac{\gamma_1 A\epsilon\breve{d}_{1(\kappa+1)}^{B}}{m_2}\right)\right.\\
&\quad\left.+\sum_{t=\kappa+2}^{T}\left(\breve{d}_{1t}^{NB}+\left(\widehat{d}_{1t}^{NB}-\breve{d}_{1t}^{NB}\right)-\breve{r}_{1t}\right)\right]\\
&\quad+(p_{1r}-c_{1r})\left[\sum_{t=1}^{\kappa}\breve{r}_{1t}+\breve{r}_{1(k+1)}+\frac{\gamma_1 A\epsilon\breve{d}_{1(\kappa+1)}^{B}}{m_2}+\sum_{t=\kappa+2}^{T}\breve{r}_{1t}\right]\\
&\quad+(p_{2n}-c_{2n})\left[\sum_{t=\tau}^{\kappa-1}\breve{s}_{2t}+\breve{s}_{2\kappa}-\epsilon+\breve{s}_{2(\kappa+1)}-\epsilon AE+\epsilon\alpha+\sum_{t=\kappa+2}^{T}\left(\breve{s}_{2t}-\left(\widehat{d}_{1t}^{NB}-\breve{d}_{1t}^{NB}\right)\right)\right]\\
&\quad+(p_{2r}-c_{2r}-p_{2n}+c_{2n})\left[\sum_{t=\tau}^{\kappa-1}\breve{r}_{2t}+\breve{r}_{2\kappa}+\breve{r}_{2(\kappa+1)}-\epsilon\left(A\left(\gamma_2+\frac{\gamma_1\breve{D}_{1(\kappa+2)}^{B}}{m_2}\right)-\alpha\right)\right.\\
&\quad\left.+\sum_{t=\kappa+2}^{T}\left(\breve{r}_{2t}-\frac{\gamma_1 A\epsilon\breve{d}_{1t}^{B}}{m_2}\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
&= (p_{1n} - c_{1n})\Bigg[\sum_{t=1}^{\kappa}\left(\breve{d}_{1t}^{NB} - \breve{r}_{1t}\right) + \left(\breve{d}_{1(\kappa+1)}^{NB} + \frac{A\epsilon\breve{d}_{1(\kappa+1)}^{B}}{m_2} - \breve{r}_{1(\kappa+1)} - \frac{\gamma_1 A\epsilon\breve{d}_{1(\kappa+1)}^{B}}{m_2}\right) \\
&\quad + \sum_{t=\kappa+2}^{T}\left(\breve{d}_{1t}^{NB} - \breve{r}_{1t}\right)\Bigg] + (p_{1r} - c_{1r})\left[\sum_{t=1}^{\kappa}\breve{r}_{1t} + \breve{r}_{1(k+1)} + \frac{\gamma_1 A\epsilon\breve{d}_{1(\kappa+1)}^{B}}{m_2} + \sum_{t=\kappa+2}^{T}\breve{r}_{1t}\right] \\
&\quad + (p_{1n} - c_{1n} - p_{2n} + c_{2n})\sum_{t=\kappa+2}^{T}\left(\widehat{d}_{1t}^{NB} - \breve{d}_{1t}^{NB}\right) \\
&\quad + (p_{2n} - c_{2n})\left[\sum_{t=\tau}^{\kappa-1}\breve{s}_{2t} + \breve{s}_{2\kappa} - \epsilon + \breve{s}_{2(\kappa+1)} - \epsilon AE + \epsilon\alpha + \sum_{t=\kappa+2}^{T}\breve{s}_{2t}\right] \\
&\quad + (p_{2r} - c_{2r} - p_{2n} + c_{2n})\Bigg[\sum_{t=\tau}^{\kappa-1}\breve{r}_{2t} + \breve{r}_{2\kappa} + \breve{r}_{2(\kappa+1)} - \epsilon\left(A\left(\gamma_2 + \frac{\gamma_1\breve{D}_{1(\kappa+2)}^{B}}{m_2}\right) - \alpha\right) \\
&\quad + \sum_{t=\kappa+2}^{T}\left(\breve{r}_{2t} - \frac{\gamma_1 A\epsilon\breve{d}_{1t}^{B}}{m_2}\right)\Bigg] \\
&\geq (p_{1n} - c_{1n})\Bigg[\sum_{t=1}^{\kappa}\left(\breve{d}_{1t}^{NB} - \breve{r}_{1t}\right) + \left(\breve{d}_{1(\kappa+1)}^{NB} + \frac{A\epsilon\breve{d}_{1(\kappa+1)}^{B}}{m_2} - \breve{r}_{1(\kappa+1)} - \frac{\gamma_1 A\epsilon\breve{d}_{1(\kappa+1)}^{B}}{m_2}\right) \\
&\quad + \sum_{t=\kappa+2}^{T}\left(\breve{d}_{1t}^{NB} - \breve{r}_{1t}\right)\Bigg] + (p_{1r} - c_{1r})\left[\sum_{t=1}^{\kappa}\breve{r}_{1t} + \breve{r}_{1(k+1)} + \frac{\gamma_1 A\epsilon\breve{d}_{1(\kappa+1)}^{B}}{m_2} + \sum_{t=\kappa+2}^{T}\breve{r}_{1t}\right] \\
&\quad + (p_{1n} - c_{1n} - p_{2n} + c_{2n})\sum_{t=\kappa+2}^{T}\frac{A\epsilon\breve{d}_{1t}^{B}}{m_2} \\
&\quad + (p_{2n} - c_{2n})\left[\sum_{t=\tau}^{\kappa-1}\breve{s}_{2t} + \breve{s}_{2\kappa} - \epsilon + \breve{s}_{2(\kappa+1)} - \epsilon AE + \epsilon\alpha + \sum_{t=\kappa+2}^{T}\breve{s}_{2t}\right] \\
&\quad + (p_{2r} - c_{2r} - p_{2n} + c_{2n})\Bigg[\sum_{t=\tau}^{\kappa-1}\breve{r}_{2t} + \breve{r}_{2\kappa} + \breve{r}_{2(\kappa+1)} - \epsilon\left(A\left(\gamma_2 + \frac{\gamma_1\breve{D}_{1(\kappa+2)}^{B}}{m_2}\right) - \alpha\right) \\
&\quad + \sum_{t=\kappa+2}^{T}\left(\breve{r}_{2t} - \frac{\gamma_1 A\epsilon\breve{d}_{1t}^{B}}{m_2}\right)\Bigg] \\
&= (p_{1n} - c_{1n})\sum_{t=1}^{T}\left(\breve{d}_{1t}^{NB} - \breve{r}_{1t}\right) + (p_{1r} - c_{1r})\sum_{t=1}^{T}\breve{r}_{1t} + (p_{2n} - c_{2n})\sum_{t=\tau}^{T}\breve{n}_{2t} + (p_{2r} - c_{2r})\sum_{t=\tau}^{T}\breve{r}_{2t} \\
&\quad + (p_{1n} - c_{1n})\frac{(1-\gamma_1)A\epsilon\breve{d}_{1(\kappa+1)}^{B}}{m_2} + (p_{1r} - c_{1r})\frac{\gamma_1 A\epsilon\breve{d}_{1(\kappa+1)}^{B}}{m_2} \\
&\quad + (p_{1n} - c_{1n} - p_{2n} + c_{2n})\sum_{t=\kappa+2}^{T}\frac{A\epsilon\breve{d}_{1t}^{B}}{m_2} - (p_{2n} - c_{2n})(\epsilon + \epsilon AE) + (p_{2r} - c_{2r})\epsilon\alpha \\
&\quad - (p_{2r} - c_{2r} - p_{2n} + c_{2n})\left(\epsilon A\left(\gamma_2 + \frac{\gamma_1\breve{D}_{1(\kappa+2)}^{B}}{m_2}\right) + \sum_{t=\kappa+2}^{T}\frac{\gamma_1 A\epsilon\breve{d}_{1t}^{B}}{m_2}\right).
\end{aligned}
$$

