ISTANBUL TECHNICAL UNIVERSITY ★ GRADUATE SCHOOL

ULTRA-RELATIVISTIC SCALING LIMITS OF MULTI-METRIC THEORIES

M.Sc. THESIS

Ertuğrul EKİZ

Department of Physics Engineering

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JUNE 2023

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Ertuğrul EKİZ
(509201115)

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Thesis Advisor: Prof. Dr. Mehmet ÖZKAN

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İSTANBUL TEKNİK ÜNİVERSİTESİ ★ LİSANSÜSTÜ EĞİTİM ENSTİTÜSÜ

ÇOK METRİKLİ TEORİLERİN AŞIRI GÖRELİ ÖLÇEKLEME LİMİTLERİ

YüKSEK LİSANS TEZİ

Ertuğrul EKİZ
(509201115)

Fizik Mühendisliği Anabilim Dalı

Fizik Mühendisliği Programı

Tez Danışmanı: Prof. Dr. Mehmet ÖZKAN

HAZİRAN 2023

Ertuğrul EKİZ, a M.Sc. student of ITU Graduate School student ID 509201115, successfully defended the thesis entitled "ULTRA-RELATIVISTIC SCALING LIMITS OF MULTI-METRIC THEORIES", which he/she prepared after fulfilling the requirements specified in the associated legislations, before the jury whose signatures are below.

Thesis Advisor :	**Prof. Dr. Mehmet ÖZKAN** Istanbul Technical University	.....
Jury Members :	**Prof. Dr. Abdurrahman Savaş ARAPOĞLU** Istanbul Technical University	.....
	Assistant Prof. Gökhan ALKAÇ Ankara Science University	.....

Date of Submission : 18 April 2023
Date of Defense : 8 June 2023

To my family,

FOREWORD

Firstly, I would like to express my sincere gratitude to my advisor Prof. Dr Mehmet Özkan for the continuous support of my study, his patience, motivation, and immense knowledge. His support and guidance helped me all the time in my study. He constantly pushed me forward to improve my knowledge from almost the beginning of my life in physics.

I had the great pleasure of working with Prof. Dr Mehmet ÖZKAN, Dr Utku ZORBA, Dr Oğuzhan KAŞIKÇI and Cemal Berfu ŞENIŞIK who are excellent collaborators. From the discussions with them, I have learned so much.

I also would like to say thank you from the deep down of my heart to the deep down of my heart. Without your continuous support, I can not be the person that I am today or can not stand where I stand today. Thanks again my beloved "Darling", İrem BAKIR.

Last but not least, I am extremely grateful to my parents. They always believed in me and helped me in the most difficult times. I always felt their unconditional love and thanks to them, I always was able to find the strength to overcome difficulties. I dedicate this thesis to them.

June 2023 Ertuğrul EKİZ

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SYMBOLS

G	: Gravitational constant
c	: Speed of light
$g_{\mu\nu}$	: Spacetime metric
$F_{\mu\nu}$	: Field strength
R_{ab}	: Curvature 2-form
e^a	: Tetrad field
$\omega^a{}_b$	: Spin connection
$\Omega^a{}_b$	: Spin connection
S	: Action
$\mathscr{L}$	: Lagrangian (density)
$\mathscr{H}$	: Hamiltonian (density)
∇_μ	: Covariant derivative
D_μ	: Gauge Covariant derivative
∂_μ	: Partial derivative
$\wedge$	: Wedge product

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ULTRA-RELATIVISTIC SCALING LIMITS
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SUMMARY

Lie algebra expansion is an elegant method to obtain higher dimensional algebras and using this method one can write some interesting ultra-relativistic gravitational theories beginning from the Poincaré algebra [1–10]. This method was first developed in [11], thanks to [12] and has been used in many other studies, for which we will provide some explanations and references for these studies in the following sections.

In this work, we will first give a brief introduction to the gauge theories, which are necessary in order to realize the gravitational theories as a gauge theory, especially the algebraic structure of gravitational theories. This is crucial for many gravity theories, such as supergravity [13].

After that, we will review the general aspects of differential geometry shortly [14–16]. This will give us the main mathematical framework to study gravity as a gauge theory. Even though we will not be able to review all the related topics, we will go through all the necessary topics in order to gather all the necessary information.

Thirdly, we will try to understand the theories of gravity, especially general relativity, as a gauge theory. After a simple introduction to the second-order formalism of GR, we will define the first-order formalism and its action. After that, we will obtain GR beginning from the Poincaré algebra and by gauging this algebra.

After that, we attempt to gain a few knowledge about bi-metric/bi-gravity [17–20] theories by examining some specific examples such as [21–23]. Of course these examination will not be enough to understand the subjects such as Massive Gravity and Bi-Gravity however, we hope that readers can follow easily. As we move on to spoken chapter, we will give a detailed reference for the people who are encouraged to learn the subject.

As a last section, we will focus on scaling limit, specifically algebraic aspects of it. We will first show that gravity theories can be obtained by using Lie Algebra Expansion. In this section we start with Poincaré algebra then divide its generators to two subspaces. After that, we will obtain Carroll Gravity [2, 7, 24] as a gauge theory. Then, we will not stop in that point and push forward to a generalization of it by giving the method, which is based on [1], to obtain the same results by contraction of a multi-metric theory by illustrating this procedure with a few examples.

ÇOK METRİKLİ TEORİLERİN AŞIRI GÖRELİ ÖLÇEKLEME LİMİTLERİ

ÖZET

Lie cebiri genişletmesi, daha yüksek boyutlu cebirler elde etmek için kullanılabilecek, zarif yöntemlerden biridir ve bu yöntemle Poincaré cebirinden başlayarak bazı ilginç aşırı göreli kütleçekim teorileri yazılabilir. Özellikle bu cebirden başlanarak, Carroll cebri, Carroll ötesi cebirler elde edilebilir [1, 2, 7–10]. Her ne kadar bu tezin kapsamı ana kapsamı içinde bulunmasa da, Bargmann cebri gibi göreli olmayan cebirler ve bu cebirlerle ilişkili kütleçekim teorileri de Lie cebri genişletmesi yöntemi kullanılarak elde edilebilir [25].

Yöntemin tarihine baktığımızda, [12] sayesinde ilk olarak [11] makalesinde kullanıldığını görürüz. Daha sonra, bu tezin giriş bölümünde örneklendireceğimiz çalışmalar da dahil olmak üzere, bir çok çalışmada bu yöntem başarıyla kullanılmıştır. Biz de bu tezin ana amacı kapsamında, hesaplamalarımıza bu yöntemi kullanarak başlayacağız.

Bu çalışmada öncelikle kütleçekimi teorilerini, özellikle de kütleçekim teorilerinin cebirsel yapısını derinlemesine anlamak için ayar teorilerine kısa bir giriş yapacağız. Bunun için basit bir örnekle başlayıp, bu örneğin eylemini ve bu eylemi değişmez bırakan yerel dönüşümleri inceleyeceğiz. Son olarak SU(N) ayar teorilerinden bahsedecek ve özel durum olarak SU(2) ayar teorisinin özelliklerini inceleyeceğiz.

Bundan sonra, kısaca diferansiyel geometrinin genel yönlerini inceleyeceğiz. Uzunca “Manifold nedir?" gibi sorulara cevap aramaktansa ulaşmak istediğimiz nesneye giden yolu açmak ve temizlemek istediğimiz için hızlıca differansiyel formlara, vektörlere ve tensörlere geçeceğiz. Ardından metrik ve tetradları tanıtacak ve dış türev operasyonunun nasıl yapılacağını aktaracağız. Daha sonra, eğrilikleri ve spin bağıntılarını nasıl elde edeceğimizi inceleyeceğiz. Bu bölümde tartışacağımız konular, bize kütleçekimini bir ayar teorisi olarak incelemek için ana matematiksel çerçeveyi sağlayacaktır.

Üçüncü olarak, yerçekimi teorilerini, özellikle genel görelilik teorilerini bir ayar teorisi olarak anlamaya çalışacağız. Bunun için ilk olarak, Genel Görelilik'in “ikinci mertebe formu”ndan tartışmamıza başlayacağız.

Bu formalizm, Einstein-Hilbert eylemi olarak da bilinen aşağıdaki eylem ile başlar:

$$S_{\mathrm{EH}} = \int \mathrm{d}^d x \sqrt{-g} R. \tag{1}$$

$S = S_{\mathrm{EH}} + S_{\mathrm{madde}}$ olarak verilebilecek eylemde, varyasyon ilkesi kullanılarak

$$G_{\mu\nu} = T_{\mu\nu} \tag{2}$$

hareket denklemi elde edilebilir. Bu denklemde $G_{\mu\nu}$ Einstein tensörü, $T_{\mu\nu}$ ise enerji-momentum tensörü olarak bilinen nesnelerdir [26].

Akabinde Genel Görelelik'in birinci mertebe formalizmi ve eylemini tanımlayacağız. Burada ise Einstein-Cartan eylemi olarak bilinen aşağıdaki eylemi kullanacağız:

$$S_{\mathrm{EC}} = \frac{1}{2}\int \varepsilon_{abcd} e^a \wedge e^b \wedge R^{cd}. \tag{3}$$

Ardından benzer bir şekilde varyasyon ilkesi kullanarak

$$\varepsilon_{abcd} e^b \wedge R^{cd} = \varepsilon_{abcd}\mathscr{T}^{bcd}, \tag{4}$$

hareket denklemini elde edeceğiz. Bu denklem ise Einstein denkleminin birinci mertebe formalizmi olarak adlandırılır [27]. Bu bölümün son kısmında, Poincaré cebirinden başlayacak ve ilgili cebri ayarlayarak Genel Görelelik'i elde etmeye çalışacağız. Bu kısım, bize eğri uzay-zamanda genel koordinat dönüşümlerinin ayarlanmış Poincaré dönüşümlerine denk olduğunu gösterecek. Ancak daha ayrıntılı incelemeler, bir kütleçekim teorisi için bunun yeterli olmadığını gösterecek ve $\delta_{\mathrm{cgct}}(\xi) = \delta_{\mathrm{gct}}(\xi) - \delta\left(\xi^\mu A_\mu\right)$ formunda verilen "kovaryant genel koordinat dönüşümleri"ni tanımlamamız gerekecek. Bir miktar hesaptan sonra görüleceği gibi, bu yeni tanım bize eğrilikler üzerine kısıtları verecektir. Böylece Poincaré cebrinden başlayarak bir kütleçekim teorisi elde edebileceğiz [19].

