

T.R.
GEBZE TECHNICAL UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

**PRACTICAL STABILITY ANALYSIS AND
CONTROLLABILITY OF PERTURBED FUZZY SYSTEM RELATED
TO UNPERTURBED FUZZY SYSTEM WITH INITIAL TIME
DIFFERENCE**

BETÜL ÖZBAY ELİBÜYÜK
**A THESIS SUBMITTED FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY
DEPARTMENT OF MATHEMATICS**

GEBZE
2023

T.R.
GEBZE TECHNICAL UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

**PRACTICAL STABILITY ANALYSIS AND
CONTROLLABILITY OF PERTURBED FUZZY
SYSTEM RELATED TO UNPERTURBED
FUZZY SYSTEM WITH INITIAL TIME
DIFFERENCE**

BETÜL ÖZBAY ELİBÜYÜK
**A THESIS SUBMITTED FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY
DEPARTMENT OF MATHEMATICS**

THESIS SUPERVISOR
PROF. DR. COŞKUN YAKAR

GEBZE
2023

T.C.
GEBZE TEKNİK ÜNİVERSİTESİ
FEN BİLİMLERİ ENSTİTÜSÜ

BAŞLANGIÇ ZAMAN FARKLI PERTÖRBE
BULANIK KONTROL SİSTEMİNİN
PERTÖRBE OLMAYAN BULANIK KONTROL
SİSTEMİNE GÖRE PRATİK KARARLILIK
ANALİZİ VE KONTROL EDİLEBİLİRLİĞİ

BETÜL ÖZBAY ELİBÜYÜK
DOKTORA TEZİ
MATEMATİK ANABİLİM DALI

DANIŞMANI
PROF. DR. COŞKUN YAKAR

GEBZE
2023



DOKTORA JÜRİ ONAY FORMU

GTÜ Fen Enstitüsü Yönetim Kurulu'nun 16/02/2023 tarih ve 2023/14 sayılı kararıyla oluşturulan jüri tarafından 28/04/2023 tarihinde tez savunma sınavı yapılan Betül ÖZBAY ELİBÜYÜK' ün tez çalışması Matematik Anabilim Dalında DOKTORA tezi olarak kabul edilmiştir.

JÜRİ

ÜYE
(TEZ DANIŞMANI) : Prof. Dr. Coşkun YAKAR

ÜYE : Prof. Dr. Emil NOVRUZ

ÜYE : Prof. Dr. Erkan ZERGEROĞLU

ÜYE : Doç. Dr. Yalçın YILMAZ

ÜYE : Dr. Öğr. Üyesi Sinem ŞİMŞEK MENGİ

ONAY

Gebze Teknik Üniversitesi Fen Bilimleri Enstitüsü Yönetim Kurulu'nun
...../...../..... tarih ve/..... sayılı kararı.

İMZA/MÜHÜR

SUMMARY

This thesis examines the practical stability and controllability properties of the perturbed fuzzy control differential equation system related to the unperturbed system with initial time and position differences. Unperturbed differential system;

$$D_{H_1}x(t) = f(t, x(t), u(t)), \quad x(t_0) = x_0 \in E^n$$

is defined as the initial value problem according to the first Hukuhara derivative of the nonlinear and non-autonomous fuzzy control differential equation.

The perturbed differential system;

$$D_{H_1}y(t) = F(t, y(t), u(t)), \quad y(\tau_0) = y_0 \in E^n$$

is defined as the initial value problem. This method aims to examine the stability solution of perturbed fuzzy control differential equation system related to unperturbed system similar to stability examination according to trivial solution in classical Lyapunov stability analysis. Necessary and sufficient conditions for practical stability, global, and practical controllability properties of the dual system are studied with initial time difference analysis. The main theorems use fuzzy control expansion of the comparison method via Lyapunov function and scalar control differential equation. The problem with the modifications has become fuzzy control Volterra-integro differential equation. The practical stability of the problem is examined, and Lyapunov's stability is compared to Ulam-Hyers on the issue. Finally, all the features given are supported with numerical examples.

Key Words: Fuzzy Control Differential Equations, Practical Stability, Lyapunov Functions, Global and Practical Controllability, Fuzzy Control Volterra-Integro Differential Equations, Ulam-Hyers Stability.

ÖZET

Bu tez, başlangıç zaman ve başlangıç konumu farklı pertörbe edilmemiş bulanık kontrol diferansiyel denklem sistemine göre pertörbe edilmiş sistemin pratik kararlılık ve kontrol edilebilirlik özelliklerini incelemektedir. Pertörbe edilmemiş diferansiyel sistemi;

$$D_{H_1}x(t) = f(t, x(t), u(t)), \quad x(t_0) = x_0 \in E^n$$

lineer ve otonom olmayan bulanık kontrol diferansiyel denkleminin birinci Hukuhara türevine göre başlangıç değer problemi olarak tanımlanmaktadır. Pertörbe diferansiyel sistem ise;

$$D_{H_1}y(t) = F(t, y(t), u(t)), \quad y(\tau_0) = y_0 \in E^n$$

başlangıç değer problem ile tanımlanmaktadır. Bu yöntemde amaç klasik Lyapunov kararlılıkta aşikâr çözüme göre yapılan kararlılık incelemesinin benzerinin pertörbe sistemin çözümünün pertörbe olmayan sistemin çözümüne göre yapılmasıdır. Bu ikili sistemin pratik kararlılık, global ve pratik kontrol edilebilirlik özellikleri için gerek ve yeter şartlar başlangıç zaman farkı analizi ile ortaya konmuştur. Ana teoremlerde, Lyapunov fonksiyonu ve skaler kontrol diferansiyel denklem aracılığıyla yapılan karşılaştırma yönteminin bulanık kontrol genişlemesi kullanılmıştır. Yapılan değişikliklerle problem, bulanık kontrol Volterra-integro diferansiyel denklemine taşınmıştır. Bu problemin çözümünün pratik kararlılığı incelenmiş, ayrıca bu problem üzerinde Ulam-Hyers ile Lyapunov kararlılığı karşılaştırılmıştır. Son olarak verilen tüm özellikler sayısal örnekler ile desteklenmiştir.

Anahtar Kelimeler: Bulanık Kontrol Diferansiyel Denklem, Pratik Kararlılık, Lyapunov Fonksiyon, Global ve Pratik Kararlılık, Bulanık Kontrol Volterra-Integro Diferansiyel Denklem, Ulam-Hyers Kararlılık.

ACKNOWLEDGEMENTS

I would like to express my deep and sincere gratitude to my supervisor, Prof. Coşkun YAKAR, who shared his profound scientific knowledge with me. I would like to thank the rest of my thesis committee, Prof. Emil NOVRUZ and Prof. Erkan ZERGEROĞLU for their support.

I am grateful to my parents, especially my mom, for their support. I stepped into the academic community with her encouragement and guidance, and I completed this Ph.D. thanks to her immense support for my son Kerem Ata. I would also like to thank my husband, Uğur, for his great support in the last period. I hope he can finish his academic career successfully and will always be with us.

Finally, I apologize for the times I stole from my son and for reflecting my stress on him. I dedicate this thesis to him.

TABLE of CONTENTS

	<u>Page</u>
SUMMARY	iv
ÖZET	v
ACKNOWLEDGMENTS	vi
TABLE of CONTENTS	vii
LIST of ABBREVIATIONS and ACRONYMS	ix
LIST of FIGURES	x
1. INTRODUCTION	1
2. FUNDAMENTALS OF FUZZY CALCULUS	6
3. STABILITY AND CONTROLLABILITY PROPERTIES FOR FCDE	
3.1. Preliminary Notes and Definitions	11
3.1.1. Controllability Properties	11
3.1.2. Stability Properties	17
3.2. Main Results	18
3.2.1. Main Results of Controllability	19
3.2.2. Main Results of Stability	22
3.3. Applications	26
4. STABILITY PROPERTIES FOR PFCDE RELATED TO UPFCDE	32
WITH ITD	
4.1. Comparing Classical Stability with ITD Stability	32
4.2. IVP for PFCDE Related to UPFCDE with ITD	33
4.3. Preliminary Notes and Definitions	35
4.4. Main Results of Stability	38
4.5. Applications	42

5. CONTROLLABILITY PROPERTIES FOR PFCDE RELATED TO UPFCDE WITH ITD	44
5.1. Preliminary Notes And Definitions	44
5.2. Main Results of Controllability	45
5.3. Applications	49
6. STABILITY PROPERTIES OF FCVIDE	53
6.1. Preliminary Notes And Definitions	53
6.2. Main Results of Stability	68
6.3. Comparison: Ulam-Hyers and Lyapunov Stability	76
6.4. Applications	82
7. CONCLUSION	85
REFERENCES	88
BIOGRAPHY	92
APPENDICES	93

LIST OF ABBREVIATIONS AND ACRONYMS

Abbreviations Explanations and Acronyms

$K_c(\mathbb{R}^n)$	: Collection of all nonempty compact, convex subsets of $\mathbb{R}^n$.
$D[A, B]$	: Hausdorff metric between the sets $A, B \in K_c(\mathbb{R}^n)$.
E^n	: Fuzzy set
E^p	: Fuzzy control input set
u	: Control input
$[A]^\alpha$	: α –level set of A
$D_0[x_1, x_2]$	: Hausdorff distance between x_1 and x_2 in E^n
θ^n	: Zero element of E^n
$D_H x$	: Hukuhara derivative
S_ρ	: Open ball with radius ρ according to D_0 metric in set E^n
κ	: Class κ function
V	: Lyapunov function
D^+V	: Dini derivative of Lyapunov function
FCDE	: Fuzzy control differential equation
FCVIDE	: Fuzzy control Volterra integro-differential equation
ITD	: Initial time difference
IVP	: Initial value problem
NLFCDE	: Nonlinear fuzzy control differential equation

LIST OF FIGURES

<u>Figure No:</u>	<u>Page</u>
3.1: The graphic of $\underline{x}^\alpha$ and $\bar{x}^\alpha$ in global controllable solution of equation (3.16) where $\lambda = 2, x_0 = \gamma$ and $x^1 = 0.5\gamma$.	28
3.2: The graphic of $\underline{x}^\alpha$ and $\bar{x}^\alpha$ in practically stable solution of equation (3.21) where $a = 1, b = 1.5$.	30
5.1: The graphic of $\underline{x}^\alpha$ and $\bar{x}^\alpha$ in global controllable solution of equation (5.7)- (5.8).	52
6.1: The graphic of $\underline{x}^\alpha$ and $\bar{x}^\alpha$ in global controllable solution of equation (6.15).	85

1. INTRODUCTION

A nonlinear dynamical system has been used to describe physical devices or processes that cannot be characterized by linear equations (containing algebraic, difference, differential, integral, functional, abstract operator equations, or a combination of these). It represents the evolution of a solution with time and/or control inputs [1]. The theory of nonlinear dynamical systems, or nonlinear control systems if control inputs are involved, which dates back to the third century BC, has been significantly advanced since the nineteenth century. Although all other discoveries marked the beginning of mathematical control and systems theory, none of them influenced the field as much as Maxwell's work, which explained the instabilities of the flyball governor by differential equations in 1868. After him, with the development of mathematical techniques, advances accelerated in optimal control in the 1950s and stochastic, robust, adaptive control methods in the 1970s. As a result of these advances, control theory applications have given a chance for the innovations such as space travel and communication satellites, aircraft, auto engines, and chemical processes.

Before discussing fuzzy control systems, it is necessary to mention about the ideas behind fuzzy set theory. In classical set theory, an element belongs to or does not belong to the cluster. If we express the membership of something to the cluster, membership is either one or zero. At this point, the fuzzy logic helps us with the concept of the degree of membership. Degree of membership actually refers to belonging degree of the set. Thanks to these ideas, Lotfi Aliaskerzadeh, as a mathematician and computer scientist, introduced impressive area called fuzzy sets theory in the 1960s [2]. Before starting his work on fuzzy set theory, Zadeh had already been a respected researcher in the field of control theory and had developed the concept of "state," which constituted the basis for modern control theory. He thought that natural processes could not be explained with finite value logic. Therefore, four years before his epochal article was published, Zadeh had stated that "a different mathematics is needed for fuzzy or uncertain quantities that cannot be defined by probability distribution". After the introduction of fuzzy sets, he proposed the concepts of fuzzy algorithms in 1968 [3], fuzzy decision-making in 1970 [4], and fuzzy ordering in 1971 [5]. In 1975, he introduced the concept of linguistic variables and suggested

using fuzzy IF-THEN rules to formulate the idea of the condition-consequence of human behaviors [6]. However, while Zadeh's work was not appreciated when it was published, it has inspired much research. For instance, Mamdani and Assilian [7], established the basic framework of the fuzzy controller and applied it to control a steam engine in 1973. Takagi and Sugeno [8] published an article asserting that their fuzzy models can highly represent almost all nonlinear dynamical systems with high precision in 1985. As a result, thanks to these studies, it has been proved that the fuzzy controller is straightforward to construct and it works remarkably well [9]-[11]. Furthermore, similar to fuzzy controllers, fuzzy logic in general has various technical applications in extensively different fields such as automobile gearboxes, dishwashers, electronic circuits, and computers using machine learning, decision-making, artificial intelligence, deep learning, and other software applications.

Differential equations, which come into prominence in engineering problems, are used in machines that facilitate daily life with fuzzy logic or fuzzy control applications. On the other hand, differential equations that contain not only a fuzzy controller but also a fuzzy derivative and fuzzy state, called fuzzy differential systems (FDS), are significant tools for modelling dynamical systems containing uncertainties and processing tentative or nominative data in mathematical models. Here, it should be noted that the fuzzy derivative was first introduced by Chang and Zadeh to work on the differential equation problem in fuzzy set theory [12]. After presenting the fuzzy derivative, the notion of differential equations in a fuzzy structure was grounded by Kaleva [13], [14]. Many authors have provided various conclusions in both the theoretical and applied fields with the help of Kaleva's works. These outcomes have been implemented in diverse real problems, such as quantum optics, gravity [15], population models, and engineering applications [16] via exact, approximate, and entirely numerical techniques for solving fuzzy initial value problems [17]- [19]. Additionally, in order to overcome such disadvantages and challenges of significant nonlinearities of most engineering systems that complicated analysis and controller design, successful approaches and criteria for understanding the system's behaviour have been developed. Stability and controllability are two of these approaches and criteria that have great importance in analyzing system behavior.

Controllability is essential in systems engineering, particularly in control systems and automation. The concept, introduced by Kalman in 1960 [20], has been

studied on the critical problems in stabilizability, trajectory tracking, and disturbance rejection. Kalman's theory successfully has constructed linear systems defined by ordinary first-order differential operators, which has meant determining the state space of lumped systems (ODEs). However, the theory has depended on a state space model depends on a particular input-output structure of the system. In this process, the variables in the description have lost their natural sense. A new behavioural theory introduced by J.C. Willems in 1991 [21], which does not depend on the state space definition, has provided significant gains in overcoming the problems related to the state space theory. Therefore, the original variables that emerged in the modelling process can be preserved with their sense. The behavioural definition does not depend on any input-output structure. Finally, with studied expansions, the theory is computationally effective even in distributed systems (PDE).

Stability theory is central of system engineering as controllability, especially in control systems and automation, regarding dynamics and control. If it is simply defined, stability means that the system outputs and its internal signals are bounded within admissible limits or, more strictly, the system outputs tend to an equilibrium state of interest. The basic concept of stability emerged by studying an equilibrium state of a mechanical system, dating back to as early as 1644 when E. Torricelli studied the equilibrium of a rigid body under the natural force of gravity. The classical stability theorem of G. Lagrange, formulated in 1788, was the well-known result about stability until the publication A. M. Lyapunov's famous thesis "The General Problem of Motion Stability" in 1992. His thesis and idea are so fundamental that they lead today to all kinds of fundamental research and practice related to the stability of dynamical systems.

The concept of stability in the sense of Lyapunov, especially the Lyapunov direct method, is an effective method to understand the system's behaviour in dynamic systems such as continuous-time/ discrete-time, linear/nonlinear, and autonomous/non-autonomous systems. It is a standard technique that allows estimation of the qualitative behaviour of differential equations and stability analysis when the behaviour of the apparent solution of the comparison system is known without solving the system in nonlinear systems. This method indicates that the system is stable if the appropriate Lyapunov function is found. However, the desired state of a system may sometimes be mathematically unstable, yet the system may oscillate

sufficiently near this state in which its performance is acceptable. It is named practical stability, which is interested in quantitative information concerning the system's behaviour, such as the estimation of trajectory bounds.

According to Lyapunov direct method, if the stability properties of a comparison system are known, then it can predict the behaviour of a differential system [22]-[26]. A problem occurs when applying Lyapunov direct method to unperturbed differential systems related to perturbed differential systems because of the difference in approach between initial time difference stability and the classical stability. The classical stability of differential equations is in terms of the null solution. However, the initial time difference stability analysis suggest to solve this problem is in terms of the unperturbed fuzzy differential system related to the perturbed fuzzy differential system with different initial positions and initial times. [27]-[34].

Another stability concept used in dynamic systems is Ulam-Hyers stability. The stability problem of functional equations originated from Stanisław Ulam's question. Then, Donald H. Hyers [35] gave a partially positive answer to the question of Ulam, which was the first significant step toward a solution in this area. In 1978, Themistocles M. Rassias [36] successfully extended Hyers's theorem and proved the stability of linear mapping. Since Rassias's expansions, generalizations of the Ulam problem and the Hyers theorem have been published in numerous papers and applied to various dynamical systems [37], [38].

This thesis aims to restate the ITD analysis method in the fuzzy control differential equations (FCDE) and improve practical stability analysis with ITD analysis in that. In addition, it intends to transfer this method, which has previously been used in stability analysis, to controllability analysis, and strengthen it with effective numerical examples. Moreover, it aims to study practical stability and to compare Ulam-Hyers and Lyapunov stability in fuzzy control Volterra-integro differential equations (FCVIDE).

The organization of the thesis is as follows; the second chapter presents fundamental definitions and features of the calculus of fuzzy set algebra. The third chapter introduces definitions of practical stability and global and practical controllability. In addition, it presents the main theorems and results of these properties and their applications on the initial value problem for FCDE. The fourth chapter introduces ITD stability analysis on the dual system, perturbed and unperturbed

differential equations. After the system introduction, it studies the practical stability property and applications of the perturbed fuzzy control differential equation system related to the unperturbed system with initial time and position differences. The fifth chapter gives the same dual system's global and practical controllability results and applications. The sixth chapter restate the main problem to the FCVIDE. It presents practical stability, Ulam-Hyers stability property and their applications and compares it with Lyapunov stability of FCVIDE.



2. FUNDAMENTALS OF FUZZY CALCULUS

After Zadeh's significant contributions to fuzzy sets, the field of fuzzy differential equations continued to progress with many studies. Lakshmikantham and Mohapatra wrote a beneficial book in 2003 [23] that both compiles the subject of fuzzy sets well and brings new approaches to stability. This study substantially uses the substructure of this book. This part gives simple definitions and some properties of fuzzy set algebra.

Let $K_c(\mathbb{R}^n)$ be the collection of all nonempty compact, convex subsets of $\mathbb{R}^n$. If $\alpha, \beta \in \mathbb{R}$ and $A, B \in K_c(\mathbb{R}^n)$ then,

$$\alpha(A + B) = \alpha A + \alpha B, \quad \alpha(\beta A) = (\alpha\beta)A, \quad 1A = A \quad (2.1)$$

If $\alpha, \beta \geq 0$, then, we have the equality $(\alpha + \beta)A = \alpha A + \beta A$. Define the Hausdorff metric as follows

$$D[A, B] = \max \left[\sup_{x \in B} d(x, A), \sup_{y \in A} d(y, B) \right] \quad (2.2)$$

where $d(x, A) = \inf[d(x, y): y \in A]$, and A, B are bounded sets in $\mathbb{R}^n$. The metric space $(K_c(\mathbb{R}^n), D)$ is a complete metric space.

Let $D[A, B]$ be the Hausdorff metric between the sets $A, B \in K_c(\mathbb{R}^n)$. Then we define the following properties:

- i) $D[A + C, B + C] = D[A, B]$
- ii) $D[A, B] = D[B, A]$
- iii) $D[\lambda A, \lambda B] = \lambda D[A, B]$
- iv) $D[A, B] \leq D[A, C] + D[C, B]$

for all $A, B, C \in K_c(\mathbb{R}^n)$ and $\lambda \in \mathbb{R}_+$. It is known that $(K_c(\mathbb{R}^n), D)$ is complete, separable and locally compact. It is defined $E^n = \{x: \mathbb{R}^n \rightarrow [0,1] \text{ such that } x(t) \text{ satisfies (i)-(iv) stated below}\}$:

- i) x maps $\mathbb{R}^n$ onto $I = [0,1]$,
- ii) $[x]^\alpha$ is compact subset of $\mathbb{R}^n$ for all $\alpha \in I$ where α -level set

$$x_\alpha = [x]^\alpha = \{z \in \mathbb{R}^n | x(z) \geq \alpha\} \quad (2.3)$$

- iii) $[x]^0$ is bounded subset of $\mathbb{R}^n$,
- vi) x is fuzzy convex, that is, for $0 \leq \lambda \leq 1$

$$x(\lambda z_1 + (1 - \lambda)z_2) \geq \min\{x(z_1), x(z_2)\} \quad \lambda \in [0,1] \quad (2.4)$$

We define the diameter of x as $\text{diam}[x]^\alpha = \bar{x}(\alpha) - \underline{x}(\alpha)$. Let $A = (a_1, a_2, a_3)$ is triangular fuzzy number. α -level set of A is

$$[A]^\alpha = [a_1 + \alpha(a_2 - a_1), a_3 - \alpha(a_3 - a_2)] \quad (2.5)$$

Let us define by $D_0[x_1, x_2] = \sup\{D([x_1]^\alpha, [x_2]^\alpha) : 0 \leq \alpha \leq 1\}$ the distance between x_1 and x_2 in E^n , where

$$D([x_1]^\alpha, [x_2]^\alpha) = \max\left\{\left|\underline{x}_1(\alpha) - \underline{x}_2(\alpha)\right|, \left|\bar{x}_1(\alpha) - \bar{x}_2(\alpha)\right|\right\} \quad (2.6)$$

is Hausdorff distance between two set $[x_1]^\alpha, [x_2]^\alpha$ of $K_c(\mathbb{R}^n)$. Then (E^n, D_0) is complete space. Some properties of metric D_0 are as follows:

- i) $D_0[x_1 + x_3, x_2 + x_3] = D_0[x_1, x_2]$
- ii) $D_0[\lambda x_1, \lambda x_2] = |\lambda| D_0[x_1, x_2]$,
- iii) $D_0[x_1, x_2] \leq D_0[x_1, x_3] + D_0[x_3, x_2]$

for all $x_1, x_2, x_3 \in E^n$ and $\lambda \in \mathbb{R}$. Let $x_1, x_2 \in E^n$, if there exists $x_3 \in E^n$ such that $x_1 = x_2 + x_3$, then x_3 is called the H-difference of x_1, x_2 and it is denoted by $x_1 - x_2$. Let us remark that $x_1 - x_2 \neq x_1 + (-1)x_2$. Let us denote $\theta^n \in E^n$ the zero element of E^n as follows: $\theta^n(z) = 1$ if $z = 0$ and $\theta^n(z) = 0$ if $z \neq 0$, where θ^n is the zero element of E^n .

It is defined the space of continuous fuzzy functions as

$$C([t_0, T], E^n) = \{x: [t_0, T] \rightarrow E^n | x \text{ is continuous}\} \quad (2.7)$$

which is complete metric space endowed with the following metric

$$D_0^*[x_1, x_2] = \sup_{t \in [t_0, T]} D_0[x_1(t), x_2(t)] \text{ for } x_1, x_2 \in C([t_0, T], E^n) \quad (2.8)$$

In the following we recall some main concepts and properties of fuzzy Hukuhara differentiability for fuzzy functions.

Definition 2.1: [23] Let $x: T \rightarrow E^n$ and $T \subset \mathbb{R}$ is compact interval. We say that x is differentiable at $t \in T$, if there exists a Hukuhara-derivative $D_H x(t) \in E^n$, such that for all $h > 0$ sufficiently small, Hukuhara-difference $x(t+h) - x(t)$, $x(t) - x(t-h)$ exist and limits

$$D_{H_1} x = \lim_{h \rightarrow 0^+} \frac{x(t+h) - x(t)}{h} \quad D_{H_2} x = \lim_{h \rightarrow 0^+} \frac{x(t) - x(t-h)}{h} \quad (2.9)$$

provided that there limits are exists. The limit by metric D_0 ,

$$\lim_{h \rightarrow 0^+} D_0 \left[\frac{x(t+h) - x(t)}{h}, D_{H_1} x(t) \right] = 0 \quad (2.10)$$

We study all theorems and operations according to the D_{H_1} derivative. Similarly, it can be studied according to the D_{H_2} derivative.

If x, y differentiable at t then,

- i) $D_{H_1}(x + y)(t) = D_{H_1}x(t) + D_{H_1}y(t)$
- ii) $D_{H_1}(\lambda x)(t) = \lambda D_{H_1}x(t), \lambda \in \mathbb{R}$

If $x: T \rightarrow E^n$ is differentiable then it is continuous.

Definition 2.2:[23] We say that a mapping $x: T \rightarrow E^n$ is strongly measurable if for all $a \in [0,1]$ the set-valued mapping $x_a: T \rightarrow P_K(\mathbb{R}^n)$ defined by $x_a(t) = [x(t)]^a$ is (Lebesgue) measurable, when $P_K(\mathbb{R}^n)$ is endowed with the topology generated by Hausdorff metric D_0 .

If x is strongly measurable, then it is measurable with respect to the topology generated by d that is defined by

$$u \in E^n, \quad \{t | d(x(t), u) \leq \varepsilon\} = \bigcap_{a \in [0,1]} \{t | D_0(x_a(t), [u]^a) \leq \varepsilon\} \quad (2.11)$$

Definition 2.3:[23] Let $x: T \rightarrow E^n$, integral of x over T , denoted $\int_T x(t)dt$ is defined levelwise for all $a \in [0,1]$ by the equation

$$\begin{aligned} \left[\int_T x(t)dt \right] &= \int_T x_a(t)dt \\ &= \left\{ \int_T \hat{x}(t)dt \mid \hat{x}: T \rightarrow \mathbb{R}^n \text{ is a measurable selection for } x_a \right\} \end{aligned} \quad (2.12)$$

If $x: T \rightarrow E^n$ is strongly measurable and integrably bounded, then x is integrable. If it is continuous, then is also integrable and for all $t \in T$ the integral $G(t) = \int_{[a,t]} x$ is differentiable and $D_{H_1} G(t) = x(t)$.

