T. R.
VAN YUZUNCU YIL UNIVERSITY
INSTITUTE OF NATURAL AND APPLIED SCIENCES
DEPARTMENT OF STATISTICS

ANALYTICAL AND APPROXIMATE SOLUTIONS FOR VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS AND IT’S APPLICATIONS

M.Sc. THESIS

PREPARED BY: Payam Mahmood MUSTAFA
SUPERVISOR: Prof. Dr. Fevzi ERDOĞAN

VAN-2022

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ACCEPTANCE AND APPROVAL PAGE

This thesis entitled “Analytical and Approximate Solutions for Volterra Integro-Differential Equations and It’s Applications” is presented by Payam Mahmood MUSTAFA under the supervision of Prof. Dr. Fevzi ERDOĞAN in the department of Statistics has accepted it as an M.Sc. thesis according to Legislations of Graduate Higher Education on 25/10/2022 with unanimity/majority of vote's members of the jury.

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THESIS STATEMENT

All information presented in the thesis obtained in the frame of ethical behavior and academic rules. In addition, all kinds of information that does not belong to me have been cited appropriately in the thesis prepared by the thesis writing rules.

Payam Mahmood MUSTAFA

ABSTRACT

ANALYTICAL AND APPROXIMATE SOLUTIONS FOR VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS AND IT'S APPLICATIONS

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The presented study major goal is to investigate and solve the integro-differential equation and Volterra integral equations as well as several scientific models for problems in real-life. These objectives can be spitted in to the following three subgoals: The first step entails categorizing and researching the topic of integral equations and providing the fundamental definitions for the Volterra integral and integrodifferential equations. In order to solve Volterra integral and integro-differential equations, we used the "Dafterdar-Jafari method (DJM)" in the second sub-objective. The third sub-objective of this thesis is to introduce the methods like the power series approach, an iterative technique to solving Volterra integral and integro-differential equations. Moreover, two different method, namely power series method and Dafterdar-Jafari method applications provided a for finding the approximation and exact solution for some scientific problems occurs in real-life, such as, logistic model, the hybrid selection model, Volterra's population model and the Riccati equation. Furthermore, it is important to highlight that Matlab software was used to develop the computer programs.

Keywords: Dafterdar-Jafari method, Fredholm integral equations, Integral equation, Power series method, Volterra integral equations.

ÖZET

VOLTERRA İNTEGRAL DİFERANSİYEL DENKLEMLERİ İÇİN ANALİTİK VE YAKLAŞIK ÇÖZÜMLER VE UYGULAMALAR

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Bu tezin temel amacı, Volterra integral denklemini ve integral diferansiyel denklemini gerçek hayat problemleri için bazı bilimsel modelleri incelemek ve çözmektir. Bu amaç aşağıdaki gibi üç alt amaca ayrılabilir: Birincisi, integral denklemler konusunu sınıflandırmak, incelemek, Volterra integral denklemi ve integro diferansiyel denklemleri ile ilgili temel tanımları vermektir. İkinci alt amaçta, Volterra integral denklemi ve integro-diferansiyel denklemlerini çözmek için Dafterdar-Jafari metodu'nu kullandık. Üçüncü alt amaç, Volterra integral denklemi ve integro-diferansiyel denklemlerinin bir çözümünü sağlamak için kuvvet serileri metodu olarak adlandırılan başka bir yinelemeli yöntemi tanıtmaktır. Son olarak, Volterra'nın popülasyon modeli, hibrit seçim modeli, Riccati denklemi ve lojistik model gibi bazı gerçek yaşam bilimsel problemlerinin analitik ve yaklaşık çözümünü bulmak için, Dafterdar-Jafari ve kuvvet serisi metodunun uygulamasını sunmaktır. Ayrıca, bilgisayar programlarının Matlab yazılımı kullanılarak kodlandığını belirtmek önemlidir.

Anahtar kelimeler: Daftardar-Jafari metodu, Fredholm integral denklemleri, İntegral denklemi, Kuvvet serileri metodu, Volterra integral denklemleri.

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Payam Mahmood MUSTAFA

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SYMBOLS AND ABBREVIATIONS

Some symbols and abbreviations used in this study are presented below, along with descriptions.

Symbols	**Description**
ADM	Adomian decomposition method
$\boldsymbol{a(x), b(x), c(x)}$	Continuous functions
DJM	Dafterdar-jafari method
HPM	Homotopy perturbation method
HSM	Hybrid selection model
IVP	Initial value problem
$\boldsymbol{K}$	Non-dimensional parameter
PSM	Power series method
$\boldsymbol{P(T)}$	The population size at time
$\boldsymbol{T.P}$	The initial population size
$\boldsymbol{u(t)}$	The scaled population of identical individuals at a time t
λ	Lagrange multiplier
M	Positive constant
VIE	Volterra integral equation
VIDE	Volterra integro-differential equation
VPM	Volterra’s population model
VIM	Variational iteration method

1. INTRODUCTION

The theory of integral equations contains a lot of overlap with other fields of mathematics. Differential equations (DE) and operator theory are at the top of the list. Any ordinary differential equation (ODE) or partial differential equation (PDE)s problems can be reformulated in integral equation form. Many results of existence and uniqueness can subsequently be determined by the integral equations comparable conclusions. Integral equations advantage can be seen in the rising use in the literature of integral equations and in different applied Mathematic domains, because some issues can be expressed mathematically simply and on a very simple and natural way using integral equations. Different problems with direct representations Integral equations substitute the auxiliary conditions of differential equations (DE) more effectively and compactly than do differential equations. In addition, many issues in mathematical physics may be expressed as integral equations, so it’s safe to conclude that integral equations play a role in practically every mathematical physics and applied mathematics area. In another sense, one may say fundamental ideas of eigen values, eigen functions and linear vector spaces will play a vital role in dealing with linear integral equations (Hochstast, 1973).

The presented thesis is organized in to many steps; first step is an introductory phase, that discusses the fundamental and main aspects of integral equation subjects to provide a broad background to the readers that interested in integral equations. The DJM is used in the second step to solve both of integrodifferential equation and Volterra integral equations. The Method of Power Series has been applied to solving integrodifferential equations and the Volterra integral equations. In the third step, several scientific models are presented. DJM and PSM methods are used, in order to assess the approximate and analytical solutions of presented models. Finally, conclusion and recommendations of the work are shown. The approximate and analytical results are provided for several illustrative examples that programmed by using Mathematica software (Brunner, 1974).

The main objectives of the purpose of the thesis is to create analytical and approximation solutions for VIDE, also to apply it to specific scientific models. The objectives were met by applying the DJM (Bhalecar et al., 2005) and the PSM (Tahmasbi et al., 2008), (Tahmasbi and Fard, 2008) to the nonlinear VIDE in a straightforward manner. In this section, we’ll go over the thesis’s main findings and give some recommendations for further research.

2. LITERATURE REVIEW

Integral Equations have many applications in science and nature. Because integral equations frequently provide fewer constraints and a more useful model than differential equations, they may be thought of as being more fundamental than differential equations Furthermore, differentiation increases inaccuracy in numerical analysis, whereas integration tends to smooth out mistakes Solving frontier problems of physics: the decomposition method (Adomian, 2013).

Even though Volterra started experimenting with integral equations in 1884, in 1896 he started to focus on the integral equations. ‘Integral equations’ was first used in 1888 by Du Bois-Reymond. In the other hand, in 1908 the Lalesco coined the moniker Volterra integral equations.

Many investigations on integrodifferential equations and the solution of integral equations have been conducted in recent years.

Burner (1974) investigated the numerical solution of the first-order Voltera integral equations.

Brunner and John (1987) studied the use of certain spline collections method for the Volterra integrodifferential equations of the certain order in 1988.

Shahmorad (2002) studied the applications of the certain spline collection methods for the Volterra integrodifferential equation of a certain order in 2003.

To obtain a numerical solution, apply the Tau approach to integro-differential equations.

Al-Jawary (2005) investigated the system’s approximation to a solution of Volterra integral equation second kind linear.

Hashim (2006) solved boundary value problems (BVP) for the integrodifferential equations in order four by using the Adomian Decomposition Method.

For solving non-linear Volterra integrodifferential equations, researchers employed Variational Iteration Method (VIM) (Abbasbandy, 2008).

Ali (2011) introduced fractional integrodifferential equations utilizing a modified variant of operator that solves same-order fractional differentiation and integration problems using the modified ADM.

Erfanian (2011) proposed an optimization method for finding a numerical solution for the set of first order and second order non-linear Volterra integral equation.

Mirazaee (2012) applied repeated Simpson quadrature approach for solving the linear equation.

Approximating the integral in first kind Volterra integral equations.

Venkatesh et al. (2012) investigated Legendre wavelets for solving boundary value issues in higher order Volterra integrodifferential equations type.

Wadeá (2012) used the VIM to study the approximate solution of fractional integrodifferential equations.

Rashidinia (2013) invented and improved the Taylors expansion technique for approximate solution of Volterra integrodifferential equation and linear Fredholm.

Alao et al. (2014) used VIM and ADM in many forms of integrodifferential equation, including the Volterra, Fredholm, and Fredholm-Volterra equation.

Both of integrodifferential and Volterra integral equation have exact and approximation solutions described in this thesis. The integral equations are solved using the method of Dafterdar-Jafari and the power series method separately. Many scholars have utilized the DJM to solve ODE and PDE's of the integer and fractional order that are linear and nonlinear

Bhalekar and Daftardar (2008) to solve partial differential equations used the DJM in his paper.

Bhalecar and Daftardar (2010) used DJM to solve the evolution equation.

Dafterdar and Bhalecar (2010) applied Dirichlet boundary conditions to fractional boundary value problems and solved them using the DJM.

It's worth noting that, if an exact solution exists, the DJM will use successive approximation method to converge it. The results of the DJM and PSM demonstrate that these methods have many advantages, including overcoming the difficulty of computations Adomian handling nonlinear variables in ADM using polynomials, being derivative-free, not requiring the calculation of the as in VIM, a Lagrange multiplier and not requiring the construction of a Homotopy and resolving the corresponding algebraic the Homotopy Perturbation Method equations (HPM). Furthermore, the suggested PSM and DJM have been applied for the first time in this thesis to address real-world problems, for example, Riccati equation, Volterra's population model, Logistic differential equation and Hybrid selection model. There have recently been numerous attempts to develop analytical and approximative methods for solving Volterra populations model. Despite the fact that such strategies have been effectively implemented, there have been certain challenges (Daftardar and Bhalekar, 2009).

Wazwaz (1999) solved the governing problem with ADM. Furthermore, to get approximation solution for the population model from Volterra

Mohyud-Din et al. (2010) used the HPM and Padé techniques.

Parand et al. (2011) developed a strategy based on the applying of the both Sinc and Rational Legndre functions in a collocation approach.

Marzaban et al. (2009) proposed an approximation Volterra’s population model can be solved by using Lagrange interpolation polynomial approximations and a hybrid function made of block-pulse and.

Khan et al. (2011) employed the New Homotopy Perturbation Method (NHPM), that is an advancement more than traditional HPM.

For the purpose of solving several mathematical populations growth models, used the ADM and Sinc-Galerkin methods (Khaled and Kemal, 2005).

Remezani et al. (2006) also used the spectral approach for solve Volterra population across semi-infinite interval.

3. MATERIALS AND METHODS

3.1. Preliminaries

Definition 3.1 (Integral equation) Equations with the unknown function u(x) appearing under the integral sign are known as integral equations (Wazwaz, 2011). An integral equation with the form u(x) has the following standard form:

$$u(x) = f(x) + \lambda \int_{g(x)}^{h(x)} k(x,t)u(t)dt, \tag{3.1}$$

Where the function $u(x)$ that will be calculated appears under the integral sign, as well as within and outside of the integral sign, and $k(x,t)$ is a function of two variables x and t that acts as the integral equation's kernel The functions $f(x)$ and $k(x,t)$ are already known $g(x)$ and $h(x)$ are two functions that indicate the limits of integration, λ is a constant parameter that acts as an eigen value. It should be noted that the integration limits $g(x)$ and $h(x)$ can be variables, constants, or mixed (Wazwaz, 2011).

3.2. Integral Equation Classification

Integral equations come in a variety of shapes and sizes. These types are mostly determined by the integration limits and the kernel of the equation (Wazwaz, 2011). Integral equations are described in two ways, both of which are dependent on the integration limits.

We will give a classification of the Fredholm and Volterra integral equations, and the four related types:

1. Fredholm integral equations
2. Volterra integral equations
3. Integro-differential equations
4. Singular integral equations
5. Volterra-Fredholm integral equations

6. Volterra-Fredholm integro-differential equations

In the following we will outline the basic definitions and properties of each type.

3.2.1. Fredholm integral equation

When the integration limits are fixed, the integral equation is known as a Fredholm integral equation and it has the form,

$$\phi(x)u(x) = f(x) + \lambda \int_a^b k(x,t)u(t)dt, x \in [a,b], \tag{3.2}$$

where a and b are constants.

1) The first kind Fredholm integral equation if $\phi(x) = 0$, then form the equation 3.2 becomes

$$f(x) + \lambda \int_a^b k(x,t)u(t)dt = 0$$

2) The second kind Fredholm integral equation if $\phi(x) = 1$, then form the equation 3.2 becomes

$$u(x) = f(x) + \lambda \int_a^b k(x,t)u(t)dt.$$

3) A Fredholm is called homogenous Fredholm integral equation if $f(x) = 0$, then form the equation 3.2 becomes

$$\phi(x)u(x) = \lambda \int_a^b k(x,t)u(t)dt,$$

(Wazwaz, 2011).

3.2.2. Volterra integral equation

If at least one limit is a variable, then the equation is called a Volterra integral equation given in the form

$$\emptyset(x)u(x) = f(x) + \lambda \int_a^x k(x,t)u(t)dt. \tag{3.3}$$

Moreover, two other distinct kinds, that depend on the appearance of the unknown function u, are defined as follows:

1) The first kind Volterra integral equation if $\phi(x) = 0$, then form the equation 3.3 becomes

$$f(x) + \lambda \int_a^x k(x,t)u(t)dt = 0.$$

2) The second kind Volterra integral equation if $\phi(x) = 1$, then form the equation 3.3 becomes

$$u(x) = f(x) + \lambda \int_a^x k(x,t)u(t)dt.$$

3) Volterra is called homogenous Volterra integral equation if $f(x) = 0$, then form the equation 3.3 becomes

$$\phi(x)u(x) = \lambda \int_a^x k(x,t)u(t)dt,$$

(Wzwaz, 2011).

3.2.3. Integro-differential equation

An integro-differential equation is the unknown function u(x) appears as the combination of the ordinary derivative and under the integral sign. A standard integro-differential equation is of the form:

$u^{(n)}(x) = f(x) + \lambda \int_{g(x)}^{h(x)} k(x,t)u(t)dt, n\epsilon N.$

where $u^{(n)}(x) = \frac{d^n u}{dx^n}$ $and\ u(0), u'(0), \cdots, u^{(n-1)}(0)$ are the initial conditions (Wazwaz, 2011).

3.2.4. Singular integral equations

A singular integral equation is defined as an integral with the infinite limits or when the kernel of the integral becomes unbounded at a certain point in the interval.

1) The integral equation of the first kind

$f(x) = \lambda \int_{\alpha(x)}^{\beta(x)} k(x,t)u(t)dt,$

2) The integral equation of the second kind

$u(x) = f(x) + \lambda \int_{\alpha(x)}^{\beta(x)} k(x,t)u(t)dt,$

(Hochstast, 1973).

3.2.5. Volterra-fredholm ıntegral equaations

The Volterra-Fredholm integral equation, which is a combination of disjoint Volterra and Fredholm integrals in one integral equation. The standard form of the Volterra-Fredholm integral equation is of the form:

$u(x) = f(x) + \int_0^x k_1(x,t)u(t)dt\ + \int_a^b k_2(x,t)u(t)dt\,,$

Where $k_1(x,t)$ and $k_2(x,t)$ are the function of two variables x and t that acts as the integral equation’s kernel (Wazwaz, 2011).

3.2.6. Volterra-fredholm ıntegro-differential equations

The Volterra-Fredholm integro-differential equation, which is a combination of disjoint Volterra and Fredholm integrals and differential of order n, in one integral equation. The standard form of the Volterra-Fredholm integro-differential equation is of the form:

$$u^{(n)}(x) = f(x) + \int_0^x k_1(x,t)u(t)dt \; + \int_a^b k_2(x,t)u(t)dt \,,$$

Where $k_1(x,t)$ and $k_2(x,t)$ are the function of two variables x and t that acts as the integral equation's kernel (Wazwaz, 2011).

Definition 3.2 If the integral operator in an integral equation is linear:

$$L\bullet = \int_{a(x)}^{b(x)} k(x,t)\bullet dt.$$

It complies with the linearity condition

$$L[c_1u_1(x) + c_2u_2(x)] = c_1L\,[u_1(x)] + c_2L[u_2(x)]$$

Where u_1, u_2 are two continuous functions and c_1, c_2 are two constants. The form of the most general linear integral equations is:

$$u(x) = f(x) + \lambda \int_a^x k(x,t)u(t)dt.$$

Nonlinear integral equations can be represented in the general form as follows:

$$u(x) = f(x) + \lambda \int_a^x k(x,t)u(t)dt$$

Where $k(x,t)u(t)dt$ is nonlinear function in $u(x)$ (Small, 1989).

A kernel function typically satisfies the following two properties:

1) If a linear integral equation's kernel k only depends on the difference $x-t$, if the kernel has the form $k(x,t)=k(x-t)$,

Then k is referred to as a difference kernel, and the integral equation is referred to as a convolution type integral equation.

2) The kernel k if it has the property $k(x,t)=k(t,x)$, is called symmetric, and kernel k if it has the property $k(x,t)=-k(t,x)$ is called antisymmetric, (Hochstast, 1973).