Bir sonraki kısımda ise varış noktamız olan çok metrikli teorilerinin temel motivasyonlarından olan kütleli yerçekimi teorilerini inceleyeceğiz. İncelememizi sürdürürken seçilmiş bazı modellere derinlemesine bakacağız. Bu süreçte kütleli yerçekimi teorileri ile çoklu metrik teoriler arasındaki ilişki ve bağıntılar açıkça görülecektir. Bu ilişki ve bağıntılara ek olarak " ölçekleme limiti" dediğimiz bir uygulamanın da ilk örneklerini göreceğiz. Kabaca özetlemek gerekirse ölçekleme limiti elinizdeki mevcut sistemde istenmeyen terimlerin yok edilmesi için kullanılan bir yöntemdir. Bu yöntemde bahsi geçen yapılar $\lambda << 1$ olacak şekilde λ parametresi ile seri açılımına tabi tutulur. İşlemlerin en son geldiği nokta λ parametresi mutlakiyet ile 0'a götürülür. Ancak her sistemde sağlıklı sonuçlar vereceği garanti değildir. Hatta bazı durumlarda sistemin kendi içerisinde tutarsızlıklara bile sebep olabilir.

En son bölümde, temel motivasyonumuz olan aşırı göreceli kütleçekim teorilerine geçiş yapacağız. Bu yolda ilerlemek için ise "Carroll Cebri" ve "Carroll Ötesi cebri"ni elde edeceğiz. Bu cebirleri elde etmek için öncelikle İnönü-Wigner daraltmasını kullanıp Poincaré cebrini parçalarına ayırarak Carroll Cebri'ni elde edeceğiz. Burada parçalara ayırmadan kasıt cebirin sahip olduğu üreteçlerin iki kümeye ayrılmasıdır. Bu kümeler tek ve çift kümeler diye adlandırılır. Bu cebrin Lie Cebri genişletmesi sayesinde Carroll Ötesi Cebri'ni elde edeceğiz. Bahsi geçen cebirleri elde ettikten sonra istediğimiz mertebedeki kütleçekim teorisine ulaşmanın kapısını aralayacağız. Bu cebirlerin her bir elemanını bir ayar alanıyla eşledikten sonra karşılık geldikleri ayarlar altında değişmez olan eylemi yazacağız. Göreceğiz ki Newtonyen kütleçekim teorilerinde görülen "boyut karmaşası" (D=3, D=4 durumları için oluşan Lagrange yoğunluğunun sırasıyla çift, tek mertebelerde oluşturulması) bizim için bir sorun teşkil etmeyecek. Çünkü açıkça hesapladığımızda aşırı göreceli kütleçekim teorileri için Lagrange yoğunluğunun daima tek mertebelerde oluştuğunu göreceğiz. Carroll Cebri'nin bir diğer güzel tarafı ise Galileo Cebri'nde olduğu gibi sisteme fazladan

eklenmesi gereken $U(1)$ yüklerine ihtiyaç duymamasıdır. Bunun sebebi cebirin üreteçleri iki kümeye ayrıldığında birebir olarak karşılık geldiği elemanlar vardır. Yani sonuç olarak elimizde birbirini izleyen iki kopya olacaktır bu sebeple arada doldurulması gereken bir boşluk olmayacaktır ve $U(1)$ yükleri sisteme müdahale edemeyecektir.

En son olarak çok metrikli teorinin daraltılmasıyla elde edeceğimiz ve cebirden elde ettiğimiz Carroll Kütleçekim Lagrange yoğunluğu:

$$\mathscr{L}_1 = -2\varepsilon_{abc}R^{ab}(\omega)\wedge e^a\wedge\tau + 2\varepsilon_{abc}R^a(\omega)\wedge e^b\wedge e^c \tag{5}$$

Bir sonraki tutarlı mertebesi olarak karşımıza çıkan durum olan Carroll Ötesi Cebri'ne de aynı süreci uyguladığımızda elde edeceğimiz Carroll Ötesi Kütleçekim Lagrange yoğunluğu ise

$$\begin{aligned}\mathscr{L}_3 \;=\; & -2\varepsilon_{abc}R^{ab}(s)\wedge e^c\wedge\tau - 2\varepsilon_{abc}R^{ab}(\omega)\wedge t^c\wedge\tau - 2\varepsilon_{abc}R^{ab}(\omega)\wedge e^c\wedge m \\ & +2\varepsilon_{abc}R^a(b)\wedge e^b\wedge e^c + 4\varepsilon_{abc}R^a(\omega)\wedge e^b\wedge t^c\,.\end{aligned} \tag{6}$$

olacaktır.

1. INTRODUCTION

There are many important moments in theoretical physics history, but only a few were as great as the construction of General Relativity (GR) by Albert Einstein at the beginning of the 20^{th} century. It drastically changed the way of research in theoretical physics.

Although Newton's gravitation description has fundamentally distinct differences from the description which is provided by the GR, most of our daily life experiences in the presence of gravity can be explained with very good precision using Newtonian mechanics. There are two ways to interpret this theory. The first one is the geometric point of view as stated in [28], Newtonian Gravity (NG) can be reformulated from a geometric point of view [29, 30]. The structure with this point of view will lead to Newton-Cartan (NC) geometry. The second one is the algebraic point of view [31, 32] which is the one we will deal with it. Roughly speaking, the algebraic point of view can be considered as constructing corresponding algebra for the model via scaling the generators of the algebra. Then taking a specific limit. For example, in the case of Newtonian Gravity, this limit corresponds to $c \to \infty$. Also, there is another limit which will be the main motivation and the topic of the thesis. Let us go one step back and see our general perspective.

As you may see from Figure 1.1, any physical model could be in one of the three regimes based on its characteristic. From our daily life experience, we accept that we are living in the Non-Relativistic Regime. Of course, we can rely on Newton and say " It is what it is " or we can try to understand and obtain this regime with the help of a geometric or algebraic point of view. As it has been told, we will focus on the algebraic point of view. In order to obtain a Non-Relativistic Regime, one starts with a general Relativistic Regime. After appropriately scaling the corresponding generators of the algebra along with matching every term in the algebra with gauge fields, the limit $c \to \infty$ case is considered and the result will be the Non-Relativistic Regime of the spoken theory.

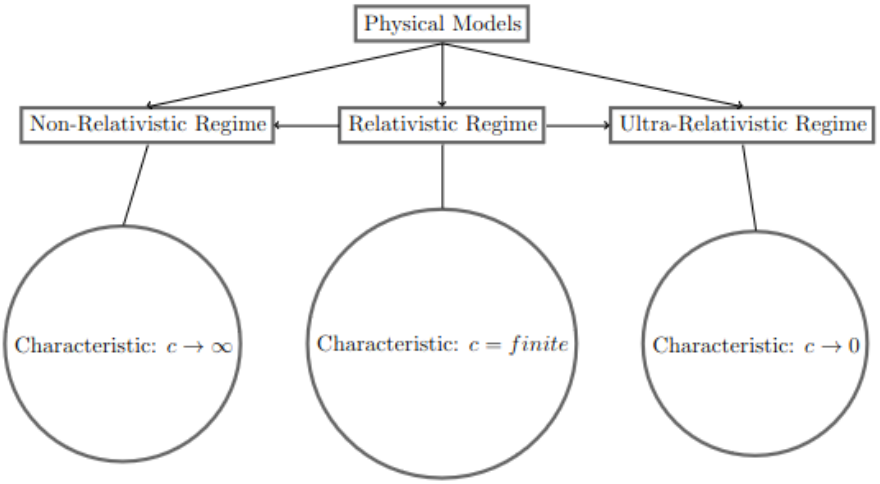

Figure 1.1 : Difference between the physical regimes based on their characteristic

However, one might wonder about the case, $c \to 0$. Again, we scale our generators of the algebra appropriately from the beginning because as you can see in Figure 1.1, the arrow has just one way to go. The topic that this limit will bring is called Carroll Algebra and it corresponds to Ultra-Relativistic Regime [33, 34]. For more details, in addition to previous references, see [35–37]. Of course, this is not the final destination. We are able to expand our algebra thanks to *Lie Algebra Expansion* and this specific methodology will provide us sunlight that will help us about finding our next-order algebra i.e. Beyond the Carroll Algebra. As we will discover, our methodology does not include some jumbo mumbo tricks such as specifying some expansion coefficient via known structures and then interpreting them like a new concept.

1.1 Purpose of the Thesis

Throughout this thesis, we will try to show how an ultra-relativistic gravity theory can be given and how can be obtained from a relativistic theory, and the concept of "gauge fields" will play a crucial role. To obtain the results, first, we should study the general structure of gauge theories and understand underlying symmetry properties. Then, we will quickly review the differential geometry topics. After gathering this knowledge, we will apply general rules to obtain gravity as a gauge theory. This result will give us a very powerful framework to study gravity which is also called tetrad formalism [27]. We will also show how GR can be obtained from an algebraic point

of view, i.e. gauging Poincaré algebra [19, 25, 31, 38]. In the 4^{th} part, we will examine and discuss some of the massive and bi-metric gravity[1] cases. There is an undeniable relation between massive gravity and bi-metric gravity as we will discuss. We will also review scaling limits[2] of these theories at the end of the section. As a last part, we will begin by constructing Carroll algebra. Furthermore, using this algebra we will obtain an action for Carroll and Beyond the Carroll Gravity, of course via matching every generator of algebra with a gauge field. In the end, we will finalize this thesis by obtaining Beyond the Carroll and Carroll Gravity action from a *multi-metric theory*.