In order to show that

$$(p_{1n}-c_{1n})\sum_{t=1}^{T}(\widehat{d}_{1t}^{NB}-\widehat{r}_{1t})+(p_{1r}-c_{1r})\sum_{t=1}^{T}\widehat{r}_{1t}+(p_{2n}-c_{2n})\sum_{t=\tau}^{T}\widehat{n}_{2t}+(p_{2r}-c_{2r})\sum_{t=\tau}^{T}\widehat{r}_{2t}$$
$$\geq (p_{1n}-c_{1n})\sum_{t=1}^{T}\left(\breve{d}_{1t}^{NB}-\breve{r}_{1t}\right)+(p_{1r}-c_{1r})\sum_{t=1}^{T}\breve{r}_{1t}+(p_{2n}-c_{2n})\sum_{t=\tau}^{T}\breve{n}_{2t}+(p_{2r}-c_{2r})\sum_{t=\tau}^{T}\breve{r}_{2t}$$

it suffices to show that

$$(p_{1n}-c_{1n})\frac{(1-\gamma_1)A\breve{d}_{1(\kappa+1)}^{B}}{m_2}+(p_{1r}-c_{1r})\frac{\gamma_1 A\breve{d}_{1(\kappa+1)}^{B}}{m_2}$$
$$+(p_{1n}-c_{1n}-p_{2n}+c_{2n})\sum_{t=\kappa+2}^{T}\frac{A\breve{d}_{1t}^{B}}{m_2}-(p_{2n}-c_{2n})(1+AE)+(p_{2r}-c_{2r})\alpha$$
$$-(p_{2r}-c_{2r}-p_{2n}+c_{2n})\left(A\left(\gamma_2+\frac{\gamma_1\breve{D}_{1(\kappa+2)}^{B}}{m_2}\right)+\sum_{t=\kappa+2}^{T}\frac{\gamma_1 A\breve{d}_{1t}^{B}}{m_2}\right)>0.$$