3.3. Converting Initial Value Problem to Volterra Integral Equation

This section will examine the process used to change an IVP into an equivalent VIE. For the sake of simplicity, we'll use a second order initial value problem provided by:

$$y''(x)+p(x)y'(x)+q(x)y(x)=g(x), x\geq 0 \tag{3.4}$$

Depending on the initial conditions $y(0)=\alpha$, $y'(0)=\beta$, where α and β are constants. The functions p and q are analytical functions, and g is continuous across the discussion interval. In order to accomplish our goal, we determined:

$$y''(x)=u(x), \tag{3.5}$$

Where the function u is continuous. Integrating equation 3.5 two sides from $0\ to\ x$ results in:

$$y'(x)-y'(0)=\int_0^x u(t)dt,$$

or equivalently:

$$y'(x)=\beta+\int_0^x u(t)dt, \tag{3.6}$$

Again, the results of integrating both sides of equation 3.6 from 0 to x are:

$$y(x) - y(0) = \beta x + \iint_0^x u(t)dtdx. \tag{3.7}$$

A formula to convert numerous integrals to a single integral should typically be described. First, by applying the following formula, we will demonstrate how One integral can be created from the double integrals.:

$$\int_0^x \int_0^{x_1} F(t)dtdx_1 = \int_0^x (x-t)\, F(t)dt.$$

There are two simple ways to demonstrate this. Setting is the first way

$$G(X) = \int_0^x (x-t)\, F(t)dt, \tag{3.8}$$

where $G(0) = 0$ differentiating both sides of equation 3.8 gives

$$G'(X) = \int_0^x F(t)\, dt.$$

The result of integrating both sides of the previous equation gotten by applying the Leibnitz rule. Integrating the two sides of the previous equation are now increased from 0 to x, yielding the values:

$$G(X) = \int_0^x \int_0^{x_1} F(t)dtdx_1. \tag{3.9}$$

As a result, the two equations' right sides equation 3.8 and 3.9 are equivalent. The proof is now complete. The second method will employ the idea of part-by-part integration. Recall that:

$$\int u\, dv = uv - \int u\, dv\; u(x_1) = \int_0^x F(t)\, dt,$$

then we find

$$\int_0^x \int_0^{x_1} F(t)dtdx_1 = x_1 \int_0^{x_1} F(t)\,dt - \int_0^x x_1 F(x_1)\,dx_1 = x \int_0^x F(t)\,dt -$$
$$\int_0^x x_1 F(x_1)\,dx_1 = \int_0^x (x-t) F(x_1)\,dx_1.$$

Achieved by setting $x_1 = t$. Given by is a single integral can be created by using the following generic formula from that convert multiple integrals

$$\int_a^x \int_a^{x_1} \dots \int_a^{x_{n-1}} u(t)dtdx_{n-1} \dots dx_1 = \frac{1}{(n-1)!} \int_a^x (x-t)^{n-1} u(t).$$

When a is a constant and n is a positive integer, allowing equation 3.7 to be rewritten as:

$$y(x) = \alpha + \beta x + \int_0^x (x-t)\,u(t)dt. \tag{3.10}$$

Substituting equation 3.5 and 3.6 and 3.10 back to equation 3.4 in the initial value problem. resulting in the following VIE:

$$u(x) + p(x)\left[\beta + \int_0^x u(t)dt\right] + q(x)\left[\alpha + \beta x + \int_0^x (x-t)\,u(t)dt\right] = g(x).$$

The final equation can be expressed as follows in the conventional VIE form:

$$u(x) = f(x) + \int_0^x k(x-t)\,u(t)dt, \tag{3.11}$$

Where

$k(x,t) = -p(x) - q(x)(x-t)$, and

$f(x) = g(x) - [\beta p(x) + \alpha q(x) + \beta x q(x)]$.

(Wazwaz, 2011).

Equation 3.3. is a VIE of the second kind, as the reader may see. In formulating the resulting integral equation, the auxiliary conditions are automatically satisfied, which is advantageous. Another advantage is in situations where neither differential nor integral forms who's exact, closed-form solutions can be expressed in terms of fundamental functions. we must take a detour the method of numerical analysis or approximation the integral representation is more appropriate for these computations (Jerri, 1985). That is to say, an integral equation and a differential equation can occasionally be used to explain the same problem. Because differentiation while integration tends to smooth out errors, numerical analysis tends to increase error, it may be important to convert a problem from an integral equation when numerical values are needed (Chambers, 1976).

In the following, we introduced the mathematical models that we encounter in our daily life and we will discussion about the solution of them in application.

3.4. The Riccati Equation

It is known that The Riccati equation is considered as one of the most famous and interested first order nonlinear differential equations. It can be written in the following form;

$$u'(x) = a(x)u(x) + b(x)u^2(x) + c(x), x \geq 0$$

When the a, c, b are continuous functions of x and $b(x), c(x) \neq 0, \forall x \in [0,\infty)$.

By taking the integration for both side of the equation from 0 to x, we obtain:

$$u(x) = u_0 + \int_0^x [a(t)u(t) + b(t)u^2(t) + c(t)]dt.$$

3.5. The Logistic Differential Equation

In this section we will discuss the logistic function, it is known that Logistic functions are the solutions of simple first-order nonlinear differential equation

$$u' = \mu u(1-u),\ u(0) = \frac{1}{2},\ \mu > 0,\ x \geq 0. \tag{3.12}$$

Logistic function has many applications in different fields such as biology, biomathematics, mathematical, Artificial neural networks, psychology, demography, sociology, political science, statistics, economics, chemistry and probability.

3.6. Volterra's Population Model

The Volterra's model for population growth of a species within a closed system will be examined in this section. The Volterra's population model (VPM) is characterized by the nonlinear Volterra integro-differential equation:

$$\frac{dp}{dt} = ap - bp^2 - cp\int_0^t p(x)dx,\ \ p(0) = p_0, \tag{3.13}$$

a, b, and c are constant and positive parameters, where a is the birth rate coefficient, b is the crowding coefficient, and c is the toxicity coefficient; where $p = p(t)$ represents the size of the population at time T and a, b, and c are constant and positive parameters, b is the crowding coefficient, and c is the toxicity coefficient. p_0 is another name for the population size at time T's beginning. The coefficient c exhibits the fundamental characteristics of the population evolution up until the point where the population level eventually hits zero over time. Equation 3.13 becomes the Malthus differential equation when b = 0 and c = 0:

$$\frac{dp}{dt} = ap\ , p(0) = p_0 \tag{3.14}$$

The Malthus equation 3.14 assumes that population growth will be proportionate to existing population, and has a separable solution provided by:

$$p(t) = p_0\, e^{aT}\ . \tag{3.15}$$

It is clear that for $a\ >\ 0$, equation 3.13 indicates population expansion, while for a, it represents population decay:

$$\frac{dp}{dt} = ap - bp^2,\ \ p(0) = p_0 \tag{3.16}$$

where $a\ <\ 0$ when $c = 0$.

Verhulst instituted the logistic growth model equation 3.16 which, by incorporating the growth-limiting term $-bp^2$, removes the unwanted impact of infinite growth. For logistic growth model equation 3.16 the answer is

$$P(T) = \frac{a\, p_0 e^{aT}}{a + bp_0(e^{aT} - 1)}\ ,$$

where

$$\lim_{T\to\infty} P(T) = \frac{a}{b}.$$

Volterra introduced an integral term $-cP\int_0^t p(x)\,dx$ to the logistic growth to get the Volterra's population growth model equation 3.13 use equation 3.16 The accumulated toxicity created since time zero is represented by the additional integral term. Many other time and population scales can be used. However, by inserting non-dimensional variables, the scale time population will be applied:

$$t = \frac{cT}{b},\ u = \frac{bP}{a},$$

to obtain the non-dimensional Volterra's population growth model:

$$k\,\frac{du}{dt} = u - u^2 - \int_0^t u(x)dx, u(0) = u_0,$$

Where $u = u(t)$ is the scaled population of identical individuals at time t, and $k = c/(\text{ab})$ is a predefined non-dimensional parameter. This model was developed by Volterra for an identical population u that exhibits crowding and is sensitive to the quantity of poisons produced. Numerous studies have been conducted in an effort to come up with numerical and analytical answers to the population increase model. equation 3.16 (Scudo, 1971), (Small, 1989; TeBeest, 1997). The analytical solution:

$$u(t) = u_0 e^{\left(\frac{1}{k}\int_0^t \left[1 - u(\tau) - \int_0^\tau u(x)dx\right] d\tau\right)}$$

If the initial population is $u_0 > 0$, then $u(t) > 0$ for every t. This closed form solution, on the other hand, cannot provide insight into population evolution behavior. As a result, study has focused on analyzing the population's quick ascent a logistic curve is followed by its long-term decline to zero. When it comes to a rapid rise to a particular amplitude followed by an exponential decrease to extinction, the non-dimensional parameter κ has a significant impact on how u behaves. The population of large is extremely sensitive to toxins, in contrast to the population of small is not sensitive to toxins (Wazwaz 1999; 2011).

3.7. Hybrid Selection Model

In this section we will introduce another numerical approach which is known as Hybrid selection model (HSM), we know that HSM with constant coefficient have the following model:

$$u' = ku(1-u)(2-u),$$
$$u(0) = 0.5 \qquad (3.17)$$

When the k is positive constant that varies according to the genetics trait. In a hybrid model, the symbol u represents the proportion of the population with a certain trait, and the symbol t represents the period in generations (Wazwaz, 2014).

4. METHODS OF SOLVING INTEGRAL DIFFERENTIAL EQUATION

4.1. The Padé Approximants

The potent Padé approximants will be studied in this section. First, we’ll go through how to construct Padé approximations for polynomials and functions. Then, several examples will be shown to show how to use Padé approximants. To approximate power series, polynomials are frequently utilized. Polynomials, on the other hand, are prone to oscillations, which can lead to approximation error boundaries. Furthermore, in a finite plane, polynomials can never explode, making the singularities invisible. For the numerical approximations to address these difficulties, Taylor series are best controlled using Padé approximants. Functions that can be approximated are exemplified by the product of two polynomials. The coefficients of the polynomials in the numerator and denominator can be found by expanding the function using the coefficients of the Taylor expansion. Padé rational approximations, which are more efficient than polynomials, are commonly employed in numerical analysis and fluid mechanics. We’ll show you how to do it in the following paragraphs using a simple and straightforward way (Wazwaz, 2009).

Assume that the Taylor series expansion of the function f is as follows:

$$f(x) = \sum_{k=0}^{\infty} c_k\, x^k$$

The process consists of looking for a logical function for the series. The function can be represented as the ratio of two polynomials using the series coefficients.
A rational function denoted by $\left[\frac{m}{n}\right]$ is known as a padé approximant:

$$\left[\frac{m}{n}\right] = \frac{a_0+a_1x+a_2x^2+\cdots+a_mx^m}{b_0+b_1x+b_2x^2+\cdots+b_nx^n}$$

The fundamental goal is to as closely match the series coefficients as you can. The limit of the function as x may be achieved even though the series’ region of convergence has a limited if $m = n$; in this case, the approximants $\left[\frac{m}{n}\right]$ are known as diagonal approximants. We observe that the denominator has $n + 1$ independent coefficients, whereas the numerator has m independent coefficients. Let $b_0 = 1$ to make the system determinable. The denominator now contains n independent coefficients, giving us a total of $m + 1$ independent coefficients. The power series can now be fitted using the $\left[\frac{m}{n}\right]$ approximant via orders $1, x,\ x^2, \dots, x^{(m+\,n)}$ In addition, if the function f doesn't contain any singularities, The entire real axis will see the convergence of the Padé approximants. It was

$$\frac{a_0+a_1x+a_2x^2+\cdots+a_nx^n}{1+b_1x+b_2x^n+\cdots+b_nx^n} = c_0 + c_1x + c_2x^2 + \cdots + c_{2n}x^{2n}.$$

Cross multiplication allows us to determine the following:

$$a_0 + a_1x + a_2x^2 + \cdots + a_nx^n = c_0 + (c_1 + b_1c_0)x + (c_2 + b_1c_1 + +b_2c_0)x^2 + (c_3 + b_1c_2 + b_2c_1 + b_3c_0)x^3 + \cdots + c_{2n}x^{2n}$$

Equating the related powers of x leads to:

coefficient of x^0: $a_0 = c_0$
coefficient of x^1: $a_1 = c_1 + b_1c_0$
coefficient of x^2: $a_2 = c_2 + b_1c_1 + b_2c_0$
coefficient of x^3: $a_3 = c_3 + b_1c_2 + b_3c_0$
$\vdots$
coefficient of x^n: $a_n = c_n + \sum_{k=1}^{n} b_kc_{n-k}$.

The constants in the numerator and denominator's polynomials have now been identified. So, here we are:

$$[1/1] = \frac{a_0 + a_1x}{b_0 + b_1x}, \lim_{x\to\infty}[1/1] = \frac{a_1}{b_1}$$

$$[2/2] = \frac{a_0 + a_1x + a_2x^2}{b_0 + b_1x + b_2x^2}, \lim_{x\to\infty}[2/2] = \frac{a_2}{b_2}$$

$$[3/3] = \frac{a_0 + a_1x + a_2x^2 + a_3x^3}{b_0 + b_1x + b_2x^2 + b_3x^3}, \lim_{x\to\infty}[3/3] = \frac{a_3}{b_3}$$

$$\vdots$$

$$[m/m] = \frac{a_0+a_1x+a_2x^2+a_3x^3+\cdots+a_mx^m}{b_0+b_1x+b_2x^2+b_3x^3+\cdots+b_mx^m}, \lim_{x\to\infty}[m/m] = \frac{a_m}{b_m}.$$

Example 4.1. Consider the function

$$f(x) = \frac{\sqrt{1+3x}}{\sqrt{1+x}},$$

where $x \in (-\infty, -1) \cup \left(-\frac{1}{3}, \infty\right)$. the formula for the Taylor series for $f(x)$ is

$$f(x) = 1 + x + \frac{3}{2}x^2 + \frac{5}{2}x^3 - \frac{37}{8}x^4 + \frac{75}{8}x^5 + \frac{327}{16}x^6 + \frac{753}{16}x^7 + O(x^8).$$

Thus, [2/2] approximates are described as;

$$[2/2] = \frac{a_0+a_1x+a_2x^2}{1+b_1x+b_2x^2}$$

The order through in $f(x)$ Taylor series of the two polynomials must satisfy the [2/2] approximants to obtain the five coefficients of the two polynomials $1, x, \dots, x^4$, therefore, we set:

$$[2/2] = \frac{a_0+a_1x+a_2x^2}{b_0+b_1x+b_2x^2} = 1 + x + \frac{3}{2}x^2 + \frac{5}{2}x^3 - \frac{37}{8}x^4.$$

Cross multiplying yields

$$a_0 + a_1x + a_2x^2 = 1 + (b_1 + 1)x + \left(b_1 + b_2 - \frac{3}{2}\right)x^2 + \left(-\frac{3}{2}b_1 + b_2\frac{5}{2}\right)x^3 + \left(\frac{5}{2}b_1 - \frac{3}{2}b_1 - \frac{37}{8}\right)x^4$$

The result of equating the powers of x is

coefficient of x^0: $a_0 = 1$
coefficient of x^1: $a_1 = b_1 + 1$
coefficient of x^2: $a_2 = b_1 + b_2 - \frac{3}{2}$
coefficient of x^3: $0 = -\frac{3}{2}b_1 + b_2 + \frac{5}{2}$
coefficient of x^4: $0 = \frac{5}{2}b_1 - \frac{3}{2}b_2 - \frac{37}{8}$

Therefore, the solution to the this system of equations is;

$$a_0 = 1, a_1 = \frac{9}{2}, a_2 = \frac{19}{4}, b_1 = \frac{7}{2}, b_2 = \frac{11}{4} .$$

The $\left[\frac{2}{2}\right]$Padé approximants are as a result

$$[2/2] = \frac{1+\frac{9}{2}x+\frac{19}{4}x^2}{1+\frac{7}{2}x+\frac{11}{4}x^2}.$$

To determine the Padé approximants [3/3], we first set

$$\frac{a_0+a_1x+a_2x^2+a_3x^3}{b_0+b_1x+b_2x^2+b_3x^3} = 1 + x + \frac{3}{2}x^2 + \frac{5}{2}x^3 - \frac{37}{8}x^4 + \frac{75}{8}x^5 + \frac{327}{16}x^6 .$$

solving the following system of equations by cross multiplying, equating, and solving the coefficient of similar powers of x results in

$$a_0 = 1, a_1 = \frac{13}{2}, a_2 = \frac{27}{2}, a_3 = \frac{71}{8},$$

$b_1 = \frac{11}{2}, b_2 = \frac{19}{2}, b_3 = \frac{41}{8},$

and from here we give

$$[3/3] = \frac{1+\frac{13}{2}x+\frac{27}{2}x^2+\frac{71}{8}x^3}{1+\frac{11}{2}x+\frac{19}{2}x^2+\frac{41}{8}x^3}.$$

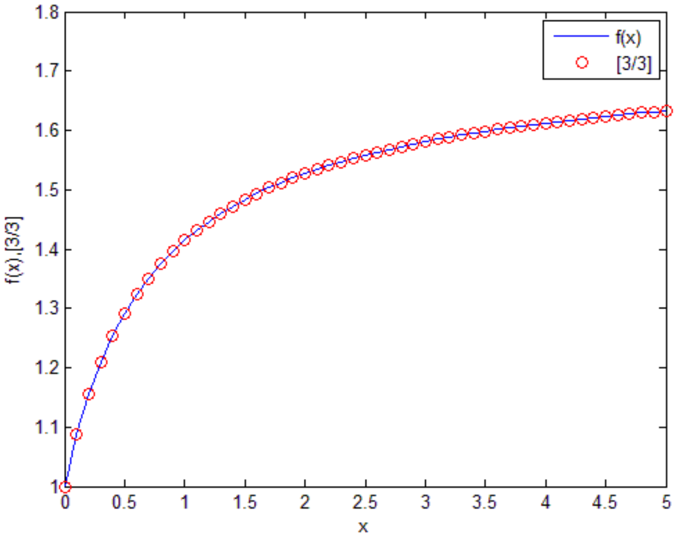

Figure 4.1. Presents the solution of $f(x)$ for $[3/3]$.

Table 4.1. Comparison of $f(x)$ and padé approximation

x	$f(x)$	$[3/3]$	$Error$
0	1.00000000	1.00000000	0.00000000
0.5	1.29099445	1.29099307	0.00000138
1	1.41421356	1.41420118	0.00001238
1.5	1.48323970	1.48320835	0.00003135
2	1.52752523	1.52747253	0.00005270
2.5	1.55838744	1.55831391	0.00007353
3	1.58113883	1.58104609	0.00009274
3.5	1.59861051	1.59850046	0.00011005
4	1.61245155	1.61232604	0.00012551
4.5	1.62368828	1.62354900	0.00013928
5	1.63299316	1.63284158	0.00015158

(Wazwaz, 2009).

Example 4.2. Considering the function

$$f(x) = \frac{\ln(1+x)}{x},$$

where $x \geq -1, x \neq 0$ the This formula creates the Taylor series for $f(x)$.

$$f(x) = 1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \frac{x^4}{5} + \frac{x^5}{6} + \frac{x^6}{7} + \frac{x^7}{8} + O(x^8).$$

To establish approximates for [3/3], we set

$$[3/3] = \frac{a_0 + a_1 x + a_2 x^2 + a_3 x^3}{b_0 + b_1 x + b_2 x^2 + b_3 x^3}$$

We proceed as before to identify the unknowns, thus we set

$$\frac{a_0 + a_1 x + a_2 x^2 + a_3 x^3}{b_0 + b_1 x + b_2 x^2 + b_3 x^3} = 1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \frac{x^4}{5} + \frac{x^5}{6} + \frac{x^6}{7}$$

Cross multiplying and proceeding as before we find

$$a_0 = 1, a_1 = \frac{17}{14}, a_2 = \frac{1}{3}, a_3 = \frac{1}{140},$$

$$b_1 = \frac{12}{7}, b_2 = \frac{6}{7}, b_3 = \frac{4}{35},$$

so that the Padé approximants is

$$[3/3] = \frac{1 + \frac{17}{14}x + \frac{1}{3}x^2 + \frac{1}{140}x^3}{1 + \frac{12}{7}x + \frac{6}{7}x^2 + \frac{4}{35}x^3}.$$

To determine the Padé approximants [4/4], we set:

$$[4/4] = \frac{a_0+a_1x+a_2x^2+a_3x^3+a_4x^4}{b_0+b_1x+b_2x^2+b_3x^3+b_4x^4},$$

Proceeding as before we obtain:

$$[4/4] = \frac{1+\frac{17}{8}x+\frac{11}{3}x^2+\frac{1}{140}x^3+\frac{13}{140}x^4}{1+\frac{12}{17}x+\frac{16}{7}x^2+\frac{14}{35}x^3+\frac{12}{90}x^4}.$$

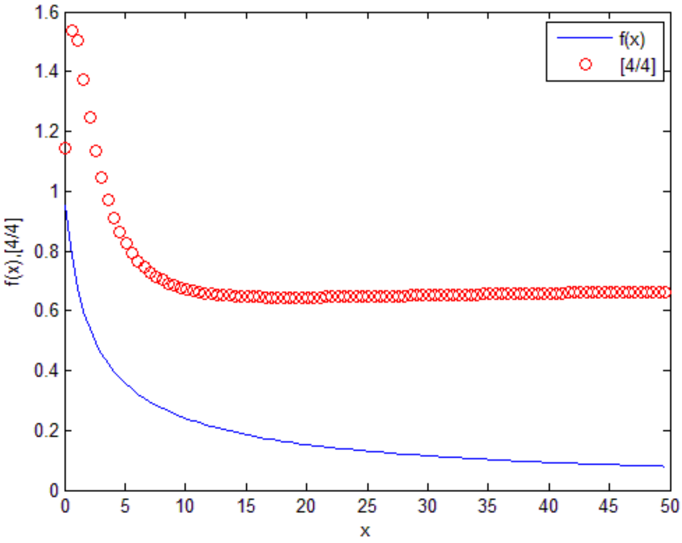

Figure 4.2. Presents the solution of $f(x)$ for $[4/4]$.