1.2 Conventions

1.2.1 Units

Unless otherwise stated, we will use the geometrized unit system, i.e. we will set $16\pi G = 1$ and $c = 1$.

1.2.2 Metric and totally anti-symmetric tensor conventions

We will use Latin scripts for internal indices or for spatial indices and Greek scripts for spacetime indices. Unless otherwise stated, we will use *mostly positive* signature convention for both spacetime and internal metrics, e.g.

$$\eta_{AB} = \begin{pmatrix} - & & & \\ & + & & \\ & & + & \\ & & & \ddots \end{pmatrix} \tag{1.1}$$

Throughout this thesis, we will use the convention for totally anti-symmetric tensor given below:

$$\varepsilon_{0abc} = -\varepsilon^{0abc} = -1 \tag{1.2a}$$

$$\varepsilon_{abc} = \varepsilon^{abc} = 1 \tag{1.2b}$$

[1] In literature, bi-metric gravity and bi-gravity or without "$-$" are used interchangeably.

[2] That part will be crucial for our last section.

2. GENERAL ASPECTS OF GAUGE THEORIES

The gauge theories have undeniable roles to describe nature [39–43]. The symmetries related to these gauge theories provide extra degrees of freedom to the system that we are trying to describe rather than being as fundamental as particles which are responsible for the system. Because one can easily fix the gauge and see the redundancies of the system.

In this section, we briefly discuss several gauge theories by focusing on their symmetries such as U(1) and $SU(2)_L$. Of course, these gauge groups were not chosen by personal pleasures. These gauge groups correspond to the gauge theories which describe Electromagnetic Interactions at the quantum level i.e. *Quantum Electrodynamics* (QED) and Weak Interactions, respectively. Since examining the Lagrangian is the easiest way to see symmetries, we will start with Lagrangian.

2.1 U(1) Gauge Theory

In this case, we will investigate QED Lagrangian for Dirac Fields:

$$\mathscr{L}_{QED} = \overline{\Psi}\left(i\gamma_\mu D^\mu - m\right)\Psi - \frac{1}{4}(F_{\mu\nu})^2 \tag{2.1}$$

where γ_μ are Dirac Matrices,

$$D_\mu \equiv \partial_\mu + ieA_\mu(x) \qquad \text{and} \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{2.2}$$

Here D_μ is called *gauge covariant derivative*. A_μ is the electromagnetic vector potential and $F_{\mu\nu}$ is the field strength of it. Also e = -$|e|$ is the electron charge. If you ever want to change it to any fermion, it is clear that $e \to Q$ with charge Q.

As it has been motivated, we first start with examining the corresponding Lagrangian. It is obvious that this Lagrangian has U(1) gauge symmetry i.e. U(1) gauge transformations leave this Lagrangian invariant of course up to a total derivative term

$$\Psi(x) \to e^{i\alpha(x)}\Psi(x) \qquad \text{and} \qquad A_\mu \to A_\mu - \frac{1}{e}\partial_\mu\alpha(x) \tag{2.3}$$

This gives us

$$\begin{aligned}\mathscr{L}' &= e^{-i\alpha(x)}\overline{\Psi}\left(i\gamma_\mu D^m + \gamma_\mu \partial^\mu \alpha(x) - m\right) e^{i\alpha(x)}\Psi - \frac{1}{4}(F_{\mu\nu})^2 \\ &= e^{-i\alpha(x)}e^{i\alpha(x)}\overline{\Psi}\left(i\gamma_\mu D^\mu - m\right)\Psi - \frac{1}{4}(F_{\mu\nu})^2 \\ &= \mathscr{L}\end{aligned} \tag{2.4}$$

As you can see, $\mathscr{L}' = \mathscr{L}$, i.e. the Lagrangian is left invariant under the local transformation (2.3). This transformation is called "gauge transformation" because the function α is an arbitrary function of coordinates [41]. Now let us calculate the equation of motions (e.o.m.) :

$$\begin{aligned}\text{for} \quad \Psi: \qquad & i\gamma_\mu D^\mu \Psi = m\Psi \\ \text{for} \quad A_\mu: \qquad & \partial_\mu F^{\mu\nu} = e\overline{\Psi}\gamma^\nu\Psi = e j^\nu\end{aligned} \tag{2.5}$$

Clearly, these two e.o.m.s are not independent of each other, i.e. for A_μ, it contains the e.o.m. for ψ. We understood the following points from this observation:

- The given system has (at least) a gauge symmetry which is based on Lie algebra.
- One of the e.o.m.s contains other.

2.2 SU(N) Gauge Theories

In this section, we will discuss the general construction of SU(N) theories. Even though we used a well-known U(1) invariant QED Lagrangian in the previous section, for this section we will step further and construct the desired theory from scratch. We will illustrate only SU(2) gauge field because this is the simplest form one can obtain with non-trivial matrix representation [13, 40]. However, the methods introduced in this chapter will be applied to all SU(N) gauge theories.[1]

2.2.1 Lie groups

The groups that are n-dimensional differential manifolds with continuous group elements are called in general *Lie groups* [40]. Since "having an identity element" is a group postulate, SU(N) Lie groups have an identity element $\mathbb{1}$ [44]. Thus one can write any group element of SU(N) groups as

$$U = \exp(i\theta^a T_a)\mathbb{1}, \tag{2.6}$$

[1] Keep in mind that for U(1) case transformation is generated just by a function.

where T_a are called the *group generators* and θ^a are some parameters. These generators form a *Lie algebra* $\mathfrak{su}$(N), which is identified by the commutation relation given as:

$$[T_a, T_b] = i f_{ab}{}^c T_c, \tag{2.7}$$

where $f_{ab}{}^c$ are called the *structure constants* [13]. As could be seen from (2.7), if $f_{ab}{}^c = 0$, this corresponds to *Abelian Group*, otherwise *Non-Abelian Group* [39,40,44]. These generators satisfy an important identity which is called *Jacobi identity* given below:

$$[T_a, [T_b, T_c]] + [T_b, [T_c, T_a]] + [T_c, [T_a, T_b]] = 0. \tag{2.8}$$

which is equivalent to

$$f_{ad}{}^e f_{bc}{}^d + f_{cd}{}^e f_{ab}{}^d + f_{bd}{}^e f_{ca}{}^d = 0 \tag{2.9}$$

2.2.2 SU(2) gauge theory

In Section 2.1, we studied a theory that has a U(1) symmetry. The transformations related to this symmetry are just phase rotations if we have just a kinetic term. But, there are more complex symmetries in the nature[2]. As an example, let us study the SU(2) gauge theory. The methods for studying this gauge theory can be generalized easily.

Let us consider a kinetic Lagrangian with 2 Dirac fermions:

$$\mathscr{L} = \sum_{j=1}^{2} \left[\overline{\psi}_j \left(i\partial\!\!\!/ - m \right) \psi_j \right] = \overline{\Psi} \left(i\partial\!\!\!/ - m \right) \Psi \tag{2.10}$$

where we used $\partial\!\!\!/ = \gamma_\mu \partial^\mu$ and

$$\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \tag{2.11}$$

is a doublet of Dirac fields.

This Lagrangian is invariant under the transformation:

$$\Psi \longrightarrow \Psi' = U\Psi, \tag{2.12}$$

where

$$U = \exp\left(i\alpha^a \frac{\sigma_a}{2} \right) \tag{2.13}$$

[2]For example as a first thing came to mind, the celebrated Standard Model's symmetry group can be given as $SU(3)_C \bigotimes SU(2)_L \bigotimes U(1)$.

which is a global SU(2) transformation [39, 40]. By gauging this transformation, i.e. making the transformation local, one can obtain a transformation such that

$$\Psi(x) \longrightarrow \Psi'(x) = U(x)\Psi(x) \tag{2.14a}$$

$$U(x) = \exp\left(i\alpha^a(x)\frac{\sigma_a}{2}\right) \tag{2.14b}$$

should the Lagrangian **(2.10)** be invariant [39, 40]. Here $a = \{1,2,3\}$, because SU(2) has $2^2 - 1 = 3$ dimensions. From (2.14b), one can easily see that the generators of this theory are simply the Pauli matrices

$$T_a = \frac{\sigma_a}{2}, \tag{2.15}$$

Furthermore, the structure constants of this theory are equal to Levi-Civita symbols because

$$\begin{aligned}[T_a, T_b] &= \left[\frac{\sigma_a}{2}, \frac{\sigma_b}{2}\right] \\ &= \frac{1}{4}[\sigma_a, \sigma_b] = \frac{1}{4}2i\varepsilon_{ab}{}^c\sigma_c = i\varepsilon_{ab}{}^c T_c \\ &= i f_{ab}{}^c T_c\end{aligned} \tag{2.16}$$

and this gives us that $f_{ab}{}^c = \varepsilon_{ab}{}^c$ for SU(2) transformations. As it has been mentioned in [40], the *Lie-algebra-valued* fields can be used appropriately instead of gauge fields. These new fields are defined as

$$A_\mu = A_\mu^a T_a \tag{2.17}$$

As you can see it is nothing but a small modification since for U(1) $T_a = \mathbb{1}$. Similar to the definition of the gauge covariant derivative in the U(1) section, one can define the gauge covariant derivative as

$$D_\mu \equiv \partial_\mu - igA_\mu(x), \tag{2.18}$$

Here g is coupling constant [45], e.g. $e \to -g$. Since this derivative is a “covariant" derivative, it has to be transformed covariantly, i.e.

$$D_\mu\Psi \longrightarrow \left(D_\mu\Psi\right)' = U(x)\left(D_\mu\Psi\right) \tag{2.19}$$

Let us make the transformation explicitly:

$$\begin{aligned}\left(D_\mu\Psi\right)' &= \partial_\mu\Psi'(x) - igA'_\mu(x)\Psi'(x) \\ &= \partial_\mu\left(U(x)\Psi(x)\right) - igA'_\mu(x)U(x)\Psi(x) \\ &= \partial_\mu U(x)\Psi(x) + U(x)\partial_\mu\Psi(x) - igA'_\mu(x)U(x)\Psi(x).\end{aligned} \tag{2.20}$$

