

ISTANBUL TECHNICAL UNIVERSITY ★ GRADUATE SCHOOL

**NOVEL SPHERICAL FUZZY AGGREGATION OPERATORS AND
SIMILARITY & DISTANCE MEASURES**



M.Sc. THESIS

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Department of Industrial Engineering

Industrial Engineering Programme

June 2022

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İSTANBUL TEKNİK ÜNİVERSİTESİ ★ LİSANSÜSTÜ EĞİTİM ENSTİTÜSÜ

**KÜRESEL BULANIK YENİ YIĞIŞTIRMA OPERATÖRLERİ VE BENZERLİK
& MESAFE ÖLÇÜLERİ**

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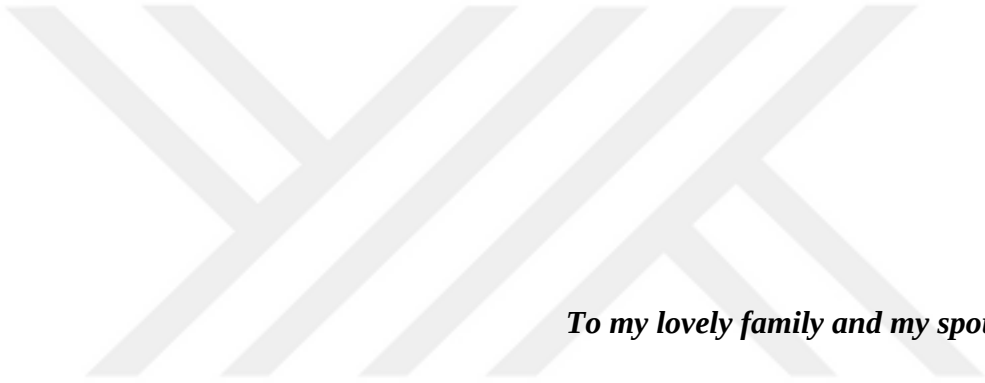
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Date of Defense : 16 06 2022





To my lovely family and my spouse Fariba,



FOREWORD

During my Ph.D. thesis period, several people have helped me in different ways. Firstly, I would like to state my sincere appreciation to my advisor Prof. Dr. Cengiz Kahraman for the valuable support of my Ph.D. research, and for his constancy, enthusiasm, willpower, and great knowledge. He guided me all the time in the research and writing of articles and this thesis. Without his support, it would have been impossible for me to complete my thesis in this area of research. I will always try my best to apply his teachings, not only in the academic area but also in life itself.

I also want to express my sincere thanks to Prof. Dr. Özgür Kabak and Assist. Prof. Dr. Tuncay Gürbüz for serving on my guidance committee and giving me valuable advice on my thesis.

Most importantly, I would like to express my special thanks to my dear mothers Roghayyeh, Shahbazi, and dear father Jafar, Donyatalab who raised me and fortify me mentally and spiritually all along the line of my life. I could not come this far without their encouragement, favor, and support. They have raised me into the person I am right now, which enabled me to have the ambition and endurance to achieve, what I've achieved so far despite my young age.

I also would like to give my special thanks to my spouse Fariba Farid, for providing me with unfailing support and continuous encouragement throughout my years of study and through the process of researching and writing this thesis. It would be never enough to thank her for all her positive and constructive thinking.

June 2022

Yaser DONYATALAB



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ABBREVIATIONS

IFS	: Intuitionistic Fuzzy Sets
IFS-2	: Intuitionistic Fuzzy Sets of Second Type
NS	: Neutrosophic Sets
HFS	: Hesitant Fuzzy Sets
PyFS	: Pythagorean Fuzzy Sets
PFS	: Picture Fuzzy Sets
qROFs	: q-Rung Orthopair Fuzzy Sets
SFS	: Spherical Fuzzy Sets
IVSFS	: Interval-Valued Spherical Fuzzy Sets
CSFRS	: Covering-based Spherical Fuzzy Rough Sets
Lt-SFN	: Linguistic Spherical Fuzzy Number
DM	: Decision Maker
MADM	: Multi-Attribute Decision Making
MCDM	: Multi-Criteria Decision Making
MAGDM	: Multi-Attribute Group Decision Making
MCGDM	: Multi-Criteria Group Decision Making
AHP	: Analytic Hierarchy Process
TOPSIS	: Technique for Order Preference by Similarity to an Ideal Solution
IF-TOPSIS	: Intuitionistic Fuzzy-Technique for Order Preference by Similarity to an Ideal Solution
IVSF-TOPSIS	: Interval-Valued Spherical Fuzzy Technique for Order Preference by Similarity to an Ideal Solution
IFWA	: Intuitionistic Fuzzy Weighted Average
SWAM	: Spherical Weighted Arithmetic Mean
SWGM	: Spherical Weighted Geometric Mean
SF-TOPSIS	: Spherical Fuzzy-Technique for Order Preference by Similarity to an Ideal Solution
NIS	: Negative Ideal Solution
PIS	: Pozitive Ideal Solution
SF-AHP	: Spherical Fuzzy-Analytic Hierarchy Process

SF-EDAS	: Spherical Fuzzy- Evaluation based on Distance from Average Solution
SF-WASPAS	: Spherical Fuzzy- Weighted Aggregated Sum Product Assessment
VIKOR	: Višekriterijumsko kompromisno rangiranje Resenje
WASPAS	: Weighted Aggregated Sum Product Assessment
HM	: Harmonic Mean
WHM	: Weighted Harmonic Mean
SFNAWHM	: Spherical Fuzzy Number Algebraic Weighted Harmonic Mean Aggregation Operator
SFNEWHM	: Spherical Fuzzy Number Einstein Weighted Harmonic Mean Aggregation Operator
SFNAOWHM	: Spherical Fuzzy Number Algebraic Ordered Weighted Harmonic Mean Aggregation Operator
SFNEOWHM	: Spherical Fuzzy Number Einstein Ordered Weighted Harmonic Mean Aggregation Operator
SFNAHOWHM	: Spherical Fuzzy Number Algebraic Hybrid Ordered Weighted Harmonic Mean Aggregation Operator
SFNEHOWHM	: Spherical Fuzzy Number Einstein Hybrid Ordered Weighted Harmonic Mean Aggregation Operator
SO₂	: Sulfur Dioxide
NO₂	: Nitrogen Dioxide
PM₁₀	: Particulate Matter 10
MD	: Minkowski Distance
MHD	: Minkowski-Hausdorff Distance
MI	: Medical Indicator
RI	: Reference Indicator
WPRI	: Weighted Positive Reference Indicator
WNRI	: Weighted Negative Reference Indicator
WRI	: Weighted Reference Indicator

SYMBOLS

μ	: Membership degree
ϑ	: Nonmembership degree
I	: Hesitancy degree
R	: Refusal degree
$\tilde{A}_s, \tilde{B}_s$	: Spherical Fuzzy Numbers
X	: Universe of discourse
$\tilde{I}$	: Intuitionistic Fuzzy Set
$\widetilde{Py}$	: Pythagorean Fuzzy Set
$\tilde{P}$	: Picture Fuzzy Set
D_n	: n^{th} Decision Maker
C_i	: j^{th} criterion
w_i	: Weight of i^{th} criterion
A_k	: k^{th} alternative
CR	: Closeness Ratio
$Md_{SFS}(\tilde{A}, \tilde{B})$	: Minkowski distance measure between $\tilde{A}$ and $\tilde{B}$
$WMd_{SFS}(\tilde{A}, \tilde{B})$	: Weighted Minkowski distance measure between $\tilde{A}$ and $\tilde{B}$
$MHd_{SFS}(\tilde{A}, \tilde{B})$	: Minkowski-Hausdorff distance measure between $\tilde{A}$ and $\tilde{B}$
$WMHd_{SFS}(\tilde{A}, \tilde{B})$	: Weighted Minkowski-Hausdorff distance measure between $\tilde{A}$ and $\tilde{B}$
$S_{SFS}(\tilde{A}, \tilde{B})$	: Similarity measure between two Spherical fuzzy sets $\tilde{A}$ and $\tilde{B}$
$S_{SFS}^{\boxed{f}}(\tilde{A}, \tilde{B})$	: Spherical fuzzy distance-based f-Similarity measure between $\tilde{A}$ and $\tilde{B}$
$S_{SFS}^{\boxed{l}}(\tilde{A}, \tilde{B})$	: Spherical fuzzy distance-based linear similarity measure between $\tilde{A}$ and $\tilde{B}$
$S_{SFS}^{\boxed{e}}(\tilde{A}, \tilde{B})$	: Spherical fuzzy distance-based exponential similarity measure between $\tilde{A}$ and $\tilde{B}$
$S_{SFS}^{\boxed{\log_2}}(\tilde{A}, \tilde{B})$	: Spherical fuzzy distance-based logarithmic similarity measure between $\tilde{A}$ and $\tilde{B}$

$S_{SFS}^{\overline{r}}(\tilde{A}, \tilde{B})$	: Spherical fuzzy distance-based rational similarity measure between $\tilde{A}$ and $\tilde{B}$
$MS_{SFS}^{\overline{l}}(\tilde{A}, \tilde{B})$	: Spherical fuzzy Minkowski based linear similarity measure between $\tilde{A}$ and $\tilde{B}$
$WMS_{SFS}^{\overline{l}}(\tilde{A}, \tilde{B})$	: Spherical fuzzy weighted Minkowski based linear similarity measure between $\tilde{A}$ and $\tilde{B}$
$MHS_{SFS}^{\overline{l}}(\tilde{A}, \tilde{B})$	: Spherical fuzzy Minkowski-Hausdorff based linear similarity measure between $\tilde{A}$ and $\tilde{B}$
$WMHS_{SFS}^{\overline{l}}(\tilde{A}, \tilde{B})$	: Spherical fuzzy weighted Minkowski-Hausdorff based linear similarity measure between $\tilde{A}$ and $\tilde{B}$
$MS_{SFS}^{\overline{e}}(\tilde{A}, \tilde{B})$	: Spherical fuzzy Minkowski based exponential similarity measure between $\tilde{A}$ and $\tilde{B}$
$WMS_{SFS}^{\overline{e}}(\tilde{A}, \tilde{B})$	: Spherical fuzzy weighted Minkowski based exponential similarity measure between $\tilde{A}$ and $\tilde{B}$
$MHS_{SFS}^{\overline{e}}(\tilde{A}, \tilde{B})$	: Spherical fuzzy Minkowski-Hausdorff based exponential similarity measure between $\tilde{A}$ and $\tilde{B}$
$WMHS_{SFS}^{\overline{e}}(\tilde{A}, \tilde{B})$	: Spherical fuzzy weighted Minkowski-Hausdorff based exponential similarity measure between $\tilde{A}$ and $\tilde{B}$
$MS_{SFS}^{\overline{\log_2}}(\tilde{A}, \tilde{B})$	: Spherical fuzzy Minkowski based logarithmic similarity measure between $\tilde{A}$ and $\tilde{B}$
$WMS_{SFS}^{\overline{\log_2}}(\tilde{A}, \tilde{B})$	: Spherical fuzzy weighted Minkowski based logarithmic similarity measure between $\tilde{A}$ and $\tilde{B}$
$MHS_{SFS}^{\overline{\log_2}}(\tilde{A}, \tilde{B})$	: Spherical fuzzy Minkowski-Hausdorff based logarithmic similarity measure between $\tilde{A}$ and $\tilde{B}$
$WMHS_{SFS}^{\overline{\log_2}}(\tilde{A}, \tilde{B})$	: Spherical fuzzy weighted Minkowski-Hausdorff based logarithmic similarity measure between $\tilde{A}$ and $\tilde{B}$
$MS_{SFS}^{\overline{r}}(\tilde{A}, \tilde{B})$	: Spherical fuzzy Minkowski based rational similarity measure between $\tilde{A}$ and $\tilde{B}$
$WMS_{SFS}^{\overline{r}}(\tilde{A}, \tilde{B})$	: Spherical fuzzy weighted Minkowski based rational similarity measure between $\tilde{A}$ and $\tilde{B}$
$MHS_{SFS}^{\overline{r}}(\tilde{A}, \tilde{B})$	: Spherical fuzzy Minkowski-Hausdorff based rational similarity measure between $\tilde{A}$ and $\tilde{B}$
$WMHS_{SFS}^{\overline{r}}(\tilde{A}, \tilde{B})$	: Spherical fuzzy weighted Minkowski-Hausdorff based rational similarity measure between $\tilde{A}$ and $\tilde{B}$

$CMS_{SFS}(\tilde{A}, \tilde{B})$	: Spherical fuzzy cosine Minkowski based similarity measure between $\tilde{A}$ and $\tilde{B}$
$WCMS_{SFS}(\tilde{A}, \tilde{B})$	: Spherical fuzzy weighted cosine Minkowski based similarity measure between $\tilde{A}$ and $\tilde{B}$
$CMHS_{SFS}(\tilde{A}, \tilde{B})$	: Spherical fuzzy cosine Minkowski-Hausdorff based similarity measure between $\tilde{A}$ and $\tilde{B}$
$WCMHS_{SFS}(\tilde{A}, \tilde{B})$	: Spherical fuzzy weighted cosine Minkowski-Hausdorff based similarity measure between $\tilde{A}$ and $\tilde{B}$
$SMS_{SFS}(\tilde{A}, \tilde{B})$	: Spherical fuzzy sine Minkowski based similarity measure between $\tilde{A}$ and $\tilde{B}$
$WSMS_{SFS}(\tilde{A}, \tilde{B})$	: Spherical fuzzy weighted sine Minkowski based similarity measure between $\tilde{A}$ and $\tilde{B}$
$SMHS_{SFS}(\tilde{A}, \tilde{B})\tilde{Q}_i$	: Spherical fuzzy sine Minkowski-Hausdorff based similarity measure between $\tilde{A}$ and $\tilde{B}$
$WSMHS_{SFS}(\tilde{A}, \tilde{B})$	: Spherical fuzzy weighted sine Minkowski-Hausdorff based similarity measure between $\tilde{A}$ and $\tilde{B}$
IFC^{α^1}	: Cosine similarity for intuitionistic fuzzy sets (IFSs)
$PyFC^{\alpha^1}$	: Cosine similarity for Pythagorean fuzzy sets (PyFSs)
SFC^{α^1}	: Cosine similarity for spherical fuzzy sets (SFSs)
S^s	: Spherical fuzzy sets (SFSs) similarity measure
JS_{SFS}	: Jaccard similarity for spherical fuzzy sets (SFSs)
d^+	: Positive Distance
d^-	: Negative Distance
$\mathcal{W}$	: Kendall's concordance coefficient
r_s	: Spearman correlations coefficient



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NOVEL SPHERICAL FUZZY AGGREGATION OPERATORS AND SIMILARITY & DISTANCE MEASURES

SUMMARY

Fuzzy Sets theory, developed by Zadeh in (Zadeh, 1965), is one of the most appropriate approaches to deal with the ambiguity of information and uncertain situations (Farrokhizadeh et al., 2021). Since the introduction of fuzzy sets by (Zadeh, 1965), they have been prevalent in almost all branches of science (Gündoğdu & Kahraman, 2019a). Many generalizations for ordinary fuzzy sets have been introduced by different researchers in the literature (K. T. Atanassov, 1986, 1999; Cuong & Kreinovich, 2014; Grattan-Guinness, 1976; Kutlu Gündoğdu & Kahraman, 2018; Smarandache, 1999a; Torra, 2010; Yager, 1986, 2014; Zadeh, 1975). Numerous researchers have utilized these extensions in recent years in the solution of multi-attribute decision-making problems (Gündoğdu & Kahraman, 2019b). These extensions are presented in chronological order, as given in Figure 1.1. One of the latest extensions of fuzzy sets is Spherical Fuzzy Sets (SFSs), recently developed (Gündoğdu & Kahraman, 2019), where the squared sum of membership, non-membership, and indeterminacy degrees is at most equal to 1.

Decision-making is the main aspect of every problem, and selecting the most appropriate alternative is fundamental for those problems and investigated by many researchers. Multi-criteria decision-making (MCDM) methods are among the most applicable and trustfulness methods to select the optimal alternative. MCDM algorithms select the best alternative from a finite set of alternatives based on multiple criteria. MCDM algorithms are methodological tools that deal with many different engineering problems. Some of those complex problems include fields of economy, businesses, insurance, medical and healthcare systems, engineering designs, sustainable supply chain, finance, actuarial, water management, energy management, agriculture and food supply chain, and environmental issues. Uncertainty is another aspect of every real-world problem that should be considered during the decision-making procedure to validate the results.

Fuzzy decision-making theories together with different modeling techniques have been studied, and many suitable approaches are introduced and studied through

different applications and case studies. Moreover, in every fuzzy MCDM problem expert or decision-maker (DM) is acting a significant role, but it also carries the uncertainty of the environment by itself. So, in most of the fuzzy MCDM problems, there would be a desire to have a number of the DMs which is called fuzzy Multi-criteria group decision-making (MCGDM) problem. A fuzzy MCGDM problem could be considered a high dimensional MCDM problem, therefore, carrying precise assessments for alternatives.

One of the primary and significant steps in every fuzzy MCGDM problem is the data fusion stage, which means there is a huge need to combine fuzzy data. Many fuzzy aggregation operators are proposed based on different characteristics for the data fusion stage of fuzzy MCGDM problems. Fuzzy distance and similarity measurements for comparison are also sitting in the framework of fuzzy data fusion, that considered in this thesis. So this manuscript, it is tried to define two main concepts of the data fusion concept: aggregation operators and distance-based similarity measurements in the spherical fuzzy environment. So, the novel concept of Harmonic mean aggregation operator for spherical fuzzy sets (SFSs) is introduced and discussed in all detail. Weighted, order weighted and hybrid order weighted Harmonic Mean operators and using two types of Algebraic and Einstein Strict Archimedean t-norms and t-conorms are types of proposed spherical fuzzy harmonic mean aggregation operators. So, 6 harmonic mean aggregators in the Spherical Fuzzy environment are introduced.

Also, this thesis investigated the concept of fuzzy distance and similarity measurements which are other types of fuzzy data fusion methods. I introduced the novel Minkowski and Minkowski-Hausdorff distance measures for spherical fuzzy sets and also studied novel spherical fuzzy f-similarity measures based on Minkowski and Minkowski-Hausdorff distances. So in this thesis, some novel fuzzy MCGDM algorithms for fuzzy data fusion extended to the spherical fuzzy environment. In these extensions, some real-life problems such as air quality evaluation problem and COVID_19 medical diagnosis problem are covered and suggesting other problems like supplier selection, energy source selection, hospital site selection, etc. problems to be covered for future works.

KÜRESEL BULANIK YENİ YIĞIŞTIRMA OPERATÖRLERİ VE BENZERLİK & MESAFE ÖLÇÜLERİ

ÖZET

Zadeh tarafından (Zadeh, 1965)'de geliştirilen Bulanık Kümeler teorisi, bilginin belirsizliği ve belirsiz durumlar (Farrokhizadeh et al., 2021) ile başa çıkmak için en uygun yaklaşımlardan biridir. Bulanık kümeler (Zadeh, 1965) tarafından tanıtıldığından beri, neredeyse tüm bilim dallarında (Gündoğdu & Kahraman, 2019a) yaygın hale gelmiştir. Literatürde farklı araştırmacılar tarafından sıradan bulanık kümeler için birçok genelleme yapılmıştır (K. T. Atanassov, 1986, 1999; Cuong & Kreinovich, 2014; Grattan-Guinness, 1976; Kutlu Gündoğdu & Kahraman, 2018; Smarandache, 1999a; Torra, 2010; Yager, 1986, 2014; Zadeh, 1975). Son yıllarda çok sayıda araştırmacı, çok özellikli karar verme problemlerinin çözümünde bu uzantıları kullanmıştır (Gündoğdu & Kahraman, 2019b). Bu uzantılar, Şekil 1.1'de verildiği gibi kronolojik sırada sunulmuştur. Bulanık kümelerin en son uzantılarından biri, yakın zamanda geliştirilen (Gündoğdu & Kahraman, 2019), üyelik, üyesizlik ve belirsizlik derecelerinin kare toplamının en fazla 1'e eşit veya daha az olduğu Küresel Bulanık Kümelerdir (SFSs).

Karar verme, her problemin ana yönüdür ve bu problemler için en uygun alternatifi seçmek esastır ve birçok araştırmacı tarafından araştırılmıştır. Çok kriterli karar verme (ÇKKV) yöntemleri, optimal alternatifi seçmek için en uygulanabilir ve güvenilir yöntemler arasındadır. ÇKKV algoritmaları, birden çok kriterle dayalı olarak sonlu bir alternatifler kümesinden en iyi alternatifi seçer. ÇKKV algoritmaları, birçok farklı mühendislik problemiyle ilgilenen metodolojik araçlardır. Bu karmaşık sorunlardan bazıları ekonomi, işletmeler, sigorta, tıbbi ve sağlık sistemleri ve mühendislik tasarımları, sürdürülebilir tedarik zinciri, finans, aktüeryal, su yönetimi, enerji yönetimi, tarım ve gıda tedarik zinciri ve çevre konularını içerir. Belirsizlik, sonuçları doğrulamak için karar verme prosedürü sırasında dikkate alınması gereken her gerçek dünya sorununun başka bir yönüdür.

Bulanık karar verme teorileri, farklı modelleme teknikleri ile birlikte çalışılmış ve farklı uygulamalar ve vaka çalışmaları yoluyla birçok uygun yaklaşım tanıtılmış ve incelenmiştir. Ayrıca, her bulanık ÇKKV probleminde uzman veya karar verici (KV) önemli bir rol oynamaktadır, ancak aynı zamanda ortamın belirsizliğini de kendi içinde taşımaktadır. Bu nedenle, bulanık ÇKKV problemlerinin çoğunda, bulanık Çok Kriterli Grup Karar Verme (ÇKGKV) problemi olarak adlandırılan çok sayıda KV'ye sahip olma isteği olacaktır. Bulanık bir ÇKKV problemi, yüksek boyutlu bir ÇKGKV problemi olarak kabul edilebilir, bu nedenle alternatifler için kesin değerlendirmeler taşır.

Her bulanık ÇKGKV problemindeki birincil ve önemli adımlardan biri veri birleştirme aşamasıdır, bu da bulanık verilerin birleştirilmesi için büyük bir ihtiyaç olduğu anlamına gelir. Bulanık ÇKGKV problemlerinin veri birleştirme aşaması için farklı özelliklere dayalı olarak birçok bulanık kümeleme operatörü önerilmiştir. Karşılaştırma için bulanık mesafe ve benzerlik ölçümleri de bu tezde ele alınan bulanık veri füzyonu çerçevesinde yer almaktadır. Bu yazıda, veri birleştirme kavramının iki ana kavramını tanımlamaya çalıştım: küresel bulanık ortamda toplama operatörleri ve mesafe tabanlı benzerlik ölçümleri. Böylece, küresel bulanık kümeler (SFSSs) için Harmonik ortalama toplama operatörünün yeni kavramını tanıttım ve kanıtladım ve tüm detayları tartıştım. Ağırlıklı, merteye ağırlıklı ve hibrit merteye ağırlıklı Harmonik Ortalama operatörleri ve iki tür Cebirsel ve Einstein Katı Arşimet t-normları ve t-conormlar, önerilen küresel bulanık harmonik ortalama toplama operatörlerinin türleridir. Böylece, Küresel Bulanık ortamında 6 harmonik ortalama toplayıcıyı tanıttım.

Ayrıca diğer bulanık veri füzyonu yöntemleri olan bulanık uzaklık ve benzerlik ölçümleri kavramını da araştırdım. Küresel bulanık kümeler için yeni Minkowski ve Minkowski-Hausdorff mesafe ölçümlerini tanıttım ve ayrıca Minkowski ve Minkowski-Hausdorff mesafelerine dayalı yeni küresel bulanık f-benzerlik ölçümlerini inceledim. Bu tezde, bulanık veri füzyonu için bazı yeni bulanık ÇKGKV algoritmaları, küresel bulanık ortama genişletildi. Bu uzantılarda, hava kalitesi değerlendirme sorunu ve COVID_19 tıbbi teşhis sorunu gibi bazı gerçek yaşam sorunları ele alınmakta ve gelecekteki çalışmalar için tedarikçi seçimi, enerji kaynağı seçimi, hastane yeri seçimi vb. sorunların ele alınması önerilmektedir.

1. INTRODUCTION

Fuzzy logic which was introduced by (Zadeh, 1965), originally is a type of multi-valued logic, which is a branch of propositional calculus. Based on Fuzzy Logic theory instead of two truth values (true or false) for propositions, has more values. Fuzzy logic derived from fuzzy set theory is capable of handling some more realistic cases that are related to approximate reasoning rather than a precise one. Since fuzzy sets are able to state propositions in more than two values, true and false, then could conclude that fuzzy sets directly state uncertainty about elements. Fuzzy sets are very popular in different branches of science due to such wonderful capability.

An obvious uncertainty in some aspects of human life that completely stems from mankind's brain in analyzing what's happening around and sometimes has just linguistic bases triggered to have a more complicated system for affirmation of such cases. For human beings facing problems, to solve them, the best and most handy way is to simplify them. But simplifying a problem by departing it into smaller parts to find a solution and then making it up is far away different from denying some parts of the fact that the problem lies in those. The world is not white or black, instead, it is grey (Beer, 1995), and even much more it is a kind of mixture of the total. Just considering the white or black is just sacrificing the fact to have simplicity, Which is the base of the first generation of fuzzy sets, ordinary fuzzy sets (Zadeh, 1965).

Fuzzy sets have been diversified into different types to express membership and non-membership, hesitancy, and refusal degrees for having a better understanding of uncertainty around a real-life event. The fuzzy logic theory has increased attention from researchers in many applications since it emerged (Zadeh, 1965). The ordinary fuzzy sets have been extended to many new types of fuzzy sets such as follows: Type-2 fuzzy sets (Zadeh, 1975), Interval-valued fuzzy sets (Grattan-Guinness, 1976; Sambuc, 1975; Zadeh, 1975), Intuitionistic fuzzy sets (K. T. Atanassov, 1986), fuzzy multisets (Yager, 1986), Neutrosophic sets (Smarandache, 2000), non-stationary fuzzy sets (Garibaldi & Ozen, 2007a), hesitant fuzzy sets (Torra, 2010), Pythagorean fuzzy sets (Yager, 2013; Yager & Abbasov, 2013), picture fuzzy sets (Cường, 2015), q-rung

ortho pair fuzzy sets (Yager, 2017) and spherical fuzzy sets (Kutlu Gündoğdu & Kahraman, 2018). Figure 1.1 presents the chronological order of the abovementioned fuzzy set extensions.

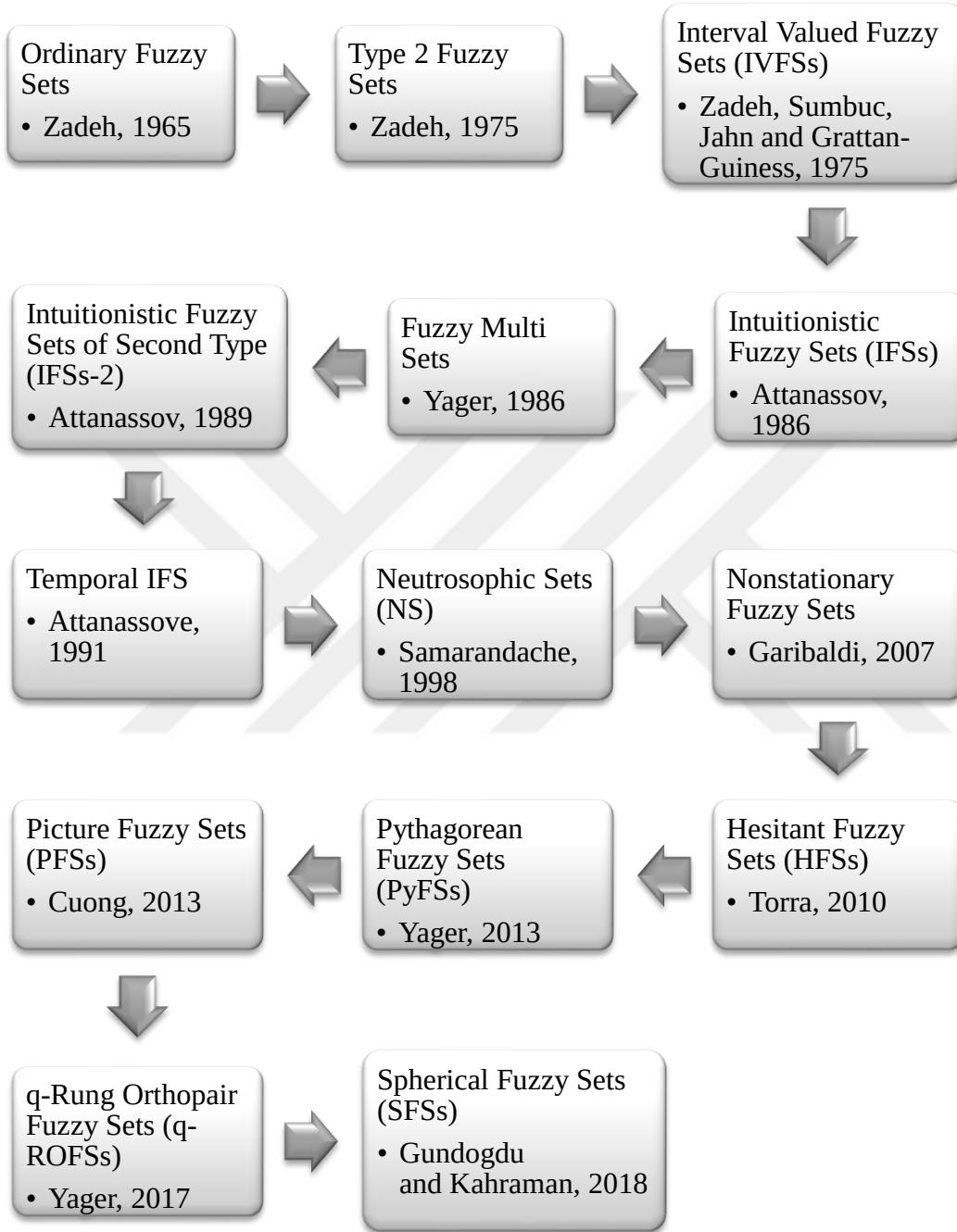


Figure 1.1: Fuzzy set extensions

1.1 Purpose of Thesis

The foremost objective of this thesis is to discuss the significant properties of the newest extension of 3D fuzzy sets and develop some novel data fusion concepts for it.

Also, in this thesis, it is proposed some decision-making methods under the novel concepts. In this thesis, the properties of the 3D fuzzy theory, the Spherical Fuzzy Sets (SFS) are discussed in detail, and then the novel concepts of Spherical Fuzzy Aggregation Operators and Spherical Fuzzy Similarity Measures are introduced and investigated in full detail. These methods are applied to different areas and through different illustrative examples and application case studies it is tried to show their validity and reliability.

After a concise introduction chapter, the rest of my thesis is developed with 3 additional chapters, which the following chapters include the presentation of my published papers in international SCI-E indexed journals. Chapter 2, is gathered comprehensive literature review of my publications, and chapter 3 includes my article about Harmonic Mean Aggregation Operators which in 2019 was published in the journal Multiple Valued Logic and Soft Computing (MVLSC). And finally, chapter 4 contains my publication about the concept of novel distance-based f-similarity measures for spherical fuzzy sets which in 2022 published in the international journal of Expert Systems With Application (ESWA).

In Chapter 2, the main concentrated concepts of the thesis are reviewed and it is tried to create a strong connection between the goal of the thesis and scientific research studies. So, in this chapter, the Fuzzy Logic theory and different extensions of fuzzy sets are reviewed comprehensively. Then provide a strong background about one of the recent 3D fuzzy sets, the Spherical Fuzzy Sets (SFSs) which in 2018 was introduced by C. Kahraman and F. K. Gundogdu (Gündoğdu & Kahraman, 2019). In Chapter 3, the concept of aggregation operators is discussed in detail and the novel Harmonic Mean Aggregation operators for spherical fuzzy sets are introduced in this thesis. To show the applicability of the proposed aggregation operators a MAGDM algorithm is proposed and evaluated through an illustrative example of the air quality evaluation problem. In Chapter 4, the novel Minkowski and Minkowski-Hausdorff distance measurements are proposed for the spherical fuzzy sets, and also some novel f-similarity measures are introduced. The proposed measurements are applied in a MAGDM algorithm for the COVID-19 diagnosis problem to show the validity and reliability of the methods.

It is tried to assure that all material is followed by some recommendations for future studies in these great areas of study and research.



2. LITERATURE REVIEW

2.1 Fuzzy Sets Theory

Ordinary Fuzzy Set (Zadeh, 1965) is introduced based on membership degree to express the ambiguity and those degrees belong to interval $[0,1]$. The other degree defined in the ordinary fuzzy sets is the non-membership degree which points to the difference between 1 and membership degree. However, in theory, Ordinary Fuzzy Sets greatly could come up in stating some complexities of uncertainties in problems, but not practically (Garibaldi & Ozen, 2007b). The inadequacy is that there is just a fixed value for membership degree to state this uncertainty. Zadeh has also addressed this problem in his paper entitled “The concept of a linguistic variable and its application to approximate reasoning-I”. Type-2 fuzzy sets have been investigated by (Prade, 1980), and (Mizumoto & Tanaka, 1976, 1981).

Attanassov extended the concept of uncertainty by introducing Intuitionistic Fuzzy Sets (IFSs) by adding the non-membership degree (K. T. Atanassov, 1986). The IFSs not only describe how much it is possible for an element to be in a fuzzy set by membership degree, but also declare how much it is not possible to have that element in the fuzzy set by non-membership degree.

Fuzzy multi-sets (Yager, 1986), which are introduced by Yager are an extension of multi-set theory in the fuzzy field. Attanassov in (K. T. Atanassov, 2016) by generalizing the IFSs to IFSs type-2, which include membership and non-membership degrees also, expands the area of uncertainty that could be considered for membership and non-membership. So IFS-2 is a higher dimension of IFSs. By using IFS-2 it could be considered much more vast amounts for membership and non-membership degrees. Then Attanassov introduced Temporal Intuitionistic Fuzzy Sets (Atanassov K., 1991), which is a time-based IFSs.

Smandrache in 1998 by defining the third dimension of uncertainty entitled hesitancy degree independently in the fuzzy set area, introduced Neutrosophic Sets in

(Smarandache, 1999b) and (Smarandache, 2003). Smarandache proved Neutrosophic Sets (NS) using the non-standard analysis method (Boyer et al., 1967) and are characterized by three dimensions: truthiness, indeterminacy degree, and falsity degrees which are similar to membership, hesitancy, and non-membership degrees, respectively. NS has a big difference from the other fuzzy sets. NS not only considers the hesitancy of the system independently but also decreases the indefiniteness of inconsistent information (Gündoğdu & Kahraman, 2019).

Garibaldi introduced the Nonstationary Fuzzy Sets (Garibaldi et al., 2008), which considers a very usual variation in human ideas and decisions over time. Torra in 2010 introduced a functional tool that allows having different membership and non-membership degrees potentially, which is called Hesitant Fuzzy Sets (HFS) (Torra, 2010); through the definition, one could conclude that IFS is a special case of the HFSs. More recently, Yager introduced Pythagorean Fuzzy Sets (PyFSs) (Yager, 2014) and proposed an application related to solving Multi-Attribute Decision Making (MADM) problems as an effective tool and mentioned that PyFSs covers a larger range of non-standard membership grade than intuitionistic membership grades; hence, it is more powerful than IFSs in handling the uncertainties (Yager, 2014).

The Picture Fuzzy Sets (PFSs) (Cường, 2015) which directly extend from Ordinary Fuzzy Sets and IFSs as an appropriate tool that involves answers like yes, abstain, no, and refusal in determining the ambiguity around events. (Gündoğdu & Kahraman, 2019) introduced the Spherical Fuzzy Sets (SFSs) which are developed based on the IFSs, thereafter they introduced some novel operators for SFSs. The idea of SFSs is to develop a 3D fuzzy set by using a sphere surface to let decision-makers have more flexibility in giving different values for uncertainty degrees vastly.

2.2 The Recent 3D Fuzzy Set: Spherical Fuzzy Sets (SFSs)

As it is discussed beforehand, researchers introduced many extensions to fuzzy sets. The Spherical Fuzzy Sets (SFSs) are one of those generalizations with specific characteristics and evaluated by many researchers in different science and study areas. SFSs is using membership, non-membership, indeterminacy, and refusal degrees which belong to the interval $[0,1]$, and also sum square of these degrees should be equal to 1 which makes sense to define the ambiguity. To be more specific, if a decision maker's idea about an element belonging or not or being in doubt will be

interpreted as a different dimension of uncertainty and affect one of those degrees. If the decision-maker believes the element neither belong (membership degree) nor does not belong (non-membership degree) and at the same time this fact is not because of his/her hesitancy (indeterminacy degree) then the refusal degree will be responsible for the uncertainty of the specific element.

Based on (Antonov, 1995) the concept of the spherical coordinates of two points in space helped to show a new way of stereometric interpretation for elements in an intuitionistic fuzzy environment. The measurable interpretation prepared the concept of geometrical conceptualization of the distance between two intuitionistic fuzzy points. In (Yang & Chiclana, 2009), defined the spherical representation of two intuitionistic fuzzy numbers. Based on (Yang & Chiclana, 2009) intuitionistic fuzzy numbers are points in an intuitionistic fuzzy environment and the defined spherical distance allowed us to have a better understanding of the concept of distance function between intuitionistic fuzzy sets.

(Gündoğdu & Kahraman, 2019) proposed spherical fuzzy sets with the union, intersection, basic operators, and aggregation operations. They provided some distance measurements as an extension of picture fuzzy sets and Pythagorean fuzzy sets. The logic behind SFSs is to allow decision-makers to generalize other extensions of fuzzy sets by assigning a wider range of degrees to determine membership possibility on a spherical surface. Sequentially, (Ashraf, Abdullah, Mahmood, et al., 2019) developed spherical fuzzy sets with some different operational rules and also proposed some aggregation operations based on Archimedean t-norm and t-conorms (Ashraf, Abdullah, Aslam, et al., 2019).

A linguistic spherical fuzzy set was developed by (H. Jin et al., 2019) and introduced linguistic spherical fuzzy weighted averaging and geometric operators. Spherical fuzzy Dombi weighted averaging and geometric aggregation operators were proposed and compared several properties of these aggregation operators for the multi-criteria decision-making problems (Ashraf, Abdullah, & Mahmood, 2019). In (Kutlu Gündoğdu & Kahraman, 2019) introduced the interval-valued spherical fuzzy sets and interval-valued spherical fuzzy TOPSIS method employed for the 3D printer selection problem. In (Ashraf, Abdullah, & Abdullah, 2019), it is presented some spherical fuzzy distance-weighted averaging operators. Jin et al. in (Y. Jin et al., 2019) proposed

some novel logarithmic operations with spherical fuzzy entropy under the spherical fuzzy environment.

Spherical fuzzy Harmonic and Bonferroni Mean Aggregation operators were presented by (Donyatalab et al., 2019) and (Farrokhizadeh et al., 2020), respectively. A new approach of Covering-based Spherical Fuzzy Rough Set (CSFRS) models to hybrid spherical fuzzy sets were adopted by (Zeng et al., 2019) and introduced the TOPSIS method for CSFRS models. The Linguistic Spherical Fuzzy Numbers (Lt-SFNs) to suggest the public's knowledge of language evaluation were developed and also TODIM and MABAC methods based on Lt-SFNs were presented by (Liu et al., 2020). A site selection problem for offshore wind power station using an integrated two-stage MCDM model using Spherical fuzzy AHP and Spherical fuzzy WASPAS methods proposed by (C. Wang et al., n.d.), and spherical fuzzy Minkowski and Minkowski-Hausdorff distance measurements and f-similarity distance-based measurement are proposed for spherical fuzzy sets (Donyatalab et al., 2022).