Table 4.2. Comparison of $f(x)$ and padé approximation (Wazwaz, 2009)

x	$f(x)$	$[4/4]$	$Error$
0.10000000	0.95310180	1.14199675	0.18889495
5.10000000	0.35456643	0.82471184	0.47014542
10.10000000	0.23831140	0.67134204	0.43303065
15.10000000	0.18402777	0.64706479	0.46303702
20.10000000	0.15170513	0.64506936	0.49336424
25.10000000	0.12995758	0.64793722	0.51797964
30.10000000	0.11419295	0.65173381	0.53754086
35.10000000	0.10217359	0.65542674	0.55325315
40.10000000	0.09266853	0.65875419	0.56608566
45.10000000	0.08494042	0.66168164	0.57674122

4.2. Method of Adomian Decomposition

We will present the Adomian Decomposition Method (ADM) and provide information, and additional details can be discovered. Consider the operator equation $Fu = G$, where G is a known function and F is a general nonlinear ordinary differential operator to introduce the ADM's basic concept in (Adonian and Rach, 1996).
Assume that the nonlinear operator F can be decomposed into the following:

$$Lu + Ru + Nu = G \tag{4.1}$$

Where L is the highest-order derivative that can be inverted, R is a linear differential operator of order less than L, and G is the nonhomogeneous term, N is a nonlinear operator. The method is founded on the formal application of the operator L to the expression:

$$Lu = G - Ru - Nu \tag{4.2}$$

As a result, applying the mentioned conditions, we get:

$$u = h + L^{-1}G - L^{-1}Ru - L^{-1}Nu \tag{4.3}$$

Where h is the solution of the initial boundary conditions for the homogeneous equation $Lu = 0$. The decomposition of the nonlinear term Nu is the current issue. Adomian created the following really elegant method to accomplish this.

According to the Adomian method, equation 4.1 solution can be approximated as an infinite series:

$$u = \sum_{n=0}^{\infty} u_n \tag{4.4}$$

and decomposing the nonlinear term Nu as:

$$f(u) = Nu = u = \sum_{n=0}^{\infty} A_n$$

Where A_n are known as Adomian polynomials of u_0 , u_1 , … , u_n these conditions are the analytical expansion of Nu, that

$$u = \sum_{n=0}^{\infty} \lambda^i u_i$$

around $\lambda = 0$. That is

$$A_n = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n} f\left(\sum_{n=0}^{\infty} \lambda^i u_i\right)\right]_{\lambda=0} .$$

The first component of the Taylor expansion of f can be used to create the Adomian polynomials, which are not unique. u_0, i.e.,

$$f(u) = \sum_{n=0}^{\infty} \frac{f^{(n)} u_0}{n!} (u - u_0).$$

The arrangement of Adomain’s polynomials results in the form:

$$A_0 = f(u_0)$$

$$A_1 = u_1 f'(u_0)$$

$$A_2 = u_1 f'(u_0) + \frac{u_1^2}{2!} f''(u_0)$$

$$A_3 = u_3 f'(u_0) + u_1 u_2 f''(u_0) \frac{u_1^3}{3!} f'''(u_0)$$

We now parameterize equation 4.3 in the following manner:

$u = h + L^{-1}G - \lambda L^{-1}Ru - \lambda L^{-1}Nu,$

Where λ is just a means to identify terms that should be gathered in a way that makes sense. A_n it depends $u_0\,,\ u_1\,, \ldots, u_n$ as well as we later set $\lambda\ = 1$

$$\sum_{n=0}^{\infty} \lambda^n u_n = h + L^{-1}G - \lambda L^{-1}R \sum_{n=0}^{\infty} \lambda^n u_n - \lambda L^{-1} \sum_{n=0}^{\infty} \lambda^n u_n\, A_n.$$

When we combine the coefficients of equal powers of λ, we get:

$$u_0 = h + L^{-1}G,$$

$$u_1 = -L^{-1}R(u_0) - L^{-1}(A_0),$$

$$u_2 = -L^{-1}R(u_1) - L^{-1}(A_1), \tag{4.5}$$

$$\vdots$$

$$u_n = -L^{-1}R(u_{n-1}) - L^{-1}(A_{n-1}).$$

The solution is roughly approximated by an n-term given by

$$\varphi_N(x) = \sum_{n=0}^{N-1} u_n\,,\ N = 1,2,3, \ldots .$$

(Adonian and Rach, 1996).

Example 4.3. Consider the linear VIDE:

$$u'(x) = 1 - \int_0^x u(t)\,dt, u(0) = 0. \tag{4.6}$$

The integral operator is applied L^{-1} defined by

$$L^{-1}(\bullet) = \int_0^x (\bullet)\,dt,$$

or both sides of equation 4.6, and using the initial condition, yields to:

$$u(x) = x - L^{-1}\left(\int_0^x u(t)\, dt\right). \tag{4.7}$$

The following will be given using the decomposition series equation 4.4 and recurrence relation equation 4.5:

$$u_0(x) = x,$$

$$u_1(x) = -L^{-1}\left(\int_0^x u(t)\, dt\right) = \frac{-1}{3!}x^3,$$

$$u_2(x) = -L^{-1}\left(\int_0^x u_1(t)\, dt\right) = \frac{1}{5!}x^5,$$

$$u_3(x) = -L^{-1}\left(\int_0^x u_2(t)\, dt\right) = \frac{-1}{7}x^7,$$

and so on. This gives the solution in a series form

$$y(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7}x^7 + \cdots$$

and as a result, the precise solution is presented by $u(x) = \sin(x)$ (Wazwaz, 2009).

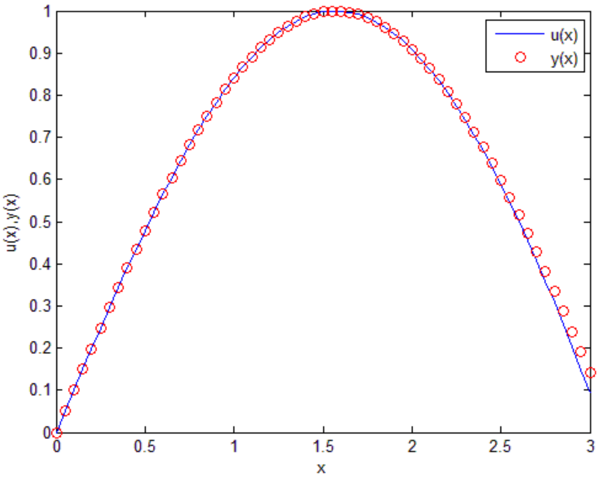

Figure 4.3. Presents the solution of linear VIDE and $u(x)$.

Table 4.3. Comparison of $u(x)$ and $y(x)$ for linear VIDE

x	$u(x)$	$y(x)$	$Error$
0.00000000	0.00000000	0.00000000	0.00000000
0.30000000	0.29552021	0.29552021	0.00000000
0.60000000	0.56464245	0.78332691	0.00000003
0.90000000	0.78332585	0.56464247	0.00000106
1.20000000	0.93202505	0.93203909	0.00001403
1.50000000	0.99739118	0.99749499	0.00010380
1.80000000	0.97331678	0.97384763	0.00053085
2.10000000	0.86110587	0.86320937	0.00210350
2.40000000	0.66855058	0.67546318	0.00691260
2.70000000	0.40769556	0.42737988	0.01968432
3.00000000	0.09107143	0.14112001	0.05004858

4.3. The Method of Iterative Variation

The Variational Iteration Method (VIM) concept will be introduced; for more information, read (He, 2007; He and Wu, 2007). Consider the differential equation:

$$Lu + Nu = g(x) \tag{4.8}$$

where $g(x)$ is the source in the homogeneous term and L and N are the respective linear and nonlinear operators. For equation 4.8 the VIM introduces a correction functional of the following form:

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(t)\big(Lu_n(t) + N\tilde{u}_n(t) - g(t)\big)dt \tag{4.9}$$

If $\tilde{u}_n(t)$ is a confined variation, which means that is a generic the variational theory can be used to determine Lagrange's multiplier λ, and as $\tilde{u}_n$ is a confined variation, $\delta\tilde{u}_n = 0$. The Lagrange multiplier can be either a constant or a function, it should be emphasized. The best way to determine the Lagrange multiplier $\lambda(t)$ is to first integrate by parts and use a constrained variation. A general formula was discovered by (Bhalecar and Daftardar-Gejii, 2005; Tatari and Dehghan, 2007). for $\lambda(t)$ for the n −th order differential equation.

$$u^{(n)} + f\left(u(x), u'(x), u''(x), \dots, u^{(n)}(x)\right) = 0 \tag{4.10}$$

can be demonstrated to be in this form:

$$\lambda(t) = (-1)^n \frac{1}{(n-1)} (t - x)^{(n-1)}. \tag{4.11}$$

This implies that, for first-order ordinary differential equations$\lambda(t) = 1$, in addition to the second-order ODEs, $\lambda(t) = (t–x)$and so forth. Despite the fact that equation 4.11 contains the proof of the formula is given in (He and Wu, 2007; Wazwaz, 2009) in detail. After determining $\lambda(t)$, for the purpose of determining the subsequent approximations, a recurrence relation or an iteration formula should be utilized $u_{n+1}(x)$, $n = 0, 1, \dots$ if the answer is $u\,(x)$. It is common knowledge that recursion requires an initial value to carry out the iteration process. This means that any selected function can be used for the zeroth approximation, u_0. However, sticking to the starting points $u(0), u'(0)$, and $u''(0)$ Usually, as will be seen later, are used for the zeroth approximation $u\,_0$ The response is as follows:

$$u(x) = \lim_{n\to\infty} u_n(x). \tag{4.12}$$

Example 4.4. Consider the nonlinear VIDE

$$u'(x) = -1 + \int_0^x u^2(t)dt \ , u(0) = 0. \tag{4.13}$$

The correction functional for equation 4.13 using VIM:

$$u_{n+1}(x) = u_n(x) + \int_0^x \left[u_n'(s) + 1 - \int_0^s u^2(t)dt\right] ds. \tag{4.14}$$

Consider the first approximation, $u_0\,(x) = -x$. Equation 4.14 yields the first three iterations, which are represented by (Batiha et al., 2008):

$$u_1(x) = -x + \frac{1}{12}x^4\,,$$

$$u_2(x) = -x + \frac{1}{12}x^4 - \frac{1}{252}\,x^7 + \frac{1}{12960}x^{10}$$

$$u_3(x) = -x + \frac{1}{12}x^4 - \frac{1}{252}\,x^7 + \frac{1}{12960}x^{10} - \frac{37}{7076160}x^{13} + \frac{109}{914457600}x^{16} - \frac{1}{558472320}x^{19} + \frac{1}{77598259200}x^{22}.$$

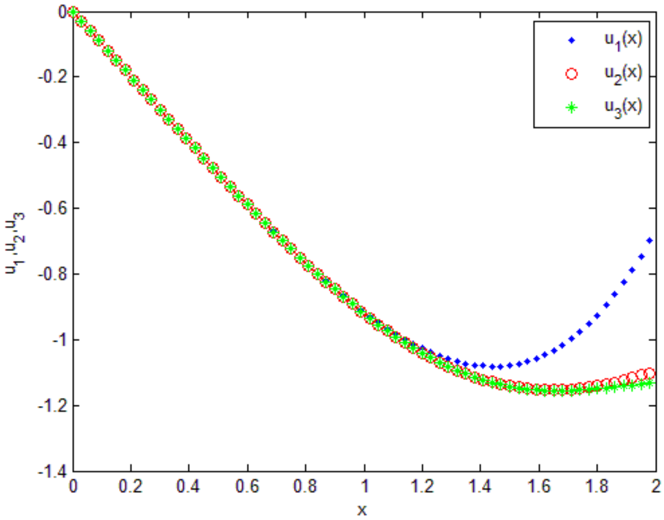

Figure 4.4. Presents the relation between both of the $u_1(x), u_2(x)$ and $u_3(x)$ iterates.

4.4. The Dafterdar-Jafari Method

In 2006 Daftardar-Gejji and Jafari proposed the Dafterdar-Jafari Method for solving both linear and nonlinear functional equations (DJM). Using consecutive approximations, the approach converges to the exact answer, if one exists. A small for numerical purposes, with a high degree of accuracy, a variety of approximations can be employed for certain problems. The DJM is easy to learn and use. With computer programs, and it produces better results without requiring any limiting assumptions for nonlinear components, as some previous systems do (Bhalecar and Daftardar-Gejii, 2005). This chapter is divided into three pieces. In section 4.4.2, the analysis of DJM for general nonlinear operator equations is covered.

4.4.1. Analysis of the dafterdar-jafari method

Consider the general functional equation (Bhalekar and Daftardar-Gejii, 2008):

$$u = N(u) + f \tag{4.15}$$

Where N is a nonlinear operator and f is a known function. We are looking for a solution u of equation 4.15 having the series form:

$$u = \sum_{i=0}^{\infty} u_i. \tag{4.16}$$

The nonlinear operator N can be decomposed as

$$N\left(\sum_{i=0}^{\infty} u_i\right) = N(u_0) + \sum_{i=0}^{\infty}\left[N\left(\sum_{j=0}^{i} u_j\right) - N\left(\sum_{j=0}^{i-1} u_j\right)\right] \tag{4.17}$$

From equation 4.16 and 4.17 which implies that equation 4.15 is comparable to

$$\sum_{i=0}^{\infty} u_i = f + N(u_0) + \sum_{i=0}^{\infty}\left[N\left(\sum_{j=0}^{i} u_j\right) - N\left(\sum_{j=0}^{i-1} u_j\right)\right] \tag{4.18}$$

We define the recurrence relation:

$$\begin{aligned} &u_0 = f, \\ &u_1 = N(u_0), \\ &u_{m+1} = N(u_0 + \cdots + u_m) - N(u_0 + \cdots + u_{m-1}), \;\; m = 1, 2, \ldots \end{aligned} \tag{4.19}$$

Then

$$u = \sum_{i=0}^{\infty} u_i + f \tag{4.20}$$

The m −term approximate solution of equation 4.15 is given by

$u = u_0 + u_1 + \cdots + u_{m-1}$.

Now, the DJM will be presented the condition to ensure convergence.

Theorem 4.1 If $N \in C^{(\infty)}$ in a neighborhood of u_0 and

$\left\|N^{(n)}(u_0)\right\| = sup\left\{N^{(n)}(u_0)(h_1, \cdots, h_n): \|h_i\| \leq 1, 1 \leq i \leq n\right\} \leq L$

For some real and for any n, $L > 0$ and $\|u_i\| \leq M < 1/e$, $i = 1,2, \ldots$ then the series $\sum_{n=0}^{\infty} G_n$ is completely convergent, and also

$\|G_n\| \leq LM^n e^{n-1}(e-1), n = 1, 2, \ldots$.

(Bhalekar and Daftardar-Gajii, 2011).

Theorem 4.2 If $N \in C^{(\infty)}$ and $N^{(n)}(u_0) \leq M \leq e^{-1}$, for all n, then the series $\sum_{n=0}^{\infty} G_n$ is absolutely convergent **(**Bhalekar and Daftardar-Gajii, 2011).

The DJM will be implemented using examples of linear and nonlinear Volterra integral and integro-differential equations in this part, and we will also compare the results using ADM and VIM (Batiha et al., 2008; Wazwaz, 2011).
Example 4.5. Consider the following linear VIE

$$u(x) = 1 + \int_0^x (t - x)\, u(t) dt \tag{4.21}$$

Following the recurrence relation provided in equation 4.21, we can solve equation 4.19 via DJM and arrive at:

$u_0 = 1$,

$$u_1 = N(u_0) = \int_0^x [(t-x)u_0(t)]dt = \frac{-1}{2!}x^2,$$

$$u_2 = N(u_0+u_1) - N(u_0) = \int_o^x \left[(t-x)\big(u_0(t)+u_1(t)\big)\right]dt - u_1 = \frac{1}{4!}x^4,$$

$$u_3 = N(u_0+u_1+u_3) - N(u_0+u_1)$$

$$= \int_o^x [(t-x)(u_0+u_1+u_3)]dt - \int_o^x [(t-x)(u_0+u_1)]dt = \frac{-1}{6!}x^6,$$

and so on. The solution in a series form is given by

$$y(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots \tag{4.22}$$

This leads to the exact solution $u(x) = \cos(x)$, which is consistent with the outcomes obtained using ADM in (Wazwaz, 2011).

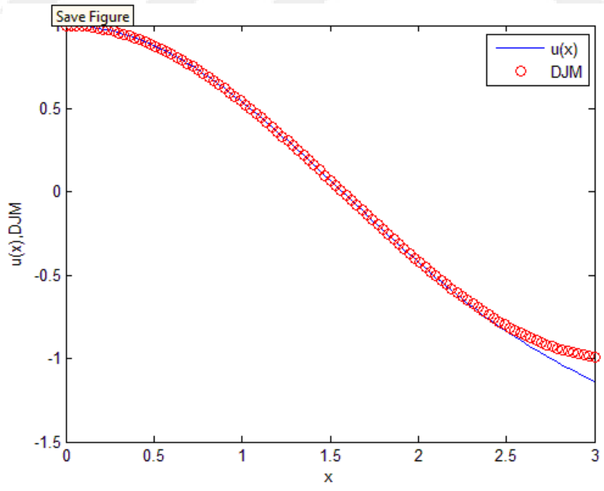

Figure 4.5. Presents the exact solution of linear VIE and DJM.