But also we have,

$$U(x)\left(D_\mu \Psi\right) = U(x)\partial_\mu \Psi(x) - igU(x)A_\mu(x)\Psi(x) \tag{2.21}$$

Since (2.20) is equal to (2.21), we obtain

$$\partial_\mu U(x) - igA'_\mu(x)U(x) = -igU(x)A_\mu(x). \tag{2.22}$$

This implies that

$$\begin{aligned} A'_\mu(x)U(x) &= U(x)A_\mu(x) - \frac{i}{g}\partial_\mu U(x) \\ \text{Then,} \qquad A'_\mu(x) &= U(x)A_\mu(x)U^{-1}(x) - \frac{i}{g}\partial_\mu U(x)U^{-1}(x). \end{aligned} \tag{2.23}$$

Let us now imply the infinitesimal form of the transformation in (2.14b) to our result. The infinitesimal form of (2.14b) can be given as

$$U(x) = \exp(i\alpha^a(x)T_a) = \mathbb{1} + i\alpha^a(x)T_a + \mathscr{O}\left(\alpha^2\right), \tag{2.24}$$

and similarly

$$U^{-1}(x) = \exp(-i\alpha^a(x)T_a) = \mathbb{1} - i\alpha^a(x)T_a + \mathscr{O}\left(\alpha^2\right). \tag{2.25}$$

If we substitute these by dropping the higher-order terms i.e. other than the first-order terms:

$$\begin{aligned} A'_\mu(x) =& (\mathbb{1} + i\alpha^a(x)T_a)A_\mu(x)\left(\mathbb{1} - i\alpha^b(x)T_b\right) \\ & - \frac{i}{g}\partial_\mu\left(\mathbb{1} + i\alpha^a(x)T_a\right)\left(\mathbb{1} - i\alpha^b(x)T_b\right) \\ =& A_\mu + i\alpha^a T_a A_\mu - i\alpha^a A_\mu T_a + \frac{1}{g}\partial_\mu \alpha^a T_a \end{aligned} \tag{2.26}$$

By writing it explicitly,

$$\begin{aligned} A'^b_\mu T_b &= A^b_\mu T_b + i\alpha^a A^b_\mu T_a T_b - i\alpha^a A^b_\mu T_b T_a + \frac{1}{g}\partial_\mu \alpha^b T_b \\ &= A^b_\mu T_b + i\alpha^a A^b_\mu [T_a, T_b] + \frac{1}{g}\partial_\mu \alpha^b T_b \\ &= A^b_\mu T_b - \alpha^a A^b_\mu f_{ab}{}^c T_c + \frac{1}{g}\partial_\mu \alpha^b T_b \\ &= A^a_\mu T_a - \alpha^b A^c_\mu f_{bc}{}^a T_a + \frac{1}{g}\partial_\mu \alpha^a T_a \\ A'^a_\mu T_a &= A^a_\mu T_a - \alpha^b A^c_\mu \varepsilon_{abc} T_a + \frac{1}{g}\partial_\mu \alpha^a T_a \end{aligned} \tag{2.27}$$

Therefore,

$$A'^a_\mu = A^a_\mu + \frac{1}{g}\partial_\mu \alpha^a - \varepsilon_{abc}\alpha^b A^c_\mu. \tag{2.28}$$

This is the transformation of the gauge field A^a_μ under SU(2) transformations [40]. Now we can define the field strength of the SU(2) gauge theory[3]. This object can be given as

$$\begin{aligned}\mathscr{F}_{\mu\nu} \equiv \frac{i}{g}\left[D_\mu, D_\nu\right] &= \frac{i}{g}\left[\left(\partial_\mu - igA_\mu\right), \left(\partial_\nu - igA_\nu\right)\right] \\ &= \partial_\mu A_\nu - \partial_\nu A_\mu - ig\left[A_\mu, A_\nu\right]\end{aligned} \tag{2.29}$$

or in components, where $\mathscr{F}_{\mu\nu} = \mathscr{F}^a_{\mu\nu} T_a$,

$$\mathscr{F}^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\varepsilon^a{}_{bc} A^b_\mu A^c_\nu, \tag{2.30}$$

and a SU(2) gauge invariant Lagrangian can be given as

$$\mathscr{L} = -\frac{1}{4}(\mathscr{F}^a_{\mu\nu})^2 + \sum_{j=1}^{2}\left[\psi_j\left(i\not{D} - m\right)\psi_j\right], \tag{2.31}$$

As can be seen easily, this field strength is very similar to the U(1) field strength. In fact, the field strengths of all gauge theories can be given in the form of (2.29). Since U(1) is an Abelian group, i.e. $T_a = \mathbb{1}$ as it has been mentioned above, the last commutator in (2.29) is trivially zero for electromagnetic field strength [40]. These results can be generalized to any SU(N) theory.

[3] If we have U(1) and SU(2) structures simultaneously, we must construct each field strength separately.

3. A BRIEF REVIEW OF DIFFERENTIAL GEOMETRY

The main reason for this chapter is to make an introduction to differential geometry, which is the main mathematical concept for understanding most of the gravity theories e.g. GR, and the relation between gauge and gravity theories. For this chapter, we are aiming to present a very brief review of the topics of differential geometry. We will not study differential geometry chronologically instead we will study it by discussing the main structures[1]. So let us start with vectors and tensors.

3.1 Beginning: The Vectors

In any physical structure, we deal with vectors, for example, position *vector* in three-dimensional vector analysis or the metric tensor[2] in General Relativity [14–16, 46]. As it is required among humankind, we need a language to understand and describe. That is the language of differential geometry. Since when we try to visualize the concept of a vector, we immediately think of an arrow connecting two points that are infinitesimally close to each other. Therefore, it is expected that it will be described by derivative in a mathematical language. So, one can start with the vector V as it is the tangent vector to some curve belonging to the manifold. Let us consider the manifold M has coordinate x^μ in some patch which resulted in a curve as $x^\mu = x^\mu(t)$, with the parameter t along the path. Since we gathered the necessary information about tangent vectors, we are able to write it as

$$V = \frac{\partial}{\partial t} \tag{3.1}$$

Keep in mind that V is coordinate-independent. But also

$$V = V^\mu \frac{\partial}{\partial x^\mu} = V^\mu \partial_\mu \tag{3.2}$$

where $V^\mu = \frac{dx^\mu}{dt}$. We observe that the components V^μ are coordinate *dependent* while the vector V is not. Also, we are able to obtain the transformation rule under general

[1] i.e. only useful parts for our theme.

[2] Since tensors are just the generalization of vectors in a manner.

coordinate transformations $x^\mu \to x'^\mu = x'^\mu(x^\nu)$ via the chain rule

$$V = V^\mu \partial_\mu = V^\mu \frac{\partial x'^\nu}{\partial x^\mu} \partial'_\nu \tag{3.3}$$

where $\partial'_\nu = \partial/\partial x'^\nu$. By definition, the coefficients of the ∂'_ν are the components of V with respect to the primed coordinate system. Then

$$V'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} V^\nu \tag{3.4}$$

That is the transformation property of a vector's components. Its generalization to multiple indices is straightforward which gives the transformation of tensors, as an example in the case of p-index tensor T,

$$T = T^{\mu_1 \cdots \mu_p} \partial_{\mu_1} \bigotimes \cdots \bigotimes \partial_{\mu_p} \tag{3.5}$$

Therefore, following the same footprints, the components of this p-index tensor transform as

$$T'^{\mu_1 \ldots \mu_p} = \frac{\partial x'^{\mu_1}}{\partial x^{\nu_1}} \ldots \frac{\partial x'^{\mu_p}}{\partial x^{\nu_p}} T^{\nu_1 \ldots \nu_p} \tag{3.6}$$

3.2 As Another Side: The Covectors

One can also consider the objects with components having lower indices as known as *covectors*. Let us consider a function f. Thanks to the chain rule, df is

$$df = \partial_\mu f dx^\mu \tag{3.7}$$

By the same logic as V, one can consider df as a geometrical, coordinate-independent object. It is easy to see that its components are $\partial_\mu f$. Actually, df is a very special case of a covector. It may be a general covector U where U_μ is its components

$$U = U_\mu dx^\mu \tag{3.8}$$

Since U is coordinate-independent, U_μ transform under general coordinate transformations via following the footprints of the Section 3.1

$$U = U_\nu \frac{\partial x^\nu}{\partial x'^\mu} dx'^\mu \tag{3.9}$$

As we said, the coefficients of dx'^μ are the components of U'_μ in the primed coordinate frame. Therefore, the transformation property of 1-form components is given by

$$U'_\mu = \frac{\partial x^\nu}{\partial x'^\mu} U_\nu \tag{3.10}$$

Again, it is straightforward to generalize to cotensors with p index:

$$U = U_{\mu_1 \ldots \mu_p} dx^{\mu_1} \bigotimes \ldots \bigotimes dx^{\mu_p} \tag{3.11}$$

The transformation rule for the components $U_{\mu_1 \cdots \mu_p}$ under general coordinate transformation is given by

$$U'_{\mu_1 \cdots \mu_p} = \frac{\partial x^{\nu_1}}{\partial x'^{\mu_1}} \cdots \frac{\partial x^{\nu_p}}{\partial x'^{\mu_p}} U_{\nu_1 \cdots \nu_p} \tag{3.12}$$

It is obvious that since w_μ transforms in the opposite way of V^μ, their combination i.e. $w_\mu V^\mu$ is invariant under general coordinate transformations,

$$\begin{aligned} w'_\mu V'^\mu =& \frac{\partial x^\nu}{\partial x'^\mu} \frac{\partial x'^\mu}{\partial x^\sigma} w_\nu V^\sigma \\ =& \delta^\nu_\sigma w_\nu V^\sigma \\ =& w_\nu V^\nu \end{aligned} \tag{3.13}$$

This is the scalar product or inner product if you would like to call it that way, of V with w. However, one can also express this relation in a geometrical way without choosing any specific coordinates i.e. $\langle w, V \rangle$. Since we have an identity

$$\langle dx^\mu, \partial_\nu \rangle = \delta^\mu_\nu \tag{3.14}$$

we obtain:

$$\begin{aligned} \langle w, V \rangle =& w_\mu V^\nu \langle dx^\mu, \partial_\nu \rangle \\ =& w_\mu V^\nu \delta^\mu_\nu = w_\mu V^\mu \end{aligned} \tag{3.15}$$

If we apply these steps to differential df:

$$\langle df, V \rangle = \partial_\mu f V^\nu \langle dx^\mu, \partial_\nu \rangle = V^\mu \partial_\mu f \tag{3.16}$$

In other words, since the original definition of V in (3.1) is a differential operator, in that case, the inner product of df and V is just the directional derivative of f throughout the curve parametrized by t: $\langle df, V \rangle = V(f) = \partial f / \partial t$.