2.3 A Background Summary

Evaluation of some alternatives to make a reasonable and at the same time optimal decision by considering all different criteria is one of the most favorite issues for humankind. Therefore, based on different methods for a decision-making process, aggregation operators have specific trends among conventional methods (Donyatalab et al., 2019). A significant foundation in every decision-making process is the way that experts state their preferences about alternatives (Lu et al., 2007). The aggregation operators are valuable models for combining and aggregating a finite set of numerical values into a single numerical while preserving the features of individuals from start to the end of the process (Bobillo & Straccia, 2013; Dülenci, 2016; Kou et al., 2016; Qin et al., 2016). So, an aggregation operator refers to the process of combining some values in a way that the final results could represent and carry the features of individuals in themselves. Therefore a mapping function from n-array to 1-array could be an aggregation operator if it satisfies some conditions. In fuzzy cases, fuzzy aggregation operators are applied to several different fuzzy sets. In fact, those are some fuzzy operations that result could be again a fuzzy set or a crisp one. Valuable research has been done by authors in this field as (Zeng et al., 2012), (W. Wang & Liu, 2014), (Zhu et al., 2016), (F. Jin et al., 2016), (Zhao et al., 2010), (Kutlu Gundogdu &

Kahraman, 2019) and (Donyatalab et al., 2019). Table 2.1, represents some of these aggregation operators based on their type of set and their features.

Table 2.1: A literature review on Fuzzy Aggregation Operators

Author	Year	Operators name	Type of Operator	Type of Fuzzy Set	Explanation
(Zeng et al., 2012)	2012	Fuzzy Generalized Ordered Weighted Averaging Distance (FGOWAD) Operator	GOWAD	Type 1 Fuzzy Sets	Generalized Ordered Weighted Triangular Fuzzy Numbers + Crisp Weight Vector + Fuzzy Weighted Hamming Distance
(Blanco-Mesa et al., 2016)	2016	Bonferroni OWA Adequacy Coefficient (BON-OWAAC)	OWA + Adequacy Coefficient (AC)	Type 1 Fuzzy Sets	The combination of OWA+AC causes aggregate inputs by comparison of two elements in a way that it neutralizes the result (Blanco-Mesa et al., 2016) (evaluation gives a real element that is higher than the ideal one.)
		Bonferroni Ordered Weighted Averaging Index of Maximum and Minimum (BON-OWAIMAM)	OWA + Index of Maximum and Minimum (IMAM)		The combination of OWA+IMAM causes to aggregate inputs by comparison of two elements in a way that operators could use both features

					of AC and Hamming distance (Blanco-Mesa et al., 2016) (Because IMAM unifies the AC and Hamming Distance in its formula)
(Ma et al., 2016)	2016	generated extended 2-tuple weighted averaging (GE2T-WA) operator	Weighted Arithmetic (WA)	Linguistic Fuzzy Sets (LFSs)	Two-tuple linguistic aggregation operators based on the subjective sensation scale and objective numerical scale
		generated extended 2-tuple ordered weighted averaging (GE2T-OWA) operator	Order Weighted Arithmetic (OWA)		
		generated extended 2-tuple weighted geometric (GE2T-WG) operator	Weighted Geometric (WG)		
		generated extended 2-tuple ordered weighted geometric (GE2T-OWG) operator	Order Weighted Geometric (OWG)		
(J. qiang Wang et al., 2014)	2014	Weighted arithmetic (IIWA) averaging operator.	WA	Intuitionistic Interval Fuzzy Sets (IVFSs)	Intuitionistic Interval Fuzzy Numbers + Crisp Weight Vector + Arithmetic Mean
		Weighted geometric (IIWG) averaging operator	WG		Intuitionistic Interval Fuzzy Numbers + Crisp Weight Vector + Geometric Mean
		Ordered weighted averaging (IIOWA) operator	OWA		Ordered Intuitionistic Interval Fuzzy

					Numbers + Crisp Weight Vector + Arithmetic Mean
		Heavy averaging (IIHA) operator	HOWA		Hybrid Ordered Intuitionistic Interval Fuzzy Numbers + Crisp Weight Vector + Arithmetic Mean
		(IIA) aggregating operator	WA		Intuitionistic Interval Fuzzy Numbers + Intuitionistic Interval Fuzzy Weights Vectors + Arithmetic Mean
(Zhou et al., 2016)	2016	Intuitionistic Hesitant Linguistic Weighted Averaging (IHLWA) Operator	WA	Intuitionistic Hesitant Linguistic Sets (IHLSs)	Intuitionistic Hesitant Linguistic Fuzzy Numbers + Crisp Weight Vector + Arithmetic Mean
		Intuitionistic Hesitant Linguistic Weighted Geometric (IHLWG) Operator	WG		Intuitionistic Hesitant Linguistic Fuzzy Numbers + Crisp Weight Vector + Geometric Mean
		Intuitionistic Hesitant Linguistic Ordered Weighted Averaging (IHLOWA) operator	OWA		Ordered Intuitionistic Hesitant Linguistic Fuzzy Numbers +

					Crisp Weight Vector + Arithmetic Mean
		Intuitionistic Hesitant Linguistic Ordered Weighted Geometric (IHLOWG) operator	OWG		Ordered Intuitionistic Hesitant Linguistic Fuzzy Numbers + Crisp Weight Vector + Geometric Mean
		Intuitionistic Hesitant Linguistic Hybrid Averaging (IHLHA)	HOWA		Hybrid Ordered Intuitionistic Hesitant Linguistic Fuzzy Numbers + Crisp Weight Vector + Arithmetic Mean
		Intuitionistic Hesitant Linguistic Hybrid Geometric (IHLHA)	HOWG		Hybrid Ordered Intuitionistic Hesitant Linguistic Fuzzy Numbers + Crisp Weight Vector + Geometric Mean
		Generalized Intuitionistic Hesitant Linguistic Hybrid Averaging (GIHLHA)	GHOWA		Generalized Hybrid Ordered Intuitionistic Hesitant Linguistic Fuzzy Numbers + Crisp Weight Vector + Arithmetic Mean
		Generalized Intuitionistic	GHOWG		Generalized Hybrid

		Hesitant Linguistic Hybrid Geometric (GIHLHA)			Ordered Intuitionistic Hesitant Linguistic Fuzzy Numbers + Crisp Weight Vector + Geometric Mean
(W. Wang & Liu, 2014)	2014	Hesitant Fuzzy Einstein Weighted Geometric (HFWG) operator	WG	Hesitant Fuzzy Sets (HFSs)	Hesitant Fuzzy Numbers + Einstein T-norm + Crisp Weight Vector
		Hesitant Fuzzy Einstein Ordered Weighted Geometric (HFOWG) Operator	OWG		Ordered Hesitant Fuzzy Numbers + Einstein T-norm + Crisp Weight Vector
		Hesitant Fuzzy Einstein Hybrid Geometric (HFOWG) Operator	HOWG		Hybrid Ordered Hesitant Fuzzy Numbers + Einstein T-norm + Crisp Weight Vector
(Peng & Yang, 2016)	2016	Pythagorean Fuzzy Choquet Integral Arithmetic (PFCIG)	Choquet Integral + WA	Pythagorean Fuzzy Sets (PyFSs)	Pythagorean Fuzzy Numbers + Choquet Integral + WA + Crisp Weight Vector
		Pythagorean Fuzzy Choquet Integral Geometric (PFCIG)	Choquet Integral + WG		Pythagorean Fuzzy Numbers + Choquet Integral + WG + Crisp Weight Vector

(Kutlu Gundogdu & Kahraman, 2019)	2019	Spherical Weighted Arithmetic Mean (SWAM)	WA	Spherical Fuzzy Sets (SFSs)	Spherical Fuzzy Numbers + WA + Crisp Weight Vector
		Spherical Weighted Geometric Mean (SWAM)	WG		Spherical Fuzzy Numbers + WG + Crisp Weight Vector
(Kutlu Gündoğdu & Kahraman, 2019)	2019	Interval-valued Spherical Weighted Arithmetic Mean (IVSWAM)	WA	Interval Valued Spherical Fuzzy Sets (IVSFSs)	Interval Valued Spherical Fuzzy Numbers + WA + Crisp Weight Vector
		Interval-valued Spherical Weighted Geometric Mean (IVSWAM)	WG		Interval Valued Spherical Fuzzy Numbers + WG + Crisp Weight Vector
(Donyatalab et al., 2019)	2019	Spherical Fuzzy Number Algebraic Weighted Harmonic Mean (SFNAWHM)	Weighted Harmonic (WH)	Spherical Fuzzy Sets (SFSs)	Spherical Fuzzy Numbers + WH + Algebraic T-norm + Crisp Weight Vector
		Spherical Fuzzy Number Einstein Weighted Harmonic Mean (SFNEWHM)	Weighted Harmonic (WH)		Spherical Fuzzy Numbers + WH + Einstein T-norm + Crisp Weight Vector
		Spherical Fuzzy Number Algebraic Order Weighted Harmonic Mean (SFNAWHM)	OWH		Spherical Fuzzy Numbers + OWH + Algebraic T-norm + Crisp

					Weight Vector
		Spherical Fuzzy Number Einstein Order Weighted Harmon Mean (SFNEWHM)	OWH		Spherical Fuzzy Numbers + OWH + Einstein T-norm + Crisp Weight Vector
		Spherical Fuzzy Number Algebraic Hybrid Order Weighted Harmonic Mean (SFNAWHM)	HOWH		Spherical Fuzzy Numbers + HOWH + Algebraic T-norm + Crisp Weight Vector
		Spherical Fuzzy Number Einstein Hybrid Order Weighted Harmon Mean (SFNEWHM)	HOWH		Spherical Fuzzy Numbers + HOWH + Einstein T-norm + Crisp Weight Vector
	(Farrokhizadeh et al., 2020)	Spherical Fuzzy Bonferroni Mean (SFBM)	-	Spherical Fuzzy Sets (SFSs)	Spherical Fuzzy Numbers + Bonn + Equal Crisp Weight Vector
		Spherical Fuzzy Normalized Weighted Bonferroni Mean (SFNWBm)	-		Spherical Fuzzy Numbers + Weighted Bonn + Crisp Weight Vector
		Spherical Fuzzy Numbers Normalized Weighted Bonferroni Mean(SFNNWBm)	-		Spherical Fuzzy Numbers + Normalized Weighted Bonn + Crisp Weight Vector

Similarity measures and distances are extensively used to determine the association between two entities in many domains composed of decision-making, network comparison, pattern recognition, and medical engineering (Chiclana et al., 2013). A similarity measure designates how similar two objects are. With the development of fuzzy sets, classical similarity measures have been developed under several fuzzy environments (Zhang et al., 2014). Some new similarity measures in the framework of spherical fuzzy sets, including cosine similarity measures, grey similarity measures, and set-theoretic similarity measures, were proposed by (Ullah et al., 2018). These similarity measures were also proposed applied to a building material recognition problem. The novel cosine similarity measures under a spherical fuzzy environment by considering the idea behind SFSs were investigated by (Rafiq et al., 2019). Ten different similarity measures for spherical fuzzy sets (SFSs) based on the cosine and cotangent functions were defined (Wei & Wei, 2018a). They also developed the weighted similarity measures of cosine and cotangent functions. They applied these similarity functions to pattern recognition and medical diagnosis problem.

3. SPHERICAL FUZZY AGGREGATION OPERATORS

3.1 Preliminaries

Spherical fuzzy sets are one of the latest extensions of intuitionistic fuzzy sets. Spherical Fuzzy Sets (SFSs) give a larger preference domain to evaluators and each of them can also determine his/her hesitancy information under a spherical fuzzy environment. In this section, preliminaries of spherical fuzzy sets are given. In the following first the definition of Intuitionistic and Pythagorean fuzzy sets are introduced and then the spherical fuzzy sets (SFSs) are discussed in detail.

Definition 3.1 For any universal set X , an Intuitionistic Fuzzy Set (IFS) is defined as (K. Atanassov, 2015):

$$\tilde{I} = \{x, \mu(x), \vartheta(x) \mid x \in X\} \quad (3.1)$$

Hence, $\mu(x), \vartheta(x) : X \rightarrow [0,1]$ represents membership degree and non-membership degree, respectively, which satisfied $0 \leq \mu(x) + \vartheta(x) \leq 1$ and $h(x) = 1 - (\mu(x) + \vartheta(x))$ are known as the degree of the hesitancy of element x in IFS $\tilde{I}$. A set of $(\mu(x), \vartheta(x))$ is known as the Intuitionistic Fuzzy Number (IFN).

Definition 3.2 For any universal set X , a Pythagorean Fuzzy Set (PyFSs) is described as (Yager, 2014):

$$\widetilde{Py} = \{x, \mu(x), \vartheta(x) \mid \forall x \in X\} \quad (3.2)$$

Here, $\mu(x), \vartheta(x) : X \rightarrow [0,1]$ define membership degree and non-membership degree, respectively, which satisfied $0 \leq \mu(x)^2 + \vartheta(x)^2 \leq 1$ and $h(x) = \sqrt{1 - (\mu(x)^2 + \vartheta(x)^2)}$ are known as the degree of the hesitancy of element x in the PyFS $\widetilde{Py}$. A duplet of $(\mu(x), \vartheta(x))$ is known as Pythagorean Fuzzy Number (PyFN).

Definition 3.3 Let X be the universe set then the Picture of Fuzzy Set (PFS) is represented as (Cường, 2015):

$$\tilde{P} = \{x, \mu(x), I(x), \vartheta(x) \mid \forall x \in X\} \quad (3.3)$$

where $\mu(x), I(x), \vartheta(x) : U \rightarrow [0,1]$ define membership degree, Indeterminacy degree, and non-membership degree respectively, which satisfied $0 \leq \mu(x) + I(x) + \vartheta(x) \leq 1$. Then $1 - (\mu(x) + I(x) + \vartheta(x))$ is said to be refusal degree of element x in $\tilde{P}$.

Definition 3.4 A single-valued Spherical Fuzzy Set (SFSs) $\tilde{A}_S$ of the universe of discourse X is given by (Gündoğdu & Kahraman, 2019):

$$\tilde{A}_s = \{x, \mu_{\tilde{A}_s}(x), \vartheta_{\tilde{A}_s}(x), I_{\tilde{A}_s}(x) \mid x \in X\} \quad (3.4)$$

where $\mu_{\tilde{A}_s}(x), \vartheta_{\tilde{A}_s}(x), I_{\tilde{A}_s}(x): X \rightarrow [0,1]$ are the degree of membership, non-membership, and indeterminacy of x to $\tilde{A}_s$, respectively, and satisfies the condition that the sum of the square of these values is between 0 and 1. Then, the refusal degree $R_{\tilde{A}_s}$ of element x in the spherical fuzzy set $\tilde{A}_s$ will be as follows:

$$S_{\tilde{A}_s}(x) = \mu_{\tilde{A}_s}^2(x) + \vartheta_{\tilde{A}_s}^2(x) + I_{\tilde{A}_s}^2(x) \rightarrow 0 \leq S_{\tilde{A}_s}(x) \leq 1 \quad (3.5)$$

$$R_{\tilde{A}_s}(x) = \sqrt{1 - S_{\tilde{A}_s}(x)} \quad (3.6)$$

Let's assume that the membership degree is 0.4, the non-membership degree is 0.2, and the indeterminacy degree is 0.1. By using equation (3.3), the refusal degree is calculated as 0.888. Membership degree is defining the possibility to have such an event, non-membership is related to the possibility of not having the event and the third dimension is determining how much it is in doubt with an expert system to define membership and non-membership degrees. The fourth degree which is introduced in equation (3.3), shows the possibility to refuse all of the previous degrees about the event.

Definition 3.5 Assume that $\tilde{A}_s$ and $\tilde{B}_s$ are any two Spherical Fuzzy sets. So the basic operations of SFSs can be defined as follows (Gündoğdu & Kahraman, 2019):

$$\begin{aligned} &\tilde{A}_s \oplus \tilde{B}_s \\ &= \left\{ \sqrt{\mu_{\tilde{A}_s}^2 + \mu_{\tilde{B}_s}^2 - \mu_{\tilde{A}_s}^2 \mu_{\tilde{B}_s}^2}, \vartheta_{\tilde{A}_s} \vartheta_{\tilde{B}_s}, \sqrt{(1 - \mu_{\tilde{B}_s}^2) I_{\tilde{A}_s}^2 + (1 - \mu_{\tilde{A}_s}^2) I_{\tilde{B}_s}^2 - I_{\tilde{A}_s}^2 I_{\tilde{B}_s}^2} \right\} \end{aligned} \quad (3.7)$$

$$\begin{aligned} &\tilde{A}_s \otimes \tilde{B}_s \\ &= \left\{ \mu_{\tilde{A}_s} \mu_{\tilde{B}_s}, \sqrt{\vartheta_{\tilde{A}_s}^2 + \vartheta_{\tilde{B}_s}^2 - \vartheta_{\tilde{A}_s}^2 \vartheta_{\tilde{B}_s}^2}, \sqrt{(1 - \vartheta_{\tilde{B}_s}^2) I_{\tilde{A}_s}^2 + (1 - \vartheta_{\tilde{A}_s}^2) I_{\tilde{B}_s}^2 - I_{\tilde{A}_s}^2 I_{\tilde{B}_s}^2} \right\} \end{aligned} \quad (3.8)$$

$$k\tilde{A}_s = \left\{ \sqrt{1 - (1 - \mu_{\tilde{A}_s}^2)^k}, \vartheta_{\tilde{A}_s}^k, \sqrt{(1 - \mu_{\tilde{A}_s}^2)^k - (1 - \mu_{\tilde{A}_s}^2 - I_{\tilde{A}_s}^2)^k} \right\}; k \geq 0 \quad (3.9)$$

$$\tilde{A}_s^k = \left\{ \mu_{\tilde{A}_s}^k, \sqrt{1 - (1 - \vartheta_{\tilde{A}_s}^2)^k}, \sqrt{(1 - \vartheta_{\tilde{A}_s}^2)^k - (1 - \vartheta_{\tilde{A}_s}^2 - I_{\tilde{A}_s}^2)^k} \right\}; k \geq 0 \quad (3.10)$$

Definition 3.6 Spherical Fuzzy Weighted Arithmetic Mean (SFWAM) with respect to, $w = (w_1, w_2, \dots, w_n)$; $w_i \in [0,1]$; $\sum_{i=1}^n w_i = 1$, SFWAM is defined as (Gündoğdu & Kahraman, 2019):

$$\begin{aligned}
SFWM_w(\tilde{A}_{S1}, \tilde{A}_{S2}, \dots, \tilde{A}_{Sn}) &= w_1 \tilde{A}_{S1} + w_2 \tilde{A}_{S2} + \dots + w_n \tilde{A}_{Sn} \\
&= \left\{ \sqrt{1 - \prod_{i=1}^n (1 - \mu_{A_s}^2)^{w_i}}, \prod_{i=1}^n \vartheta_{A_s}^{w_i}, \sqrt{\prod_{i=1}^n (1 - \mu_{A_s}^2)^{w_i} - \prod_{i=1}^n (1 - \mu_{A_s}^2 - I_{A_s}^2)^{w_i}} \right\} \quad (3.11)
\end{aligned}$$

Definition 3.7 Spherical Fuzzy Weighted Geometric Mean (SFWGM) with respect to, $w = (w_1, w_2, \dots, w_n)$; $w_i \in [0,1]$; $\sum_{i=1}^n w_i = 1$, SFWGM is defined as (Gündoğdu & Kahraman, 2019):

$$\begin{aligned}
SFWGM_w(\tilde{A}_{S1}, \tilde{A}_{S2}, \dots, \tilde{A}_{Sn}) &= \tilde{A}_{S1}^{w_1} + \tilde{A}_{S2}^{w_2} + \dots + \tilde{A}_{Sn}^{w_n} \\
&= \left\{ \prod_{i=1}^n \mu_{A_s}^{w_i}, \sqrt{1 - \prod_{i=1}^n (1 - \vartheta_{A_s}^2)^{w_i}}, \sqrt{\prod_{i=1}^n (1 - \vartheta_{A_s}^2)^{w_i} - \prod_{i=1}^n (1 - \vartheta_{A_s}^2 - I_{A_s}^2)^{w_i}} \right\} \quad (3.12)
\end{aligned}$$

The rest of this section will discuss more familiarized ideas, which are utilized in the following analysis. So, it will review some very basic definitions and notations of the concepts including definitions of aggregation operations, Harmonic Mean (HM) aggregation operator, and defining of SFSs operations based on t-norms and t-conorms.

3.1.1 Aggregation operators

Aggregation Operation, a mapping function to direct an n-dimension array to a 1-dimension array while through the process of aggregation the final results could represent the features of the n-dimensional array.

Definition 3.8 An aggregation function is a function of $n > 1$ argument $f: [0,1]^n \rightarrow [0,1]$ with the properties (Ashraf & Abdullah, 2019):

- Boundary conditions:

Based on this property it is concentrated on the behavior of the aggregator in the best and the worst cases. It is expected that an aggregation operator satisfies:

$$f_{Agg}(0, 0, \dots, 0) = 0 \quad (3.13)$$

$$f_{Agg}(1, 1, \dots, 1) = 1 \quad (3.14)$$

Condition (3.13), $f_{Agg}(0, 0, \dots, 0) = 0$ means that if it is observed only completely bad, false, or not satisfactory criteria the total aggregation amount has to be also completely bad, false, or not satisfactory.

The second in equation (3.14) $f_{Agg}(1, 1, \dots, 1) = 1$ translates that if it is observed only true or completely satisfactory values then the total aggregation amount has to be also completely true or satisfactory.

- Monotonicity condition:

If an argument increases then the final aggregation increases or at least does not decrease and remains equal, so:

$$x_i \geq y_i \rightarrow Agg(x_1, x_2, \dots, x_n) \geq Agg(y_1, y_2, \dots, y_n) \quad (3.15)$$

The two above-mentioned properties are two fundamentals but there are some other properties like idempotency, commutativity, associativity, and so on.

3.1.2 Harmonic mean aggregation operators

Definition 3.9 Suppose that $a_i = \{a_1, a_2, \dots, a_n\}$ be a collection of real numbers, so the harmonic mean aggregation operator could be defined as follows (Sun & Sun, 2012):

$$HM(a_1, a_2, \dots, a_n) = \frac{n}{\sum_{i=1}^n \frac{1}{a_i}} \quad (3.16)$$

Definition 3.10 Suppose that $a_i = \{a_1, a_2, \dots, a_n\}$ be a collection of real numbers and $w_i = (w_1, w_2, \dots, w_n)^T$ be a weight vector where $w_i \in [0,1]$; $\sum_{i=1}^n w_i = 1$. So the weighted harmonic mean aggregation operator could be defined as follows (Sun & Sun, 2012):

$$WHM(a_1, a_2, \dots, a_n) = \left(\sum_{i=1}^n \frac{w_i}{a_i} \right)^{-1} = \left(\sum_{i=1}^n w_i * (a_i)^{-1} \right)^{-1} \quad (3.17)$$

3.1.3 Triangular norm and conorm

The triangular norm was introduced by Menger (Menger, 1942) to generalize the triangular inequality of a metric. The current notion of a t-norm and its dual operation (t-conorm) which is developed by Schweizer and Sklar (Schweizer & Sklar, 1960). Both are functional as a natural interpretation of the conjunction and disjunction in the semantics of mathematical fuzzy logic and are applied by many researchers to incorporate criteria in multi-criteria decision-making problems (Höhle, 1978). The t-

norms and t-conorm generalize the conjunctive “AND” and disjunctive “OR” operators, respectively. It explains their capability to define the intersection and union operations in the fuzzy logic area which was first noted by Hohle (Höhle, 1978).

Definition 3.11 Let $UI = [0, 1]$ be a unit interval. Then the mapping $T: UI \times UI \rightarrow UI$ is said to be t-norm if $\forall a, b, c, d \in UI$ satisfied following 4 axioms (Cuong et al., 2016):

- Commutativity $\leftrightarrow T(a, b) = T(b, a)$
- Monotonicity $\leftrightarrow T(a, b) \leq T(c, d)$ if $a \leq c$ & $b \leq d$
- Associativity $\leftrightarrow T(a, T(b, c)) = T(T(a, b), c)$
- One as identity $\leftrightarrow T(a, 1) = a$

Definition 3.12 Let $UI = [0, 1]$ be a unit interval. Then the mapping $S: UI \times UI \rightarrow UI$ is said to be t-conorm if $\forall a, b, c, d \in UI$ satisfied following 4 axioms (Cuong et al., 2016):

- Commutativity $\leftrightarrow S(a, b) = S(b, a)$
- Monotonicity $\leftrightarrow S(a, b) \leq S(c, d)$ if $a \leq c$ & $b \leq d$
- Associativity $\leftrightarrow S(a, S(b, c)) = S(S(a, b), c)$
- Zero as identity $\leftrightarrow S(a, 0) = a$

Archimedean components of triangular norms are a convex subset of the unit interval, including element x if and only if each n^{th} power of x is also an element of it.

Definition 3.13 A t-norm function $T(a, b)$ is called Archimedean t-norm if it is (Ashraf & Abdullah, 2019):

- Continuous.
- $\forall a \in (0, 1): T(a, a) < a$.

And Archimedean t-norm is called strictly Archimedean t-norm if it is strictly increasing in each variable $\forall a, b \in [0, 1]$.

Definition 3.14 A t-conorm function $S(a, b)$ is called Archimedean t-conorm if it is (Ashraf & Abdullah, 2019):

- Continuous.
- $\forall a \in (0, 1): S(a, a) > a$.

And Archimedean t-conorm is called strictly Archimedean t-conorm if it is strictly increasing in each variable $\forall a, b \in [0, 1]$.

It is well known (Rahman et al., 2017) that a strict Archimedean t-norm and t-conorm is expressed via its additive generator k as: $T(a, b) = k^{-1}(k(a) + k(b))$, and similarly, applied to its dual, for t-conorm: $S(a, b) = l^{-1}(l(a) + l(b))$, with $l(a) = k(1 - a)$.

To develop Harmonic Mean Aggregations for Spherical Fuzzy Sets, applied the additive generator of Algebraic Strict Archimedean t-norms and t-conorms and their inverse functions are respectively represented as:

$$k(x_i) = -\log x_i \quad (3.18)$$

$$l(x_i) = -\log(1 - x_i) \quad (3.19)$$

$$k^{-1}(x_i) = \frac{1}{10^{x_i}} \quad (3.20)$$

$$l^{-1}(x_i) = 1 - \frac{1}{10^{x_i}} \quad (3.21)$$

Similarly, it is use the additive generator of Einstein Strict Archimedean t-norms and t-conorms, and their inverse functions are respectively represented as:

$$k(x_i) = \log \frac{2 - x_i}{x_i} \quad (3.22)$$

$$l(x_i) = \log \frac{1 + x_i}{1 - x_i} \quad (3.23)$$

$$k^{-1}(x_i) = \frac{2}{1 + 10^{x_i}} \quad (3.24)$$

$$l^{-1}(x_i) = \frac{10^{x_i} - 1}{10^{x_i} + 1} \quad (3.25)$$

Definition 3.15 Propose that $\tilde{A}_s$ and $\tilde{B}_s$ be any two Spherical Fuzzy Sets (SFSs). So the particular operations of SFNs based on strict Archimedean triangular norm and conorm can be illustrated as follow:

$$\begin{aligned} \tilde{A}_s \oplus \tilde{B}_s = & \left\{ \sqrt{s^{-1} \left(s \left(\mu_{\tilde{A}_s}^2(u) \right) + s \left(\mu_{\tilde{B}_s}^2(u) \right) \right)}, t^{-1} \left(t \left(I_{\tilde{A}_s}(u) \right) \right. \right. \\ & \left. \left. + t \left(I_{\tilde{B}_s}(u) \right) \right), t^{-1} \left(t \left(\vartheta_{\tilde{A}_s}(u) \right) + t \left(\vartheta_{\tilde{B}_s}(u) \right) \right) \right\} \end{aligned} \quad (3.26)$$

$$k\tilde{A}_s = \left\{ \sqrt{s^{-1} \left(ks \left(\mu_{\tilde{A}_s}^2(u) \right) \right)}, t^{-1} \left(kt \left(I_{\tilde{A}_s}(u) \right) \right), t^{-1} \left(kt \left(\vartheta_{\tilde{A}_s}(u) \right) \right) \right\} \quad (3.27)$$

$$\begin{aligned} \tilde{A}_s \otimes \tilde{B}_s = & \left\{ t^{-1} \left(t \left(\mu_{\tilde{A}_s}(u) \right) + t \left(\mu_{\tilde{B}_s}(u) \right) \right), t^{-1} \left(t \left(I_{\tilde{A}_s}(u) \right) \right. \right. \\ & \left. \left. + t \left(I_{\tilde{B}_s}(u) \right) \right), \sqrt{s^{-1} \left(s \left(\vartheta_{\tilde{A}_s}^2(u) \right) + s \left(\vartheta_{\tilde{B}_s}^2(u) \right) \right)} \right\} \end{aligned} \quad (3.28)$$

$$\tilde{A}_s^k = \left\{ t^{-1} \left(kt \left(\mu_{\tilde{A}_s}(u) \right) \right), t^{-1} \left(kt \left(I_{\tilde{A}_s}(u) \right) \right), \sqrt{s^{-1} \left(ks \left(\vartheta_{\tilde{A}_s}^2(u) \right) \right)} \right\} \quad (3.29)$$

$$\begin{aligned} \tilde{A}_s^{-k} &= \frac{1}{\tilde{A}_s^k} \\ &= \left\{ t^{-1} \left(kt \left(\vartheta_{\tilde{A}_s}(u) \right) \right), t^{-1} \left(kt \left(I_{\tilde{A}_s}(u) \right) \right), \sqrt{s^{-1} \left(ks \left(\mu_{\tilde{A}_s}^2(u) \right) \right)} \right\} \end{aligned} \quad (3.30)$$

If it is allocated $k(x_i) = -\log x_i$ and $l(x_i) = -\log(1 - x_i)$ to generate the t-norm and t-conorm, respectively. So the above-defined operations will be the same as follows (Ullah et al., 2018):

$$\begin{aligned} \tilde{A}_s \oplus \tilde{B}_s = & \left\{ \sqrt{\mu_{\tilde{A}_s}^2(u) + \mu_{\tilde{B}_s}^2(u) - \mu_{\tilde{A}_s}^2(u) \cdot \mu_{\tilde{B}_s}^2(u)}, I_{\tilde{A}_s}(u) \cdot I_{\tilde{B}_s}(u), \right. \\ & \left. \vartheta_{\tilde{A}_s}(u) \cdot \vartheta_{\tilde{B}_s}(u) \right\} \end{aligned} \quad (3.31)$$

$$k\tilde{A}_s = \left\{ \sqrt{1 - (1 - \mu_{\tilde{A}_s}^2(u))^k}, \left(I_{\tilde{A}_s}(u) \right)^k, \left(\vartheta_{\tilde{A}_s}(u) \right)^k \right\} \quad (3.32)$$

$$\begin{aligned} \tilde{A}_s \otimes \tilde{B}_s \\ = & \left\{ \mu_{\tilde{A}_s}(u) \cdot \mu_{\tilde{B}_s}(u), I_{\tilde{A}_s}(u) \cdot I_{\tilde{B}_s}(u), \sqrt{\vartheta_{\tilde{A}_s}^2(u) + \vartheta_{\tilde{B}_s}^2(u) - \vartheta_{\tilde{A}_s}^2(u) \cdot \vartheta_{\tilde{B}_s}^2(u)} \right\} \end{aligned} \quad (3.33)$$

$$\tilde{A}_s^k = \left\{ \left(\mu_{\tilde{A}_s}(u) \right)^k, \left(I_{\tilde{A}_s}(u) \right)^k, \sqrt{1 - (1 - \vartheta_{\tilde{A}_s}^2(u))^k} \right\} \quad (3.34)$$

Definition 3.16 Suppose that $\tilde{A}_s$ and $\tilde{B}_s$ be two Spherical Fuzzy sets. Then to compare these SFSs Ashraf et al. proposed the score function (SC), accuracy function (AC), and comparison rules which are defined as follows:

$$SC(\tilde{A}_s) = \frac{1}{3} \left(2 + \mu_{\tilde{A}_s}(u) - I_{\tilde{A}_s}(u) - \vartheta_{\tilde{A}_s}(u) \right) \quad (3.35)$$

$$AC(\tilde{A}_s) = \left(\mu_{\tilde{A}_s}(u) - \vartheta_{\tilde{A}_s}(u) \right) \quad (3.36)$$

If $SC(\tilde{A}_s) > SC(\tilde{B}_s)$ then $\tilde{A}_s > \tilde{B}_s$

If $SC(\tilde{A}_s) = SC(\tilde{B}_s)$ and $AC(\tilde{A}_s) > AC(\tilde{B}_s)$ then $\tilde{A}_s > \tilde{B}_s$

3.2 Spherical Fuzzy Harmonic Mean Aggregation Operators

This section presents some discussion about Harmonic Mean Aggregation Operators in Spherical Fuzzy Sets. The proposed aggregations are organized in three subsections: weighted, order weighted and hybrid order weighted Harmonic Mean operators and used two types of Algebraic and Einstein Strict Archimedean t-norms and t-conorms. So, 6 harmonic mean aggregation operators are introduced in the Spherical Fuzzy environment. In the following subsections, our proposed operators are defined and proved and discussed properties in detail.

3.2.1 Spherical fuzzy number weighted harmonic mean aggregation operator

Definition 3.17 If $\tilde{A}_{si} = \{u, \langle \mu_{\tilde{A}_{si}}(u), I_{\tilde{A}_{si}}(u), \vartheta_{\tilde{A}_{si}}(u) \rangle \mid u \in U\}$ be Spherical Fuzzy Numbers and weight vector $w_i = \{w_1, w_2, \dots, w_i\}$ while $\sum_{i=1}^n w_i = 1, w_i \geq 0$ and $i = 1, \dots, n$. So the Algebraic Weighted Harmonic Mean Aggregation Operators to aggregate these SFNs are described as follows:

$$SFNAWHM(\tilde{A}_{s1}, \tilde{A}_{s2}, \dots, \tilde{A}_{sn}) = \left[\prod_{i=1}^n ((\mu_{\tilde{A}_{si}})^{w_i}), \prod_{i=1}^n ((I_{\tilde{A}_{si}})^{w_i}), \sqrt{1 - \prod_{i=1}^n ((1 - \vartheta_{\tilde{A}_{si}}^2)^{w_i})} \right] \quad (3.37)$$

Proof: According to Definition 3.17, will prove the Eq. (3.37) as follows:

$$SFNAWHM(\tilde{A}_{s1}, \tilde{A}_{s2}, \dots, \tilde{A}_{sn}) = \left(\sum_{i=1}^n w_i * (\tilde{A}_{si})^{-1} \right)^{-1}$$

Where by using Eq. (3.30), have:

$$(\tilde{A}_{si})^{-1} = \langle \vartheta_{\tilde{A}_{si}}, I_{\tilde{A}_{si}}, \mu_{\tilde{A}_{si}} \rangle$$

For $\left(\sum_{i=1}^n w_i * (\tilde{A}_{si})^{-1} \right)^{-1}$ using Definition 3.15 will have:

Membership degree:

$$\begin{aligned} \vartheta_{\tilde{A}_{si}}^2 \xrightarrow{\wedge^2} (\vartheta_{\tilde{A}_{si}})^2 &= \vartheta_{\tilde{A}_{si}}^2 \xrightarrow{S_A} -\log(1 - \vartheta_{\tilde{A}_{si}}^2) \xrightarrow{\sum_{i=1}^n w_i} -\sum_{i=1}^n w_i \cdot \log(1 - \vartheta_{\tilde{A}_{si}}^2) = \\ &= -\sum_{i=1}^n \log(1 - \vartheta_{\tilde{A}_{si}}^2)^{w_i} = -\log \prod_{i=1}^n ((1 - \vartheta_{\tilde{A}_{si}}^2)^{w_i}) \xrightarrow{S_A^{-1}} 1 - \frac{1}{10^{(-\log \prod_{i=1}^n ((1 - \vartheta_{\tilde{A}_{si}}^2)^{w_i}))}} = \end{aligned}$$

$$1 - (\prod_{i=1}^n ((1 - \vartheta_{\tilde{A}si}^2)^{w_i})) \xrightarrow{\sqrt{X}} \sqrt{1 - (\prod_{i=1}^n ((1 - \vartheta_{\tilde{A}si}^2)^{w_i}))} \xrightarrow{\wedge(-1)} \prod_{i=1}^n ((\mu_{\tilde{A}si})^{w_i})$$

$$\xrightarrow{t_A} -\log(\prod_{i=1}^n ((\mu_{\tilde{A}si})^{w_i})) \xrightarrow{* (1)} -\log(\prod_{i=1}^n ((\mu_{\tilde{A}si})^{w_i})) \xrightarrow{t_A^{-1}} \frac{1}{10^{-\log(\prod_{i=1}^n ((\mu_{\tilde{A}si})^{w_i}))}} =$$

$$\prod_{i=1}^n ((\mu_{\tilde{A}si})^{w_i})$$

Indeterminacy degree:

$$I_{\tilde{A}si} \xrightarrow{t_A} -\log(I_{\tilde{A}si}) \xrightarrow{\sum_{i=1}^n w_i} -\sum_{i=1}^n w_i \cdot \log(I_{\tilde{A}si}) = -\sum_{i=1}^n \log(I_{\tilde{A}si})^{w_i} =$$

$$-\log \prod_{i=1}^n ((I_{\tilde{A}si})^{w_i}) \xrightarrow{t_A^{-1}} \frac{1}{10^{(-\log \prod_{i=1}^n ((I_{\tilde{A}si})^{w_i}))}} = \prod_{i=1}^n ((I_{\tilde{A}si})^{w_i})$$

$$\xrightarrow{t_A} -\log \prod_{i=1}^n ((I_{\tilde{A}si})^{w_i}) \xrightarrow{* (1)} -\log \prod_{i=1}^n ((I_{\tilde{A}si})^{w_i}) \xrightarrow{t_A^{-1}} \frac{1}{10^{-\log \prod_{i=1}^n ((I_{\tilde{A}si})^{w_i})}} =$$

$$\prod_{i=1}^n ((I_{\tilde{A}si})^{w_i})$$

Nonmembership degree:

$$\mu_{\tilde{A}si} \xrightarrow{t_A} -\log(\mu_{\tilde{A}si}) \xrightarrow{\sum_{i=1}^n w_i} -\sum_{i=1}^n w_i \cdot \log(\mu_{\tilde{A}si}) = -\sum_{i=1}^n \log(\mu_{\tilde{A}si})^{w_i} =$$

$$-\log(\prod_{i=1}^n ((\mu_{\tilde{A}si})^{w_i})) \xrightarrow{t_A^{-1}} \frac{1}{10^{(-\log(\prod_{i=1}^n ((\mu_{\tilde{A}si})^{w_i}))}} =$$

$$\prod_{i=1}^n ((\mu_{\tilde{A}si})^{w_i}) \xrightarrow{\wedge(-1)} \sqrt{1 - (\prod_{i=1}^n ((1 - \vartheta_{\tilde{A}si}^2)^{w_i}))}$$

$$\xrightarrow{\wedge^2} \left(\sqrt{1 - (\prod_{i=1}^n ((1 - \vartheta_{\tilde{A}si}^2)^{w_i}))} \right)^2 = 1 - (\prod_{i=1}^n ((1 - \vartheta_{\tilde{A}si}^2)^{w_i})) \xrightarrow{s_A} -\log(1 - 1 +$$

$$(\prod_{i=1}^n ((1 - \vartheta_{\tilde{A}si}^2)^{w_i})) = -\log((\prod_{i=1}^n ((1 - \vartheta_{\tilde{A}si}^2)^{w_i})) \xrightarrow{* (1)} -\log((\prod_{i=1}^n ((1 -$$

$$\vartheta_{\tilde{A}si}^2)^{w_i})) \xrightarrow{s_A^{-1}} 1 - \frac{1}{10^{(-\log((\prod_{i=1}^n ((1 - \vartheta_{\tilde{A}si}^2)^{w_i}))}} = 1 - \prod_{i=1}^n ((1 - \vartheta_{\tilde{A}si}^2)^{w_i})$$

$$\xrightarrow{\sqrt{X}} \sqrt{1 - \prod_{i=1}^n ((1 - \vartheta_{\tilde{A}si}^2)^{w_i})}$$

Where t_A and s_A are respectively Additive Generator of Algebraic Strict Archimedean t-norm and t-conorm and t_A^{-1} and s_A^{-1} are respectively their inverse are defined in Eqs. (3.18)-(3.21).