If $x: T \rightarrow E^n$ is differentiable and assume that the derivative $D_{H_1} x$ is integrable over T . Then for each $s \in T$, we have $x(s) = x(a) + \int_{(a,s]} D_{H_1} x dt$.

If $x, y: T \rightarrow E^n$ are integrable, then the following properties of the integral are valid. $D_0[x(\cdot), y(\cdot)]: T \rightarrow \mathbb{R}$ is integrable;

- i) $D_0 \left[\int_T x(t)dt, \int_T y(t)dt \right] \leq \int_T D_0[x(t), y(t)]dt$
- ii) $\int_a^b x(s)ds = \int_a^c x(s)ds + \int_c^b x(s)ds$ for $a \leq c \leq b$.

By using the Hausdorff metric, it follows that for $A \in E^n$,

$$D_0[A, \theta] = \|A\| = \sup_{a \in A} \|a\| \quad (2.13)$$

where θ is the zero element of $\mathbb{R}^n$ which is a one-point set.



3. CONTROLLABILITY AND STABILITY PROPERTIES OF FCDE

After presenting the initial value problem for FCDE, this section introduces some definitions of stability in Lyapunov's sense and controllability. The Lyapunov function, which has an important place in Lyapunov's direct method, and its Dini derivative are presented. The comparison method via Lyapunov function and scalar differential equation is applied to obtain sufficient conditions for stability. The main theorems of controllability are revealed with the help of the Lyapunov function.

In fuzzy metric space E^n , let's consider the initial value problem of fuzzy control differential equations (IVP for FCDE's) in $[t_0, T]$;

$$D_H x(t) = f(t, x(t), u(t)) \quad x(t_0) = x_0 \in E^n, u(t_0) = u_0 \in \Omega \quad t \geq t_0 \geq 0 \quad (3.1)$$

where $f: [t_0, T] \times S_\rho \times \Omega \rightarrow E^n$, $S_\rho = \{x(t) \in E^n; D_0[x(t), \theta^n] < \rho < \infty\}$
admissible control $u(t) \in \Omega$ where $\Omega = \{u(t) \in E^p; U(t, u(t)) \leq v(t), t \geq t_0\}$
where $U(t, u) \in C[[t_0, T] \times E^p, \mathbb{R}_+]$ and $v(t) \in \mathbb{R}_+$.

3.1. Preliminary Notes and Definitions

Definition 3.1:[23] A fuzzy function $x: [t_0, T] \rightarrow E^n$ is called solution on $[t_0, T]$, if x is differentiable fuzzy function and satisfies IVP for FCDEs (3.1).

For the local existence and uniqueness of fuzzy solution for IVP for FCDEs (3.1) with admissible control we consider the set X of all functions $x \in C([t_0, T], E^n)$, that imply (3.3). So it imply for $\forall \alpha \in [0, 1]$:

$$\text{diam}[f(s, x(s), u(s))]^\alpha ds \leq \text{diam}[x_0]^\alpha \quad \forall t \in [t_0, T] \quad (3.2)$$

where $\text{diam}[x(t)]^\alpha$ denotes the diameter of set $[x(t)]^\alpha \in K_C(\mathbb{R}^n)$.

Lemma 3.1:[23] Assume that $f: [t_0, T] \times S_\rho \times E^p \rightarrow E^n$ is continuous. A fuzzy mapping $x: [t_0, T] \rightarrow E^n$ is called to be a local solution to IVP for FCDEs (3.1) on $[t_0, T]$ if and only if x is a continuous fuzzy mapping and it satisfies to following fuzzy integral equations:

$$x(t) = x_0 + \int f(s, x(s), u(s)) ds \quad t \in [t_0, T] \quad (3.3)$$

If x is a differentiable fuzzy function. Moreover, if $x: [t_0, T] \rightarrow E^n$ is a solution to fuzzy integral equation (3.3) then the function $t \rightarrow \text{diam}[x(t)]^\alpha$ is nondecreasing for all $\alpha \in [0, 1]$.

We can show that the continuity of f guarantees that X is closed set in $C([t_0, T], E^n)$, so that X is a complete metric considering the distance D_0^* , where

$$D_0^*[\underline{x}, \bar{x}] = \sup_{t \in [t_0, T]} D_0[\underline{x}, \bar{x}] \text{ for } \underline{x}, \bar{x} \in C([t_0, T], E^n) \quad (3.4)$$

where $D_0[\underline{x}, \bar{x}] = \sup_{\alpha \in [0, 1]} D([\underline{x}]^\alpha, [\bar{x}]^\alpha)$.

Assume that the IVP for FCDEs (3.1) satisfy;

- i) $f \in C([t_0, T] \times B(t_0, \rho) \times E^p, E^n)$,
- ii) There exists $L > 0$ such that

$$D_0[f(t, x_1, u_1), f(t, x_2, u_2)] \leq L(D_0[x_1, x_2] + D_0[u_1, u_2]) \quad (3.5)$$

for all $x_1, x_2 \in E^n, u_1, u_2 \in E^p$ and $t \in [t_0, T]$.

- iii) X is nonempty; there is $x \in C([t_0, T], E^n)$ such that

$$x(t) = x_0 + \int f(s, x(s), u(s)) ds \quad (3.6)$$

exists for $t \in [t_0, T]$.

- vi) If $x \in C([t_0, T], E^n)$ then, $x(t) = x_0 + \int f(s, x(s), u(s)) ds$ exists for

$t \in [t_0, T]$.

Define the operator $P: X \rightarrow \{x|x: [t_0, T] \rightarrow E^n\}$ by:

$$(Px)(t) = x_0 + \int f(s, x(s), u(s)) ds, \quad t \in [t_0, T] \quad (3.7)$$

and the sequence of successive approximations $(x^m)_{m \in \mathbb{N}} \subset X$ given by:

$$x^0(t) = x_0, \text{ and } x^m(t) = (Px^{m-1})(t), t \in [t_0, T] \quad (3.8)$$

So for $\underline{x}, \bar{x} \in X$ and $t \in [t_0, T]$, we have

$$\begin{aligned} D_0[(P\underline{x})(t), (P\bar{x})(t)] &= \int D_0[f(t, \underline{x}(s), u(s)), f(t, \bar{x}(s), u(s))] ds \\ &\leq L \int D_0[\underline{x}(s), \bar{x}(s)] ds \leq L_p D_0^*[\underline{x}, \bar{x}] \end{aligned} \quad (3.9)$$

Theorem 3.1: Let $u \in C([t_0, T], E^p)$ be an admissible control. Suppose that the assumption i)-iv) hold. Moreover, if $L_p < 1$ in inequality (3.9).

Then,

- i) the IVP for FCDE's (3.1) has solution on $[t_0, T]$.*
- ii) The fuzzy solution of IVP for FCDE's is bounded and Lipschitzian.*

Proof: See in [19].

Theorem 3.2: Let $u \in C([t_0, T], E^p)$ be an admissible control. Suppose that the following conditions hold:

- i) Let $f: ([t_0, t_0 + p] \times B(t_0, \rho) \times E^p \rightarrow E^n$ be a continuous function such that $D_0[f(t, x, u), \theta^n] \leq M_1, \forall (t, x, u) \in [t_0, t_0 + p] \times B(t_0, \rho) \times E^p$.*
- ii) Let $g \in C([t_0, t_0 + p] \times \rho \times \mathbb{R}^+, \mathbb{R}^+)$, such that $g(t, 0, v) \equiv 0$ and $0 \leq z \leq \rho$*

$0 \leq g(t, z, v) \leq M_2, \forall t \in [t_0, t_0 + p]$, such that $g(t, z, v)$ is nondecreasing in z and $v(t) \in C([t_0, T], \mathbb{R})$ such that the initial value problem

$$\frac{dz}{dt} = g(t, z(t), v(t)), \quad z(t_0) = z_0 = 0 \quad (3.10)$$

has only the solution $z(t) \equiv 0$ on $[t_0, t_0 + p]$;

iii) $\forall (t, x_1, u_1), (t, x_2, u_2) \in [t_0, t_0 + p] \times B(x_0, \rho) \times E^p$ and $D_0[x_1, x_2] \leq M_3$;

$$D_0[f(t, x_1, u_1), f(t, x_2, u_2)] \leq g(t, D_0[x_1, x_2], D_0[u_1, u_2]) \quad (3.11)$$

vi) There exists $q > 0$ such that for $t \in [t_0, t_0 + p]$ the sequence given by $x^0(t) = x_0, x^m: [t_0, t_0 + p] \rightarrow E^n, x^{m+1}(t) = x_0 + \int f(t, x^m(s), u(s))ds$ is well defined for any $m \in \mathbb{N}$.

Then, IVP for FCDEs has unique solutions $x: [t_0, t_0 + r] \rightarrow B(x_0, \rho)$ where $r = \min\{p, \rho/M_1, \rho/M_2\}$ and successive iterations

$$x^0(t) = x_0, \text{ and } x^{m+1}(t) = x_0 + \int f(s, x^m(s), u(s))ds, \quad t \in [t_0, T] \quad (3.12)$$

converge to solution $x(t)$ provided that the IVP for FCDEs exists.

Proof: See in [19].

Theorem 3.3: Assume that $f \in C([t_0, T] \times B(t_0, \rho) \times E^p, E^n)$

$$D_0[f(t, x_1, u_1), f(t, x_2, u_2)] \leq g(t, D_0[x_1, x_2], D_0[u_1, u_2]) \quad (3.13)$$

where $g \in C([t_0, t_0 + p] \times \rho \times \mathbb{R}^+, \mathbb{R}^+)$ is nondecreasing in z for each $t \in \mathbb{R}_+$ and the maximal solution $r(t, t_0, z_0, v)$ of equation (3.10) exist on $[t_0, +\infty)$. Suppose further that f is smooth enough to guarantee local existence of solution of the IVP for FCDEs for any $(t_0, x_0) \in ([t_0, T], E^n)$.

Then, there exist the solution $x(t)$ of the IVP for FCDEs such that $D_0[x_0, \theta^n] \leq z_0$ in largest interval $[t_0, +\infty)$.

Proof: See in [39].

Definition 3.2: [23] A function $\phi(r)$ is said to belong to the class $\mathcal{K}$ if $\phi(0) = 0$, $\phi \in C[(0, \rho), \mathbb{R}_+]$ and $\phi(r)$ is strictly monotone increasing in r . It is said to belong to class $\mathcal{K}_\infty$ if $\rho = \infty$ and $\phi(r) \rightarrow \infty$ as $r \rightarrow \infty$.

Example 3.1: $\phi(r) = \tan^{-1}(r)$ is strictly increasing since: $\phi'(r) = \frac{1}{1+r^2} > 0$. It belongs to class $\mathcal{K}$, but not to class $\mathcal{K}_\infty$ since $\lim_{r \rightarrow \infty} \phi(r) = \frac{\pi}{2}$.

Definition 3.3: [23] For Lyapunov function $V(t, x(t)) \in C[\mathbb{R}_+ \times E^n, \mathbb{R}_+]$ we define the Dini derivatives as follows:

$$\begin{aligned} D^+V(t, x) &\equiv \limsup_{h \rightarrow 0^+} \frac{1}{h} [V(t+h, x+hf(t, x, u)) - V(t, x)] \\ D_-V(t, x) &\equiv \liminf_{h \rightarrow 0^-} \frac{1}{h} [V(t+h, x+hf(t, x, u)) - V(t, x)] \end{aligned} \quad (3.14)$$

for $(t, x) \in \mathbb{R}_+ \times E^n$.

Example 3.2: Let Lyapunov function $V(t, x(t)) = m^2(t)x^2$ where $m \in C^1(\mathbb{R}_+, \mathbb{R})$. Then, Dini derivative of $V(t, x(t))$,

$$\begin{aligned} D^+V(t, x) &= \limsup_{h \rightarrow 0^+} \frac{1}{h} [m^2(t+h)(x+hf(t, x, u))^2 - m^2(t)x^2] \\ &= \limsup_{h \rightarrow 0^+} \frac{1}{h} [m^2(t+h)(x^2 + 2xhf(t, x, u) + h^2f^2(t, x, u)) - m^2(t)x^2] \end{aligned} \quad (3.15a)$$

$$\begin{aligned}
&= \lim_{h \rightarrow 0^+} \sup \frac{1}{h} [(m^2(t+h) - m^2(t))x^2] \\
&\quad + \lim_{h \rightarrow 0^+} \sup \frac{1}{h} [(m^2(t+h))2xf(t, x, u)] \\
&\quad + \lim_{h \rightarrow 0^+} \sup \frac{1}{h} [(m^2(t+h))h^2f^2(t, x, u)] \\
&= x^2 D^+ m(t) + 2xf(t, x, u)m^2(t)
\end{aligned} \tag{3.15b}$$

Definition 3.4: [23] Lyapunov function $V(t, x(t)) \in C[\mathbb{R}_+ \times E^n, \mathbb{R}_+]$ is said to be positive definite if there exists a function $\phi \in \mathcal{K}$ such that

$$\phi(D_0[x(t), \theta^n]) \leq V(t, x(t)) \tag{3.16}$$

Definition 3.5: [23] Lyapunov function $V(t, x(t)) \in C[\mathbb{R}_+ \times E^n, \mathbb{R}_+]$ is said to be decrescent if there exists a function $\psi \in \mathcal{K}$ such that

$$V(t, x(t)) \leq \psi(D_0[x(t), \theta^n]) \tag{3.17}$$

Lemma 3.2: Suppose that, the Lyapunov-like function $V(t, x(t))$ satisfies the followings:

- i) $V \in C(\mathbb{R}_+ \times E^n, \mathbb{R}_+)$;
- ii) $|V(t, x) - V(t, y)| \leq L(D_0[x, y])$
- iii) $D^+ V(t, x) \leq 0$
- iv) $b(D_0[x(t), \theta^n]) \leq V(t, x) \leq a(t, D_0[x(t), \theta^n])$ for $t, L \in \mathbb{R}_+$, $x, y \in E^n$, $u, u_1, u_2 \in E^p$, where $b(\cdot), a(t, \cdot) \in \kappa$

$$\kappa = \{\sigma \in C[\mathbb{R}_+, \mathbb{R}_+]: \sigma(w) \text{ is increasing } w \text{ and } \sigma(w) \rightarrow \infty \text{ as } w \rightarrow \infty\}. \tag{3.18}$$

Then, for every $\beta_1 > 0$, $t_0 \in \mathbb{R}_+$ be given, we can choose $\beta_2 = \beta(t, \beta_1) > 0$ such that $D_0[x_0, \theta^n] < \beta_1$ implies $D_0[x(t), \theta^n] < \beta_2$ for $\forall t > t_0$.

Proof: See in [19].

We consider again solution of NLFCDE (3.1) through (t_0, x_0) and present comparison theorem.

Theorem 3.4 : Assume that $f: [t_0, T] \times E^n \times E^p \rightarrow E^n$,

i) $D^+V(t, x) \leq g(t, V(t, x), U(t, u))$ where $g \in C[[t_0, T] \times \mathbb{R}_+, \mathbb{R}_+, \mathbb{R}]$ is monotone nondecreasing in U and in V ,

ii) $r(t) = r(t, t_0, z_0, v)$ is maximal solution of the scalar differential equation exists on $[t_0, T]$ and $v(t) \in C([t_0, T], \mathbb{R})$ is control function,

$$z' = g(t, z, v), \quad z(t_0) = z_0 \geq 0 \text{ for } t \geq t_0 \quad (3.19)$$

Then, if $x(t)$ is solution of (3.1) through (t_0, x_0) on $[t_0, T]$. We have

$$V(t, x) \leq r(t, t_0, z_0, v) \text{ provided that } V_0 = V(t_0, x_0) \leq z_0. \quad (3.20)$$

Proof: See in [23] for similar proof.

Corollary 3.1: The function $g(t, V(t, x), U(t, u)) = 0$ is admissible in Theorem 3.4 to yield the estimate $V(t, x) \leq V(t_0, x_0)$.

3.1.1. Controllability Properties

Definition 3.6:[18] If there exists $t_1 > t_0$ such that $x(t_1) = x(t_1, t_0, x_0, u(t_1)) = \bar{x}$ for any $\bar{x} \in E^n$ while we have $x(t_0) = x(t_0, t_0, x, u(t_0)) = x_0 \in E^n$, $u(t) \in E^p$ will be an admissible fuzzy control set. We say that the pair of fuzzy states $x_0, \bar{x} \in E^n$ is controllable and the IVP for FCDEs (3.1) is fuzzy controllable by $u(t)$.

Definition 3.7: [18] The IVP for NLFCDEs (3.1) is said to be global controllable if every pair of fuzzy state $x_0, \bar{x} \in E^n$ will be a controllable by $u(t)$. Suppose that;

- i) $R_t(x_0) = \{x(t) \in E^n | \exists u(t) \in E^p: x(t_0, x_0, t, u(t)) = x(t)\}$
- ii) $R(x_0) = \bigcup_{t>0} R_t(x_0)$

Theorem 3.5: The IVP for NLFCDEs (3.1) is said to be global controllable if for every fuzzy state $x_0 \in E^n$, we have $R(x_0) = E^n$.

Definition 3.8: [24] The IVP for NLFCDEs (3.1) is said to be practically controllable if given $0 < \lambda < A$ and $0 < \beta < A$, all solutions $x(t) = x(t, t_0, x_0, u(t))$ starting in

$$\Omega_1 = \{x \in E^n; D_0[x(t), \theta^n] < \lambda\} \quad (3.21)$$

are transferred to

$$\Omega_2 = \{x \in E^n; D_0[x(t), \theta^n] \leq \beta\} \quad (3.22)$$

in a finite time $T = T(t_0, x_0)$.

3.1.2. Stability Properties

We assume that NLFCDE in equation (3.1) has the trivial solution, which means $f(t, \theta^n, u(t)) = \theta^n$.

Definition 3.9: [23] The trivial NLFCDE solution $x(t) = x(t, t_0, x_0, u(t))$ of equation (3.1) through (t_0, x_0) is said to be:

- i) *stable by Lyapunov's mean if: for each $\varepsilon, t_0 > 0$ there exist a $\delta = \delta(t_0, \varepsilon)$ such that $D_0[x_0, \theta^n] < \delta$ implies $D_0[x(t), \theta^n] < \varepsilon$ for $t \geq t_0$.*
- ii) *uniformly stable by Lyapunov's mean if: if δ in (S) is independent of $t_0 \in \mathbb{R}_+$.*
- iii) *asymptotically stable by Lyapunov's mean if: it is stable and*

$$\lim_{t \rightarrow \infty} D_0[x(t), \theta^n] = 0 \quad (3.23)$$

iv) exponentially stable by Lyapunov's mean if:

$$D_0[x(t), \theta^n] \leq \beta(D_0[x_0, \theta^n], t_0) \exp[-a(t - t_0)], t \geq t_0 \quad (3.24)$$

where $\beta(D_0[\cdot, \cdot], t_0): [0, 1] \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ and $a > 0$.

v) practical stable by Lyapunov's mean if: given (λ, A) with $0 < \lambda < A$ there exists a $\sigma(\lambda, A) \geq 0$ such that $D_0[x_0, \theta^n] < \lambda$ implies $D_0[x(t), \theta^n] < A$, $t \geq t_0$ for some $t_0 \in \mathbb{R}_+$.

vi) uniformly practical stable by Lyapunov's mean if: (PS) holds for every $t_0 \in \mathbb{R}_+$.

vii) practically quasi stable by Lyapunov's mean if: given $(\lambda, B, T) > 0$, $0 < \lambda < B$ and $t_0 \in \mathbb{R}_+$ such that $D_0[x_0, \theta^n] < \lambda$ implies $D_0[x(t), \theta^n] < B$, $t \geq t_0 + T$.

3.2. Main Results

3.2.1. Main Results of Controllability

Theorem 3.6: Suppose that for IVP of nonlinear fuzzy control differential equation (IVP for NLFCEs) (3.1) satisfy the followings:

- i) The Lyapunov-like function $V(t, x(t))$ satisfies the conditions of Lemma 3.2;
- ii) $f \in C(\mathbb{R}_+ \times E^n \times E^p, E^n)$, $L > 0$ such that

$$D_0[f(t, x, u_1), f(t, y, u_2)] \leq L(D_0[x, y] + D_0[u_1, u_2]); \quad (3.25)$$

- iii) For every pair of set states $(x_0, x_1) \in E^n$, for $\forall t_1 \geq t_0$, $\exists u(t) \in E^p$ such that

$$\int_{t_0}^{t_1} D_0[u(s), \theta^p] ds = \frac{D_0[x_1, \theta^n] \exp[-L(t_1 - t_0)] - D_0[x_0, \theta^n]}{L} \quad (3.26)$$

Then, the IVP for NLFCEs (3.1) is global controllable.

Proof: Before we choose the controllable set $u(t) \in E^p$ satisfied (3.26). We have the fuzzy solution in $x(t) = x_0 + \int f(s, x(s), u(s))ds$, $t \in (t_0, +\infty)$, if x is differentiable.

The fuzzy solution of the IVP for NLFCEs (3.1) is written in the form $x(t) = x_0 + \int f(s, x(s), u(s))ds$, $t \in (t_0, +\infty)$, if x is differentiable. Then

$$x(t_1) = x(t_0, x_0, t_1, u(t_1)) = x_0 + \int_{t_0}^{t_1} f(s, x(s), u(s))ds \quad (3.27)$$

implies that

$$\begin{aligned} D_0[x(t_1), \theta^n] &\leq D_0[x_0, \theta^n] + \int_{t_0}^{t_1} D_0[f(s, x(s), u(s)), \theta^n]ds \\ &\leq D_0[x_0, \theta^n] + \int_{t_0}^{t_1} L(D_0[x(s), \theta^n] + D_0[u(s), \theta^p])ds \quad (3.28) \\ &\leq D_0[x_0, \theta^n] + L \int_{t_0}^{t_1} D_0[x(s), \theta^n]ds + L \int_{t_0}^{t_1} D_0[u(s), \theta^p]ds \end{aligned}$$

Let's substitute equation (3.26) in the last inequality. By the Gronwall-Belman's lemma [22] we received $D_0[x(t_1), \theta^n] \leq D_0[x_1, \theta^n]$.

The proof is complete, if we prove that $D_0[x(t_1), \theta^n] = D_0[x_1, \theta^n]$. If this is not true, that if $D_0[x(t_1), \theta^n] < D_0[x_1, \theta^n]$. But in this case we have by Lemma 3.2 $D_0[x(t_1), \theta^n] = \beta_2$ implies

$$\begin{aligned} b(\beta_2) &\leq b(D_0[x(t_1), \theta^n]) \leq V(t_1, x(t_1)) \leq V(t_1, x_1) \\ &\leq a(t_1, D_0[x(t_1), \theta^n]) < a(t_1, \beta_2) < b(\beta_2) \end{aligned} \quad (3.29)$$

This contradiction proves $D_0[x(t_1), \theta^n] = D_0[x_1, \theta^n]$ and that means the state pair $(x_0, x_1) \in E^n$ will be controllable, the IVP for NLFCEs (3.1) is global controllable.

Theorem 3.7: Assume that following assumptions;

i) $0 < \lambda < A$ and $0 < \beta < A$ are given;

ii) The Lyapunov-like function $V(t, x(t))$ is locally Lipschitzian in x , the function $Q(V(t, x))$ where $Q: \mathbb{R}_+^N \rightarrow \mathbb{R}_+$ is continuous, strictly increasing and $Q(0) = 0$ and for $b(\cdot), a(t, \cdot) \in \kappa$.

$$b(D_0[x(t), \theta^n]) \leq Q(V(t, x)) \leq a(t, D_0[x(t), \theta^n]) \quad (3.30)$$

iii) $D^+V(t, x) \leq g(t, V(t, x), U(t, u))$ where $g \in C[[t_0, T] \times \mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}]$ and $g(t, z, v)$ is quasimonotone nondecreasing in w and nondecreasing in v ;

iv) $a(t, \lambda) < b(A)$ holds;

v) There exists a control function $v(t) \in \mathbb{R}_+$ such that any solution $z(t, t_0, z_0, v)$ of

$$z' = g(t, z, v(t)) \quad z(t_0) = z_0 \geq 0 \quad (3.31)$$

satisfies $Q(z_0) < a(\lambda)$ implies $Q(z(t, t_0, z_0, v)) < b(A)$ and

$$Q(z(t_0 + T, t_0, z_0, v)) \leq b(\beta) \quad (3.32)$$

for some $T = T(t_0, z_0) > 0$.

Then, there exist admissible control $u(t) = u \in \Omega$ such that IVP for NLFCEs (3.1) is practically controllable.

Proof: $D_0[x_0, \theta^n] < \lambda$, $U(t, u) \leq v(t)$ $t \geq t_0$. Then we claim that for any solution $x(t, t_0, x_0, u(t))$ solution of FCDE (3.1) we have $D_0[x_0, \theta^n] < \lambda$ implies $D_0[x(t), \theta^n] < A$ for $t \geq t_0$.

If this is false, there would exist a $u = u(t) \in \Omega$ $D_0[x_0, \theta^n] < \lambda$ and $D_0[x(t_1), \theta^n] = A$ and $D_0[x(t), \theta^n] \leq A$ for $t_1 \geq t \geq t_0$.

Setting $m(t) = V(t, x(t))$ for $t_1 \geq t \geq t_0$ and using (iii), we get $t_1 \geq t \geq t_0$ $D^+m(t) \leq g(t, m(t), v(t))$ which implies by Comparison Theorem 3.4. the estimate $D^+m(t) \leq r(t, t_0, z_0, v)$ $t_1 \geq t \geq t_0$ where $r(t, t_0, z_0, v)$ is the maximal solution of (3.31). Choosing $z_0 = V(t_0, x_0)$, we then get, because of (ii), the relation

$$b(D_0[x(t), \theta^n]) \leq Q(V(t, x(t))) \leq Q(r(t, t_0, z_0, v)) \quad (3.33)$$

for $t_1 \geq t \geq t_0$. Now we are lead to the following contradiction, in view of the relations (3.30),

$$b(A) = b(D_0[x(t_1), \theta^n]) \leq Q(V(t_1, x(t_1))) \leq Q(r(t_1, t_0, z_0, v)) < b(A) \quad (3.34)$$

As a result, for all $t \geq t_0$ and $D_0[x(t_0 + T^*), \theta^n] \leq \beta$ $T^*(t_0, x_0) = T(t_0, V(t_0, x_0))$. The proof is complete.

3.2.2. Main Results of Stability

Theorem 3.8: Assume that the Lyapunov-like function $V(t, x(t))$ satisfies the conditions of Lemma 3.2 with

$$D^+V(t, x) \leq 0 \quad (3.35)$$

and $a(\lambda) < b(A)$.