Table 4.4. Comparison of $u(x)$ and $y(x)$ for linear VIE

x	$u(x)$	$y(x)$	$Error$
0.00000000	1.00000000	1.00000000	0.00000000
0.60000000	0.82533520	0.82533561	0.00000041
1.20000000	0.36225280	0.36235775	0.00010495
1.80000000	-0.22983920	-0.22720209	0.00263711
2.40000000	-0.76302080	-0.73739372	0.02562708
3.00000000	-1.13750000	-0.98999250	0.14750750

Example 4.6. Consider the following nonlinear VIE

$$u(x) = x + \int_0^x [u^2(t)]\, dt \tag{4.23}$$

Follow the recurrence connection in equation 4.19 to solve equation 4.23 using DJM, then we have:

$$u_0 = x,$$

$$u_1 = N(u_0) = \int_0^x [u_0^2(t)] dt = \frac{1}{3}x^3$$

$$u_2 = N(u_0 + u_1) - N(u_0) = \int_0^x [(u_0 + u_1)^2] dt - u_1 = \frac{2}{15}x^2 + \frac{1}{63}x^7,$$

$$u_3 = N(u_0 + u_1 + u_2) - N(u_0 + u_1)$$

$$= \int_0^x [(u_0 + u_1 + u_2)^2] dt - \int_0^x [(u_0 + u_1)^2]\, dt = \frac{4}{105}x^7 + \frac{38}{2835}x^9$$

$$+ \cdots up\ to\ x^{15}$$

and so on. In series form, the following is obtained to be:

$$y(x) = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{1}{63}x^7 + \cdots$$

Which is the same result attained through ADM in (Wazwaz, 2011) that converges to the exact solution $u(x) = \tan(x)$.

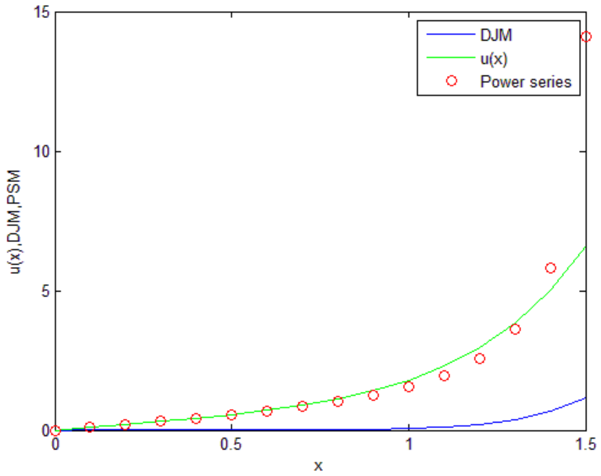

Figure 4.6. Presents the solution of nonlinear VIE and DJM and PSM.

Table 4.5. Comparison of $u(x)$, $y(x)$ and Power series for nonlinear VIE

x	DJM	Exact solution	Power series	$Error$
0.00000000	0.00000000	0.00000000	0.00000000	0.00000000
0.30000000	0.00000860	0.30933625	0.30998380	0.30932765
0.60000000	0.00120150	0.68413681	0.70461477	0.68293531
0.90000000	0.02341377	1.26015822	1.40500885	1.23674445
1.20000000	0.20566320	2.57215162	2.96470601	2.36648842
1.50000000	1.16618304	14.10141995	6.58459821	12.93523691

Example 4.7. Consider the linear VIDE

$$u'(x) = 1 - \int_0^x u(t)dt, u(0) = 0 \tag{4.24}$$

To solve equation 4.24, Integrate, as we must convert it into an integral equation. 4.24 applying the initial conditions from 0 to x, then yields to:

$$u(x) = x - \int_0^x \int_0^x u(t)dtdx = x - \int_0^x (x-t)u(t)\,dt \tag{4.25}$$

Using DJM to equation 4.25, we discover the following:

$u_0 = x,$

$u_1 = N(u_0) = \int_0^x [(x-t)u_0(t)]dt = -\frac{1}{3!}x^3,$

$u_2 = N(u_0 + u_1) - N(u_o) = \int_0^x [(x-t)(u_0 + u_1)]dt - u_1 = \frac{1}{5!}x^5,$

$$u_3 = N(u_0 + u_1 + u_2) - N(u_o + u_1) = \int_0^x [(x-t)(u_0 + u_1 + u_2)]dt - \int_0^x [(x - t)(u_0 + u_1)]\, dt = \frac{1}{7!}x^7,$$

and so on. The series-form response is given by

$$y(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots$$

Which is the identical result as those attained by ADM in (Wazwaz, 2011), and as a result, the exact solution is provided by $u(x) = \sin(x)$ (Wazwaz, 2011).

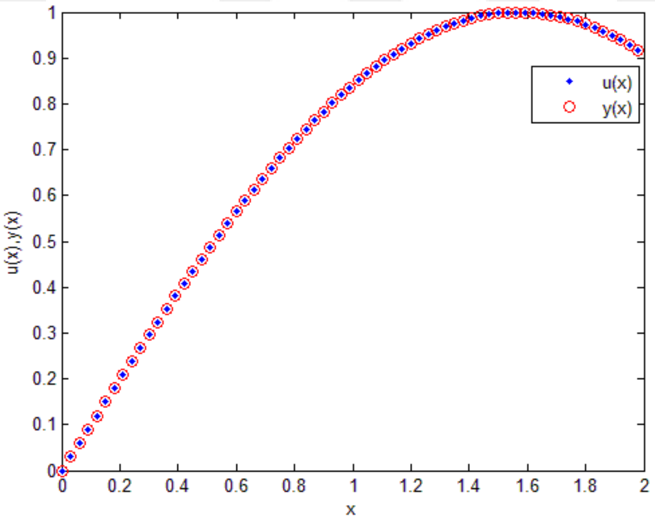

Figure 4.7. Presents the solution of linear VIDE, DJM and PSM.

Table 4.6. Comparison of $u(x)$ and $y(x)$ for linear VIDE

x	$u(x)$	$y(x)$	$Error$
0.00000000	0.00000000	0.00000000	0.00000000
0.50000000	0.47942553	0.47942554	0.00000001
1.00000000	0.84146825	0.84147098	0.00000273
1.50000000	0.99739118	0.99749499	0.00010380
2.00000000	0.90793651	0.90929743	0.00136092

Example 4.8. Consider the nonlinear VIDE

$$u'(x) = -1 + \int_0^x u^2(t)dt, u(0) = 0 \tag{4.26}$$

To solve equation 4.26, we must transform it to an integral equation, thus after integrating equation 4.26 from o to x, applying the initial conditions, and getting the result, we get:

$$u(x) = -x + \int_0^x \int_0^x u^2(t)dtdx = -x + \int_0^x (x-t)u^2(t)\, dt. \tag{4.27}$$

Now, using the DJM to solve equation 4.27, we obtain

$$u_0 = x,$$

$$u_1 = N(u_0) = \int_0^x [(x-t)u_0^2 t]\, dtdt = \frac{1}{12}x^4,$$

$$u_2 = N(u_0+u_1) - N(u_0) = \int_0^x [(x-t)(u_0+u_1)^2]\, dt - u_1 = -\frac{1}{252}x^7 + \frac{1}{12960}x^{10},$$

$$u_3 = N(u_0+u_1+u_2) - N(u_0+u_1) = \int_0^x [(x-t)(u_0+u_1+u_2)^2]\, dt - \int_0^x [(x-t)(u_0+u_1)^2]\, dtu_1 = \frac{1}{11340}x^{10} - \frac{37}{7076160}x^{13} + \frac{109}{914457600}x^{16} - \frac{1}{558472320}x^{19} + \frac{1}{77598259200}x^{22}$$

and so on. The formula for the series-form answer is

$$u(x) = -x + \frac{1}{12}x^4 - \frac{1}{252}x^7 + \frac{1}{6048}x^{10} - \frac{37}{7076160}x^{13} + \frac{109}{914457600}x^{16} - \frac{1}{558472320}x^{19} + \frac{1}{77598259200}x^{22}$$

which is the same results obtained by VIM in (Batiha et al., 2008).

4.5. The Power Series Method

To solve the integral and integro-differential equations of Volterra developed the power series method (PSM) (Tahmasbi and Fard, 2008a; 2008b). The results of employing the PSM to create analytical approximations solutions are perfectly consistent with those of other methodologies reported in the literature. Using illustrated examples, the PSM was applied to Volterra integral and inegro-differential equations in section 4.5.1.

Consider the equation (Tahmasbi and Fard, 2008a; 2008b).

$$u(x) = f(x) + \int_0^x k(x,t)[u(t)]^p dt \tag{4.28}$$

In Eq (4.31), the functions $f(x)$ an $k(x,t)$ are known, $u(x)$ is the unknown function to be determined, also $p \geq 1$ is a positive integer. Suppose the initial condition $e_0 = u(0) = f(0)$ results in the following solution to equation 4.28:

$$u(x) = e_0 + e_1 x. \tag{4.32}$$

where, e_1 is an unknown parameter.

If we substitute equation 4.32 into equation 4.33 the following linear algebraic equation will be obtained:

$$(a_1 e_1 - b_1)x + Q(x^2) = 0 \tag{4.33}$$

where $Q(x^2)$ is a polynomial of order greater than unity and a and b are known constants.

By neglecting $Q(x^2)$ in equation 4.33 and solving the equation $a_1 e_1 = b_1$, the unknown parameter e_1, the coefficient of x in equation 4.32 are obtained.

We assume that equation 4.31's solution is as follows in the following step:

$$u(x) = e_0 + e_1 x + e_2 x^2 \tag{4.34}$$

here, e_0 and e_1 both are known, however the parameter e_2 is unidentified.

By using instead equation 4.34 into equation 4.28, we have the following system,

$$(a_2 e_2 - b_2)x^2 + Q(x^3) = 0 \tag{4.35}$$

By neglecting $Q(x^3)$ in equation 4.35 and solving the algebraic equation $a_2e_2 = b_2$, the unknown parameter e_2 and therefore the coefficient of x^2 in equation 4.34 obtains. One can derive a power series of the following type by repeating the aforementioned process m iteration:

$$u(x) = e_0 + e_1x + e_2x^2 + \cdots + e_mx^m, \tag{4.36}$$

The exact solution u(x) of integral equation 4.32 is approximated by equation 4.36.

The following theorems illustrate how PSM converges for the nonlinear Volterra integral equation and integro-differential equation.

Theorem 4.1. Let $u = u(x)$ be the exact solution of the following VIE:

$$u(x) = f(x) + \int_0^x k(x,t)[u(t)]^p dt \tag{4.37}$$

The Taylor expansion obtained via the power series approach. $u(x)$ (Tahmasbi and Fard, 2008).

Corollary 4.2

The power series method will give the exact solution if the polynomial is the exact solution to equation 4.37. (Tahmasbi and Fard, 2008).

Theorem 4.3

Let $u = u(x)$ be the exact solution to the following Volterra integro differential equation

$$u'(x) = g\big(x, u(x)\big) + \int_0^x k\big(x, t, u(t), u'(t)\big)dt, u(0) = a.$$

Furthermore, assume that $u(x)$ has a power series representation as well. The Taylor expansion is then obtained using the power series method of $u(x)$ (Erfanian and Mostahsan, 2011).

The PSM will be used to some instances in this section, including Both linear and nonlinear VIDE and VIE are examples. We also contrast our results with those attained by ADM and VIM.

Example 4.9. A linear VIE is as follows (Wazwaz, 2011)

$$u(x) = 1 + \int_0^x (t - x)u(t)\,dt \tag{4.38}$$

We assume that the solution of equation 4.38 obtained by applying the PSM with, $u_0(x) = u(0) = e_0 = 1$ is $u_1(x) = e_0 + e_1 x$, and hence

$$u_1(x) = 1 + e_1 x\,. \tag{4.39}$$

Substitute equation 4.39 in equation 4.38, then,

$$1 + e_1 x = 1 + \int_0^x (t - x)(1 + e_1 t)\,dt, \tag{4.40}$$

After simplifying, we get, $e_1 = -\frac{x}{2}$ and hence,

$$u_1(x) = 1 - \frac{1}{2}x^2, \tag{4.41}$$

Which is the first approximation for the solution to equation 4.38. The solution of equation 4.38 can be obtained as:

$$u(x) = 1 - \frac{1}{2}x^2 + e_2 x^2, \tag{4.42}$$

Substitute equation 4.42 in equation 4.38, we have,

$$1 - \frac{1}{2}x^2 + e_2 x^2 = 1 + \int_0^x (t - x) - \frac{1}{2}t^2 + e_2 t^2 dt,$$

After simplifying, we get, $e_2 = \frac{x^2}{24}$ and hence,

$$u_2(x) = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 \ .$$

By continuing in this way, we obtain the series-form solution provided by

$$y(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots$$

Which is the exact same results discovered by DJM and achieved by ADM in (Wazwaz, 2011), which converges to the answer $u(x) = \cos(x)$.

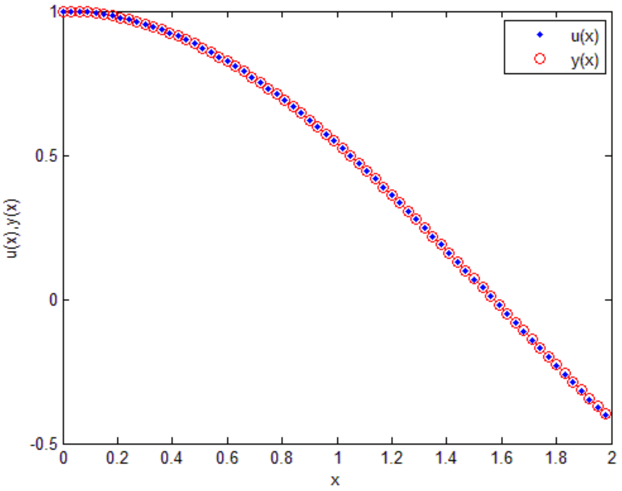

Figure 4.8. Presents of the solution in a series form of linear VIE and exact solution.

Table 4.7 Comparison of $u(x)$ and $y(x)$ for linear VIE

x	$u(x)$	$y(x)$	$Error$
0.00000000	1.00000000	1.00000000	0.00000000
0.50000000	0.87758247	0.87758256	0.00000010
1.00000000	0.54027778	0.54030231	0.00002453
1.50000000	0.07011719	0.07073720	0.00062001
2.00000000	-0.42222222	-0.41614684	0.00607539

Example 4.10. Consider the following nonlinear VIE (Wazwaz, 2011)

$$u(x) = x + \int_0^x u^2(t)\,dt \tag{4.43}$$

By using the PSM to solve equation 4.43, We assume that equation 4.43 has a solution with $u_0(x) = u(0) = e_0$ is $u_1(x) = e_0 + e_1 x$, and hence

$$u_1(x) = 0 + e_1 x = e_1 x \tag{4.44}$$

Substitute equation 4.44 in equation 4.43, we have,

$$e_1 x \ = x + \int_0^x (e_1 t)^2\,dt \tag{4.45}$$

We obtain $e_1 = 1$ after simplification. That's why,

$$u_1(x) = x, \tag{4.46}$$

which is the first approximation for the solution of equation 4.43. The solution of equation 4.43 can be considered to:

$$u(x) = x + e_2 x^2, \tag{4.47}$$

Now we substitute equation 4.47 in equation 4.43 and we obtain,

$$x + e_2 x^2 = x + \int_0^x (t + e_2 t^2)^2\,dt,$$

and we get, $e_2 = \frac{1}{3}x$ and hence,

$$u_2(x) = x - \frac{1}{3}x^3.$$

By continuing in this way, we discover that the series-form solution provided by

$$y(x) = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \cdots$$

This converges to the same exact solution as those discovered by DJM and acquired by ADM. $u(x) = \tan(x)$ (Wazwaz, 2011).

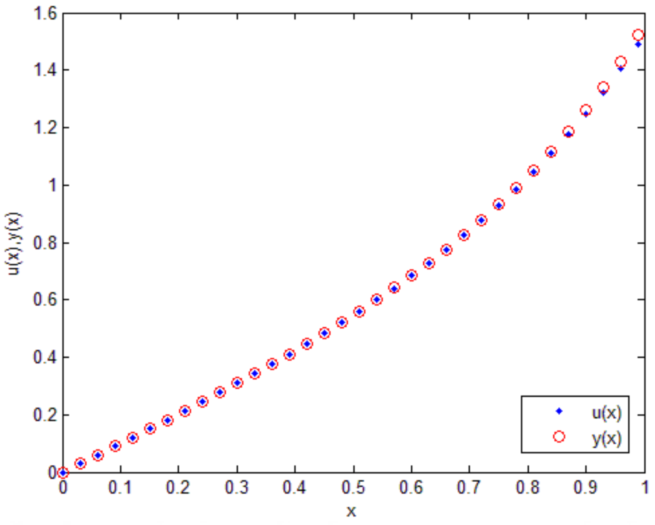

Figure 4.9. Presents the relation between both of $u(x)$ and $y(x)$.

Table 4.8. Comparison of $u(x)$ and $y(x)$ for nonlinear VIE

x	$u(x)$	y	$Error$
0.00000000	1.00000000	0.00000000	0.00000000
0.10000000	0.10033467	0.10033467	0.00000000
0.20000000	0.20271002	0.20271004	0.00000001
0.30000000	0.30933580	0.30933625	0.00000045
0.40000000	0.42278709	0.42279322	0.00000613
0.50000000	0.54625496	0.54630249	0.00004753
0.60000000	0.68387877	0.68413681	0.00025804
0.70000000	0.84118718	0.84228838	0.00110120
0.80000000	1.02567530	1.02963856	0.00396326
0.90000000	1.24754485	1.26015822	0.01261337
1.00000000	1.52063492	1.55740772	0.03677280

Example 4.11. Consider the linear VIDE (Wazwaz, 2011)

$$u'(x) = 1 - \int_0^x u(t)dt, \;\; u(0) = 0 \tag{4.48}$$

To solve equation 4.48, integrate equation 4.48 applying the initial condition and going from 0 to x, then we have:

$$u(x) = x - \int_0^x \int_0^x u(t)dtdx = x - \int_0^x (x-t)u(t)dt \tag{4.49}$$

By using the PSM to equation 4.49, we assume that, the solution of equation 4.49 is $u_0(x) = u(0) = e_0 = 0$ and $u_1(x) = e_0 + e_1 x,$ hence

$$u_1(x) = 0 + e_1 x = e_1 x \tag{4.50}$$

Substitute equation 4.50 in equation 4.50, we have,

$$e_1 x = x + \int_0^x (t-x)e_1 t \, dt\,, \tag{4.51}$$

By integrating from equation 4.51, we get

$$(e_1 - x)x - \frac{e_1 x^3}{6} = 0,$$

by neglecting $\left(-\frac{e_1 x^3}{6}\right)$, therefore $e_1 = 1$ and hence

$$u_1(x) = x, \tag{4.52}$$

which is the first approximation for the solution of equation 4.49. The solution of equation 4.49 can be regarded as:

$$u(x) = x + e_2 x^2, \tag{4.53}$$

Substitute equation 4.53 in equation 4.49, we get,

$$x+e_2x^2=x+\int_0^x(x-t)(t+e_2t^2)\,dt\ .$$

By integrating and solving, we get $\left(e_2+\frac{x}{6}\right)x^2-\frac{e_2x^4}{12}=0$ and by neglecting $\left(-\frac{e_2x^4}{12}\right)$, therefore $e_2=-\frac{1}{6}x$ and hence,

$$u_2(x)=x-\frac{1}{6}x^3,$$

so, the series form of the solution is provided by

$$y(x)=x-\frac{1}{3!}x^3+\frac{1}{5!}x^5-\frac{1}{7!}x^7+\cdots$$

Which are the same results discovered by DJM and discovered by ADM in (Wazwaz, 2011); as a result, the exact solution is provided by $u(x)=\sin(x)$.