■

As it is mentioned in Definition 3.17 for the proposed harmonic mean aggregation operators, could demonstrate that the conditions in Definition 3.8 are true as follows:

- i. Boundary Condition: It is obvious this condition refers to the fact that if it is observed only completely bad, false, or not satisfactory criteria the total aggregation has to be also completely bad, false, or not satisfactory and when it is observed only true or completely satisfactory criteria then the total aggregation has to be also completely true or satisfactory.

$$SFNAWHM(0, 0, \dots, 0) = 0$$

$$SFNAWHM(1, 1, \dots, 1) = 1$$

- ii. Monotonicity condition: If $\tilde{A}_{si} \subset \tilde{B}_{si} \rightarrow SFNAWHM(\tilde{A}_{s1}, \tilde{A}_{s2}, \dots, \tilde{A}_{sn}) \leq SFNAWHM(\tilde{B}_{s1}, \tilde{B}_{s2}, \dots, \tilde{B}_{sn})$.

Definition 3.18 If $\tilde{A}_{si} = \{u, \mu_{\tilde{A}_{si}}(u), I_{\tilde{A}_{si}}(u), \vartheta_{\tilde{A}_{si}}(u) \mid u \in U\}$ be Spherical Fuzzy Numbers and weight vector $w_i = \{w_1, w_2, \dots, w_i\}$ while $\sum_{i=1}^n w_i = 1$, $w_i \geq 0$ and $i = 1, \dots, n$. So the Einstein Weighted Harmonic Mean Aggregation Operators to aggregate these SFNs are described as follows:

$$SFNEWHM(\tilde{A}_{s1}, \tilde{A}_{s2}, \dots, \tilde{A}_{sn}) = \left[\frac{2}{1 + \prod_{i=1}^n \left(\left(\frac{2 - \mu_{\tilde{A}_{si}}}{\mu_{\tilde{A}_{si}}} \right)^{w_i} \right)}, \frac{2}{1 + \prod_{i=1}^n \left(\left(\frac{2 - I_{\tilde{A}_{si}}}{I_{\tilde{A}_{si}}} \right)^{w_i} \right)}, \frac{\left(\prod_{i=1}^n \left(\left(\frac{1 + \vartheta_{\tilde{A}_{si}}^2}{1 - \vartheta_{\tilde{A}_{si}}^2} \right)^{w_i} \right) \right)^{-1}}{\left(\prod_{i=1}^n \left(\left(\frac{1 + \vartheta_{\tilde{A}_{si}}^2}{1 - \vartheta_{\tilde{A}_{si}}^2} \right)^{w_i} \right) \right)^{+1}} \right] \quad (3.38)$$

Proof: According to Definition 3.18, will prove the Eq. (3.38) as follows:

$$SFNEWHM(\tilde{A}_{s1}, \tilde{A}_{s2}, \dots, \tilde{A}_{sn}) = \left(\sum_{i=1}^n w_i * (\tilde{A}_{si})^{-1} \right)^{-1}$$

By using Eq. (3.30):

$$(\tilde{A}_{si})^{-1} = \langle \vartheta_{\tilde{A}_{si}}, I_{\tilde{A}_{si}}, \mu_{\tilde{A}_{si}} \rangle$$

Then $\left(\sum_{i=1}^n w_i * (\tilde{A}_{si})^{-1} \right)^{-1}$ by using Definition 3.15:

Membership degree:

$$\begin{aligned}
\vartheta_{\tilde{A}_{Si}} &\xrightarrow{\wedge^2} (\vartheta_{\tilde{A}_{Si}})^2 \xrightarrow{S_E} \log \frac{1+\vartheta_{\tilde{A}_{Si}}^2}{1-\vartheta_{\tilde{A}_{Si}}^2} \xrightarrow{\sum_{i=1}^n w_i} \sum_{i=1}^n w_i \cdot \log \frac{1+\vartheta_{\tilde{A}_{Si}}^2}{1-\vartheta_{\tilde{A}_{Si}}^2} = \sum_{i=1}^n \log \left(\frac{1+\vartheta_{\tilde{A}_{Si}}^2}{1-\vartheta_{\tilde{A}_{Si}}^2} \right)^{w_i} = \\
&\log \prod_{i=1}^n \left(\left(\frac{1+\vartheta_{\tilde{A}_{Si}}^2}{1-\vartheta_{\tilde{A}_{Si}}^2} \right)^{w_i} \right) \xrightarrow{S_E^{-1}} \frac{\log \prod_{i=1}^n \left(\left(\frac{1+\vartheta_{\tilde{A}_{Si}}^2}{1-\vartheta_{\tilde{A}_{Si}}^2} \right)^{w_i} \right)^{-1}}{\log \prod_{i=1}^n \left(\left(\frac{1+\vartheta_{\tilde{A}_{Si}}^2}{1-\vartheta_{\tilde{A}_{Si}}^2} \right)^{w_i} \right)^{+1}} = \frac{\left(\prod_{i=1}^n \left(\left(\frac{1+\vartheta_{\tilde{A}_{Si}}^2}{1-\vartheta_{\tilde{A}_{Si}}^2} \right)^{w_i} \right) \right)^{-1}}{\left(\prod_{i=1}^n \left(\left(\frac{1+\vartheta_{\tilde{A}_{Si}}^2}{1-\vartheta_{\tilde{A}_{Si}}^2} \right)^{w_i} \right) \right)^{+1}} \\
&\xrightarrow{\sqrt{X}} \frac{\left(\prod_{i=1}^n \left(\left(\frac{1+\vartheta_{\tilde{A}_{Si}}^2}{1-\vartheta_{\tilde{A}_{Si}}^2} \right)^{w_i} \right) \right)^{-1}}{\left(\prod_{i=1}^n \left(\left(\frac{1+\vartheta_{\tilde{A}_{Si}}^2}{1-\vartheta_{\tilde{A}_{Si}}^2} \right)^{w_i} \right) \right)^{+1}} \xrightarrow{\wedge(-1)} \frac{2}{1+\prod_{i=1}^n \left(\left(\frac{2-\mu_{\tilde{A}_{Si}}}{\mu_{\tilde{A}_{Si}}} \right)^{w_i} \right)} \xrightarrow{t_E} \log \frac{2 - \frac{2}{1+\prod_{i=1}^n \left(\left(\frac{2-\mu_{\tilde{A}_{Si}}}{\mu_{\tilde{A}_{Si}}} \right)^{w_i} \right)}}{\frac{2}{1+\prod_{i=1}^n \left(\left(\frac{2-\mu_{\tilde{A}_{Si}}}{\mu_{\tilde{A}_{Si}}} \right)^{w_i} \right)}} \\
&\xrightarrow{t_E^{-1}} \frac{2}{\frac{2 - \frac{2}{1+\prod_{i=1}^n \left(\left(\frac{2-\mu_{\tilde{A}_{Si}}}{\mu_{\tilde{A}_{Si}}} \right)^{w_i} \right)}}{\log \frac{2}{1+\prod_{i=1}^n \left(\left(\frac{2-\mu_{\tilde{A}_{Si}}}{\mu_{\tilde{A}_{Si}}} \right)^{w_i} \right)}}} = \frac{2}{\frac{2 - \frac{2}{1+\prod_{i=1}^n \left(\left(\frac{2-\mu_{\tilde{A}_{Si}}}{\mu_{\tilde{A}_{Si}}} \right)^{w_i} \right)}}{1 + \frac{2}{1+\prod_{i=1}^n \left(\left(\frac{2-\mu_{\tilde{A}_{Si}}}{\mu_{\tilde{A}_{Si}}} \right)^{w_i} \right)}}} = \frac{2}{1+\prod_{i=1}^n \left(\left(\frac{2-\mu_{\tilde{A}_{Si}}}{\mu_{\tilde{A}_{Si}}} \right)^{w_i} \right)}
\end{aligned}$$

Indeterminacy degree:

$$\begin{aligned}
I_{\tilde{A}_{Si}} &\xrightarrow{t_E} \log \frac{2-I_{\tilde{A}_{Si}}}{I_{\tilde{A}_{Si}}} \xrightarrow{\sum_{i=1}^n w_k} \sum_{i=1}^n w_k \cdot \log \frac{2-I_{\tilde{A}_{Si}}}{I_{\tilde{A}_{Si}}} = \sum_{i=1}^n \log \left(\frac{2-I_{\tilde{A}_{Si}}}{I_{\tilde{A}_{Si}}} \right)^{w_i} = \\
&\log \prod_{i=1}^n \left(\left(\frac{2-I_{\tilde{A}_{Si}}}{I_{\tilde{A}_{Si}}} \right)^{w_i} \right) \xrightarrow{t_E^{-1}} \frac{2}{\log \prod_{i=1}^n \left(\left(\frac{2-I_{\tilde{A}_{Si}}}{I_{\tilde{A}_{Si}}} \right)^{w_i} \right)} = \frac{2}{1+\prod_{i=1}^n \left(\left(\frac{2-I_{\tilde{A}_{Si}}}{I_{\tilde{A}_{Si}}} \right)^{w_i} \right)} \\
&\xrightarrow{t_E} \log \frac{2 - \frac{2}{1+\prod_{i=1}^n \left(\left(\frac{2-I_{\tilde{A}_{Si}}}{I_{\tilde{A}_{Si}}} \right)^{w_i} \right)}}{\frac{2}{1+\prod_{i=1}^n \left(\left(\frac{2-I_{\tilde{A}_{Si}}}{I_{\tilde{A}_{Si}}} \right)^{w_i} \right)}} \xrightarrow{t_E^{-1}} \frac{2}{\frac{2 - \frac{2}{1+\prod_{i=1}^n \left(\left(\frac{2-I_{\tilde{A}_{Si}}}{I_{\tilde{A}_{Si}}} \right)^{w_i} \right)}}{\log \frac{2}{1+\prod_{i=1}^n \left(\left(\frac{2-I_{\tilde{A}_{Si}}}{I_{\tilde{A}_{Si}}} \right)^{w_i} \right)}}} = \frac{2}{\frac{2 - \frac{2}{1+\prod_{i=1}^n \left(\left(\frac{2-I_{\tilde{A}_{Si}}}{I_{\tilde{A}_{Si}}} \right)^{w_i} \right)}}{1 + \frac{2}{1+\prod_{i=1}^n \left(\left(\frac{2-I_{\tilde{A}_{Si}}}{I_{\tilde{A}_{Si}}} \right)^{w_i} \right)}}} = \frac{2}{1+\prod_{i=1}^n \left(\left(\frac{2-I_{\tilde{A}_{Si}}}{I_{\tilde{A}_{Si}}} \right)^{w_i} \right)} \\
&\xrightarrow{1+10} \frac{\frac{2}{1+\prod_{i=1}^n \left(\left(\frac{2-I_{\tilde{A}_{Si}}}{I_{\tilde{A}_{Si}}} \right)^{w_i} \right)} + \frac{2}{1+\prod_{i=1}^n \left(\left(\frac{2-I_{\tilde{A}_{Si}}}{I_{\tilde{A}_{Si}}} \right)^{w_i} \right)}}{\frac{2}{1+\prod_{i=1}^n \left(\left(\frac{2-I_{\tilde{A}_{Si}}}{I_{\tilde{A}_{Si}}} \right)^{w_i} \right)}} = \frac{\frac{2}{1+\prod_{i=1}^n \left(\left(\frac{2-I_{\tilde{A}_{Si}}}{I_{\tilde{A}_{Si}}} \right)^{w_i} \right)}}{\left(\frac{2}{1+\prod_{i=1}^n \left(\left(\frac{2-I_{\tilde{A}_{Si}}}{I_{\tilde{A}_{Si}}} \right)^{w_i} \right)} \right)} = \frac{2}{1+\prod_{i=1}^n \left(\left(\frac{2-I_{\tilde{A}_{Si}}}{I_{\tilde{A}_{Si}}} \right)^{w_i} \right)}
\end{aligned}$$

Nonmembership degree:

$$\begin{aligned}
& \mu_{\tilde{A}_{Si}} \xrightarrow{t_E} \log \frac{2 - \mu_{\tilde{A}_{Si}}}{\mu_{\tilde{A}_{Si}}} \xrightarrow{\sum_{i=1}^n w_i} \sum_{i=1}^n w_i \cdot \log \frac{2 - \mu_{\tilde{A}_{Si}}}{\mu_{\tilde{A}_{Si}}} = \sum_{i=1}^n \log \left(\frac{2 - \mu_{\tilde{A}_{Si}}}{\mu_{\tilde{A}_{Si}}} \right)^{w_i} = \\
& \log \prod_{i=1}^n \left(\left(\frac{2 - \mu_{\tilde{A}_{Si}}}{\mu_{\tilde{A}_{Si}}} \right)^{w_i} \right) \xrightarrow{t_A^{-1}} \frac{2}{\log \prod_{i=1}^n \left(\left(\frac{2 - \mu_{\tilde{A}_{Si}}}{\mu_{\tilde{A}_{Si}}} \right)^{w_i} \right)} = \frac{2}{1 + \prod_{i=1}^n \left(\left(\frac{2 - \mu_{\tilde{A}_{Si}}}{\mu_{\tilde{A}_{Si}}} \right)^{w_i} \right)} \\
& \xrightarrow{\wedge(-1)} \sqrt{\frac{\left(\prod_{i=1}^n \left(\left(\frac{1 + \vartheta_{\tilde{A}_{Si}}^2}{1 - \vartheta_{\tilde{A}_{Si}}^2} \right)^{w_i} \right) \right)^{-1}}{\left(\prod_{i=1}^n \left(\left(\frac{1 + \vartheta_{\tilde{A}_{Si}}^2}{1 - \vartheta_{\tilde{A}_{Si}}^2} \right)^{w_i} \right) \right)^{+1}}} \xrightarrow{S_E} \log \frac{1 + \left(\frac{\left(\prod_{i=1}^n \left(\left(\frac{1 + \vartheta_{\tilde{A}_{Si}}^2}{1 - \vartheta_{\tilde{A}_{Si}}^2} \right)^{w_i} \right) \right)^{-1}}{\left(\prod_{i=1}^n \left(\left(\frac{1 + \vartheta_{\tilde{A}_{Si}}^2}{1 - \vartheta_{\tilde{A}_{Si}}^2} \right)^{w_i} \right) \right)^{+1}} \right)^2}{1 - \left(\frac{\left(\prod_{i=1}^n \left(\left(\frac{1 + \vartheta_{\tilde{A}_{Si}}^2}{1 - \vartheta_{\tilde{A}_{Si}}^2} \right)^{w_i} \right) \right)^{-1}}{\left(\prod_{i=1}^n \left(\left(\frac{1 + \vartheta_{\tilde{A}_{Si}}^2}{1 - \vartheta_{\tilde{A}_{Si}}^2} \right)^{w_i} \right) \right)^{+1}} \right)^2} \xrightarrow{S_E^{-1}} \frac{10}{\log \frac{1 + \left(\frac{\left(\prod_{i=1}^n \left(\left(\frac{1 + \vartheta_{\tilde{A}_{Si}}^2}{1 - \vartheta_{\tilde{A}_{Si}}^2} \right)^{w_i} \right) \right)^{-1}}{\left(\prod_{i=1}^n \left(\left(\frac{1 + \vartheta_{\tilde{A}_{Si}}^2}{1 - \vartheta_{\tilde{A}_{Si}}^2} \right)^{w_i} \right) \right)^{+1}} \right)^2}{1 - \left(\frac{\left(\prod_{i=1}^n \left(\left(\frac{1 + \vartheta_{\tilde{A}_{Si}}^2}{1 - \vartheta_{\tilde{A}_{Si}}^2} \right)^{w_i} \right) \right)^{-1}}{\left(\prod_{i=1}^n \left(\left(\frac{1 + \vartheta_{\tilde{A}_{Si}}^2}{1 - \vartheta_{\tilde{A}_{Si}}^2} \right)^{w_i} \right) \right)^{+1}} \right)^2}} = \\
& \frac{10}{\log \frac{1 + \left(\frac{\left(\prod_{i=1}^n \left(\left(\frac{1 + \vartheta_{\tilde{A}_{Si}}^2}{1 - \vartheta_{\tilde{A}_{Si}}^2} \right)^{w_i} \right) \right)^{-1}}{\left(\prod_{i=1}^n \left(\left(\frac{1 + \vartheta_{\tilde{A}_{Si}}^2}{1 - \vartheta_{\tilde{A}_{Si}}^2} \right)^{w_i} \right) \right)^{+1}} \right)^2}{1 - \left(\frac{\left(\prod_{i=1}^n \left(\left(\frac{1 + \vartheta_{\tilde{A}_{Si}}^2}{1 - \vartheta_{\tilde{A}_{Si}}^2} \right)^{w_i} \right) \right)^{-1}}{\left(\prod_{i=1}^n \left(\left(\frac{1 + \vartheta_{\tilde{A}_{Si}}^2}{1 - \vartheta_{\tilde{A}_{Si}}^2} \right)^{w_i} \right) \right)^{+1}} \right)^2}} + 1
\end{aligned}$$

$$\begin{aligned}
& \frac{1+ \left(\frac{\prod_{i=1}^n \left(\frac{1+\vartheta_{\tilde{A}_{Si}}^2}{1-\vartheta_{\tilde{A}_{Si}}^2} \right)^{w_i}}{\prod_{i=1}^n \left(\frac{1+\vartheta_{\tilde{A}_{Si}}^2}{1-\vartheta_{\tilde{A}_{Si}}^2} \right)^{w_i}} \right)^{-1}}{\sqrt{\left(\frac{\prod_{i=1}^n \left(\frac{1+\vartheta_{\tilde{A}_{Si}}^2}{1-\vartheta_{\tilde{A}_{Si}}^2} \right)^{w_i}}{\prod_{i=1}^n \left(\frac{1+\vartheta_{\tilde{A}_{Si}}^2}{1-\vartheta_{\tilde{A}_{Si}}^2} \right)^{w_i}} \right)^{-1}}}^{2-1} \\
& \frac{1- \left(\frac{\prod_{i=1}^n \left(\frac{1+\vartheta_{\tilde{A}_{Si}}^2}{1-\vartheta_{\tilde{A}_{Si}}^2} \right)^{w_i}}{\prod_{i=1}^n \left(\frac{1+\vartheta_{\tilde{A}_{Si}}^2}{1-\vartheta_{\tilde{A}_{Si}}^2} \right)^{w_i}} \right)^{-1}}{\sqrt{\left(\frac{\prod_{i=1}^n \left(\frac{1+\vartheta_{\tilde{A}_{Si}}^2}{1-\vartheta_{\tilde{A}_{Si}}^2} \right)^{w_i}}{\prod_{i=1}^n \left(\frac{1+\vartheta_{\tilde{A}_{Si}}^2}{1-\vartheta_{\tilde{A}_{Si}}^2} \right)^{w_i}} \right)^{-1}}}^{2-1} \\
& \frac{1+ \left(\frac{\prod_{i=1}^n \left(\frac{1+\vartheta_{\tilde{A}_{Si}}^2}{1-\vartheta_{\tilde{A}_{Si}}^2} \right)^{w_i}}{\prod_{i=1}^n \left(\frac{1+\vartheta_{\tilde{A}_{Si}}^2}{1-\vartheta_{\tilde{A}_{Si}}^2} \right)^{w_i}} \right)^{-1}}{\sqrt{\left(\frac{\prod_{i=1}^n \left(\frac{1+\vartheta_{\tilde{A}_{Si}}^2}{1-\vartheta_{\tilde{A}_{Si}}^2} \right)^{w_i}}{\prod_{i=1}^n \left(\frac{1+\vartheta_{\tilde{A}_{Si}}^2}{1-\vartheta_{\tilde{A}_{Si}}^2} \right)^{w_i}} \right)^{-1}}}^{2+1} \\
& \frac{1- \left(\frac{\prod_{i=1}^n \left(\frac{1+\vartheta_{\tilde{A}_{Si}}^2}{1-\vartheta_{\tilde{A}_{Si}}^2} \right)^{w_i}}{\prod_{i=1}^n \left(\frac{1+\vartheta_{\tilde{A}_{Si}}^2}{1-\vartheta_{\tilde{A}_{Si}}^2} \right)^{w_i}} \right)^{-1}}{\sqrt{\left(\frac{\prod_{i=1}^n \left(\frac{1+\vartheta_{\tilde{A}_{Si}}^2}{1-\vartheta_{\tilde{A}_{Si}}^2} \right)^{w_i}}{\prod_{i=1}^n \left(\frac{1+\vartheta_{\tilde{A}_{Si}}^2}{1-\vartheta_{\tilde{A}_{Si}}^2} \right)^{w_i}} \right)^{-1}}}^{2+1} \\
& \xrightarrow{\sqrt{X}} \frac{\left(\frac{\prod_{i=1}^n \left(\frac{1+\vartheta_{\tilde{A}_{Si}}^2}{1-\vartheta_{\tilde{A}_{Si}}^2} \right)^{w_i}}{\prod_{i=1}^n \left(\frac{1+\vartheta_{\tilde{A}_{Si}}^2}{1-\vartheta_{\tilde{A}_{Si}}^2} \right)^{w_i}} \right)^{-1}}{\sqrt{\left(\frac{\prod_{i=1}^n \left(\frac{1+\vartheta_{\tilde{A}_{Si}}^2}{1-\vartheta_{\tilde{A}_{Si}}^2} \right)^{w_i}}{\prod_{i=1}^n \left(\frac{1+\vartheta_{\tilde{A}_{Si}}^2}{1-\vartheta_{\tilde{A}_{Si}}^2} \right)^{w_i}} \right)^{-1}}}
\end{aligned}$$

■

Boundary limits: because of the harmonic mean's nature, since the denominator could not take zero and if it takes zero value then it is not defined in real numbers, so it is assumed below boundary limits throughout the study:

Membership degree $\mu_{\tilde{A}_{Si}}$: $\begin{cases} \text{if } \mu_{\tilde{A}_{Si}} \neq 0 \text{ then } \varepsilon = 0 \text{ thus } \mu_{\tilde{A}_{Si}} = \mu_{\tilde{A}_{Si}} \\ \text{if } \mu_{\tilde{A}_{Si}} = 0 \text{ then } \varepsilon \rightarrow 0 \text{ thus } \mu_{\tilde{A}_{Si}} \rightarrow \mu_{\tilde{A}_{Si}} + \varepsilon \end{cases}$

Indeterminacy degree $I_{\tilde{A}_{Si}}$: $\begin{cases} \text{if } I_{\tilde{A}_{Si}} \neq 0 \text{ then } \varepsilon = 0 \text{ thus } I_{\tilde{A}_{Si}} = I_{\tilde{A}_{Si}} \\ \text{if } I_{\tilde{A}_{Si}} = 0 \text{ then } \varepsilon \rightarrow 0 \text{ thus } I_{\tilde{A}_{Si}} \rightarrow I_{\tilde{A}_{Si}} + \varepsilon \end{cases}$

Non-membership degree $\vartheta_{\tilde{A}_{Si}}$: $\begin{cases} \text{if } \vartheta_{\tilde{A}_{Si}} \neq 1 \text{ then } \varepsilon = 0 \text{ thus } \vartheta_{\tilde{A}_{Si}} = \vartheta_{\tilde{A}_{Si}} \\ \text{if } \vartheta_{\tilde{A}_{Si}} = 1 \text{ then } \varepsilon \rightarrow 0 \text{ thus } \vartheta_{\tilde{A}_{Si}} \rightarrow \vartheta_{\tilde{A}_{Si}} - \varepsilon \end{cases}$

As it is mentioned in Definition (3.18) and based on defined boundary limits for our harmonic mean aggregation operators could demonstrate that the conditions in Definition (3.8) are true for our operators as follows:

- i. Boundary condition: It is obvious this condition refers to the fact that if it is observed only completely bad, false, or not satisfactory criteria the total aggregation has to be also completely bad, false, or not satisfactory and when

it is observed only true or completely satisfactory criteria then the total aggregation has to be also completely true or satisfactory.

$$SFNEWHM(0,0 \dots, 0) = SFNEWHM(0^+, 0^+ \dots, 0^+) = 0^+ = 0 + \varepsilon \cong 0; (\varepsilon \rightarrow 0)$$

$$SFNEWHM(1,1 \dots, 1) = SFNEWHM(1^-, 1^- \dots, 1^-) = 1^- = 1 - \varepsilon \cong 1; (\varepsilon \rightarrow 0)$$

And the logical fact behind it is that:

$$\tilde{A}_s^{min} \leq SFNEWHM(\tilde{A}_{s1}, \tilde{A}_{s2}, \dots, \tilde{A}_{sn}) \leq \tilde{A}_s^{max}$$

And we know that:

$$\tilde{A}_s^{min} \text{ and } \tilde{A}_s^{max} \subset \{\tilde{A}_{s1}, \tilde{A}_{s2}, \dots, \tilde{A}_{sn}\}$$

Also, it is clear that:

$$\tilde{A}_s^{min} = \langle \min \mu_{\tilde{A}_{si}}, \min I_{\tilde{A}_{si}}, \max \vartheta_{\tilde{A}_{si}} \rangle \text{ and } \tilde{A}_s^{max} = \langle \max \mu_{\tilde{A}_{si}}, \min I_{\tilde{A}_{si}}, \min \vartheta_{\tilde{A}_{si}} \rangle \in [0,1]$$

So, it is clearly understood that when it is observed completely negative criteria the total aggregation will return us something completely negative or the least possible amount, $\tilde{A}_s^{min}$ from our set, it means may be $SFNEWHM(0,0 \dots, 0)$ is not exactly equal to 0 but it is going to return us the least possible value (like 0^+) that exist in our set. And know that it is true for the other side.

$$\text{ii. Monotonicity: If } \tilde{A}_{si} \subset B_{si} \rightarrow SFNEWHM(\tilde{A}_{s1}, \tilde{A}_{s2}, \dots, \tilde{A}_{sn}) \leq SFNEWHM(\tilde{B}_{s1}, \tilde{B}_{s2}, \dots, \tilde{B}_{sn}).$$

Up to now, weighted harmonic mean aggregation based on Algebraic and Einstein Strict Archimedean t-norm and t-conorms are introduced. Now it is going to have another type of aggregation operator whose weights depend on the order of placement of the evaluation values.

3.2.2 Spherical fuzzy number ordered weighted harmonic mean aggregation operators

The ordered weighted mean, also called the ordered statistical mean is the variant of the weighted mean, when the weights are not dependent on the source but on the place of the evaluation value on the order. Evaluation values have to be first ordered in the ascending or descending order by the Permutation of indices $1, 2, \dots, n$ (such that $\tilde{A}_{s1} \leq \tilde{A}_{s2} \leq \dots \leq \tilde{A}_{sn}$ or $\tilde{A}_{s1} \geq \tilde{A}_{s2} \geq \dots \geq \tilde{A}_{sn}$) and therefore the ordered statistical mean is calculated depending on the given weights vector $\vec{w} = \{w_1, w_2, w_3, \dots, w_n\} \in (0,1)$ and $\sum_{i=1}^n w_i = 1, w_i \geq 0$. The difference between

weighted mean and ordered weighted mean consists in the fact, that for the ordered weighted mean we first have to order the input aspect of its content and then determine weights depending on this order.

Definition 3.19 If $\tilde{A}_{si} = \{u, \mu_{\tilde{A}_{si}}(u), I_{\tilde{A}_{si}}(u), \vartheta_{\tilde{A}_{si}}(u) \mid u \in U\}$ be Spherical Fuzzy Numbers and weight vector $w_i = \{w_1, w_2, \dots, w_i\}$ while $\sum_{i=1}^n w_i = 1, w_i \geq 0$, $\tilde{A}_{\Delta s1} \leq \tilde{A}_{\Delta s2} \leq \dots \leq \tilde{A}_{\Delta sn}$ and $i = 1, \dots, n$. So the Algebraic Ordered Weighted Harmonic Mean Aggregation Operator to aggregate these SFNs are described as follows:

$$\begin{aligned} & SFNAOWHM(\tilde{A}_{s1}, \tilde{A}_{s2}, \dots, \tilde{A}_{sn}) \\ &= \left[\prod_{i=1}^n ((\mu_{\tilde{A}_{\Delta si}})^{w_i}), \prod_{i=1}^n ((I_{\tilde{A}_{\Delta si}})^{w_i}), \sqrt{1 - \prod_{i=1}^n ((1 - \vartheta_{\tilde{A}_{\Delta si}}^2)^{w_i})} \right] \end{aligned} \quad (3.39)$$

Where $\mu_{\tilde{A}_{\Delta si}}, I_{\tilde{A}_{\Delta si}}$ and $\vartheta_{\tilde{A}_{\Delta si}}$ are ordered $\mu_{\tilde{A}_{si}}, I_{\tilde{A}_{si}}$ and $\vartheta_{\tilde{A}_{si}}$ for example $\mu_{\tilde{A}_{\Delta si}}$ is the i th largest $\mu_{\tilde{A}_{si}}$.

Proof: Proof is similar to Definition (3.17). ■

Similarly what developed and showed for SFNAOWHM the conditions of Definition (3.17) are satisfying our new aggregation SFNAOWHM.

Definition 3.20 If $\tilde{A}_{si} = \{u, \mu_{\tilde{A}_{si}}(u), I_{\tilde{A}_{si}}(u), \vartheta_{\tilde{A}_{si}}(u) \mid u \in U\}$ be Spherical Fuzzy Numbers and weight vector $w_i = \{w_1, w_2, \dots, w_i\}$ while $\sum_{i=1}^n w_i = 1, w_i \geq 0$, $\tilde{A}_{\Delta s1} \leq \tilde{A}_{\Delta s2} \leq \dots \leq \tilde{A}_{\Delta sn}$ and $i = 1, \dots, n$. So the Einstein Ordered Weighted Harmonic Mean Aggregation Operators to aggregate these SFNs are described as follows:

$$\begin{aligned} & SFNEOWHM(\tilde{A}_{s1}, \tilde{A}_{s2}, \dots, \tilde{A}_{sn}) = \\ & \left[\frac{2}{1 + \prod_{i=1}^n \left(\left(\frac{2 - \mu_{\tilde{A}_{\Delta si}}}{\mu_{\tilde{A}_{\Delta si}}} \right)^{w_i} \right)}, \frac{2}{1 + \prod_{i=1}^n \left(\left(\frac{2 - I_{\tilde{A}_{\Delta si}}}{I_{\tilde{A}_{\Delta si}}} \right)^{w_i} \right)}, \sqrt{\frac{\left(\prod_{i=1}^n \left(\left(\frac{1 + \vartheta_{\tilde{A}_{\Delta si}}^2}{1 - \vartheta_{\tilde{A}_{\Delta si}}^2} \right)^{w_i} \right) \right)^{-1}}{\left(\prod_{i=1}^n \left(\left(\frac{1 + \vartheta_{\tilde{A}_{\Delta si}}^2}{1 - \vartheta_{\tilde{A}_{\Delta si}}^2} \right)^{w_i} \right) \right)^{+1}}} \right] \end{aligned} \quad (3.40)$$

Where $\mu_{\tilde{A}_{\Delta si}}, I_{\tilde{A}_{\Delta si}}$ and $\vartheta_{\tilde{A}_{\Delta si}}$ are ordered $\mu_{\tilde{A}_{si}}, I_{\tilde{A}_{si}}$ and $\vartheta_{\tilde{A}_{si}}$ for example $\mu_{\tilde{A}_{\Delta si}}$ is the i th largest $\mu_{\tilde{A}_{si}}$.

Proof: Proof is similar to Definition (3.18). ■

Boundary limits: because of the harmonic mean's nature, since the denominator could not take zero and if it takes zero value then it is not defined in real numbers, so it is assumed below boundary limits throughout the study:

$$\text{Membership degree } \mu_{\tilde{A}_{si}} : \begin{cases} \text{if } \mu_{\tilde{A}_{si}} \neq 0 \text{ then } \varepsilon = 0 \text{ thus } \mu_{\tilde{A}_{si}} = \mu_{\tilde{A}_{si}} \\ \text{if } \mu_{\tilde{A}_{si}} = 0 \text{ then } \varepsilon \rightarrow 0 \text{ thus } \mu_{\tilde{A}_{si}} \rightarrow \mu_{\tilde{A}_{si}} + \varepsilon \end{cases}$$

$$\text{Indeterminacy degree } I_{\tilde{A}_{si}} : \begin{cases} \text{if } I_{\tilde{A}_{si}} \neq 0 \text{ then } \varepsilon = 0 \text{ thus } I_{\tilde{A}_{si}} = I_{\tilde{A}_{si}} \\ \text{if } I_{\tilde{A}_{si}} = 0 \text{ then } \varepsilon \rightarrow 0 \text{ thus } I_{\tilde{A}_{si}} \rightarrow I_{\tilde{A}_{si}} + \varepsilon \end{cases}$$

$$\text{Non-membership degree } \vartheta_{\tilde{A}_{si}} : \begin{cases} \text{if } \vartheta_{\tilde{A}_{si}} \neq 1 \text{ then } \varepsilon = 0 \text{ thus } \vartheta_{\tilde{A}_{si}} = \vartheta_{\tilde{A}_{si}} \\ \text{if } \vartheta_{\tilde{A}_{si}} = 1 \text{ then } \varepsilon \rightarrow 0 \text{ thus } \vartheta_{\tilde{A}_{si}} \rightarrow \vartheta_{\tilde{A}_{si}} - \varepsilon \end{cases}$$

Similarly what developed and showed for SFNEWHM the conditions of Definition (3.18) are satisfying our new aggregation SFNEOWHM.

3.2.3 Spherical fuzzy number hybrid ordered weighted harmonic mean aggregation operators

Definition 3.21 If $\tilde{A}_{si} = \{u, \mu_{\tilde{A}_{si}}(u), I_{\tilde{A}_{si}}(u), \vartheta_{\tilde{A}_{si}}(u) \mid u \in U\}$ be Spherical Fuzzy Numbers and weight vector $\tau_i = (\tau_1, \tau_2, \dots, \tau_n)$, $\sum_{i=1}^n \tau_i = 1, \tau_i \geq 0$, the associated weight vector $w_i = \{w_1, w_2, \dots, w_i\}$, $\sum_{i=1}^n w_i = 1, w_i \geq 0$, and $i = 1, \dots, n$. So the Algebraic Hybrid Ordered Weighted Harmonic Mean Aggregation Operator to aggregate these SFNs are described as follows:

$$\begin{aligned} & SFNAHOWHM(\tilde{A}_{s1}, \tilde{A}_{s2}, \dots, \tilde{A}_{sn}) \\ &= \left[\prod_{i=1}^n ((\mu_{\tilde{A}_{\Delta si}})^{w_i}), \prod_{i=1}^n ((I_{\tilde{A}_{\Delta si}})^{w_i}), \sqrt{1 - \prod_{i=1}^n ((1 - \vartheta_{\tilde{A}_{\Delta si}}^2)^{w_i})} \right] \end{aligned} \quad (3.41)$$

Where $\tilde{A}'_{\Delta si} = \langle \mu_{\tilde{A}_{\Delta si}}, I_{\tilde{A}_{\Delta si}}, \vartheta_{\tilde{A}_{\Delta si}} \rangle$ is the i th largest of the weight spherical fuzzy value and $\tilde{A}'_{\Delta si} = n\tau_i \tilde{A}_{\Delta si}$.

Proof: Proof is similar to Definition 3.17. ■

Similarly what developed and showed for SFNAWHM the properties of Definition 3.17 are satisfying our new aggregation SFNAHOWHM.

Definition 3.22 If $\tilde{A}_{si} = \{u, \mu_{\tilde{A}_{si}}(u), I_{\tilde{A}_{si}}(u), \vartheta_{\tilde{A}_{si}}(u) \mid u \in U\}$ be Spherical Fuzzy Numbers and weight vector $\tau_i = (\tau_1, \tau_2, \dots, \tau_n)$, $\sum_{i=1}^n \tau_i = 1, \tau_i \geq 0$, the associated weight vector $w_i = \{w_1, w_2, \dots, w_i\}$, $\sum_{i=1}^n w_i = 1, w_i \geq 0$, and $i = 1, \dots, n$. So the Einstein Hybrid Order Weighted Harmonic Mean Aggregation Operators to aggregate these SFNs are described as follows:

$$SFNEHOWHM(\tilde{A}_{s1}, \tilde{A}_{s2}, \dots, \tilde{A}_{sn}) = \left[\frac{2}{1 + \prod_{i=1}^n \left(\left(\frac{2 - \mu_{\tilde{A}_{\Delta si}}}{\mu_{\tilde{A}_{\Delta si}}} \right)^{w_i} \right)}, \frac{2}{1 + \prod_{i=1}^n \left(\left(\frac{2 - I_{\tilde{A}_{\Delta si}}}{I_{\tilde{A}_{\Delta si}}} \right)^{w_i} \right)}, \frac{\left(\prod_{i=1}^n \left(\left(\frac{1 + \vartheta_{\tilde{A}_{\Delta si}}^2}{1 - \vartheta_{\tilde{A}_{\Delta si}}^2} \right)^{w_i} \right) \right)^{-1}}{\left(\prod_{i=1}^n \left(\left(\frac{1 + \vartheta_{\tilde{A}_{\Delta si}}^2}{1 - \vartheta_{\tilde{A}_{\Delta si}}^2} \right)^{w_i} \right) \right)^{+1}} \right] \quad (3.42)$$

Where $\tilde{A}'_{\Delta si} = \langle \mu_{\tilde{A}_{\Delta si}}, I_{\tilde{A}_{\Delta si}}, \vartheta_{\tilde{A}_{\Delta si}} \rangle$ is the i th largest of the weight spherical fuzzy value and $\tilde{A}'_{\Delta si} = n\tau_i \tilde{A}_{\Delta si}$.

Proof: Proof is similar to Definition 3.18. ■

Boundary limits: because of the harmonic mean's nature, since the denominator could not take zero and if it takes zero value then it is not defined in real numbers, so it is assumed below boundary limits throughout the study:

Membership degree $\mu_{\tilde{A}_{si}} : \begin{cases} \text{if } \mu_{\tilde{A}_{si}} \neq 0 \text{ then } \varepsilon = 0 \text{ thus } \mu_{\tilde{A}_{si}} = \mu_{\tilde{A}_{si}} \\ \text{if } \mu_{\tilde{A}_{si}} = 0 \text{ then } \varepsilon \rightarrow 0 \text{ thus } \mu_{\tilde{A}_{si}} \rightarrow \mu_{\tilde{A}_{si}} + \varepsilon \end{cases}$

Indeterminacy degree $I_{\tilde{A}_{si}} : \begin{cases} \text{if } I_{\tilde{A}_{si}} \neq 0 \text{ then } \varepsilon = 0 \text{ thus } I_{\tilde{A}_{si}} = I_{\tilde{A}_{si}} \\ \text{if } I_{\tilde{A}_{si}} = 0 \text{ then } \varepsilon \rightarrow 0 \text{ thus } I_{\tilde{A}_{si}} \rightarrow I_{\tilde{A}_{si}} + \varepsilon \end{cases}$

Non-membership degree $\vartheta_{\tilde{A}_{si}} : \begin{cases} \text{if } \vartheta_{\tilde{A}_{si}} \neq 1 \text{ then } \varepsilon = 0 \text{ thus } \vartheta_{\tilde{A}_{si}} = \vartheta_{\tilde{A}_{si}} \\ \text{if } \vartheta_{\tilde{A}_{si}} = 1 \text{ then } \varepsilon \rightarrow 0 \text{ thus } \vartheta_{\tilde{A}_{si}} \rightarrow \vartheta_{\tilde{A}_{si}} - \varepsilon \end{cases}$

Similarly what developed and showed for SFNEWHM the conditions of Definition 3.18 are satisfying our new aggregation SFNEHOWHM.

3.3 Application Of Proposed Aggregation Operators in MAGDM

Algorithm

In this section, it is going to apply our proposed Harmonic Mean Aggregation Operators in the MAGDM method based on Spherical Fuzzy Sets. And then will solve a numerical example by using the MAGDM method and HM aggregations to illustrate our method.