Then, $x(t, t_0, x_0, u(t))$ solution of FCDE (3.1) is practical stable for $t \geq t_0$.

Proof: Let us assume that given (λ, A) with $0 < \lambda < A$. Practical stability holds such that $D_0[x(t, t_0, x_0, u(t)), \theta^n] < A$ provided that $D_0[x_0, \theta^n] < \lambda$.

If the fuzzy control differential equation (3.1) is not practical stable, there would exist a $x(t, t_0, x_0, u(t))$ solution of fuzzy control differential equation (3.1) for $t \geq t_0$ and exist $t_1 > t_0$ with $D_0[x_0, \theta^n] < \lambda$ satisfying

$$D_0[x(t_1, t_0, x_0, u(t)), \theta^n] = A \quad t \in [t_0, t_1] \quad (3.36)$$

So that we have, because of (3.35) and assumption iv) of Lemma 3.2

$$b(A) \leq V\left(t_1, x(t_1, t_0, x_0, u(t))\right) \text{ for } t_1 > t_0 \quad (3.37)$$

This means that $D_0[x(t_1, t_0, x_0, u(t)), \theta^n] < \rho$ for $t \in [t_0, t_1]$ and hence we get from the assumptions (3.35), the estimate $(t_1, x(t_1, t_0, x_0, u(t))) \leq V(t_0, x_0)$ $t \geq t_0$. We get

$$\begin{aligned} b(A) &= b(D_0[x(t_1, t_0, x_0, u(t)), \theta^n]) \leq V(t_1, x(t_1, t_0, x_0)) \leq V(t_0, x_0) \\ &\leq a(D_0[x_0, \theta^n]) \leq a(\lambda) < b(A) \end{aligned} \quad (3.38)$$

which contradicts. Hence, practical stability holds and we have $x(t, t_0, x_0, u(t))$ the solution of (3.1) is practical stable. If $V(t, x(t))$ is only decrescent, we get by $V(t, x_0) \leq a(D_0[x_0, \theta^n])$ and choose $\lambda = \lambda(A) > 0$ such that $a(\lambda) < b(A)$. Since λ is now dependent of t_0 , we have uniformly practical stability of $x(t, t_0, x_0, u(t))$ solution of equation (3.1).

Theorem 3.9: Assume that the Lyapunov-like function $V(t, x(t))$ satisfies the conditions of Lemma 3.2 with

$$D^+V(t, x) \leq \mu V(t_0, x_0) \quad (3.39)$$

and $a(\lambda) < b(A)$.

Then, $x(t, t_0, x_0, u(t))$ solution of FCDE (3.1) is practically quasi stable for $t \geq t_0$.

Proof: It is clear from (3.39) that we have practical stability of $x(t, t_0, x_0, u(t))$ solution of the system (3.1) for $t \geq t_0$. Hence, we have from Theorem 3.8, taking $B = \rho$ and designating $\lambda_0 = \lambda_0(t_0, \rho) = \lambda > 0$. We have practical stability with this λ_0 , practical stability holds such that $D_0[x(t, t_0, x_0, u(t)), \theta^n] < \rho$ provided that $D_0[x_0, \theta^n] < \lambda$ for $t \geq t_0 + T$.

If the solution $x(t, t_0, x_0, u(t))$ is not practically quasi-stable and then there would exist a solution of $x(t, t_0, x_0, u(t))$ for $t_1 > t_0 + T$ and with $D_0[x_0, \theta^n] < \lambda$ satisfying $D_0[x(t_1, t_0, x_0, u(t)), \theta^n] = B$ $t \in [t_0 + T, t_1]$. Consequently, we get from assumption (3.17), the estimate

$$V[t, x(t, t_0, x_0)] < V(t_0, x_0) \exp[-\mu(t - t_0)], \quad t \geq t_0 + T \quad (3.40)$$

Given $B > 0$, we choose $T = T(t_0, B) = \frac{1}{\mu} \ln \left[\frac{a(t_0, \lambda_0)}{b(B)} \right] + 1$. Then we have from (3.39), assumption of iv) of Lemma 3.2, and last inequality, we get

$$\begin{aligned} b(D_0[x(t_1, t_0, x_0, u(t)), \theta^n]) &\leq V(t_1, x(t_1, t_0, x_0)) \\ &\leq V(t_0, x_0) \exp[-\mu(t - t_0)] \leq a(\lambda) \exp[-\mu(t - t_0)] < b(B) \end{aligned} \quad (3.41)$$

This contradiction gives us practical stability of $x(t, t_0, x_0, u(t))$ solution of the system (3.1) for $t \geq t_0$. If $V(t, x(t))$ is only decrescent, we get

$$V(t, x_0) \leq a(D_0[x_0, \theta^n]). \quad (3.42)$$

Since λ is now dependent of t_0 , we have uniformly practical stability of $x(t, t_0, x_0, u(t))$ solution of equation (3.1).

Theorem 3.10: Assume that

i) Lyapunov-like function $V(t, x(t))$ satisfies the conditions of Lemma 3.2 with

$$D^+V(t, x) \leq g(t, V(t, x(t)), U(t, u(t))) \quad (3.43)$$

ii) Let $r(t) = r(t, t_0, z_0, v)$ be the maximal solution of the scalar differential equation

$$z' = g(t, z, v), z(t_0) = z_0 \geq 0 \text{ for } t \geq t_0 \quad (3.44)$$

and $a(\lambda) < b(A)$.

Then, the practical stability properties of the comparison differential equation imply the corresponding practical stability properties of $x(t, t_0, x_0, u(t))$ solution of the system (3.1) for $t \geq t_0$.

Proof: Suppose that comparison equation is practically stable, then for given any (λ, A) with $0 < \lambda < A$ and there exists $\lambda = \lambda(t_0, A)$ and $b \in \kappa$ such that $z(t, t_0, z_0, v) < b(A)$ provided that $d_s[z_0, 0] < \lambda, t \geq t_0$. We claim that with this λ , practical stability holds such that $D_0[x(t, t_0, x_0, u(t)), \theta^n] < A$ provided that $D_0[x_0, \theta^n] < \lambda$ for $t \geq t_0$.

If the solution of fuzzy control differential equation (3.1) is not practically stable, there would exist $x(t, t_0, x_0, u(t))$ solution of the system (3.1) for $t \geq t_0$ and exist a $t_1 > t_0$ and with $D_0[x_0, \theta^n] < \lambda$ for $t \geq t_0$ satisfying $D_0[x(t_1, t_0, x_0, u(t)), \theta^n] = A$ for $t \in [t_0, t_1]$. Choose $z_0 = a(D_0[x_0, \theta^n])$, we get the inequality $V(t, x(t)) \leq r(t, t_0, z_0, v)$ $t \in [t_0, t_1]$.

So that we have $b(A) \leq V(t_1, x(t_1, t_0, x_0, u(t)))$, $t_1 > t_0$. This means that $D_0[x(t), \theta^n] < \rho$ for $t \in [t_0, t_1]$ and hence we have the inequality $V(t_1, x(t_1, t_0, x_0, u(t))) \leq r(t_1, t_0, z_0, v)$ $t \geq t_0$. We get

$$\begin{aligned}
 b(A) &= b(D_0[x(t_1, t_0, x_0, u(t)), \theta^n]) \\
 &\leq V(t_1, x(t_1, t_0, x_0, u(t))) \\
 &\leq r(t_1, t_0, z_0, v) \\
 &\leq r(t_1, t_0, a(D_0[x_0, \theta^n]), v) \\
 &\leq r(t, t_0, a(\lambda_1), v) < b(A)
 \end{aligned} \tag{3.45}$$

This contradiction gives us practical stability properties of $x(t, t_0, x_0, u(t))$ solution of the system (3.1) for $t \geq t_0$. If $V(t, x(t))$ is only decrescent, we get

$$V(t_0, x_0) \leq a(D_0[x_0, \theta^n]) \tag{3.46}$$

Assume that the comparison system is practically quasi-stable. Given any $b(B) > 0$, there exists a $T = T(t_0, \epsilon) > 0$ such that

$$0 < z_0 = a(d_s[z_0, 0]) < \lambda \text{ implies } z(t, t_0, z_0, v) < b(B), t \geq t_0 + T. \quad (3.47)$$

Setting $z_0 = a(d_s[z_0, 0]) < \lambda$ and using (i) $V(t_0, x_0) \leq a(d_s[z_0, 0]) = z_0 < \lambda$. By using (i)

$$b(D_0[x(t, t_0, x_0, u(t)), \theta^n]) \leq V(t, x(t, t_0, x_0, u(t))) \leq r(t, t_0, z_0, v) < b(B) \quad (3.48)$$

for $t \geq t_0 + T$. We obtain $D_0[x(t, t_0, x_0, u(t)), \theta^n] < B, t \geq t_0 + T$ since $b \in \kappa$. This proves the practical quasi-stability.

3.3. Application

Application 3.1: We consider basic fuzzy control differential equation for example of (3.1). We show that this system is global controllable with Theorem 3.6.

$$\begin{aligned} D_H x(t) &= -\lambda x(t) + u(t) & x(t_0) &= a\gamma \in E^n \\ t \in [0, 0.3] \quad \lambda, a, b &\in \mathbb{R}_+ \quad [\gamma]^\alpha = [\alpha - 1, 1 - \alpha] & x(t_1) &= b\gamma \in E^n \end{aligned} \quad (3.49)$$

It is easy showing that $f(t, x(t), u(t)) = -\lambda x(t) + u(t)$ satisfy assumption ii) of Theorem 3.6 with $L = 1$ if $\lambda \leq 1$, $L = \lambda$ if $1 \leq \lambda$. Let's choose λ as greater, if not, similar operations can be studied. We can choose that The Lyapunov-like function $V(t, x(t)) = D_0[x, \theta^n]$ that provides the conditions of Lemma 3.2. If we can find an equation that satisfies (3.26), we show that the system is global controllable. We

consider the solution of (3.49) like following: $x(t) = x_0 + \int_0^t -\lambda x(s) + u(s) ds$, $t \in [0, 0.3]$, $x_0 \in E^n, u_0 \in E^p$. We use this equation in fuzzy Laplace transform [40] of solution of (3.49). By Theorem 3.6, for every pair of fuzzy states $(x_0, x^1) \in E^n$, $\forall t_1 > t_0, \exists u(t) \in E^p$ such that

$$\begin{aligned}
\int_0^t D_0[u(s), \theta^p] ds &= \sup_{\alpha \in [0,1]} \int_0^t \max\{|\underline{u}^\alpha|, |\bar{u}^\alpha|\} \\
&= \sup_{\alpha \in [0,1]} \left(\frac{be^{-\lambda t} - a}{\lambda} \right) \max\{|\alpha - 1|, |1 - \alpha|\}.
\end{aligned} \tag{3.50}$$

We want to show that with this $u(t)$,

$$\begin{aligned}
[x]^\alpha &= [\underline{x}_0]^\alpha + \int_0^t -\lambda \underline{x}^\alpha(s) ds + \int_0^t \underline{u}^\alpha(s) ds, \\
\bar{x}_0^\alpha &+ \int_0^t -\lambda \bar{x}^\alpha(s) ds + \int_0^t \bar{u}^\alpha(s) ds
\end{aligned} \tag{3.50}$$

Laplace fuzzy transform of these equation,

$$\begin{aligned}
L\{\underline{x}^\alpha(t)\} &= L\{\underline{x}_0^\alpha\} + L\left\{\int_0^t -\lambda \underline{x}^\alpha(s) ds\right\} + L\left\{\int_0^t \underline{u}^\alpha(s) ds\right\} \\
L\{\bar{x}^\alpha(t)\} &= L\{\bar{x}_0^\alpha\} + L\left\{\int_0^t -\lambda \bar{x}^\alpha(s) ds\right\} + L\left\{\int_0^t \bar{u}^\alpha(s) ds\right\}
\end{aligned} \tag{3.51}$$

The solution of (3.16), $[x]^\alpha = \gamma e^{-\lambda t} \left(\frac{a\lambda - a + b}{\lambda} + bt \right)$. If this solution satisfy

$$x(t_1) = x^1 = b\gamma = \gamma e^{-\lambda t_1} \left(\frac{a\lambda - a + b}{\lambda} + bt_1 \right), \tag{3.52}$$

the system is global controllable.

$$b(e^{\lambda t_1} - t_1 - 1/\lambda) = a(1 - 1/\lambda) \quad a = e^{\lambda t_1} - t_1 - 1/\lambda, \quad b = 1 - 1/\lambda. \tag{3.53}$$

Thus, for any t_1 and for every pair of set states $(x_0, x_1) \in E^n$, $\exists u(t) \in E^p$ such that

$$\int_0^t D_0[u(s), \theta^p] ds = \sup_{\alpha \in [0,1]} \left(\frac{be^{-\lambda t} - a}{\lambda} \right) \max\{|\alpha - 1|, |1 - \alpha|\}.$$

Application 3.2: Linear fuzzy control system $x(t) \in E^n$ and $u(t) \in E^p$

$$D_H x(t) = -x(t) + 3u(t) \quad x(t_0) = x_0 \quad (3.54)$$

We can choose that Lyapunov like function

$$V(t, x) = D_0[x(t), \theta^n] \text{ and } U(t, u) = D_0[u(t), \theta^n] \leq v(t) \quad (3.55)$$

for defined region Ω and we find that

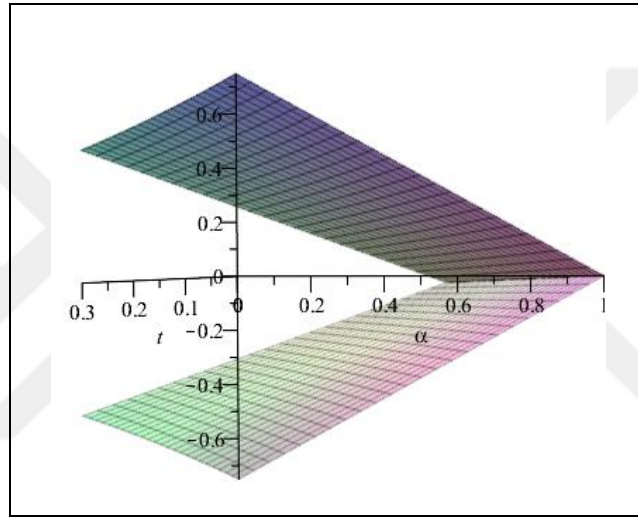


Figure 3.1: The graphic of $\underline{x}^\alpha$ and $\bar{x}^\alpha$ in global controllable solution of equation (3.49) where $\lambda = 2$, $x_0 = \gamma$ and $x^1 = 0.5\gamma$.

$$D^+V(t, x) \leq 3(D_0[x(t), \theta^n] + D_0[u(t), \theta^n]) = 3V(t, x) + 3U(t, u) \quad (3.56)$$

Then we can choose linear control system as comparison system

$$z = 3z + 3v \quad z(t_0) = z_0 \quad (3.57)$$

$v(t) \in \mathbb{R}_+$ is control function. Suppose that $Y(t) = c_1 e^{3t}$ is the fundamental solution of $z' = 3z$. We shall show that we can find suitable admissible controls $v(t)$ to assure practically controllability of the system (3.57). The transformation $z = Y(t)w$ reduces

(3.57) to $w' = \varphi(t)v$ $w(t_0) = z_0$. Then with necessary processes and assumptions, we can find easily that $|w(t)| < A$, for $t \geq t_0$, provided $|z_0| < \lambda$. Because $|z| = |Y(t)||w|$, if $|Y(t)| \leq 1$ for $t \geq t_0$, then we have $|z(t)| < A$ and if there exists $T = T(t_0, z_0) > 0$ such that $|Y(t_0 + T)| \leq \frac{\beta}{A}$.

So $|z(t_0 + T)| \leq \beta < A$ which shows that the system (3.57) is practically controllable.

We should determine that equation (3.54) is practical controllable by comparison with equation (3.57). Using Theorem 3.10, let's show that equation (3.54) is practical controllable. We can choose

$$Q(V(t, x)) = V(t, x), a(t, V(t, x)) = 2V(t, x) \quad b(V(t, x)) = \frac{1}{2}V(t, x). \quad (3.58)$$

Then choosing $Q(z_0) = z_0 < 2\lambda$ implies $Q(z(t, t_0, z_0, v)) = z(t, t_0, z_0, v) < \frac{1}{2}A$ and $Q(z(t_0 + T, t_0, z_0, v)) = z(t_0 + T, t_0, z_0, v) \leq b(\beta) = \frac{1}{2}\beta < \frac{1}{2}A$ for some $T = T(t_0, z_0) > 0$. So we can say that problem (3.18) is practically controllable.

Application 3.3: Let us consider fuzzy control differential equation $x(t) \in E^n$ and $u(t) \in \Omega$,

$$D_H x(t) = a(t - 4)x(t) + u(t) \quad x(0) = (-2, 0, 2) \quad t = [0, 1] \quad (3.59)$$

In this example, we consider the feedback control

$$u(t) = b \sin(t)x(t) \quad \text{where } a \in (0, 1] \text{ and } b \in (0, 2]. \quad (3.60)$$

We can easily find that α -level set of triangular fuzzy number

$$[x_0]^\alpha = [(2\alpha - 2), (2 - 2\alpha)] = \gamma \quad (3.61)$$

The solution of equation (3.59) is as follows,

$$\begin{aligned}
[x(t)]^\alpha &= \gamma \left[\exp \left(\int_0^t as - 4a + b \sin(s) ds \right) \right] \\
&= \gamma \left[\exp \left(\frac{at^2}{2} - 4at - b \cos(t) + b \right) \right]
\end{aligned} \tag{3.62}$$

$$\begin{aligned}
D_0[x_0, \theta^n] &= \sup\{\max\{|(2\alpha - 2)|, |(2 - 2\alpha)|\}: 0 \leq a \leq 1\} < \lambda \\
\max_{t \in [0,1]} \left\{ \exp \left(\frac{at^2}{2} - 4at - b \cos(t) + b \right) \right\} &= k
\end{aligned} \tag{3.63}$$

$D_0[x(t), \theta^n] = \sup\{\max\{|(2\alpha - 2)k|, |(2 - 2\alpha)k|\}: 0 \leq a \leq 1\} \leq A$. So equation (3.59) is practically stable.

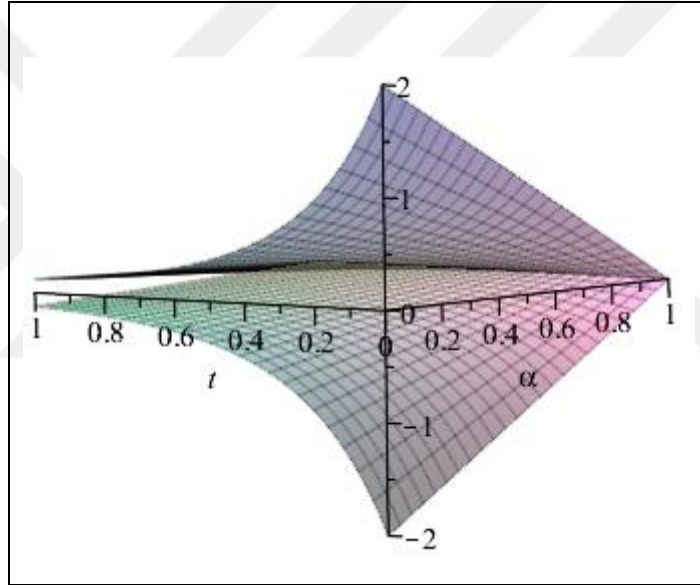


Figure 3.2: The graphic of $\underline{x}^\alpha$ and $\bar{x}^\alpha$ in practically stable solution of equation (3.59) where $a = 1$, $b = 1.5$.

We can determine that it is a practically stable using the Theorem 3.10 without choosing this feedback control. We can choose $V(t, x) = D_0[x(t), \theta^n]$ and calculate $D^+V(t, x) \leq 4(V(t, x) + U(t, u))$. So we can choose $g(t, w, v) = 4z + 4v$ and scalar differential equation $z' = 4z + 4v$. Suppose that $Y(t) = c_1 e^{4t}$ is the fundamental solution of $z' = 4z$. We shall show that we can find suitable admissible controls $v(t)$ to assure practical stability of scalar differential equation. The transformation $z = Y(t)w$ reduces scalar differential equation to

$$w' = 4Y^{-1}v \quad w(t_0) = z_0 \quad (3.64)$$

Then with necessary processes and assumptions, we can find easily that $|w(t)| < A$ $t \geq t_0$ provided $|z_0| < \lambda$. Therefore, if $|Y(t)| \leq 1$ $t \geq t_0$, then we have $|z(t)| < A$ which shows that the system scalar differential equation is practically stable.

We should determine that equation (3.59) is practical stable by scalar differential equation. Using Theorem 3.10, let's show that equation (3.59) is practical stable. We can choose $a(t, V(t, x)) = 2V(t, x)$ and $b(V(t, x)) = \frac{1}{2}V(t, x)$. Then choosing $x_0 < \lambda$ implies $x(t, t_0, x_0, u) \leq A$ for some $t \geq t_0$ for choosing that

$$a(\lambda) = 2\lambda < \frac{1}{2}A = b(A) \quad (3.65)$$

4. STABILITY PROPERTIES FOR PFCDE RELATED TO UPFCDE WITH ITD

This chapter, where the main work begins, presents two fuzzy control differential equations with different initial times and initial positions, one is perturbed, and the other is unperturbed. These two equations are handled with a new analysis method called initial time difference. The stability of the solution of the perturbed equation is determined related to the solution of the unperturbed system. Particular emphasis was placed on practical stability, and necessary and sufficient conditions were presented for this feature of the comparative system. Finally, the conclusions of the theorems are reinforced with scalar examples.

4.1. Comparing Classical Stability with ITD Stability

Let $x(t, t_0, x_0)$ and $x(t, t_0, y_0)$ be any solutions of the fuzzy control differential system respectively with $x' = f(t, x)$, $x(t_0) = x_0$ and $x(t_0) = y_0$ for $t \geq t_0$ where $f \in C[\mathbb{R}_+ \times S\rho, E^n]$, $S\rho = [x \in E^n: d[x, 0] < \rho < \infty]$.

It is seem to study the classical stability of a given solution $x(t, t_0, y_0)$ of the system, it is convenient to make a change of variable. Let $x(t, t_0, y_0)$ and $x(t, t_0, x_0)$ be the unique solutions and set

$$w_1(t, t_0, y_0 - x_0) = x(t, t_0, y_0) - x(t, t_0, x_0) \text{ for } t \geq t_0 \quad (4.1)$$

Then,

$$\begin{aligned} w_1'(t, t_0, y_0 - x_0) &= x'(t, t_0, y_0) - x'(t, t_0, x_0) \\ &= f(t, w_1(t, t_0, y_0 - x_0) + x(t, t_0, x_0)) \\ &\quad - f(t, x(t, t_0, x_0)) \\ &= f(t, w_1(t, t_0, y_0 - x_0)) \end{aligned} \quad (4.2)$$

$w_1(t, t_0, y_0 - x_0) \equiv 0$ is a solution of the conversion system if $y_0 - x_0 = 0$ which implies $f(t, 0) = 0$. Since $w_1'(t) = 0$ and $w_1(t, t_0, 0) = 0$ is the null

solution, $x(t, t_0, x_0)$ the solution of corresponds to the identically null solution of $w_1' = f(t, w_1)$ where

$$f(t, w_1(t, t_0, y_0 - x_0)) = f(t, w_1(t, t_0, y_0 - x_0) + x(t, t_0, x_0)) - f(t, x(t, t_0, x_0)) \quad (4.3)$$

Hence, $x(t, t_0, x_0) = 0$ is the null solution, and classical stability uses this null solution. However, it is not possible conversion for initial time difference stability. Let $x(t, \tau_0, y_0)$ be a solution of $x' = f(t, x)$, $x(\tau_0) = y_0$ and $x(t, \tau_0, x_0) = x(t - \eta, t_0, x_0)$ where $x(t, t_0, x_0)$ is any solution of FCDE with $x(t_0) = x_0$ for $t \geq t_0 \geq 0$. Let us make a same conversion. Set,

$$w_2(t, \tau_0, y_0 - x_0) = x(t, \tau_0, y_0) - x(t - \eta, t_0, x_0), \eta = \tau_0 - t_0 \quad (4.4)$$

for $t \geq \tau_0$. Then,

$$\begin{aligned} w_2'(t, \tau_0, y_0 - x_0) &= x'(t, \tau_0, y_0) - x'(t - \eta, t_0, x_0) \\ &= f(\eta; t, w_2(t, \tau_0, y_0 - x_0)) \end{aligned} \quad (4.5)$$

It can seem that even if $y_0 = x_0$, $w_2(t, \tau_0, 0)$ is not zero, and is not the null solution of the conversion system, and the solution $x(t - \eta, t_0, x_0)$ does not correspond to the identically zero solution of $w_2' = f(\eta; t, w_2)$. Therefore, more than the classical stability approach is needed in order to find (ITD) stability properties.

4.2. IVP for PFCDE Related to UPFCDE with ITD

In fuzzy metric space E^n , let's consider the initial valued problems of nonlinear fuzzy control differential equations (IVP for NLFCDE's) in $[t_0, T]$;

$$\begin{aligned} D_{H_1}x(t) &= f(t, x(t), u(t)), & x(t_0) &= x_0 \in E^n \\ D_{H_1}x(t) &= f(t, x(t), u(t)), & x(\tau_0) &= y_0 \in E^n \end{aligned} \quad (4.6)$$

the perturbed system of (4.6) in $[\tau_0, T]$.

$$\begin{aligned} D_{H_1}y(t) &= F(t, y(t), u(t)), & y(\tau_0) &= y_0 \in E^n, \\ u(\tau_0) &= u_0 \in E^p \end{aligned} \quad (4.7)$$

where $f, F: [t_0, T] \times E^n \times E^p \rightarrow E^n$, admissible control $u(t) \in \Omega$ where

$$\Omega = \{u(t) \in E^p; U(t, u(t)) \leq v(t), t \geq t_0\} U(t, u) \in C[[t_0, T] \times E^p, \mathbb{R}_+] \quad (4.8)$$

$v(t) \in \mathbb{R}_+$ for $\tau_0 > t_0$ and $\eta = \tau_0 - t_0$. We have a special case of (4.7) that is perturbation equation of (4.6) if $F(t, y(t), u(t)) = f(t, y(t), u(t)) + R(t, y(t), u(t))$ where $R(t, y(t), u(t))$ is the perturbation term. The above assumptions imply the existence of trivial solutions of (4.6) and (4.7) through (t_0, x_0, u_0) and (τ_0, y_0, u_0) respectively.