3.3 Differential Forms

Now let us focus on the class of cotensors whose components are totally antisymmetric[3]

$$U_{\mu_1 \ldots \mu_p} = U_{[\mu_1 \ldots \mu_p]} \tag{3.17}$$

[3]They will be crucial for our last chapter.

where $[\cdots]$ antisymmetrize anything inside of itself:

$$U_{[\mu\nu]} = \frac{1}{2!}\left(U_{\mu\nu} - U_{\nu\mu}\right) \tag{3.18}$$

If the indices of the cotensor is antisymmetric then it will lead to making an antisymmetric projection, as one can easily guess, on the tensor product of 1-forms dx^{μ}:

$$dx^{\mu} \wedge dx^{\nu} = dx^{\mu} \bigotimes dx^{\nu} - dx^{\nu} \bigotimes dx^{\mu} \tag{3.19}$$

where $\wedge$ symbol is called *wedge* and left-hand side of (3.19) is called *wedge product*. Cotensors with p indices are called p-forms. And they are given below,

$$A = \frac{1}{p!} A_{\mu_1 \ldots \mu_p} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_p} \tag{3.20}$$

As a result if A is a p-form and B is q-form,

$$A \wedge B = (-1)^{pq} B \wedge A \tag{3.21}$$

3.4 Taking the Derivatives in a Geometric Way

In this section, we will understand the structure that came by using the *exterior derivative* which is symbolized as d. It produces a $(p+1)$-form when it is used on a p-form field. If it operates on a 0-form i.e. function, it just takes the differential of it as in (3.7). As a more general case e.g. on a p-form:

$$\begin{aligned} \omega &= \frac{1}{p!} \omega_{\mu_1 \ldots \mu_p} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_p} \\ d\omega &= \frac{1}{p!} \left(\partial_{\nu} \omega_{\mu_1 \ldots \mu_p}\right) dx^{\nu} \wedge dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_p} \end{aligned} \tag{3.22}$$

Keep in mind that from our definition of p-form, the components of the $(p+1)$ -form $d\omega$ are

$$(d\omega)_{\nu\mu_1 \ldots \mu_p} = (p+1)\, \partial_{[\nu} \omega_{\mu_1 \ldots \mu_p]} \tag{3.23}$$

It is obvious that if A is a p-form and B is a q-form then, thanks to the Leibnitz rule, we obtain

$$d(A \wedge B) = dA \wedge B + (-1)^p A \wedge dB \tag{3.24}$$

As you may see from **(3.22)**, it gives zero when it is used twice. Because d is an *antisymmetric* derivative while partial derivatives are *symmetric*. Let us focus on Maxwell Electrodynamics since it will be a good example for the use of not

only differential forms but also exterior derivative. The vector potential is a 1-form, $A = A_\mu dx^\mu$. The Maxwell field strength is a 2-form, $F = \frac{1}{2}F_{\mu\nu}dx^\mu \wedge dx^\nu$, and one can easily construct it from A by taking the exterior derivative as

$$F = dA = \partial_\mu A_\nu dx^\mu \wedge dx^\nu = \frac{1}{2}F_{\mu\nu}dx^\mu \wedge dx^\nu \tag{3.25}$$

We read off the components as $F_{\mu\nu} = 2\partial_{[\mu}A_{\nu]}$. Note that $dF = 0$ trivially since $dF = d^2A$.

3.5 Spin Connection and Curvature 2-Forms

In this section, let us begin by considering what might come out when we take the square root of metric $g_{\mu\nu}$. In order to do so we need to introduce some structure as known as vielbein. It is the basis of 1-forms $e^a = e^a_\mu dx^\mu$ and its components e^a_μ which satisfy

$$g_{\mu\nu} = \eta_{ab}e^a_\mu e^b_\nu \tag{3.26}$$

where the Latin indices represent the local-Lorentz indices or tangent-space indices. Here, η_{ab} is a flat metric whose components are constant. The signature of η_{ab} must be the same as that of $g_{\mu\nu}$ since we will be studying the cases in GR with Minkowskian signature

$$\eta_{ab} = \text{diag}\,(-1, 1, ..., 1) \tag{3.27}$$

Likewise, if we attempt to study in Euclidean signature, we must change the first component's sign.

Let us introduce the spin connection i.e. connection 1-forms, $\omega^a{}_b = \omega^a{}_{b\mu}dx^\mu$ and the torsion 2-forms $T^a = \frac{1}{2}T^a_{\mu\nu}dx^\mu \wedge dx^\nu$. They have a bonding relationship as

$$T^a = de^a + \omega^a{}_b \wedge e^b \tag{3.28}$$

As a further step, let us define curvature 2-forms $\Theta^a{}_b$ as

$$\Theta^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b \tag{3.29}$$

Also, metric compatibility requires that

$$\omega_{ab} = -\omega_{ba} \tag{3.30}$$

where $\omega_{ab} = \eta_{ac}\omega^{c}{}_{b}$. In addition to the requirement of metric compatibility, we usually choose a torsion-free spin connection, i.e. vanishing torsion 2-forms T^{a} in (3.28). These two determine $w^{a}{}_{b}$ completely:

$$de^{a} = -\omega^{a}{}_{b} \wedge e^{b}, \quad \omega_{ab} = -\omega_{ba} \tag{3.31}$$

Let us continue with curvature 2-form $\Theta^{a}{}_{b}$. Its indices are antisymmetric under the exchange of each other just like the Riemann tensor:

$$\Theta^{a}{}_{b} = \frac{1}{2} R^{a}{}_{b\mu\nu} dx^{\mu} \wedge dx^{\nu} \tag{3.32}$$

One can also free to use the vielbein e^{a}_{μ} to make a transition between different types of indices such as coordinate indices ν and tangent-space indices b. However, in order to be able to do that inverse of the vielbein, E^{μ}_{a}, is required. Also, it satisfies the given below,

$$E^{\mu}_{a} e^{a}_{\nu} = \delta^{\mu}_{\nu}, \qquad E^{\mu}_{a} e^{b}_{\mu} = \delta^{a}_{b} \tag{3.33}$$

Then, we are ready to read off Riemann tensor components completely constructed on the tangent-frame basis as

$$R^{a}{}_{bcd} = E^{\mu}_{c} E^{\nu}_{d} R^{a}{}_{b\mu\nu} \tag{3.34}$$

Be careful that we distinguish these tensors by examining the indices that they carry. In terms of $R^{a}{}_{bcd}$, it is easy to see

$$\Theta^{a}{}_{b} = \frac{1}{2} R^{a}{}_{bcd} e^{c} \wedge e^{d} \tag{3.35}$$

One can also construct two further quantities based on the Riemann tensor which are the Ricci tensor R_{ab} and Ricci scalar R:

$$R_{ab} = R^{c}{}_{acb}, \qquad R = \eta^{ab} R_{ab} \tag{3.36}$$

Of course, some *familiar* concepts of GR can be obtained easily. For example, the Riemann tensor can be calculated as

$$R^{\mu}{}_{\nu\rho\sigma} = E^{\mu}_{a} e^{b}_{\nu} R^{a}{}_{b\rho\sigma} \tag{3.37}$$

The coordinate analog of **(3.36)**:

$$R_{\mu\nu} = R^{\rho}{}_{\mu\rho\nu}, \qquad R = g^{\mu\nu} R_{\mu\nu} \tag{3.38}$$

Also obviously, similar to the **(3.34)** with the help of the vielbein, we see that

$$R_{\mu\nu} = e^{a}_{\mu} e^{b}_{\nu} R_{ab} \tag{3.39}$$

4. GRAVITY AS A GAUGE THEORY

In the previous chapter, we mainly focused on developing of the necessary mathematical objects for "gravity as a gauge theory". The main focus of this chapter will be GR, its first and second-order formalisms, and how it can be acquired GR via gauging Poincaré algebra. Through finalizing this chapter, we will shortly discuss the generalizations of the GR to the multi-metric theories, etc., and its gauge theoretic formalisms.

4.1 Second Order Formalism of GR

Almost all the introductory books for GR begin with studying the second-order formalism of GR. So let us follow our origins and begin with second-order formalism. For a detailed study, see [16, 26, 27].