3.3.1 Evaluation: Multi-Attribute Group Decision Making (MAGDM)

algorithm

A group of decision-makers tries to select the best alternative from all possible alternatives which are evaluated on multiple attributes. This kind of decision problem is called the MAGDM problem. Suppose that a group of n decision-makers $D = \{D_1, D_2, \dots, D_n\}$ with related weight vector $w = \{w_1, w_2, \dots, w_n\}$, $\sum_{i=1}^n w_i = 1, w_i \geq 0$ is going to select the best alternative among m alternatives $A = \{A_1, A_2, \dots, A_m\}$ based on p criteria $C = \{C_1, C_2, \dots, C_p\}$ with corresponding weight vector $\omega_i = (\omega_1, \omega_2, \dots, \omega_p)$, $\sum_{i=1}^p \omega_i = 1, \omega_i \geq 0$. For the k^{th} decision-maker matrix of alternatives based on criteria is represented by Spherical Fuzzy Numbers. The steps of the MAGDM method based on proposed aggregation operators are defined as follows:

Step 1. Prepare all Alternative-Criteria matrices for each decision-maker together with weight vectors for decision-makers and criteria.

Step 2. Normalize the established decision matrices by considering the positive or negative form of criteria, if they are not uniform.

Step 3. Aggregate the established decision matrix including Spherical fuzzy Numbers for all decision-makers concerning the corresponding weight vector, into one Alternative-Criteria decision matrix by using one of the proposed aggregation operators (SFNAWHM, SFNEWHM, SFNAOWHM, SFNEOWHM, SFNAHOWHM, and SFNEHOWHM).

Step 4. Aggregate the obtained decision matrix from step 3, based on criteria, by considering the related weight vector of each criterion by using one of the proposed aggregation operators (SFNAWHM, SFNEWHM, SFNAOWHM, SFNEOWHM, SFNAHOWHM, and SFNEHOWHM).

Step 5. Calculate the comparison functions defined in Definition (3.16) for obtained decision matrix from the previous step.

Step 6. Rank the alternatives based on comparison rules defined in Definition 3.16.

Step 7. Select the best alternative.

3.3.2 Results and discussion: Illustrative example in air quality analysis

This section will present an example by using a proposed MAGDM method to show the effectiveness of the new algorithm. In this part, an example from Ashraf and Abdullah's article (Ashraf & Abdullah, 2019) which describes the evaluation of air quality is selected. In this example, there are three stations which are analyzed the air quality by (α, β, γ) as Decision Makers with corresponding weight vector $w = \{0.314, 0.355, 0.331\}$ and consider SO_2, NO_2 and PM_{10} as criteria for each alternative. The values are shown in Table 3.1, 3.2, and 3.3.

Table 3.1: Air quality data derived from the station α .

	$c_1 (SO_2)$			$c_2 (NO_2)$			$c_3 (PM_{10})$		
e_1	0.5	0.8	0.3	0.8	0.4	0.3	0.7	0.5	0.4
e_2	0.7	0.6	0.2	0.3	0.9	0.1	0.5	0.3	0.7
e_3	0.4	0.8	0.4	0.5	0.8	0.2	0.7	0.3	0.2
e_4	0.8	0.3	0.5	0.6	0.6	0.3	0.3	0.6	0.5

Table 3.2: Air quality data derived from the station β .

	$c_1 (SO_2)$			$c_2 (NO_2)$			$c_3 (PM_{10})$		
e_1	0.7	0.5	0.1	0.6	0.3	0.4	0.3	0.8	0.6
e_2	0.8	0.4	0.4	0.5	0.7	0.1	0.7	0.2	0.4
e_3	0.3	0.9	0.2	0.7	0.1	0.4	0.4	0.6	0.3
e_4	0.8	0.4	0.3	0.4	0.6	0.5	0.5	0.1	0.8

Table 3.3: Air quality data derived from the station γ .

	$c_1 (SO_2)$			$c_2 (NO_2)$			$c_3 (PM_{10})$		
e_1	0.8	0.2	0.4	0.4	0.4	0.3	0.6	0.4	0.5
e_2	0.7	0.5	0.2	0.6	0.5	0.4	0.8	0.5	0.2
e_3	0.5	0.4	0.6	0.9	0.3	0.1	0.3	0.1	0.9
e_4	0.7	0.3	0.5	0.8	0.5	0.2	0.2	0.8	0.4

Apply the algorithm mentioned in section 3.3.1 as follows:

Step 1. Matrixes are shown in above Tables 3.1, 3.2, and 3.3.

Step 2. Data are uniform so there is no need to normalize decision matrices.

Step 3. Use SFNAWHM to aggregate the assessment of three stations in one matrix as given in Table 3.4.

Table 3.4: Aggregated air quality data for the stations α, β, γ (SFNAWHM).

	$c_1 (SO_2)$			$c_2 (NO_2)$			$c_3 (PM_{10})$		
e_1	0.658	0.427	0.294	0.574	0.361	0.339	0.492	0.548	0.515
e_2	0.733	0.489	0.290	0.452	0.677	0.249	0.658	0.307	0.499
e_3	0.388	0.663	0.441	0.684	0.276	0.273	0.433	0.266	0.670
e_4	0.765	0.332	0.443	0.571	0.564	0.367	0.314	0.349	0.632

Step 4. Use SFNAWHM to aggregate the previous matrix based on the criteria for all alternatives as given in Table 3.5.

Table 3.5: Aggregated data for alternatives (SFNAWHM).

	$\mu_{\tilde{e}_i}$	$I_{\tilde{e}_i}$	$\vartheta_{\tilde{e}_i}$
e_1	0.570303	0.456901	0.410223
e_2	0.637884	0.433699	0.38773
e_3	0.454767	0.386693	0.537395
e_4	0.505817	0.376949	0.523971

Step 5. Calculate comparison functions for all alternatives as given in Table 3.6.

Table 3.6: Obtained comparison functions (SFNAWHM).

	score	accuracy	certainty
e_1	0.5677	0.1134	0.5703
e_2	0.6055	0.2042	0.6379
e_3	0.5102	0.0681	0.4548
e_4	0.5350	0.1289	0.5058

Step 6. Rank and select the best alternative as given in Table 3.7.

Table 3.7: Ranking for alternative (SFNEWHM).

SFNAWHM
$Sc(e_2) > Sc(e_1) > Sc(e_4) > SC(e_3)$
$e_2 > e_1 > e_4 > e_3$

So the best alternative is e_2 .

Similarly, it resolved the example by using the SFNEWHM aggregation operator and the final results are shown in Tables (3.8 – 3.11).

Table 3.8: Aggregated air quality data for the stations α, β, γ (SFNEWHM).

	$c_1 (SO_2)$			$c_2 (NO_2)$			$c_3 (PM_{10})$		
e_1	0.664	0.447	0.291	0.583	0.361	0.339	0.503	0.558	0.512
e_2	0.734	0.491	0.287	0.457	0.687	0.245	0.664	0.311	0.484
e_3	0.39	0.68	0.432	0.694	0.296	0.27	0.441	0.278	0.628
e_4	0.766	0.332	0.44	0.581	0.565	0.362	0.319	0.381	0.618

Table 3.9: Aggregated data for alternatives (SFNEWHM).

	$\mu_{\tilde{e}_i}$	$I_{\tilde{e}_i}$	$\vartheta_{\tilde{e}_i}$
e_1	0.5809	0.470487	0.404166
e_2	0.6461	0.444245	0.374013
e_3	0.4642	0.414908	0.501995
e_4	0.5247	0.392881	0.51005

Table 3.10: Obtained comparison functions (SFNEWHM).

	score	accuracy	Certainty
e_1	0.5687	0.1104	0.5809
e_2	0.6093	0.2018	0.6461
e_3	0.5158	0.0493	0.4642
e_4	0.5406	0.1318	0.5247

Table 3.11: Ranking for alternative (SFNEWHM).

SFNEWHM
$Sc(e_2) > Sc(e_1) > Sc(e_4) > SC(e_3)$
$e_2 > e_1 > e_4 > e_3$

So the best alternative is e_2 . Based on results obtained from SFNAWHM and SFNEWHM the most optimal and conservative alternative is e_2 .

3.4 Conclusion

The essence of using an aggregation operator in decision-making problems is brilliantly clear, but the issue is which type of operator is suitable for which type of problem. So, the selection of an appropriate aggregation operator in accordance with the problem type is of the first degree of importance. Harmonic mean operator despite having general features that are common among aggregation operators has critical features and central tendency. Since there is always some out-of-range data in every studied case, harmonic mean aggregation operators can pass up this kind of data to select the best alternative. One of the last and most important generations of fuzzy sets is Spherical Fuzzy Sets (SFSs), which have a realistic base of data type. Based on Algebraic and Einstein Strict Archimedean t-norm and t-conorm are developed harmonic mean aggregation operators in SFNs and then are established a MAGDM method to make use of the advantages of these aggregation operators in decision-making problems. The other advantage of this MAGDM method is its simplicity and effectiveness. For future studies suggest extending the proposed aggregation operators in other fuzzy environments like T-Spherical Fuzzy Sets and expanding its application in various fields of decision-making problems like supply chain selection, healthcare systems, transportation systems, material selection, and disaster management.

4. SPHERICAL FUZZY DISTANCE AND SIMILARITY MEASUREMENTS

4.1 Preliminaries

In this section, first, the Minkowski and Minkowski-Hausdorff distances are defined mathematically. Then novel Spherical Fuzzy Minkowski and Minkowski-Hausdorff distance measures. Afterward, the novel Spherical Fuzzy f-Similarity measures based on Minkowski and Minkowski-Hausdorff distance measurements are introduced and discussed in detail. At the end of this section, the proposed distance and similarity measurements are applied to the medical diagnosis problem for the case study of the COVID-19 patient diagnosis.

4.1.1 Distance measurements

Definition 4.1 Let $X = [x_1, x_2, \dots, x_n]$ and $Y = [y_1, y_2, \dots, y_n]$ be two crisp vectors. Then normalized Minkowski distance $Md(X, Y)$ is given as (Merigó & Casanovas, 2011):

$$Md(X, Y) = \sqrt[\alpha]{\frac{1}{2n} \sum_{i=1}^n |x_i - y_i|^\alpha} \quad (4.1)$$

Where $\alpha \geq 1$.

Definition 4.2 Let $X = (x_1, x_2, \dots, x_n)$ and $Y = (y_1, y_2, \dots, y_n)$ be two crisp vectors with corresponding weight vector $W = \{(w_1, w_2, \dots, w_n)\}$ where $w_i \in [0, 1]$ and $\sum_{i=1}^n w_i = 1, \forall i = 1, 2, \dots, n$. Then weighted Minkowski distance $WMd(X, Y)$ is given as (Merigó & Casanovas, 2011):

$$WMd(X, Y) = \sqrt[\alpha]{\frac{1}{2} \sum_{i=1}^n w_i |x_i - y_i|^\alpha} \quad (4.2)$$

Where $\alpha \geq 1$.

Definition 4.3 Let $X = [x_1, x_2, \dots, x_n]$ and $Y = [y_1, y_2, \dots, y_n]$ be two crisp vectors. Then normalized Minkowski-Hausdorff distance $MHd(X, Y)$ is given as (Merigó & Casanovas, 2011):

$$MHd(X, Y) = \sqrt[\alpha]{\frac{1}{2n} \sum_{i=1}^n \max_i |x_i - y_i|^\alpha} \quad (4.3)$$

Where $\alpha \geq 1$.

Definition 4.4 Let $X = (x_1, x_2, \dots, x_n)$ and $Y = (y_1, y_2, \dots, y_n)$ be two crisp vectors with corresponding weight vector $W = \{(w_1, w_2, \dots, w_n)\}$ where $w_i \in [0,1]$ and $\sum_{i=1}^n w_i = 1, \forall i = 1, 2, \dots, n..$ Then weighted Minkowski-Hausdorff distance $WMHd(X, Y)$ is given as (Merigó & Casanovas, 2011):

$$WMHd(X, Y) = \sqrt[\alpha]{\frac{1}{2} \sum_{i=1}^n w_i \max_i |x_i - y_i|^\alpha} \quad (4.4)$$

Where $\alpha \geq 1$.

4.2 The Novel Distance Measurements For Spherical Fuzzy Sets

In this section, some novel distance measurements are developed under a spherical fuzzy environment.

Definition 4.5 Let X be the universal set and $x_i \in X; \forall i = 1, 2, \dots, n$ then $\tilde{A} = \langle \mu_{\tilde{A}}(x_i), \vartheta_{\tilde{A}}(x_i), I_{\tilde{A}}(x_i) \rangle$ and $\tilde{B} = \langle \mu_{\tilde{B}}(x_i), \vartheta_{\tilde{B}}(x_i), I_{\tilde{B}}(x_i) \rangle$ be two sample spherical fuzzy sets. The distance between $\tilde{A}$ and $\tilde{B}$ will be presented as $d_{SFS}(\tilde{A}, \tilde{B})$ and have to satisfy the following properties:

- i. $0 \leq d_{SFS}(\tilde{A}, \tilde{B}) \leq 1$
- ii. $d_{SFS}(\tilde{A}, \tilde{B}) = d_{SFS}(\tilde{B}, \tilde{A})$
- iii. $d_{SFS}(\tilde{A}, \tilde{B}) = 0$ iff $\tilde{A} = \tilde{B}$.

4.2.1 Spherical fuzzy Minkowski distance

Minkowski distance also known as the generalized distance is based on L_p norm is a metric measure because it satisfies non-negativity $d(i, j) \geq 0$, the identity of indiscernible $d(i, j) = 0$ and symmetry $d(i, j) = d(j, i)$ (R. Şahin & P. Liu, 2016). Minkowski distance is also used in fuzzy sets in fields of clustering, decision making, and pattern recognition (Brouwer, 2009; Merigo & Casanovas, 2011; Xu et al., 2008). Definition 4.1 is used to introduce spherical fuzzy normalized Minkowski distance as follows.

Definition 4.6 Let X be the universal set and $x_i \in X; \forall i = 1, 2, \dots, n$ then $\tilde{A} = \langle \mu_{\tilde{A}}(x_i), \vartheta_{\tilde{A}}(x_i), I_{\tilde{A}}(x_i) \rangle$ and $\tilde{B} = \langle \mu_{\tilde{B}}(x_i), \vartheta_{\tilde{B}}(x_i), I_{\tilde{B}}(x_i) \rangle$ be two sample spherical fuzzy sets. Then normalized Minkowski distance $Md_{SFS}(\tilde{A}, \tilde{B})$ is given as:

$$Md_{SFS}(\tilde{A}, \tilde{B}) = \sqrt[\alpha]{\frac{1}{2n} \sum_{i=1}^n [|\mu_{\tilde{A}}^2(x_i) - \mu_{\tilde{B}}^2(x_i)|^\alpha + |\vartheta_{\tilde{A}}^2(x_i) - \vartheta_{\tilde{B}}^2(x_i)|^\alpha + |I_{\tilde{A}}^2(x_i) - I_{\tilde{B}}^2(x_i)|^\alpha]} \quad (4.5)$$

Where $\alpha \geq 1$.

Proof: To prove Definition 4.6, have shown that Eq. (4.5) satisfies properties in Definition 4.5 as follows:

$$i. \quad 0 \leq Md_{SFS}(\tilde{A}, \tilde{B}) \leq 1$$

we know that:

$$|\mu_{\tilde{A}}^2(x_i) - \mu_{\tilde{B}}^2(x_i)| \geq 0, \quad |\vartheta_{\tilde{A}}^2(x_i) - \vartheta_{\tilde{B}}^2(x_i)| \geq 0 \text{ and } |I_{\tilde{A}}^2(x_i) - I_{\tilde{B}}^2(x_i)| \geq 0$$

So, will have:

$$\sum_{i=1}^n [|\mu_{\tilde{A}}^2(x_i) - \mu_{\tilde{B}}^2(x_i)|^\alpha + |\vartheta_{\tilde{A}}^2(x_i) - \vartheta_{\tilde{B}}^2(x_i)|^\alpha + |I_{\tilde{A}}^2(x_i) - I_{\tilde{B}}^2(x_i)|^\alpha] \geq 0$$

Thus, have that:

$$Md_{SFS}(\tilde{A}, \tilde{B}) = \sqrt[\alpha]{\frac{1}{2n} \sum_{i=1}^n [|\mu_{\tilde{A}}^2(x_i) - \mu_{\tilde{B}}^2(x_i)|^\alpha + |\vartheta_{\tilde{A}}^2(x_i) - \vartheta_{\tilde{B}}^2(x_i)|^\alpha + |I_{\tilde{A}}^2(x_i) - I_{\tilde{B}}^2(x_i)|^\alpha]} \geq 0$$

The other side of the point is that:

$$|\mu_{\tilde{A}}^2(x_i) - \mu_{\tilde{B}}^2(x_i)| \leq 1, \quad |\vartheta_{\tilde{A}}^2(x_i) - \vartheta_{\tilde{B}}^2(x_i)| \leq 1 \text{ and } |I_{\tilde{A}}^2(x_i) - I_{\tilde{B}}^2(x_i)| \leq 1$$

So, will have:

$$\frac{1}{2n} \sum_{i=1}^n [|\mu_{\tilde{A}}^2(x_i) - \mu_{\tilde{B}}^2(x_i)|^\alpha + |\vartheta_{\tilde{A}}^2(x_i) - \vartheta_{\tilde{B}}^2(x_i)|^\alpha + |I_{\tilde{A}}^2(x_i) - I_{\tilde{B}}^2(x_i)|^\alpha] \leq 1$$

Thus, have that:

$$Md_{SFS}(\tilde{A}, \tilde{B}) = \sqrt[\alpha]{\frac{1}{2n} \sum_{i=1}^n [|\mu_{\tilde{A}}^2(x_i) - \mu_{\tilde{B}}^2(x_i)|^\alpha + |\vartheta_{\tilde{A}}^2(x_i) - \vartheta_{\tilde{B}}^2(x_i)|^\alpha + |I_{\tilde{A}}^2(x_i) - I_{\tilde{B}}^2(x_i)|^\alpha]} \leq 1$$

So, it is proved that:

$$0 \leq Md_{SFS}(\tilde{A}, \tilde{B}) \leq 1$$

$$ii. \quad Md_{SFS}(\tilde{A}, \tilde{B}) = Md_{SFS}(\tilde{B}, \tilde{A})$$

Also, we know that:

$$|\mu_{\tilde{A}}^2(x_i) - \mu_{\tilde{B}}^2(x_i)| = |\mu_{\tilde{B}}^2(x_i) - \mu_{\tilde{A}}^2(x_i)|, \quad |\vartheta_{\tilde{A}}^2(x_i) - \vartheta_{\tilde{B}}^2(x_i)| = |\vartheta_{\tilde{B}}^2(x_i) - \vartheta_{\tilde{A}}^2(x_i)| \text{ and } |I_{\tilde{A}}^2(x_i) - I_{\tilde{B}}^2(x_i)| = |I_{\tilde{B}}^2(x_i) - I_{\tilde{A}}^2(x_i)|$$

Thus, have that:

$$\begin{aligned}
Md_{SFS}(\tilde{A}, \tilde{B}) &= \\
&\sqrt[\alpha]{\frac{1}{2n} \sum_{i=1}^n [|\mu_{\tilde{A}}^2(x_i) - \mu_{\tilde{B}}^2(x_i)|^\alpha + |\vartheta_{\tilde{A}}^2(x_i) - \vartheta_{\tilde{B}}^2(x_i)|^\alpha + |I_{\tilde{A}}^2(x_i) - I_{\tilde{B}}^2(x_i)|^\alpha]} = \\
&\sqrt[\alpha]{\frac{1}{2n} \sum_{i=1}^n [|\mu_{\tilde{B}}^2(x_i) - \mu_{\tilde{A}}^2(x_i)|^\alpha + |\vartheta_{\tilde{B}}^2(x_i) - \vartheta_{\tilde{A}}^2(x_i)|^\alpha + |I_{\tilde{B}}^2(x_i) - I_{\tilde{A}}^2(x_i)|^\alpha]} = \\
Md_{SFS}(\tilde{B}, \tilde{A})
\end{aligned}$$

$$\text{iii. } Md_{SFS}(\tilde{A}, \tilde{B}) = 0 \text{ iff } \tilde{A} = \tilde{B}$$

Know that:

If $\tilde{A} = \tilde{B}$ then $\mu_{\tilde{A}}(x_i) = \mu_{\tilde{B}}(x_i)$, $\vartheta_{\tilde{A}}(x_i) = \vartheta_{\tilde{B}}(x_i)$ and $I_{\tilde{A}}(x_i) = I_{\tilde{B}}(x_i)$ so will have:

$$|\mu_{\tilde{A}}^2(x_i) - \mu_{\tilde{B}}^2(x_i)| = 0, |\vartheta_{\tilde{A}}^2(x_i) - \vartheta_{\tilde{B}}^2(x_i)| = 0 \text{ and } |I_{\tilde{A}}^2(x_i) - I_{\tilde{B}}^2(x_i)| = 0$$

So, will have:

$$\sum_{i=1}^n [|\mu_{\tilde{A}}^2(x_i) - \mu_{\tilde{B}}^2(x_i)|^\alpha + |\vartheta_{\tilde{A}}^2(x_i) - \vartheta_{\tilde{B}}^2(x_i)|^\alpha + |I_{\tilde{A}}^2(x_i) - I_{\tilde{B}}^2(x_i)|^\alpha] = 0$$

Thus, have that:

$$\begin{aligned}
Md_{SFS}(\tilde{A}, \tilde{B}) &= \\
&\sqrt[\alpha]{\frac{1}{2n} \sum_{i=1}^n [|\mu_{\tilde{A}}^2(x_i) - \mu_{\tilde{B}}^2(x_i)|^\alpha + |\vartheta_{\tilde{A}}^2(x_i) - \vartheta_{\tilde{B}}^2(x_i)|^\alpha + |I_{\tilde{A}}^2(x_i) - I_{\tilde{B}}^2(x_i)|^\alpha]} = 0
\end{aligned}$$

So, based on the above properties it proved that $Md_{SFS}(\tilde{A}, \tilde{B})$ is a distance measure.

Remark 4.1

- i. If $\alpha = 1$ spherical fuzzy Minkowski distance will turn into spherical fuzzy Hamming distance:

$$\begin{aligned}
Hd_{SFS}(\tilde{A}, \tilde{B}) &= \frac{1}{2n} \sum_{i=1}^n [|\mu_{\tilde{A}}^2(x_i) - \mu_{\tilde{B}}^2(x_i)| + |\vartheta_{\tilde{A}}^2(x_i) - \vartheta_{\tilde{B}}^2(x_i)| + \\
&|I_{\tilde{A}}^2(x_i) - I_{\tilde{B}}^2(x_i)|]
\end{aligned} \tag{4.6}$$

- ii. If $\alpha = 2$ spherical fuzzy Minkowski distance will turn into spherical fuzzy Euclidean distance:

$$\begin{aligned}
Ed_{SFS}(\tilde{A}, \tilde{B}) &= \\
&\sqrt{\frac{1}{2n} \sum_{i=1}^n [|\mu_{\tilde{A}}^2(x_i) - \mu_{\tilde{B}}^2(x_i)|^2 + |\vartheta_{\tilde{A}}^2(x_i) - \vartheta_{\tilde{B}}^2(x_i)|^2 + |I_{\tilde{A}}^2(x_i) - I_{\tilde{B}}^2(x_i)|^2]}
\end{aligned} \tag{4.7}$$

4.2.2 Spherical fuzzy weighted Minkowski distance

Definition 4.7 Let X be the universal set and $x_i \in X; \forall i = 1, 2, \dots, n$ then $\tilde{A} = \langle \mu_{\tilde{A}}(x_i), \vartheta_{\tilde{A}}(x_i), I_{\tilde{A}}(x_i) \rangle$ and $\tilde{B} = \langle \mu_{\tilde{B}}(x_i), \vartheta_{\tilde{B}}(x_i), I_{\tilde{B}}(x_i) \rangle$ be two sample spherical fuzzy sets with corresponding weight vector $w = \{w_1, w_2, \dots, w_n\}$, where $w_i \in [0, 1]$ and $\sum_{i=1}^n w_i = 1, \forall i = 1, 2, \dots, n$. Then Weighted Minkowski distance $Md_{SFS}(\tilde{A}, \tilde{B})$ is given as:

$$WMd_{SFS}(\tilde{A}, \tilde{B}) = \sqrt[\alpha]{\frac{1}{2} \sum_{i=1}^n w_i [|\mu_{\tilde{A}}^2(x_i) - \mu_{\tilde{B}}^2(x_i)|^\alpha + |\vartheta_{\tilde{A}}^2(x_i) - \vartheta_{\tilde{B}}^2(x_i)|^\alpha + |I_{\tilde{A}}^2(x_i) - I_{\tilde{B}}^2(x_i)|^\alpha]} \quad (4.8)$$

Where $\alpha \geq 1$.

Proof: Proof of Definition 4.7 is similar to the proof of Definition 4.6, hence, omit this proof.

Remark 4.2 If $w_i = \frac{1}{n}; \forall i$ then WMd_{SFS} will turn into Md_{SFS} .

4.2.3 Spherical fuzzy Minkowski-Hausdorff distance

Definition 4.8 Let X be the universal set and $x_i \in X; \forall i = 1, 2, \dots, n$ then $\tilde{A} = \langle \mu_{\tilde{A}}(x_i), \vartheta_{\tilde{A}}(x_i), I_{\tilde{A}}(x_i) \rangle$ and $\tilde{B} = \langle \mu_{\tilde{B}}(x_i), \vartheta_{\tilde{B}}(x_i), I_{\tilde{B}}(x_i) \rangle$ be two sample spherical fuzzy sets. Then normalized Minkowski-Hausdorff distance $MHd_{SFS}(\tilde{A}, \tilde{B})$ is given as

$$MHd_{SFS}(\tilde{A}, \tilde{B}) = \sqrt[\alpha]{\frac{1}{2n} \sum_{i=1}^n \max_i [|\mu_{\tilde{A}}^2(x_i) - \mu_{\tilde{B}}^2(x_i)|^\alpha, |\vartheta_{\tilde{A}}^2(x_i) - \vartheta_{\tilde{B}}^2(x_i)|^\alpha, |I_{\tilde{A}}^2(x_i) - I_{\tilde{B}}^2(x_i)|^\alpha]} \quad (4.9)$$

Where $\alpha \geq 1$.

Proof: Proof of Definition 4.8 is similar to the proof of Definition 4.6, hence, omit this proof.

Remark 4.3

- i. If $\alpha = 1$ spherical fuzzy Minkowski-Hausdorff distance will turn into spherical fuzzy Hamming-Hausdorff distance:

$$HHd_{SFS}(\tilde{A}, \tilde{B}) = \frac{1}{2n} \sum_{i=1}^n \max_i [|\mu_{\tilde{A}}^2(x_i) - \mu_{\tilde{B}}^2(x_i)|, |\vartheta_{\tilde{A}}^2(x_i) - \vartheta_{\tilde{B}}^2(x_i)|, |I_{\tilde{A}}^2(x_i) - I_{\tilde{B}}^2(x_i)|] \quad (4.10)$$

- ii. If $\alpha = 2$ spherical fuzzy Minkowski- Hausdorff distance will turn into spherical fuzzy Euclidean-Hausdorff distance:

$$EHd_{SFS}(\tilde{A}, \tilde{B}) = \sqrt{\frac{1}{2n} \sum_{i=1}^n \max_i [|\mu_{\tilde{A}}^2(x_i) - \mu_{\tilde{B}}^2(x_i)|^2, |\vartheta_{\tilde{A}}^2(x_i) - \vartheta_{\tilde{B}}^2(x_i)|^2, |I_{\tilde{A}}^2(x_i) - I_{\tilde{B}}^2(x_i)|^2]} \quad (4.11)$$

4.2.4 Spherical fuzzy weighted Minkowski-Hausdorff distance

Definition 4.9 Let X be the universal set and $x_i \in X; \forall i = 1, 2, \dots, n$ then $\tilde{A} = \langle \mu_{\tilde{A}}(x_i), \vartheta_{\tilde{A}}(x_i), I_{\tilde{A}}(x_i) \rangle$ and $\tilde{B} = \langle \mu_{\tilde{B}}(x_i), \vartheta_{\tilde{B}}(x_i), I_{\tilde{B}}(x_i) \rangle$ be two sample spherical fuzzy sets with corresponding weight vector $w = \{w_1, w_2, \dots, w_n\}$, where $w_i \in [0, 1]$ and $\sum_{i=1}^n w_i = 1, \forall i = 1, 2, \dots, n$. Then Weighted Minkowski-Hausdorff distance $WMHd_{SFS}(\tilde{A}, \tilde{B})$ is given as

$$WMHd_{SFS}(\tilde{A}, \tilde{B}) = \sqrt[\alpha]{\frac{1}{2} \sum_{i=1}^n w_i \max_i [|\mu_{\tilde{A}}^2(x_i) - \mu_{\tilde{B}}^2(x_i)|^\alpha, |\vartheta_{\tilde{A}}^2(x_i) - \vartheta_{\tilde{B}}^2(x_i)|^\alpha, |I_{\tilde{A}}^2(x_i) - I_{\tilde{B}}^2(x_i)|^\alpha]} \quad (4.12)$$

Where $\alpha \geq 1$.

Proof: Proof of Definition 4.9 is similar to the proof of Definition 4.6, hence, omit this proof.

Remark 4.4 If $w_i = \frac{1}{n}; \forall i$ then $WMHd_{SFS}$ will turn into MHd_{SFS} .

4.3 The Novel Similarity Measurements for Spherical Fuzzy Sets

In this section, some spherical fuzzy similarity measures will introduce as follows;

Definition 4.10 Let X be the universal set and $x_i \in X; \forall i = 1, 2, \dots, n$ then $\tilde{A} = \langle \mu_{\tilde{A}}(x_i), \vartheta_{\tilde{A}}(x_i), I_{\tilde{A}}(x_i) \rangle$ and $\tilde{B} = \langle \mu_{\tilde{B}}(x_i), \vartheta_{\tilde{B}}(x_i), I_{\tilde{B}}(x_i) \rangle$ be two sample spherical fuzzy sets. The similarity measurement between $\tilde{A}$ and $\tilde{B}$ will be presented as $S_{SFS}(\tilde{A}, \tilde{B})$ and have to satisfy the following properties:

- i. $0 \leq S_{SFS}(\tilde{A}, \tilde{B}) \leq 1$

- ii. $S_{SFS}(\tilde{A}, \tilde{B}) = S_{SFS}(\tilde{B}, \tilde{A})$
- iii. $S_{SFS}(\tilde{A}, \tilde{B}) = 1$ iff $\tilde{A} = \tilde{B}$

Based on the definitions of distance and similarity, it is clear that distance and similarity are two sides of a coin. In other words, distance and similarity are dual concepts of each other, which means that when two sets have a smaller distance from each other, they are close and similar to each other (Hussian & Yang, 2019). This is also true for spherical fuzzy sets as it is defined above. In the following, will introduce some spherical fuzzy similarity measures based on distance.

4.3.1 Spherical fuzzy distance-based f-similarity measure

Definition 4.11 Suppose that there are two sets of SFSs $\tilde{A}_s = \{x_i \langle \mu_{\tilde{A}}(x_i), \vartheta_{\tilde{A}}(x_i), I_{\tilde{A}}(x_i) \rangle | x_i \in X\}; \forall i = 1, 2, \dots, n$ and $\tilde{B}_s = \{x_i \langle \mu_{\tilde{B}}(x_i), \vartheta_{\tilde{B}}(x_i), I_{\tilde{B}}(x_i) \rangle | x_i \in X\}; \forall i = 1, 2, \dots, n$ be two SFSs so that $\mu(x_i)$, $\vartheta(x_i)$ and $I(x_i)$ are membership, non-membership, and indeterminacy degrees, respectively. Then, Spherical Fuzzy Similarity based on normalized Minkowski distance is given as follows:

$$S_{SFS}^{\boxed{f}}(\tilde{A}, \tilde{B}) = \frac{f(d_{SFS}(\tilde{A}_s, \tilde{B}_s)) - f(d_{max})}{f(d_{min}) - f(d_{max})} \quad (4.13)$$

Based on Definition 4.11, it is possible to use different spherical fuzzy distance measurements in Eq (4.13), then will get different spherical fuzzy f-similarity measures based on different distances. As a common feature among all distance measures which is referred to in Definition 4.5, the maximum and minimum amount of distance between two spherical fuzzy sets $\tilde{A}_s$ and $\tilde{B}_s$ is $0 \leq d_{SFS}(\tilde{A}, \tilde{B}) \leq 1$, so will have $d_{max} = 1$ and $d_{min} = 0$. Then Eq. (4.13) could be written as follows:

$$S_{SFS}^{\boxed{f}}(\tilde{A}, \tilde{B}) = \frac{f(d_{SFS}(\tilde{A}_s, \tilde{B}_s)) - f(1)}{f(0) - f(1)} \quad (4.14)$$

In Definition 4.11, by using different linear and non-linear functions, a vast variety of similarity measures could be obtained from Eq. (4.13). In the following part, will introduce some of these spherical fuzzy distance-based f-similarities in Remark 4.5.

Remark 4.5

- i. The first simplest function that could be used in Definition 4.11 is linear function $f(x) = 1 - x$ which declares the duality concept of distance and

similarity measures so will have the most popular spherical fuzzy distance-based linear similarity measure as below:

$$S_{SFS}^{\square}(\tilde{A}, \tilde{B}) = 1 - d_{SFS}(\tilde{A}_s, \tilde{B}_s) \quad (4.15)$$

- ii. Exponential function $f(x) = e^{-x}$ is another function that could be used in Definition 4.11 and then we will get spherical fuzzy distance-based exponential similarity measure as follows:

$$S_{SFS}^{\square}(\tilde{A}, \tilde{B}) = \frac{e^{-d_{SFS}(\tilde{A}_s, \tilde{B}_s)} - e^{-1}}{e^0 - e^{-1}} \quad (4.16)$$

- iii. Logarithm function $f(x) = \log_2(2 - x)$ is another simple function that could be used in Definition 4.11 and then will get spherical fuzzy distance-based logarithmic similarity measure as follows:

$$\begin{aligned} S_{SFS}^{\square}(\tilde{A}, \tilde{B}) &= \frac{\log_2(2 - d_{SFS}(\tilde{A}_s, \tilde{B}_s)) - \log_2 1}{\log_2 2 - \log_2 1} \\ &= \log_2(2 - d_{SFS}(\tilde{A}_s, \tilde{B}_s)) \end{aligned} \quad (4.17)$$

- iv. Simple rational function $f(x) = \frac{1}{1+x}$ is another function that could be used in Definition 4.11 and then will get spherical fuzzy distance-based rational similarity measure as follows:

$$S_{SFS}^{\square}(\tilde{A}, \tilde{B}) = \frac{1 - d_{SFS}(\tilde{A}_s, \tilde{B}_s)}{1 + d_{SFS}(\tilde{A}_s, \tilde{B}_s)} \quad (4.18)$$

4.3.1.1 Spherical fuzzy distance-based f-similarity (linear function) measure

Definition 4.12 Suppose that there are two sets of SFSs $\tilde{A}_s = \{x_i \langle \mu_{\tilde{A}}(x_i), \vartheta_{\tilde{A}}(x_i), I_{\tilde{A}}(x_i) \rangle | x_i \in X\}; \forall i = 1, 2, \dots, n$ and $\tilde{B}_s = \{x_i \langle \mu_{\tilde{B}}(x_i), \vartheta_{\tilde{B}}(x_i), I_{\tilde{B}}(x_i) \rangle | x_i \in X\}; \forall i = 1, 2, \dots, n$ be two SFSs so that $\mu(x_i)$, $\vartheta(x_i)$ and $I(x_i)$ are membership, non-membership, and indeterminacy degrees, respectively. Then, Spherical Fuzzy Distance-based f-Similarity (Linear function) is given as follows:

$$S_{SFS}^{\square}(\tilde{A}, \tilde{B}) = 1 - d_{SFS}(\tilde{A}_s, \tilde{B}_s) \quad (4.19)$$

Proof: To prove Definition 4.12, will use properties in Definition 4.10 to show that Eq. (4.19) satisfies the properties, so will have:

$$\text{i. } 0 \leq S_{SFS}^{\square}(\tilde{A}, \tilde{B}) \leq 1$$

We know that $0 \leq d_{SFS}(\tilde{A}, \tilde{B}) \leq 1$ then $0 \leq S_{SFS}^{\square}(\tilde{A}, \tilde{B}) = 1 - d_{SFS}(\tilde{A}, \tilde{B}) \leq 1$.

$$\text{ii. } S_{SFS}^{\square}(\tilde{A}, \tilde{B}) = S_{SFS}^{\square}(\tilde{B}, \tilde{A})$$

We know that $d_{SFS}(\tilde{A}, \tilde{B}) = d_{SFS}(\tilde{B}, \tilde{A})$, then it is straightforward that $S_{SFS}^{\square}(\tilde{A}, \tilde{B}) = S_{SFS}^{\square}(\tilde{B}, \tilde{A})$.

$$\text{iii. } S_{SFS}^{\square}(\tilde{A}, \tilde{B}) = 1 \text{ iff } \tilde{A} = \tilde{B}$$

We know that if $\tilde{A} = \tilde{B}$ then $d_{SFS}(\tilde{A}, \tilde{B}) = 0$ and $S_{SFS}^{\square}(\tilde{A}, \tilde{B}) = 1 - d_{SFS}(\tilde{A}, \tilde{B}) = 1$.

Spherical fuzzy normalized Minkowski distance-based f-similarity (linear function) measure

Definition 4.13 Suppose that there are two sets of SFSs $\tilde{A}_s = \{x_i \langle \mu_{\tilde{A}}(x_i), \vartheta_{\tilde{A}}(x_i), I_{\tilde{A}}(x_i) \rangle | x_i \in X\}; \forall i = 1, 2, \dots, n$ and $\tilde{B}_s = \{x_i \langle \mu_{\tilde{B}}(x_i), \vartheta_{\tilde{B}}(x_i), I_{\tilde{B}}(x_i) \rangle | x_i \in X\}; \forall i = 1, 2, \dots, n$ be two SFSs so that $\mu(x_i)$, $\vartheta(x_i)$ and $I(x_i)$ are membership, non-membership, and indeterminacy degrees, respectively. Then, the Spherical Fuzzy Normalized Minkowski Distance-based f-Similarity (Linear function) measure is given as follows:

$$MS_{SFS}^{\square}(\tilde{A}, \tilde{B}) = 1 - Md_{SFS}(\tilde{A}, \tilde{B}) = 1 - \sqrt[\alpha]{\frac{1}{2n} \sum_{i=1}^n [|\mu_{\tilde{A}}^2(x_i) - \mu_{\tilde{B}}^2(x_i)|^{\alpha} + |\vartheta_{\tilde{A}}^2(x_i) - \vartheta_{\tilde{B}}^2(x_i)|^{\alpha} + |I_{\tilde{A}}^2(x_i) - I_{\tilde{B}}^2(x_i)|^{\alpha}]} \quad (4.20)$$

Where $\alpha \geq 1$.

Proof: Proof of Definition 4.13 is similar to the proof of Definition 4.12, hence, omit this proof.