Thus the corresponding with the IVP for NLFCDEs of (4.6), (4.7) are the followings respectively

$$\begin{aligned} x(t) &= x_0 + \int_{t_0}^T f(s, x(s), u(s))ds, & t \in [t_0, T], & & x_0 \in E^n, u_0 \in E^p \\ y(t) &= y_0 + \int_{\tau_0}^T F(s, y(s), u(s))ds, & t \in [\tau_0, T], & & y_0 \in E^n, u_0 \in E^p \end{aligned} \quad (4.9)$$

Set $u(t) \in E^p$ will be an admissible fuzzy control set, we have

$$y(t) = y(t, \tau_0, y_0, u(t)) = y_0 \quad (4.10)$$

solutions of IVP for NLFCDEs of (4.7) with ITD related to

$$x^*(t) = x^*(t, \tau_0, x_0, u(t)) = x(t - \eta, t_0, x_0, u(t)) \quad (4.11)$$

where $x(t, t_0, x_0, u(t))$ solution of the system (4.6).

4.3. Preliminary Notes and Definitions

Definition 4.1: [25] For a real-valued function $V \in C[\mathbb{R}_+ \times S_\rho, R_+]$, it is defined the generalized derivatives (Dini-like derivatives) as follows

$$\begin{aligned} D_*^+ V(t, y - x^*) \\ = \limsup_{h \rightarrow 0^+} \frac{1}{h} [V(t + h, y - x^* + h[F(t, y, u) - f(t, x^*, u)]) - V(t, y - x^*)] \end{aligned} \quad (4.12)$$

where $y(t) = y(t, \tau_0, y_0, u(t))$ solution of the system (4.7) and

$$x^*(t) = x(t - \eta, t_0, x_0, u(t)) \quad (4.13)$$

where $x(t, t_0, x_0, u(t))$ solution of the system (4.6) for $t \geq \tau_0$.

Definition 4.2: [31] We assume that NLFCDE the solution $y(t) = y(t, \tau_0, y_0, u(t))$ of system (4.7) through (τ_0, y_0) for $t \geq \tau_0$,

i) stable with ITD related to the solution $x^*(t) = x(t - \eta, t_0, x_0, u(t))$ where $x(t, t_0, x_0, u(t))$ solution of the system (4.6) for $t \geq t_0$ if and only if given any $\varepsilon > 0$ and

$\tau_0 > 0$ there exist a $\delta = \delta(\tau_0, \varepsilon)$ such that $D_0[y_0 - x_0, \theta^n] < \delta$ implies

$D_0[y(t) - x^*(t), \theta^n] < \varepsilon$ for $t \geq \tau_0$,

ii) uniformly stable with ITD related to the solution $x^*(t, \tau_0, x_0, u(t))$ if δ in i) is independent of $\tau_0 \in \mathbb{R}_+$.

iii) asymptotically stable with ITD related to the solution $x^*(t, \tau_0, x_0, u(t))$ if i) holds and there exists $\gamma(\tau_0) > 0$ such that $\lim_{t \rightarrow \infty} D_0[y(t) - x^*(t), \theta^n] = 0$ with

$$D_0[y_0 - x_0, \theta^n] < \gamma(\tau_0). \quad (4.14)$$

iv) exponentially stable with ITD related to the solution $x^*(t, \tau_0, x_0, u(t))$ if there exists an estimate such that

$$D_0[y(t) - x^*(t), \theta^n] \leq D_0[y_0 - x_0, \theta^n] \exp[-a(t - \tau_0)], \quad (4.15)$$

for all $a > 0$.

v) practical stable with ITD related to the solution $x^*(t, \tau_0, x_0, u(t))$ if given (λ, A) with $0 < \lambda < A$ there exists a $\sigma(\lambda, A) \geq 0$ such that $D_0[y_0 - x_0, \theta^n] < \lambda$ implies $D_0[y(t) - x^*(t), \theta^n] < A$, $t \geq \tau_0$ for some $\tau_0 \in \mathbb{R}_+$.

vi) uniformly practical stable with ITD related to the solution $x^*(t, \tau_0, x_0, u(t))$ if v) holds for every $\tau_0 \in \mathbb{R}_+$.

vii) practically quasi stable with ITD related to the solution $x^*(t, \tau_0, x_0, u(t))$ if given $(\lambda, B, T) > 0$ with $0 < \lambda < B$ such that $D_0[y_0 - x_0, \theta^n] < \lambda$ implies

$$D_0[y(t) - x^*(t), \theta^n] < B \quad (4.16)$$

$t \geq \tau_0 + T$ for some $\tau_0 \in \mathbb{R}_+$.

Lemma 4.1: Suppose that, the Lyapunov-like function $V(t, y - x^*)$ satisfies the followings:

i) $V \in C(\mathbb{R}_+ \times E^n, \mathbb{R}_+)$;

ii) $|V(t, y_1 - x_1^*) - V(t, y_2 - x_2^*)| \leq L(D_0[y_1 - x_1^*, y_2 - x_2^*])$

iii) $D^+V(t, y - x^*) \leq 0$

iv) $b(D_0[y(t) - x^*(t), \theta^n]) \leq V(t, y(t) - x^*(t)) \leq a(t, D_0[y(t) - x^*(t), \theta^n])$

for $t, L \in \mathbb{R}_+$, $x, y \in E^n$, where $b(\cdot), a(t, \cdot) \in \kappa = \{\sigma \in C[\mathbb{R}_+, \mathbb{R}_+]: \sigma(w) \text{ is increasing } w \text{ and } \sigma(w) \rightarrow \infty \text{ as } w \rightarrow \infty\}$.

Then, for every $\beta_1 > 0$, $t_0 \in \mathbb{R}_+$ be given, we can choose $\beta_2 = \beta(t, \beta_1) > 0$ such that $D_0[y_0 - x_0, \theta^n] < \beta_1$ implies $D_0[y(t) - x^*(t), \theta^n] < \beta_2$ for $\forall t > t_0$.

Proof: For the detail of the proof, please see [19].

Example 4.1: Let Lyapunov function $(t, y(t) - x^*(t)) = D_0[y - x^*, \theta^n]$. Then, Dini derivative of $V(t, y(t) - x^*(t))$,

$$\begin{aligned} D^+V(t, y(t) - x^*(t)) \\ = \lim_{h \rightarrow 0^+} \sup \frac{1}{h} [D_0[y - x^* + h[F(s, y(s), u(s)) - f(t, x^*, u(t))], \theta^n] - D_0[y - x^*, \theta^n] \end{aligned} \quad (4.17a)$$

$$\begin{aligned} &\leq \lim_{h \rightarrow 0^+} \sup \frac{1}{h} [D_0[y - x^*, \theta^n] \\ &\quad + hD_0[F(s, y(s), u(s)) - f(t, x^*, u(t)), \theta^n] - D_0[y - x^*, \theta^n]] \\ &\leq \sup [D_0[F(s, y(s), u(s)) - f(t, x^*, u(t)), \theta^n]] \end{aligned} \quad (4.17b)$$

We consider comparison theorem to predict the stability properties of $y(t)$ solution of the system (4.7) and $x^*(t)$ where $x(t, t_0, x_0, u(t))$ solution of the system (4.7) for $t \geq \tau_0$.

Theorem 4.1: Assume that $f, F : [t_0, T] \times E^n \times E^p \rightarrow E^n$ and

i) $D^+V(t, y - x^*) \leq g(t, V(t, y - x^*), U(t, u))$ where $g \in C[[\tau_0, T] \times \mathbb{R}_+, \mathbb{R}_+]$ is monotone nondecreasing in u and in v ,

ii) $r(t) = r(t, \tau_0, z_0, v)$ is maximal solution of the scalar differential equation exists on $[\tau_0, T]$ and $v(t) \in C([\tau_0, T], \mathbb{R})$ is control function,

$$z' = g(t, z, v), \quad z(\tau_0) = z_0 \geq 0 \text{ for } t \geq \tau_0 \quad (4.18)$$

Then, if $x^*(t)$ ve $y(t)$ are solution of (4.6) and (4.7) through (t_0, x_0) and (τ_0, y_0) respectively on $[\tau_0, T]$. We have

$$V\left(t, y(t, \tau_0, y_0, u(t)) - x^*(t, \tau_0, x_0, u(t))\right) \leq r(t, \tau_0, z_0, v) \quad (4.19)$$

provided that $V(t, y_0 - x_0) \leq z_0$.

Proof: For the detail of the proof, please see [23].

Corollary 4.1: The function $g(t, z) = 0$ is admissible in Theorem 4.1 to yield the estimate

$$V\left(t, y(t, \tau_0, y_0, u(t)) - x^*(t, \tau_0, x_0, u(t))\right) \leq V(t, y_0 - x_0) \quad (4.20)$$

4.4. Main Results of Stability

We assume that unperturbed and perturbed NLFCDE (4.6) and (4.7) have the trivial solution, which means $f(t, \theta^n, u(t)) = \theta^n$ and $F(t, \theta^n, u(t)) = \theta^n$.

Theorem 4.2: Assume that Lyapunov-like function $V(t, y - x^*)$ satisfies the conditions of Lemma 4.1 with

$$D^+V(t, y - x^*) \leq 0 \quad (4.21)$$

Then $y(t, \tau_0, y_0, u(t))$ the solution of FCDE (4.7) is practical stable with ITD related to $x^*(t, \tau_0, x_0, u(t))$ solution of FCDE (4.1) for $t \geq \tau_0$.

Proof: Let us assume that given (λ, A) with $0 < \lambda < A$. Then it is possible to find a and b such that $a(\lambda) < b(A)$. Practical stability holds such that

$$D_0[y(t, \tau_0, y_0, u(t)) - x^*(t, \tau_0, x_0, u(t)), \theta^n] < A \quad (4.22)$$

provided that $D_0[y_0 - x_0] < \lambda$. If $y(t, \tau_0, y_0, u(t))$ the solution of FCDE (4.7) is not practical stable with ITD related to $x^*(t, \tau_0, x_0, u(t))$ solution of FCDE (4.6), there would exist $t_1 > \tau_0$ with

$$\begin{aligned} D_0[x_0 - y_0, \theta^n] &< \lambda \\ D_0[y(t_1, \tau_0, y_0, u(t)) - x^*(t_1, \tau_0, x_0, u(t)), \theta^n] &= A \quad t \in [\tau_0, t_1] \end{aligned} \quad (4.23)$$

So that we have, $b(A) \leq V(t_1, y(t_1, \tau_0, y_0, u(t)) - x^*(t_1, \tau_0, x_0, u(t)))$ for $t_1 > \tau_0$. This means that $D_0[y(t_1, \tau_0, y_0, u(t)) - x^*(t_1, \tau_0, x_0, u(t)), \theta^n] < \rho$ for $t \in [\tau_0, t_1]$ and hence we get from the assumptions (4.21) and Corollary 4.1, the estimate

$$V(t_1, y(t_1, \tau_0, y_0, u(t)) - x^*(t_1, \tau_0, x_0, u(t))) \leq V(t_0, y_0 - x_0) \quad t \geq \tau_0 \quad (4.24)$$

We get

$$\begin{aligned} b(A) &= b(D_0[y(t_1, \tau_0, y_0, u(t)) - x^*(t_1, \tau_0, x_0, u(t)), \theta^n]) \\ &\leq V(t_1, y(t_1, \tau_0, y_0, u(t)) - x^*(t_1, \tau_0, x_0, u(t))) \\ &\leq V(t_0, y_0 - x_0) \\ &\leq a(D_0[y_0 - x_0, \theta^n]) \\ &\leq a(\lambda) < b(A) \end{aligned} \quad (4.25)$$

This result is contradiction. Hence we have $y(t, \tau_0, y_0, u(t))$ the solution of (4.7) is practical stable with ITD related to $x^*(t, \tau_0, x_0, u(t))$. If $V(t, y(t) - x^*(t))$ is only decrescent, we get $V(t, y_0 - x_0) \leq a(D_0[y_0 - x_0, \theta^n])$ and choose $\lambda = \lambda(A) > 0$ such that $a(\lambda) < b(A)$. Since λ is now dependent of τ_0 , we have uniformly practical stability of $y(t, \tau_0, y_0, u(t))$ the solution of (4.7) with ITD related to $x^*(t, \tau_0, x_0, u(t))$ solution of (4.6).

Theorem 4.3: Assume that Lyapunov-like function $V(t, y - x^*)$ satisfies the conditions of Lemma 4.1 with

$$D^+V(t, y - x, u) \leq -\mu V(t_0, y_0 - x_0, u(t)) \quad (4.26)$$

Then $y(t, \tau_0, y_0, u(t))$ the solution of FCDE (4.7) is practically quasi stable with ITD related to $x^*(t, \tau_0, x_0, u(t))$ solution of FCDE (4.6) for $t \geq \tau_0$.

Proof: It is clear from (4.26) that we have practical stability of $y(t, \tau_0, y_0, u(t))$ the solution of (4.7) with respect to solution of the system (4.6) for $t \geq \tau_0$. Hence taking $B = \rho$ and designating $\lambda_0 = \lambda_0(\tau_0, \rho) > 0$. We have practical stability with this λ_0 , practical stability holds such that

$$D_0[y(t, \tau_0, y_0, u(t)) - x^*(t, \tau_0, x_0, u(t)), \theta^n] < \rho \text{ providing that} \quad (4.27)$$

$$D_0[y_0 - x_0, \theta^n] < \lambda_0$$

for $t \geq \tau_0 + T$. If $y(t, \tau_0, y_0, u(t))$ the solution of FCDE (4.7) is not practically quasi stable with ITD related to $x^*(t, \tau_0, x_0, u(t))$ solution of FCDE (4.6), there would exist $t_1 > \tau_0 + T$ and with $D_0[y_0 - x_0, \theta^n] < \lambda_0$ satisfying

$$D_0[y(t_1, \tau_0, y_0, u(t)) - x^*(t_1, \tau_0, x_0, u(t)), \theta^n] = B \text{ for } t \in [\tau_0 + T, t_1] \quad (4.28)$$

Consequently, we get from assumption (4.26), the estimate

$$V[t, y(t, \tau_0, y_0, u(t)) - x^*(t, \tau_0, x_0, u(t))] < V(\tau_0, y_0 - x_0) \exp[-\mu(t - \tau_0)] \quad (4.29)$$

$$t \geq \tau_0 + T. \quad \text{Given } B > 0, \text{ we choose } T = T(\tau_0, B) = \frac{1}{\mu} \ln \left[\frac{a(\tau_0, \lambda_0)}{b(B)} \right] + 1.$$

Then we get,

$$\begin{aligned} b(D_0[y(t_1, \tau_0, y_0, u(t)) - x^*(t_1, \tau_0, x_0, u(t)), \theta^n]) &\leq V(t_1, y(t, \tau_0, y_0) \\ &\quad - x^*(t_1, \tau_0, x_0, u(t))) \\ &\leq V(\tau_0, y_0 - x_0) \exp[-\mu(t - \tau_0)] \quad (4.30) \\ &\leq a(\lambda) \exp[-\mu(t - \tau_0)] \\ &< b(B) \end{aligned}$$

This contradiction gives us practically quasi stability of $y(t, \tau_0, y_0, u(t))$ the solution of FCDE (4.7) with ITD related to $x^*(t, \tau_0, x_0, u(t))$ solution of FCDE (4.6). If $V(t, y(t) - x^*(t))$ is only decrescent, we get by, $V(t, y_0 - x_0) \leq a(D_0[y_0 - x_0, \theta^n])$ and choose $\lambda_0 = \lambda_0(A) > 0$ such that $a(\lambda_0) < b(A)$. Since λ is now dependent of τ_0 , we have uniformly practically quasi stability of $y(t, \tau_0, y_0, u(t))$ the solution of FCDE (4.7) with ITD related to $x^*(t, \tau_0, x_0, u(t))$ solution of FCDE (4.6).

Theorem 4.4: Assume that

i) Lyapunov-like function $V(t, y - x^*)$ satisfies the conditions of Lemma 4.1 with

$$D^+V(t, y - x^*) \leq g(t, V(t, y - x^*), U(t, u)) \quad (4.31)$$

ii) $r(t) = r(t, t_0, z_0, v)$ is maximal solution of the scalar differential equation exists on $[t_0, T]$ and $v(t) \in C([t_0, T], \mathbb{R})$ is control function,

$$z' = g(t, z, v), \quad z(t_0) = z_0 \geq 0 \text{ for } t \geq t_0 \quad (4.32)$$

Then, the practical stability properties of the comparison differential equation imply the corresponding practical stability properties of $y(t, \tau_0, y_0, u(t))$ the solution of (4.7) with ITD related to $x^*(t, \tau_0, x_0, u(t))$ solution of (4.6) for $t \geq \tau_0$.

Proof: Suppose that comparison equation (4.32) is practically stable, then for given any (λ, A) with $0 < \lambda < A$ and there exists $\lambda = \lambda(\tau_0, A)$ and $b \in \kappa$ such that $z(t, \tau_0, z_0) < b(A)$ provided that $d_s[z_0, 0] < \lambda, t \geq \tau_0$. We claim that with this λ , practical stability holds such that $D_0[y(t, \tau_0, y_0, u(t)) - x^*(t, \tau_0, x_0, u(t)), \theta^n] < A$ provided that $D_0[y_0 - x_0, \theta^n] < \lambda$ for

$t \geq \tau_0$. If $y(t, \tau_0, y_0, u(t))$ the solution of (4.7) with ITD related to $x^*(t, \tau_0, x_0, u(t))$ solution of (4.6) is not practically stable, there would exist $t_1 > \tau_0$ and with

$D_0[y_0 - x_0, \theta^n] < \lambda$ for $t \geq \tau_0, D_0[y(t_1, \tau_0, y_0, u(t)) - x^*(t_1, \tau_0, x_0, u(t)), \theta^n] = A$ for $t \in [\tau_0, t_1]$.

Choose $z_0 = a(D_0[y_0 - x_0, \theta^n])$, we get the inequality,

$$V(t, y(t) - x^*(t)) \leq r(t, t_0, z_0, v) \quad t \in [\tau_0, t_1] \quad (4.33)$$

So that we have, $b(A) \leq V(t_1, y(t_1, \tau_0, y_0, u(t)) - x^*(t_1, \tau_0, x_0, u(t)))$ $t_1 > \tau_0$. This means that $D_0[y(t) - x^*(t), \theta^n] < \rho$ for $t \in [\tau_0, t_1]$ and hence we have the inequality $V(t_1, y(t_1, \tau_0, y_0, u(t)) - x^*(t_1, \tau_0, x_0, u(t))) \leq r(t_1, \tau_0, z_0, v)$ $t \geq \tau_0$. Then we get

$$\begin{aligned} b(A) &= b(D_0[y(t_1, \tau_0, y_0, u(t)) - x^*(t_1, \tau_0, x_0, u(t)), \theta^n]) \\ &\leq V(t_1, y(t_1, \tau_0, y_0, u(t)) - x^*(t_1, \tau_0, x_0, u(t))) \\ &\leq r(t_1, \tau_0, z_0, v) \\ &\leq r(t_1, \tau_0, a(D_0[y_0 - x_0, \theta^n]), v) \\ &\leq r(t, \tau_0, a(\lambda), v) < b(A) \end{aligned} \quad (4.34)$$

This contradiction gives us practical stability properties of the comparison differential equation imply the corresponding practical stability properties of $y(t, \tau_0, y_0, u(t))$ the solution of (4.7) with ITD related to $x^*(t, \tau_0, x_0, u(t))$ solution of (4.6) for $t \geq \tau_0$. If $V(t, y(t) - x^*(t))$ is only decrescent, we get by

$$\begin{aligned} V(t, y(t) - x^*(t)) &\leq a(D_0[y(t) - x^*(t), \theta^n]); \\ V(t, y_0 - x_0) &\leq a(D_0[y_0 - x_0, \theta^n]) \end{aligned} \quad (4.35)$$

and choose $\lambda = \lambda(A) > 0$ such that $a(\lambda) < b(A)$. Assume that the comparison system is practically quasi-stable. Given any $b(B) > 0$, there exists a $T = T(\tau_0, \epsilon) > 0$ such that

$$0 < z_0 = a(d_s[z_0, 0]) < \lambda \quad \text{implies} \quad z(t, \tau_0, z_0) < b(B), t \geq \tau_0 + T.$$

$$\text{Setting } z_0 = a(d_s[z_0, 0]) < \lambda \text{ and } V(t_0, y_0 - x_0) \leq a(d_s[z_0, 0]) = z_0 < \lambda.$$

$$\begin{aligned} b(D_0[y(t, \tau_0, y_0, u(t)) - x(t, \tau_0, x_0, u(t)), \theta^n]) &\leq V(t, y(t, \tau_0, y_0, u(t)) \\ &\quad - x^*(t, \tau_0, x_0, u(t))) \\ &\leq r(t, \tau_0, z_0) < b(B) \end{aligned} \quad (4.36)$$

for $t \geq \tau_0 + T$. We obtain $D_0[y(t, \tau_0, y_0, u(t)) - x^*(t, \tau_0, x_0, u(t)), \theta^n] < B$ for $t \geq \tau_0 + T$ since $b \in \kappa$. This proves the practically quasi-stability.

4.5. Applications

Application 4.1: We consider unperturbed fuzzy control differential equation (4.6) and perturbed fuzzy control differential equation (4.7). We show that this comparative system is practically stable with ITD by Theorem 4.4.

$$\begin{aligned} D_H x(t) &= -ax(t) + u(t) \\ t \in [0, 0.5] \quad [\gamma]^\alpha &= [\alpha - 1, 1 - \alpha] \quad x(0) = x_0 = 0.3\gamma \end{aligned} \quad (4.37)$$

$$\begin{aligned} D_H y(t) &= -(a + b)y(t) + 2u(t) \\ 1 &\leq (a + b), \quad y(0.1) = y_0 = 0.5\gamma \end{aligned} \quad (4.38)$$

It is easy to see that

$$\begin{aligned} F(s, y(s), u(s)) - f(t, x^*, u(t)) &= -(a + b)y(t) + 2u(t) + ax^*(t) - u(t) \\ &= -a(y(t) - x^*(t)) - by(t) + u(t) \end{aligned} \quad (4.39)$$

We can choose that The Lyapunov-like function $V(t, y(t) - x^*(t)) = D_0[y - x^*, \theta^n]$ and $U(t, u) = D_0[u(t), \theta^n]$ that provides the conditions of Lemma 4.1. It is easily found that

$$\begin{aligned} D^+ V(t, y(t) - x^*(t)) &\leq \sup \left[D_0[F(s, y(s), u(s)) - f(t, x^*, u(t)), \theta^n] \right] \\ &\leq \sup \left[D_0[-a(y(t) - x^*(t)) - by(t) + u(t), \theta^n] \right] \quad (4.40) \\ &= -a(y(t) - x^*(t)) - by(t) + u(t) \\ &\leq aD_0[y(t) - x^*(t), \theta^n] + bD_0[y(t) - x^*(t), \theta^n] \\ &\quad + D_0[u(t), \theta^n] \\ &\leq (a + b)D_0[y(t) - x^*(t), \theta^n] + D_0[u(t), \theta^n] \quad (4.41) \\ &\leq (a + b)(V(t, x) + U(t, u)) \end{aligned}$$

Then we can choose linear control system as comparison equation

$$z' = (a + b)(z + v) \quad z(\tau_0) = z_0 \quad (4.42)$$

$v(t) \in \mathbb{R}_+$ is control function. It can easily be shown that this system is practically stable as the linear control system in Application 3.3. We can choose

$$a(t, V(t, x)) = 2(a + b)V(t, x) \text{ and } b(V(t, x)) = \frac{(a+b)}{2}V(t, x) \quad (4.43)$$

Using Theorem 4.4, we should determine that comparative system (4.37) and (4.38) is practically stable by comparison equation (4.42). It seems that $y(t, \tau_0, y_0, u(t))$ the solution of FCDE (4.38) related to $x^*(t, \tau_0, x_0, u(t)) = x(t - \eta, t_0, x_0, u(t))$ solution of FCDE (4.37) is practically stable with ITD.

5. CONTROLLABILITY PROPERTIES FOR PFCDE RELATED TO UPFCDE WITH ITD

This chapter presents the global and practical controllability of perturbed fuzzy control differential equation related to unperturbed fuzzy control differential equation via Lyapunov function and applications of this comparative system.

5.1. Preliminary Notes and Definitions

Definition 5.1: If there exists $t_1 > t_0$ such that

$$y(t_1) - x(t_1) = y(t_1, \tau_0, y_0, u(t_1)) - x(t_1 - \eta, t_0, x_0, u(t_1)) = \bar{y} - \bar{x}^* \quad (5.1)$$

for any $\bar{y} - \bar{x}^* \in E^n$ while we have have $y(\tau_0) = y(\tau_0, \tau_0, y_0, u(t_0)) = y_0$ solutions of the system (4.7) with ITD related to $x^*(\tau_0) = x(\tau_0 - \eta, t_0, x_0, u(t)) = x_0$ solution of the system (4.6), $u(t) \in E^p$ will be an admissible fuzzy control set. We say that the pair of fuzzy states $(y_0 - x_0, \bar{y} - \bar{x}) \in E^n$ is controllable with ITD by $u(t)$. This comparative system is said to be global controllable with ITD if every pair of fuzzy state $(y_0 - x_0, \bar{y} - \bar{x}) \in E^n$ is a controllable by $u(t)$.

Definition 5.2: $y(t, \tau_0, y_0, u(t))$ the solution of FCDE (4.7) is practical controllable with ITD related to $x^*(t, \tau_0, x_0, u(t)) = x(t - \eta, t_0, x_0, u(t)) = x^*(t)$ solution of FCDE (4.6), if all solutions $y(t) - x^*(t)$ starting in

$$\Omega_1 = \{y_0 - x_0^* \in E^n; D_0[y_0 - x_0^*, \theta^n] < \lambda\} \quad (5.2)$$

are transferred to the region

$$\Omega_2 = \{y(t) - x^*(t) \in E^n; D_0[y(t) - x^*(t), \theta^n] \leq \beta\} \quad (5.3)$$

in a time $T = T(\tau_0, x_0)$ with the given (λ, β) such that $0 < \lambda < A$ and $0 < \beta < A$.