4.2 Einstein Tensor

General relativity is a geometric theory and we need some crucial objects such as metric, Riemann curvature tensor (and related curvature objects), covariant derivative operator, and connection. Since GR is a torsionless theory, we use the Christoffel connection. Using the equation (A.1.3), the contraction of its indices gives us that

$$\Gamma^{\rho}{}_{\mu\rho} = \frac{1}{2} g^{\rho\sigma} \left(\partial_\mu g_{\sigma\rho} + \partial_\rho g_{\sigma\mu} - \partial_\sigma g_{\mu\rho} \right) = \frac{1}{2} g^{\rho\sigma} \partial_\mu g_{\rho\sigma} = \frac{1}{\sqrt{-g}} \partial_\mu \sqrt{-g}. \tag{4.1}$$

In general, the curvature can be given as a commutator of two covariant derivatives acted on a vector, i.e.

$$\left[\nabla_\mu, \nabla_\nu \right] v^\rho = R^\rho{}_{\alpha\mu\nu} v^\alpha, \tag{4.2}$$

where $v \in T(\mathscr{M})$ and $R^\rho{}_{\alpha\mu\nu}$ are the components of Riemann tensor. Its components can be given as

$$R^\sigma{}_{\rho\mu\nu} = \partial_\mu \Gamma^\sigma{}_{\rho\nu} - \partial_\nu \Gamma^\sigma{}_{\rho\mu} + \Gamma^\alpha{}_{\rho\nu} \Gamma^\sigma{}_{\alpha\mu} - \Gamma^\alpha{}_{\rho\mu} \Gamma^\sigma{}_{\alpha\nu} \tag{4.3}$$

If we contract the first and the third indices of the Riemann tensor, we obtain the Ricci tensor

$$R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu}, \tag{4.4}$$

and contraction of the Ricci tensor with metric gives us the Ricci scalar

$$R = g^{\mu\nu} R_{\mu\nu}. \tag{4.5}$$

After some simple calculations, one can also see that the Riemann tensor satisfies two Bianchi identities

$$R^{\sigma}{}_{\rho\mu\nu} + R^{\sigma}{}_{\mu\nu\rho} + R^{\sigma}{}_{\nu\rho\mu} = 0, \tag{4.6a}$$

$$\nabla_{\alpha} R^{\sigma}{}_{\rho\mu\nu} + \nabla_{\mu} R^{\sigma}{}_{\rho\nu\alpha} + \nabla_{\nu} R^{\sigma}{}_{\rho\alpha\mu} = 0 \tag{4.6b}$$

From the second Bianchi identity, we can obtain

$$\begin{aligned} &\nabla_{\alpha} R_{\sigma\rho\mu\nu} + \nabla_{\mu} R_{\sigma\rho\nu\alpha} + \nabla_{\nu} R_{\sigma\rho\alpha\mu} = 0 \\ \Rightarrow &-\nabla^{\alpha} R_{\alpha\sigma\mu\nu} + \nabla_{\mu} R_{\sigma\nu} - \nabla_{\nu} R_{\sigma\mu} = 0 \\ \Rightarrow &-\nabla^{\alpha} R_{\alpha\mu} + \nabla_{\mu} R - \nabla^{\alpha} R_{\alpha\mu} = 0 \\ \Rightarrow &\nabla^{\alpha} R_{\alpha\mu} - \frac{1}{2} \nabla_{\mu} R = 0. \end{aligned} \tag{4.7}$$

Using the metric, and putting the terms in common parentheses of the derivative, we can obtain

$$\begin{aligned} &\nabla^{\mu} R_{\mu\nu} - \frac{1}{2} \nabla^{\mu} \left(g_{\mu\nu} R \right) = 0 \\ &\nabla^{\mu} \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) = 0. \end{aligned} \tag{4.8}$$

Let us denote

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \tag{4.9}$$

which is called the *Einstein tensor* [16]. One can read also from **(4.8)** that the Einstein tensor is a divergenceless quantity.

4.2.1 Einstein-Hilbert action

Most of our physical theories can be interpreted in the Lagrangian formalism. Fortunately, GR is one of these theories which we know the interpretation in the Lagrangian formalism. However, there is a subtle detail in its formulation. For all non-gravitational theories which can be formulated as a Lagrangian theory, one can

obtain a scalar, which can be interpreted as a Lagrangian density, using the dynamic objects of the theory and their first derivatives.

For a gravitational theory, e.g. GR, it is not a true statement; it is not possible to obtain a Lagrangian from the metric, which is the dynamic object for a gravitational theory in second-order formalism, and its first derivatives. As stated in Krasnov, the simplest scalar object obtained from a geometrical point of view is the Ricci scalar [27]. One can see from the definitions given by **(4.1)** and **(4.3)** that the Ricci scalar contains the metric and its second-order derivatives. Thus we can write the simplest action for a gravitational theory as

$$S_{\mathrm{EH}} = \int \mathrm{d}^d x \sqrt{-g} R, \tag{4.10}$$

which is called the *Einstein-Hilbert action*. By varying this action, we reach the famous *Einstein equations* as we are about to do so.

4.2.2 Einstein equations

Similar to obtaining the e.o.m. of a Dirac field in Chapter 2, one can obtain the e.o.m. for the gravitational theory which we are interested in, by varying the action **(4.10)**:

$$\delta(S_{\mathrm{EH}}) = \int \mathrm{d}^d x \left[\delta\left(\sqrt{-g}\right) R + \sqrt{-g}\,\delta(R)\right]. \tag{4.11}$$

The first variation is given in (A.2.5), thus we need to calculate the variation of the Ricci scalar. For this calculation, we should first obtain the variation of the Riemann tensor. After a short calculation, one can show that

$$\delta\left(R^{\rho}{}_{\sigma\mu\nu}\right) = \boldsymbol{\nabla}_{\mu}\,\delta(\Gamma^{\rho}{}_{\nu\sigma}) - \boldsymbol{\nabla}_{\nu}\,\delta\left(\Gamma^{\rho}{}_{\mu\sigma}\right), \tag{4.12}$$

which transforms as a tensor. So we should now find the variation of the Christoffel connection, which is given as

$$\begin{aligned}\delta\left(\Gamma^{\mu}{}_{\nu\rho}\right) = &\frac{1}{2}\,\delta(g^{\mu\sigma})\left(\partial_{\nu} g_{\sigma\rho} + \partial_{\rho} g_{\sigma\nu} - \partial_{\sigma} g_{\nu\rho}\right) \\ &+ \frac{1}{2} g^{\mu\sigma}\left(\partial_{\nu}\delta\left(g_{\sigma\rho}\right) + \partial_{\rho}\delta(g_{\sigma\nu}) - \partial_{\sigma}\delta\left(g_{\nu\rho}\right)\right).\end{aligned} \tag{4.13}$$

If we substitute the metric variation (A.2.3), we see that the derivatives covariantize:

$$\delta\left(\Gamma^{\mu}{}_{\nu\rho}\right) = \frac{1}{2} g^{\mu\sigma}\left(\boldsymbol{\nabla}_{\nu}\delta\left(g_{\sigma\rho}\right) + \boldsymbol{\nabla}_{\rho}\delta(g_{\sigma\nu}) - \boldsymbol{\nabla}_{\sigma}\delta\left(g_{\nu\rho}\right)\right). \tag{4.14}$$

As a result, we have

$$\delta(R) = \delta\left(g^{\mu\nu}R_{\mu\nu}\right) = \delta(g^{\mu\nu})R_{\mu\nu} + g^{\mu\nu}\,\delta\left(R^{\rho}{}_{\mu\rho\nu}\right), \tag{4.15}$$

and after some algebra, we obtain

$$\delta(R) = \left(R_{\mu\nu} - \boldsymbol{\nabla}_{\mu}\boldsymbol{\nabla}_{\nu} + g_{\mu\nu}\boldsymbol{\nabla}^{\rho}\boldsymbol{\nabla}_{\rho}\right)\delta(g^{\mu\nu}), \tag{4.16}$$

where the second and third terms will give us some boundary terms in the variation of the action integral. Putting all these results together, we obtain

$$\delta(S_{\mathrm{EH}}) = \int \mathrm{d}^{d}x\sqrt{-g}\left[R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + (\mathrm{B.T.})\right] = 0 \tag{4.17}$$

If we assume that the boundary terms vanish at the boundaries, we conclude that the e.o.m. for our gravitational theory is

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 0 \tag{4.18}$$

which are called the *Einstein equations* without matter. Note that the left-hand side of **(4.18)** is equal to the Einstein tensor. Now, if we add a matter Lagrangian to our action, the e.o.m. modifies as

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = T_{\mu\nu}, \tag{4.19}$$

where $T_{\mu\nu}$ is called the *energy-momentum tensor* [16,26]. Note that the Einstein tensor is symmetric since $R_{\mu\nu}$ and $g_{\mu\nu}$ are both symmetric tensors. Also from Subsection 4.2.2, we are aware that the Einstein tensor is divergenceless, thus one can see from **(4.19)** that the energy-momentum tensor should also be symmetric and divergenceless.

4.3 First Order Formalism of GR

Now, we will study the first order formalism of GR for $d = 4$, since there are some technical differences for $2+1$ dimensions and the generalization to the higher dimensional cases are trivial[1]. Let us begin with the action of the theory. The Einstein-Cartan action can be given as

$$S_{\mathrm{EC}} = \frac{1}{2}\int \varepsilon_{abcd}e^{a}\wedge e^{b}\wedge R^{cd}, \tag{4.20}$$

[1]Of course, this statement is valid only for the bosonic case. There are some major differences for the supergravity theories.

where $R^{cd} = R^{cd}(\omega)$ is the curvature of the ω. Varying **(4.20)** with respect to ω, one can obtain

$$\delta_\omega(S_{\mathrm{EC}}) = \frac{1}{2}\int \varepsilon_{abcd} e^a \wedge e^b \wedge \delta_\omega(R^{cd}). \tag{4.21}$$

So we need to calculate the variation of curvature, which can be found after some index manipulation as

$$\delta_\omega(R^{cd}) = \mathrm{d}\left(\delta\omega^{cd}\right) - \omega^d{}_f \wedge \delta\omega^{fc} + \omega^c{}_f \wedge \delta\omega^{fd}. \tag{4.22}$$

Substituting this into the variation of the curvature and using the integration by parts, one can find that

$$\begin{aligned}
\delta_\omega(S_{\mathrm{EC}}) &= \frac{1}{2}\int \varepsilon_{abcd} e^a \wedge e^b \wedge \left[\mathrm{d}\left(\delta\omega^{cd}\right) - \omega^d{}_f \wedge \delta\omega^{fc} + \omega^c{}_f \wedge \delta\omega^{fd}\right] \\
&= \frac{1}{2}\int \varepsilon_{abcd} e^a \wedge e^b \wedge \left[\mathrm{d}\left(\delta\omega^{cd}\right) + \omega^c{}_f \wedge \delta\omega^{fd} + \omega^c{}_f \wedge \delta\omega^{df}\right] \\
&= \frac{1}{2}\int \varepsilon_{abcd} e^a \wedge e^b \wedge D\left(\delta\omega^{cd}\right) \\
&= \frac{1}{2}\int \left[D\left(\varepsilon_{abcd} e^a \wedge e^b \wedge \delta\omega^{cd}\right) - \varepsilon_{abcd} D\left(e^a \wedge e^b\right) \wedge \delta\omega^{cd}\right].
\end{aligned} \tag{4.23}$$

As can be seen, the first term is a boundary term, thus we have

$$\delta_\omega(S_{\mathrm{EC}}) \simeq -\frac{1}{2}\int \varepsilon_{abcd} D\left(e^a \wedge e^b\right) \wedge \delta\omega^{cd}, \tag{4.24}$$

and for $\delta_\omega(S_{\mathrm{EC}}) = 0$, we obtain that

$$De^a = 0, \tag{4.25}$$

which is the zero torsion condition.