The above inequality holds as we assume

$$\alpha>\left(\frac{p_{2n}-c_{2n}}{p_{2r}-c_{2r}}\right)(1+AE)-\left(\frac{p_{1n}-c_{1n}}{p_{2r}-c_{2r}}\right)\frac{(1-\gamma_1)A\breve{d}_{1(\kappa+1)}^{B}}{m_2}$$
$$-\left(\frac{p_{1r}-c_{1r}}{p_{2r}-c_{2r}}\right)\frac{\gamma_1 A\breve{d}_{1(\kappa+1)}^{B}}{m_2}-\left(\frac{p_{1n}-c_{1n}-p_{2n}+c_{2n}}{p_{2r}-c_{2r}}\right)\sum_{t=\kappa+2}^{T}\frac{A\breve{d}_{1t}^{B}}{m_2}$$
$$+\left(1-\frac{p_{2n}-c_{2n}}{p_{2r}-c_{2r}}\right)\left(A\left(\gamma_2+\frac{\gamma_1\breve{D}_{1(\kappa+2)}^{B}}{m_2}\right)+\sum_{t=\kappa+2}^{T}\frac{\gamma_1 A\breve{d}_{1t}^{B}}{m_2}\right).$$

(c) Suppose that the conditions in part (a) or (b) hold and T is sufficiently large. Thus the partial-fulfillment policy is optimal. Also, suppose that the partial-fulfillment policy is initiated in period $\kappa'>\kappa$ at optimality. We use the tilde ($\tilde{}$) to denote the variables of this sales plan. The total cumulative demand for second-generation remanufactured items is given by $\gamma_1 m_1+\gamma_2 m_2$ in period T. The total remanufacturing volume for the second-generation product is $\sum_{t=\tau}^{\kappa}\breve{e}_{2t}+\sum_{t=\kappa+1}^{T}\left(\gamma_2\breve{d}_{2t}^{B}+\frac{\gamma_1\breve{d}_{1t}^{B}\breve{D}_{2(t+1)}^{B}}{m_2}+\frac{\gamma_1\breve{D}_{1t}^{B}\breve{d}_{2t}^{B}}{m_2}\right)$ under the immediate-fulfillment policy, while it cannot exceed $\sum_{t=\tau}^{\kappa}\widetilde{e}_{2t}+\sum_{t=\kappa+1}^{T}\left(\gamma_2\widetilde{d}_{2t}^{B}+\frac{\gamma_1\widetilde{d}_{1t}^{B}\widetilde{D}_{2(t+1)}^{B}}{m_2}+\frac{\gamma_1\widetilde{D}_{1t}^{B}\widetilde{d}_{2t}^{B}}{m_2}\right)=\sum_{t=\tau}^{\kappa}\breve{e}_{2t}+\sum_{t=\kappa+1}^{\kappa'}\left(\gamma_2\breve{d}_{2t}^{B}+\frac{\gamma_1\breve{d}_{1t}^{B}\breve{D}_{2(t+1)}^{B}}{m_2}+\frac{\gamma_1\breve{D}_{1t}^{B}\breve{d}_{2t}^{B}}{m_2}\right)+\sum_{t=\kappa'+1}^{T}\left(\gamma_2\widetilde{d}_{2t}^{B}+\frac{\gamma_1\widetilde{d}_{1t}^{B}\widetilde{D}_{2(t+1)}^{B}}{m_2}+\frac{\gamma_1\widetilde{D}_{1t}^{B}\widetilde{d}_{2t}^{B}}{m_2}\right)$ under the partial-fulfillment policy. Since $\sum_{t=\kappa'+1}^{T}\left(\gamma_2\widetilde{d}_{2t}^{B}+\frac{\gamma_1\widetilde{d}_{1t}^{B}\widetilde{D}_{2(t+1)}^{B}}{m_2}+\right.$

$\frac{\gamma_1 \widetilde{D}_{1t}^B \widetilde{d}_{2t}^B}{m_2}\Big) = \sum_{t=\kappa'+1}^{T} \left(\gamma_2 \breve{d}_{2t}^B + \frac{\gamma_1 \breve{d}_{1t}^B \breve{D}_{2(t+1)}^B}{m_2} + \frac{\gamma_1 \breve{D}_{1t}^B \breve{d}_{2t}^B}{m_2}\right)$, the total remanufacturing volume for the second-generation product cannot be increased with the partial-fulfillment policy. □