5.2. Main Results of Controllability

Theorem 5.1: Suppose that for IVP of nonlinear fuzzy control differential equation (IVP for NLFCEs) (4.7) and (4.6) satisfy the followings:

i) *The Lyapunov-like function $V(t, y(t) - x^*(t))$ satisfies the conditions of Lemma 4.6;*

ii) *$f, F \in C(\mathbb{R}_+ \times S_\rho \times E^p, E^n)$, $L > 0$ such that*

$$\begin{aligned} D_0[F(t, y_1, u_1) - f(t, x_1, u_1), F(t, y_2, u_2) - f(t, x_2, u_2)] \\ \leq L(D_0[y_1 - x_1, y_2 - x_2] + D_0[u_1, u_2]) \end{aligned} \quad (5.4)$$

iii) *For every pair of set states $(y_0 - x_0, \bar{y} - \bar{x}^*) \in E^n$, $\forall t_1 \geq \tau_0$, $\exists u(t) \in E^p$ such*

$$\int_{\tau_0}^{t_1} D_0[u(s), \theta^p] ds = \frac{D_0[\bar{y} - \bar{x}^*, \theta^n] \exp[-L(t_1 - \tau_0)] - D_0[y_0 - x_0, \theta^n]}{L} \quad (5.5)$$

Then, $y(t, \tau_0, y_0, u(t))$ the solution of FCDE (4.7) is global controllable with ITD related to $x^*(t, \tau_0, x_0, u(t))$ solution of FCDE (4.6).

Proof: Before we choose the controllable set $u(t) \in E^p$ satisfied (5.5), the fuzzy solution of IVP for NLFCEs (4.6) and (4.7) is written in the form

$$x^*(t) = x_0 + \int_{\tau_0}^T f(s, x(s), u(s)) ds, \quad t \in [\tau_0, T], \quad x_0 \in E^n, u_0 \in E^p \quad (5.6)$$

$$y(t) = y_0 + \int_{\tau_0}^T F(s, y(s), u(s)) ds, \quad t \in [\tau_0, T], \quad y_0 \in E^n, u_0 \in E^p \quad (5.7)$$

Then for $t_1 > \tau_0$, $x^*(t_1) = x^*(t_1, \tau_0, x_0, u(t_1)) = x_0 + \int_{\tau_0}^{t_1} f(s, x(s), u(s)) ds$ and

$$y(t_1) = y(t_1, \tau_0, y_0, u(t_1)) = y_0 + \int_{\tau_0}^{t_1} F(s, y(s), u(s)) ds \quad (5.8)$$

$$\begin{aligned}
D_0[y(t_1) - x(t_1), \theta^n] &= D_0[y_0 - x_0, \theta^n] \\
&+ D_0 \left[\int_{\tau_0}^{t_1} F(s, y(s), u(s)) ds - \int_{\tau_0}^{t_1} f(s, x(s), u(s)) ds, \theta^n \right] \\
&\leq D_0[y_0 - x_0, \theta^n] \\
&+ \int_{\tau_0}^{t_1} D_0[F(s, y(s), u(s)) - f(s, x(s), u(s)), \theta^n] ds, \theta^n \\
&\leq D_0[y_0 - x_0, \theta^n] \\
&+ \int_{\tau_0}^{t_1} L(D_0[y(s) - x^*(s), \theta^n] + D_0[u(s), \theta^n]) ds \\
&\leq D_0[y_0 - x_0, \theta^n] + L \left(\int_{\tau_0}^{t_1} D_0[y(s) - x^*(s), \theta^n] ds \right. \\
&\quad \left. + \int_{\tau_0}^{t_1} D_0[u(s), \theta^n] ds \right)
\end{aligned} \quad (5.9)$$

We use equation (5.5) in last inequality then we receive

$$D_0[y(t_1) - x^*(t_1), \theta^n] \leq D_0[\bar{y} - \bar{x}^*, \theta^n] \quad (5.10)$$

The proof is complete, if we prove that $D_0[y(t_1) - x^*(t_1), \theta^n] = D_0[\bar{y} - \bar{x}^*, \theta^n]$. If this is not true, it should be $D_0[y(t_1) - x^*(t_1), \theta^n] < D_0[\bar{y} - \bar{x}^*, \theta^n]$. But in this case we have by Lemma 5.1 $D_0[y(t_1) - x^*(t_1), \theta^n] = \beta_2$. Let $\beta_1 > 0$ and we can choose $\beta_2 = \beta(t, \beta_1)$ such that $a(t_0, \beta_1) < \beta_2$

$$\begin{aligned}
b(D_0[y(t_1) - x^*(t_1), \theta^n]) &= b(\beta_2) \leq V(t_1, y(t_1) - x^*(t_1)) \\
&\leq V(t_0, y(t_0) - x(t_0)) \\
&\leq a(t_0, D_0[y_0 - x_0, \theta^n]) \\
&= a(t_1, \beta_1) < b(\beta_2)
\end{aligned} \quad (5.11)$$

This contradiction proves that $D_0[y(t_1) - x^*(t_1), \theta^n] = D_0[\bar{y} - \bar{x}^*, \theta^n]$. That means the state pair $(y_0 - x_0, \bar{y} - \bar{x}^*) \in E^n$ is controllable by $u(t)$ and $y(t, \tau_0, y_0, u(t))$ the solution of FCDE (4.7) is global controllable with ITD related to $x(t, \tau_0, x_0, u(t))$ solution of FCDE (4.6).

Theorem 5.2 : Assume that

i) $0 < \lambda < A$ and $0 < \beta < A$ are given,

ii) The Lyapunov-like function $V(t, y(t) - x^*(t))$ is locally Lipschitzian in x , the function $Q(V(t, y(t) - x^*(t)))$ where $Q: \mathbb{R}_+^N \rightarrow \mathbb{R}_+$ is continuous, strictly increasing and $Q(0) = 0$ and for $b(\cdot), a(t, \cdot) \in \kappa$

$$b(D_0[y(t) - x^*(t), \theta^n]) \leq Q(V(t, y(t) - x^*(t))) \leq a(t, D_0[y(t) - x^*(t), \theta^n]) \quad (5.12)$$

iii) $g \in C[[\tau_0, T] \times \mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}]$, $g(t, z, v)$ is quasimonotone nondecreasing in w and in v ;

$$D^+V(t, y(t) - x^*(t)) \leq g(t, V(t, y(t) - x^*(t)), U(t, u)) \quad (5.13)$$

iv) $a(t, \lambda) < b(A)$ holds,

v) There exists a control function $v(t) \in \mathbb{R}_+$ such that any solution $z(t, \tau_0, z_0, v)$

$$z' = g(t, z, v(t)) \quad z(\tau_0) = z_0 \geq 0 \quad (5.14)$$

satisfies $Q(z_0) < a(\lambda)$ implies $Q(z(t, \tau_0, z_0, v)) < b(A)$

$$Q(z(\tau_0 + T, \tau_0, z_0, v)) \leq b(\beta) \text{ for some } T = T(\tau_0, z_0) > 0 \quad (5.15)$$

Then, there exist admissible control $u(t) = u \in \Omega$ such that $y(t, \tau_0, y_0, u(t))$ the solution of FCDE (4.7) is practically controllable with ITD related to $x^*(t) = x(t - \eta, t_0, x_0, u(t))$ solution of FCDE (4.6).

Proof: $D_0[y_0 - x_0, \theta^n] < \lambda$, $U(t, u) \leq v(t)$ $t \geq \tau_0$. Then we claim that for any solution $y(t, \tau_0, y_0, u(t))$ the solution of FCDE (4.7) related to $x^*(t)$ solution of FCDE (4.6) we have

$$D_0[y_0 - x_0, \theta^n] < \lambda \text{ implies } D_0[y(t) - x^*(t), \theta^n] < A \text{ for } t \geq \tau_0. \quad (5.16)$$

If this is false, there would exist $u(t) \in \Omega$, $D_0[x_0, \theta^n] < \lambda$ and $D_0[x(t_1), \theta^n] = A$, $D_0[x(t), \theta^n] \leq A$ for $t_1 \geq t \geq \tau_0$.

Setting $m(t) = V(t, y(t) - x^*(t))$ for $t_1 \geq t \geq \tau_0$ and using (iii), we get

$$D^+m(t) \leq g(t, m(t), v(t)) \quad t_1 \geq t \geq \tau_0 \quad (5.17)$$

which implies by Comparison Theorem 4.1. the estimate

$$D^+m(t) \leq r(t, \tau_0, z_0, v) \quad t_1 \geq t \geq \tau_0 \quad (5.18)$$

where $r(t, \tau_0, z_0, v)$ is the maximal solution of (5.14).

Choosing $z_0 = V(\tau_0, y_0 - x_0)$, we then get, because of (ii), the relation

$$b(D_0[y(t) - x^*(t), \theta^n]) \leq Q(V(t, y(t) - x^*(t))) \leq Q(r(t, \tau_0, z_0, v)) \quad (5.19)$$

for $t_1 \geq t \geq \tau_0$. Now we are lead to the following contradiction, in view of the relations (5.12),(5.14).

$$\begin{aligned} b(A) &= b(D_0[y(t_1) - x^*(t_1), \theta^n]) \leq Q(V(t_1, y(t_1) - x^*(t_1))) \\ &\leq Q(r(t_1, \tau_0, z_0, v)) < b(A) \end{aligned} \quad (5.20)$$

for all $t \geq \tau_0$ and $D_0[x(\tau_0 + T^*), \theta^n] \leq \beta$ where $T^*(\tau_0, x_0) = T(\tau_0, V(\tau_0, x_0))$. The proof is complete.

5.3. Applications

Application 5.1: We again consider unperturbed fuzzy control differential equation (4.6) and perturbed fuzzy control differential equation (4.7). We show that this comparative system is global controllable with ITD by Theorem 5.1.

$$\begin{aligned} D_H x(t) &= -\alpha x(t) + u(t) & x(0) &= x_0 = m_1 \gamma \in E^n \\ t &\in [0, 0.3] & [\gamma]^\alpha &= [\alpha - 1, 1 - \alpha] & x^1 &= n_1 \gamma \end{aligned} \quad (5.21)$$

$$\begin{aligned} D_H y(t) &= -(\alpha + \beta)y(t) + 3u(t) & y(\tau_0) &= y_0 = m_2 \gamma \in E^n \\ t &\in [0.1, 0.3] & y^1 &= n_2 \gamma \end{aligned} \quad (5.22)$$

It is easy seen that

$$F(t, y(t), u(t)) - f(t, x^*(t), u(t)) = -\alpha(y(t) - x^*(t)) - \beta y(t) + 2u(t) \quad (5.23)$$

satisfy assumption ii) of Theorem 5.1 with $L = \alpha + \beta$. We can choose that The Lyapunov-like function $V(t, y(t) - x^*(t), u(t)) = D_0[y - x^*, \theta^n]$ that provides the conditions of Lemma 5.1. If we can find an equation that satisfies (5.5), we show that the comparative system is global controllable. We use this equation in fuzzy Laplace transform [40] of solution of (5.21) and (5.22). By Theorem 5.1, for every pair of fuzzy states $(y_0 - x_0, y^1 - x^1) \in E^n, \forall t_1 > \tau_0, \exists u(t) \in E^p$ such that

$$\int_0^t D_0[u(s), \theta^p] ds = \gamma \frac{(n_2 - n_1)e^{-(\alpha+\beta)t} - m_2 - m_1}{(\alpha + \beta)} \quad (5.24)$$

The solutions are follows,

$$\begin{aligned} [x^*]^\alpha &= [\underline{x}_0]^\alpha - \alpha \int_{0.1}^t \underline{x}^{*\alpha}(s) ds + \int_{0.1}^t \underline{u}^\alpha(s) ds, \\ &\quad \overline{x}_0^\alpha - \alpha \int_{0.1}^t \overline{x}^{*\alpha}(s) ds + \int_{0.1}^t \overline{u}^\alpha(s) ds \end{aligned} \quad (5.25a)$$

$$\begin{aligned}
[y]^\alpha &= [\underline{y}_0]^\alpha - (\alpha + \beta) \int_{0.1}^t \underline{y}^\alpha(s) ds + 3 \int_{0.1}^t \underline{u}^\alpha(s) ds, \\
&\quad \overline{y}_0^\alpha - (\alpha + \beta) \int_{0.1}^t \overline{y}^\alpha(s) ds - 3 \int_{0.1}^t \overline{u}^\alpha(s) ds] \quad (5.25b)
\end{aligned}$$

Then, we find solution with Laplace fuzzy transforms [40],

$$\begin{aligned}
[y - x^*]^\alpha &= \gamma [e^{-(\alpha+\beta)t} (m_2 - \frac{3(m_2-m_1)}{\alpha+\beta} - \frac{\alpha}{(\alpha+\beta)\beta} + \frac{(n_2-n_1)t}{\alpha+\beta}) \\
&\quad - e^{-\alpha t} (m_1 - \frac{(m_2-m_1)}{\alpha+\beta} - \frac{\alpha}{(\alpha+\beta)\beta})] \quad (5.26)
\end{aligned}$$

If this solution satisfy $y(t_1) - x^*(t_1) = y^1 - x^1 = \gamma(n_2 - n_1)$, the system is global controllable. We have

$$n_2 - n_1 \cong e^{-3t_1} [6m_1 + 2(n_2 - n_1)t_1 - 1] - e^{-t_1} (8m_1 - 2m_2 - 1) \quad (5.27)$$

where $\alpha = 1$, $\beta = 2$. If we take $m_1 = 0.1, m_2 = 0.9$ and we can find $n_2 - n_1 = 0.2$ for

$t_1 = 0.3$. Thus, we find the global solution

$$[y - x^*]^\alpha = \gamma [e^{-3t} [0.4t - 0.4] + 2e^{-t}] \quad (5.28)$$

for any t_1 and for every pair of set states $(y_0 - x_0, y^1 - x^1) \in E^n$, for $\forall t_1 \geq t_0$,

$$\exists u(t) \in E^p \text{ such that } \int_0^t D_0[u(s), \theta^p] ds = \gamma \frac{(n_2-n_1)e^{-3t}-m_2-m_1}{3}.$$

Application 5.2: We show that this comparative system is practically controllable with ITD by Theorem 5.2

$$\begin{aligned}
D_H x(t) &= -\alpha x(t) + u(t) & x(0) &= x_0 = 0.2\gamma \in E^n \\
t &\in [0, 0.3] & [\gamma]^\alpha &= [\alpha - 1, 1 - \alpha] & x^1 &= 0.6\gamma \quad (5.29)
\end{aligned}$$

$$\begin{aligned}
D_H y(t) &= -(\alpha + \beta)y(t) + 3u(t) & y(\tau_0) &= y_0 = 0.4\gamma \in E^n \\
t &\in [0.1, 0.3] & y^1 &= 0.9\gamma \quad \alpha, \beta \in \mathbb{R}_+ \quad 2 \leq \alpha + \beta
\end{aligned} \tag{5.30}$$

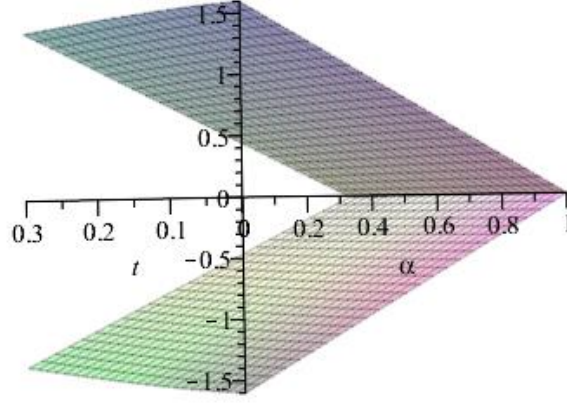


Figure 5.1: The graphic of $\underline{x}^\alpha$ and $\bar{x}^\alpha$ in global controllable solution of equation (5.21)- (5.22).

It is easy showing that

$$F(t, y(t), u(t)) - f(t, x^*(t), u(t)) = -\alpha(y(t) - x^*(t)) - \beta y(t) + 2u(t) \tag{5.31}$$

We can choose that Lyapunov like function $V(t, y - x^*) = D_0[y(t) - x^*(t), \theta^n]$ and $U(t, u) = D_0[u(t), \theta^n]$. It is easily found that

$$\begin{aligned}
D^+ V(t, y(t) - x^*(t)) &\leq (\alpha + \beta)(D_0[y(t) - x^*(t), \theta^n] \\
&\quad + D_0[u(t), \theta^n])
\end{aligned} \tag{5.32}$$

Then we can choose linear control system as comparison equation

$$z' = (\alpha + \beta)z + (\alpha + \beta)v \quad z(\tau_0) = z_0 \tag{5.33}$$

$v(t) \in \mathbb{R}_+$ is control function. It can easily be shown that this system is practical controllable as the linear control system in Application 3.2. Using Theorem 5.2, we should determine that comparative system (5.29) and (5.30) is practical controllable

by comparison equation (5.33). After that, the process can be proceed in a similar way like Application 3.2. It can easily be shown that $y(t, \tau_0, y_0, u(t))$ the solution of FCDE (5.30) related to $x(t - \eta, t_0, x_0, u(t)) = x^*(t)$ solution of FCDE (5.29) is practical controllable with ITD.



6. STABILITY PROPERTIES OF FCVIDE

In this section, we will slightly change our problem, contrary to previous sections. We will work on fuzzy control Volterra-integro differential equations. An integro-differential equation is an equation that involves both integrals and derivatives of a function. They can model many situations from science and engineering, such as circuit analysis [41], [42]. We will first introduce the IVP for fuzzy control Volterra-integro differential equations. Later, we will give the conditions for this problem to be practically stable and examine the Ulam-Hyers stability mentioned earlier in this equation system. In the last part, we will compare the Ulam-Hyers and Lyapunov stability on Fuzzy control Volterra-integro differential equations like Obloza's comparison in 1997. [43].

6.1. Preliminary Notes and Definitions

We handle the initial valued problems of fuzzy control Volterra integro-differential equations (IVP for FCVIDEs) in fuzzy metric space E^n .

$$D_H x(t) = f(t, x(t), u(t)) + \int_{t_0}^t g(t, s, x(s), u(s)) ds \quad x(t_0) = x_0 \in E^n \quad (6.1)$$

where $x(t) \in E^n$, $f: [t_0, T] \times E^n \times E^m \rightarrow E^n$ and $g: [t_0, T] \times [t_0, T] \times E^n \times E^m \rightarrow E^n$ the admissible control $u(t) \in \Omega$ where $\Omega = \{u(t) \in E^p; U(t, u(t)) \leq v(t), t \geq t_0\}$ where $U(t, u) \in C[[t_0, T] \times E^p, \mathbb{R}_+]$ and $v(t) \in \mathbb{R}_+$.

Definition 6.1: [37] We say that problem (6.1) is Ulam-Hyers stable if there exists a real number K_f such that for $\varepsilon > 0$ and for each $v \in C([t_0, T] \times E^n \times E^p, E^n)$ to the problem

$$D_0[D_H v(t), f(t, v(t), u(t))) + \int_{t_0}^t g(t, s, x(s), u(s)) ds] \leq \varepsilon \quad (6.2)$$

There exist a solution to problem (6.1) with $D_0[v(t), x(t)] \leq K_f \varepsilon$ for all $t \in [t_0, T]$.
We call K_f is Ulam-Hyers stability constant of (6.1).

Definition 6.2: [37] We say that a function $v \in C([t_0, T] \times E^n \times E^p, E^n)$ is solution of (6.2) if and only if there exists a function,

- i) $D_0[\delta(t), \theta^n] \leq \varepsilon$ for any $t \in [t_0, T]$.
- ii) $D_H v(t) = f(t, v(t), u(t)) + \int_{t_0}^t g(t, s, x(s), u(s)) ds + \delta(t)$ for $t \in [t_0, T]$.

Lemma 6.1: Consider that

$$f: [t_0, T] \times E^n \times E^m \rightarrow E^n \text{ and } g: [t_0, T] \times [t_0, T] \times E^n \times E^m \rightarrow E^n \quad (6.3)$$

is continuous. The local solution to IVP for FCVIDEs (6.1) on $[t_0, T]$ is a fuzzy mapping $x: [t_0, T] \rightarrow E^n$ if and only if x is a continuous fuzzy mapping and it provide to fuzzy integral equations as follows:

$$x(t) = x_0 + \int_{t_0}^t f(s, x(s), u(s)) ds + \int_{t_0}^t \int_{t_0}^s g(s, z, x(z), u(z)) dz ds \quad t \in [t_0, T] \quad (6.4)$$

We consider the set X of all functions $x \in C([t_0, T], E^n)$ that imply (6.4) for the local existence and uniqueness of fuzzy solution for IVP for FCVIDEs (6.1) with admissible control.

Let the IVP for FCVIDEs (6.1) satisfy;

- i) $f \in C(Q, E^n)$ is levelwise continuous, there exists $L_f > 0$ such that for

$$D_0[f(t, x, u), f(t, y, v)] \leq L_f(D_0[x, y] + D_0[u, v]) \quad t \in [t_0, T] \quad (6.5)$$

for all $x, y \in E^n$, $u, v \in E^m$ and $t \in [t_0, T]$.

ii) B is fixed real number for $\forall u(t) \in E^m$ provide $\int_{t_0}^t D_0[u(s), \theta^n] ds < B$.

iii) $g \in C(\varphi, E^n)$ is levelwise continuous, there exists $L_g > 0$ such that

$$D_0[g(t, x, u), g(t, y, v)] \leq L_g(D_0[x, y] + D_0[u, v]) \quad (6.6)$$

for all $x, y \in E^n$, $u, v \in E^m$ and $t \in [t_0, T]$.

$$Q = [t_0, T] \times B(x_0, a) \times B(u_0, b), \varphi = [t_0, T] \times [t_0, T] \times B(x_0, a) \times B(u_0, b) \quad (6.7)$$

where $a, b > 0$, $B(x_0, a) = \{x(t) \in E^n : D_0[x, x_0] \leq a\}$

$$B(u_0, b) = \{u(t) \in E^m : D_0[u, u_0] \leq b\}, \quad T = \{t : |t - t_0| \leq \sigma \leq s \leq \rho\} \quad (6.8)$$

Theorem 6.2: There exist a unique solution $x(t)$ of (6.1) defined $|t - t_0| \leq \sigma$, if the conditions i)-iii) holds. Besides, there exist $x(t) : [t_0, T] \rightarrow E^n$ such that $D_0[x_n(t), x(t)] \rightarrow 0$ on $|t - t_0| \leq \delta$ as $n \rightarrow \infty$.

Proof: A sequence $x_n(t) \in E^n$ $n = 0, 1, 2, \dots$ of successive approximations is defined with $x(t_0) = x_0$ as follows for $t \in [t_0, T]$:

$$x_n(t) = x_0 + \int_{t_0}^t f(s, x_{n-1}(s), u(s)) ds + \int_{t_0}^t \int_{t_0}^s g(s, z, x_{n-1}(z), u(z)) dz ds \quad (6.9)$$

For $n = 1$,

$$x_1(t) = x_0 + \int_{t_0}^t f(s, x_0(s), u(s)) ds + \int_{t_0}^t \int_{t_0}^s g(s, z, x_0(z), u(z)) dz ds \quad (6.10)$$

$t \in [t_0, T]$, $x(t_0) = x_0$. Moreover we observe that,

$$\begin{aligned}
\int_{t_0}^t D_0[f(t, x_0(s), u(s)), \theta^n] ds &\leq \int_{t_0}^t L_f(D_0[x_0, \theta^n] + D_0[u(s), \theta^n]) ds \\
&\leq |t - t_0| L_f(D_0[x_0, \theta^n] + B) \\
\int_{t_0}^t \int_{t_0}^s D_0[g(t, s, x_0(z), u(z)), \theta^n] ds dz & \\
&\leq \int_{t_0}^t \int_{t_0}^s L_g(D_0[x_0, \theta^n] + D_0[u(z), \theta^n]) dr ds \\
&\leq \frac{(t-t_0)^2}{2!} L_g(D_0[x, \theta^n] + B)
\end{aligned} \tag{6.11}$$

We have the following evaluation from these results and by Lemma 3.1 [44].

$$\begin{aligned}
D_0[x_1, x_0] &= D_0[x_0 + \int_{t_0}^t f(s, x_0(s), u(s)) ds \\
&\quad + \int_{t_0}^t \int_{t_0}^s g(s, z, x_0(z), u(z)) dz ds, x_0] \\
&\leq \int_{t_0}^t D_0[f(t, x_0(s), u(s)), \theta^n] ds \\
&\quad + \int_{t_0}^t \int_{t_0}^s D_0[g(t, s, x_0(z), u(z)), \theta^n] dr ds \\
&\leq |t - t_0| L_f(D_0[x_0, \theta^n] + B) + \frac{(t-t_0)^2}{2!} L_g(D_0[x, \theta^n] + B) \\
&\leq M L_{fg} \left(|t - t_0| + \frac{(t-t_0)^2}{2!} \right)
\end{aligned} \tag{6.12}$$

where $L_{f,g} = \text{Max}\{L_f, L_g\}$ and $M = D_0[x_0, \theta^n] + B$. Similarly,

$$\begin{aligned}
D_0[x_2, x_1] &\leq M(L_{fg})^2 \left(\frac{(t-t_0)^2}{2!} + 2 \frac{(t-t_0)^3}{3.2!} + \frac{(t-t_0)^4}{4.3.2!} \right) \\
D_0[x_3, x_2] &\leq M(L_{fg})^3 \left(\frac{(t-t_0)^3}{3!} + 3 \frac{(t-t_0)^4}{4!} + 3 \frac{(t-t_0)^5}{5!} + \frac{(t-t_0)^6}{6!} \right)
\end{aligned} \tag{6.13}$$

and consequently

$$\begin{aligned}
D_0[x_{n+1}, x_n] &\leq M(L_{fg})^{n+1} \left(\binom{n+1}{0} \frac{(t-t_0)^{n+1}}{n+1!} + \binom{n+1}{1} \frac{(t-t_0)^{n+2}}{n+2!} \right. \\
&\quad \left. + \cdots + \binom{n+1}{n+1} \frac{(t-t_0)^{2n+2}}{2n+2!} \right) \\
&= M(L_{fg})^{n+1} \sum_{k=0}^{n+1} \binom{n+1}{k} \frac{(t-t_0)^{n+1+k}}{n+1+k!} \\
&\leq M\lambda(L_{fg})^{n+1} \frac{(t-t_0)^{n+1}}{n+1!} \sum_{k=0}^{n+1} \frac{(t-t_0)^k}{k!} \\
&\leq M\lambda(L_{fg})^{n+1} \frac{\sigma^{n+1}}{n+1!} \sum_{k=0}^{n+1} \frac{\sigma^k}{k!}
\end{aligned} \tag{6.14}$$

where $\text{Max}\left\{\binom{n+1}{k}\right\} = \lambda$. For every $w \in \mathbb{R}$, $\sum_{m=1}^{\infty} \frac{w^m}{m!}$ is convergent, so $D_0[x_{n+1}, x_n] \rightarrow 0$ uniformly converges on $|t - t_0| \leq \sigma$ as $n \rightarrow \infty$. Then we should prove that limit is solution to (6.1). If $\{x_n(t)\}$ satisfies (6.1), we have

$$\begin{aligned}
D_0 \left[x(t), x_0 + \int_{t_0}^t f(s, x(s), u(s)) ds + \int_{t_0}^t \int_{s_0}^s g(s, z, x(z), u(z)) dz ds \right] &= I \\
I &\leq D_0[x(t), x_n(t)] + D_0[x_n(t), x_0 + \int_{t_0}^t f(s, x_{n-1}(s), u(s)) ds \\
&\quad + \int_{t_0}^t \int_{s_0}^s g(s, z, x_{n-1}(z), u(z)) dz ds] \tag{6.15} \\
&\quad + D_0[x_0 + \int_{t_0}^t f(s, x_{n-1}(s), u(s)) ds + \int_{t_0}^t \int_{s_0}^s g(s, z, x_{n-1}(z), u(z)) dz ds, \\
&\quad x_0 + \int_{t_0}^t f(s, x(s), u(s)) ds + \int_{t_0}^t \int_{s_0}^s g(s, z, x(z), u(z)) dz ds]
\end{aligned}$$

We see $D_0[x(t), x_n(t)] \rightarrow 0$ uniformly converges and

$$D_0 \left[x_n(t), x_0 + \int_{t_0}^t f(s, x_{n-1}(s), u(s)) ds + \int_{t_0}^t \int_{s_0}^s g(s, z, x_{n-1}(z), u(z)) dz ds \right] \rightarrow 0 \quad (6.16)$$

Let's look at the last term, similarly

$$\begin{aligned} D_0 [x_0 + \int_{t_0}^t f(s, x_{n-1}(s), u(s)) ds + \int_{t_0}^t \int_{s_0}^s g(s, z, x_{n-1}(z), u(z)) dz ds, \\ x_0 + \int_{t_0}^t f(s, x(s), u(s)) ds + \int_{t_0}^t \int_{s_0}^s g(s, z, x(z), u(z)) dz ds] \quad (6.17) \\ \leq 2L_{fg} \left(\frac{(t-t_0)}{1!} + \frac{(t-t_0)^2}{2!} \right). \end{aligned}$$

$$D_0[x(t), x_{n-1}(t)] \rightarrow 0 \quad (6.18)$$

$$\text{Therefore, } D_0 \left[x(t), x_0 + \int_{t_0}^t f(s, x(s), u(s)) ds + \int_{t_0}^t \int_{s_0}^s g(s, z, x(z), u(z)) dz ds \right] = 0.$$

For uniqueness, assume that we have two solutions $x(t), y(t)$ of (6.1) with $x_0 = y_0$. We have to prove that $D_0[x(t), y(t)] = 0$. We have,

$$\begin{aligned} D_0[x(t), y(t)] &= D_0 \left[x_0 + \int_{t_0}^t f(s, x(s), u(s)) ds \right. \\ &\quad \left. + \int_{t_0}^t \int_{s_0}^s g(s, z, x(z), u(z)) dz ds, \right. \\ &\quad \left. y_0 + \int_{t_0}^t f(s, y(s), u(s)) ds + \int_{t_0}^t \int_{s_0}^s g(s, z, y(z), u(z)) dz ds \right] \quad (6.19) \\ &\leq D_0[x_0, y_0] + L_f \int_{t_0}^t D_0[x(s), y(s)] ds + L_g \int_{t_0}^t D_0[x(s), y(s)] ds \\ &\leq D_0[x_0, y_0] + 2L_{fg} \int_{t_0}^t D_0[x(s), y(s)] ds \end{aligned}$$

We get $D_0[x(t), y(t)] \leq D_0[x_0, y_0] \exp 2L_{fg}$ by using Gronwall's inequality, then $D_0[x_0, y_0] = 0$ implies $D_0[x(t), y(t)] = 0$.