We can also vary **(4.20)** with respect to the tetrad field. With a very simple calculation, one can get the e.o.m. for the tetrad as

$$\varepsilon_{abcd} e^b \wedge R^{cd} = 0. \tag{4.26}$$

Similar to the previous section, if we add a matter Lagrangian to our action, we conclude that the e.o.m. modifies as

$$\varepsilon_{abcd} e^b \wedge R^{cd} = \varepsilon_{abcd} \mathscr{T}^{bcd}, \tag{4.27}$$

which is called the Einstein equations in first-order formalism.

4.4 Poincaré Algebra Gauge Theory

In theoretical physics, understanding the symmetries underlying the physical theories are crucial. At late 19th, we have used Newton's laws to formulate the dynamics of the particles and systems. This theory is invariant under Galilean transformation, which can be generically given as

$$\begin{aligned} \mathbf{x} &\to \mathbf{x}' = R\mathbf{x} + t\mathbf{x} + \mathbf{a}, \\ t &\to t' = t + s, \end{aligned} \tag{4.28}$$

where $R \in \mathrm{SO}(3)$ is a rotation matrix, $\mathbf{x}$ is known as the position vector, $\mathbf{a}$ is a constant vector and t is the time. The group constructed by these transformations is called the *Galilean group* $\mathrm{Gal}(3)$, also the corresponding algebra is called the *Galilean algebra*, which is a Lie algebra. However, Maxwell's equations are not invariant under these transformations. With the works of Lorentz, Poincaré, and others, it was shown that the correct transformation is the Lorentz transformation which leaves invariant Maxwell's equations. This transformation can be given as

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu, \tag{4.29}$$

where $\Lambda^\mu{}_\nu \in \mathrm{SO}(1,d-1)$ is a rotation matrix restricted by the condition

$$\Lambda^\alpha{}_\mu \Lambda^\beta{}_\nu \eta_{\alpha\beta} = \eta_{\mu\nu}, \tag{4.30}$$

$\eta_{\mu\nu}$ is the Minkowski metric and a^μ is a constant d-vector. The corresponding group is called the *Poincaré group* $\mathrm{ISO}(1,d-1) \equiv \mathbb{R}^{(1,d-1)} \rtimes \mathrm{SO}(1,d-1)^2$ [47], and the corresponding algebra is called the *Poincaré algebra*, which is also a Lie algebra. Now we will examine this group in more detail.

A Poincaré transformation consists of a Lorentz transformation, which acts as a rotation in $(D+1)$-dimensional spacetime [3], and a spacetime translation a^μ

$$T(a,\Lambda)x^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu = x'^\mu. \tag{4.31}$$

[2] This group is also called the *inhomogeneous Lorentz group*.

[3] Here $D = d-1$, where d is the spacetime dimension.

Using this definition, one can easily check whether these transformations indeed form a group:

$$\begin{aligned} T(a_2,\Lambda_2)T(a_1,\Lambda_1)x &= T(a_2,\Lambda_2)\left(\Lambda_1 x + a_1\right) \\ &= \Lambda_2\Lambda_1 x + \Lambda_2 a_1 + a_2 \\ &= \Lambda_3 x + a_3 \end{aligned} \tag{4.32}$$

where the indices are suppressed. This gives us the result that the Poincaré transformations establish indeed a group and its composition rule can be given as

$$(a_2,\Lambda_2)\circ(a_1,\Lambda_1) = (\Lambda_2 a_1 + a_2, \Lambda_2\Lambda_1) \tag{4.33}$$

Here if we take $a^\mu = 0$, we simply obtain a Lorentz transformation, and because of the same reason in **(4.32)**, these transformations form also a group which is known as the *Lorentz group*, also the corresponding algebra is known as the *Lorentz algebra*. We will begin the study of the gauge theory of Poincaré algebra by studying Lorentz algebra.

4.4.1 Lorentz algebra

First of all, we should note that we know from relativistic physics that the *global* spacetime symmetries without boost can be expressed using the Lorentz transformation. More clearly, the symmetry transformations can be given as follows. Let Λ be a Lorentz transformation

$$x^m \to x'^\mu = \Lambda^\mu{}_\nu x^\nu, \tag{4.34}$$

and let $\phi(x)$ be a scalar, $\mathscr{L}(x)$ be a Lagrangian, $V^\mu(x)$ be a vector and $\eta_{\mu\nu}$ the Minkowski metric. For our purposes, we only give the transformations of scalars, vectors, and tensors. For the transformations of other objects such as spinors, see [13, 39, 40] etc.

Our motivated transformations can be given as

$$\begin{aligned} \eta^{\mu\nu} &\to \Lambda^\rho{}_\mu \Lambda^\sigma{}_\nu \eta^{\mu\nu} = \eta^{\rho\sigma}, \\ \phi(x) &\to \phi(\Lambda^{-1}x), \\ \partial_\mu\phi(x) &\to (\Lambda^{-1})^\nu{}_\mu \partial_\nu\phi(\Lambda^{-1}x), \\ \mathscr{L}(x) &\to \mathscr{L}(\Lambda^{-1}x), \\ V^\mu(x) &\to \Lambda^\mu{}_\nu V^\nu(\Lambda^{-1}x). \end{aligned} \tag{4.35}$$

Now let us obtain the corresponding algebra by expanding the transformation around the identity. Let $\omega^{\mu}{}_{\nu}$ be taken an infinitesimal object. If we expand the Lorentz transformation Λ around identity i.e. $\delta^{\mu}{}_{\nu}$, we find that

$$\Lambda^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + \omega^{\mu}{}_{\nu} + \cdots . \tag{4.36}$$

From **(4.35)**

$$\begin{aligned} \eta^{\rho\sigma} &= \left(\delta^{\rho}{}_{\mu} + \omega^{\rho}{}_{\mu}\right)\left(\delta^{\sigma}{}_{\nu} + \omega^{\sigma}{}_{\nu}\right)\eta^{\mu\nu} \\ &= \eta^{\rho\sigma} + \eta^{\rho\nu}\omega^{\sigma}{}_{\nu} + \eta^{\mu\sigma}\omega^{\rho}{}_{\mu} + \mathcal{O}\left(\omega^2\right) \\ &\simeq \eta^{\rho\sigma} + \omega^{\sigma\rho} + \omega^{\rho\sigma}, \end{aligned} \tag{4.37}$$

thus ω should be antisymmetric, i.e. $\omega^{\mu\nu} = -\omega^{\nu\mu}$. In d dimensions, this object has $\frac{1}{2}d(d-1)$ elements, which are the parameters for the generators of the related Lie algebra. As described in Freedman and van Proeyen, this Lie algebra is a real linear space spanned by these generators with the commutator $[m,m']$ [13]. The corresponding generators are $J^{\mu\nu}$, where the group elements can be given as an exponential such that

$$\begin{aligned} T(0,\bar{\Lambda}) &= T(0,\mathbb{1} + \omega + \mathcal{O}\left(\omega^2\right)) \\ &= \mathbb{1} + \frac{i}{2}\omega_{\mu\nu}J^{\mu\nu} + \mathcal{O}\left(\omega^2\right) \\ &= e^{\frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}}. \end{aligned} \tag{4.38}$$

Note that this transformation should be *unitary*, thus J must be *Hermitian*. One can also see from above, J must be antisymmetric, since ω is antisymmetric [4]. Let us now investigate the Lorentz transformation properties of the generators $J^{\mu\nu}$. We begin by considering the product given below:

$$T(\Lambda)T(\mathbb{1} + \omega)T^{-1}(\Lambda), \tag{4.39}$$

where we denote $T(\Lambda) = T(0,\Lambda)$ for simplicity. Note also that

$$T(\Lambda)T^{-1}(\Lambda) = \mathbb{1} = T(\mathbb{1}) = T(\Lambda\Lambda^{-1}) \tag{4.40}$$

We see that

$$T^{-1}(\Lambda) = T(\Lambda^{-1}). \tag{4.41}$$

[4]The generator $J^{\mu\nu}$ is called the *angular momentum operator*.