Example 4.12. Consider the following nonlinear VIDE

$$u'(x)=-1+\int_0^x u^2(t)dt\ ,u(0)=0. \tag{4.54}$$

To solve equation 4.54, integrate equation 4.54 if we extend the initial condition from 0 to x, we get:

$$u(x)=-x+\int_0^x\int_0^x u^2(t)dtdx=-x+\int_0^x(x-t)u^2\,(t)dt \tag{4.55}$$

we applying PSM to equation 4.55 and assume that the solution of equation 4.55 with $u_0(x)=u(0)=e_0=0$ is $u_1(x)=e_0+e_1x$, and therefore

$$u_1=0+e_1x=e_1x\ . \tag{4.56}$$

by Substituting equation 4.56 in equation 4.55, we get,

$$e_1 x = -x + \int_0^x (x-t)(e_1 t)^2\, dt,$$

and by integrating and solving the equation, we get,

$$(e_1+1)x - \frac{e_1^2 x^4}{12} = 0,$$

by neglecting $\left(-\frac{e_1 x^4}{12}\right)$ term, we obtain $e_1 = -1$ and hence,

$$u_1(x) = -x,$$

Which is the first approximation for the solution of equation 4.55. The solution of equation 4.55 can be obtained as:

$$u(x) = -x + e_2 x^2 \tag{4.57}$$

We substituting the equation 4.57 in equation 4.55, we get,

$$x + e_2 x^2 = -x + \int_0^x (x-t)(t+e_2 t^2)^2 dt,$$

and by integrating and solving the equation, we have,

$$\left(e_2 - \frac{x^2}{12}\right)x^2 - \left(\frac{e_2 x^5}{10} - \frac{e_2^2 x^6}{30}\right) = 0$$

by neglecting $\left(\frac{e_2 x^5}{10} - \frac{e_2^2 x^6}{30}\right)$ term, therefor $e_2 = \frac{x^2}{12}$, and hence,

$$u_2(x) = -x + \frac{1}{12}x^4,$$

so, we discover that the series form of the solution is provided by

$$u(x) = -x + \frac{1}{12}x^4 - \frac{1}{252}x^7 + \frac{1}{6048}x^{10} - \frac{37}{707616}x^{13} + \frac{109}{9144576}x^{16} - \frac{1}{55847232}x^{19} + \frac{1}{775982592}x^{22}$$

Which are the same results obtained by DJM and gotten by VIM in (Batiha et al., 2008).

5. APPLICATION

5.1. Approximate Solutions for Volterra’s Population Model

Apart from the Dafterdar-Jafari Method, the Power Series Method, the Adomian Decomposition Method, the Variational Iteration Method It is performed independently of the model in the literature (Wazwaz, 1999; 2011). In this chapter, there have been some approximations approaches there have developed to achieve to get answers for Volterra’s population model. In terms population increase in a closed system of a species. Analytic and Approximations of Some Real-Life Model Solutions in the Padé in this part the research effectively uses approximants, which typically outperform series approximants in capturing the essential behavior to the population u(t) of identical individuals (Wazwaz, 2011).

5.1.2. ADM’ solution of Volterra’s population model

The nonlinear VIDE population growth model is discussed in this section.

$$\kappa \frac{du}{dt} = u - u^2 - u \int_0^t u(x)dx \, , u(0) = u_0$$

Which have been considered, and is also studied in (Mohyud-Din et al., 2010; Wazwaz, 2010)

$$\frac{du}{dt} = 10u(t) - 10u^2(t) - 10u(t) \int_0^t u(x)\, dx \, , u(0) = 0.1 \tag{5.1}$$

In which (Wazwaz, 1999) used the initial condition $u(0) = 0.1$ and the nondimensional parameter $\kappa = 0.1$ for simplicity.

Equation 5.1 can be written more conveniently in an operator form as follows to start:

$$Lu(t) = 10u(t) - 10u^2(t) - 10u(t)\int_0^t u(t)u(x)\,dx\,, u(0) = 0.1 \tag{5.2}$$

Where $L. = \frac{d}{dt}.$ serves as the definition of the differential operator. L must obviously be invertible for the integral operator to be defined as:

$$L^{-1}(\bullet) = \int_0^t (\bullet)dt$$

Using the initial condition leads after applying L^{-1} to both parties equation 5.2 then using the initial condition leads to:

$$u(t) = 0.1 + L^{-1}\left(10u(t) - 10u^2(t) - 10\int_0^t u(t)u(x)dx\right). \tag{5.3}$$

By using this this method, we usually the decomposition series represent u in equation 5.3:

$$u(t) = \sum_{n=0}^{\infty} u_n(t) \tag{5.4}$$

Therefore, the main objective of this work is formalized identify the components $n(t)$ and $u(0)$. To obtain, we enter equation 5.4 into both sides of equation 5.3.

$$\sum_{n=0}^{\infty} u_n(t) = 0.1 + L^{-1}\left(10\sum_{n=0}^{\infty} u_n(t) - 10\sum_{n=0}^{\infty} A_n(t) - 10\int_0^{10} B_n(x,t)dx\right)$$

Adomian polynomials are used to express the nonlinear terms $u^2(t)$ and $u(x)u(t)$ are represented $A_n(t)$ and $B_n(x,t)$ respectively. As a result, we set

$$u^2(t) = \sum_{n=0}^{\infty} A_n(t),$$

$$u(x)u(t) = \sum_{n=0}^{\infty} B_n(x,t),$$

Analyzing the various nonlinearity classes proposed for the analysis of the Adomian polynomials $A_n(t)$, $B_n(x,t)$.

Analyzing the Adomian polynomials $A_n(t)$ and $B_n(x,t)$ has been covered for a variety of nonlinearity classes and using the techniques described in (Adomian, 1994), we list a handful of the Adomian polynomials for convenience below $A_n(t)$ and $B_n(x,t)$ for $A_n(t)$, we discover:

$$A_0(t) = u_0^2(t), A_1(t) = 2u_0(t)u_1(t), A_2(t) = u_1^2(t) + 2u_0(t)u_2(t),$$
$$A_3(t) = 2u_1(t)u_2(t) + 2u_0(t)u_3(t),$$
$$A_4(t) = u_2^2(t) + 2u_1(t)u_3(t) + 2u_0(t)u_4(t), \ldots$$

For $B_n(x,t)$, we obtain

$$B_0(x,t) = u_0(x)u_0(t), B_1(x,t) = u_0(x)u_1(t) + u_1(x)u_0(t),$$
$$B_2(x,t) = u_0(x)u_2(t) + u_1(x)u_1(t) + u_2(x)u_0(t),$$
$$B_3(x,t) = u_0(x)u_3(t) + u_1(x)u_2(t) + u_2(x)u_1(t) + u_3(x)u_0(t)$$
$$B_4(x,t) = u_0(x)u_4(t) + u_1(x)u_3(t) + u_2(x)u_2(t) + u_3(x)u_1(t) + u_4(x)u_0(t), \ldots$$

To determine the components $u_0(t), u_1(t), u_2(t), \ldots$ of $u(t)$, The recursive relationship described below will be used

$$u_0(t) = 0.1,$$
$$u_{k+1}(t) = L^{-1}\left(10u_k(t) - 10A_k(t) - \int_0^t B_k(x,t)dx\right), k \geq 0$$

Practically, this can be quite helpful in assessing the other elements. with $u_0(t)$ defined as shown above, the next step is

$$u_0(t) = 0.1,$$
$$u_1(t) = L^{-1}\left(10u_0(t) - 10A_0(t) - 10\int_0^t B_0(x,t)dx\right) = 0.9t + 0.05t^2,$$

$$u_2(t) = L^{-1}\left(10u_1(t) - 10A_1(t) - 10\int_0^t B_1(x,t)dx\right) = 3.6t^2 - \frac{7}{12}t^3 + \frac{1}{60}t^4,$$

$$u_3(t) = L^{-1}\left(10u_2(t) - 10A_2(t) - 10\int_0^t B_2(x,t)dx\right)$$

$$= 6.9t^3 - 3.1541667t^4 - 0.0047222t^6\ ,$$

$$u_4(t) = L^{-1}\left(10u_3(t) - 10A_3(t) - 10\int_0^t B_3(x,t)dx\right) = -2.4t^4 - 9.3516667t^5 + 1.6631944t^6 - 0.08273809t^7 + 0.00123016t^8,$$

The components $u_5(t), u_6(t), u_7(t)$ and $u_8(t)$ were also determined and will be utilized, but not noted for brevity. The approximate value of u provided has now been formally determined by

$$u(t) = 0.1 + 0.9t + 3.55t^2 + 6.31666667t^3 - 5.5375t^4 - 63.70916667t^5 - 156.0804167t^6 - 18.47323411t^7 + 1056.288569t^8 + O(t^9) \quad (5.5)$$

(Wazwaz, 1999).

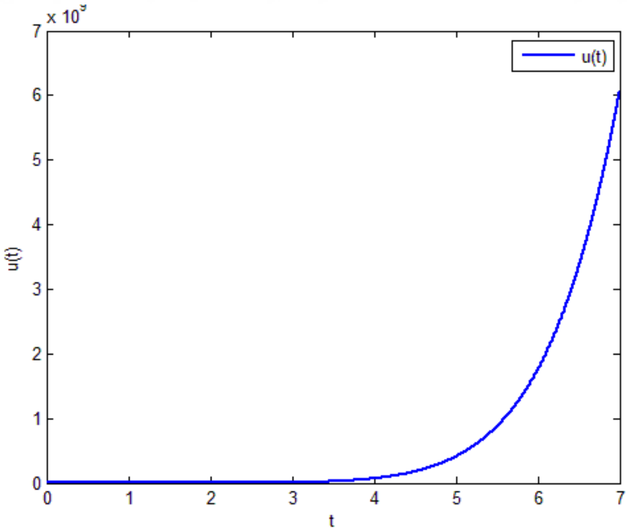

Figure 5.1. Presents the relation between both of $u(t)$ and t.

5.1.3. VIM’s solution to the Volterra population model

The VIM (Wazwaz, 2011), which is defined by the nonlinear VIDE equation 5.1, is used to solve Volterra’s population model in this subsection equation 5.1 To solve equation 5.1 with VIM, we went through the following steps equation 5.1 has the following correction functional:

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda(t)\left[u_n'(t) - 10u_n(t) + 10u_n^2(t) + 10u_n(t)\int_0^t u_n(x)dx\right]dt \tag{5.6}$$

In which we used $\lambda(t) = -1$ for the integro-differential equation of first order equation 5.1. The initial value of $u(0) = 0.1$ can be used to select the zeroth approximation, or u_0. Using the parameters $u_0(t) = 0.1$ and equation 5.6, the successive approximations shown below have been obtained:

$$u_0(t) = 0.1,$$

$$u_1(t) = u_0 - \int_0^t \left[u_0'(t) - 10u_0(t) + 10u_0^2(t) + 10u_0(t)\int_0^t u_0(x)dx\right]dt = 0.1 + 0.9t - 0.05t^2,$$

$$u_2(t) = u_1 - \int_0^t \left[u_1'(t) - 10u_1(t) + 10u_1^2(t) + 10u_1(t)\int_0^t u_1(x)dx\right]dt = 0.1 + 0.9t + 3.55t^2 - 3.2833333t^3 - 0.7708333t^4 + 0.0699999t^5 - 0.0013888t^6,$$

$$u_3(t) = u_2 - \int_0^t \left[u_2'(t) - 10u_2(t) + 10u_2^2(t) + 10u_2(t)\int_0^t u_2(x)dx\right]dt$$

$$= 0.1 + 0.9t + 3.55t^2 + 6.31667t^3 - 24.7375t^4 - 19.1225t^5 + \cdots$$

$$u_4(t) = u_3 - \int_0^t \left[u_3'(t) - 10u_3(t) + 10u_3^2(t) + 10u_3(t)\int_0^t u_3(x)dx\right]dt = 0.1 + 0.9t + 3.55t^2 + 6.31667t^3 - 5.5375t^4 - 94.4292t^5 + \cdots$$

The components $u_5(t), u_6(t), u_7(t)$ and $u_8(t)$ were also evaluted as well as used but not listed for brevity is now formalized in the following way as the approximate value of $u(t)$ provided by

$$u(t) = 0.1 + 0.9t + 3.55t^2 + 6.31667t^3 - 5.5375t^4 - 63.709167t^5 - 156.080417t^6 - 18.473234t^7 + 1056.2887t^8 + O(t^9) \tag{5.7}$$

The results obtained using ADM are in very good agreement with the approximation in equation 5.7.

To learn more about the mathematical form of the $u(t)$, it is possible to debate the results of the work that has been presented $u(t)$. The rapid ascent within the logistic curve, which will eventually reach its peak and then undergo a slow exponential decay, are of great interest to us (Wazwaz,1999). We use Padé approximations to investigate the solution of $u(t)$. Using the approximation in equation 5.7, we can get:

$$[4/4] = \frac{0.1 + 0.46879318\ \mathrm{t} + 0.9249573t^2 + 0.92312949\ t^3 + 0.40042348\ t^4}{1 - 4.3120682\ \mathrm{t} + 12.55819\ t^2 - 13.880639\ t^3 + 10.86831\ t^4}$$

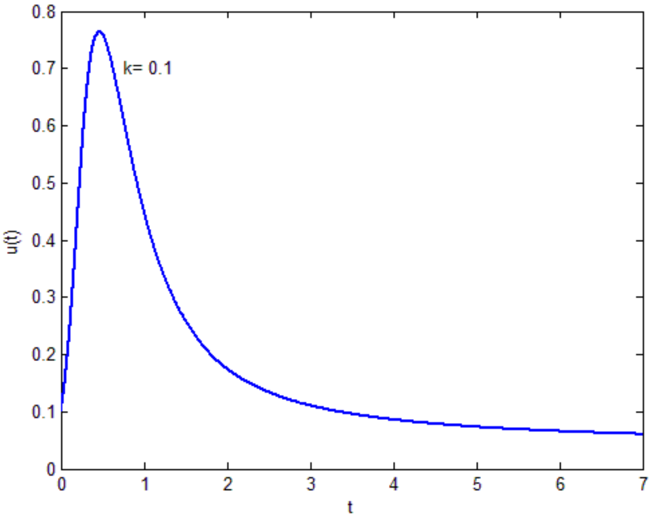

Figure 5.2. Relation between [4/4] of $u(t)$ and t for $u(0) = 0.1, \kappa = 0.1$.

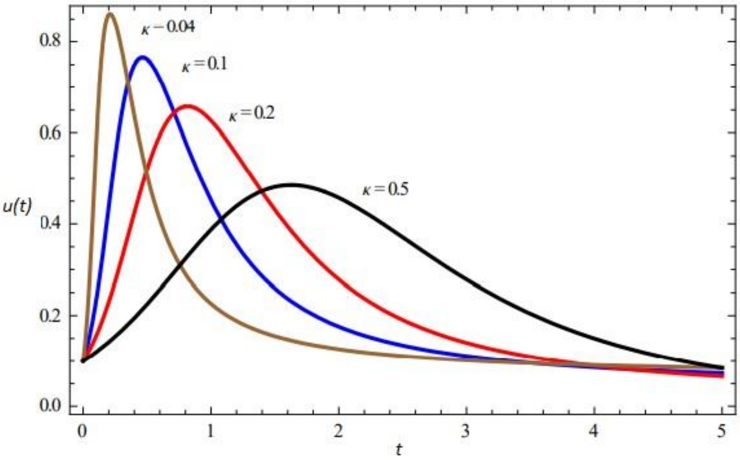

Figure 5.3. Presents the relation between both of [4/4] of $u(t)$ and t for $u(0) = 0.1$, $\kappa = 0.04, 0.1, 0.2$ *and* 0.5.

Which is the similar to that obtained by the Method of Adomian Decomposition. From the Figure 5.2, can understand which for $u(0) = 0.1$ and $\kappa = 0.1$,we achieve $u_{max} = 0.76511309$ occurs at $t_{\text{critical}} = 0.46447674$. On the other hand, in the next figure, we understand that the figure shows the Padé approximant $[4/4]$ of $u(t)$ for $u(0) = 0.1$ and for $\kappa = 0.04, 0.1, 0.2$ and 0.5, which is the same as ADM's result. The most important result in this the amplitudes of the $u(t)$ decrease as the graph develops, while the exponential decay increases. Table 5.2 provides an overview. The relationship between k and t_{critical}. With the aid of, the precise values of u_{max} were

$$u_{max} = 1 + \kappa \ln\left(\frac{\kappa}{1+\kappa-u_0}\right) \tag{5.8}$$

obtained by (TeBeest, 1997).

Table 5.1. Results of the numerical solution of u_{max}d analytical value of u_{max} for $\kappa = 0.04, 0.1, 0.2$ and 0.5

κ	Critical t	Approx. u_{max}	Exact u_{max}
0.04	0.210246	0.8612401	0.873719
0.1	0.464476	0.7651130	0.769741
0.2	0.816858	0.6579123	0.659050
0.5	1.626662	0.4852823	0.485190

5.1.4. DJM Solution for the Volterra's population model

DJM solves the model of Volterra population, in this part. Remember the nonlinear population growth model is characterized by VIDE in the equation 5.1, for solving it using DJM, we have applied integration for the equation 5.1 using the initial condition and the integration calculated with respect to the variable t, in the interval $[0,t]$the following is the result:

$$u(t)=0.1+\int_0^t\left[10u(t)-1ou^2(t)-10u(t)\int_0^t u(x)dx\right]dt \tag{5.9}$$

Now, VIM will be applied to equation 5.9,

$u_0(t)=0.1,$

$\mathrm{u}_1(t)=N(u_0)=\int\limits_0^t\left[10u_0(t)-1ou^2{}_0(t)-10u_0(t)\int\limits_0^t u_0(x)dx\right]dt$

$=\int_0^t\left[10(0.1)-1(0.1)-10(0.1)\int_0^t 0.1(x)dx\right]dt=0.9t-0.05t^2,$

$u_2(t)=N(u_0+u_1)-N(u_0)=\int_0^t\Big[10\big(u_{\,0}(t)\ +\ u_{\,1}(t)\big)-10(u_{\,0}(t)+u_{\,1}(t)^2)-$

$10\big(u_{\,0}(t)+u_1(t)\big)\int_0^t(u_{\,0}(x)+u_1(x)dx)\Big]dt-u_1,$

$=\int_0^t\Big[10\big(0.1+(0.9t-0.05t^2)\big)-10\big(0.1+(0.9t-0.05t^2)\big)^2-10\big(0.1+(0.9t-$

$0.05t^2)\big)\int_0^t\big(0.1+(0.9x-0.05x^2)\big)dx\Big]dt-(0.9t-0.05t^2)=3.6t^2-3.28333t^3-$

$0.770833t^4,$

$u_3(t)=N(\,u_0+u_1+u_2\,)-N(u_0+u_1)=\int_0^t\Big[10(u_0+u_1+u_2)-10(u_0+u_1+$

$u_2)^2-10(u_0+u_1+u_2)\int_0^t\big(u_0(x)+u_1(x)u_2(x)\big)dx\Big]-\int_0^t\Big[10(u_{\,0}+u_{\,1})-$

$10(u_{\,0}+u_{\,1})^{\,2}-10(u_{\,0}+u_{\,1})\int_0^t(\,u_{\,0}\,(x)+u_{\,1}(x)\,)\,dx\Big]dt-u_1,$

$u_4(t)=N(u_0+u_1+u_2+u_3)-N(u_0+u_1+u_2)=\int_0^t\Big[10(u_0+u_1+u_2+u_3)-$

$10(u_0+u_1+u_2+u_3)^2-10(u_0+u_1+u_2+u_3)\int_0^t\big(u_0(x)+u_1(x)+u_2(x)+$

$$u_3(x)\big)dx\Big] - \int_0^t \Big[10(u_0 + u_1 + u_2) - 10(u_0 + u_1 + u_2)^2 - 10(u_0 + u_1 + u_2)\int_0^t \big(u_0(x) + u_1(x) + u_2(x)\big)dx\Big] = 19.2\, t^{4} + 75.3067\, t^{5} + 290.762\, t^{7} + 540.771\, t^{8}$$

The $u_5(t), u_6(t), u_7(t)$ and $u_8(t)$ components were evaluated, but we have not showed them because of brevity, the process is similar. This complete that the formal determination of approximations of the $u(t)$ are presented by

$$u(t) = 0.1 + 0.9t + 3.55t^2 + 6.31666667t^3 - 5.5375t^4 - 63.7091667t^5 - 156.08041667t^6 - 18.47323413t^7 + 1056.288569t^8 + O(t^9) \qquad (5.10)$$

The findings obtained by employing ADM, HPM, and VIM in (Mohyud-Din et al., 2010), (Wazwaz, 1999; 2011) are quite close to the approximation given in equation 5.10. It's worth noting that the DJM is simpler than the ADM in that it doesn't need computing Adomian polynomials to handle nonlinear terms, constructing a homotope like the HPM, or calculating the Lagrange multiplier like the VIM.