Theorem 6.3: Let $u(t) \in \Omega$ be an admissible control. Suppose that the following conditions hold:

i) Let $f \in C([t_0, t_0 + p] \times B(x_0, a) \times B(u_0, b), E^n)$,

$$g \in C([t_0, t_0 + p] \times [t_0, t_0 + p] \times B(x_0, a) \times B(u_0, b), E^n) \quad (6.20)$$

be a continuous function such that

$$D_0[f(t, x, u), \theta^n] \leq M_1, D_0[g(t, s, x, u), \theta^n] \leq M_2 \quad (6.21)$$

ii) Let

$$F \in C([t_0, t_0 + p] \times [0, a], \mathbb{R}^+), G \in C([t_0, t_0 + p] \times [t_0, t_0 + p] \times [0, a], \mathbb{R}^+) \quad (6.22)$$

such that $F(t, 0) \equiv 0$ and $G(t, s, 0) \equiv 0$ and

$$0 \leq F(t, k) \leq M_3 \text{ and } 0 \leq G(t, s, k) \leq M_4 \quad \forall t \in [t_0, t_0 + p], 0 \leq k \leq a \quad (6.23)$$

such that $F(t, k), G(t, s, k)$ is nondecreasing in k and the initial value problem

$$\frac{dk}{dt} = F(t, k(t)) + \int_{t_0}^t G(t, s, k(s)) ds, \quad k(t_0) = 0 \quad (6.24)$$

has only the solution $k(t) \equiv 0$ on $[t_0, T]$,

iii) $D_0[f(t, x, u), f(t, y, v)] \leq F(t, D_0[x, y] + D_0[u, v])$,

$$D_0[g(s, t, x, u), g(s, t, y, v)] \leq G(t, s, D_0[x, y] + D_0[u, v]) \quad D_0[x, y] \leq M_5, \quad (6.25)$$

vi) There exists $a > 0$ such that for $t \in [t_0, t_0 + p]$ given by $x_0(t) = x_0$
the sequence $x_m: [t_0, t_0 + p] \rightarrow E^n$

$$x_{m+1}(t) = x_0 + \int_{t_0}^t f(s, x_m(s), u(s)) ds + \int_{t_0}^t \int_{t_0}^s g(s, z, x_m(z), u(z)) dz ds \quad (6.26)$$

is defined for $m \in \mathbb{N}$.

$$x: [t_0, t_0 + l] \rightarrow B(x_0, a) \text{ where } l = \min\{p, a/M_1, a/M_2, a/M_3, a/M_4\} \quad (6.27)$$

Then, there is unique solution of IVP and successive iterations $x_{m+1}(t)$ converge to solution $x(t)$ satisfied that the IVP for FCDEs exists.

Proof: By inductive method, we get for $t \in [t_0, t_0 + l]$ $m = 0, 1, 2, \dots$

$$\begin{aligned} D_0[x_{m+1}(t), x_0] &\leq \int_{t_0}^t D_0[f(s, x_m(s), u(s)), \theta^n] ds \\ &\quad + \int_{t_0}^t \int_{t_0}^s D_0[g(s, z, x_m(z), u(z)), \theta^n] dz ds \quad (6.28) \\ &\leq M_1(t - t_0) + \frac{M_2}{2!} (t - t_0)^2 \leq \max\{M_1, M_2\} \left[(t - t_0) + \frac{(t - t_0)^2}{2!} \right] \end{aligned}$$

For $t_0 \leq t_1 \leq t_2 \leq t_0 + l$, $m = 0, 1, 2, \dots$ we have

$$\begin{aligned}
D_0[\tilde{x}_m(t_1), \tilde{x}_m(t_2)] &\leq D_0[x_0 + \int_{t_0}^{t_1} f(s, x_{m-1}(s), u(s))ds \\
&\quad + \int_{t_0}^{t_1} \int_{t_0}^s g(s, z, x_{m-1}(z), u(z))dzds, \\
&\quad x_0 + \int_{t_0}^{t_2} f(s, x_{m-1}(s), u(s))ds \\
&\quad + \int_{t_0}^{t_2} \int_{t_0}^s g(s, z, x_{m-1}(z), u(z))dzds] \quad (6.29) \\
&\leq \int_{t_1}^{t_2} D_0[f(s, x_{m-1}(s), u(s)), \theta^n]ds \\
&\quad + \int_{t_1}^{t_2} \int_{t_0}^s D_0[g(s, z, x_m(z), u(z)), \theta^n]dzds \\
&\leq M_1(t_2 - t_1) + \frac{M_2}{2!}(t_2 - t_1)^2 \\
&\leq \max\{M_1, M_2\} \left[(t_2 - t_1) + \frac{(t_2 - t_1)^2}{2!} \right] \leq \varepsilon
\end{aligned}$$

provided $(t_2 - t_1) < \delta$ and $x_m(t)$ is continuous on $[t_0, t_0 + l]$.

Hence $x_{m+1} \in C([t_0, t_0 + l], B(x_0, a))$. The Hukuhara differences exist $x_{m+1}(t + h) - x_{m+1}(t)$ and

$$\begin{aligned}
x_{m+1}(t + h) - x_{m+1}(t) &= \left[\int_t^{t+h} f(s, x_m(s), u(s))ds \right. \\
&\quad \left. + \int_t^{t+h} \int_{t_0}^s g(s, z, x_m(z), u(z))dzds \right] \quad (6.30)
\end{aligned}$$

Hence we deduce that $(x_{m+1})'(t) = f(s, x_m(t), u(t)) + \int_{t_0}^t g(s, z, x_m(z), u(z))dzds$.

Let $M = \max\{M_1, M_2, M_3, M_4\}$. It is defined that the sequence $\{k^m\}_{m=1}^\infty$, $k^m: [t_0, t_0 + l] \rightarrow [0, a]$ of successive approximations of (6.23) as following:

$$k^0(t) = M \left[(t - t_0) + \frac{(t - t_0)^2}{2!} \right] \quad (6.31)$$

$$k^{m+1}(t) = \int_{t_0}^t F(t, k^m(s)) ds + \int_{t_0}^t \int_{t_0}^s G(s, z, k^m(z)) dz ds, t \in [t_0, t_0 + l] \quad (6.32)$$

We take

$$\begin{aligned} k^1(t) &= \int_{t_0}^t F(t, k^0(s)) ds + \int_{t_0}^t \int_{t_0}^s G(s, z, k^0(z)) dz ds \\ &\leq M_3(t - t_0) + M_4 \frac{(t - t_0)^2}{2!} \\ &\leq M \left[(t - t_0) + \frac{(t - t_0)^2}{2!} \right] \\ &= k^0(t) \leq a \end{aligned} \quad (6.33)$$

Therefore, the function $F(t, k)$ and $G(t, s, k)$ is non-decreasing on k by the inductive method, we obtain $0 \leq k^{m+1}(t) \leq k^m(t) \leq a$. Besides,

$$\left| \frac{dk^{m+1}(t)}{dt} \right| = \left| F(t, k(t)) + \int_{t_0}^t G(t, s, k(s)) ds \right| \leq M_3 + M_4(t - t_0) \quad (6.34)$$

and moreover, we can obtain by Ascoli-Arzela theorem $\{k^m(t)\}$ that $\lim_{m \rightarrow \infty} k^m(t) = k(t)$ uniformly on $[t_0, t_0 + l]$ and

$$k(t) = \int_{t_0}^t F(s, k(s)) ds + \int_{t_0}^t \int_{t_0}^s G(s, z, k(z)) dz ds \quad (6.35)$$

Thus, $k \in C([t_0, T], [0, a])$ and $k(t)$ is the solution of the IVP (6.4). From condition (b), we get $k(t) \equiv 0$. Moreover, we have

$$\begin{aligned}
D_0[x_1(t), x_0(t)] &\leq \int_{t_0}^t D_0[f(s, x_0(s), u(s)), \theta^n] ds \\
&\quad + \int_{t_0}^t \int_{t_0}^s D_0[g(s, z, x_0(z), u(z)), \theta^n] dz ds \quad (6.36) \\
&\leq M \left[(t - t_0) + \frac{(t - t_0)^2}{2!} \right] \leq k^0(t)
\end{aligned}$$

$$\begin{aligned}
D_0[x_2(t), x_1(t)] &\leq D_0 \left[\int_{t_0}^t f(s, x_1(s), u(s)) ds \right. \\
&\quad + \int_{t_0}^t \int_{t_0}^s g(s, z, x_1(z), u(z)) dz ds, \int_{t_0}^t f(s, x_0(s), u(s)) ds \\
&\quad \left. + \int_{t_0}^t \int_{t_0}^s g(s, z, x_0(z), u(z)) dz ds \right] \\
&\leq D_0 \left[\int_{t_0}^t f(s, x_1(s), u(s)) ds, \int_{t_0}^t f(s, x_0(s), u(s)) ds \right] \\
&\quad + D_0 \left[\int_{t_0}^t \int_{t_0}^s g(s, z, x_1(z), u(z)) dz ds, \int_{t_0}^t \int_{t_0}^s g(s, z, x_0(z), u(z)) dz ds \right] \quad (6.37) \\
&\leq \int_{t_0}^t D_0[f(s, x_1(s), u(s)), f(s, x_0(s), u(s))] ds \\
&\quad + \int_{t_0}^t \int_{t_0}^s D_0[g(s, z, x_1(z), u(z)), g(s, z, x_0(z), u(z))] dz ds \\
&\leq \int_{t_0}^t F(t, D_0[x_1, x_0]) ds + \int_{t_0}^t \int_{t_0}^s G(t, s, D_0[x_1, x_0]) dz ds \\
&\leq \int_{t_0}^t F(t, k^0(t)) ds + \int_{t_0}^t \int_{t_0}^s G(t, s, k^0(t)) dz ds = k^1(t)
\end{aligned}$$

Suppose that $D_0[x_i(t), x_{i-1}(t)] \leq k^{i-1}(t)$ and

$$\begin{aligned}
D_0[x_{i+1}(t), x_i(t)] &\leq \int_{t_0}^t D_0[f(s, x_i(s), u(s)), f(s, x_{i-1}(s), u(s))]ds \\
&\quad + \int_{t_0}^t \int_{t_0}^s D_0[g(s, z, x_i(z), u(z)), g(s, z, x_{i-1}(z), u(z))]dzds \\
&\leq \int_{t_0}^t F(t, D_0[x_i, x_{i-1}])ds + \int_{t_0}^t \int_{t_0}^s G(t, s, D_0[x_i, x_{i-1}])dzds \\
&\leq \int_{t_0}^t F(t, k^{i-1}(t))ds + \int_{t_0}^t \int_{t_0}^s G(t, s, k^{i-1}(t))dzds = k^i(t)
\end{aligned} \tag{6.38}$$

So, we obtain by the inductive method: $D_0[x_{m+1}(t), x_m(t)] \leq k^m(t)$. Thus, we have

$$\begin{aligned}
D_0[(x_{m+1})'(t), (x_m)'(t)] &\leq D_0[f(t, x_m(t), u(t)), f(t, x_{m-1}(t), u(t))] \\
&\quad + \int_{t_0}^t D_0[g(t, s, x_m(s), u(s)), g(t, s, x_{m-1}(s), u(s))]ds \\
&\leq F(t, D_0[x_m, x_{m-1}]) + \int_{t_0}^t G(t, s, D_0[x_m, x_{m-1}]) \\
&\leq F(t, k^{m-1}(t)) + \int_{t_0}^t G(t, s, k^{m-1}(t))ds
\end{aligned} \tag{6.39}$$

Consider $m \geq n$, the one easily get

$$\begin{aligned}
D_0[(x_n)'(t), (x_m)'(t)] &\leq D_0[(x_n)'(t), (x_{n+1})'(t)] \\
&\quad + D_0[(x_{n+1})'(t), (x_{m+1})'(t)] \\
&\quad + D_0[(x_{m+1})'(t), (x_m)'(t)] \\
&\leq \left[F(t, k^{n-1}(t)) + \int_{t_0}^t G(t, s, k^{n-1}(t))ds \right] \\
&\quad + F(t, D_0[x_n, x_m]) + \int_{t_0}^t G(t, s, D_0[x_n, x_m])ds \\
&\quad + \left[F(t, k^{m-1}(t)) + \int_{t_0}^t G(t, s, k^{m-1}(t))ds \right]
\end{aligned} \tag{6.40}$$

Moreover, we get the Dini derivative D^+ of the function $D_0[x_n(t), x_m(t)]$ as following:

$$\begin{aligned}
D^+ D_0[x_n(t), x_m(t)] &\leq \lim_{h \rightarrow 0^+} \frac{1}{h} D_0[x_n(t) - x_n(t+h), x_m(t) - x_m(t+h)] \\
&\leq \lim_{h \rightarrow 0^+} D_0 \left[\frac{x_n(t) - x_n(t+h)}{h}, \frac{x_m(t) - x_m(t+h)}{h} \right] \\
&= D_0[(x_n)'(t), (x_m)'(t)] \\
&\leq F(t, k^{n-1}(t)) + \int_{t_0}^t G(t, s, k^{n-1}(t)) ds \\
&\quad + F(t, D_0[x_n(t), x_m(t)]) + \int_{t_0}^t G(t, s, D_0[x_n(t), x_m(t)]) ds \\
&\quad + \left[F(t, k^{m-1}(t)) + \int_{t_0}^t G(t, s, k^{m-1}(t)) ds \right]
\end{aligned} \tag{6.41}$$

Since,

$$F(t, k^{n-1}(t)) + \int_{t_0}^t G(t, s, k^{n-1}(t)) ds \quad F(t, k^{m-1}(t)) + \int_{t_0}^t G(t, s, k^{m-1}(t)) ds \tag{6.42}$$

uniformly converges to 0, there exists all number n_0 such that for arbitrary $\varepsilon > 0$,

$$\begin{aligned}
D^+ D_0[x_n(t), x_m(t)] &\leq F(t, D_0[x_n(t), x_m(t)]) \\
&\quad + \int_{t_0}^t G(t, s, D_0[x_n(t), x_m(t)]) ds + \varepsilon
\end{aligned} \tag{6.43}$$

$m \geq n \geq n_0$ From the fact that $D_0[x_n(t_0), \tilde{x}_m(t_0)] = 0 < \varepsilon$ and by proposition 2.3 in [44] we have $D_0[x_n(t), x_m(t)] \leq k_\varepsilon(t)$ where $k_\varepsilon(t)$ is the maximal solution to the next IVP for each m :

$$\frac{dk_\varepsilon(t)}{dt} = F(t, k_\varepsilon(t)) + \int_{t_0}^t G(t, s, k_\varepsilon(s)) ds + \varepsilon \tag{6.44}$$

By Proposition 2.2 in [44] one can deduce that $\{k_\varepsilon(\cdot)\}$ converges uniformly to the maximal solution $k(t) \equiv 0$ as $\varepsilon \rightarrow 0$. Thus, we obtain that there exist a fuzzy function $x: [t_0, T] \rightarrow B(x_0, a)$ such that $D_0[x_n(t), x_m(t)]$ uniformly converges to 0 as $m \rightarrow \infty$. We prove the uniqueness of solution of the IVP for FCVIDES (6.1). Assume $y(t)$ is solution of IVP for FCVIDES (6.1) on $[t_0, T]$ too. Set $m(t) = D_0[x(t), y(t)]$, then $m(t_0) = 0$ and

$$\begin{aligned} D^+m(t) &= D^+D_0[(x)'(t), (y)'(t)] \leq F(t, D_0[x(t), y(t)]) \\ &\quad + \int_{t_0}^t G(t, s, D_0[x(t), y(t)])ds \\ &\leq F(t, m(t)) + \int_{t_0}^t G(t, s, m(t))ds \end{aligned} \quad (6.45)$$

Thus from Proposition 2.3 in [44], we have $m(t) \leq k(t)$ for all $t \in [t_0, T]$, where $k(t)$ maximal solution of (6.44). By condition we have $k(t) \equiv 0$. Thus $x(t) \equiv y(t)$. The proof is done.

We take comparison system to estimate the stability properties of $x(t, t_0, x_0, u(t))$ is solution of the system (6.1).

Theorem 6.4:

i) Let $f: [t_0, T] \times E^n \times E^m \rightarrow E^n$ and $g: [t_0, T] \times [t_0, T] \times E^n \times E^m \rightarrow E^n$ is continuous.

$$\begin{aligned} \limsup_{h \rightarrow 0^+} \frac{1}{h} [D_0x + h((f(t, x, u) + \int_{t_0}^t g(t, s, x(s), u(s))ds)), \theta^n] \\ - D_0[x, \theta^n] \leq F(t, D_0[x, \theta^n]) + \int_{t_0}^t G(t, s, D_0[x, \theta^n])ds \end{aligned} \quad (6.46)$$

where

$$F \in C([t_0, t_0 + p] \times [0, a], \mathbb{R}^+) \quad G \in C([t_0, t_0 + p] \times [t_0, t_0 + p] \times [0, a], \mathbb{R}^+) \quad (6.47)$$

such that $F(t, 0) \equiv 0$ and $G(t, s, 0) \equiv 0$.

ii) Let $\forall t \in [t_0, t_0 + p], 0 \leq k \leq a$, such that $F(t, k), G(t, s, k)$ is nondecreasing in k and the initial value problem

$$\frac{dk}{dt} = F(t, k(t)) + \int_{t_0}^t G(t, s, k(s))ds, \quad k(t_0) = 0 \quad (6.48)$$

has only the solution $k(t) \equiv 0$ on $[t_0, T]$.

Then, we have,

$$D_0[x(t, t_0, x_0, u(t)), \theta^n] \leq r(t, t_0, k_0) \text{ providing that } D_0[x_0, \theta^n] \leq r_0 \quad (6.49)$$

Proof: The H -difference of $x(t + h) - x(t)$ exist for small h . Define for $t \in [t_0, T]$ $m(t) = D_0[x, \theta^n]$ and we have,

$$m(t + h) - m(t) = D_0[x(t + h), \theta^n] - D_0[x(t), \theta^n] \quad (6.50)$$

For D_0 , by using the triangular inequality, we conclude that

$$\begin{aligned} D_0[x(t + h), \theta^n] &\leq D_0[x(t + h), x(t)] \\ &\quad + h((f(t, x, u) + \int_{t_0}^t g(t, s, x(s), u(s))ds)) \quad (6.51) \\ &\quad + D_0[x(t) + h((f(t, x, u) + \int_{t_0}^t g(t, s, x(s), u(s))ds)), \theta^n] \end{aligned}$$

Thus,

$$\begin{aligned}
\frac{m(t+h)-m(t)}{h} &\leq \frac{1}{h} D_0 \left[x(t+h) - x(t), h \left(\left(f(t, x, u) + \int_{t_0}^t g(t, s, x(s), u(s)) ds \right) \right) \right] \\
&+ \frac{1}{h} [D_0 \left[x(t) + h \left(\left(f(t, x, u) + \int_{t_0}^t g(t, s, x(s), u(s)) ds \right) \right), \theta^n \right] - \frac{1}{h} D_0[x(t), \theta^n]]
\end{aligned} \tag{6.52}$$

Because of the properties of D_0 , we get,

$$\begin{aligned}
D^+m(t) &= \limsup_{h \rightarrow 0^+} \frac{m(t+h) - m(t)}{h} \\
&\leq \limsup_{h \rightarrow 0^+} D_0 \left[\frac{x(t+h)-x(t)}{h}, \left(f(t, x, u) + \int_{t_0}^t g(t, s, x(s), u(s)) ds \right) \right] \\
&+ \limsup_{h \rightarrow 0^+} \frac{1}{h} D_0 \left[x(t) + h \left(f(t, x, u) + \int_{t_0}^t g(t, s, x(s), u(s)) ds \right), \theta^n \right] \\
&\quad - \limsup_{h \rightarrow 0^+} \frac{1}{h} D_0[x(t), \theta^n]
\end{aligned} \tag{6.53}$$

So,

$$\begin{aligned}
D^+m(t) &\leq F(t, D_0[x(t), \theta^n]) + \int_{t_0}^t G(t, s, D_0[x(t), \theta^n]) ds \\
&= F(t, m(t)) + \int_{t_0}^t G(t, s, m(t)) ds
\end{aligned} \tag{6.54}$$

and by comparison result in Theorem 1.4.1 [23]. Therefore, we have

$$m(t) = D_0[x(t, t_0, x_0, u(t)), \theta^n] \leq r(t, t_0, k_0) \text{ providing } D_0[x_0, \theta^n] \leq k_0 \tag{6.55}$$

Corollary 6.1: The function $F(t, k) \equiv 0$ and $G(t, s, k) \equiv 0$ is acceptable in Theorem 6.3 to estimate $D_0[x(t, t_0, x_0, u(t)), \theta^n] \leq D_0[x_0, \theta^n]$.

6.2. Main Results of Stability

We consider that FCVIDE in equation (6.1) has the trivial solution, is that case $f(t, \theta^n, u(t)) = \theta^n$ and $g(t, s, \theta^n, u(t)) = \theta^n$.

Theorem 6.5: Let $f: [t_0, T] \times E^n \times E^m \rightarrow E^n$ and $g: [t_0, T] \times [t_0, T] \times E^n \times E^m \rightarrow E^n$ is continuous and where $S_\rho = [x \in E^n: D_0[x, \theta^n] < \rho]$.

i) Let $f: [t_0, T] \times E^n \times E^m \rightarrow E^n$ and $g: [t_0, T] \times [t_0, T] \times E^n \times E^m \rightarrow E^n$ is continuous.

$$\begin{aligned} \lim_{h \rightarrow 0^+} \sup \frac{1}{h} [D_0[x + h(f(t, x, u) + \int_{t_0}^t g(t, s, x(s), u(s))ds), \theta^n] - D_0[x, \theta^n]] \\ \leq F(t, D_0[x, \theta^n]) + \int_{t_0}^t G(t, s, D_0[x, \theta^n])ds \end{aligned} \quad (6.56)$$

where

$$F \in C([t_0, t_0 + p] \times [0, a], \mathbb{R}^+) \quad G \in C([t_0, t_0 + p] \times [t_0, t_0 + p] \times [0, a], \mathbb{R}^+) \quad (6.57)$$

such that $F(t, 0) \equiv 0$ and $G(t, s, 0) \equiv 0$

ii) $0 \leq F(t, k) \leq M_3$ and $0 \leq G(t, s, k) \leq M_4 \forall t \in [t_0, t_0 + p], 0 \leq k \leq a$, such that $F(t, k), G(t, s, k)$ is nondecreasing in k and the initial value problem

$$\frac{dk}{dt} = F(t, k(t)) + \int_{t_0}^t G(t, s, k(s))ds, \quad k(t_0) = 0 \quad (6.58)$$

has only the solution $k(t) \equiv 0$ on $[t_0, T]$

Then, the stability properties of trivial solution of scalar integro-differential equation show the same stability features of trivial solution of fuzzy differential equation (6.1).