Remark 4.6 It is obvious Spherical Fuzzy Normalized Hamming Distance-based f-Similarity (Linear function) and Spherical Fuzzy Normalized Euclidean Distance based f-Similarity (Linear function) could result in Spherical Fuzzy Normalized Minkowski Distance-based f-Similarity (Linear function) $MS_{SFS}^{\square}$, by changing the value α in Eq. (4.20) as follows:

- i. If $\alpha = 1$ Spherical Fuzzy Normalized Minkowski Distance-based f-Similarity (Linear function) $MS_{SFS}^{[l]}$ will turn into Spherical Fuzzy Normalized Hamming Distance-based f-Similarity (Linear function) $HS_{SFS}^{[l]}$:

$$HS_{SFS}^{[l]}(\tilde{A}, \tilde{B}) = 1 - \left(\frac{1}{2n} \sum_{i=1}^n [|\mu_{\tilde{A}}^2(x_i) - \mu_{\tilde{B}}^2(x_i)| + |\vartheta_{\tilde{A}}^2(x_i) - \vartheta_{\tilde{B}}^2(x_i)| + |I_{\tilde{A}}^2(x_i) - I_{\tilde{B}}^2(x_i)|] \right) \quad (4.21)$$

- ii. If $\alpha = 2$ Spherical Fuzzy Normalized Minkowski Distance-based f-Similarity (Linear function) $MS_{SFS}^{[l]}$ will turn into Spherical Fuzzy Normalized Euclidean Distance based f-Similarity (Linear function) $ES_{SFS}^{[l]}$:

$$ES_{SFS}^{[l]}(\tilde{A}, \tilde{B}) = 1 - \sqrt{\frac{1}{2n} \sum_{i=1}^n \left[\left(\mu_{\tilde{A}}^2(x_i) - \mu_{\tilde{B}}^2(x_i) \right)^2 + \left(\vartheta_{\tilde{A}}^2(x_i) - \vartheta_{\tilde{B}}^2(x_i) \right)^2 + \left(I_{\tilde{A}}^2(x_i) - I_{\tilde{B}}^2(x_i) \right)^2 \right]} \quad (4.22)$$

Spherical fuzzy weighted Minkowski distance-based f-similarity (linear function) measure

Definition 4.14 Suppose that there are two sets of SFSs $\tilde{A}_s = \{x_i \langle \mu_{\tilde{A}}(x_i), \vartheta_{\tilde{A}}(x_i), I_{\tilde{A}}(x_i) \rangle | x_i \in X\}; \forall i = 1, 2, \dots, n$ and $\tilde{B}_s = \{x_i \langle \mu_{\tilde{B}}(x_i), \vartheta_{\tilde{B}}(x_i), I_{\tilde{B}}(x_i) \rangle | x_i \in X\}; \forall i = 1, 2, \dots, n$ be two SFSs with corresponding weight vector $w = \{w_1, w_2, \dots, w_n\}$, where $w_i \in [0, 1]$ and $\sum_{i=1}^n w_i = 1, \forall i = 1, 2, \dots, n$, so that $\mu(x_i)$, $\vartheta(x_i)$ and $I(x_i)$ are membership, non-membership, and indeterminacy degrees, respectively. Then, Spherical Fuzzy Weighted Minkowski Distance-based f-Similarity (Linear function) $WMS_{SFS}^{[l]}$ given as follows:

$$WMS_{SFS}^{[l]}(\tilde{A}, \tilde{B}) = 1 - WMD_{SFS}(\tilde{A}, \tilde{B}) = 1 - \sqrt[\alpha]{\frac{1}{2} \sum_{i=1}^n w_i [|\mu_{\tilde{A}}^2(x_i) - \mu_{\tilde{B}}^2(x_i)|^\alpha + |\vartheta_{\tilde{A}}^2(x_i) - \vartheta_{\tilde{B}}^2(x_i)|^\alpha + |I_{\tilde{A}}^2(x_i) - I_{\tilde{B}}^2(x_i)|^\alpha]} \quad (4.23)$$

Where $\alpha \geq 1$.

Proof: Proof of Definition 4.14 is similar to the proof of Definition 4.12, hence, omit this proof.

Remark 4.7 It is obvious that Spherical Fuzzy Weighted Hamming Distance-based f-Similarity (Linear function) $WHS_{SFS}^{[l]}$, and Spherical Fuzzy Weighted Euclidean

Distance based f-Similarity (Linear function) $WES_{SFS}^{\square}$ could result from Spherical Fuzzy Weighted Minkowski Distance-based f-Similarity (Linear function) $WMS_{SFS}^{\square}$, by changing the value α in Eq. (4.23) as follows:

- i. If $\alpha = 1$ Spherical Fuzzy Weighted Minkowski Distance-based f-Similarity (Linear function) $WMS_{SFS}^{\square}$ will turn into Spherical Fuzzy Weighted Hamming Distance-based f-Similarity (Linear function) $WHS_{SFS}^{\square}$:

$$WHS_{SFS}^{\square}(\tilde{A}, \tilde{B}) = 1 - \left(\frac{1}{2} \sum_{i=1}^n w_i [|\mu_{\tilde{A}}^2(x_i) - \mu_{\tilde{B}}^2(x_i)| + |\vartheta_{\tilde{A}}^2(x_i) - \vartheta_{\tilde{B}}^2(x_i)| + |I_{\tilde{A}}^2(x_i) - I_{\tilde{B}}^2(x_i)|] \right) \quad (4.24)$$

- ii. If $\alpha = 2$ Spherical Fuzzy Weighted Minkowski Distance-based f-Similarity (Linear function) $WMS_{SFS}^{\square}$ will turn into Spherical Fuzzy Weighted Euclidean Distance based f-Similarity (Linear function) $WES_{SFS}^{\square}$:

$$WES_{SFS}^{\square}(\tilde{A}, \tilde{B}) = 1 - \sqrt{\frac{1}{2} \sum_{i=1}^n w_i \left[\left(\mu_{\tilde{A}}^2(x_i) - \mu_{\tilde{B}}^2(x_i) \right)^2 + \left(\vartheta_{\tilde{A}}^2(x_i) - \vartheta_{\tilde{B}}^2(x_i) \right)^2 + \left(I_{\tilde{A}}^2(x_i) - I_{\tilde{B}}^2(x_i) \right)^2 \right]} \quad (4.25)$$

Spherical fuzzy weighted Minkowski-Hausdorff distance-based f-similarity (linear function) measure

Definition 4.15 Suppose that there are two sets of SFSs $\tilde{A}_s = \{x_i \langle \mu_{\tilde{A}}(x_i), \vartheta_{\tilde{A}}(x_i), I_{\tilde{A}}(x_i) \rangle | x_i \in X\}; \forall i = 1, 2, \dots, n$ and $\tilde{B}_s = \{x_i \langle \mu_{\tilde{B}}(x_i), \vartheta_{\tilde{B}}(x_i), I_{\tilde{B}}(x_i) \rangle | x_i \in X\}; \forall i = 1, 2, \dots, n$ be two SFSs with corresponding weight vector $w = \{w_1, w_2, \dots, w_n\}$, where $w_i \in [0, 1]$ and $\sum_{i=1}^n w_i = 1, \forall i = 1, 2, \dots, n$, so that $\mu(x_i)$, $\vartheta(x_i)$ and $I(x_i)$ are membership, non-membership, and indeterminacy degrees, respectively. Then, Spherical Fuzzy Weighted Minkowski-Hausdorff Distance-based f-Similarity (Linear function) is given as follows:

$$WMHS_{SFS}^{\square}(\tilde{A}, \tilde{B}) = 1 - WMHd_{SFS}(\tilde{A}, \tilde{B}) = 1 - \sqrt[\alpha]{\frac{1}{2} \sum_{i=1}^n w_i \left(\max_i [|\mu_{\tilde{A}}^2(x_i) - \mu_{\tilde{B}}^2(x_i)|^\alpha, |\vartheta_{\tilde{A}}^2(x_i) - \vartheta_{\tilde{B}}^2(x_i)|^\alpha, |I_{\tilde{A}}^2(x_i) - I_{\tilde{B}}^2(x_i)|^\alpha] \right)} \quad (4.26)$$

Where $\alpha \geq 1$.

Proof: Proof of Definition 4.15 is similar to the proof of Definition 4.12, hence, omit this proof.

Remark 4.7 It is obvious that Spherical Fuzzy Weighted Hamming-Hausdorff Distance-based f-Similarity (Linear function) $WHHS_{SFS}^{\square}$ and Spherical Fuzzy Weighted Euclidean-Hausdorff Distance-based f-Similarity (Linear function) $WEHS_{SFS}^{\square}$ could result from Spherical Fuzzy Weighted Minkowski-Hausdorff Distance-based f-Similarity (Linear function) $WMHS_{SFS}^{\square}$, by changing the value for α in Eq. (4.26) as follows:

- i. If $\alpha = 1$ Spherical Fuzzy Weighted Minkowski-Hausdorff Distance-based f-Similarity (Linear function) $WMHS_{SFS}^{\square}$ will turn into Spherical Fuzzy Weighted Hamming-Hausdorff Distance-based f-Similarity (Linear function) $WHHS_{SFS}^{\square}$:

$$WHHS_{SFS}^{\square}(\tilde{A}, \tilde{B}) = 1 - \left(\frac{1}{2} \sum_{i=1}^n w_i \left(\max_i \left[|\mu_{\tilde{A}}^2(x_i) - \mu_{\tilde{B}}^2(x_i)|, |\vartheta_{\tilde{A}}^2(x_i) - \vartheta_{\tilde{B}}^2(x_i)|, |I_{\tilde{A}}^2(x_i) - I_{\tilde{B}}^2(x_i)| \right] \right) \right) \quad (4.27)$$

- ii. If $\alpha = 2$ Spherical Fuzzy Weighted Minkowski-Hausdorff Distance-based f-Similarity (Linear function) $WMHS_{SFS}^{\square}$ will turn into Spherical Fuzzy Weighted Euclidean-Hausdorff Distance-based f-Similarity (Linear function) $WEHS_{SFS}^{\square}$:

$$WEHS_{SFS}^{\square}(\tilde{A}, \tilde{B}) = 1 - \sqrt{\frac{1}{2} \sum_{i=1}^n w_i \left(\max_i \left[\left(\mu_{\tilde{A}}^2(x_i) - \mu_{\tilde{B}}^2(x_i) \right)^2, \left(\vartheta_{\tilde{A}}^2(x_i) - \vartheta_{\tilde{B}}^2(x_i) \right)^2, \left(I_{\tilde{A}}^2(x_i) - I_{\tilde{B}}^2(x_i) \right)^2 \right] \right)} \quad (4.28)$$

Spherical fuzzy normalized Minkowski-Hausdorff distance-based f-similarity (linear function) measure

Definition 4.16 Suppose that there are two sets of SFSs $\tilde{A}_s = \{x_i \langle \mu_{\tilde{A}}(x_i), \vartheta_{\tilde{A}}(x_i), I_{\tilde{A}}(x_i) \rangle | x_i \in X\}; \forall i = 1, 2, \dots, n$ and $\tilde{B}_s = \{x_i \langle \mu_{\tilde{B}}(x_i), \vartheta_{\tilde{B}}(x_i), I_{\tilde{B}}(x_i) \rangle | x_i \in X\}; \forall i = 1, 2, \dots, n$ be two SFSs so that $\mu(x_i)$, $\vartheta(x_i)$ and $I(x_i)$ are membership, non-membership, and indeterminacy degrees, respectively.

Then, Spherical Fuzzy Normalized Minkowski-Hausdorff Distance-based f-Similarity (Linear function) is given as follows:

$$MHS_{SFS}^{\square}(\tilde{A}, \tilde{B}) = 1 - Mhd_{SFS}(\tilde{A}, \tilde{B}) = 1 - \sqrt[\alpha]{\frac{1}{2n} \sum_{i=1}^n \max_i [|\mu_{\tilde{A}}^2(x_i) - \mu_{\tilde{B}}^2(x_i)|^\alpha, |\vartheta_{\tilde{A}}^2(x_i) - \vartheta_{\tilde{B}}^2(x_i)|^\alpha, |I_{\tilde{A}}^2(x_i) - I_{\tilde{B}}^2(x_i)|^\alpha]} \quad (4.29)$$

Where $\alpha \geq 1$.

Proof: Proof of Definition 4.16 is similar to the proof of Definition 4.12, hence, omit this proof.

Remark 4.8 It is obvious that Spherical Fuzzy Normalized Hamming-Hausdorff Distance-based f-Similarity (Linear function) $HHS_{SFS}^{\square}$ and Spherical Fuzzy Normalized Euclidean-Hausdorff Distance-based f-Similarity (Linear function) $EHS_{SFS}^{\square}$ could result from Spherical Fuzzy Normalized Minkowski-Hausdorff Distance-based f-Similarity (Linear function) $MHS_{SFS}^{\square}$, by changing the value for α in Eq. (4.29) as follows:

- i. If $\alpha = 1$ Spherical Fuzzy Normalized Minkowski-Hausdorff Distance-based f-Similarity (Linear function) $MHS_{SFS}^{\square}$ will turn into Spherical Fuzzy Normalized Hamming-Hausdorff Distance-based f-Similarity (Linear function) $HHS_{SFS}^{\square}$:

$$HHS_{SFS}^{\square}(\tilde{A}, \tilde{B}) = 1 - \left(\frac{1}{2n} \sum_{i=1}^n \max_i [|\mu_{\tilde{A}}^2(x_i) - \mu_{\tilde{B}}^2(x_i)|, |\vartheta_{\tilde{A}}^2(x_i) - \vartheta_{\tilde{B}}^2(x_i)|, |I_{\tilde{A}}^2(x_i) - I_{\tilde{B}}^2(x_i)|] \right) \quad (4.30)$$

- ii. If $\alpha = 2$ Spherical Fuzzy Normalized Minkowski-Hausdorff Distance-based f-Similarity (Linear function) $MHS_{SFS}^{\square}$ will turn into Spherical Fuzzy Normalized Euclidean-Hausdorff Distance-based f-Similarity (Linear function) $EHS_{SFS}^{\square}$:

$$EHS_{SFS}^{\square}(\tilde{A}, \tilde{B}) = 1 - \sqrt{\frac{1}{2n} \sum_{i=1}^n \max_i \left[\left(\mu_{\tilde{A}}^2(x_i) - \mu_{\tilde{B}}^2(x_i) \right)^2, \left(\vartheta_{\tilde{A}}^2(x_i) - \vartheta_{\tilde{B}}^2(x_i) \right)^2, \left(I_{\tilde{A}}^2(x_i) - I_{\tilde{B}}^2(x_i) \right)^2 \right]} \quad (4.31)$$

4.3.1.2 Spherical fuzzy distance-based f-similarity (exponential function) measure

Definition 4.17 Suppose that there are two sets of SFSs $\tilde{A}_s = \{x_i \langle \mu_{\tilde{A}}(x_i), \vartheta_{\tilde{A}}(x_i), I_{\tilde{A}}(x_i) \rangle | x_i \in X\}; \forall i = 1, 2, \dots, n$ and $\tilde{B}_s = \{x_i \langle \mu_{\tilde{B}}(x_i), \vartheta_{\tilde{B}}(x_i), I_{\tilde{B}}(x_i) \rangle | x_i \in X\}; \forall i = 1, 2, \dots, n$ be two SFSs so that $\mu(x_i)$, $\vartheta(x_i)$ and $I(x_i)$ are membership, non-membership, and indeterminacy degrees, respectively. Then, the Spherical Fuzzy Distance-based f-Similarity (Exponential function) measure is given as follows:

$$S_{SFS}^{\square}(\tilde{A}, \tilde{B}) = \frac{e^{-d_{SFS}(\tilde{A}_s, \tilde{B}_s)} - e^{-1}}{e^0 - e^{-1}} \quad (4.32)$$

Proof: To prove Definition 4.17, use properties in Definition 4.10 to show that Eq. (4.32) satisfies the properties, so will have:

$$i. \quad 0 \leq S_{SFS}^{\square}(\tilde{A}, \tilde{B}) \leq 1$$

We know that $0 \leq d_{SFS}(\tilde{A}, \tilde{B}) \leq 1$ then $e^{-1} \leq e^{-d_{SFS}(\tilde{A}_s, \tilde{B}_s)} \leq e^0$, will have:

$$0 \leq e^{-d_{SFS}(\tilde{A}_s, \tilde{B}_s)} - e^{-1} \leq e^0 - e^{-1}, \text{ so:}$$

$$0 \leq S_{SFS}^{\square}(\tilde{A}, \tilde{B}) \leq 1$$

$$ii. \quad S_{SFS}^{\square}(\tilde{A}, \tilde{B}) = S_{SFS}^{\square}(\tilde{B}, \tilde{A})$$

We know that $d_{SFS}(\tilde{A}, \tilde{B}) = d_{SFS}(\tilde{B}, \tilde{A})$, then it is straightforward that $S_{SFS}^{\square}(\tilde{A}, \tilde{B}) = S_{SFS}^{\square}(\tilde{B}, \tilde{A})$.

$$iii. \quad S_{SFS}^{\square}(\tilde{A}, \tilde{B}) = 1 \text{ iff } \tilde{A} = \tilde{B}$$

We know that if $\tilde{A} = \tilde{B}$ then $d_{SFS}(\tilde{A}, \tilde{B}) = 0$ and $S_{SFS}^{\square}(\tilde{A}, \tilde{B}) = \frac{e^{-d_{SFS}(\tilde{A}_s, \tilde{B}_s)} - e^{-1}}{e^0 - e^{-1}} = 1$.

Spherical fuzzy normalized Minkowski distance-based f-similarity (exponential function) measure

Definition 4.18 Suppose that there are two sets of SFSs $\tilde{A}_s = \{x_i \langle \mu_{\tilde{A}}(x_i), \vartheta_{\tilde{A}}(x_i), I_{\tilde{A}}(x_i) \rangle | x_i \in X\}; \forall i = 1, 2, \dots, n$ and $\tilde{B}_s = \{x_i \langle \mu_{\tilde{B}}(x_i), \vartheta_{\tilde{B}}(x_i), I_{\tilde{B}}(x_i) \rangle | x_i \in X\}; \forall i = 1, 2, \dots, n$ be two SFSs so that $\mu(x_i)$, $\vartheta(x_i)$ and $I(x_i)$ are membership, non-membership, and indeterminacy degrees, respectively.

Then, Spherical Fuzzy Normalized Minkowski Distance-based f-Similarity (Exponential function) is given as follows:

$$MS_{SFS}^{[e]}(\tilde{A}, \tilde{B}) = \frac{e^{-Md_{SFS}(\tilde{A}, \tilde{B})} - e^{-1}}{e^0 - e^{-1}} \quad (4.33)$$

$$= \frac{e^{\alpha \sqrt{\frac{1}{2n} \sum_{i=1}^n \left[\left| \mu_{\tilde{A}}^2(x_i) - \mu_{\tilde{B}}^2(x_i) \right|^\alpha + \left| \vartheta_{\tilde{A}}^2(x_i) - \vartheta_{\tilde{B}}^2(x_i) \right|^\alpha + \left| I_{\tilde{A}}^2(x_i) - I_{\tilde{B}}^2(x_i) \right|^\alpha \right]}} - e^{-1}}{e^0 - e^{-1}}$$

where $\alpha \geq 1$.

Proof: Proof of Definition 4.18 is similar to the proof of Definition 4.17, hence, omit this proof.

Remark 4.9 It is obvious Spherical Fuzzy Normalized Hamming Distance-based f-Similarity (Exponential function) and Spherical Fuzzy Normalized Euclidean Distance based f-Similarity (Exponential function) could result in Spherical Fuzzy Normalized Minkowski Distance-based f-Similarity (Exponential function) $MS_{SFS}^{[e]}$, by changing the value α in Eq. (4.33) as follows:

- i. If $\alpha = 1$ Spherical Fuzzy Normalized Minkowski Distance-based f-Similarity (Exponential function) $MS_{SFS}^{[e]}$ will turn into Spherical Fuzzy Normalized Hamming Distance-based f-Similarity (Exponential function) $HS_{SFS}^{[e]}$:

$$HS_{SFS}^{[e]}(\tilde{A}, \tilde{B}) = \frac{e^{\left(\frac{1}{2n} \sum_{i=1}^n \left[\left| \mu_{\tilde{A}}^2(x_i) - \mu_{\tilde{B}}^2(x_i) \right| + \left| \vartheta_{\tilde{A}}^2(x_i) - \vartheta_{\tilde{B}}^2(x_i) \right| + \left| I_{\tilde{A}}^2(x_i) - I_{\tilde{B}}^2(x_i) \right| \right] \right)} - e^{-1}}{e^0 - e^{-1}} \quad (4.34)$$

- ii. If $\alpha = 2$ Spherical Fuzzy Normalized Minkowski Distance-based f-Similarity (Exponential function) $MS_{SFS}^{[e]}$ will turn into Spherical Fuzzy Normalized Euclidean Distance based f-Similarity (Exponential function) $ES_{SFS}^{[e]}$:

$$ES_{SFS}^{[e]}(\tilde{A}, \tilde{B}) = \frac{e^{\sqrt{\frac{1}{2n} \sum_{i=1}^n \left[\left(\mu_{\tilde{A}}^2(x_i) - \mu_{\tilde{B}}^2(x_i) \right)^2 + \left(\vartheta_{\tilde{A}}^2(x_i) - \vartheta_{\tilde{B}}^2(x_i) \right)^2 + \left(I_{\tilde{A}}^2(x_i) - I_{\tilde{B}}^2(x_i) \right)^2 \right]}} - e^{-1}}{e^0 - e^{-1}} \quad (4.35)$$

Spherical fuzzy weighted Minkowski distance-based f-similarity (exponential function) measure

Definition 4.19 Suppose that there are two sets of SFSs $\tilde{A}_S = \{x_i \langle \mu_{\tilde{A}}(x_i), \vartheta_{\tilde{A}}(x_i), I_{\tilde{A}}(x_i) \rangle | x_i \in X\}; \forall i = 1, 2, \dots, n$ and $\tilde{B}_S = \{x_i \langle \mu_{\tilde{B}}(x_i), \vartheta_{\tilde{B}}(x_i), I_{\tilde{B}}(x_i) \rangle | x_i \in X\}; \forall i = 1, 2, \dots, n$ be two SFSs with corresponding weight vector $w = \{w_1, w_2, \dots, w_n\}$, where $w_i \in [0, 1]$ and $\sum_{i=1}^n w_i = 1, \forall i =$

$1, 2, \dots, n$, so that $\mu(x_i)$, $\vartheta(x_i)$ and $I(x_i)$ are membership, non-membership, and indeterminacy degrees, respectively. Then, Spherical Fuzzy Weighted Minkowski Distance-based f-Similarity (Exponential function) $WMS_{SFS}^{[e]}$ given as follows:

$$WMS_{SFS}^{[e]}(\tilde{A}, \tilde{B}) = \frac{e^{-WMS_{SFS}(\tilde{A}, \tilde{B}) - e^{-1}}}{e^0 - e^{-1}} = \frac{e^{\sqrt[\alpha]{\frac{1}{2} \sum_{i=1}^n w_i \left[|\mu_{\tilde{A}}^2(x_i) - \mu_{\tilde{B}}^2(x_i)|^\alpha + |\vartheta_{\tilde{A}}^2(x_i) - \vartheta_{\tilde{B}}^2(x_i)|^\alpha + |I_{\tilde{A}}^2(x_i) - I_{\tilde{B}}^2(x_i)|^\alpha \right]} - e^{-1}}}{e^0 - e^{-1}} \quad (4.36)$$

Where $\alpha \geq 1$.

Proof: Proof of Definition 4.19 is similar to the proof of Definition 4.17, hence, omit this proof.

Remark 4.10 It is obvious that Spherical Fuzzy Weighted Hamming Distance-based f-Similarity (Exponential function) $WHS_{SFS}^{[e]}$ and Spherical Fuzzy Weighted Euclidean Distance based f-Similarity (Exponential function) $WES_{SFS}^{[e]}$ could result from Spherical Fuzzy Weighted Minkowski Distance-based f-Similarity (Exponential function) $WMS_{SFS}^{[e]}$, by changing the value α in Eq. (4.36) as follows:

- i. If $\alpha = 1$ Spherical Fuzzy Weighted Minkowski Distance-based f-Similarity (Exponential function) $WMS_{SFS}^{[e]}$ will turn into Spherical Fuzzy Weighted Hamming Distance-based f-Similarity (Exponential function) $WHS_{SFS}^{[e]}$:

$$WHS_{SFS}^{[e]}(\tilde{A}, \tilde{B}) = \frac{e^{\left(\frac{1}{2} \sum_{i=1}^n w_i \left[|\mu_{\tilde{A}}^2(x_i) - \mu_{\tilde{B}}^2(x_i)| + |\vartheta_{\tilde{A}}^2(x_i) - \vartheta_{\tilde{B}}^2(x_i)| + |I_{\tilde{A}}^2(x_i) - I_{\tilde{B}}^2(x_i)| \right] \right) - e^{-1}}}{e^0 - e^{-1}} \quad (4.37)$$

- ii. If $\alpha = 2$ Spherical Fuzzy Weighted Minkowski Distance-based f-Similarity (Exponential function) $WMS_{SFS}^{[e]}$ will turn into Spherical Fuzzy Weighted Euclidean Distance based f-Similarity (Exponential function) $WES_{SFS}^{[e]}$:

$$WES_{SFS}^{[e]}(\tilde{A}, \tilde{B}) = \frac{e^{\sqrt{\frac{1}{2} \sum_{i=1}^n w_i \left[\left(\mu_{\tilde{A}}^2(x_i) - \mu_{\tilde{B}}^2(x_i) \right)^2 + \left(\vartheta_{\tilde{A}}^2(x_i) - \vartheta_{\tilde{B}}^2(x_i) \right)^2 + \left(I_{\tilde{A}}^2(x_i) - I_{\tilde{B}}^2(x_i) \right)^2 \right]} - e^{-1}}}{e^0 - e^{-1}} \quad (4.38)$$

Spherical fuzzy normalized Minkowski-Hausdorff distance-based f-similarity (exponential function) measure

Definition 4.20 Suppose that there are two sets of SFSs $\tilde{A}_s = \{x_i \langle \mu_{\tilde{A}}(x_i), \vartheta_{\tilde{A}}(x_i), I_{\tilde{A}}(x_i) \rangle | x_i \in X\}; \forall i = 1, 2, \dots, n$ and $\tilde{B}_s = \{x_i \langle \mu_{\tilde{B}}(x_i), \vartheta_{\tilde{B}}(x_i), I_{\tilde{B}}(x_i) \rangle | x_i \in X\}; \forall i = 1, 2, \dots, n$ be two SFSs so that $\mu(x_i)$, $\vartheta(x_i)$ and $I(x_i)$ are membership, non-membership, and indeterminacy degrees, respectively. Then, Spherical Fuzzy Normalized Minkowski-Hausdorff Distance-based f-Similarity (Exponential function) is given as follows:

$$MHS_{SFS}^{\boxed{e}}(\tilde{A}, \tilde{B}) = \frac{e^{-MHd_{SFS}(\tilde{A}_s, \tilde{B}_s) - e^{-1}}}{e^0 - e^{-1}} = \frac{e^{\sqrt{\frac{1}{2n} \sum_{i=1}^n \max_i [|\mu_{\tilde{A}}^2(x_i) - \mu_{\tilde{B}}^2(x_i)|^\alpha, |\vartheta_{\tilde{A}}^2(x_i) - \vartheta_{\tilde{B}}^2(x_i)|^\alpha, |I_{\tilde{A}}^2(x_i) - I_{\tilde{B}}^2(x_i)|^\alpha] - e^{-1}}}}}{e^0 - e^{-1}} \quad (4.39)$$

Where $\alpha \geq 1$.

Proof: Proof of Definition 4.20 is similar to the proof of Definition 4.17, hence, omit this proof.

Remark 4.11 It is obvious that Spherical Fuzzy Normalized Hamming-Hausdorff Distance-based f-Similarity (Exponential function) $HHS_{SFS}^{\boxed{e}}$ and Spherical Fuzzy Normalized Euclidean-Hausdorff Distance-based f-Similarity (Exponential function) $EHS_{SFS}^{\boxed{e}}$ could result from Spherical Fuzzy Normalized Minkowski-Hausdorff Distance-based f-Similarity (Exponential function) $MHS_{SFS}^{\boxed{e}}$, by changing the value for α in Eq. (4.39) as follows:

- i. If $\alpha = 1$ Spherical Fuzzy Normalized Minkowski-Hausdorff Distance-based f-Similarity (Exponential function) $MHS_{SFS}^{\boxed{e}}$ will turn into Spherical Fuzzy Normalized Hamming-Hausdorff Distance-based f-Similarity (Exponential function) $HHS_{SFS}^{\boxed{e}}$:

$$HHS_{SFS}^{\boxed{e}}(\tilde{A}, \tilde{B}) = \frac{e^{(\frac{1}{2n} \sum_{i=1}^n \max_i [|\mu_{\tilde{A}}^2(x_i) - \mu_{\tilde{B}}^2(x_i)|, |\vartheta_{\tilde{A}}^2(x_i) - \vartheta_{\tilde{B}}^2(x_i)|, |I_{\tilde{A}}^2(x_i) - I_{\tilde{B}}^2(x_i)|]) - e^{-1}}}{e^0 - e^{-1}} \quad (4.40)$$

- ii. If $\alpha = 2$ Spherical Fuzzy Normalized Minkowski-Hausdorff Distance-based f-Similarity (Exponential function) $MHS_{SFS}^{\boxed{e}}$ will turn into Spherical Fuzzy Normalized Euclidean-Hausdorff Distance-based f-Similarity (Exponential function) $EHS_{SFS}^{\boxed{e}}$:

$$\begin{aligned}
& EHS_{SFS}^{[e]}(\tilde{A}, \tilde{B}) \\
&= \frac{e^{\sqrt{\frac{1}{2n} \sum_{i=1}^n \max_i \left[\left(\mu_{\tilde{A}}^2(x_i) - \mu_{\tilde{B}}^2(x_i) \right)^2, \left(\vartheta_{\tilde{A}}^2(x_i) - \vartheta_{\tilde{B}}^2(x_i) \right)^2, \left(I_{\tilde{A}}^2(x_i) - I_{\tilde{B}}^2(x_i) \right)^2 \right]} } - e^{-1}}{e^0 - e^{-1}} \quad (4.41)
\end{aligned}$$

Spherical fuzzy weighted Minkowski-Hausdorff distance-based f-similarity (exponential function) measure

Definition 4.21 Suppose that there are two sets of SFSs $\tilde{A}_s = \{x_i \langle \mu_{\tilde{A}}(x_i), \vartheta_{\tilde{A}}(x_i), I_{\tilde{A}}(x_i) \rangle | x_i \in X\}; \forall i = 1, 2, \dots, n$ and $\tilde{B}_s = \{x_i \langle \mu_{\tilde{B}}(x_i), \vartheta_{\tilde{B}}(x_i), I_{\tilde{B}}(x_i) \rangle | x_i \in X\}; \forall i = 1, 2, \dots, n$ be two SFSs with corresponding weight vector $w = \{w_1, w_2, \dots, w_n\}$, where $w_i \in [0, 1]$ and $\sum_{i=1}^n w_i = 1, \forall i = 1, 2, \dots, n$, so that $\mu(x_i)$, $\vartheta(x_i)$ and $I(x_i)$ are membership, non-membership, and indeterminacy degrees, respectively. Then, Spherical Fuzzy Weighted Minkowski-Hausdorff Distance-based f-Similarity (Exponential function) is given as follows:

$$\begin{aligned}
WMHS_{SFS}^{[e]}(\tilde{A}, \tilde{B}) &= \frac{e^{-WMHd_{SFS}(\tilde{A}_s, \tilde{B}_s) - e^{-1}}}{e^0 - e^{-1}} = \\
&= \frac{e^{\sqrt{\frac{1}{2} \sum_{i=1}^n w_i \left(\max_i \left[\left| \mu_{\tilde{A}}^2(x_i) - \mu_{\tilde{B}}^2(x_i) \right|^\alpha, \left| \vartheta_{\tilde{A}}^2(x_i) - \vartheta_{\tilde{B}}^2(x_i) \right|^\alpha, \left| I_{\tilde{A}}^2(x_i) - I_{\tilde{B}}^2(x_i) \right|^\alpha \right] \right)} } - e^{-1}}{e^0 - e^{-1}} \quad (4.42)
\end{aligned}$$

Where $\alpha \geq 1$.

Proof: Proof of Definition 4.21 is similar to the proof of Definition 4.17, hence, omit this proof.

Remark 4.12 It is obvious that Spherical Fuzzy Weighted Hamming-Hausdorff Distance-based f-Similarity (Exponential function) $WHHS_{SFS}^{[e]}$ and Spherical Fuzzy Weighted Euclidean-Hausdorff Distance-based f-Similarity (Exponential function) $WEHS_{SFS}^{[e]}$ could result from Spherical Fuzzy Weighted Minkowski-Hausdorff Distance-based f-Similarity (Exponential function) $WMHS_{SFS}^{[e]}$, by changing the value for α in Eq. (4.42) as follows:

- i. If $\alpha = 1$ Spherical Fuzzy Weighted Minkowski-Hausdorff Distance-based f-Similarity (Exponential function) $WMHS_{SFS}^{[e]}$ will turn into Spherical Fuzzy Weighted Hamming-Hausdorff Distance-based f-Similarity (Exponential function) $WHHS_{SFS}^{[e]}$:

$$WHHS_{SFS}^{\boxed{e}}(\tilde{A}, \tilde{B}) = \frac{e^{\left(\frac{1}{2} \sum_{i=1}^n w_i (\max_i [|\mu_{\tilde{A}}^2(x_i) - \mu_{\tilde{B}}^2(x_i)|, |\vartheta_{\tilde{A}}^2(x_i) - \vartheta_{\tilde{B}}^2(x_i)|, |I_{\tilde{A}}^2(x_i) - I_{\tilde{B}}^2(x_i)|])\right)} - e^{-1}}{e^0 - e^{-1}} \quad (4.43)$$

- ii. If $\alpha = 2$ Spherical Fuzzy Weighted Minkowski-Hausdorff Distance-based f-Similarity (Exponential function) $WMHS_{SFS}^{\boxed{e}}$ will turn into Spherical Fuzzy Weighted Euclidean-Hausdorff Distance-based f-Similarity (Exponential function) $WEHS_{SFS}^{\boxed{e}}$:

$$WEHS_{SFS}^{\boxed{e}}(\tilde{A}, \tilde{B}) = \frac{e^{\sqrt{\frac{1}{2} \sum_{i=1}^n w_i \left(\max_i \left[\left(\mu_{\tilde{A}}^2(x_i) - \mu_{\tilde{B}}^2(x_i) \right)^2, \left(\vartheta_{\tilde{A}}^2(x_i) - \vartheta_{\tilde{B}}^2(x_i) \right)^2, \left(I_{\tilde{A}}^2(x_i) - I_{\tilde{B}}^2(x_i) \right)^2 \right] \right)}} - e^{-1}}{e^0 - e^{-1}} \quad (4.44)$$

4.3.1.3 Spherical fuzzy distance-based f-similarity (logarithm function) measure

Definition 4.22 Suppose that there are two sets of SFSs $\tilde{A}_s = \{x_i \langle \mu_{\tilde{A}}(x_i), \vartheta_{\tilde{A}}(x_i), I_{\tilde{A}}(x_i) \rangle | x_i \in X\}; \forall i = 1, 2, \dots, n$ and $\tilde{B}_s = \{x_i \langle \mu_{\tilde{B}}(x_i), \vartheta_{\tilde{B}}(x_i), I_{\tilde{B}}(x_i) \rangle | x_i \in X\}; \forall i = 1, 2, \dots, n$ be two SFSs so that $\mu(x_i)$, $\vartheta(x_i)$ and $I(x_i)$ are membership, non-membership, and indeterminacy degrees, respectively. Then, Spherical Fuzzy Distance-based f-Similarity (Logarithm function) is given as follows:

$$S_{SFS}^{\boxed{\log_2}}(\tilde{A}, \tilde{B}) = \log_2 \left(2 - d_{SFS}(\tilde{A}_s, \tilde{B}_s) \right) \quad (4.45)$$

Proof: To prove Definition 4.22, we use properties in Definition 4.10 to show that Eq. (4.45) satisfies the properties, so will have:

$$i. \quad 0 \leq S_{SFS}^{\boxed{\log_2}}(\tilde{A}, \tilde{B}) \leq 1$$

We know that $0 \leq d_{SFS}(\tilde{A}, \tilde{B}) \leq 1$ then $1 \leq 2 - d_{SFS}(\tilde{A}_s, \tilde{B}_s) \leq 2$, so will have:

$$0 \leq S_{SFS}^{\boxed{\log_2}}(\tilde{A}, \tilde{B}) \leq 1$$

$$ii. \quad S_{SFS}^{\boxed{\log_2}}(\tilde{A}, \tilde{B}) = S_{SFS}^{\boxed{\log_2}}(\tilde{B}, \tilde{A})$$

We know that $d_{SFS}(\tilde{A}, \tilde{B}) = d_{SFS}(\tilde{B}, \tilde{A})$, then it is straightforward that $S_{SFS}^{\lceil \log_2 \rceil}(\tilde{A}, \tilde{B}) = S_{SFS}^{\lceil \log_2 \rceil}(\tilde{B}, \tilde{A})$.

$$\text{iii. } S_{SFS}^{\lceil \log_2 \rceil}(\tilde{A}, \tilde{B}) = 1 \text{ iff } \tilde{A} = \tilde{B}$$

We know that if $\tilde{A} = \tilde{B}$ then $d_{SFS}(\tilde{A}, \tilde{B}) = 0$ and $S_{SFS}^{\lceil \log_2 \rceil}(\tilde{A}, \tilde{B}) = \log_2(2 - d_{SFS}(\tilde{A}, \tilde{B})) = 1$.

Spherical fuzzy normalized Minkowski distance-based f-similarity (logarithm function) measure

Definition 4.23 Suppose that there are two sets of SFSs $\tilde{A}_s = \{x_i \langle \mu_{\tilde{A}}(x_i), \vartheta_{\tilde{A}}(x_i), I_{\tilde{A}}(x_i) \rangle | x_i \in X\}; \forall i = 1, 2, \dots, n$ and $\tilde{B}_s = \{x_i \langle \mu_{\tilde{B}}(x_i), \vartheta_{\tilde{B}}(x_i), I_{\tilde{B}}(x_i) \rangle | x_i \in X\}; \forall i = 1, 2, \dots, n$ be two SFSs so that $\mu(x_i)$, $\vartheta(x_i)$ and $I(x_i)$ are membership, non-membership, and indeterminacy degrees, respectively. Then, Spherical Fuzzy Normalized Minkowski Distance-based f-Similarity (Logarithm function) is given as follows:

$$MS_{SFS}^{\lceil \log_2 \rceil}(\tilde{A}, \tilde{B}) = \log_2(2 - Md_{SFS}(\tilde{A}_s, \tilde{B}_s)) = \log_2\left(2 - \sqrt[\alpha]{\frac{1}{2n} \sum_{i=1}^n [|\mu_{\tilde{A}}^2(x_i) - \mu_{\tilde{B}}^2(x_i)|^\alpha + |\vartheta_{\tilde{A}}^2(x_i) - \vartheta_{\tilde{B}}^2(x_i)|^\alpha + |I_{\tilde{A}}^2(x_i) - I_{\tilde{B}}^2(x_i)|^\alpha]}\right) \quad (4.46)$$

Where $\alpha \geq 1$.

Proof: Proof of Definition 4.23 is similar to the proof of Definition 4.22, hence, omit this proof.