We desire to investigate the quick increase within logistic curve following by slow exponential decay that reaches the peak, when the $u(t) \to 0\ as\ t \to \infty$, (Wazwaz, 1999), to better understand the mathematical form of u as illustrated in the above figures. To approximate power series, polynomials are frequently utilized. Which is prone to fluctuations that could lead to approximation error boundaries. Furthermore, polynomials can be never blowup in the finite-plane, making the singularities invisible. The Taylor series has been controlled best by using Padé approximations for the numerical approximations to solve these issues Padé Approximants (Wazwaz, 2009) have the advantage of being able to manipulate the polynomial approximations in to the rational function in order to learn more about $u(t)$. Using the coefficient from the function's Taylor expansion, efficient are founded on numerator and denominator. We can find the following by using the approximation derived for $u(t)$ in equations 5.10:

$$[4/4] = \frac{0.1 + 0.4687931832\mathrm{t} + 0.924957398t^2 + 0.9231294892t^3 + 0.4004234788t^4}{1 - 4.312068168\mathrm{t} + 12.5581875t^2 - 13.88063927t^3 + 10.8683047t^4}$$

Which has an excellent accuracy obtained the use of ADM and VIM. From the Figure 5.2, we can conclude that this figure depicts the evolution of u(t) and investigates the quick rise that reaches a peak before slowing exponentially decaying. ADM and VIM both came up with the same conclusions. Figure 5.3 shows the same results that ADM and VIM obtained. We may plainly see from Fig 5.2 that for $u(0) = 0.1$and $\kappa = 0.1$, we get $u_{max} = 0.7651130891$ which occur at $t_{\mathrm{critical}} = 0.4644767409$.

Table 5.2 summarizes the relations between t_{critical} and κ. The analytical value of u_{max} were calculated by the using of the equation 5.8 presented in (TeBeest, 1997).

Table 5.2. Presents the exact value and the Numerical of the u_{max}, for $k = 0.04, 0.1, 0.2\ and\ 0.5$

κ	Critical t	Approx. u_{max}	Exactu_{ma}
0.04	0.210246	0.861240	0.873719
0.1	0.464476	0.765113	0.769741
0.2	0.816858	0.657912	0.659050
0.5	1.626662	0.485282	0.485190

Padé approximants' relationship to one another [4/4] of $u(t)$ and t, identifying the identical conclusions reached by the ADM, the VIM, and the DJM, is shown in Figure 5.2 Figure 5.3 also shows similar outcomes to those attained by ADM, VIM, and NIM. We can quickly see from Figure 5.2 that for $u(0) = 0.1$ and $\kappa = 0.1$, we get $u_{max} = 0.76511309$, which happens at $t_{critical} = 0.4644767409$. Table 5.3 summarizes how they are connected κ , u_{max} and $t_{critical}$. Additionally, the Padé approximants and [4/4] findings of the corresponding absolute

$$Errors\ = |exact\ solution - approximate\ solution|$$

are also shown.

The results produced by PSM are in excellent agreement, as can be seen in Table 5.3. With the exact solution, the absolute errors are reduced as $u(t)$ becomes smooth and as soon as the κ values rise

Table 5.3. Approximation of u_{max}, exact value of u_{max}, absolute errors for $\kappa = 0.04, 0.1, 0.2, 0.5$, with the Padé approximants [4 / 4]

κ	Critical t	Approx. $max\ u$	Exact $max\ u$	ε for [4/4]
0.04	0.2102464442	0.8612401810	0.8737199832	4.62836×10^{-3}
0.1	0.4644767409	0.7651130891	0.7697414491	1.13807×10^{-3}
0.2	0.8168581213	0.657912310	0.6590503816	9.20576×10^{-5}
0.5	1.6266622270	0.4852823490	0.4851902914	1.24798×10^{-2}

5.1.5. Applying power series method (PSD) for Solution of Volterra's population model

The Volterra population model have been solved by using the PSM in this part. Recall that the nonlinear VIE equation 5.9, which the populations growth model is characterized by, and by applying the PSM to equation 5.9, the population growth model may be described as follows equation 5.9, Consider that the answer to equation 5.9 with $u_0(t) = u(0) = e_0 = 0.1$ is $u_1(t) = e_0 + e_1 t$ and hence

$$u_1(t) = 0.1 + e_1 t \tag{5.11}$$

By putting the values of equation 5.11 in to the equation 5.9, we will obtain:

$$0.1 + e_1 t = 0.1 + \int_0^t \left[10(0.1 + e_1 t) - 10(0.1 + e_1 t)^2 - 10(0.1 + e_1 t) \int_0^t (0.1 + e_1 t) dx\right] dt \tag{5.12}$$

By simplifying a few steps, we obtain $e_1 = 0.9$, then

$$u_1(t) = 0.1 + 0.9t \tag{5.13}$$

That is first approximation for the solution equation 5.9, Also, the second approximation. Solution of the equation 5.9, can be proposed as follow:

$$u(t) = 0.1 + 0.9t + e_2 t^2 \tag{5.14}$$

and putting equation 5.14 in equation 5.9, will obtain:

$$0.1 + e_1 t + e_2 t^2 = 0.1 + \int_0^t \Big[10(0.1 + e_1 t + e_2 t^2) - 10(0.1 + e_1 t + e_2 t^2)^2 - 10(0.1 + e_1 t + e_2 t^2) \int_0^t (0.1 + e_1 x + e_2 x^2) dx \Big] dt \tag{5.15}$$

After a few steps of simplification, the value of the second constant, e_2, was discovered to be $e_2 = 3.55$ as a result,

$$u_2(t) = 0.1 + 0.9t + 3.55t^2. \tag{5.16}$$

In the same way, the approximation solution has been obtained as follow,

$$u(t) = 0.1 + 0.9t + 3.55t^2 - 6.3166667t^3 - 5.5375t^4 - 63.709167t^5 - 156.0804167t^6 - 18.47323413t^7 + 1056.288569t^8 + O(t^9) \tag{5.17}$$

To investigate the mathematical formula $u(t)$, the Padé approximants introduced and constructed in the first section are utilized. By using approximation of $u(t)$ derived in equation 5.17, we find

$$[4/4] = \frac{o.1 + \mathrm{o.}4687931832\mathrm{t} + 0.924957398t^2 + 0.923l294892t^3 + 0.4004234788t^4}{1 - 4.312068168.\mathrm{t} + 12.5581875.t^2 - 13.88063927t^3 + 10.8683047t^4}$$

Figure 5.2 depicts the graph of the relationships between the Padé approximant $[4/4]$ of t and $u(t)$ above. That have similar results as obtained by the methods DJM, VIM and ADM. However, the next figure, Figure 5.3, have similar results as obtained by the methods DJM, VIM and ADM from Figure 5.2, We can quickly notice that for $\kappa = 0.1$ and $u\,(0) = 0.1$, we get $u_{max} = 0.7651130891$, that is happens at $t_{critical} = 0.4644767409$. The relation between t_{critical} , κ and u_{max} are Table 5.3 summarized in the Table 5.3. Furthermore, what corresponds as a result absolute errors are also presented for Padé approximations and $[4/4]$. The analytical values of the u_{max} are investigated by using the equation 5.8 obtained by Tee Best in the reference (TeBeest, 1997). It is easy to notice that from the Table 5.3 which the obtained a result of Power Series Method has a good accuracy within as $u(t)$ becomes are increases, the value of grows as both the exact solution and absolute error value decrease.

Table 5.4. Presents the Exact and Numerical values of u_{max}, and the exact error for$\kappa = 0.04, 0.1, 0.2, 0.5$ and with the Padé approximant [4/4]

κ	Critical t	Approx. max u	Exact max u	ε for [4/4]
0.04	0.2102464442	0.8612401810	0.8737199832	4.62836x10^{-3}
0.1	0.4644767409	0.7651130891	0.7697414491	1.13807x10^{-3}
0.2	0.8168581213	0.657912310	0.6590503816	9.20576x10^{-5}
0.5	1.6266622270	0.4852823490	0.4851902914	1.24798x10^{-2}

5.1.6. PSM convergence for the Volterra's population model

In this section we will focus on the convergence of PSM, it is easy to see that the Volterra population model has been given in equation 5.1, and the following integral-differential equation can be used as a generalization:

$$u'(x) = g\big(x, u(x)\big) + u(x)\int_0^x k\big(x, t, u(t), u'(t)\big)dt, u(0) = a. \qquad (5.18)$$

We demonstrate PSM convergence for Volterra's population model in the subsequent theorem. A variation to the theorem's proof is the proof of the convergence theorem 4.3.

Theorem 5.1. Consider as the generalized Volterra's population model's analytical resolution. in equation 5.18 and that u is represented by power series. Then, the PSM's approximation solution converges to an exact answer.

Proof: to prove this theorem, we will start by assuming the approximation solution to equation 5.18 be as follow

$$\tilde{u}(x) = e_0 + e_1 x + e_2 x^2 + \cdots$$

Hence, it suffices to prove that $e_m = \frac{u^m(0)}{m!}$, $m = 1,2,3, \ldots$. First case is that, if $m = 0$, and using the initial conditions, we get

$$a = \tilde{u}(0) = e_0 + e_1 0 + e_2 0 + \cdots = e_0 \rightarrow e_0 = a$$

In the other hand, if we select $m = 1$, because of that the analytical answer to the equation 5.18 is $u(x)$, then it's satisfied this integrodifferential equation. i.e.

$$u'(x) = g\big(x, u(x)\big) + u(x) \int_0^x k\big(x, t, u(t), u'(t)\big) dt, \tag{5.19}$$

Then by substituting the value of $x = 0$ into the equation 5.19, it yields

$$u'(0) = g\big(0, u(0)\big) + u(0) \int_0^x k\big(x, t, u(t), u'(t)\big) dt = g\big(0, u(0)\big) \tag{5.20}$$

and since

$$u'(x) = \tilde{u}'(0) = e_1 + 2e_2 x + 3e_3 x^2 + \cdots$$

and therefore, $u'(0) = \tilde{u}'(0)$, then $g\big(0, u(0)\big) = u'(0)$.

If we select, $m = 2$, then by differentiating the equation 5.18 reference to variable x and then substituting, u by $u(x)$ we obtain

$$u''(x) = \frac{\partial}{\partial x} g\big(x, u(x)\big) + \frac{\partial}{\partial u} g\big(x, u(x)\big) u'(x) +$$
$$u'(x) \int_0^x k(x,t,u,u')dt + u(x) \left[k(x,t,u,u') + \int_o^x \frac{\partial}{\partial x} k(x,t,u,u')dt + \right] dt \qquad (5.21)$$

By evaluating the equation 5.21 at the value $x = 0$, it yields

$$u''(x) = \frac{\partial}{\partial x} g\big(0, u(0)\big) + \frac{\partial}{\partial u} g\big(o, u(0)\big) u'(0) +$$
$$u'(0) \int_0^x k(x,t,u,u')dt + u(0) \left[k(x,t,u,u') + \int_o^x \frac{\partial}{\partial x} k(x,t,u,u')dt + \right] dt =$$
$$\frac{\partial}{\partial x} g\big(0, u(0)\big) + \frac{\partial}{\partial u} g\big(0, u(0)\big) u'(0) + u'(0) + u(0) k\big(0, u(0), u'(0)\big) \qquad (5.22)$$

Hence, $u''(x) = \tilde{u}''(x)$, then $u''(0) = \tilde{u}''(0)$. Then,

$$2e_2 = \frac{\partial}{\partial x} g\big(0, u(0)\big) + \frac{\partial}{\partial u} g\big(0, u(0)\big) u'(0) + u(0) k\big(0, u(0)\big) u'(0),$$

and hence

$$2e_2 = \frac{\partial}{\partial x} g(0, e_0) + \frac{\partial}{\partial u} g(0, e_0) e_1 + e_0 k(0, e_0, e_1). \qquad (5.23)$$

By making a comparition between equation 5.22 and 5.23, we obtain

$$2e_2 = u''(0) \rightarrow e_2 = \frac{u''(0)}{2!}\,.$$

Hence, by inducting, it is easy to show that $e_3 = \frac{u''(0)}{3!}$, $e_4 = \frac{u''(0)}{4!}$. Generally, by using the mathematical induction method, we obtain

$$e_m = \frac{u^m(0)}{m!},\ m = 1,2,3, \ldots$$

This satisfies the theorem 5.1.

5.2. Using DJM for Solving the Hybrid Selection Model

For solving the equation 3.17, we will integrate both sides of the differential equation from 0 to t, also we will apply the initial conditions to that, then we will obtain the following nonlinear VIE:

$$u(t) = 0.5 + \int_0^t [ku(1-u)(2-u)]dt \tag{5.24}$$

Now, by applying the DJM to the equation 5.24 yields:

$$u_0 = \frac{1}{2},$$

$$u_1 = N(u_0) = \int_0^t [ku_0(1-u_0)(2-u_0)]\,dt = \frac{3}{8}kt,$$

$$u_2 = N(u_0+u_1) - N(u_0) = \int_0^t \left[k(u_0+u_1)\big(1-(u_0+u_1)\big)\big(2-(u_0+u_1)\big)\right]dt - u_1 = \frac{3}{64}(kt)^2 - \frac{9}{128}(kt)^3 - \frac{27}{2048}(kt)^4,$$

$$u_3 = N(u_0+u_1+u_2) - N(u_0+u_1) = \int_0^t \left[k(u_0+u_1+u_2)\big(1-(u_0+u_1+u_2)\big)\big(2-(u_0+u_1+u_2)\big)\right]dt - \int_0^t \left[k(u_0+u_1)\big(1-(u_0+u_1)\big)\big(2-(u_0+u_1)\big)\right]dt = \frac{17}{256}(kt)^3 - \frac{63}{2048}(kt)^4 + \frac{27}{2560}(kt)^5,$$

and so on. This gives

$$y(t) = 0.5 + \frac{3}{8}kt - \frac{3}{64}(kt)^2 - \frac{17}{256}(kt)^3 - \frac{125}{4096}(kt)^4 + \frac{721}{81920}(kt)^5 + \cdots \tag{5.25}$$

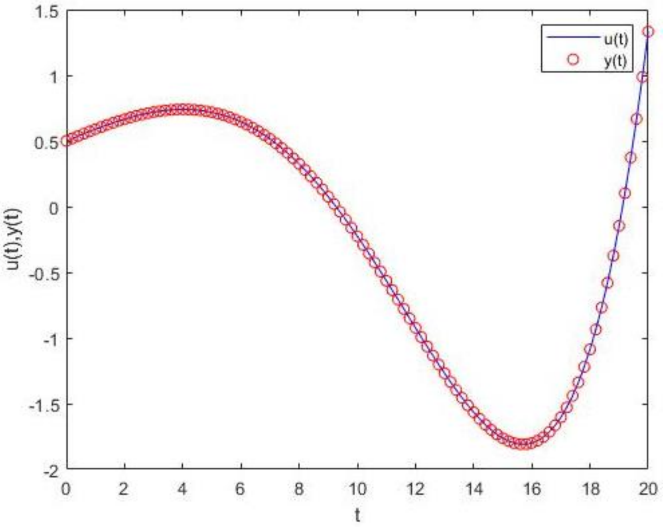

Figure 5.4. Presents the solution $u(t)$ and $y(t)$ for k = 0.25, where $0 \leq t \leq 20$.

Table 5.5. Comparison of $u(t)$ and $y(t)$ for the Hybrid Selection Model

t	$u(t)$	$y(t)$	$Error$
0	0.50000000	0.50000000	0.00000000
2	0.66584816	0.73688827	0.07104011
4	0.74000244	0.87223158	0.13222914
6	0.64524956	0.93926009	0.29401053
8	0.32460937	0.97126729	0.64665792
10	-0.22566032	0.98642327	1.21208360
12	-0.92305908	0.99358635	1.91664543
14	-1.56583900	0.99697036	2.56280936
16	-1.80000000	0.99856889	2.79856889
18	-1.08628502	0.99932399	2.08560901
20	1.33282471	0.99968068	0.33314403

Which gives

$$u(t) = \frac{\sqrt{1+3e^{3kt}-1}}{\sqrt{1+3e^{3kt}}}$$

Which was analytical solution of the giving problem and it is result is similar to obtained results by VIM (Wazwaz, 2014).

5.2.1. Solving the hybrid selection model by applying PSM

Here, we will apply Power Series technique for HSM, we start by applying the PSM for solving the equation 5.24, We assume that Equation's solution with

$$u_0(t) = u(0) = e_0 = 0.5$$

is $u_1(t) = e_0 + e_1 t$, which implies

$$u_1(t) = 0.5 + e_1 t \tag{5.26}$$

Putting the value of equation 5.26 in to equation 5.24, leads to:

$$0.5 + e_1 t = 0.5 + \int_0^t \left[k(0.5 + e_1 t)\left(1 - (0.5 + e_1 t)\right)\left(2 - (0.5 + e_1 t)\right)\right] dt, \tag{5.27}$$

After simplifying, we get, $e_1 = \frac{3}{8}k$ and hence:

$$u_1(t) = 0.5 + \frac{3}{8}k. \tag{5.28}$$

That is $u_1(t)$ is the first approximation for solution of the equation 5.24. In similar manner, the equation 5.24 solution can be proposed as follow:

$$u_1(t) = 0.5 + \frac{3}{8}k + e_2 t^2, \tag{5.29}$$

Now, by putting the value of equation 5.29 inside the equation 5.24, we get:

$$0.5 + \frac{3}{8}k + e_2 t^2 = 0.5 + \int_0^t \left[k\left(0.5 + \frac{3}{8}kt + e_2 t^2\right)\left(1 - \left(0.5 + \frac{3}{8}kt + e_2 t^2\right)\right)\left(2 - \left(0.5 + \frac{3}{8}kt + e_2 t^2\right)\right)\right] \text{dt}. \tag{5.30}$$

After simplifying, we get, $e_2 = \frac{3}{64}k^2$ and then we can rewrite,

$$u_2(t) = 0.5 + \frac{3}{8}kt + \frac{3}{64}(kt)^2, \tag{5.31}$$

By proceeding it in similar manner, we obtain:

$$u(t) = 0.5 + \frac{3}{8}kt + \frac{3}{64}(kt)^2 - \frac{17}{256}(kt)^3 - \frac{125}{4096}(kt)^4 + \frac{721}{81920}(kt)^5 + \cdots \tag{5.32}$$

Hence, we obtained the analytical solution to the problem, however it yields the same result as VIM and DJM in (Wazwaz, 2014).