Proof: Assume that the trivial solution of (6.9) with $F(t, k(t)) = G(t, s, k(s)) = 0$, be stable. So, given $\epsilon > 0$ and $t_0 \in \mathbb{R}_+$, there exists a positive $\delta = \delta(t_0, \epsilon)$ with the property $0 \leq w_0 \leq \delta$ provide $w(t, t_0, w_0) < \epsilon$, $t \geq t_0$ where $w(t, t_0, w_0)$ is any solution of scalar integro-differential equation. Let claim that δ the trivial solution $x(t) = 0$ of (6.1) is stable with these ϵ . If we get it wrong, with $D_0[x_0, \theta^n] < \delta$ and $t_1 > t_0$ such that $D_0[x_1, \theta^n] = \epsilon$ and $D_0[x, \theta^n] \leq \epsilon < \rho$, $t_0 \leq t \leq t_1$ there would exist a solution $x(t) = x(t, t_0, x_0, u(t))$ of (6.1). For $[t_0, t_1]$ by Theorem 6.3 and the estimate $D_0[x, \theta^n] \leq r(t, t_0, D_0[x_0, \theta^n]) < \epsilon$ so our assumption is correct.

Theorem 6.6:

i) Let $f: [t_0, T] \times E^n \times E^m \rightarrow E^n$ and $g: [t_0, T] \times [t_0, T] \times E^n \times E^m \rightarrow E^n$ is continuous.

$$\begin{aligned} \lim_{h \rightarrow 0^+} \sup \frac{1}{h} \left[D_0 \left[x + h \left(f(t, x, u) + \int_{t_0}^t g(t, s, x(s), u(s)) ds \right), \theta^n \right] - D_0[x, \theta^n] \right] \\ \leq F(t, D_0[x, \theta^n]) + \int_{t_0}^t G(t, s, D_0[x, \theta^n]) ds \end{aligned} \quad (6.59)$$

Where

$$F \in C([t_0, t_0 + p] \times [0, a], \mathbb{R}^+) \quad G \in C([t_0, t_0 + p] \times [t_0, t_0 + p] \times [0, a], \mathbb{R}^+) \quad (6.60)$$

such that $F(t, 0) \equiv 0$ and $G(t, s, 0) \equiv 0$

ii) $0 \leq F(t, k) \leq M_3$ and $0 \leq G(t, s, k) \leq M_4 \quad \forall t \in [t_0, t_0 + p], 0 \leq k \leq a$, such that $F(t, k), G(t, s, k)$ is nondecreasing in k and the initial value problem

$$\frac{dk}{dt} = F(t, k(t)) + \int_{t_0}^t G(t, s, k(s)) ds, \quad k(t_0) = 0 \quad (6.61)$$

has only the solution $k(t) \equiv 0$ on $[t_0, T]$ and $F(t, k(t)) = G(t, s, k(s)) = 0$

Then, for $t \geq t_0$ trivial solution $x(t, t_0, x_0, u(t))$ of FCVIDE (6.1) is practically stable.

Proof: It is given (λ, A) with $0 < \lambda < A$. Then it is found $a, b > 0$, $a, b \in \kappa$ such that $a(t, \lambda) < b(A)$. Practical stability property satisfy such that $D_0[x(t, t_0, x_0, u(t)), \theta^n] < A$ implies that $D_0[x_0] < \lambda$. If we get it wrong and then for $t \geq t_0$ there would exist $x(t, t_0, x_0, u(t))$ solution of (6.1) and exist $t_1 > t_0$ with $D_0[x_0, \theta^n] < \lambda$ providing $D_0[x(t_1, t_0, x_0, u(t)), \theta^n] = A, t \in [t_0, t_1]$.

Then, $b(A) \leq V(t_1, x(t_1, t_0, x_0, u(t)), u(t))$ for $t_1 > t_0$, is that case $D_0[x(t_1, t_0, x_0, u(t)), \theta^n] < \rho$ for $t \in [t_0, t_1]$ and for $t \geq t_0$

$$V(t_1, x(t_1, t_0, x_0, u(t)), u(t)) \leq V(t_0, x_0, u(t)) \quad (6.62)$$

We conclude ,

$$\begin{aligned} b(A) = b(D_0[x(t_1, t_0, x_0, u(t)), \theta^n]) &\leq V(t_1, x(t_1, t_0, x_0, u(t)), u(t)) \\ &\leq V(t_0, x_0, u(t)) \\ &\leq a(t_0, D_0[x_0, \theta^n]) \\ &\leq a(t_0, \lambda) < b(A) \end{aligned} \quad (6.63)$$

which is contrast. Thus, we understand that $x(t, t_0, x_0, u(t))$ is practical stable. If λ is dependent of t_0 , we get uniformly practically stability property of solution $x(t, t_0, x_0, u(t))$.

We gave a beneficial comparison theorem revealing that practical stability of fuzzy control integro-differential systems via scalar integro-differential equation.

Theorem 6.7: Assume that

$$F \in C([t_0, t_0 + p] \times [0, a], \mathbb{R}^+) \quad G \in C([t_0, t_0 + p] \times [t_0, t_0 + p] \times [0, a], \mathbb{R}^+) \quad (6.64)$$

such $F(t, 0) \equiv 0$ and $G(t, s, 0) \equiv 0$ and $0 \leq F(t, k) \leq M_3$ and $0 \leq G(t, s, k) \leq M_4$

$\forall t \in [t_0, t_0 + p], 0 \leq k \leq a$, such that $F(t, k), G(t, s, k)$ is nondecreasing in k and the initial value problem

$$\frac{dk}{dt} = F(t, k(t)) + \int_{t_0}^t G(t, s, k(s))ds, \quad k(t_0) = 0 \quad (6.65)$$

has only the solution $k(t) \equiv 0$ on $[t_0, T]$ and $F(t, k(t)) = G(t, s, k(s)) = 0$

ii) Assume that $(t, x, u) \in \mathbb{R}_+ \times S_\rho \times E^m$ where $S_\rho = [x \in E^n: D_0[x, \theta^n] < \rho]$

Lyapunov-like function $V(t, x(t), u(t)) \in C(\mathbb{R}_+ \times E^n \times E^m, E^n)$,

$$|V(t, x, u_1) - V(t, y, u_2)| \leq L(D_0[x, y] + D_0[u_1, u_2]) \quad L > 0 \quad (6.66)$$

bounded Lipschitz constant and Dini derivative of Lyapunov-like function as following,

iii) Let $V(t, x(t), u(t)) \in C(\mathbb{R}_+ \times E^n \times E^m, E^n)$ and $a, b \in \kappa$,

$$D^+V(t, x, u) \leq F(t, V(t, x, u)) + \int_{t_0}^t G(t, s, V(t, x, u))ds \quad (6.67)$$

$$\text{iv)} \quad b(D_0[x(t), \theta^n]) \leq V(t, x(t), u(t)) \leq a(t, D_0[x(t), \theta^n]) \quad (6.68)$$

Then, for $t \geq t_0$, the practical stability properties of the comparison differential equation show the same practical stability features of $x(t, t_0, x_0, u(t))$ solution of the system (6.1).

Proof: Let consider that comparison equation is practically stable, there exists $b \in \kappa$ $b = b(A)$ for given any (λ, A) with $0 < \lambda < A$ such that $k(t, t_0, k_0) < b(A)$ satisfies that $d_s[k_0, 0] < \lambda, t \geq t_0$. We assume that practical stability property provides with λ such that $D_0[x(t, t_0, x_0, u(t)), \theta^n] < A$ satisfies that $D_0[x_0, \theta^n] < \lambda$. If we get it wrong and then for $t \geq t_0$, there would exist solution of fuzzy control integro-differential equation and with $D_0[x_0, \theta^n] < \lambda$ providing for $t_1 > t_0$

$$D_0[x(t_1, t_0, x_0, u(t)), \theta^n] = A \quad (6.69)$$

Set $k_0 = a(t_0, D_0[x_0, \theta^n])$, we have $V(t, x(t), u(t)) \leq r(t, t_0, k_0)$ $t \in [t_0, t_1]$. Then for $t_1 > t_0$,

$$b(A) \leq V(t_1, x(t_1, t_0, x_0, u(t)), u(t)) \quad (6.70)$$

Because of $D_0[x(t), \theta^n] < \rho$ and thus we get

$$V(t_1, x(t_1, t_0, x_0, u(t)), u(t)) \leq r(t_1, t_0, k_0) \quad (6.71)$$

Then we have

$$\begin{aligned} b(A) &= b(D_0[x(t_1, t_0, x_0, u(t)), \theta^n]) \leq V(t_1, x(t_1, t_0, x_0, u(t)), u(t)) \\ &\leq r(t_1, t_0, k_0) \\ &\leq r(t_1, t_0, a(t_0, D_0[x_0, \theta^n])) \\ &\leq r(t_1, t_0, a(t_0, \lambda_1)) < b(A) \end{aligned} \quad (6.72)$$

which is contrast. Thus, we understand that $x(t, t_0, x_0, u(t))$ is practical stable.

Theorem 6.8: Assume that

i) $f \in C(R_+ \times E^n \times E^p, E^n)$ is levelwise continuous, there exists $L_f > 0$ such that $D_0[f(t, x, u), f(t, y, v)] \leq L_f(D_0[x, y] + D_0[u, v])$ for all $x, y \in E^n$, $u, v \in E^p$ and $t \in [t_0, T]$.

ii) $g \in C(R_+ \times E^n \times E^p, E^n)$ is levelwise continuous, there exists $L_g > 0$ such that $D_0[g(t, x, u), g(t, y, v)] \leq L_g(D_0[x, y] + D_0[u, v])$ for all $x, y \in E^n$, $u, v \in E^p$, $t \in [t_0, T]$ and $L_{fg} = \max\{L_f, L_g\}$.

iii) There is a function $v \in C([t_0, T] \times E^n \times E^p, E^n)$ is solution of (6.2).

Then, there exists a solution to problem (6.1) with $D_0[v(t), x(t)] \leq K_f \varepsilon$ and

$x_0 = v_0$ where $K_f = (T - t_0)[1 + (L_{fg}/(L_{fg} + 1))(e^{(L_{fg}+1)(T-t_0)} - 1)]$. That is the solution of problem (6.1) is Ulam-Hyers stable.

Proof: Assume that for each $\varepsilon > 0$ providing (6.2) for any $t \in [t_0, T]$ there is continuously differentiable function $v \in C([t_0, T] \times E^n \times E^p, E^n)$ with $D_0[\delta(t), \theta^n] \leq \varepsilon$.

We have $D_H v(t) = f(t, v(t), u(t)) + \int_{t_0}^t g(t, s, x(s), u(s))ds + \delta(t)$. If

function v continuous and differentiable on $[t_0, T]$, then

$$v(t) = v_0 + \int_{t_0}^t f(s, v(s), u(s))ds + \int_{t_0}^t \int_{t_0}^s g(s, r, v(r), u(r))drds + \int_{t_0}^t \delta(s)ds \quad (6.73)$$

Set sequence function $x_n: [t_0, T] \rightarrow E^n$ successive approximation with $x_0 = v_0$ as following:

$$x_n(t) = v_0 + \int_{t_0}^t f(s, x_{n-1}(s), u(s))ds + \int_{t_0}^t \int_{t_0}^s g(s, r, x_{n-1}(r), u(r))drds \quad (6.74)$$

Then,

$$\begin{aligned} D_0[x_1, x_0] &= D_0\left[\int_{t_0}^t f(s, x_0(s), u(s))ds + \int_{t_0}^t \int_{t_0}^s g(s, r, x_0(r), u(r))drds, \theta^n\right] \\ &= D_0\left[\int_{t_0}^t f(s, v_0(s), u(s))ds + \int_{t_0}^t \int_{t_0}^s g(s, r, v_0(r), u(r))drds, \theta^n\right] \\ &= D_0\left[-\int_{t_0}^t \delta(s)ds, \theta^n\right] \leq \int_{t_0}^t D_0[\delta(s), \theta^n]ds \leq \varepsilon(t - t_0) \end{aligned} \quad (6.75)$$

For second step

$$\begin{aligned}
D_0[x_2, x_1] &= D_0\left[\int_{t_0}^t f(s, x_1(s), u(s))ds + \int_{t_0}^t \int_{t_0}^s g(s, r, x_1(r), u(r))drds, \right. \\
&\quad \left. \int_{t_0}^t f(s, x_0(s), u(s))ds + \int_{t_0}^t \int_{t_0}^s g(s, r, x_0(r), u(r))drds\right] \\
&\leq \int_{t_0}^t L_{fg} D_0[x_1, x_0]ds + \int_{t_0}^t \int_{t_0}^s L_{fg} D_0[x_1, x_0]drds \\
&\leq \varepsilon L_{fg} \left[\frac{(t-t_0)^2}{2!} + \frac{(t-t_0)^3}{3!} \right]
\end{aligned} \tag{6.76}$$

If it continues like this,

$$\begin{aligned}
D_0[x_n, x_{n-1}] &\leq \varepsilon (L_{fg})^{n-1} \sum_{k=0}^n \binom{n}{k} \frac{(t-t_0)^{n+k}}{n+k!} \leq \frac{\varepsilon \lambda}{L_{fg}} (L_{fg})^n \sum_{k=0}^{\infty} \frac{(t-t_0)^{n+k}}{n+k!} \\
&\leq \frac{\varepsilon \lambda}{L_{fg}} \frac{(L_{fg}(t-t_0))^n}{n!} \sum_{k=0}^{\infty} \frac{(t-t_0)^k}{k!} = \frac{\varepsilon \lambda}{L_{fg}} \frac{(L_{fg}(t-t_0))^n}{n!} e^{(t-t_0)} \\
&\leq \frac{\varepsilon \lambda}{L_{fg}} \frac{(L_{fg}(T-t_0))^n}{n!} e^{(T-t_0)}
\end{aligned} \tag{6.77}$$

where $\lambda = \max\left\{\binom{n}{k}\right\}$ $k = 0, 1, 2, \dots$ and $e^{t-t_0} = \sum_{k=0}^{\infty} \frac{(t-t_0)^k}{k!}$. Then $D_0[x_n, x_{n-1}] \rightarrow 0$ uniformly as $n \rightarrow \infty$. Then $D_0[x_n, x_{n-1}] \rightarrow 0$ uniformly as $n \rightarrow \infty$.

$$\begin{aligned}
D_0[x_n, x] &\leq D_0\left[\int_{t_0}^t f(s, x_{n-1}(s), u(s))ds \right. \\
&\quad \left. + \int_{t_0}^t \int_{t_0}^s g(s, r, x_{n-1}(r), u(r))drds, \right. \\
&\quad \left. \int_{t_0}^t f(s, x(s), u(s))ds + \int_{t_0}^t \int_{t_0}^s g(s, r, x(r), u(r))drds\right] \tag{6.78} \\
&\leq L_{fg} \int_{t_0}^t D_0[x_{n-1}(s), x(s)]ds + L_{fg} \int_{t_0}^t \int_{t_0}^s D_0[x_{n-1}(r), x(r)]drds \\
&\leq \xi(t) \leq L_{fg} \left(\int_{t_0}^t \xi(s)ds + \int_{t_0}^t \int_{t_0}^s \xi(r)drds \right)
\end{aligned}$$

$\xi(t) \leq L_{fg} \left(\int_{t_0}^t \xi(s) ds + \int_{t_0}^t \int_{t_0}^s \xi(r) dr ds \right)$ where $\xi(t) = D_0[x_n, x]$. From [44], $\xi(t) = 0$. So $D_0[x_n, x] = 0$. Then,

$$\begin{aligned}
D_0[x(t), v(t)] &= D_0[v_0 + \int_{t_0}^t f(s, v(s), u(s)) ds \\
&\quad + \int_{t_0}^t \int_{t_0}^s g(s, r, v(r), u(r)) dr ds + \int_{t_0}^t \delta(s) ds, \\
&\quad v_0 + \int_{t_0}^t f(s, x(s), u(s)) ds + \int_{t_0}^t \int_{t_0}^s g(s, r, x(r), u(r)) dr ds] \\
&\leq L_{fg} \int_{t_0}^t D_0[x(s), v(s)] ds + L_{fg} \int_{t_0}^t \int_{t_0}^s D_0[x(r), v(r)] dr ds \\
&\quad + \int_{t_0}^t D_0[\delta(s), \theta^n] ds \\
&\leq L_{fg} \left(\int_{t_0}^t D_0[x(s), v(s)] ds \right) + L_{fg} \int_{t_0}^t \int_{t_0}^s D_0[x(r), v(r)] dr ds + \varepsilon(T - t_0)
\end{aligned} \tag{6.79}$$

With same process again, $\xi(t) \leq \varepsilon(T - t_0) + \int_{t_0}^t L_{fg} \left(\xi(t) + \int_{t_0}^s \xi(t) dz \right) ds$ where

$\xi(t) = D_0[x(t), v(t)]$ with same process again by Pachpatte's inequality [44]. So

$$\begin{aligned}
D_0[x(t), v(t)] &\leq \varepsilon(T - t_0) \left[1 + \frac{L_{fg}}{L_{fg} + 1} (\exp[(L_{fg} + 1)(T - t_0)] - 1) \right] \\
&= \varepsilon K_f
\end{aligned} \tag{6.80}$$

where $K_f = ((T - t_0) \left[1 + \frac{L_{fg}}{L_{fg} + 1} (\exp[(L_{fg} + 1)(T - t_0)] - 1) \right])$.

Thus, the solution of problem (6.1) is Ulam-Hyers stable.

6.3. Comparison: Ulam-Hyers and Lyapunov Stability

Earlier, we gave our definitions of stability over null solution. Because Lyapunov second method works on null solution. But we will rearrange the concept of Lyapunov stability so that it is easier to compare the concepts of Ulam-Hyers stability and Lyapunov stability.

Definition 6.3:[46] We consider that for each $\varepsilon > 0$ and $t_0 > 0$ any solution $x(t)$ of equation (6.1) through (t_0, x_0) . The solution $x(t)$ is said to be stable by Lyapunov's mean, if for $t \geq t_0$ there exist a $\delta = \delta(t_0, \varepsilon)$ such that $D_0[x_0, x_1] < \delta$ provided that $D_0[x(t, t_0, x_0, u(t)), x(t, t_0, x_1, u(t))] < \varepsilon$. If each solution is stable by Lyapunov's mean, then the equation (6.1) is stable by Lyapunov's mean.

Theorem 6.9: Assume that,

i) $f \in C(R_+ \times E^n \times E^p, E^n)$ is levelwise continuous, there exists $L_f > 0$ such that $D_0[f(t, x, u), f(t, y, v)] \leq L_f(D_0[x, y] + D_0[u, v])$ for all $x, y \in E^n$, $u, v \in E^p$ and $t \in [t_0, T]$.

ii) $g \in C(R_+ \times E^n \times E^p, E^n)$ is levelwise continuous, there exists $L_g > 0$ such that $D_0[g(t, x, u), g(t, y, v)] \leq L_g(D_0[x, y] + D_0[u, v])$ for all $x, y \in E^n$, $u, v \in E^p$, $t \in [t_0, T]$ and $L_{fg} = \max\{L_f, L_g\}$.

iii) $x^0(t) = x^0(t, t_0, x_0, u(t))$ and $x^1(t) = x^1(t, t_0, x_1, u(t))$ solution of problem (6.1) such that $D_0[x_0, x_1] < \delta$ with $x_0 = x^0(t_0)$ and $x_1 = x^1(t_0)$.

Then $D_0[x^0, x^1] \leq \delta \left[1 + \frac{L_{fg}}{L_{fg}+1} (\exp[(L_{fg} + 1)(T - t_0)] - 1) \right]$ for all $t \in [t_0, T]$.

Proof :

$$\begin{aligned}
D_0[x^0, x^1] &= D_0[x_0 + \int_{t_0}^t f(s, x^0(s), u(s))ds \\
&\quad + \int_{t_0}^t \int_{t_0}^s g(s, z, x^0(z), u(z))dzds, \\
&\quad x_1 + \int_{t_0}^t f(s, x^1(s), u(s))ds + \int_{t_0}^t \int_{t_0}^s g(s, z, x^1(z), u(z))dzds] \\
&\leq D_0[x_0, x_1] + D_0 \left[\int_{t_0}^t f(s, x^0(s), u(s))ds, \int_{t_0}^t f(s, x^1(s), u(s))ds \right] \\
&\quad + D_0 \left[\int_{t_0}^t \int_{t_0}^s g(s, z, x^0(z), u(z))dzds, \int_{t_0}^t \int_{t_0}^s g(s, z, x^1(z), u(z))dzds \right] \\
&\leq D_0[x_0, x_1] + \int_{t_0}^t D_0[f(s, x^0(s), u(s)), f(s, x^1(s), u(s))]ds \\
&\quad + \int_{t_0}^t \int_{t_0}^s D_0[g(s, z, x^0(z), u(z)), g(s, z, x^1(z), u(z))]dzds \\
&\leq \delta + \int_{t_0}^t D_0[f(s, x^0(s), u(s)), f(s, x^1(s), u(s))]ds \\
&\quad + \int_{t_0}^t \int_{t_0}^s D_0[g(s, z, x^0(z), u(z)), g(s, z, x^1(z), u(z))]dzds \\
&\leq \delta + L_{fg} \left(\int_{t_0}^t D_0[x^0, x^1]ds + \int_{t_0}^t \int_{t_0}^s D_0[x^0, x^1]dzds \right)
\end{aligned} \tag{6.81}$$

$$D_0[x^0, x^1] = m(t) \leq \delta + L_{fg} \left(\int_{t_0}^t m(s)ds + \int_{t_0}^t \int_{t_0}^s m(r)dzds \right) \tag{6.82}$$

By Gronwall inequality, $m(t) \leq \delta + \delta L_{fg} \int_{t_0}^t \exp(\int_{t_0}^s (L_{fg} + 1)dz)ds$

$$D_0[x^0, x^1] \leq \delta \left[1 + \frac{L_{fg}}{L_{fg} + 1} (\exp[(L_{fg} + 1)(T - t_0)] - 1) \right] \tag{6.83}$$

Corollary 6.2: Assume that i)-iii) of Theorem 6.9, then there exist $d > 0$ such that $D_0[x^0, x^1] < 2\delta$ provided for all $t \in [t_0 - d, t_0 + d]$.

Proof: We get from Theorem 6.9 that

$$D_0[x^0, x^1] \leq \delta \left[1 + \frac{L_{fg}}{L_{fg} + 1} (\exp[(L_{fg} + 1)(T - t_0)] - 1) \right] \quad (6.84)$$

Hence if $1 + \frac{L_{fg}}{L_{fg} + 1} (\exp[(L_{fg} + 1)(T - t_0)] - 1) \leq 2$. So our assertion is correct.

It is sufficient to take $d := \ln \left[2 + \left(\frac{1}{L_{fg}} \right) \right] / (L_{fg} + 1)$.

Theorem 6.10: If the problem (6.1) is Ulam-Hyers stable, then it is stable by Lyapunov's mean.