Remark 4.13 It is obvious Spherical Fuzzy Normalized Hamming Distance-based f-Similarity (Logarithm function) $HS_{SFS}^{\lceil \log_2 \rceil}$ and Spherical Fuzzy Normalized Euclidean Distance based f-Similarity (Logarithm function) $ES_{SFS}^{\lceil \log_2 \rceil}$ could result in Spherical Fuzzy Normalized Minkowski Distance-based f-Similarity (Logarithm function) $MS_{SFS}^{\lceil \log_2 \rceil}$, by changing the value α in Eq. (4.46) as follows:

- i. If $\alpha = 1$ Spherical Fuzzy Normalized Minkowski Distance-based f-Similarity

(Logarithm function) $MS_{SFS}^{\lceil \log_2 \rceil}$ will turn into Spherical Fuzzy Normalized Hamming Distance-based f-Similarity (Logarithm function) $HS_{SFS}^{\lceil \log_2 \rceil}$:

$$HS_{SFS}^{\lceil \log_2 \rceil}(\tilde{A}, \tilde{B}) = \log_2 \left(2 - \left(\frac{1}{2n} \sum_{i=1}^n [|\mu_{\tilde{A}}^2(x_i) - \mu_{\tilde{B}}^2(x_i)| + |\vartheta_{\tilde{A}}^2(x_i) - \vartheta_{\tilde{B}}^2(x_i)| + |I_{\tilde{A}}^2(x_i) - I_{\tilde{B}}^2(x_i)|] \right) \right) \quad (4.47)$$

- ii. If $\alpha = 2$ Spherical Fuzzy Normalized Minkowski Distance-based f-Similarity

(Logarithm function) $MS_{SFS}^{\lceil \log_2 \rceil}$ will turn into Spherical Fuzzy Normalized Euclidean Distance based f-Similarity (Logarithm function) $ES_{SFS}^{\lceil \log_2 \rceil}$:

$$ES_{SFS}^{\lceil \log_2 \rceil}(\tilde{A}, \tilde{B}) = \log_2 \left(2 - \sqrt{\frac{1}{2n} \sum_{i=1}^n \left[\left(\mu_{\tilde{A}}^2(x_i) - \mu_{\tilde{B}}^2(x_i) \right)^2 + \left(\vartheta_{\tilde{A}}^2(x_i) - \vartheta_{\tilde{B}}^2(x_i) \right)^2 + \left(I_{\tilde{A}}^2(x_i) - I_{\tilde{B}}^2(x_i) \right)^2 \right]} \right) \quad (4.48)$$

Spherical Fuzzy Weighted Minkowski Distance Based f-Similarity (Logarithm function) Measure

Definition 4.24 Suppose that there are two sets of SFSs $\tilde{A}_s = \{x_i \langle \mu_{\tilde{A}}(x_i), \vartheta_{\tilde{A}}(x_i), I_{\tilde{A}}(x_i) \rangle | x_i \in X\}; \forall i = 1, 2, \dots, n$ and $\tilde{B}_s = \{x_i \langle \mu_{\tilde{B}}(x_i), \vartheta_{\tilde{B}}(x_i), I_{\tilde{B}}(x_i) \rangle | x_i \in X\}; \forall i = 1, 2, \dots, n$ be two SFSs with corresponding weight vector $w = \{w_1, w_2, \dots, w_n\}$, where $w_i \in [0, 1]$ and $\sum_{i=1}^n w_i = 1, \forall i = 1, 2, \dots, n$, so that $\mu(x_i)$, $\vartheta(x_i)$ and $I(x_i)$ are membership, non-membership, and indeterminacy degrees, respectively. Then, Spherical Fuzzy Weighted Minkowski

Distance-based f-Similarity (Logarithm function) $WMS_{SFS}^{\lceil \log_2 \rceil}$ given as follows:

$$WMS_{SFS}^{\lceil \log_2 \rceil}(\tilde{A}, \tilde{B}) = \log_2 \left(2 - WMD_{SFS}(\tilde{A}_s, \tilde{B}_s) \right) = \log_2 \left(2 - \sqrt[\alpha]{\frac{1}{2} \sum_{i=1}^n w_i [|\mu_{\tilde{A}}^2(x_i) - \mu_{\tilde{B}}^2(x_i)|^\alpha + |\vartheta_{\tilde{A}}^2(x_i) - \vartheta_{\tilde{B}}^2(x_i)|^\alpha + |I_{\tilde{A}}^2(x_i) - I_{\tilde{B}}^2(x_i)|^\alpha]} \right) \quad (4.49)$$

Where $\alpha \geq 1$.

Proof: Proof of Definition 4.24 is similar to the proof of Definition 4.22, hence, omit this proof.

Remark 4.14 It is obvious that Spherical Fuzzy Weighted Hamming Distance-based f-Similarity (Logarithm function) $WHS_{SFS}^{\log_2}$ and Spherical Fuzzy Weighted Euclidean Distance based f-Similarity (Logarithm function) $WES_{SFS}^{\log_2}$ could result from Spherical Fuzzy Weighted Minkowski Distance-based f-Similarity (Logarithm function) $WMS_{SFS}^{\log_2}$, by changing the value α in Eq. (4.49) as follows:

- i. If $\alpha = 1$ Spherical Fuzzy Weighted Minkowski Distance-based f-Similarity (Logarithm function) $WMS_{SFS}^{\log_2}$ will turn into Spherical Fuzzy Weighted Hamming Distance-based f-Similarity (Logarithm function) $WHS_{SFS}^{\log_2}$:

$$WHS_{SFS}^{\log_2}(\tilde{A}, \tilde{B}) = \log_2 \left(2 - \left(\frac{1}{2} \sum_{i=1}^n w_i [|\mu_{\tilde{A}}^2(x_i) - \mu_{\tilde{B}}^2(x_i)| + |\vartheta_{\tilde{A}}^2(x_i) - \vartheta_{\tilde{B}}^2(x_i)| + |I_{\tilde{A}}^2(x_i) - I_{\tilde{B}}^2(x_i)|] \right) \right) \quad (4.50)$$

- ii. If $\alpha = 2$ Spherical Fuzzy Weighted Minkowski Distance-based f-Similarity (Logarithm function) $WMS_{SFS}^{\log_2}$ will turn into Spherical Fuzzy Weighted Euclidean Distance based f-Similarity (Logarithm function) $WES_{SFS}^{\log_2}$:

$$WES_{SFS}^{\log_2}(\tilde{A}, \tilde{B}) = \log_2 \left(2 - \sqrt{\frac{1}{2} \sum_{i=1}^n w_i \left[\left(\mu_{\tilde{A}}^2(x_i) - \mu_{\tilde{B}}^2(x_i) \right)^2 + \left(\vartheta_{\tilde{A}}^2(x_i) - \vartheta_{\tilde{B}}^2(x_i) \right)^2 + \left(I_{\tilde{A}}^2(x_i) - I_{\tilde{B}}^2(x_i) \right)^2 \right]} \right) \quad (4.51)$$

Spherical fuzzy normalized Minkowski-Hausdorff distance-based f-similarity (logarithm function) measure

Definition 4.25 Suppose that there are two sets of SFSs $\tilde{A}_s = \{x_i \langle \mu_{\tilde{A}}(x_i), \vartheta_{\tilde{A}}(x_i), I_{\tilde{A}}(x_i) \rangle | x_i \in X\}; \forall i = 1, 2, \dots, n$ and $\tilde{B}_s = \{x_i \langle \mu_{\tilde{B}}(x_i), \vartheta_{\tilde{B}}(x_i), I_{\tilde{B}}(x_i) \rangle | x_i \in X\}; \forall i = 1, 2, \dots, n$ be two SFSs so that $\mu(x_i)$, $\vartheta(x_i)$ and $I(x_i)$ are membership, non-membership, and indeterminacy degrees, respectively. Then, Spherical Fuzzy Normalized Minkowski-Hausdorff Distance-based f-Similarity (Logarithm function) is given as follows:

$$MHS_{SFS}^{\lceil \log_2 \rceil}(\tilde{A}, \tilde{B}) = \log_2 \left(2 - Mhd_{SFS}(\tilde{A}_s, \tilde{B}_s) \right) = \log_2 \left(2 - \sqrt[\alpha]{\frac{1}{2n} \sum_{i=1}^n \max_i \left[|\mu_{\tilde{A}}^2(x_i) - \mu_{\tilde{B}}^2(x_i)|^\alpha, |\vartheta_{\tilde{A}}^2(x_i) - \vartheta_{\tilde{B}}^2(x_i)|^\alpha, |I_{\tilde{A}}^2(x_i) - I_{\tilde{B}}^2(x_i)|^\alpha \right]} \right) \quad (4.52)$$

Where $\alpha \geq 1$.

Proof: Proof of Definition 4.25 is similar to the proof of Definition 4.22, hence, omit this proof.

Remark 4.15 It is obvious that Spherical Fuzzy Normalized Hamming-Hausdorff Distance-based f-Similarity (Logarithm function) $HHS_{SFS}^{\lceil \log_2 \rceil}$ and Spherical Fuzzy Normalized Euclidean-Hausdorff Distance-based f-Similarity (Logarithm function) $EHS_{SFS}^{\lceil \log_2 \rceil}$ could result from Spherical Fuzzy Normalized Minkowski-Hausdorff Distance-based f-Similarity (Logarithm function) $MHS_{SFS}^{\lceil \log_2 \rceil}$, by changing the value for α in Eq. (4.52) as follows:

- i. If $\alpha = 1$ Spherical Fuzzy Normalized Minkowski-Hausdorff Distance-based f-Similarity (Logarithm function) $MHS_{SFS}^{\lceil \log_2 \rceil}$ will turn into Spherical Fuzzy Normalized Hamming-Hausdorff Distance-based f-Similarity (Logarithm function) $HHS_{SFS}^{\lceil \log_2 \rceil}$:

$$HHS_{SFS}^{\lceil \log_2 \rceil}(\tilde{A}, \tilde{B}) = \log_2 \left(2 - \left(\frac{1}{2n} \sum_{i=1}^n \max_i \left[|\mu_{\tilde{A}}^2(x_i) - \mu_{\tilde{B}}^2(x_i)|, |\vartheta_{\tilde{A}}^2(x_i) - \vartheta_{\tilde{B}}^2(x_i)|, |I_{\tilde{A}}^2(x_i) - I_{\tilde{B}}^2(x_i)| \right] \right) \right) \quad (4.53)$$

- ii. If $\alpha = 2$ Spherical Fuzzy Normalized Minkowski-Hausdorff Distance-based f-Similarity (Logarithm function) $MHS_{SFS}^{\lceil \log_2 \rceil}$ will turn into Spherical Fuzzy Normalized Euclidean-Hausdorff Distance-based f-Similarity (Logarithm function) $EHS_{SFS}^{\lceil \log_2 \rceil}$:

$$EHS_{SFS}^{\lceil \log_2 \rceil}(\tilde{A}, \tilde{B}) = \log_2 \left(2 - \sqrt{\frac{1}{2n} \sum_{i=1}^n \max_i \left[\left(\mu_{\tilde{A}}^2(x_i) - \mu_{\tilde{B}}^2(x_i) \right)^2, \left(\vartheta_{\tilde{A}}^2(x_i) - \vartheta_{\tilde{B}}^2(x_i) \right)^2, \left(I_{\tilde{A}}^2(x_i) - I_{\tilde{B}}^2(x_i) \right)^2 \right]} \right) \quad (4.54)$$

Spherical fuzzy weighted Minkowski-Hausdorff distance-based f-similarity (logarithm function) measure

Definition 4.26 Suppose that there are two sets of SFSs $\tilde{A}_s = \{x_i \langle \mu_{\tilde{A}}(x_i), \vartheta_{\tilde{A}}(x_i), I_{\tilde{A}}(x_i) \rangle | x_i \in X\}; \forall i = 1, 2, \dots, n$ and $\tilde{B}_s = \{x_i \langle \mu_{\tilde{B}}(x_i), \vartheta_{\tilde{B}}(x_i), I_{\tilde{B}}(x_i) \rangle | x_i \in X\}; \forall i = 1, 2, \dots, n$ be two SFSs with corresponding weight vector $w = \{w_1, w_2, \dots, w_n\}$, where $w_i \in [0, 1]$ and $\sum_{i=1}^n w_i = 1, \forall i = 1, 2, \dots, n$, so that $\mu(x_i)$, $\vartheta(x_i)$ and $I(x_i)$ are membership, non-membership, and indeterminacy degrees, respectively. Then, Spherical Fuzzy Weighted Minkowski-Hausdorff Distance-based f-Similarity (Logarithm function) is given as follows:

$$WMHS_{SFS}^{\lceil \log_2 \rceil}(\tilde{A}, \tilde{B}) = \log_2 \left(2 - WMHd_{SFS}(\tilde{A}_s, \tilde{B}_s) \right) = \log_2 \left(2 - \sqrt[\alpha]{\frac{1}{2} \sum_{i=1}^n w_i (\max_i [|\mu_{\tilde{A}}^2(x_i) - \mu_{\tilde{B}}^2(x_i)|^\alpha, |\vartheta_{\tilde{A}}^2(x_i) - \vartheta_{\tilde{B}}^2(x_i)|^\alpha, |I_{\tilde{A}}^2(x_i) - I_{\tilde{B}}^2(x_i)|^\alpha])} \right) \quad (4.55)$$

Where $\alpha \geq 1$.

Proof: Proof of Definition 4.26 is similar to the proof of Definition 4.22, hence, omit this proof.

Remark 4.16 It is obvious that Spherical Fuzzy Weighted Hamming-Hausdorff Distance-based f-Similarity (Logarithm function) $WHHS_{SFS}^{\lceil \log_2 \rceil}$ and Spherical Fuzzy Weighted Euclidean-Hausdorff Distance-based f-Similarity (Logarithm function) $WEHS_{SFS}^{\lceil \log_2 \rceil}$ could result from Spherical Fuzzy Weighted Minkowski-Hausdorff Distance-based f-Similarity (Logarithm function) $WMHS_{SFS}^{\lceil \log_2 \rceil}$, by changing the value for α in Eq. (4.55) as follows:

- i. If $\alpha = 1$ Spherical Fuzzy Weighted Minkowski-Hausdorff Distance-based f-Similarity (Logarithm function) $WMHS_{SFS}^{\lceil \log_2 \rceil}$ will turn into Spherical Fuzzy Weighted Hamming-Hausdorff Distance-based f-Similarity (Logarithm function) $WHHS_{SFS}^{\lceil \log_2 \rceil}$.

$$WHHS_{SFS}^{\lceil \log_2 \rceil}(\tilde{A}, \tilde{B}) = \log_2 \left(2 - \left(\frac{1}{2} \sum_{i=1}^n w_i (\max_i [|\mu_{\tilde{A}}^2(x_i) - \mu_{\tilde{B}}^2(x_i)|, |\vartheta_{\tilde{A}}^2(x_i) - \vartheta_{\tilde{B}}^2(x_i)|, |I_{\tilde{A}}^2(x_i) - I_{\tilde{B}}^2(x_i)|]) \right) \right) \quad (4.56)$$

- ii. If $\alpha = 2$ Spherical Fuzzy Weighted Minkowski-Hausdorff Distance-based f-Similarity (Logarithm function) $WMHS_{SFS}^{\lceil \log_2 \rceil}$ will turn into Spherical Fuzzy Weighted Euclidean-Hausdorff Distance-based f-Similarity (Logarithm function) $WEHS_{SFS}^{\lceil \log_2 \rceil}$.

$$WEHS_{SFS}^{\lceil \log_2 \rceil}(\tilde{A}, \tilde{B}) = \log_2 \left(2 - \sqrt{\frac{1}{2} \sum_{i=1}^n w_i \left(\max_i \left[\left(\mu_{\tilde{A}}^2(x_i) - \mu_{\tilde{B}}^2(x_i) \right)^2, \left(\vartheta_{\tilde{A}}^2(x_i) - \vartheta_{\tilde{B}}^2(x_i) \right)^2, \left(I_{\tilde{A}}^2(x_i) - I_{\tilde{B}}^2(x_i) \right)^2 \right] \right)} \right) \quad (4.57)$$

4.3.1.4 Spherical fuzzy distance-based f-similarity (rational function) measure

Definition 4.27 Suppose that there are two sets of SFSs $\tilde{A}_s = \{x_i \langle \mu_{\tilde{A}}(x_i), \vartheta_{\tilde{A}}(x_i), I_{\tilde{A}}(x_i) \rangle | x_i \in X\}; \forall i = 1, 2, \dots, n$ and $\tilde{B}_s = \{x_i \langle \mu_{\tilde{B}}(x_i), \vartheta_{\tilde{B}}(x_i), I_{\tilde{B}}(x_i) \rangle | x_i \in X\}; \forall i = 1, 2, \dots, n$ be two SFSs so that $\mu(x_i)$, $\vartheta(x_i)$ and $I(x_i)$ are membership, non-membership, and indeterminacy degrees, respectively. Then, Spherical Fuzzy Distance-based f-Similarity (Rational function) is given as follows:

$$S_{SFS}^{\lceil r \rceil}(\tilde{A}, \tilde{B}) = \frac{1 - d_{SFS}(\tilde{A}_s, \tilde{B}_s)}{1 + d_{SFS}(\tilde{A}_s, \tilde{B}_s)} \quad (4.58)$$

Proof: To prove Definition 4.27, use properties in Definition 4.10 to show that Eq. (4.58) satisfies the properties, so will have:

$$i. \quad 0 \leq S_{SFS}^{\lceil r \rceil}(\tilde{A}, \tilde{B}) \leq 1$$

We know that $0 \leq d_{SFS}(\tilde{A}, \tilde{B}) \leq 1$ then $0 \leq S_{SFS}^{\lceil r \rceil}(\tilde{A}, \tilde{B}) \leq 1$.

$$ii. \quad S_{SFS}^{\lceil r \rceil}(\tilde{A}, \tilde{B}) = S_{SFS}^{\lceil r \rceil}(\tilde{B}, \tilde{A})$$

We know that $d_{SFS}(\tilde{A}, \tilde{B}) = d_{SFS}(\tilde{B}, \tilde{A})$, then it is straightforward that $S_{SFS}^{\lceil r \rceil}(\tilde{A}, \tilde{B}) = S_{SFS}^{\lceil r \rceil}(\tilde{B}, \tilde{A})$.

$$iii. \quad S_{SFS}^{\lceil r \rceil}(\tilde{A}, \tilde{B}) = 1 \text{ iff } \tilde{A} = \tilde{B}$$

We know that if $\tilde{A} = \tilde{B}$ then $d_{SFS}(\tilde{A}, \tilde{B}) = 0$ and $S_{SFS}^{\lceil r \rceil}(\tilde{A}, \tilde{B}) = 1$.

Spherical fuzzy normalized Minkowski distance-based f-similarity (rational function) measure

Definition 4.28 Suppose that there are two sets of SFSs $\tilde{A}_s = \{x_i \langle \mu_{\tilde{A}}(x_i), \vartheta_{\tilde{A}}(x_i), I_{\tilde{A}}(x_i) \rangle | x_i \in X\}; \forall i = 1, 2, \dots, n$ and $\tilde{B}_s = \{x_i \langle \mu_{\tilde{B}}(x_i), \vartheta_{\tilde{B}}(x_i), I_{\tilde{B}}(x_i) \rangle | x_i \in X\}; \forall i = 1, 2, \dots, n$ be two SFSs so that $\mu(x_i)$, $\vartheta(x_i)$ and $I(x_i)$ are membership, non-membership, and indeterminacy degrees, respectively. Then, Spherical Fuzzy Normalized Minkowski Distance-based f-Similarity (Rational function) is given as follows:

$$MS_{SFS}^{[r]}(\tilde{A}, \tilde{B}) = \frac{1 - Md_{SFS}(\tilde{A}, \tilde{B})}{1 + Md_{SFS}(\tilde{A}, \tilde{B})} = \frac{1 - \sqrt[\alpha]{\frac{1}{2n} \sum_{i=1}^n \left[|\mu_{\tilde{A}}^2(x_i) - \mu_{\tilde{B}}^2(x_i)|^\alpha + |\vartheta_{\tilde{A}}^2(x_i) - \vartheta_{\tilde{B}}^2(x_i)|^\alpha + |I_{\tilde{A}}^2(x_i) - I_{\tilde{B}}^2(x_i)|^\alpha \right]}}{1 + \sqrt[\alpha]{\frac{1}{2n} \sum_{i=1}^n \left[|\mu_{\tilde{A}}^2(x_i) - \mu_{\tilde{B}}^2(x_i)|^\alpha + |\vartheta_{\tilde{A}}^2(x_i) - \vartheta_{\tilde{B}}^2(x_i)|^\alpha + |I_{\tilde{A}}^2(x_i) - I_{\tilde{B}}^2(x_i)|^\alpha \right]}} \quad (4.59)$$

Where $\alpha \geq 1$.

Proof: Proof of Definition 4.28 is similar to the proof of Definition 4.27, hence, omit this proof.

Remark 4.17 It is obvious Spherical Fuzzy Normalized Hamming Distance-based f-Similarity (Rational function) and Spherical Fuzzy Normalized Euclidean Distance based f-Similarity (Rational function) could result in Spherical Fuzzy Normalized Minkowski Distance-based f-Similarity (Rational function) $MS_{SFS}^{[r]}$, by changing the value α in Eq. (4.59) as follows:

- i. If $\alpha = 1$ Spherical Fuzzy Normalized Minkowski Distance-based f-Similarity (Rational function) $MS_{SFS}^{[r]}$ will turn into Spherical Fuzzy Normalized Hamming Distance-based f-Similarity (Rational function) $HS_{SFS}^{[r]}$:

$$HS_{SFS}^{[r]}(\tilde{A}, \tilde{B}) = \frac{1 - \left(\frac{1}{2n} \sum_{i=1}^n [|\mu_{\tilde{A}}^2(x_i) - \mu_{\tilde{B}}^2(x_i)| + |\vartheta_{\tilde{A}}^2(x_i) - \vartheta_{\tilde{B}}^2(x_i)| + |I_{\tilde{A}}^2(x_i) - I_{\tilde{B}}^2(x_i)|] \right)}{1 + \left(\frac{1}{2n} \sum_{i=1}^n [|\mu_{\tilde{A}}^2(x_i) - \mu_{\tilde{B}}^2(x_i)| + |\vartheta_{\tilde{A}}^2(x_i) - \vartheta_{\tilde{B}}^2(x_i)| + |I_{\tilde{A}}^2(x_i) - I_{\tilde{B}}^2(x_i)|] \right)} \quad (4.60)$$

- ii. If $\alpha = 2$ Spherical Fuzzy Normalized Minkowski Distance-based f-Similarity (Rational function) $MS_{SFS}^{[r]}$ will turn into Spherical Fuzzy Normalized Euclidean Distance based f-Similarity (Rational function) $ES_{SFS}^{[r]}$:

$$ES_{SFS}^{[r]}(\tilde{A}, \tilde{B}) = \frac{1 - \sqrt{\frac{1}{2n} \sum_{i=1}^n \left[\left(\mu_{\tilde{A}}^2(x_i) - \mu_{\tilde{B}}^2(x_i) \right)^2 + \left(\vartheta_{\tilde{A}}^2(x_i) - \vartheta_{\tilde{B}}^2(x_i) \right)^2 + \left(I_{\tilde{A}}^2(x_i) - I_{\tilde{B}}^2(x_i) \right)^2 \right]}}{1 + \sqrt{\frac{1}{2n} \sum_{i=1}^n \left[\left(\mu_{\tilde{A}}^2(x_i) - \mu_{\tilde{B}}^2(x_i) \right)^2 + \left(\vartheta_{\tilde{A}}^2(x_i) - \vartheta_{\tilde{B}}^2(x_i) \right)^2 + \left(I_{\tilde{A}}^2(x_i) - I_{\tilde{B}}^2(x_i) \right)^2 \right]}} \quad (4.61)$$

Spherical fuzzy weighted Minkowski distance-based f-similarity (rational function) measure

Definition 4.29 Suppose that there are two sets of SFSs $\tilde{A}_s = \{x_i \langle \mu_{\tilde{A}}(x_i), \vartheta_{\tilde{A}}(x_i), I_{\tilde{A}}(x_i) \rangle | x_i \in X\}; \forall i = 1, 2, \dots, n$ and $\tilde{B}_s = \{x_i \langle \mu_{\tilde{B}}(x_i), \vartheta_{\tilde{B}}(x_i), I_{\tilde{B}}(x_i) \rangle | x_i \in X\}; \forall i = 1, 2, \dots, n$ be two SFSs with corresponding weight vector $w = \{w_1, w_2, \dots, w_n\}$, where $w_i \in [0, 1]$ and $\sum_{i=1}^n w_i = 1, \forall i = 1, 2, \dots, n$, so that $\mu(x_i)$, $\vartheta(x_i)$ and $I(x_i)$ are membership, non-membership, and indeterminacy degrees, respectively. Then, Spherical Fuzzy Weighted Minkowski Distance-based f-Similarity (Rational function) $WMS_{SFS}^{[r]}$ given as follows:

$$WMS_{SFS}^{[r]}(\tilde{A}, \tilde{B}) = \frac{1 - WMD_{SFS}(\tilde{A}, \tilde{B})}{1 + WMD_{SFS}(\tilde{A}, \tilde{B})} = \frac{1 - \sqrt[\alpha]{\frac{1}{2} \sum_{i=1}^n w_i \left[\left| \mu_{\tilde{A}}^2(x_i) - \mu_{\tilde{B}}^2(x_i) \right|^\alpha + \left| \vartheta_{\tilde{A}}^2(x_i) - \vartheta_{\tilde{B}}^2(x_i) \right|^\alpha + \left| I_{\tilde{A}}^2(x_i) - I_{\tilde{B}}^2(x_i) \right|^\alpha \right]}}{1 + \sqrt[\alpha]{\frac{1}{2} \sum_{i=1}^n w_i \left[\left| \mu_{\tilde{A}}^2(x_i) - \mu_{\tilde{B}}^2(x_i) \right|^\alpha + \left| \vartheta_{\tilde{A}}^2(x_i) - \vartheta_{\tilde{B}}^2(x_i) \right|^\alpha + \left| I_{\tilde{A}}^2(x_i) - I_{\tilde{B}}^2(x_i) \right|^\alpha \right]}} \quad (4.62)$$

Where $\alpha \geq 1$.

Proof: Proof of Definition 4.29 is similar to the proof of Definition 4.27, hence, omit this proof.

Remark 4.18 It is obvious that Spherical Fuzzy Weighted Hamming Distance-based f-Similarity (Rational function) $WHS_{SFS}^{[r]}$ and Spherical Fuzzy Weighted Euclidean Distance based f-Similarity (Rational function) $WES_{SFS}^{[r]}$ could result from Spherical Fuzzy Weighted Minkowski Distance-based f-Similarity (Rational function) $WMS_{SFS}^{[r]}$, by changing the value α in Eq. (4.62) as follows:

- i. If $\alpha = 1$ Spherical Fuzzy Weighted Minkowski Distance-based f-Similarity (Rational function) $WMS_{SFS}^{[r]}$ will turn into Spherical Fuzzy Weighted Hamming Distance-based f-Similarity (Rational function) $WHS_{SFS}^{[r]}$:

$$WHS_{SFS}^{[r]}(\tilde{A}, \tilde{B}) = \frac{1 - \left(\frac{1}{2} \sum_{i=1}^n w_i \left[\left| \mu_{\tilde{A}}^2(x_i) - \mu_{\tilde{B}}^2(x_i) \right| + \left| \vartheta_{\tilde{A}}^2(x_i) - \vartheta_{\tilde{B}}^2(x_i) \right| + \left| I_{\tilde{A}}^2(x_i) - I_{\tilde{B}}^2(x_i) \right| \right] \right)}{1 + \left(\frac{1}{2} \sum_{i=1}^n w_i \left[\left| \mu_{\tilde{A}}^2(x_i) - \mu_{\tilde{B}}^2(x_i) \right| + \left| \vartheta_{\tilde{A}}^2(x_i) - \vartheta_{\tilde{B}}^2(x_i) \right| + \left| I_{\tilde{A}}^2(x_i) - I_{\tilde{B}}^2(x_i) \right| \right] \right)} \quad (4.63)$$

- ii. If $\alpha = 2$ Spherical Fuzzy Weighted Minkowski Distance-based f-Similarity (Rational function) $WMS_{SFS}^{[r]}$ will turn into Spherical Fuzzy Weighted Euclidean Distance based f-Similarity (Rational function) $WES_{SFS}^{[r]}$:

$$WES_{SFS}^{[r]}(\tilde{A}, \tilde{B}) = \frac{1 - \sqrt{\frac{1}{2} \sum_{i=1}^n w_i \left[(\mu_A^2(x_i) - \mu_B^2(x_i))^2 + (\vartheta_A^2(x_i) - \vartheta_B^2(x_i))^2 + (I_A^2(x_i) - I_B^2(x_i))^2 \right]}}{1 + \sqrt{\frac{1}{2} \sum_{i=1}^n w_i \left[(\mu_A^2(x_i) - \mu_B^2(x_i))^2 + (\vartheta_A^2(x_i) - \vartheta_B^2(x_i))^2 + (I_A^2(x_i) - I_B^2(x_i))^2 \right]}} \quad (4.64)$$

Spherical fuzzy normalized Minkowski-Hausdorff distance-based f-similarity (rational function) measure

Definition 4.30 Suppose that there are two sets of SFSs $\tilde{A}_s = \{x_i \langle \mu_{\tilde{A}}(x_i), \vartheta_{\tilde{A}}(x_i), I_{\tilde{A}}(x_i) \rangle | x_i \in X\}; \forall i = 1, 2, \dots, n$ and $\tilde{B}_s = \{x_i \langle \mu_{\tilde{B}}(x_i), \vartheta_{\tilde{B}}(x_i), I_{\tilde{B}}(x_i) \rangle | x_i \in X\}; \forall i = 1, 2, \dots, n$ be two SFSs so that $\mu(x_i)$, $\vartheta(x_i)$ and $I(x_i)$ are membership, non-membership, and indeterminacy degrees, respectively. Then, Spherical Fuzzy Normalized Minkowski-Hausdorff Distance-based f-Similarity (Rational function) is given as follows:

$$MHS_{SFS}^{[r]}(\tilde{A}, \tilde{B}) = \frac{1 - MHS_{SFS}(\tilde{A}, \tilde{B})}{1 + MHS_{SFS}(\tilde{A}, \tilde{B})} = \frac{1 - \sqrt[\alpha]{\frac{1}{2n} \sum_{i=1}^n \max_i \left[|\mu_A^2(x_i) - \mu_B^2(x_i)|^\alpha, |\vartheta_A^2(x_i) - \vartheta_B^2(x_i)|^\alpha, |I_A^2(x_i) - I_B^2(x_i)|^\alpha \right]}}{1 + \sqrt[\alpha]{\frac{1}{2n} \sum_{i=1}^n \max_i \left[|\mu_A^2(x_i) - \mu_B^2(x_i)|^\alpha, |\vartheta_A^2(x_i) - \vartheta_B^2(x_i)|^\alpha, |I_A^2(x_i) - I_B^2(x_i)|^\alpha \right]}} \quad (4.65)$$

Where $\alpha \geq 1$.

Proof: Proof of Definition 4.30 is similar to the proof of Definition 4.27, hence, omit this proof.

Remark 4.19 It is obvious that Spherical Fuzzy Normalized Hamming-Hausdorff Distance-based f-Similarity (Rational function) $HHS_{SFS}^{[r]}$ and Spherical Fuzzy Normalized Euclidean-Hausdorff Distance-based f-Similarity (Rational function) $EHS_{SFS}^{[r]}$ could result from Spherical Fuzzy Normalized Minkowski-Hausdorff Distance-based f-Similarity (Rational function) $MHS_{SFS}^{[r]}$, by changing the value for α in Eq. (4.65) as follows:

- i. If $\alpha = 1$ Spherical Fuzzy Normalized Minkowski-Hausdorff Distance-based f-Similarity (Rational function) $MHS_{SFS}^{[r]}$ will turn into Spherical Fuzzy

Normalized Hamming-Hausdorff Distance-based f-Similarity (Rational function) $HHS_{SFS}^{[r]}$:

$$HHS_{SFS}^{[r]}(\tilde{A}, \tilde{B}) = \frac{1 - \left(\frac{1}{2n} \sum_{i=1}^n \max_i [|\mu_{\tilde{A}}^2(x_i) - \mu_{\tilde{B}}^2(x_i)|, |\vartheta_{\tilde{A}}^2(x_i) - \vartheta_{\tilde{B}}^2(x_i)|, |I_{\tilde{A}}^2(x_i) - I_{\tilde{B}}^2(x_i)|] \right)}{1 + \left(\frac{1}{2n} \sum_{i=1}^n \max_i [|\mu_{\tilde{A}}^2(x_i) - \mu_{\tilde{B}}^2(x_i)|, |\vartheta_{\tilde{A}}^2(x_i) - \vartheta_{\tilde{B}}^2(x_i)|, |I_{\tilde{A}}^2(x_i) - I_{\tilde{B}}^2(x_i)|] \right)} \quad (4.66)$$

- ii. If $\alpha = 2$ Spherical Fuzzy Normalized Minkowski-Hausdorff Distance-based f-Similarity (Rational function) $MHS_{SFS}^{[r]}$ will turn into Spherical Fuzzy Normalized Euclidean-Hausdorff Distance-based f-Similarity (Rational function) $EHS_{SFS}^{[r]}$:

$$EHS_{SFS}^{[r]}(\tilde{A}, \tilde{B}) = \frac{1 - \sqrt{\frac{1}{2n} \sum_{i=1}^n \max_i \left[\left(\mu_{\tilde{A}}^2(x_i) - \mu_{\tilde{B}}^2(x_i) \right)^2, \left(\vartheta_{\tilde{A}}^2(x_i) - \vartheta_{\tilde{B}}^2(x_i) \right)^2, \left(I_{\tilde{A}}^2(x_i) - I_{\tilde{B}}^2(x_i) \right)^2 \right]}}{1 + \sqrt{\frac{1}{2n} \sum_{i=1}^n \max_i \left[\left(\mu_{\tilde{A}}^2(x_i) - \mu_{\tilde{B}}^2(x_i) \right)^2, \left(\vartheta_{\tilde{A}}^2(x_i) - \vartheta_{\tilde{B}}^2(x_i) \right)^2, \left(I_{\tilde{A}}^2(x_i) - I_{\tilde{B}}^2(x_i) \right)^2 \right]}} \quad (4.67)$$

Spherical fuzzy weighted Minkowski-Hausdorff distance-based f-similarity (rational function) measure

Definition 4.31 Suppose that there are two sets of SFSs $\tilde{A}_s = \{x_i \langle \mu_{\tilde{A}}(x_i), \vartheta_{\tilde{A}}(x_i), I_{\tilde{A}}(x_i) \rangle | x_i \in X\}; \forall i = 1, 2, \dots, n$ and $\tilde{B}_s = \{x_i \langle \mu_{\tilde{B}}(x_i), \vartheta_{\tilde{B}}(x_i), I_{\tilde{B}}(x_i) \rangle | x_i \in X\}; \forall i = 1, 2, \dots, n$ be two SFSs with corresponding weight vector $w = \{w_1, w_2, \dots, w_n\}$, where $w_i \in [0, 1]$ and $\sum_{i=1}^n w_i = 1, \forall i = 1, 2, \dots, n$, so that $\mu(x_i)$, $\vartheta(x_i)$ and $I(x_i)$ are membership, non-membership, and indeterminacy degrees, respectively. Then, Spherical Fuzzy Weighted Minkowski-Hausdorff Distance-based f-Similarity (Rational function) is given as follows:

$$WMHS_{SFS}^{[r]}(\tilde{A}, \tilde{B}) = \frac{1 - WMHd_{SFS}(\tilde{A}, \tilde{B})}{1 + WMHd_{SFS}(\tilde{A}, \tilde{B})} = \frac{1 - \sqrt[\alpha]{\frac{1}{2} \sum_{i=1}^n w_i \left(\max_i \left[\left| \mu_{\tilde{A}}^2(x_i) - \mu_{\tilde{B}}^2(x_i) \right|^\alpha, \left| \vartheta_{\tilde{A}}^2(x_i) - \vartheta_{\tilde{B}}^2(x_i) \right|^\alpha, \left| I_{\tilde{A}}^2(x_i) - I_{\tilde{B}}^2(x_i) \right|^\alpha \right] \right)}}{1 + \sqrt[\alpha]{\frac{1}{2} \sum_{i=1}^n w_i \left(\max_i \left[\left| \mu_{\tilde{A}}^2(x_i) - \mu_{\tilde{B}}^2(x_i) \right|^\alpha, \left| \vartheta_{\tilde{A}}^2(x_i) - \vartheta_{\tilde{B}}^2(x_i) \right|^\alpha, \left| I_{\tilde{A}}^2(x_i) - I_{\tilde{B}}^2(x_i) \right|^\alpha \right] \right)}} \quad (4.68)$$

Where $\alpha \geq 1$.

Proof: Proof of Definition 4.31 is similar to the proof of Definition 4.27, hence, omit this proof.

Remark 4.20 It is obvious that Spherical Fuzzy Weighted Hamming-Hausdorff Distance-based f-Similarity (Rational function) $WHHS_{SFS}^{[r]}$ and Spherical Fuzzy

Weighted Euclidean-Hausdorff Distance-based f-Similarity (Rational function) $WEHS_{SFS}^{\overline{r}}$ could result from Spherical Fuzzy Weighted Minkowski-Hausdorff Distance-based f-Similarity (Rational function) $WMHS_{SFS}^{\overline{r}}$, by changing the value for α in Eq. (4.68) as follows:

- i. If $\alpha = 1$ Spherical Fuzzy Weighted Minkowski-Hausdorff Distance-based f-Similarity (Rational function) $WMHS_{SFS}^{\overline{r}}$ will turn into Spherical Fuzzy Weighted Hamming-Hausdorff Distance-based f-Similarity (Rational function) $WHHS_{SFS}^{\overline{r}}$:

$$WHHS_{SFS}^{\overline{r}}(\tilde{A}, \tilde{B}) = \frac{1 - \left(\frac{1}{2} \sum_{i=1}^n w_i (\max_i [|\mu_{\tilde{A}}^2(x_i) - \mu_{\tilde{B}}^2(x_i)|, |\vartheta_{\tilde{A}}^2(x_i) - \vartheta_{\tilde{B}}^2(x_i)|, |I_{\tilde{A}}^2(x_i) - I_{\tilde{B}}^2(x_i)|]) \right)}{1 + \left(\frac{1}{2} \sum_{i=1}^n w_i (\max_i [|\mu_{\tilde{A}}^2(x_i) - \mu_{\tilde{B}}^2(x_i)|, |\vartheta_{\tilde{A}}^2(x_i) - \vartheta_{\tilde{B}}^2(x_i)|, |I_{\tilde{A}}^2(x_i) - I_{\tilde{B}}^2(x_i)|]) \right)} \quad (4.69)$$

- ii. If $\alpha = 2$ Spherical Fuzzy Weighted Minkowski-Hausdorff Distance-based f-Similarity (Rational function) $WMHS_{SFS}^{\overline{r}}$ will turn into Spherical Fuzzy Weighted Euclidean-Hausdorff Distance-based f-Similarity (Rational function) $WEHS_{SFS}^{\overline{r}}$:

$$WEHS_{SFS}^{\overline{r}}(\tilde{A}, \tilde{B}) = \frac{1 - \sqrt{\frac{1}{2} \sum_{i=1}^n w_i (\max_i [(\mu_{\tilde{A}}^2(x_i) - \mu_{\tilde{B}}^2(x_i))^2, (\vartheta_{\tilde{A}}^2(x_i) - \vartheta_{\tilde{B}}^2(x_i))^2, (I_{\tilde{A}}^2(x_i) - I_{\tilde{B}}^2(x_i))^2])}}{1 + \sqrt{\frac{1}{2} \sum_{i=1}^n w_i (\max_i [(\mu_{\tilde{A}}^2(x_i) - \mu_{\tilde{B}}^2(x_i))^2, (\vartheta_{\tilde{A}}^2(x_i) - \vartheta_{\tilde{B}}^2(x_i))^2, (I_{\tilde{A}}^2(x_i) - I_{\tilde{B}}^2(x_i))^2])}} \quad (4.70)$$

4.3.2 Spherical fuzzy trigonometric similarity measure

4.3.2.1 Spherical fuzzy cosine Minkowski similarity measure

Definition 4.32 Suppose that there are two sets of SFSs $\tilde{A}_s = \{x_i \langle \mu_{\tilde{A}}(x_i), \vartheta_{\tilde{A}}(x_i), I_{\tilde{A}}(x_i) \rangle | x_i \in X\}; \forall i = 1, 2, \dots, n$ and $\tilde{B}_s = \{x_i \langle \mu_{\tilde{B}}(x_i), \vartheta_{\tilde{B}}(x_i), I_{\tilde{B}}(x_i) \rangle | x_i \in X\}; \forall i = 1, 2, \dots, n$ be two SFSs so that $\mu(x_i)$, $\vartheta(x_i)$ and $I(x_i)$ are membership, non-membership, and indeterminacy degrees, respectively. Then, Spherical Fuzzy Cosine Minkowski Similarity Measure is given as follows:

$$CMS_{SFS}(\tilde{A}, \tilde{B}) = \sqrt[\alpha]{\frac{1}{n} \sum_{i=1}^n \cos \frac{\pi}{4} [|\mu_{\tilde{A}}^2(x_i) - \mu_{\tilde{B}}^2(x_i)|^\alpha + |\vartheta_{\tilde{A}}^2(x_i) - \vartheta_{\tilde{B}}^2(x_i)|^\alpha + |I_{\tilde{A}}^2(x_i) - I_{\tilde{B}}^2(x_i)|^\alpha]} \quad (4.71)$$

Where $\alpha \geq 1$.