5.3. Applying DJM for Solving the Riccati Equation

It is known that the Riccati equation can be stated in different forms, let's Consider it in the following form:

$$u'(t) = u^2(t) - 2xu(t) + u(0) = 0.5. \tag{5.33}$$

For solving the equation 5.33, we can integrate both sides from 0 to x , and applying the initial condition, we obtain:

$$u(t) = 0.5 + \int_0^x [u^2 - 2tu + x^2 + 1]dt. \tag{5.34}$$

Here, by applying DJM to the equation 5.34 we get:

$u_0 = \frac{1}{2}$,

$u_1 = N(u_0) = \int_0^x [u_0^2 - 2tu_0 + x^2 + 1]dt = u_1 = \frac{5}{4}x + \frac{1}{2}x^2 - \frac{1}{3}x^3$,

$u_2 = N(u_0 + u_1) - N(u_0) = \int_0^x [(u_0 + u_1)^2 - 2t(u_0 + u_1) + x^2 + 1]dt - u_1 =$
$\frac{1}{8}x^2 + \frac{7}{48}x^3 - \frac{1}{28}x^4$,

$u_3 = N(u_0+u_1+u_2) - N(u_0+u_1) = \int_0^x [(u_0+u_1+u_2)^2 - 2t(u_0+u_1+u_2) + x^2+1]dt - \int_0^x [(u_0+u_1)^2 - 2t(u_0+u_1) + x^2+1]dt \quad = \frac{1}{16}x^3 + \frac{1}{48}x^4 - \frac{7}{960}x^5,$

and so on, this gives

$$y(x) = x + \frac{1}{2}\left(1 + \frac{1}{2}x + \frac{1}{4}x^2 + \frac{1}{8}x^3 + \frac{1}{16}x^4 + \frac{1}{32}x^5 + \cdots\right) \qquad (5.35)$$

which results

$$u(x) = x + \frac{1}{2-x},$$

where $x \in (-2,2)$. Which was analytical solution of the giving problem and it is result is similar to obtained results by VIM (Wazwaz, 2014).

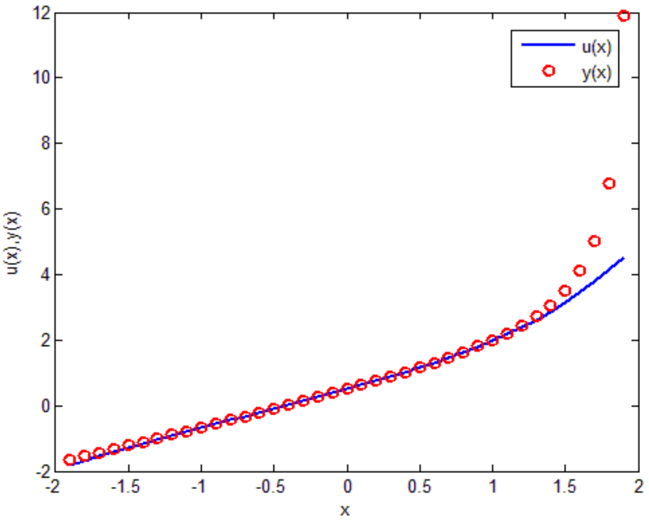

Figure 5.5. Presents the solution of $u(x), y(x)$ for DJM for Solving the Riccati Equation.

Table 5.6. Comparison of $u(t)$ and $y(t)$ for solving the Riccati equation

x	$u(x)$	$y(x)$	$Error$
-1.9	-1.83269387	-1.64358974	0.18910412
-1.3	-1.01991673	-0.99696970	0.02294703
-0.7	-0.33031467	-0.32962963	0.00068504
-0.1	0.37619047	0.37619048	0.00000001
0.5	1.16650469	1.16666667	0.00016198
1.1	2.18039511	2.21111111	0.03071600
1.7	3.77652325	0.97384763	1.25681009

5.3.1. Using PSM to solve the Riccati equation

Currently, using the PSM to equation 5.34, and suppose the first approximate solution of (5.34) with $u_0(x) = u(0) = e_0 = 0.5$ is $u_1(x) = e_0 + e_1 x$, and hence

$$u_1(x) = 0.5 + e_1 x \tag{5.36}$$

Substitute equation 5.36 in equation 5.34, to get,

$$0.5 + e_1 x = 0.5 + \int_o^x [(0.5 + e_1 t)^2 - 2t(0.5 + e_1 t) + x^2 + 1]\, dt\ . \tag{5.37}$$

After being simplified, we have $e_1 = \frac{5}{4}$,

$$u_1(x) = 0.5 + \frac{5}{4}x. \tag{5.38}$$

This represents the answer of equation 5.34’s first approximation. The solution of equation 5.34 can be supposed as:

$$u(x) = 0.5 + \frac{5}{4}x + e_2 x^2, \tag{5.39}$$

By Substituting equation 5.39 in equation 5.34, we get,

$$0.5+\frac{5}{4}x+e_2x^2=0.5+\int_0^x\left[\left(0.5+\frac{5}{4}t+e_2t^2\right)^2-2t\left(0.5+\frac{5}{4}t+e_2t^2\right)+x^2+1\right]dt \tag{5.40}$$

so, we get, $e_2=\frac{1}{8}$ and hence,

$$u_2(x)=0.5+\frac{5}{4}x+\frac{1}{8}x^2. \tag{5.41}$$

Proceeding in this way, we get,

$$u(x)=x+\frac{1}{2}\left(1+\frac{1}{2}x+\frac{1}{4}x^2+\frac{1}{8}x^3+\frac{1}{16}x^4+\frac{1}{32}x^5+\cdots\right) \tag{5.42}$$

That $u(x)$ converges to

$$u(x)=x+\frac{1}{2-x},$$

where $x\in(-2,2)$. This provides the exact solution to the problem and yields the same result as DJM and discover VIM.

5.4. Using DJM for Solving the Logistic Differential Equation

Here, we will focus on equation 3.12 and we will apply DJM method for solving it. We start by integrating both sides from 0 to x and apply the initial conditions, we will apply DJM method for solving it, then we obtain

$$u(x)=0.5+\int_0^x\left[\mu u(t)\left(1-u(t)\right)\right]dt \tag{5.43}$$

Now, we applying the DJM for equation 3.12, it yields

$u_0 = \frac{1}{2},$

$u_1 = \int_0^x [\mu u_0(t)(1 - u_0(t))]dt = \frac{\mu}{4}x,$

$u_2 = \int_0^x [\mu(u_0 + u_1)(t)(1 - (u_0 + u_1))(t)]dt = \frac{\mu^3}{48}x^3,$

$u_3 = \int_0^x [\mu(u_0 + u_1 + u_2)(t)(1 - (u_0 + u_1 + u_2))(t)]dt - \int_0^x [\mu(u_0 + u_1)(t)(1 -$

$(u_0 + u_1))(t)]\, dt = \frac{\mu^5}{480}x^5 - \frac{\mu^7}{16128}x^7,$

and so on, which gives

$$y(x) = \frac{1}{2} + \frac{\mu}{4}x - \frac{\mu^3}{48}x^3 + \frac{\mu^5}{480}x^5 - \frac{17\mu^7}{80640}x^7 - \frac{31\mu^9}{1451520}x^9 + \cdots \tag{5.44}$$

While, the exact solution is presented in (Venkatesh et al., 2012).

$$u(x) = \frac{e^{\mu x}}{1+e^{\mu x}}$$

which are the same results that VIM obtained (Wazwaz, 2009).

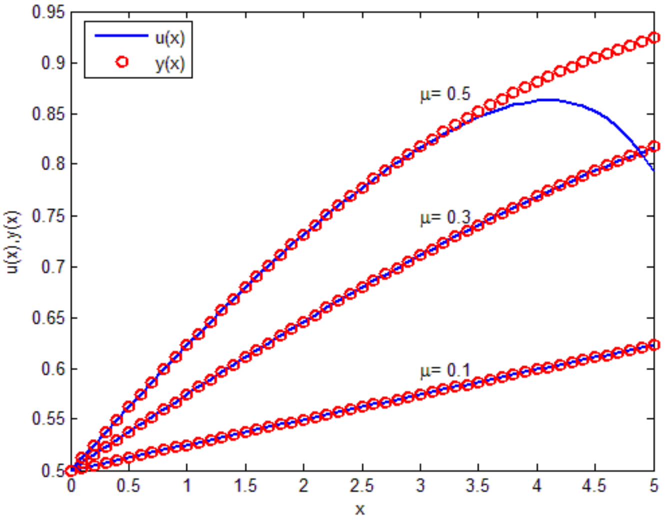

Figure 5.7. Presents the solution of $u(x), y(x)$ for using PSM to solve the Riccati equation.

Table 5.7. Comparison of $u(t)$ and $y(t)$ for μ $= 0.1$

x	$u(x)$	$y(x)$	$Error$
0.00000000	0.50000000	0.50000000	0.00000000
0.50000000	0.51249740	0.51249740	0.00000000
1.00000000	0.52497919	0.52497919	0.00000000
1.50000000	0.53742985	0.53742985	0.00000000
2.00000000	0.54983400	0.54983400	0.00000000
2.50000000	0.56217650	0.56217650	0.00000000
3.00000000	0.57444252	0.57444252	0.00000000
3.50000000	0.58661758	0.58661758	0.00000000
4.00000000	0.59868765	0.59868766	0.00000001
4.50000000	0.61063920	0.61063923	0.00000003
5.00000000	0.62245925	0.62245933	0.00000008

Table 5.8. Comparison of $u(t)$ and $y(t)$ for μ $= 0.3$

x	$u(x)$	$y(x)$	$Error$
0.00000000	0.50000000	0.50000000	0.00000000
0.50000000	0. 53742985	0. 53742985	0.00000000
1.00000000	0. 57444252	0. 57444252	0.00000000
1.50000000	0. 61063920	0. 61063923	0.00000003
2.00000000	0. 64565588	0. 64565631	0.00000042
2.50000000	0. 67917558	0. 67917870	0.00000312
3.00000000	0. 71093358	0. 71094950	0.00001592
3.50000000	0. 74071197	0. 4077490	0. 00006293
4.00000000	0. 76831842	0. 76852478	0. 00020636
4.50000000	0. 79354304	0. 79412963	0. 00058659
5.00000000	0.81608483	0.81757448	0.00148964

Table 5.9. Comparison of $u(t)$ and $y(t)$ for μ $= 0.5$

x	$u(x)$	$y(x)$	$Error$
0.00000000	0.50000000	0.50000000	0.00000000
0.50000000	0.56217650	0.56217650	0.00000000
1.00000000	0.62245925	0.62245933	0.00000008
1.50000000	0.67917558	0.67917870	0.00000312
2.00000000	0.73101783	0.73105858	0.00004075
2.50000000	0.77700337	0.77729986	0.00029650
3.00000000	0.81608483	0.81757448	0.00148964
3.50000000	0.84615611	0.85195280	0.00579669
4.00000000	0.86208113	0.88079708	0.01871595
4.50000000	0.85222496	0.90465054	0.05242557
5.00000000	0.79278915	0.92414182	0.13135267

5.5. Using PSM for Solving the Logistic Differential Equation

Power Series Method we can start by supposing that the solution of (3.26) can be applied for solving equation 3.12, with $u_0(x) = u(0) = e_0 = 0.5$ is, $u_1(x) = e_0 + e_1 x$, and hence:

$$u_1(x) = 0.5 + e_1 x \tag{5.46}$$

Putting equation 5.46 in to equation 3.12, then:

$$0.5 + e_1 x = 0.5 + \int_o^x \left[\mu(0.5 + e_1 t)\big(1 - (0.5 + e_1 t)\big)\right] dt \tag{5.47}$$

After simplifying, we get, $e_1 = \frac{\mu}{4}$ and hence:

$$u_1(x) = 0.5 + \frac{\mu}{4} x, \tag{5.48}$$

$u_1(x)$ is an initial approximation for the equation's solution equation 3.12. Additionally, the answer to equation 3.12 can be written as follows:

$$u_1(x) = 0.5 + \frac{\mu}{4} x + e_2 x^2 \tag{5.49}$$

Putting equation 5.49 in to the equation 3.12, we obtain,

$$0.5 + \frac{\mu}{4} x + e_2 x^2 = 0.5 + \int_0^x \left[\mu\left(0.5 + \frac{\mu}{4} t + e_2 t^2\right)\left(1 - \left(0.5 + \frac{\mu}{4} t + e_2 t^2\right)\right)\right] dt. \tag{5.50}$$

Therefore, we get, $e_2 = \frac{\mu^3}{48}$ and hence,

$$u(x) = 0.5 + \frac{\mu}{4} x + \frac{\mu^3}{48} x^2 \tag{5.51}$$

Continuing in this way, we are able to

$$y(x) = \frac{1}{2} + \frac{\mu}{4}x - \frac{\mu^3}{48}x^3 + \frac{\mu^5}{480}x^5 - \frac{17\mu^7}{80640}x^7 - \frac{31\mu^9}{1451520}x^9 + \cdots . \tag{5.52}$$

The exact solution of the Logistic differential equation is given by (Wazwaz, 2009)

$$u(x) = \frac{e^{\mu x}}{1+e^{\mu x}}$$

Finally, by VIM and DJM It yields the same result. Obtained, above value of $u(x)$ is the exact solution of the giving problem. Moreover in (Wazwaz, 2009).

6. CONCLUSION

6.1. Conclusion

In this work, there are three crucial aspects to keep in mind. First, unlike TeeBest's classic grid points techniques (TeBeest, 1997), which solution was defined solely at grid points, the investigated solution in the presented thesis is on the series form. In the other hand, the Second point is that, the approach in (TeBeest, 1997) used 20 iteration stages, however in this thesis, results shows that there were fewer than ten iterations applied. Third, there is a clear and notable agreement in the approximations of the solution between the approaches used in this study. This advantage over previous techniques indicates the solutions' dependability and efficiency. Furthermore, when using the DJM or the PSM, these procedures efficiency can be greatly improved by calculating additional terms or component of solution. Using the Padé approximant, the solution's mathematical form was successfully improved, producing a result that is valid within the definitional constraints of this thesis. For identical purposes, Padé approximants are a useful tool and a promising technique, as they frequently outperform series approximations. The following are some of the conclusions drawn from this thesis:

1. Volterra's population model has been effectively utilized with the DJM and PSM. The numerical result show that the presented approaches are simple and straight forward, and that they produce the same outcomes as ADM, HPM, and VIM in (Wazwaz, 2011;1999), (Mohyud-Din et al., 2010).
2. Unlike mesh point systems, the DJM and PSM do not provides a linear or non-linear equation systems.
3. Because the PSM and DJM methods do not requires any linearization, discretization, or minor perturbations, they can significantly reduce the number of calculations while keeping good numerical accuracy.
4. It's worth noting that the Adomain Decomposition Method (ADM) necessarily requires the evaluations of Adomian polynomials, which are typically requires time-consuming algebraic operations.

5. Furthermore, in comparison to other numerical techniques, both of PSM and DJM are efficient and powerful for the nonlinear equations that frequently appear for expressing nonlinear phenomena, facilitating computing labor and providing a solution quickly.
6. The approaches are also applied for solving several scientific models, such as the hybrids selection model, the logistic model, and Riccati model, in order to present an analytic solution, and the findings revealed that the PSM and DJM presented same results as VIM, (Wazwaz, 2009;2014).
7. The suggested approaches' key advantages over other current methods are their simplicity, ease of application, cost-effectiveness in terms of both of the computer's memory and power, and speed of the algorithm.

Table 6.1. Presents a comparison between absolute error of each of DJM, VIM, ADM, PSM

κ	ADM	VIM	PSM	DJM
0.1	4.628×10^{-3}	4.6283×10^{-3}	4.6283×10^{-3}	4.6283×10^{-3}
0.2	1.13807×10^{-3}	1.13807×10^{-3}	1.13807×10^{-3}	1.13807×10^{-3}
0.5	9.20568×10^{-5}	9.20576×10^{-5}	9.20576×10^{-5}	9.20568×10^{-5}
0.04	1.24798×10^{-2}	1.24798×10^{-2}	1.24798×10^{-2}	1.24798×10^{-2}

The approximation in equations 5.30 and 3.35 is the same for all methods, Since the absolute errors of VIM, and ADM are identical, as can be shown from the above Table, the absolute errors of PSM and DJM are as well.

6.2. Recommendations

We have suggestions for further work are as follows:

1. The mixed Volterra-Fredholm integral equations can be solved using DJM or PSM.
2. Solving linked Burger equations, two-dimensional (2D) Burger's equations, and the Burger equation using DJM or PSM.
3. Using PSM to solve Bratu-type equations.
4. Using DJM or PSM to solve the Fokker-Planck equations.

5. solving two different iterations of Blasiu’s equations on a domain that is half infinite using PSM or DJM.
6. Using PSM or DJM to solve non-linear PDE problems

REFERENCES

Abbasbandy, S., Shivanian, E., 2008. Application of the variational iteration method to nonlinear Volterra's integro-differential equations. ***Zeitschrift für Naturforschung A***, **63** (9): 538-542.

Adomian, G., 1994. ***Integral Equations In Solving frontier problems of physics: the Decomposition Method***. pp. 224-227. Springer, Dordrecht.

Adomian, G., Rach, R., 1996. Modified Adomian polynomials. ***Mathematical and Computer Modelling***, **24** (11): 39-46.

Alao, S., Akinboro, F. S., Akinpelu, F.,Oderinu, R., 2014. Numerical solution of integro-differential equation using Adomian decomposition and variational iteration methods. ***IOSR Journal of Mathematics***, ***10***(4), 18-22.

Ali, S. M., 2011. ***Some Approximate Solutions of Fractional Integro-Differential Equations***. Doctoral dissertation, (Msc. Thesis, unpublished), College of Science, Al-Nahrian University.

Al-Khaled, K., 2005. Numerical approximations for population growth models. ***Applied Mathematics and Computation***, **160** (3): 865-873.

Al-Jawary M. A., 2005. ***Numerical Methods for System of Integral Equations*** (M.Sc. thesis, unpublished). Baghdad University.

Batiha, B., Noorani, M. S. M., Hashim, I., 2008. Numerical solutions of the nonlinear integro-differential equations. ***International Journal of Open Problems in Computer Science and Mathematics. IJOPCM***, **1** (1): 34-42.

Bhalekar, S., Daftardar-Gejji, V., 2008. New iterative method: application to partial differential equations. ***Applied Mathematics and Computation***, **203** (2): 778-783.

Bhalekar, S., Daftardar-Gejji, V., 2010. Solving evolution equations using a new iterative method. Numerical methods for partial differential equations***: An International Journal***, **26** (4): 906-916.

Bhalekar, S., Daftardar-Gejji, V., 2011. Convergence of the new iterative method. ***International Journal of Differential Equations***.

Brunner, H., 1974. On the approximate solution of first-kind integral equations of Volterra type. ***Computing***, **13** (1): 67-79.