Proof: If the problem (6.1) is Ulam-Hyers stable, then conditions of Theorem 6.8 holds. Let us denote $x^0(t) = x^0(t, t_0, x_0, u(t))$ and $x^1(t) = x^1(t, t_0, x_1, u(t))$ solution of problem (6.1) with $D_0[x_0, x_1] < \delta$. Since f satisfies $D_0[f(t, x, u), f(t, y, v)] \leq L_f(D_0[x, y] + D_0[u, v])$, by Theorem 6.9 we have $D_0[x^0, x^1] < 2\delta$ for $t \in [t_0 - d, t_0 + d]$. We take the function $q: \mathbb{R} \rightarrow [0, 1]$ satisfying following conditions;

- i) $q \in C^1(\mathbb{R})$
- ii) $q(t) = 0$ for $t \leq t_0$, $q(t) = 1$ for $t \geq t_0 + d$ $d > 0$
- iii) $q'(t) \geq 0$ for $\forall t \in \mathbb{R}$

We define the function $v \in C([t_0, T] \times E^n \times E^m, E^n)$

$$v(t) = x^0(t) + q(t)[x^1 - x^0] \quad t \in \mathbb{R} \quad (6.85)$$

where $x^1 - x^0$ is Hukuhara distance and exist. Note that $v(t) = x^0(t)$ for $t \leq t_0$ and $v(t) = x^1(t)$ for $t \geq t_0 + d$. We get for $t \in [t_0, t_0 + d]$

$$\begin{aligned}
& D_0 \left[D_H v(t), f(t, v(t), u(t)) + \int_{t_0}^t g(t, s, v(s), u(s)) ds \right] = D_0[I_1, I_2] \\
& I_1 = D_H(x^0(t) + q(t)[x^1 - x^0]) \\
& I_2 = f(t, x^0(t) + q(t)[x^1 - x^0], u(t)) \\
& \quad + \int_{t_0}^t g(t, s, x^0(t) + q(t)[x^1 - x^0], u(s)) ds \\
& I_1 = D_H(x^0(t) + q(t)[x^1 - x^0]) \\
& \quad = D_H x^0(t) + q'(t)[x^1 - x^0] + q(t)[D_H x^1 - D_H x^0] \tag{6.86} \\
& \quad = f(t, x^0(t), u(t)) + \int_{t_0}^t g(t, s, x^0(s), u(s)) ds \\
& \quad \quad + q'(t)[x^1 - x^0] + q(t)[f(t, x^1(t), u(t)) \\
& \quad \quad + \int_{t_0}^t g(t, s, x^1(s), u(s)) ds - f(t, x^0(t), u(t)) \\
& \quad \quad + \int_{t_0}^t g(t, s, x^0(s), u(s)) ds]
\end{aligned}$$

By Lemma 3.1 [44]

$$\begin{aligned}
D_0[I_1, I_2] & \leq D_0[f(t, x^0(t), u(t)) + \int_{t_0}^t g(t, s, x^0(s), u(s)) ds, \\
& \quad f(t, x^0(t) + q(t)[x^1 - x^0], u(t)) \\
& \quad + \int_{t_0}^t g(t, s, x^0(t) + q(t)[x^1 - x^0], u(s)) ds] \\
& \quad + D_0[q'(t)[x^1 - x^0], \theta^n] \tag{6.87a} \\
& \quad + D_0[q(t)[f(t, x^1(t), u(t)) + \int_{t_0}^t g(t, s, x^1(s), u(s)) ds \\
& \quad - f(t, x^0(t), u(t)) + \int_{t_0}^t g(t, s, x^0(s), u(s)) ds], \theta^n]
\end{aligned}$$

$$\begin{aligned}
&\leq D_0[f(t, x^0(t), u(t)), f(t, x^0(t) + q(t)[x^1 - x^0], u(t))] \\
&\quad + D_0\left[\int_{t_0}^t g(t, s, x^0(s), u(s))ds, \right. \\
&\quad \left. \int_{t_0}^t g(t, s, x^0(s) + q(s)[x^1 - x^0], u(s))ds\right] \\
&\quad + D_0[q'(t)[x^1 - x^0], \theta^n] + D_0[q(t)(f(t, x^1(t), u(t)) \\
&\quad + \int_{t_0}^t g(t, s, x^1(s), u(s))ds - f(t, x^0(t), u(t)) \\
&\quad + \int_{t_0}^t g(t, s, x^0(s), u(s))ds), \theta^n]
\end{aligned} \tag{6.87b}$$

We define $\sup_{t \in t_0, t_0+d] } q(t) = Q$ and $\sup_{t \in t_0, t_0+d] } q'(t) = Q'$

$$\begin{aligned}
D_0[I_1, I_2] &\leq L_f D_0[x^0(t), x^0(t) + q(t)(x^1 - x^0)] \\
&\quad + L_g \int_{t_0}^t D_0[x^0(s), x^0(s) + q(s)[x^1 - x^0]]ds \\
&\quad + Q' D_0[x^1, x^0] + Q D_0[f(t, x^1(t), u(t)) \\
&\quad + \int_{t_0}^t g(t, s, x^1(s), u(s))ds - f(t, x^0(t), u(t)) \\
&\quad + \int_{t_0}^t g(t, s, x^0(s), u(s))ds, \theta^n]
\end{aligned} \tag{6.88}$$

The last Hukuhara distance is less and equal to following;

$$\begin{aligned}
&Q D_0[f(t, x^1(t), u(t)), f(t, x^0(t), u(t))] \\
&\quad + Q D_0\left[\int_{t_0}^t g(t, s, x^1(s), u(s))ds, \int_{t_0}^t g(t, s, x^0(s), u(s))ds\right] \\
&\leq Q L_f D_0[x^1(t), x^0(t)] + Q L_g \int_{t_0}^t D_0[x^1(s), x^0(s)]ds
\end{aligned} \tag{6.89}$$

Consequently,

$$\begin{aligned}
D_0[I_1, I_2] &\leq L_f D_0[x^0(t), x^0(t) + q(t)(x^1 - x^0)] \\
&\quad + L_g \int_{t_0}^t D_0[x^0(s), x^0(s) + q(s)[x^1 - x^0]] ds + Q' D_0[x^1, x^0] \\
&\quad + 6QL_f D_0[x^1(t), x^0(t)] + QL_g \int_{t_0}^t D_0[x^1(s), x^0(s)] ds \\
&\leq L_f Q D_0[x^1(t), x^0(t)] + QL_g \int_{t_0}^t D_0[x^1(s), x^0(s)] ds \\
&\quad + Q' D_0[x^1, x^0] + QL_f D_0[x^1(t), x^0(t)] \\
&\quad + QL_g \int_{t_0}^t D_0[x^1(s), x^0(s)] ds \\
&\leq 2QL_f D_0[x^1(t), x^0(t)] \\
&\quad + 2QL_g \int_{t_0}^t D_0[x^1(s), x^0(s)] ds + Q' D_0[x^1, x^0] \\
&\leq 4QL_f \delta + 4QL_g \delta' + 2Q' \delta \leq 4QL_{fg}(\delta + \delta') + 2Q' \delta = \varepsilon
\end{aligned} \tag{6.90}$$

where

$$\begin{aligned}
D_0[x^0(t), x^0(t) + q(t)(x^1 - x^0)] &= D_0[q(t)(x^1 - x^0), \theta^n] \\
&\leq Q D_0[x^1(t), x^0(t)]
\end{aligned} \tag{6.91}$$

and

$$\int_{t_0}^t D_0[x^1(s), x^0(s)] ds \leq 2 \int_{t_0}^t \delta ds = \delta' \tag{6.92}$$

Consequently $D_0[D_H v(t), f(t, v(t), u(t)) + \int_{t_0}^t g(t, s, v(s), u(s)) ds] \leq \varepsilon$ so $v(t)$ is ε -approximate solution of (6.1). There exists a solution to problem (6.1) provided that $D_0[v(t), x(t)] \leq K_f \varepsilon$ and $x_0 = v_0$. So the solution of problem (6.1) is Ulam-Hyers stable. We have $v(t_0) = x^0(t_0)$ and

$$\begin{aligned}
v(t_0 + d) &= x^1(t_0 + d) \quad D_0[v(t), x^0(t)] \leq K_f \varepsilon_0 \\
D_0[v(t), x^1(t)] &\leq K_f \varepsilon_1 \\
D_0[x^0(t), x^1(t)] &\leq D_0[v(t), x^0(t)] + D_0[v(t), x^1(t)] \\
&\leq K_f \varepsilon_0 + K_f \varepsilon_1 = \varepsilon^*
\end{aligned} \tag{6.93}$$

Consequently solution of (6.1) is said to be stable by Lyapunov's mean.

6.4. Applications

Application 6.1: We consider the initial valued problems of fuzzy control Volterra integro-differential equations (IVP for FCVIDEs) like as equation (6.1)

$$\begin{aligned}
D_H x(t) &= -3x + \int_0^t e^{s-t} x(s) ds \quad x(t_0) = x_0 \\
[x_0]^\alpha &= \varphi(\alpha, t) = [\alpha - 1, 1 - \alpha] \quad t \in [0, 3]
\end{aligned} \tag{6.94}$$

α -level set of equation (6.94) as following;

$$D_H[\underline{x}^\alpha(t), \bar{x}^\alpha(t)] = -3[\underline{x}^\alpha, \bar{x}^\alpha] + \int_0^t e^{s-t} [\underline{x}^\alpha(s), \bar{x}^\alpha(s)] ds \tag{6.95}$$

$$i) \quad D_H \underline{x}^\alpha(t) = -3\underline{x}^\alpha + \int_0^t e^{s-t} \underline{x}^\alpha(s) ds$$

$$ii) \quad D_H \bar{x}^\alpha(t) = -3\bar{x}^\alpha + \int_0^t e^{s-t} \bar{x}^\alpha(s) ds$$

and by Fuzzy Laplace Transform. For first equation;

$$s \underline{F}^\alpha(s) - (\alpha - 1) = -3 \underline{F}^\alpha(s) + \frac{F^\alpha(s)}{s} \rightarrow \underline{F}^\alpha(s) = \frac{(\alpha - 1)s}{s^2 + 3s - 1} \tag{6.96}$$

For second equation;

$$s\overline{F^\alpha}(s) - (1 - \alpha) = -3\overline{F^\alpha}(s) + \frac{\overline{F^\alpha}(s)}{s} \rightarrow \overline{F^\alpha}(s) = \frac{(1 - \alpha)s}{s^2 + 3s - 1} \quad (6.97)$$

by inverse Fuzzy Laplace Transform.

$$i) \underline{x}^\alpha(t) = (\alpha - 1) \left[\left(\frac{\sqrt{13}+3}{2\sqrt{13}} \right) e^{-(3/2+\sqrt{13}/2)t} + \left(\frac{\sqrt{13}-3}{2\sqrt{13}} \right) e^{(\sqrt{13}/2-3/2)t} \right]$$

$$ii) \overline{x}^\alpha(t) = (1 - \alpha) \left[\left(\frac{\sqrt{13}+3}{2\sqrt{13}} \right) e^{-(3/2+\sqrt{13}/2)t} + \left(\frac{\sqrt{13}-3}{2\sqrt{13}} \right) e^{(\sqrt{13}/2-3/2)t} \right]$$

For Ulam-Hyers stability, we can easily find that $L_f = 3$, $L_g = 1$ and $L_{fg} = 3$.

$$D_0[D_H v(t), -3v(t)] + \int_{t_0}^t e^{s-t} v(s) ds \leq \varepsilon, D_0[v(t), x(t)] \leq K_f \varepsilon \quad (6.98)$$

and $x_0 = v_0$ where

$$K_f = (T - t_0) \left[1 + \frac{L_{fg}}{L_{fg}+1} \left(\exp[(L_{fg} + 1)(T - t_0)] - 1 \right) \right] = 3 \left[1 + \frac{3}{4} \cdot (e^{12} - 1) \right] \quad (6.99)$$

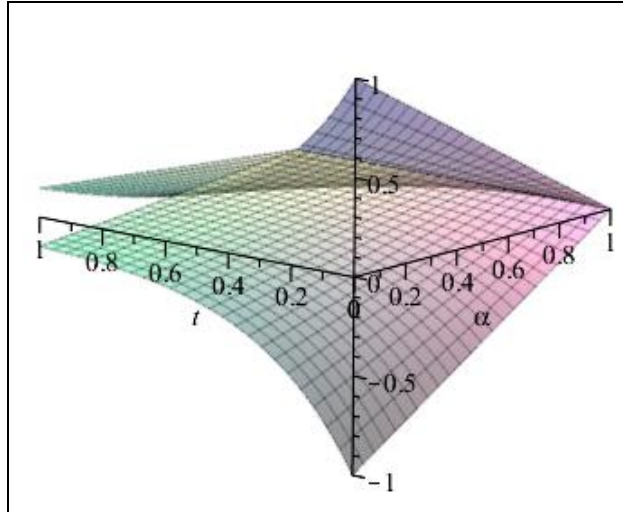


Figure 6.1: The graphic of $\underline{x}^\alpha$ and $\overline{x}^\alpha$ in global controllable solution of equation (6.15).

For Lyapunov stability, We can take $x_1 = \psi(\alpha, t) = 6[\alpha - 1, 1 - \alpha]$ while $x_0 = \varphi(\alpha, t) = [\alpha - 1, 1 - \alpha]$ and such that $D_0[x_0, x_1] < \delta$ then we show $D_0[x^0, x^1] < \varepsilon$ provided $\delta = \delta(\varepsilon)$ where $x^0 = x(t, t_0, x_0)$, $x^1 = x(t, t_0, x_1)$,

$$\begin{aligned} \varphi(t) &= \left[\left(\frac{\sqrt{13}+3}{2\sqrt{13}} \right) e^{-\left(\frac{3}{2}+\frac{\sqrt{13}}{2}\right)t} + \left(\frac{\sqrt{13}-3}{2\sqrt{13}} \right) e^{\left(\frac{\sqrt{13}}{2}-\frac{3}{2}\right)t} \right] \text{ and } \max_t \varphi(t) = k \\ D_0[x^0, x^1] &= \sup_{\alpha \in [0,1]} \max \left\{ |6(\alpha - 1)\varphi(t) - (\alpha - 1)\varphi(t)|, |6(1 - \alpha)\varphi(t) - (1 - \alpha)\varphi(t)| \right\} \\ &= \sup_{\alpha \in [0,1]} \max \left\{ |5(\alpha - 1)\varphi(t)|, |5(1 - \alpha)\varphi(t)| \right\} \\ &\leq \sup_{\alpha \in [0,1]} \max \{ k|5(\alpha - 1)|, k|5(1 - \alpha)| \} \\ &= kD_0[\varphi(t), \psi(t)] = kD_0[x_0, x_1] = k\delta \end{aligned} \quad (6.100)$$

Then $D_0[x^0, x^1] \leq k\delta = \varepsilon$, $\delta = \frac{\varepsilon}{k}$.

Application 6.2: Let see whether Example 1 is practically stable with help of Theorem 4.1.2.

We can choose $V(t, x(t)) = D_0[x(t), \theta^n]$ and $U(t, u(t)) = D_0[u(t), \theta^n]$ calculate $D^+V(t, x) \leq 4V(t, x) + U(t, u(t))$. After this inequality, the scalar differential equation can be chosen as follows,

$$\frac{dw}{dt} = F(t, w(t), v(t)) = 4w(t) + v(t) \quad w(t_0) = w_0 \quad \text{for } t \geq t_0 \quad (6.101)$$

$v(t) \in \mathbb{R}_+$ is control function. Suppose that $Y(t) = w_0 e^{4(t-t_0)}$ is the fundamental solution of $w' = 4w$. We shall show that we can find suitable admissible controls $v(t)$ to assure practically stable of the system (6.16).

The transformation $w = Y(t)z$ reduces (6.16) to

$$z' = 4(w_0)^{-1} e^{-4(t-t_0)} v(t) \quad z(t_0) = w_0 \quad (6.102)$$

therefore the solution $z(t)$ of (6.17) is given by $z(t) = w_0 + \frac{4}{w_0} \int_{t_0}^t v(s) e^{-4(t-t_0)} ds$.

Then with necessary processes and assumptions, we can find easily that $|z(t)| < A \quad t \geq t_0$, provided $|w_0| < \lambda$.

But $|w| = |Y(t)||z(t)|$ and therefore, if $|Y(t)| \leq 1 \quad t \geq t_0$ and there exists a $T > 0$ such that $|Y(t)| \leq \frac{\beta}{A} < 1$, then we have $|w(t)| = |Y(t)||z(t)| < A$ which shows that the system (6.15) is practically stable. Let $b(\cdot), a(t, \cdot) \in \kappa$ and be chosen so that $a(t, \lambda) < b(A)$ and $b(D_0[x(t), \theta^n]) \leq V(t, x) \leq a(t, D_0[x(t), \theta^n])$.

We can choose , $a(t, V(t, x)) = 2V(t, x)$ and $b(V(t, x)) = \frac{1}{2}V(t, x)$ is satisfied that $2\lambda < \frac{1}{2}A$. Consequently, all assumption of Theorem 4.1.2 hold and IVP for NLFCDEs (6.15) is said to be practically controllable like linear control differential equation (6.16).



7. CONCLUSION

In this thesis, we have examined the practical stability and practical and global controllability properties of the perturbed fuzzy control differential equation system related to the unperturbed system with initial time and position differences. We have dealt with to show that when a single system is not stable and controllable on its own, it can be stable and controllable related to a more straightforward system. Thus, we have extended the set of stable and controllable fuzzy control differential equation systems.

In chapter 2, we have presented fundamental definitions and features of fuzzy set algebra to understand fuzzy calculus. We have introduced the Hausdorff metric and Hukuhara derivative and given some properties used in the proof.

In chapter 3, we have introduced a single system, the initial value problem for FCDE. We have given the existence, local and global uniqueness theorem of IVP for FCDE. The definitions of practical stability in the sense of Lyapunov and global and practical controllability are presented. The Lyapunov function, essential in Lyapunov's direct method, and its Dini derivative are given. We have applied fuzzy control extension of the comparison method via the Lyapunov function and scalar differential equation to obtain sufficient conditions for stability and controllability of FCDE. At the end of chapter 3, we have given the main results of practical stability, global and practical controllability, and their applications. In the given three applications, initial value problems of the Hukuhara derivative linear fuzzy control differential equation are included. An example has been solved for each practical stability, global and practical controllability property. The solutions are generally found by fuzzy Laplace transform, and their graphs are given.

In chapter 4, we have transferred the studies on the singular system in chapter 3 to the dual system with ITD stability analysis method. Here we have meant that the dual system is IVP for perturbed fuzzy control differential equations related to unperturbed ones with ITD. This new method was needed since it is not possible to keep measurements with the expected initial time in real-world applications. We have redefined practical stability property, the Lyapunov function, and its Dini derivative with the ITD stability analysis method. We have applied the ITD method to the

comparison method via the Lyapunov function and scalar differential equation. At the end of chapter 4, we have given the main results of the practical stability behaviour of the perturbed system related to the unperturbed system with ITD and its application. In the given application, we have given dual system, unperturbed fuzzy control differential equation and perturbed ones for ITD practical stability property. We have determined whether the perturbed fuzzy control differential equation system related to the unperturbed system with ITD is practical stable via comparison equation and have given the solution graph.

In chapter 5, we have studied the global and practical controllability properties of the dual system. We have redefined these properties with the ITD analysis method. We have given the main results of the global and practical controllability behaviour of the perturbed system related to the unperturbed system with ITD and its application. In the given two applications, an example has been solved for each global and practical controllability property of the perturbed fuzzy control differential equation system related to the unperturbed system with ITD. The solutions are generally found by fuzzy Laplace transform and have determined whether the dual system is practically controllable via comparison equation, and their graphs are given.

In chapter 6, we have restated the main problem to the FCVIDE. We have given the existence and uniqueness theorem and presented the main results of the practical stability behaviour of FCVIDE. We have defined Ulam-Hyers stability property and compared it with Lyapunov stability on IVP of FCVIDE. In the given application, the example has been first solved by fuzzy Laplace transform, then showed that it is Lyapunov and Ulam-Hyers stable. In the second application, we have determined the problem is practically stable via comparison equation, and their graphs are given.

Consequently, we have accomplished what we previously intended. We have interpreted the ITD analysis method in the FCDE and developed practical stability analysis with ITD analysis. Moreover, we have applied this method to controllability analysis and supported it with influential applications. In addition, we have examined practical stability and compared Ulam-Hyers and Lyapunov stability in FCVIDE.

In future studies, additional stability and controllable properties not addressed in this thesis can be handled with ITD analysis. Although all theorems and results are on nonlinear problems, our applications contain linear examples because we have difficulty finding an application sample. Therefore it can be analyzed with ITD on

nonlinear fuzzy control differential equations used in engineering problems. During the movement of two or more robots, one can be selected as the central robot, and the stability and controllability of the others can be examined accordingly. Doing such a study with engineers can move forward ITD analysis. In addition, using ITD analysis in artificial intelligence applications will be a great success.



REFERENCES

- [1] Nijmeijer H., Schaft A. V. D., (1990), “Nonlinear Dynamical Control Systems”, 3rd Edition, Springer.
- [2] Zadeh L.A., (1965), “Fuzzy Sets”, Information And Control, 8, 338-353.
- [3] Zadeh L. A., (1968), “Fuzzy Algorithms Information and Control”, 12(2), 94-102.
- [4] Bellman R. E., Zadeh L. A., (1970) “Decision Making In A Fuzzy Environment”, Management Sciences, 17, 141-164.
- [5] Zadeh L. A., (1971) “Similarity Relations and Fuzzy Orderings”, Information Sciences, 3(2), 177-200.
- [6] Zadeh L. A., (1975), “The Concept of a Linguistic Variable and Its Application to Approximate Reasoning-I”, Information Sciences, 8, 199-249.
- [7] Mamdani E. H., Assilian S., (1973), “An Experiment in Linguistic Synthesis with a Fuzzy Logic Controller”, Int. J. Man-Machine Studies, 7, 1-13.
- [8] Takagi T., Sugeno M., (1985) “Fuzzy Identification of Systems and Its Applications to Modeling and Control”, IEEE Trans. Syst., 15(1), 1620-1723.
- [9] Cheng J., Wang Y., Park J. H., Cao J., Shi K., (2021), “Static Output Feedback Quantized Control for Fuzzy Markovian Switching Singularly Perturbed Systems with Deception Attacks”, IEEE Transactions On Fuzzy Systems.
- [10] Hwang G. C., Lin S. C., (1992) “A Stability Approach to Fuzzy Control Design for Nonlinear Systems”, Fuzzy Sets and Systems, 48(3), 279-287.
- [11] Kandel A., Luo Y., Zhang Q., (1999), “Stability Analysis of Fuzzy Control Systems”, Fuzzy Sets and Systems, 105(1), 33-48.
- [12] Chang S. L., Zadeh L. A., (1972), “On Fuzzy Mapping and Control”, IEEE Transactions on Systems Man. and Cybernetics, 2, 330-340.
- [13] Kaleva O., (1987), “Fuzzy Differential Equations”, Fuzzy Sets and Systems; 24(3), 301-317.
- [14] Kaleva O., (1990), “The Cauchy Problem for Fuzzy Differential Equations”, Fuzzy Sets and Systems; 35, 389-396.
- [15] El Naschie M. S., (2005), “From Experimental Quantum Optics to Quantum Gravity via a Fuzzy Kahler Manifold”, Chaos, Solitons and Fractals, 25, 69-977.
- [16] Hanss M., (2005), “Applied Fuzzy Arithmetic: an Introduction with Engineering Applications”, 3rd Edition, Springer.

- [17] Tanaka K., Wang H.O., (2001), "Fuzzy Control Systems Design and Analysis", 3rd Edition, John Wiley and Sons.
- [18] Phu N.D., Dung L.Q., (2011), "On The Stability and Controllability of Fuzzy Control Set Differential Equations", IJRS (International Journal of Reliability and Safety); 5(3/4): 141-183.
- [19] Phu N. D., Tri P. V., (2018), Salahshour S., Ahmadian A., Baleanu D., "Some Kinds of Controllable Problems for Fuzzy Control Dynamic Systems", Journal of Dynamic Systems Measurement and Control, 140(9).
- [20] Kalman, R., (1960), "A New Approach to Linear Filtering and Prediction Problems", ASME Journal of Basic Engineering, 82, 35-45.
- [21] Willems J. C., (1991), "Paradigms and Puzzles in the Theory of Dynamical Systems", IEEE Transactions on Automatic Control, 36(3).
- [22] Lakshmikantham V., Leela S., (1969), Differential and Integral Inequalities, 3rd Edition, Academic Press.
- [23] Lakshmikantham V., Mohapatra R. N., (2003), "Theory of Fuzzy Differential Equations and Inclusions", 3rd Edition, Taylor And Francis Inc.
- [24] Lakshmikantham V., Leela S., Martynyuk A. A., (1990), "Practical Stability of Nonlinear System", 3rd Edition, World Scientific Publishing.
- [25] Lakshmikantham V., Leela S., (1999), "Stability Theory of Fuzzy Differential Equations via Differential Inequalities", Mathematical Inequalities and Applications.
- [26] Lakshmikantham V., Leela S., Martynyuk A. A., (1989), "Stability Analysis of Nonlinear System", 3rd Edition, Marcel Dekker.
- [27] Yakar C., Gücen M. B., (2015), "Initial Time Difference Stability of Causal Differential Systems in Terms of Lyapunov Function and Lyapunov Functionals", Journal of Applied Mathematics.
- [28] Liu X., Shaw M. D., (2001), "Boundedness in Terms of Two Measures for Perturbed Systems by Generalized Variation of Parameters." Communications in Applied Analysis, 5, 435-444.
- [29] Yakar C., (2007), "Strict Stability Criteria of Perturbed Systems with respect to Unperturbed Systems in Terms of Initial Time Diference", Complex Analysis and Potential Theory.
- [30] Yakar C., Şimşek M., Gücen M. B., (2012), "Practical Stability, Boundedness Criteria and Lagrange Stability of Fuzzy Differential Systems", Computers and Mathematics With Applications; 64(6), 2118-2127.

- [31] Yakar C., Şimşek M., Gücen M. B., (2011), “Boundedness and Lagrange Stability of Fractional Order Perturbed System related to Unperturbed Systems with Initial Time Difference in Caputo’s Sense”, *Advances in Difference Equations*, 54(1).
- [32] Yakar C., Gücen M. B., (2018), “Strict Stability of Fuzzy Differential Equations by Lyapunov Functions”, *International Journal of Computer and Information Engineering*, 12(5), 315-319.
- [33] Yakar C., Shaw M. D., (2009), “Practical Stability in Terms of Two Measures with Initial Time Difference”, *Nonlinear Analysis: Theory, Methods and Applications*, 71(12), 781-785.
- [34] Yakar C., Özbay Elibüyük B., (2021), “Practical Stability Analysis of Perturbed Fuzzy Control System related to Unperturbed Fuzzy Control System”, *Turkish Journal of Mathematics*.
- [35] Hyers D. H., (1941), “On The Stability of the Linear Functional Equation”, *Proc. Natl. Acad. Sci.*, 27, 222-224.
- [36] Rassias T. M., (1978), “On the Stability of The Linear Mapping in Banach Spaces”, *Proc. Amer. Math. Soc.*, 72.
- [37] Phung N. N., Ta B. Q., Vu H., (2019), “Ulam-Hyers Stability and Ulam-Hyers-Rassias Stability for Fuzzy Integro-Differential Equation”, *Complexity*.
- [38] Vu H., Hoa N. V., (2013), “Stability Results for Set Solution of Fuzzy Integro-Differential Systems”, *Int. J. Industrial Mathematics*, 5 (1).
- [39] Phu N. D. and Tung T. T., (2007) “Existence of Solution of Fuzzy Control Differential Equations”, *VNUHCM Journal of Science and Technology Development*.
- [40] Allahviranloo T., Ahmadi M. B., (2010), “Fuzzy Laplace Transforms”, *Soft Comput*, 14, 235-243.
- [41] Phuong P., Phung N. N., Vu H., (2013), “Existence and Uniqueness of Fuzzy Control Integro-Differential Equation with Perturbed”, *Bulletin of Mathematical Sciences and Applications*, 3, 37-44.
- [42] Hajighasemi S., Allahviranloo T., Khezerloo M., Khorasany M., Salahshour S., (2010), “Existence and Uniqueness of Solutions of Fuzzy Volterra Integro-Differential Equations”, *International Conference on Information Processing and Management of Uncertainty in Knowledge-Based Systems*.
- [43] Obloza M., (1997), “Connections between Hyers and Lyapunov Stability of the Ordinary Differential Equations”, *Prace Matematyczne IV*.
- [44] Phu N. D., Hoa N. V., Vu H., (2012), “On Comparisons of Set Solutions for Fuzzy

Control Integro-Differential Systems”, Journal of Advanced Research in Applied Mathematics, 4(1), 84-101.

[45] Pachpatte B. G., (1998), “Inequalities for Differential and Integral Equations”, 3rd Edition, Academic Press.

[46] Agarwal R. P., O’regan D., (2000), “An Introduction to Ordinary Differential Equations”, 3rd Edition, Springer.



BIOGRAPHY

Betül ÖZBAY ELİBÜYÜK completed primary school and science high school in KIRKLARELİ. She graduated from Yıldız Technical University, Department of Mathematics Engineering, in 2012 and has her Master's Degree at Gebze Institute of Technology, Department of Mathematics, in 2015. Before her academic career, she worked as a systems analyst in an insurance company. She has been working as a research assistant at the Department of Mathematics at Kırklareli University since 2015.



APPENDICES

Appendix A: Publications Based on the Thesis

Yakar C., Özbay Elibüyük B., (2021), “Practical Stability Analysis of Perturbed Fuzzy Control System related to Unperturbed Fuzzy Control System”, Turkish Journal of Mathematics.

Yakar C., Özbay Elibüyük B., (2023), “Practical and Global Controllability of Perturbed Fuzzy Control System Related to Unperturbed Fuzzy Control System with Initial Time Difference”, preprint.

Yakar C., Özbay Elibüyük B., (2023), “Practically and Ulam-Hyers Stability of Fuzzy Control Volterra-Integro Differential System: A Stability Comparison”, preprint.