Proof: To prove Definition 4.32, have shown that Eq. (4.71) satisfies the properties in Definition 4.10.

$$i. \quad 0 \leq CMS_{SFS}(\tilde{A}, \tilde{B}) \leq 1$$

We know that

$$0 \leq |\mu_{\tilde{A}}^2(x_i) - \mu_{\tilde{B}}^2(x_i)| \leq 1, \quad 0 \leq |\vartheta_{\tilde{A}}^2(x_i) - \vartheta_{\tilde{B}}^2(x_i)| \leq 1 \quad \text{and} \quad 0 \leq |I_{\tilde{A}}^2(x_i) - I_{\tilde{B}}^2(x_i)| \leq 1$$

will have:

$$[|\mu_{\tilde{A}}^2(x_i) - \mu_{\tilde{B}}^2(x_i)|^\alpha + |\vartheta_{\tilde{A}}^2(x_i) - \vartheta_{\tilde{B}}^2(x_i)|^\alpha + |I_{\tilde{A}}^2(x_i) - I_{\tilde{B}}^2(x_i)|^\alpha] = \mathbb{A} \rightarrow 0 \leq \mathbb{A} \leq 2 \rightarrow \cos \frac{\pi \mathbb{A}}{4} \in [0, 1]$$

So, will have:

$$0 \leq$$

$$\sqrt[\alpha]{\frac{1}{n} \sum_{i=1}^n \cos \frac{\pi}{4} [|\mu_{\tilde{A}}^2(x_i) - \mu_{\tilde{B}}^2(x_i)|^\alpha + |\vartheta_{\tilde{A}}^2(x_i) - \vartheta_{\tilde{B}}^2(x_i)|^\alpha + |I_{\tilde{A}}^2(x_i) - I_{\tilde{B}}^2(x_i)|^\alpha]} \leq 1$$

Thus, it is proved that $0 \leq CMS_{SFS}(\tilde{A}, \tilde{B}) \leq 1$.

$$ii. \quad CMS_{SFS}(\tilde{A}, \tilde{B}) = CMS_{SFS}(\tilde{B}, \tilde{A})$$

We know that:

$$|\mu_{\tilde{A}}^2(x_i) - \mu_{\tilde{B}}^2(x_i)| = |\mu_{\tilde{B}}^2(x_i) - \mu_{\tilde{A}}^2(x_i)|, \quad |\vartheta_{\tilde{A}}^2(x_i) - \vartheta_{\tilde{B}}^2(x_i)| = |\vartheta_{\tilde{B}}^2(x_i) - \vartheta_{\tilde{A}}^2(x_i)| \quad \text{and} \\ |I_{\tilde{A}}^2(x_i) - I_{\tilde{B}}^2(x_i)| = |I_{\tilde{B}}^2(x_i) - I_{\tilde{A}}^2(x_i)|$$

Then, will have:

$$[|\mu_{\tilde{A}}^2(x_i) - \mu_{\tilde{B}}^2(x_i)|^\alpha + |\vartheta_{\tilde{A}}^2(x_i) - \vartheta_{\tilde{B}}^2(x_i)|^\alpha + |I_{\tilde{A}}^2(x_i) - I_{\tilde{B}}^2(x_i)|^\alpha] = [|\mu_{\tilde{B}}^2(x_i) - \mu_{\tilde{A}}^2(x_i)|^\alpha + |\vartheta_{\tilde{B}}^2(x_i) - \vartheta_{\tilde{A}}^2(x_i)|^\alpha + |I_{\tilde{B}}^2(x_i) - I_{\tilde{A}}^2(x_i)|^\alpha]$$

Thus, have that:

$$CMS_{SFS}(\tilde{A}, \tilde{B}) =$$

$$\sqrt[\alpha]{\frac{1}{n} \sum_{i=1}^n \cos \frac{\pi}{4} [|\mu_{\tilde{A}}^2(x_i) - \mu_{\tilde{B}}^2(x_i)|^\alpha + |\vartheta_{\tilde{A}}^2(x_i) - \vartheta_{\tilde{B}}^2(x_i)|^\alpha + |I_{\tilde{A}}^2(x_i) - I_{\tilde{B}}^2(x_i)|^\alpha]} =$$

$$\sqrt[n]{\frac{1}{n} \sum_{i=1}^n \cos \frac{\pi}{4} [|\mu_{\tilde{B}}^2(x_i) - \mu_{\tilde{A}}^2(x_i)|^\alpha + |\vartheta_{\tilde{B}}^2(x_i) - \vartheta_{\tilde{A}}^2(x_i)|^\alpha + |I_{\tilde{B}}^2(x_i) - I_{\tilde{A}}^2(x_i)|^\alpha]} = CMS_{SFS}(\tilde{B}, \tilde{A})$$

iii. $CMS_{SFS}(\tilde{A}, \tilde{B}) = 1$ iff $\tilde{A} = \tilde{B}$

We know that:

If $\tilde{A} = \tilde{B}$ then $\mu_A(x_i) = \mu_B(x_i)$, $\vartheta_B(x_i) = \vartheta_A(x_i)$ and $I_A(x_i) = I_B(x_i)$ so will have:

$$|\mu_{\tilde{A}}^2(x_i) - \mu_{\tilde{B}}^2(x_i)| = 0, |\vartheta_{\tilde{A}}^2(x_i) - \vartheta_{\tilde{B}}^2(x_i)| = 0 \text{ and } |I_{\tilde{A}}^2(x_i) - I_{\tilde{B}}^2(x_i)| = 0$$

So, will have:

$$\begin{aligned} & [|\mu_{\tilde{B}}^2(x_i) - \mu_{\tilde{A}}^2(x_i)|^\alpha + |\vartheta_{\tilde{B}}^2(x_i) - \vartheta_{\tilde{A}}^2(x_i)|^\alpha + |I_{\tilde{B}}^2(x_i) - I_{\tilde{A}}^2(x_i)|^\alpha] = 0 \rightarrow \\ & \cos \frac{\pi}{4} [|\mu_{\tilde{B}}^2(x_i) - \mu_{\tilde{A}}^2(x_i)|^\alpha + |\vartheta_{\tilde{B}}^2(x_i) - \vartheta_{\tilde{A}}^2(x_i)|^\alpha + |I_{\tilde{B}}^2(x_i) - I_{\tilde{A}}^2(x_i)|^\alpha] = \\ & \cos \frac{\pi}{4} (0) = 1 \end{aligned}$$

Thus, it is proved that:

$$\text{if } \tilde{A} = \tilde{B} \rightarrow CMS_{SFS}(\tilde{A}, \tilde{B}) = 1$$

4.3.2.2 Spherical fuzzy weighted cosine Minkowski similarity measure

Definition 4.33 Suppose that there are two sets of SFSs $\tilde{A}_s = \{x_i \langle \mu_{\tilde{A}}(x_i), \vartheta_{\tilde{A}}(x_i), I_{\tilde{A}}(x_i) \rangle | x_i \in X\}; \forall i = 1, 2, \dots, n$ and $\tilde{B}_s = \{x_i \langle \mu_{\tilde{B}}(x_i), \vartheta_{\tilde{B}}(x_i), I_{\tilde{B}}(x_i) \rangle | x_i \in X\}; \forall i = 1, 2, \dots, n$ be two SFSs with corresponding weight vector $w = \{w_1, w_2, \dots, w_n\}$, where $w_i \in [0, 1]$ and $\sum_{i=1}^n w_i = 1, \forall i = 1, 2, \dots, n$, so that $\mu(x_i)$, $\vartheta(x_i)$ and $I(x_i)$ are membership, non-membership, and indeterminacy degrees, respectively. Then, Spherical Fuzzy Weighted Cosine Minkowski Similarity Measure is given as follows:

$$WCMS_{SFS}(\tilde{A}, \tilde{B}) = \sqrt[n]{\sum_{i=1}^n w_i \cos \frac{\pi}{4} [|\mu_{\tilde{A}}^2(x_i) - \mu_{\tilde{B}}^2(x_i)|^\alpha + |\vartheta_{\tilde{A}}^2(x_i) - \vartheta_{\tilde{B}}^2(x_i)|^\alpha + |I_{\tilde{A}}^2(x_i) - I_{\tilde{B}}^2(x_i)|^\alpha]} \quad (4.72)$$

Where $\alpha \geq 1$.

Proof: Proof of Definition 4.33 is similar to the proof of Definition 4.32, hence, omit this proof.

Remark 4.21 If $w_i = \frac{1}{n}; \forall i$ then $WCMS_{SFS}$ will turn into CMS_{SFS} .

4.3.2.3 Spherical fuzzy normalized sine Minkowski similarity measure

Definition 4.34 Suppose that there are two sets of SFSs $\tilde{A}_s = \{x_i \langle \mu_{\tilde{A}}(x_i), \vartheta_{\tilde{A}}(x_i), I_{\tilde{A}}(x_i) \rangle | x_i \in X\}; \forall i = 1, 2, \dots, n$ and $\tilde{B}_s = \{x_i \langle \mu_{\tilde{B}}(x_i), \vartheta_{\tilde{B}}(x_i), I_{\tilde{B}}(x_i) \rangle | x_i \in X\}; \forall i = 1, 2, \dots, n$ be two SFSs so that $\mu(x_i)$, $\vartheta(x_i)$ and $I(x_i)$ are membership, non-membership, and indeterminacy degrees, respectively. Then, Spherical Fuzzy Sine Minkowski Similarity Measure is given as follows:

$$SMS_{SFS}(\tilde{A}, \tilde{B}) = 1 - \sqrt[\alpha]{\frac{1}{n} \sum_{i=1}^n \sin \frac{\pi}{4} [|\mu_{\tilde{A}}^2(x_i) - \mu_{\tilde{B}}^2(x_i)|^\alpha + |\vartheta_{\tilde{A}}^2(x_i) - \vartheta_{\tilde{B}}^2(x_i)|^\alpha + |I_{\tilde{A}}^2(x_i) - I_{\tilde{B}}^2(x_i)|^\alpha]} \quad (4.73)$$

Where $\alpha \geq 1$.

Proof: Proof of Definition 4.34 is similar to the proof of Definition 4.32, hence, omit this proof.

4.3.2.4 Spherical fuzzy weighted sine Minkowski similarity measure

Definition 4.35 Suppose that there are two sets of SFSs $\tilde{A}_s = \{x_i \langle \mu_{\tilde{A}}(x_i), \vartheta_{\tilde{A}}(x_i), I_{\tilde{A}}(x_i) \rangle | x_i \in X\}; \forall i = 1, 2, \dots, n$ and $\tilde{B}_s = \{x_i \langle \mu_{\tilde{B}}(x_i), \vartheta_{\tilde{B}}(x_i), I_{\tilde{B}}(x_i) \rangle | x_i \in X\}; \forall i = 1, 2, \dots, n$ be two SFSs with corresponding weight vector $w = \{w_1, w_2, \dots, w_n\}$, where $w_i \in [0, 1]$ and $\sum_{i=1}^n w_i = 1, \forall i = 1, 2, \dots, n$, so that $\mu(x_i)$, $\vartheta(x_i)$ and $I(x_i)$ are membership, non-membership, and indeterminacy degrees, respectively. Then, Spherical Fuzzy Weighted Sine Minkowski Similarity Measure is given as follows:

$$WSMS_{SFS}(\tilde{A}, \tilde{B}) = 1 - \sqrt[\alpha]{\sum_{i=1}^n w_i \sin \frac{\pi}{4} [|\mu_{\tilde{A}}^2(x_i) - \mu_{\tilde{B}}^2(x_i)|^\alpha + |\vartheta_{\tilde{A}}^2(x_i) - \vartheta_{\tilde{B}}^2(x_i)|^\alpha + |I_{\tilde{A}}^2(x_i) - I_{\tilde{B}}^2(x_i)|^\alpha]} \quad (4.74)$$

Where $\alpha \geq 1$.

Proof: Proof of Definition 4.35 is similar to the proof of Definition 4.32, hence, omit this proof.

Remark 4.22 If $w_i = \frac{1}{n}; \forall i$ then $WSMS_{SFS}$ will turn into SMS_{SFS} .

4.3.2.5 Spherical fuzzy normalized cosine Minkowski-Hausdorff similarity measure

Definition 4.36 Suppose that there are two sets of SFSs $\tilde{A}_s = \{x_i \langle \mu_{\tilde{A}}(x_i), \vartheta_{\tilde{A}}(x_i), I_{\tilde{A}}(x_i) \rangle | x_i \in X\}; \forall i = 1, 2, \dots, n$ and $\tilde{B}_s = \{x_i \langle \mu_{\tilde{B}}(x_i), \vartheta_{\tilde{B}}(x_i), I_{\tilde{B}}(x_i) \rangle | x_i \in X\}; \forall i = 1, 2, \dots, n$ be two SFSs so that $\mu(x_i)$, $\vartheta(x_i)$ and $I(x_i)$ are membership, non-membership, and indeterminacy degrees, respectively. Then, Spherical Fuzzy Cosine Minkowski-Hausdorff Similarity Measure is given as follows:

$$CMHS_{SFS}(\tilde{A}, \tilde{B}) = \sqrt[\alpha]{\frac{1}{n} \sum_{i=1}^n \cos \frac{\pi}{2} \max_i [|\mu_{\tilde{A}}^2(x_i) - \mu_{\tilde{B}}^2(x_i)|^\alpha, |\vartheta_{\tilde{A}}^2(x_i) - \vartheta_{\tilde{B}}^2(x_i)|^\alpha, |I_{\tilde{A}}^2(x_i) - I_{\tilde{B}}^2(x_i)|^\alpha]} \quad (4.75)$$

Where $\alpha \geq 1$.

Proof: To prove Definition 4.36, have shown that Eq. (4.75) satisfies the properties in Definition 4.10.

$$i. \quad 0 \leq CMHS_{SFS}(\tilde{A}, \tilde{B}) \leq 1$$

We know that:

$$0 \leq |\mu_{\tilde{A}}^2(x_i) - \mu_{\tilde{B}}^2(x_i)| \leq 1, \quad 0 \leq |\vartheta_{\tilde{A}}^2(x_i) - \vartheta_{\tilde{B}}^2(x_i)| \leq 1 \quad \text{and} \quad 0 \leq |I_{\tilde{A}}^2(x_i) - I_{\tilde{B}}^2(x_i)| \leq 1$$

Will have:

$$\max_i [|\mu_{\tilde{A}}^2(x_i) - \mu_{\tilde{B}}^2(x_i)|^\alpha, |\vartheta_{\tilde{A}}^2(x_i) - \vartheta_{\tilde{B}}^2(x_i)|^\alpha, |I_{\tilde{A}}^2(x_i) - I_{\tilde{B}}^2(x_i)|^\alpha] = \mathbb{A} \rightarrow 0 \leq \mathbb{A} \leq 1 \rightarrow \cos \frac{\pi \cdot \mathbb{A}}{2} \in [0, 1]$$

So, will have:

$$0 \leq \sqrt[\alpha]{\frac{1}{n} \sum_{i=1}^n \cos \frac{\pi}{2} \max_i [|\mu_{\tilde{A}}^2(x_i) - \mu_{\tilde{B}}^2(x_i)|^\alpha, |\vartheta_{\tilde{A}}^2(x_i) - \vartheta_{\tilde{B}}^2(x_i)|^\alpha, |I_{\tilde{A}}^2(x_i) - I_{\tilde{B}}^2(x_i)|^\alpha]} \leq 1$$

Thus, it is proved that $0 \leq CMHS_{SFS}(\tilde{A}, \tilde{B}) \leq 1$.

$$ii. \quad CMHS_{SFS}(\tilde{A}, \tilde{B}) = CMHS_{SFS}(\tilde{B}, \tilde{A})$$

We know that:

$$|\mu_{\tilde{A}}^2(x_i) - \mu_{\tilde{B}}^2(x_i)| = |\mu_B^2(x_i) - \mu_A^2(x_i)|, |\vartheta_{\tilde{A}}^2(x_i) - \vartheta_{\tilde{B}}^2(x_i)| = |\vartheta_B^2(x_i) - \vartheta_A^2(x_i)| \text{ and } |I_{\tilde{A}}^2(x_i) - I_{\tilde{B}}^2(x_i)| = |I_B^2(x_i) - I_A^2(x_i)|$$

Then, will have:

$$\begin{aligned} \max_i [|\mu_{\tilde{A}}^2(x_i) - \mu_{\tilde{B}}^2(x_i)|^\alpha, |\vartheta_{\tilde{A}}^2(x_i) - \vartheta_{\tilde{B}}^2(x_i)|^\alpha, |I_{\tilde{A}}^2(x_i) - I_{\tilde{B}}^2(x_i)|^\alpha] = \\ \max_i [|\mu_B^2(x_i) - \mu_A^2(x_i)|^\alpha, |\vartheta_B^2(x_i) - \vartheta_A^2(x_i)|^\alpha, |I_B^2(x_i) - I_A^2(x_i)|^\alpha] \end{aligned}$$

Thus, have that:

$$\begin{aligned} CMHS_{SFS}(\tilde{A}, \tilde{B}) = \\ \sqrt[\alpha]{\frac{1}{n} \sum_{i=1}^n \cos \frac{\pi}{2} \max_i [|\mu_{\tilde{A}}^2(x_i) - \mu_{\tilde{B}}^2(x_i)|^\alpha, |\vartheta_{\tilde{A}}^2(x_i) - \vartheta_{\tilde{B}}^2(x_i)|^\alpha, |I_{\tilde{A}}^2(x_i) - I_{\tilde{B}}^2(x_i)|^\alpha]} = \\ \sqrt[\alpha]{\frac{1}{n} \sum_{i=1}^n \cos \frac{\pi}{2} \max_i [|\mu_B^2(x_i) - \mu_A^2(x_i)|^\alpha, |\vartheta_B^2(x_i) - \vartheta_A^2(x_i)|^\alpha, |I_B^2(x_i) - I_A^2(x_i)|^\alpha]} = \\ CMHS_{SFS}(\tilde{B}, \tilde{A}) \end{aligned}$$

$$\text{iii. } CMHS_{SFS}(\tilde{A}, \tilde{B}) = 1 \text{ iff } \tilde{A} = \tilde{B}$$

We know that:

If $\tilde{A} = \tilde{B}$ then $\mu_A(x_i) = \mu_B(x_i)$, $\vartheta_B(x_i) = \vartheta_A(x_i)$ and $I_A(x_i) = I_B(x_i)$ so will have:

$$|\mu_{\tilde{A}}^2(x_i) - \mu_{\tilde{B}}^2(x_i)| = 0, |\vartheta_{\tilde{A}}^2(x_i) - \vartheta_{\tilde{B}}^2(x_i)| = 0 \text{ and } |I_{\tilde{A}}^2(x_i) - I_{\tilde{B}}^2(x_i)| = 0$$

So, will have:

$$\begin{aligned} \max_i [|\mu_{\tilde{A}}^2(x_i) - \mu_{\tilde{B}}^2(x_i)|^\alpha, |\vartheta_{\tilde{A}}^2(x_i) - \vartheta_{\tilde{B}}^2(x_i)|^\alpha, |I_{\tilde{A}}^2(x_i) - I_{\tilde{B}}^2(x_i)|^\alpha] = 0 \rightarrow \\ \cos \frac{\pi}{2} \max_i [|\mu_{\tilde{A}}^2(x_i) - \mu_{\tilde{B}}^2(x_i)|^\alpha, |\vartheta_{\tilde{A}}^2(x_i) - \vartheta_{\tilde{B}}^2(x_i)|^\alpha, |I_{\tilde{A}}^2(x_i) - I_{\tilde{B}}^2(x_i)|^\alpha] = \\ \cos \frac{\pi}{2}(0) = 1 \end{aligned}$$

Thus, it is proved that:

$$\text{if } \tilde{A} = \tilde{B} \rightarrow CMHS_{SFS}(\tilde{A}, \tilde{B}) = 1$$

4.3.2.6 Spherical fuzzy weighted cosine Minkowski-Hausdorff similarity measure

Definition 4.37 Suppose that there are two sets of SFSs $\tilde{A}_s = \{x_i \langle \mu_{\tilde{A}}(x_i), \vartheta_{\tilde{A}}(x_i), I_{\tilde{A}}(x_i) \rangle | x_i \in X\}; \forall i = 1, 2, \dots, n$ and $\tilde{B}_s = \{x_i \langle \mu_{\tilde{B}}(x_i), \vartheta_{\tilde{B}}(x_i), I_{\tilde{B}}(x_i) \rangle | x_i \in X\}; \forall i = 1, 2, \dots, n$ be two SFSs with corresponding weight vector $w = \{w_1, w_2, \dots, w_n\}$, where $w_i \in [0, 1]$ and $\sum_{i=1}^n w_i = 1, \forall i = 1, 2, \dots, n$, so that $\mu(x_i)$, $\vartheta(x_i)$ and $I(x_i)$ are membership, non-membership, and

indeterminacy degrees, respectively. Then, Spherical Fuzzy Weighted Cosine Minkowski-Hausdorff Similarity Measure is given as follows:

$$WCMHS_{SFS}(\tilde{A}, \tilde{B}) = \sqrt[\alpha]{\sum_{i=1}^n w_i \cos \frac{\pi}{2} \max_i [|\mu_{\tilde{A}}^2(x_i) - \mu_{\tilde{B}}^2(x_i)|^\alpha, |\vartheta_{\tilde{A}}^2(x_i) - \vartheta_{\tilde{B}}^2(x_i)|^\alpha, |I_{\tilde{A}}^2(x_i) - I_{\tilde{B}}^2(x_i)|^\alpha]} \quad (4.76)$$

Where $\alpha \geq 1$.

Proof: Proof of Definition 4.37 is similar to the proof of Definition 4.36, hence, omit this proof.

Remark 4.23 If $w_i = \frac{1}{n}; \forall i$ then $WCMHS_{SFS}$ will turn into $CMHS_{SFS}$.

4.3.2.7 Spherical fuzzy normalized sine Minkowski-Hausdorff similarity measure

Definition 4.38 Suppose that there are two sets of SFSs $\tilde{A}_s = \{x_i \langle \mu_{\tilde{A}}(x_i), \vartheta_{\tilde{A}}(x_i), I_{\tilde{A}}(x_i) \rangle | x_i \in X\}; \forall i = 1, 2, \dots, n$ and $\tilde{B}_s = \{x_i \langle \mu_{\tilde{B}}(x_i), \vartheta_{\tilde{B}}(x_i), I_{\tilde{B}}(x_i) \rangle | x_i \in X\}; \forall i = 1, 2, \dots, n$ be two SFSs so that $\mu(x_i)$, $\vartheta(x_i)$ and $I(x_i)$ are membership, non-membership, and indeterminacy degrees, respectively. Then, Spherical Fuzzy Sine Minkowski-Hausdorff Similarity Measure is given as follows:

$$SMHS_{SFS}(\tilde{A}, \tilde{B}) = 1 - \sqrt[\alpha]{\frac{1}{n} \sum_{i=1}^n \sin \frac{\pi}{2} \max_i [|\mu_{\tilde{A}}^2(x_i) - \mu_{\tilde{B}}^2(x_i)|^\alpha, |\vartheta_{\tilde{A}}^2(x_i) - \vartheta_{\tilde{B}}^2(x_i)|^\alpha, |I_{\tilde{A}}^2(x_i) - I_{\tilde{B}}^2(x_i)|^\alpha]} \quad (4.77)$$

Proof: Proof of Definition 4.38 is similar to the proof of Definition 4.36, hence, omit this proof.

4.3.2.8 Spherical fuzzy weighted sine Minkowski-Hausdorff similarity measure

Definition 4.39 Suppose that there are two sets of SFSs $\tilde{A}_s = \{x_i \langle \mu_{\tilde{A}}(x_i), \vartheta_{\tilde{A}}(x_i), I_{\tilde{A}}(x_i) \rangle | x_i \in X\}; \forall i = 1, 2, \dots, n$ and $\tilde{B}_s = \{x_i \langle \mu_{\tilde{B}}(x_i), \vartheta_{\tilde{B}}(x_i), I_{\tilde{B}}(x_i) \rangle | x_i \in X\}; \forall i = 1, 2, \dots, n$ be two SFSs with corresponding weight vector $w = \{w_1, w_2, \dots, w_n\}$, where $w_i \in [0, 1]$ and $\sum_{i=1}^n w_i = 1, \forall i = 1, 2, \dots, n$, so that $\mu(x_i)$, $\vartheta(x_i)$ and $I(x_i)$ are membership, non-membership, and

indeterminacy degrees, respectively. Then, Spherical Fuzzy Weighted Sine Minkowski-Hausdorff Similarity Measure is given as follows:

$$WSMHS_{SFS}(\tilde{A}, \tilde{B}) = 1 - \sqrt[\alpha]{\sum_{i=1}^n w_i \sin \frac{\pi}{2} \max_i [|\mu_{\tilde{A}}^2(x_i) - \mu_{\tilde{B}}^2(x_i)|^\alpha, |\vartheta_{\tilde{A}}^2(x_i) - \vartheta_{\tilde{B}}^2(x_i)|^\alpha, |I_{\tilde{A}}^2(x_i) - I_{\tilde{B}}^2(x_i)|^\alpha]} \quad (4.78)$$

Where $\alpha \geq 1$.

Proof: Proof of Definition 4.39 is similar to the proof of Definition 4.36, hence, omit this proof.

Remark 4.24 If $w_i = \frac{1}{n}; \forall i$ then $WSMHS_{SFS}$ will turn into $SMHS_{SFS}$.

4.4 Application of The Proposed Similarity Measures

In this section, to show the applicability of the proposed novel Spherical Fuzzy distance and Similarity measurements are applied to the medical diagnosis problem. The COVID-19 virus which spread in a short period all over the world and affected many aspects of human life is selected as a case study in this section. So, in this section, the medical diagnosis problem for detecting COVID-19 patients is defined and applied the introduced distance and similarity measures. At the end of this section comparative study of new similarity measures was established and some advantages of the proposed work are discussed.

4.4.1 Evaluation: Medical diagnosis algorithms based on similarity measures for COVID-19 diagnosis problem

Similarity measures have been applied to the different types of problems in the literature. The applicability of similarity measures is illustrated by many authors in decision making, pattern recognition, data mining, clustering, and medical diagnosis areas. Fuzzy similarity measures are much more applicable than classical measures when the uncertainties in real-world problems are considered. In this thesis, it will use the proposed similarity measures in the medical diagnosis problem to show their efficiency. Medical diagnosis is one of the significant and interesting areas of fuzzy application because there are so many possibilities in the given judgments. An increase in the number of diseases that threaten human life requires establishing a medical diagnosis system for detecting the infected people as fast as possible under this

uncertainty. One of the recent diseases is Coronavirus disease (COVID-19) spreading at a high speed and becoming a global problem. One of the possible sources of spreading this virus is hospitals and health centers when the patients do not care about the precautions. It is also reported that many of the people who just want to pass the first test of COVID-19 for having relief are unfortunately infected in test centers in hospitals. Because of that, prepare a very simple method of detecting the infected person by using our proposed similarity measures from his/her home without going to test centers. The proposed method can be programmed in a mobile application form and used all over the world to make it easy to determine how much the symptoms are strong and in which level of infection they are, and what they should have to do first. The applicability of this method simply consists of a questionnaire on mobile phones, which is composed of some easy questions related to the symptoms of COVID-19.

Based on different researches on COVID-19 which recognize its symptoms, the most common symptoms are fever, cough, and shortness of breath. But some other symptoms like muscle pain, sputum production, diarrhea, sore throat, loss of smell, and abdominal pain are also possible. It is considered three of these symptoms to determine which patient with which degree of possibility could have Coronavirus disease (COVID-19). It is also recommended by infectious disease specialists that some symptoms have high priorities over others, which causes us to give larger weights to those symptoms than the others. The suggested symptoms from infectious disease specialists are fever (MI_1), cough (MI_2) and shortness of breath (MI_3). Thus, defined a medical indicator set $MI = \{Fever(MI_1), Cough(MI_2), Shortness\ of\ breath(MI_3)\}$. To show the reliability of the proposed methods, this indicator set is applied to determine the patients carrying the COVID-19 virus with high possibility. For this, the symptom-diagnose matrix is constructed for the ten patients which are selected by three physicians who are in close contact with them. Physicians are first informed about our method and it is decided to give some tricky data for a better analysis of these methods. Three of ten patients are infected and the conditions of two patients are uncertain and five patients are not infected. Symptom-diagnosis matrix is constructed by those physicians' comments on some predefined linguistic terms each corresponding to a spherical fuzzy number. These linguistic terms empower the capability of this method in handling the uncertainty of real-world cases in a 3D work-frame. Linguistic terms based on spherical fuzzy numbers are presented in Table 4.4. Let us assume $S =$

$\{No(s_1), Mild(s_2), Severe(s_3)\}$ is a set of physicians' comments to describe the state of medical indicators based on those linguistic terms. Using these sets, the pairwise medical indicators MI_j and physicians' comments S_j will be the indicator factor (M_j) to satisfy $M_j = (MI_j, S_j)$. Then, a reference indicator ($\widetilde{RI}$) matrix which indicates that a patient has the most possible symptoms of the COVID-19 is defined. Then, the similarity measures between medical indicators of each patient and reference indicator ($\widetilde{RI}$) matrix are evaluated. If the similarity measure value is close to 1, the possibility of a person suffering from COVID-19 is more. Now suppose the symptom-diagnose matrix for all 10 patients based on 3 medical indicators is constructed as shown in Tables 4.1, 4.2, and 4.3. The following steps will represent the procedure of solution to detect the infected patient in medical diagnosis problem.

Step 1: Take the symptom-diagnosis matrix based on medical indicators of patients for every physician's comments. These matrices are presented in Tables 4.1, 4.2, and 4.3.

Step 2: Construct the spherical fuzzy symptom-diagnosis matrix by using Table 4.4 to change linguistic terms into spherical fuzzy numbers. The reference indicator ($\widetilde{RI}$) matrix is defined as in Table 4.5.

Step 3: Aggregate the symptom-diagnosis matrices of physicians to obtain the final symptom-diagnosis matrix using SFWAM which is introduced in Eq. (3.11). It is assumed that physicians have equal weights in process of aggregation.

Step 4: Evaluate the similarity measure between each patient and ($\widetilde{RI}$) matrix to obtain the value of similarity based on our proposed similarity measures. Since it is assumed that all indicator factors (M_j), have equal weights so we will apply the normalized similarity measures in section 4.3, and results are presented for different similarity measurements in Tables 4.6, 4.7, 4.8, and 4.9.

Table 4.1: Symptom-Diagnose matrix by physician 1.

	(MI_1, s_1)	(MI_1, s_2)	(MI_1, s_3)	(MI_2, s_1)	(MI_2, s_2)	(MI_2, s_3)	(MI_3, s_1)	(MI_3, s_2)	(MI_3, s_3)
P_1	VLP	MP	EP	LP	LP	VMP	MP	EP	VLP
P_2	EP	EP	VLP	VMP	EP	VLP	VLP	LP	VMP
P_3	MP	LP	VLP	VLP	LP	VMP	VMP	LP	VLP
P_4	LP	LP	VMP	VLP	EP	VMP	VLP	LP	VMP
P_5	VMP	LP	VLP	EP	EP	VLP	LP	LP	VMP
P_6	MP	EP	VLP	VMP	VLP	VLP	VMP	VLP	VLP
P_7	EP	LP	VLP	VLP	LP	VMP	LP	VMP	LP
P_8	LP	MP	LP	VLP	VMP	LP	VLP	LP	VMP
P_9	VMP	LP	VLP	MP	LP	VLP	LP	LP	VMP
P_{10}	VLP	EP	VMP	LP	VMP	VLP	VLP	VLP	VMP

Table 4.2: Symptom-Diagnose matrix by physician 2.

	(MI_1, s_1)	(MI_1, s_2)	(MI_1, s_3)	(MI_2, s_1)	(MI_2, s_2)	(MI_2, s_3)	(MI_3, s_1)	(MI_3, s_2)	(MI_3, s_3)
P_1	LP	MP	EP	VLP	LP	VMP	MP	EP	VLP
P_2	EP	LP	VLP	VMP	EP	VLP	LP	LP	VMP
P_3	MP	LP	VLP	LP	LP	VMP	VMP	LP	LP
P_4	VLP	LP	VMP	VLP	MP	VMP	VLP	LP	VMP
P_5	MP	LP	VLP	EP	MP	VLP	LP	EP	VMP
P_6	MP	LP	VLP	VMP	LP	VLP	VMP	LP	VLP
P_7	MP	LP	VLP	LP	LP	VMP	LP	VMP	VLP
P_8	VLP	MP	LP	LP	VMP	LP	LP	LP	VMP
P_9	MP	VLP	VLP	VMP	LP	VLP	VLP	LP	VMP
P_{10}	VLP	LP	VMP	VLP	VMP	VLP	LP	VLP	VMP

Table 4.3: Symptom-Diagnose matrix by physician 3.

	(MI_1, s_1)	(MI_1, s_2)	(MI_1, s_3)	(MI_2, s_1)	(MI_2, s_2)	(MI_2, s_3)	(MI_3, s_1)	(MI_3, s_2)	(MI_3, s_3)
P_1	LP	VMP	LP	VLP	VLP	VMP	MP	LP	VLP
P_2	MP	LP	VLP	VMP	LP	VLP	VLP	VLP	VMP
P_3	MP	LP	VLP	LP	LP	VMP	VMP	LP	VLP
P_4	VLP	EP	VMP	VLP	MP	VMP	VLP	VLP	VMP
P_5	MP	LP	VLP	EP	MP	VLP	VLP	EP	VMP
P_6	VMP	VLP	VLP	VMP	LP	VLP	VMP	LP	VLP
P_7	MP	VLP	VLP	VLP	LP	VMP	EP	MP	VLP
P_8	VLP	MP	LP	VLP	VMP	LP	VLP	LP	VMP
P_9	VMP	VLP	VLP	VMP	LP	VLP	VLP	VLP	VMP
P_{10}	LP	LP	MP	VLP	MP	VLP	LP	VLP	MP

Table 4.4: Linguistic terms of physicians' diagnosis.

Linguistic terms	SFNs (μ, ϑ, I)
Very Low Possible (VLP)	(0.1, 0.9, 0.0)
Low Possible (LP)	(0.3, 0.7, 0.2)
Equally Possibility (EP)	(0.5, 0.5, 0.5)
More Possible (MP)	(0.7, 0.3, 0.2)
Very More Possible (VMP)	(0.9, 0.1, 0.0)

Table 4.5: Reference Indicators (RI) of each M_j .

	M_1			M_2			M_3		
	(MI_1, s_1)	(MI_1, s_2)	(MI_1, s_3)	(MI_2, s_1)	(MI_2, s_2)	(MI_2, s_3)	(MI_3, s_1)	(MI_3, s_2)	(MI_3, s_3)
$\widetilde{RI}$	(0.1,0.9,0)	(0.7,0.3,0.2)	(0.9,0.1,0)	(0.1,0.9,0)	(0.1,0.9,0)	(0.9,0.1,0)	(0.1,0.9,0)	(0.7,0.3,0.2)	(0.9,0.1,0)

Table 4.6: Spherical Fuzzy Similarity Measures based on Minkowski Distance.

	$MS_{SFS}^{\lfloor \rfloor}(\widetilde{P}_v, \widetilde{RI})$			$MS_{SFS}^{\lceil \rceil}(\widetilde{P}_v, \widetilde{RI})$			$MS_{SFS}^{\log_2}(\widetilde{P}_v, \widetilde{RI})$			$MS_{SFS}^{\lceil \rceil}(\widetilde{P}_v, \widetilde{RI})$		
	$\alpha = 1$	$\alpha = 2$	$\alpha = 3$	$\alpha = 1$	$\alpha = 2$	$\alpha = 3$	$\alpha = 1$	$\alpha = 2$	$\alpha = 3$	$\alpha = 1$	$\alpha = 2$	$\alpha = 3$
P_1	0.6883	0.8427	0.8983	0.5763	0.7697	0.8470	0.7555	0.8818	0.9247	0.5247	0.7281	0.8154
P_2	0.5213	0.7041	0.7917	0.3982	0.5948	0.7026	0.6053	0.7690	0.8414	0.3526	0.5433	0.6553
P_3	0.5422	0.7122	0.7989	0.4189	0.6043	0.7118	0.6250	0.7758	0.8471	0.3720	0.5530	0.6651
P_4	0.8314	0.9288	0.9649	0.7545	0.8913	0.9454	0.8729	0.9477	0.9744	0.7114	0.8671	0.9321
P_5	0.5179	0.7121	0.8033	0.3949	0.6042	0.7176	0.6021	0.7758	0.8507	0.3494	0.5529	0.6713
P_6	0.3648	0.5498	0.6611	0.2562	0.4265	0.5452	0.4487	0.6321	0.7321	0.2231	0.3791	0.4937
P_7	0.6252	0.7932	0.8648	0.5055	0.7044	0.7999	0.7006	0.8425	0.8990	0.4548	0.6573	0.7618
P_8	0.7011	0.8249	0.8819	0.5913	0.7459	0.8238	0.7665	0.8678	0.9122	0.5398	0.7020	0.7887
P_9	0.5095	0.6714	0.7623	0.3867	0.5569	0.6653	0.5941	0.7411	0.8175	0.3419	0.5053	0.6159
P_{10}	0.6729	0.8061	0.8626	0.5586	0.7211	0.7968	0.7423	0.8529	0.8973	0.5070	0.6752	0.7583

Table 4.7: Spherical Fuzzy Similarity Measures based on Minkowski Distance.

	$CMS_{SFS}(\tilde{P}_l, \tilde{R}\tilde{I})$			$SMS_{SFS}(\tilde{P}_l, \tilde{R}\tilde{I})$		
	$\alpha = 1$	$\alpha = 2$	$\alpha = 3$	$\alpha = 1$	$\alpha = 2$	$\alpha = 3$
P_1	0.8115	0.9203	0.9557	0.5755	0.7756	0.8508
P_2	0.6566	0.8250	0.8931	0.3788	0.5912	0.7011
P_3	0.6794	0.8335	0.9021	0.4021	0.5987	0.7081
P_4	0.9084	0.9795	0.9940	0.7597	0.8911	0.9453
P_5	0.6622	0.8393	0.9074	0.3674	0.5958	0.7137
P_6	0.5021	0.7032	0.8129	0.2072	0.3908	0.5170
P_7	0.7612	0.8942	0.9422	0.4949	0.7049	0.8009
P_8	0.8018	0.9074	0.9495	0.6018	0.7506	0.8258
P_9	0.6359	0.7977	0.8787	0.3732	0.5472	0.6569
P_{10}	0.7807	0.8848	0.9318	0.5700	0.7307	0.8006

Table 4.8: Spherical Fuzzy Similarity Measures based on Minkowski-Hausdorff Distance.

	$MHS_{SFS}^{\lfloor}(\tilde{P}_l, \tilde{R}\tilde{I})$			$MHS_{SFS}^{\lfloor e \rfloor}(\tilde{P}_l, \tilde{R}\tilde{I})$			$MHS_{SFS}^{\lfloor \log_2 \rfloor}(\tilde{P}_l, \tilde{R}\tilde{I})$			$MHS_{SFS}^{\lfloor r \rfloor}(\tilde{P}_l, \tilde{R}\tilde{I})$		
	$\alpha = 1$	$\alpha = 2$	$\alpha = 3$	$\alpha = 1$	$\alpha = 2$	$\alpha = 3$	$\alpha = 1$	$\alpha = 2$	$\alpha = 3$	$\alpha = 1$	$\alpha = 2$	$\alpha = 3$
P_1	0.8256	0.6885	0.5993	0.7468	0.5766	0.4777	0.8684	0.7558	0.6774	0.7030	0.5250	0.4278
P_2	0.7525	0.8378	0.8857	0.6531	0.7631	0.8291	0.8094	0.8780	0.9151	0.6032	0.7209	0.7948
P_3	0.7541	0.8408	0.8870	0.6551	0.7671	0.8310	0.8107	0.8803	0.9161	0.6053	0.7253	0.7970
P_4	0.9083	0.9540	0.9743	0.8614	0.9288	0.9598	0.9323	0.9664	0.9813	0.8320	0.9120	0.9498

P_5	0.7473	0.8359	0.8848	0.6468	0.7606	0.8279	0.8052	0.8765	0.9144	0.5966	0.7181	0.7935
P_6	0.6757	0.7660	0.8227	0.5619	0.6700	0.7430	0.7448	0.8205	0.8661	0.5102	0.6208	0.6989
P_7	0.7878	0.8744	0.9166	0.6976	0.8132	0.8735	0.8382	0.9064	0.9386	0.6499	0.7768	0.8461
P_8	0.8325	0.8953	0.9262	0.7561	0.8428	0.8874	0.8738	0.9224	0.9457	0.7131	0.8105	0.8625
P_9	0.7391	0.8220	0.8701	0.6367	0.7420	0.8073	0.7983	0.8655	0.9031	0.5862	0.6977	0.7701
P_{10}	0.8197	0.8899	0.9205	0.7390	0.8351	0.8791	0.8637	0.9183	0.9415	0.6945	0.8017	0.8527

Table 4.9: Spherical Fuzzy Similarity Measures based on Minkowski-Hausdorff Distance.