Brunner, H., 1988. The approximate solution of initial-value problems for general Volterra integro-differential equations. ***Computing***, **40** (2): 125-137.

Daftardar-Gejji, V., Jafari, H., 2006. An iterative method for solving nonlinear functional equations. ***Journal of Mathematical Analysis and Applications***, **316** (2): 753-763.

Daftardar-Gejji, V., Bhalekar, S., 2010. Solving fractional boundary value problems with Dirichlet boundary conditions using a new iterative method. ***Computers and Mathematics with Applications***, **59** (5): 1801-1809.

Dehghan, M., Tatari, M., 2006. The use of He's variational iteration method for solving a Fokker–Planck equation. ***Physica Scripta***, **74** (3): 310.

Erfanian, H. R., Mostahsan, T., 2011. Approximate solution of a class of nonlinear Volterra integral equations. ***J Journal of Mathematics and Computer Science***, **3** (3): 278-286.

Gachpazan, M., 2009. Numerical Scheme to Solve Integro-Differentia Equations Systeml. ***Journal of Advanced Research in Scientific Computing***, **1**: 11-21.

Hashim, I., 2006. Adomian decomposition method for solving BVPs for fourth-order integro-differential equations. ***Journal of Computational and Applied Mathematics***, **193** (2): 658-664.

He, J. H., 2007. Variational iteration method-some recent results and new interpretations. ***Journal of Computational and Applied Mathematics***, **207** (1): 3-17.

He, J. H., Wu, X. H., 2007. Variational iteration method: new development and applications. ***Computers and Mathematics with Applications***, **54** (7-8): 881-894.

Hochstast H., 1973. ***Integral Equations***; Polytechnic Institute of Brooklyn, New York.

Hosseini, S. M., Shahmorad, S., 2003. Numerical solution of a class of integro-differential equations by the Tau method with an error estimation. ***Applied Mathematics and Computation***, **136** (2-3): 559-570.

Jerri, A. J., 1985. ***Introduction to İntegral Equations with Applications***. Marcel Dekker, Inc NY.

Joshi, M. C., Bose, R. K., 1985. ***Some Topics in Nonlinear Functional Analysis***. John Wiley and Sons.

Khan, N. A., Ara, A., Jamil, M., 2011. Approximations of the nonlinear Volterra’s population model by an efficient numerical method. ***Mathematical Methods in the Applied Sciences***, **34** (14): 1733-1738.

Marzban, H. R., Hoseini, S. M., Razzaghi, M., 2009. Solution of Volterra’s population model via block-pulse functions and Lagrange-interpolating polynomials. ***Mathematical Methods in the Applied Sciences***, **32** (2): 127-134.

Mohyud-Din, S. T., Yıldırım, A., Gülkanat, Y., 2010. Analytical solution of Volterra’s population model. ***Journal of King Saud University-Science***, **22** (4): 247-250.

Mirzaee, F., 2012. Numerical solution for Volterra integral equations of the first kind via quadrature rule. ***Applied Mathematical Sciences***, **6** (20): 969-974.

Parand, K., Delafkar, Z., Pakniat, N., Pirkhedri, A., Haji, M. K., 2011. Collocation method using sinc and Rational Legendre functions for solving Volterra’s population model. ***Communications in Nonlinear Science and Numerical Simulation***, **16** (4): 1811-1819.

Ramezani, M., Razzaghi, M., Dehghan, M., 2007. Composite spectral functions for solving Volterra’s population model. ***Chaos, Solitons and Fractals***, **34** (2): 588-593.

Rashidinia, J., Tahmasebi, A., 2013. Approximate solution of linear integro-differential equations by using modified Taylor expansion method. ***World Journal of Modelling and Simulation***, **9** (4): 289-301.

Scudo, F. M., 1971. Vito Volterra and theoretical ecology. ***Theoretical Population Biology***, **2** (1): 1-23.

Schubert, H., Chambers, LG., 1978. ***Integral Equations***, A Short Course London. International Textbook Company Limited.

Small, R. D., 1989. Population growth in a closed model. ***Mathematical Modelling: Classroom Notes in Applied Mathematics, SIAM, Philadelphia***, **PA:** 484-493.

Tatari, M., Dehghan, M., 2007. Solution of problems in calculus of variations via He’s variational iteration method. ***Physics Letters A***, **362** (5-6): 401-406.

Tahmasbi, A., Fard, O. S., 2008. Numerical solution of linear Volterra integral equations system of the second kind. ***Applied Mathematics and Computation***, **201** (1-2): 547-552.

Tahmasbi, A., Fard, O. S., 2008. Numerical solution of nonlinear Volterra integral equations of the second Kind by power series. ***Journal of Information and Computing Science***, **3** (1): 57-61.

TeBeest, K. G., 1997. Classroom Note: numerical and analytical solutions of Volterra's population model. ***SIAM Review***, **39** (3): 484-493.

Venkatesh, S. G., Ayyaswamy, S. K., Balachandar, S. R., 2012. Legendre approximation solution for a class of higher-order Volterra integro-differential equations. ***Ain Shams Engineering Journal***, **3** (4): 417-422.

Wadea, M. H., 2012. ***Variational İteration Method for Solving Fractional Order İntegro-Differential Equations***. (M. Sc. Thesis, unpublished) college of eduction, Al-Mustansiriya University.

Wazwaz, A. M., 2011. ***Linear and Nonlinear İntegral Equations***. Berlin: Springer, **639**: 35-36.

Wazwaz, A. M., 1999. Analytical approximations and Padé approximants for Volterra's population model. ***Applied Mathematics and Computation***, **100** (1): 13-25.

Wazwaz, A. M., 2010. The variational iteration method for solving linear and nonlinear Volterra integral and integro-differential equations. ***International Journal of Computer Mathematics***, **87** (5): 1131-1141.

Wazwaz, A. M., 2009. Solitary waves theory. In ***Partial Differential Equations and Solitary Waves Theory***. Springer, Berlin, Heidelberg, 479-502.

Wazwaz, A. M., 2014. The variational iteration method for solving linear and nonlinear ODEs and scientific models with variable coefficients. ***Central European Journal of Engineering***, **4** (1): 64-71.

Wazwaz, A. M., 2009. The variational iteration method for analytic treatment for linear and nonlinear ODEs. ***Applied Mathematics and Computation***, **212** (1): 120-134.

EXTENDED TURKISH SUMMARY
(GENİŞLETİLMİŞ TÜRKÇE ÖZET)

VOLTERRA İNTEGRAL DİFERANSİYEL DENKLEMLERİ İÇİN ANALİTİK VE YAKLAŞIK ÇÖZÜMLER VE UYGULAMALAR

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Bu tezin temel amacı, Volterra integral denklemini ve integral diferansiyel denklemini gerçek hayat problemleri için bazı bilimsel modelleri incelemek ve çözmektir. Bu amaç aşağıdaki gibi üç alt amaca ayrılabilir: Birincisi, integral denklemler konusunu sınıflandırmak, incelemek, Volterra integral denklemi ve integro diferansiyel denklemleri ile ilgili temel tanımları vermektir. İkinci alt amaçta, Volterra integral denklemi ve integro-diferansiyel denklemlerini çözmek için Dafterdar-Jafari metodu'nu kullandık. Üçüncü alt amaç, Volterra integral denklemi ve integro-diferansiyel denklemlerinin bir çözümünü sağlamak için kuvvet serileri metodu olarak adlandırılan başka bir yinelemeli yöntemi tanıtmaktır. Son olarak, Volterra'nın popülasyon modeli, hibrit seçim modeli, Riccati denklemi ve lojistik model gibi bazı gerçek yaşam bilimsel problemlerinin analitik ve yaklaşık çözümünü bulmak için, Dafterdar-Jafari ve kuvvet serisi metodunun uygulamasını sunmaktır. Ayrıca, bilgisayar programlarının Mathematica bilgisayar yazılımı kullanılarak kodlandığını belirtmek önemlidir.

Anahtar kelimeler: Daftardar-Jafari metodu, Fredholm integral denklemleri, İntegral denklemi, Kuvvet serileri metodu, Volterra integral denklemleri.

1. GİRİŞ

İntegral denklemler teorisi, matematiğin diğer alanlarıyla pek çok örtüşme içerir. İntegral denklemlerin incelenmesi, diferansiyel denklemlerden daha temel bir öneme sahip olarak kabul edilebilir. Çünkü bir integral denklem genellikle diferansiyel denklemlerden daha az kısıtlama ve daha kullanışlı bir model sağlar. Ayrıca, sayısal analizde farklılaşma hatayı artırır, ancak integrasyon hataları yumuşatma eğiliminde olacaktır. Literatürde ve uygulamalı matematiğin birçok alanında integral denklemlerin artan sıklığı, integral denklemlerin avantajına tanıklık eder, çünkü bazı problemlerin matematiksel temsilleri doğrudan ve çok doğal bir şekilde integral denklemler cinsinden ortaya çıkar. Doğrudan gösterimi diferansiyel denklemler cinsinden olan diğer problemlerin yardımcı koşulları, diferansiyel denklemlerden daha zarif ve kompakt bir şekilde integral denklemlerle değiştirilir. Tarihsel olarak Volterra, integral denklemler üzerinde çalışmaya 1884'te başladı, ancak ciddi çalışması 1896'da başlamıştır. İntegral denklemler adı 1888'de Bois-Reymond tarafından verilmiştir. Son yıllarda birçok araştırma, integral denklemlerin ve tam diferansiyel denklemlerin çözümü ile ilgili çalışmalar olmuştur.

İntegral denklemlerin avantajı, integral denklemlerin literatürde ve farklı uygulamalı Matematik alanlarında artan kullanımında görülebilir. Çünkü bazı konular integral denklemler kullanılarak matematiksel olarak basit ve doğal bir şekilde ifade edilebilir. Doğrudan gösterimli farklı problemler integral denklemler, diferansiyel denklemlerin yardımcı koşullarını diferansiyel denklemlerden daha etkili ve kompakt bir şekilde değiştirir. Ek olarak, matematiksel fizikteki birçok konu integral denklemler olarak ifade edilebilir, bu nedenle integral denklemlerin hemen hemen her matematiksel fizik ve uygulamalı matematik alanında rol oynadığı sonucuna varmak güvenlidir. Başka bir anlamda, özdeğerler, özfonksiyonlar ve lineer vektör uzaylarının temel fikirlerinin lineer integral denklemlerle uğraşırken hayati bir rol oynayacağı söylenebilir (Hochstast, 1973).

Bu tezde, Volterra integrali ve tamsayı diferansiyel denklemleri için analitik ve yaklaşık çözümler sunulmaktadır. Dafterdar-Jafari metodu (DJM) ve kuvvet serisi metodu (KSM), integral denklemlere bağımsız olarak uygulanır. DJM, pek çok araştırmacı tarafından tamsayılı ve kesirli sıralı doğrusal ve doğrusal olmayan adi ve kısmi diferansiyel denklemlerin tedavisinde yaygın olarak kullanılmaktadır. DJM'nin, eğer varsa, ardışık yaklaşımlar yoluyla kesin çözüme yakınsayacağını fark etmek önemlidir. KSM ve DJM sonuçları, yöntemin türevsiz olması, Adomian decomposition metodu'nda (ADM) doğrusal olmayan terimleri ele almak için Adomian polinomlarının hesaplanmasında ortaya çıkan zorluğun üstesinden gelmesi, Variational iteration methodu'ndaki (VIM) gibi Lagrange çarpanı hesaplaması gerektirmemesi ve ihtiyaç olmaması gibi birçok avantajı olduğunu göstermektedir. Homotopi Pertürbasyon Metodunda (HPM) olduğu gibi bir homotopi oluşturmak ve karşılık gelen cebirsel denklemleri çözmektir.

Bu tezdeki çalışma altı bölümden oluşmaktadır. Birinci bölümde, konuyla ilgili genel bilgiler ve uygulama alanları verilmektedir. İkinci bölümde, integro-diferensiyel denklemler için litüratür taraması yapılıp ve bilgiler verrilmektedir. Üçüncü bölümde temel tanım ve kavramlar, konuyla ilgili ele alınan problemlerın sınıfadırılması yaılıp ve tanımları verilmiştir. Dördüncü bölümde tezde kullanılan Volterra integralini ve integro-diferansiyel denklemlerini çözmek için Padé Yaklaşımları, Adomian Ayrıştırma Yöntemi, İteratif Varyasyon Metodu, Dafterdar-Jafari yöntemi ve kuvvet seri metodunu kullanmak için genel bigiler verilip ve örnekler üzerinde gösterilmiştir. Beşinci bölümde, nüfus modeli, popülasyon modeli, hibrit seçim modeli, Riccati Denklemi ve Lojistik diferansiyel denklemlerin çözümleri için verilen metodlar uygulanmıştır ve altıncı bölümde ele alınan problemler için metodları karşılaştırarak sonuçlar değerlendirilmiştir. Ayrıca, bilgisayar programlarının Matlab 2014a yazılımını kullanılarak grafikler ve tablolar elde edilmiştir.

2. MATERYAL VE YÖNTEM

Tez kapsamındaki bu çalışmada doğrusal ve doğrusal olmayan fonksiyonel denklemleri çözmek için Dafterdar-Jafari Metodu (DJM), ardışık varyasonel metodu, ADM, Kuvvet serisi ve Padé yaklaşım metodunu verilmiştir. Bu metotlar somut problemler için yüksek derecede problemlerin çözümünü bulmak için kullanılmıştır.

Ayrıca, doğal hayatta birçok problemlerle karşılaşan Riccati denklemleri, Lojistik diferansiyel denklemleri, Volterra popülasyona ve Hibrid şeçim modelleri için metotlar uygulanıp ve bilgisayar desteği ile sonuçlar grafik ve tablo olarak sunulmuştur.

3. BULGULAR

VIDE'ler, çeşitli fiziksel, mühendislik ve biyolojik modellerin matematiksel modellemesinden, örneğin kapalı bir sistemdeki bir türün nüfus artışından ortaya çıkar. Bu modeller, popülasyon evriminin belirli bir süre boyunca davranışı gibi farklı faktörleri anlamamıza yardımcı olur.

Tezde, yaklaşık çözümleri bulmak için Volterra'nın popülasyon modeli için DJM ve KSM uygulamaları sunulmuştur. Mevcut yöntemlerin başlıca çekici özellikleri, türevsiz olmaları, mevcut bazı tekniklerdeki zorlukların üstesinden gelmeleri, kolay anlaşılır olmalarıdır. Bilgisayar gücü ve belleği açısından ekonomiktir ve ADM'eki Adomian polinomları gibi sıkıcı hesaplamalar içermez. HPM'deki gibi bir homotopi oluşturur ve karşılık gelen cebirsel denklemleri çözer ve VIM'deki gibi Lagrange çarpanını hesaplar. Sayısal çözüm, DJM ve KSM'nin Padé yaklaşımı tekniği birleştirilmesiyle elde edilmiştir. Ayrıca, elde edilen sonuçların karşılaştırılması mevcut sonuçlarla ADM tarafından da karşılaştırılmıştır. Ayrıca, bazı bilimsel modellerin çözümünü değerlendirmek için DJM ve KSM uygulanmıştır. Çalışmamızı iyi bilinen üç doğrusal olmayan denkleme, yani Hibrit seçilim modeli, Riccati denklemi ve Lojistik denklemi ele alınmıştır.

Volterra denkleminin kapalı bir sistemdeki bir türün popülasyon büyümesinin model çözümün elde etmek için yaklaşık bir yöntem uygulanmış, DJM ve KSM'nin yanı sıra literatürdeki modelden bağımsız olarak ADM ve VIM uygulanması yapılmıştır. Genellikle seri yaklaşımlara göre üstün performans gösteren Padé yaklaşımları, aynı bireylerden oluşan $u(t)$ popülasyonunun temel davranışını yakalamak için analizde etkili bir şekilde kullanılmıştır. Böylece, benzer bir şekilde diğer modeller için uygun metodları kullanarak sonuçlar elde edilip ve elde edilen çözümler kesin çözümlerle karşılaştırarak hataları incelenmiştir.

4. TARTIŞMA VE SONUÇ

Bu çalışmada dikkat edilmesi gereken üç önemli nokta vardır. İlk olarak, 1997'de TeeBest tarafından kullanılan klasik şebeke noktaları tekniklerinden farklı olarak, çözüm yalnızca şebeke noktalarında tanımlanır. Burada çözüm bir seri formunda verilir. İkincisi, 1997'de TeeBest çalışmasında şema 20 yineleme adımı kullanırken, bu tezde 10'dan az yineleme adımı kullanılmıştır. Üçüncüsü, bu çalışmada kullanılan yöntemler arasındaki çözüm yaklaşımlarının yüksek uyumu açık ve dikkat çekicidir. Mevcut tekniklere göre bu avantaj, bu yöntemlerin güvenilirliğini ve etkinliğini göstermektedir. Ayrıca, bu yaklaşımların verimliliği, sırasıyla DJM veya KSM kullanıldığında çözümün diğer terimleri veya bileşenleri hesaplanarak önemli ölçüde artırılabilir. Bu tezde tanım alanında geçerli olan bir çözüm elde edilmiş ve çözümün matematiksel yapısı Padé yaklaşımları kullanılarak başarılı bir şekilde geliştirilmiştir. Genellikle seri yaklaşımlara göre üstün performans gösteren Padé yaklaşımları, benzer uygulamalar için başarılı bir araç ve gelecek vaat eden bir şema sağlar.

DJM ve KSM, Volterra'nın popülasyon modeline başarıyla uygulandı. Sayısal sonuçlar, önerilen yöntemlerin çok basit, anlaşılır olduğunu ADM, HPM ve VIM tarafından elde edilen sonuçların aynısını sağladığını ortaya koymaktadır. Şebeke noktaları şemalarından farklı olarak DJM ve KSM, herhangi bir doğrusal veya doğrusal olmayan denklem sistemini sağlamaz. DJM ve KSM, herhangi bir ayrıklaştırma, doğrusallaştırma veya küçük pertürbasyon gerektirmez ve bu nedenle sayısal çözümün yüksek doğruluğunu korurken hesaplamaların boyutunu büyük ölçüde azaltabilir. ADM'nin çoğunlukla sıkıcı cebirsel hesaplamalar gerektiren Adomian polinomlarının değerlendirilmesini gerektirdiğini belirtmekte fayda vardır. Ayrıca VIM, Lagrange çarpanını hesaplamayı gerektirir. Yineleme doğrudan ve anlaşılır olduğundan, DJM ve KSM'nin mevcut tekniklerle karşılaştırıldığında hesaplama hacmini azalttığını belirtmek ilginçtir. Ayrıca, doğrusal olmayan fenomeni ifade etmek için sıklıkla ortaya çıkan doğrusal olmayan denklemler için, DJM ve KSM güçlü ve verimlidir, hesaplama işini kolaylaştırır ve diğer sayısal tekniklerle karşılaştırıldığında çözümü hızlı bir şekilde verir. Ayrıca, analitik çözümü sağlamak için hibrit seçim modeli, Riccati modeli ve lojistik model gibi bazı bilimsel modelleri çözmek için yöntemler kullanılır ve sonuçlar DJM ve

KSM’nin VIM tarafından elde edilen sonuçların aynısını sağladığını gösterdi. Önerilen yöntemlerin basit, uygulaması kolay, bilgisayar gücü/belleği açısından ekonomik ve hızlı algoritması mevcut diğer yöntemlere göre başlıca avantajlarıdır.

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