	$CMHS_{SFS}(\tilde{P}_i, \tilde{R}I)$			$SMHS_{SFS}(\tilde{P}_i, \tilde{R}I)$		
	$\alpha = 1$	$\alpha = 2$	$\alpha = 3$	$\alpha = 1$	$\alpha = 2$	$\alpha = 3$
P_1	0.7828	0.9455	0.9798	0.5301	0.4768	0.4267
P_2	0.6405	0.8985	0.9603	0.3603	0.3296	0.3090
P_3	0.6477	0.8985	0.9596	0.3636	0.3372	0.3121
P_4	0.8939	0.9816	0.9951	0.7437	0.6298	0.5715
P_5	0.6360	0.8978	0.9606	0.3453	0.3248	0.3063
P_6	0.4889	0.8290	0.9287	0.1937	0.2059	0.2042
P_7	0.7185	0.9295	0.9740	0.4345	0.4043	0.3752
P_8	0.7679	0.9355	0.9755	0.5631	0.4602	0.4007
P_9	0.6080	0.8809	0.9520	0.3365	0.3019	0.2805
P_{10}	0.7575	0.9285	0.9710	0.5291	0.4509	0.3888

Step 5: Compare the similarity measures of each patient. Any patient who has a higher similarity measure (so close to 1) is considered an infected person. The ranking of patients results in Tables 4.10 and 4.11.

Table 4.10: Ranking for every patient by using the similarity measure based on Minkowski Distance

	$MS_{SFS}^{\lfloor \cdot \rfloor}(\tilde{P}_v, \tilde{R}\tilde{I})$			$MS_{SFS}^{\lceil \cdot \rceil}(\tilde{P}_v, \tilde{R}\tilde{I})$			$MS_{SFS}^{\lfloor \log_2 \cdot \rfloor}(\tilde{P}_v, \tilde{R}\tilde{I})$			$MS_{SFS}^{\lceil \cdot \rceil}(\tilde{P}_v, \tilde{R}\tilde{I})$			$CM_{SFS}(\tilde{P}_v, \tilde{R}\tilde{I})$			$SM_{SFS}(\tilde{P}_v, \tilde{R}\tilde{I})$		
	$\alpha = 1$	$\alpha = 2$	$\alpha = 3$	$\alpha = 1$	$\alpha = 2$	$\alpha = 3$	$\alpha = 1$	$\alpha = 2$	$\alpha = 3$	$\alpha = 1$	$\alpha = 2$	$\alpha = 3$	$\alpha = 1$	$\alpha = 2$	$\alpha = 3$	$\alpha = 1$	$\alpha = 2$	$\alpha = 3$
P_1	3	2	2	3	2	2	3	2	2	3	2	2	2	2	2	3	2	2
P_2	7	8	8	7	8	8	7	8	8	7	8	8	8	8	8	7	8	8
P_3	6	6	7	6	6	7	6	6	7	6	6	7	6	7	7	6	6	7
P_4	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
P_5	8	7	6	8	7	6	8	7	6	8	7	6	7	6	6	9	7	6
P_6	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10
P_7	5	5	4	5	5	4	5	5	4	5	5	4	5	4	4	5	5	4
P_8	2	3	3	2	3	3	2	3	3	2	3	3	3	3	3	2	3	3
P_9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	8	9	9
P_{10}	4	4	5	4	4	5	4	4	5	4	4	5	4	5	5	4	4	5

Table 4.11: Ranking for every patient by using the similarity measure based on Minkowski-Hausdorff Distance

	$MHS_{SFS}^{\lfloor \cdot \rfloor}(\tilde{P}_v, \tilde{R}\tilde{I})$			$MHS_{SFS}^{\lceil \cdot \rceil}(\tilde{P}_v, \tilde{R}\tilde{I})$			$MHS_{SFS}^{\lfloor \log_2 \cdot \rfloor}(\tilde{P}_v, \tilde{R}\tilde{I})$			$MHS_{SFS}^{\lceil \cdot \rceil}(\tilde{P}_v, \tilde{R}\tilde{I})$			$CMHS_{SFS}(\tilde{P}_v, \tilde{R}\tilde{I})$			$SMHS_{SFS}(\tilde{P}_v, \tilde{R}\tilde{I})$		
	$\alpha = 1$	$\alpha = 2$	$\alpha = 3$	$\alpha = 1$	$\alpha = 2$	$\alpha = 3$	$\alpha = 1$	$\alpha = 2$	$\alpha = 3$	$\alpha = 1$	$\alpha = 2$	$\alpha = 3$	$\alpha = 1$	$\alpha = 2$	$\alpha = 3$	$\alpha = 1$	$\alpha = 2$	$\alpha = 3$
P_1	3	2	2	3	2	2	3	2	2	3	2	2	2	2	2	3	2	2
P_2	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7
P_3	6	6	6	6	6	6	6	6	6	6	6	6	6	6	8	6	6	6
P_4	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
P_5	8	8	8	8	8	8	8	8	8	8	8	8	8	8	6	8	8	8

P_6	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10
P_7	5	5	5	5	5	5	5	5	5	5	5	5	5	4	4	5	5	5
P_8	2	3	3	2	3	3	2	3	3	2	3	3	3	3	3	2	3	3
P_9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9
P_{10}	4	4	4	4	4	4	4	4	4	4	4	4	4	4	5	5	4	4

4.4.2 Discussion

In this section, the results of applying the proposed SFSs f-similarity measures to COVID-19 medical diagnosis are presented in more detail. As previously discussed in section 4.2, some novel distances based on Minkowski and Minkowski-Hausdorff distances are introduced in a spherical fuzzy environment. Then in section 4.3, it is introduced the novel SFSs f-similarity measures based on Minkowski and Minkowski-Hausdorff distances, and some properties are discussed in detail. Afterward, an application of the proposed SFSs f-similarity measures to COVID-19 medical diagnosis is introduced. So, the goal of the application is to recognize patients with the highest possibility of infection. The proposed algorithm based on the SFSs f-similarity measures for the application part is illustrated and discussed through the five steps. In this application there exist three physicians as an expert in epidemic infectious diseases. In step 1, symptom-diagnosis matrices based on medical indicators of patients for each physician are collected and shown in Tables 4.1, 4.2, and 4.3. In step 2, spherical fuzzy symptom-diagnosis matrices will be established by changing linguistic terms into SFNs by using Table 4.4, also reference indicator ($\widetilde{RI}$) matrix is introduced as in Table 4.5. Step 3, symptom-diagnosis matrices of physicians will be aggregated using the SFWAM aggregation operator in Eq. (3.11). Then, in step 4, proposed SFSs f-similarity measures in section 4.3 are calculated for each patient with ($\widetilde{RI}$) matrix.

Also, it is assumed that indicator factors (M_j), have equal weights, and the results of this step are presented in Tables 4.6, 4.7, 4.8, and 4.9. Finally, in step 5, compare the similarity values related to each patient, and patients with high similarity values are identified as the most possible infected people in the set of patients. So, in this way the goal of the application, which is recognizing the patients with high infectious risk of COVID-19 based on the designed expert system in the spherical fuzzy environment, is satisfied. Any patients in the set of patients will be ranked based on the SFSs similarity values, in descending way and the results are shown in Tables 4.10, and 4.11.

4.4.3 Results analysis

Proposed f-similarity measures are based on Minkowski and Minkowski-Hausdorff distances which are the generalized formats of some well-known distances like Hamming and Euclidean distances. So, $\alpha = 1, 2 \text{ and } 3$ values are assumed in computations in the proposed solution method. So, two main categories of the proposed f-similarity measures, spherical fuzzy Minkowski distance-based f-similarities, and spherical fuzzy Minkowski-Hausdorff distance-based f-similarities are introduced, also it is considered different functions like linear, exponential, logarithmic, rational, and trigonometric as f-similarity functions. To sum it up it could refer that in this study 24 different SFSs similarity measures are introduced and could consider different α for each of those. Similarity measure values for different patients (alternatives) are compared to each other and turn out ranking of the possible COVID-19 infectious risk for those patients (alternatives). The Computation results of introduced SFSs f-similarity measures are presented in Tables 4.6, 4.7, 4.8, and 4.9 and related rankings are given in Tables 4.10, and 4.11. Based on referred tables, it is determined that the risk of COVID-19 infectious in patient P_4 based on all SFSs f-similarities are too high and stand in the first place as an infected person. The recognized order based on decreasing the risk of COVID-19 virus among the set of patients is given as: $P_4 > P_1 > P_8 > P_{10} > P_7 > P_3 > P_2 > P_5 > P_9 > P_6$. Also, in the rest of this part, more details about the results of SFSs f-similarity measures are discussed in Figures 4.1 and 4.2.

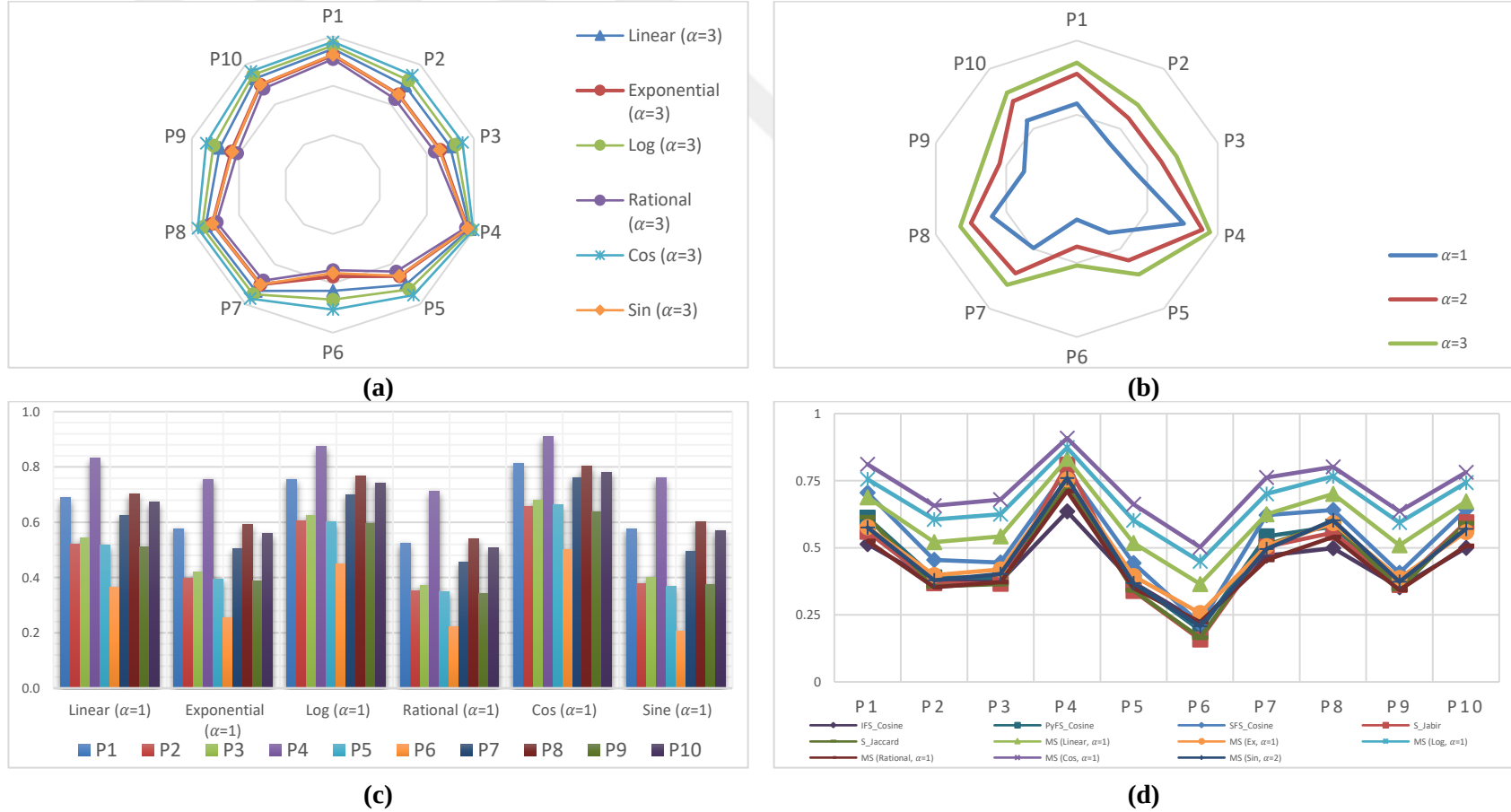


Figure 4.1: Spherical fuzzy Minkowski distance-based f-similarity measures: **(a):** Radar diagram of SFSs Minkowski distance-based f-similarities for all patients when $\alpha = 3$. **(b):** Radar diagram of SMS_{SFS} for all patients based on different $\alpha = 1, 2$ and 3 . **(c):** SFSs Minkowski distance-based f-similarities for all patients when $\alpha = 1$. **(d):** Comparison between SFSs Minkowski distance-based f-similarities ($\alpha = 1$) and selected comparative similarities IFC^{α^1} , $PyFC^{\alpha^1}$, SFC^{α^1} , S^s and JS_{SFS} .

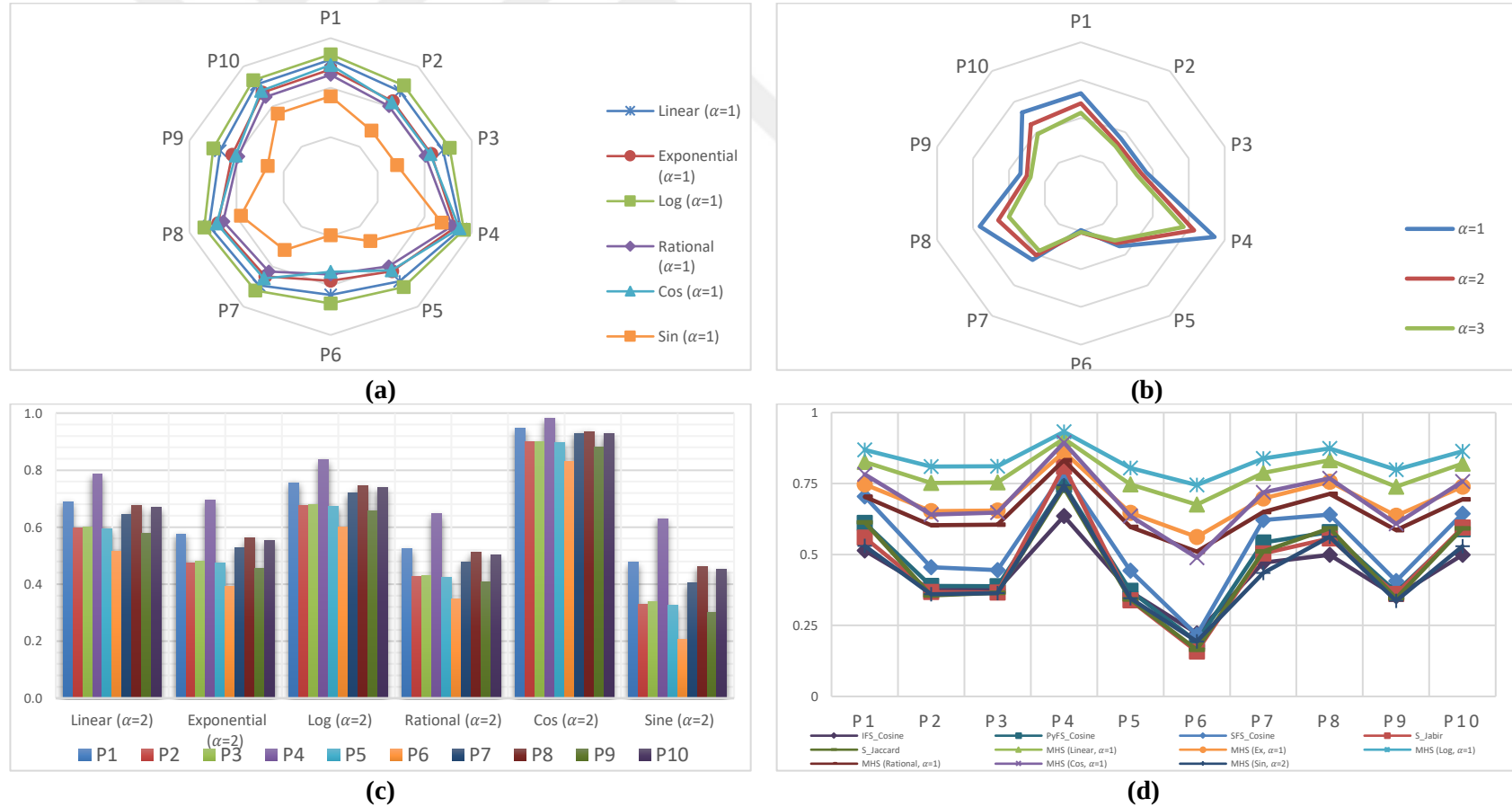


Figure 4.2: Minkowski-Hausdorff Spherical fuzzy Minkowski-Hausdorff distance-based f-similarity measures: (a): Radar diagram of SFSs Minkowski-Hausdorff distance-based f-similarities for all patients when $\alpha = 1$. (b): Radar diagram of $SMHS_{SFS}$ for all patients based on different $\alpha = 1, 2$ and 3 . (c): SFSs Minkowski-Hausdorff distance-based f-similarities for all patients when $\alpha = 2$. (d): Comparison between SFSs Minkowski-Hausdorff distance-based f-similarities ($\alpha = 1$) and selected comparative similarities IFC^{α^1} , $PyFC^{\alpha^1}$, SFC^{α^1} , S^s and JS_{SFS} .

When the similarity measures for each patient under different α values are compared, patient 4 has a higher similarity measure (so close to 1) and is considered an infected person in terms of all proposed similarity measures. Undoubtedly, patient 6 has lower similarity measure values and can be categorized as a healthy person or less possibly infected by COVID-19. However, the ranking of other patients varies according to the type of the similarity measure functions under different α values. For example, patient 6 stands in 10th rank for all similarity measures, but its $MS_{SFS}^{\square}(\tilde{P}_i, \tilde{RI}) = 0.6729$ for $\alpha = 1$ and $MS_{SFS}^{\square}(\tilde{P}_i, \tilde{RI}) = 0.8626$ for $\alpha = 3$ and it is higher than other values, but in comparison, it is still in the 10th ranking. Increasing or decreasing alfa values, can control the process of how much will be optimistic or pessimistic. High alfa values could apply to more pessimistic areas. An applicable advantage of these SFSs f-similarities based on Minkowski and Minkowski-Hausdorff distances is that could control the degree of being optimistic and pessimistic. So, in real-world applications, decision-makers could consider different types of SFSs distance-based f-similarities with different α values. For example, assume in a decision process supposed to apply SFSs Minkowski distance-based f-similarity measure (linear function) $MHS_{SFS}^{\square}$, while determined above 0.6 is suspicious for COVID-19 disease. But point is that for $MHS_{SFS}^{\square}$ if the $\alpha = 1$ in Tables 4.8, patients $P_4, P_1, P_8, P_{10}, P_7$ are suspicious patients while patients P_3, P_2, P_5, P_6, P_9 and P_6 in the safety zone. But if for $MHS_{SFS}^{\square}$ if $\alpha = 2$ is considered then based on Tables 4.8, all patients except P_6 are suspicious of COVID-19, and whenever $\alpha = 3$, then all patients are suspicious of the COVID-19 Virus. So, this level of being optimistic and pessimistic for different SFSs f-similarities are controllable by α values. To have illustrative example could refer to some places such as airports which are under a rigorous and very pessimistic point of view for controlling COVID-19, so airports could perfectly control and increase the similarity degree to a very pessimistic level for a very high level of COVID-19 diagnosis of passengers.

4.4.4 Comparative analysis

In this part of section 4.4, to show the performance of the proposed similarities, it is compared the proposed SFSs f-similarities with some methods selected from the

literature review part. Comparative analysis constitutes of two types of comparison methods, the first one is some existed SFSs similarities and the second one is an MCDM preference selection method applied in the COVID-19 medical diagnosis field of study. The following sub-parts of this section will explain the selected comparison methods.

4.4.4.1 Comparative analysis based on similarity measures

As the first comparative analysis method, some existed similarity measures from the literature review part in the framework of different fuzzy extensions are selected to indicate the feasibility of proposed SFSs f-similarities to the COVID-19 medical diagnosis problem. In the literature review, some of the research studies together with significant contributions of those are referred to. Cosine similarity with membership, non-membership, and indeterminacy degrees (Shi & Ye, 2013) are introduced for vague sets to solve fault diagnosis of turbines. (Wei & Wei, 2018b) are introduced cosine similarity measures for Pythagorean fuzzy sets (PyFSs) together with some applications in pattern recognition and medical diagnosis problems. In (Rafiq et al., 2019) some cosine similarities in the spherical fuzzy environment are introduced, then (Khan et al., 2020) proposed some new similarity measures together with their application in large project selection. The next concentrated article for comparative analysis is our previous research which introduces some novel spherical fuzzy similarities (Shishavan et al., 2020). So, to prepare a comprehensive comparative analysis by using existed similarities, previously mentioned articles are selected. In this comparative analysis, five different types of similarities in the spherical fuzzy environment are selected to compare with proposed novel SFSs distance-based f-similarities in this manuscript. Cosine similarity for IFSs (IFC^{α^1}), PyFSs ($PyFC^{\alpha^1}$) and SFSs (SFC^{α^1}) together with SFSs similarity (S^s) and Jaccard similarity for SFSs (JS_{SFS}) are the selected similarity measures for comparative analysis, and results of applying these similarities to COVID-19 medical diagnosis problem are presented in Table 4.12. Also, in Figures 4.1 and 4.2 it is tried to show a graphical comparison among above selected similarities and proposed SFSs distance-based f-similarities. Table 4.12 confirms that patient P_4 stands in first place which means that has the highest risk of the COVID-19 infectious risk, also, patient P_6 stands have the least risk of COVID-19 infectious. The total ranking based on the comparative methods are

significantly close to ranking in Table 4.10 and 4.11 which are related to our proposed similarities.

4.4.4.2 Comparative analysis based on integrated TOPSIS with SFSs entropy weighting method

To show the validity and feasibility of the proposed novel SFSs distance-based f-similarities, prepared another comparative analysis with a different basis than similarity measures. So, for the second part of the comparative analysis, an integrated TOPSIS with SFSs entropy weighting method is proposed by (Ashraf & Abdullah, 2020) to solve the COVID-19 emergency decision process. In this part, the integrated TOPSIS with SFSs entropy weighting method is hired with some adaptations and applied to the COVID-19 case study. In what follows, will briefly discuss the steps of the comparison method taken from (Ashraf & Abdullah, 2020). Results of the applying this method to COVID-19 medical diagnosis application is presented in Table 4.14.

Steps 1-3: of the integrated TOSIS with SFSs entropy weighting method, are the same as our proposed methodology in this manuscript, and the results of these steps are shown in Tables 4.1, 4.2, and 4.3.

Remark 4.25 After step 2, it is suggested to normalize the constructed spherical fuzzy symptom-diagnosis matrices from step 2. But in this application, there is no need for normalization, because the decision matrices are in a uniform format and all medical indicators are cost-type criteria.

Step 4: Calculate weights of indicator factor (w_j) by using the entropy measure which is defined in Eq (4.79). The result of this step is shown in Table 4.13.

$$w_j = \frac{1 + \frac{1}{m} \sum_{j=1}^m (\mu_i \ln \mu_i + \vartheta_i \ln \vartheta_i + I_i \ln I_i)}{\sum_{i=1}^k \left(1 + \frac{1}{m} \sum_{j=1}^m (\mu_i \ln \mu_i + \vartheta_i \ln \vartheta_i + I_i \ln I_i) \right)} \quad (4.79)$$

Where $i = \{1, 2, \dots, k\}$ and $j = \{1, 2, \dots, m\}$ are respectively sets of alternatives and indicator factors.

Table 4.12: Comparative analysis based on similarity measures (Cosine similarity for IFSs (IFC^{α^1}), PyFSs ($PyFC^{\alpha^1}$), SFSs (SFC^{α^1}), SFSs similarity (S^S) and Jaccard similarity for SFSs (JS_{SFS})).

	$IFC^{\alpha^1}(\tilde{P}_v, \tilde{RI})$	Rank	$PyFC^{\alpha^1}(\tilde{P}_v, \tilde{RI})$	Rank	$SFC^{\alpha^1}(\tilde{P}_v, \tilde{RI})$	Rank	$S^S(\tilde{P}_v, \tilde{RI})$	Rank	$JS_{SFS}(\tilde{P}_v, \tilde{RI})$	Rank
P_1	0.513470704	2	0.6113932	2	0.704485	2	0.559592	3	0.6136	2
P_2	0.374692661	7	0.3888509	6	0.454724	6	0.367865	6	0.3560	7
P_3	0.375624724	6	0.3882693	7	0.444513	7	0.366141	7	0.3647	6
P_4	0.635061965	1	0.7448390	1	0.790171	1	0.810337	1	0.7349	1
P_5	0.369461499	8	0.3720901	8	0.443313	8	0.338896	9	0.3414	9
P_6	0.222342397	10	0.1928396	10	0.216859	10	0.158255	10	0.1645	10
P_7	0.471027571	5	0.5419088	5	0.621463	5	0.504129	5	0.5105	5
P_8	0.498639659	4	0.5787702	4	0.640562	4	0.556422	4	0.5923	3
P_9	0.352643117	9	0.3683467	9	0.406198	9	0.361206	8	0.3461	8
P_{10}	0.499164262	3	0.5891232	3	0.643275	3	0.594518	2	0.5915	4

Table 4.13: Weights of indicator factors (w_j).

	M_1			M_2			M_3		
	(MI_1, s_1)	(MI_1, s_2)	(MI_1, s_3)	(MI_2, s_1)	(MI_2, s_2)	(MI_2, s_3)	(MI_3, s_1)	(MI_3, s_2)	(MI_3, s_3)
w_j	0.060909	0.023052	0.175373	0.121302	0.05079	0.209323	0.108736	0.057828	0.192689

Step 5: Calculate the weighted aggregated symptom-diagnosis matrix by using Eq (3.11) to multiply the weights of indicators (w_j) in aggregated symptom-diagnosis matrix resulted in step 3.

Step 6: Establish the weighted positive reference indicator ($\widetilde{WPRI}$) and weighted negative reference indicator ($\widetilde{WNRi}$), so we will have:

Step 6.1: Establish the positive ideal solution ($\widetilde{PIS}$) and negative ideal solution ($\widetilde{NIS}$) based on (Ashraf & Abdullah, 2020).

Step 6.2: Establish the weighted reference indicator ($\widetilde{WRI}$) by using Eq (3.8) and multiplying the reference indicator matrix ($\widetilde{RI}$) with calculated weights of indicators (w_j) in step 4.

Step 6.3: By using Eq (3.8), multiply the ($\widetilde{PIS}$) and ($\widetilde{NIS}$) with ($\widetilde{WRI}$), then ($\widetilde{WPRI}$) and ($\widetilde{WNRi}$) are calculated respectively.

Step 7: Evaluate the positive (d^+) and negative (d^-) distance measurements, by calculating the distances among weighted aggregated symptom diagnosis matrix with ($\widetilde{WPRI}$) and ($\widetilde{WNRi}$), based on (Ashraf & Abdullah, 2020), and results are shown in Table 4.14.

Step 8: Evaluate the closeness ratio (CR_i) of each patient as follows:

$$CR_i = \frac{d_i^-}{d_i^+ + d_i^-} \quad (4.80)$$

Step 9: Rank the alternatives based on evaluated values for closeness ratio (CR_i) in descending way. Results of steps 8 and 9 are presented in Table 4.14.

Table 4.14: Comparative analysis based on integrated TOPSIS with SFSs entropy weighting method (results of steps 7, 8, and 9).

	d_i^-	d_i^+	CR_i	Rank
P_1	0.1738164	0.1789449	0.507269126	3
P_2	0.1958518	0.1910604	0.493808235	6
P_3	0.1995623	0.1948621	0.494041661	5
P_4	0.2426759	0.2643976	0.521418711	2
P_5	0.1730116	0.1676052	0.492063737	9
P_6	0.1616027	0.1558066	0.490869755	10
P_7	0.1800077	0.1747487	0.492587944	8
P_8	0.1681087	0.1683783	0.50040063	4
P_9	0.1922564	0.1873986	0.493602342	7
P_{10}	0.1650996	0.1919693	0.537625327	1

Based on Table 4.14 confirms that patient P_4 stands in second place which means that the second comparative analysis model based on integrated TOPSIS with SFSs entropy weighting method is confirming that P_4 has a high risk of the COVID-19 infectious, also, patient P_6 stands have the least risk of COVID-19 infectious. The total ranking based on the comparative methods is significantly close to the ranking in Tables 4.10, 4.11, and 4.12 which are related to our proposed similarities and first comparison method.

Also, to have a better comparison between ranks, it is evaluated the concordance coefficient of the rankings. Kendall's coefficient and spearman correlation coefficient which are proposed in (Field, 2014) calculated, and the results are shown in Table 4.15. Based on (Field, 2014) Kendall's concordance coefficient ($\mathcal{W}$) is between 0 and 1, how much $\mathcal{W}$ is close to 1 the concordance between rankings set is higher. Spearman correlations coefficient (r_s) is ranging between -1 and 1, where -1 shows the complete disagreement and 1 shows a complete agreement among rankings. In this study it is assumed two cases to complete the comparative analysis part of the discussion, case 1: considers the rankings related to the SFSs similarities proposed in this manuscript, and the case 2 considers the resulted rankings of comparative analysis methods (similarity measures: IFC^{α^1} , $PyFC^{\alpha^1}$, SFC^{α^1} , S^s , JS_{SFS} and integrated TOPSIS with SFSs entropy weighting method). Both cases resulted high Kendall's coefficient ($\mathcal{W}$) and Spearman correlations coefficient (r_s) which indicates a high degree of agreement among different rankings. And by the concordance of proposed SFSs similarity methods with each other and also with comparative methods shows high reliability of the proposed SFSs similarity measures. So, based on Table 4.15, the feasibility and reliability of proposed SFSs similarity measures are illustrated.

Table 4.15: Kendall's concordance coefficient $\mathcal{W}$ and spearman correlation coefficient r_s .

	Case 1: Considering rankings of Proposed SFSs similarities	Case 2: Considering rankings of Proposed SFSs similarities together with comparative methods
$\mathcal{W}$	0.975514403	0.961341304
r_s	0.974814815	0.960398409

In Figures 4.1 and 4.2, vertical axes show the similarity degree of each patient and horizontal axes is showing the patient number. These figures give details of

comparison among the proposed SFSs similarities with each other and with comparative methods. Also, based on Figures 4.1 and 4.2 could have a better understanding of the pessimistic and optimistic comparison between proposed similarity measures. Figures 4.1.a and 4.2.a are comparing the similarity values of SFSs Minkowski distance-based f-similarities for all patients when $\alpha = 3$ and SFSs Minkowski-Hausdorff distance-based f-similarities for all patients when $\alpha = 1$, respectively. Figures 4.1.b and 4.2.b give details about for example SM_{SFS} and $SMHS_{SFS}$ respectively, for all patients based on different $\alpha = 1, 2 \text{ and } 3$. Figure 4.1.c present the comparison similarity measures based on SFSs Minkowski distance-based f-similarities for all patients when $\alpha = 1$ and Figure 4.2.c give the comparison details about SFSs Minkowski-Hausdorff distance-based f-similarities for all patients when $\alpha = 2$. Figures 4.1.d and 4.2.d illustrate the comparison between proposed SFSs similarity measures with selected comparative similarities IFC^{α^1} , $PyFC^{\alpha^1}$, SFC^{α^1} , S^s and JS_{SFS} .

4.4.4.3 Advantages of proposed SFSs distance-based f-similarity measures

- i. Spherical fuzzy sets (SFSs) are significant extensions of fuzzy sets with more capability in explaining the uncertainty based on a 3-dimensional framework. SFSs has three membership, non-membership, and indeterminacy degrees determined by an expert-based system. SFSs let decision-makers have more flexibility in giving different values for uncertainty degrees (membership, non-membership, and indeterminacy degrees).
- ii. Minkowski and Minkowski-Hausdorff distances proposed for SFSs in this manuscript are generalized from many other distances like Hamming and Euclidean distances. Also, some properties of the proposed novel distances are discussed in detail. Introduced distance-based similarities are also contained beneficial of those distances while some properties of those are discussed.
- iii. One of the most significant advantages of the proposed algorithm based on SFSs distance-based f-similarities to solve the COVID-19 medical diagnosis problem is controlling the optimistic and pessimistic levels. Based on the proposed similarity measures, changing its value can control the optimistic and

pessimistic levels. To have a more optimistic point of view could choose a low value and to have a more pessimistic perspective, could select a high value.

- iv. Another significant advantage of proposed SFSs distance-based f-similarities in this manuscript relates to functions that could be used in the base formula of f-similarity. Different functions like linear, exponential, logarithmic, rational, cosine and sine functions are used and discussed in detail in this study. The more important point is that in introduced SFSs distance-based f-similarities are available to use many different functions, and each function has unique features itself. It could refer that SFSs, distance-based f-similarities could be more generalized and adapted according to problem types by using different functions that satisfy the properties in Definition 4.13.
- v. Almost all existed similarity measures for intuitionistic fuzzy sets (IFSs), Pythagorean fuzzy sets (PyFSs), and spherical fuzzy sets (SFSs) are some exceptional cases of the proposed novel SFSs Minkowski and Minkowski-Hausdorff distance-based f-similarity measures.
- vi. Finally, it could refer to the introduced MAGDM algorithm based on the study's proposed similarity measure, which comparative analysis fully supports the feasibility and approves the method's performance.

4.5 Conclusion

Spherical fuzzy sets are the latest extension of ordinary fuzzy sets. In the literature, there is still a shortage of development of the theory of these sets. In this paper, comprehensive research on distance and similarity measures of SFSs have been investigated. The existing distance and similarity measures of SFSs are mainly based on the geometric distance model, and the most commonly used distance measurements are the Hamming, Euclidean, and spherical distance measures. It has been proposed some novel distance and similarity measures of spherical fuzzy sets such as Minkowski, Minkowski-Hausdorff, Weighted Minkowski, and Weighted Minkowski-Hausdorff distances. In addition, trigonometric and f-similarity measures are developed based on the proposed distance measurements in this paper. The newly defined similarity measures are applied to the medical diagnosis problem of the COVID-19 virus and results are compared. The generalization of new similarity and

distance measures was discussed with the help of some remarks. The new similarity measures of SFSs were employed for a problem of medical diagnosis for the COVID-19 virus. Comparative analysis was performed within proposed similarity measures.

For future research, the proposed similarity measures and distances can be applied to many real-life complex applications, including linguistic summarization risk analysis, pattern recognition, and image processing.



5. CONCLUSIONS AND FUTURE DIRECTIONS

The uncertainty is the most significant aspect in defining and considering a problem in a realistic uncertain world since that will comprehensively make it more complex to model and solve. The fuzzy logic theory is one of the strongest tools in defining uncertainty and many methods for different problems are investigated in the fuzzy environments. Fuzzy logic is also progressively extended both conceptually and methodologically. Both aspects are hot topics of many research studies all over the world. Different generalizations are introduced and each one tried to complement the previous versions. Based on the literature review section, some additional dimensions are introduced for DMs to express their ambiguity also the domain of those degrees is enlarged to increase the capacity of ambiguity handled by those new generations of fuzzy sets. To exemplify could refer to IFS, IFS-2, HFS, PyFS, PFS, qROFS, and SFS. The SFS discussed in detail in previous sections of this thesis, is one of the recent high-dimensional fuzzy sets that consider membership, non-membership, and indeterminacy degrees to express and evaluate the uncertainty. SFS also has a 4th degree which is called the refusal degree and is discussed in detail in this thesis. The other feature that makes SFSs more significant in evaluating the uncertainty is the large domain that provides which DMs could express their ideas more freely because the sum square of those degrees should belong to the interval $[0, 1]$.

Many novel methodologies are proposed for SFS by different researchers. Also, many complex problems are evaluated based on the SFS framework to utilize the SFS novel eligibilities in solving the uncertainty. In the literature review section, some of those methods are referred to and tried to cover some applications in different research fields like healthcare and medical intelligence, supply chain management, disaster management, and many other fields. This thesis, it is tried to create a strong connection between different sections, to comprehensively deliver the linkage between the introduced concepts and the goal of the thesis. In this thesis, the main goal is to introduce some novel data fusion methods in the SFS environment and investigate those methods in detail. Afterward, to show the applicability of the introduced methods some novel MAGDM algorithms are suggested and then evaluated through an application case study. Also to show the validity and reliability of the proposed methods some comparative analyses have been done to compare the results.

In chapter 3 of this thesis, some novel data fusion methods for spherical fuzzy sets to aggregate the uncertain data are proposed. The novel Harmonic mean aggregation operators are designed based on 2 types of Algebraic and Einstein t-norms and t-conorms, then three kinds of weighted, ordered weighted, hybrid ordered weighted operators are proposed. The proposed SFNAWHM, SFNEWHM, SFNAOWHM, SFNEOWHM, SFNAHOWHM, and SFNEHOWHM are discussed in detail in section 3.2, and also a novel MAGDM algorithm is proposed based on these operators to show the applicability of the proposed operators. Thereafter, the illustrative example of air quality from (Ashraf & Abdullah, 2019) is selected and solved by using the proposed aggregation operators. The results are completely compatible and comprehensively refer to the reliability of the proposed aggregation methods.

In chapter 4 of the thesis, other concepts of spherical fuzzy data fusion, the novel spherical fuzzy distance, and similarity measures are introduced. In this chapter, of the novel Minkowski and Minkowski-Hausdorff distance measurements are introduced and proved. Also, all details are investigated about the parameters of the proposed novel methods. In the second part of this chapter, the novel concept of spherical fuzzy f-similarity measures based on the previously introduced Minkowski and Minkowski-Hausdorff distance measures is proposed. The novel spherical fuzzy f-similarity has the ability to use different functions to utilize specific function characteristics suitable for different problems. This thesis proposed 4 main linear, exponential, logarithmic, and rational functions. Also, Minkowski and Minkowski-Hausdorff distances could take different values which means have the ability to turn into some important distances like hamming, and Euclidean distances.

In the final part of this chapter, to show the applicability of the proposed methods an application of the COVID-19 diagnosis problem is investigated to show the sensitivity and validity of the proposed methods. Also, in this part, the results are discussed in detail together with a comparative analysis. The comparison analysis part includes two different methods, one based on other similarity measures by different researchers, and one based on the well-known TOSIS method proposed (Ashraf & Abdullah, 2020). Then results of both methods are evaluated and compared with the results of my proposed measurements. The methods completely confirm each other and to show this confirmation, the concordance coefficient of the rankings is evaluated. Kendall's coefficient and spearman correlation coefficient (Field, 2014), are the concordance

coefficients. Both of the coefficients have high values close to one to show the reliability of the proposed methods.

For future studies, it is suggested the application of the proposed methods in different problems like supply chain management, alternative material/machine selection, disaster management, allocation and assignment problems, and healthcare and medical intelligent systems. All the above-mentioned fields and many problems in each field could be the subject of new research based on the proposed methods of this thesis. Another future research direction could be the investigation of new methods like new aggregation operators, distance, and similarity measures in the spherical fuzzy environment. Each of the proposed methods has specific characteristics and is appropriate for a group of specific problems.





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