Now, using **(4.33)**, we can obtain

$$\begin{aligned} T(\Lambda)T(\mathbb{1}+\omega)T^{-1}(\Lambda) &= T(\Lambda)T(\mathbb{1}+\omega)T(\Lambda^{-1}) \\ &= T\left(\Lambda(\mathbb{1}+\omega)\Lambda^{-1}\right). \end{aligned} \tag{4.42}$$

To the first order in ω, we obtain from LHS

$$T(\Lambda)\left(\mathbb{1}+\frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}\right)T(\Lambda^{-1}) = \mathbb{1}+T(\Lambda)\left(\frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}\right)T(\Lambda^{-1}), \tag{4.43}$$

and from RHS

$$T\left(\Lambda(\mathbb{1}+\omega)\Lambda^{-1}\right) = T(\mathbb{1}+\Lambda\omega\Lambda^{-1}) = \mathbb{1}+\frac{i}{2}\left(\Lambda\omega\Lambda^{-1}\right)_{\mu\nu}J^{\mu\nu}. \tag{4.44}$$

Equating both sides, we find that

$$\frac{i}{2}\omega_{\rho\sigma}T(\Lambda)J^{\rho\sigma}T(\Lambda^{-1}) = \frac{i}{2}\left(\Lambda_{\mu}{}^{\rho}\omega_{\rho\sigma}\left(\Lambda^{-1}\right)^{\sigma}{}_{\nu}\right)J^{\mu\nu}. \tag{4.45}$$

Simplifying this relation, we obtain the transformation rule for $J^{\mu\nu}$:

$$T(\Lambda)J^{\rho\sigma}T(\Lambda^{-1}) = \Lambda_{\mu}{}^{\rho}\Lambda_{\nu}{}^{\sigma}J^{\mu\nu}, \tag{4.46}$$

where $\left(\Lambda^{-1}\right)^{\sigma}{}_{\nu} = \Lambda_{\nu}{}^{\sigma}$. Now, let $T(\Lambda)$ be also an infinitesimal transformation, i.e.

$$\begin{aligned} \Lambda_{\mu}{}^{\nu} &= \delta_{\mu}{}^{\nu} + \bar{\omega}_{\mu}{}^{\nu}, \\ T(\Lambda) &= \mathbb{1}+\frac{i}{2}\bar{\omega}_{\mu\nu}J^{\mu\nu}+\mathscr{O}\left(\bar{\omega}^2\right). \end{aligned} \tag{4.47}$$

Thus we have,

$$\left(\mathbb{1}+\frac{i}{2}\bar{\omega}_{\mu\nu}J^{\mu\nu}\right)J^{\rho\sigma}\left(\mathbb{1}-\frac{i}{2}\bar{\omega}_{\mu\nu}J^{\mu\nu}\right) = J^{\rho\sigma}+\frac{i}{2}\bar{\omega}_{\mu\nu}[J^{\mu\nu},J^{\rho\sigma}]+\mathscr{O}\left(\bar{\omega}^2\right), \tag{4.48}$$

and

$$\left(\delta_{\mu}{}^{\rho}+\bar{\omega}_{\mu}{}^{\rho}\right)\left(\delta_{\nu}{}^{\sigma}+\bar{\omega}_{\nu}{}^{\sigma}\right)J^{\mu\nu} = J^{\rho\sigma}+\bar{\omega}_{\mu}{}^{\rho}J^{\mu\sigma}+\bar{\omega}_{\nu}{}^{\sigma}J^{\rho\nu}+\mathscr{O}\left(\bar{\omega}^2\right). \tag{4.49}$$

Equating both relations, we obtain

$$\frac{i}{2}\bar{\omega}_{\mu\nu}[J^{\mu\nu},J^{\rho\sigma}] = \bar{\omega}_{\mu}{}^{\rho}J^{\mu\sigma}+\bar{\omega}_{\nu}{}^{\sigma}J^{\rho\nu}, \tag{4.50}$$

and eliminating $\bar{\omega}$, we find that

$$[J^{\mu\nu},J^{\rho\sigma}] = -i\left[\eta^{\mu\rho}J^{\nu\sigma}-\eta^{\mu\sigma}J^{\nu\rho}-\eta^{\nu\rho}J^{\mu\sigma}+\eta^{\nu\sigma}J^{\mu\rho}\right], \tag{4.51}$$

which is the Lorentz algebra. We can also rewrite this commutator as in **(2.7)**

$$[J^{\mu\nu},J^{\rho\sigma}] = \frac{i}{2}f^{[\mu\nu][\rho\sigma]}{}_{[\kappa\tau]}J^{\kappa\tau}, \tag{4.52}$$

where the structure constants can be given as

$$f^{[\mu\nu][\rho\sigma]}{}_{[\kappa\tau]} = 8\eta^{[\rho[\nu}\delta^{\mu]}{}_{[\kappa}\delta^{\sigma]}{}_{\tau]} \tag{4.53}$$

4.4.2 Poincaré algebra

Now we can generalize the results beginning with **(4.36)** and **(4.37)** by adding $a^\mu = \xi^\mu$ where ξ^μ be taken an infinitesimal object. As before, this object has d elements in d dimensions, which are the parameters for the generators (of course, with $\omega^{\mu\nu}$) of the Poincaré algebra. As defined earlier, the corresponding generators for ω are $J^{\mu\nu}$, and the corresponding generators for ξ are P^μ. Thus the elements of the group are given by

$$\begin{aligned} T(\bar{a},\bar{\Lambda}) &= T(\xi, \mathbb{1}+\omega+\mathscr{O}(\omega^2)) \\ &= \mathbb{1} + \frac{i}{2}\omega_{\mu\nu}J^{\mu\nu} - i\xi_\mu P^\mu + \mathscr{O}(\omega^2) + \mathscr{O}(\xi^2) \\ &= e^{\frac{i}{2}\omega_{\mu\nu}J^{\mu\nu} - i\xi_\mu P^\mu}. \end{aligned} \tag{4.54}$$

As stated earlier, this transformation should be unitary, so P must be Hermitian, as well as J [5]. Similar to the previous section, we should investigate the transformation properties of the generators J and P. Using the fact that

$$T^{-1}(a,\Lambda) = T(\Lambda^{-1}, -\Lambda^{-1}a) \tag{4.55}$$

with the help of **(4.33)**, we can find that

$$T(a,\Lambda)T(\xi,\mathbb{1}+\omega)T^{-1}(a,\Lambda) = T\left(\Lambda\xi - \Lambda\omega\Lambda^{-1}a, \Lambda(\mathbb{1}+\omega)\Lambda^{-1}\right) \tag{4.56}$$

If we use the same methodology for calculating the transformation rules, we can obtain

$$T(a,\Lambda)J^{\rho\sigma}T^{-1}(a,\Lambda) = \Lambda_\mu{}^\rho\Lambda_\nu{}^\sigma\left(J^{\mu\nu} - a^\mu P^\nu + a^\nu P^\mu\right), \tag{4.57}$$

and

$$T(a,\Lambda)P^\rho T^{-1}(a,\Lambda) = \Lambda_\mu{}^\rho P^\mu. \tag{4.58}$$

After some simple calculations, we can see that the Lie algebra of the Poincaré group can be given by

$$\begin{aligned} [J^{\mu\nu}, J^{\rho\sigma}] &= -i\left[\eta^{\mu\rho}J^{\nu\sigma} - \eta^{\mu\sigma}J^{\nu\rho} - \eta^{\nu\rho}J^{\mu\sigma} + \eta^{\nu\sigma}J^{\mu\rho}\right], \\ [P^\mu, J^{\rho\sigma}] &= -i\left[\eta^{\mu\rho}P^\sigma - \eta^{\mu\sigma}P^\rho\right], \\ [P^\mu, P^\nu] &= 0. \end{aligned} \tag{4.59}$$

Infinitesimally, these generators can also be given in the form of

$$\begin{aligned} J_{\mu\nu} &= x_\mu\partial_\nu - x_\nu\partial_\mu, \\ P_\mu &= \partial_\mu. \end{aligned} \tag{4.60}$$

[5] The generator P^μ is called the *momentum operator*

So, the infinitesimal transformations of a scalar field can be given as

$$\begin{aligned}\delta(\omega)\,\phi(x) &= \frac{i}{2}\omega^{\mu\nu}J_{\mu\nu}\phi(x)\\ &= \frac{i}{2}\omega^{\mu\nu}\left(x_\mu\partial_\nu - x_\nu\partial_\mu\right)\phi(x)\\ &= i\omega^{\mu\nu}x_\mu\partial_\nu\phi(x),\end{aligned} \tag{4.61}$$

and

$$\begin{aligned}\delta(\xi)\,\phi(x) &= -i\xi^\mu P_\mu\phi(x)\\ &= -i\xi^\mu\partial_\mu\phi(x).\end{aligned} \tag{4.62}$$

4.4.3 Gauging the Poincaré algebra

In GR, the geometric objects are transformed covariantly under the general coordinate transformations (gct) [13], thus we have

$$\begin{aligned}\phi'(x') &= \phi(x),\\ V'_\mu(x') &= \frac{\partial x^\nu}{\partial x'^\mu}V_\nu(x),\\ &\text{etc.}\end{aligned} \tag{4.63}$$

Now, we can investigate the infinitesimal gct, which can be given as

$$x^\mu \to x'^\mu = x^\mu - \varepsilon^\mu(x). \tag{4.64}$$

Using this definition, we can calculate the infinitesimal gct of a scalar field as

$$\begin{aligned}\delta_{\text{gct}}\phi(x) &= \phi'(x) - \phi(x)\\ &= \phi'(x'+\varepsilon) - \phi(x)\\ &\simeq \phi'(x') + \varepsilon^\mu(x)\partial_\mu\phi'(x') - \phi(x)\\ &= \phi(x) + \varepsilon^\mu(x)\partial_\mu\phi(x) - \phi(x)\\ &= \varepsilon^\mu(x)\partial_\mu\phi(x)\\ &= \mathscr{L}_\varepsilon\phi(x),\end{aligned} \tag{4.65}$$

which is known as the Lie derivative of the ϕ with respect to ε, and with similar calculations, we can obtain the infinitesimal gct of the vector field:

$$\begin{aligned}\delta_{\text{gct}}V_\mu(x) &= V'_\mu(x) - V_\mu(x)\\ &= \varepsilon^\rho\partial_\rho V_\mu(x) + V_\rho\partial_\mu\varepsilon^\rho\\ &= \mathscr{L}_\varepsilon V_\mu(x),\end{aligned} \tag{4.66}$$

which is the Lie derivative of the vector V with respect to ε [45]. Let us now gauging the Poincaré algebra, i.e. make the parameters local $(\xi,\omega)\to(\xi(x),\omega(x))$.

Calculating the gauged Poincaré transformation of a scalar field gives us

$$\begin{aligned}\delta(\xi,\omega)\,\phi(x) &= \left(i\omega^{\mu\nu}x_\mu\partial_\nu - i\xi^\mu\partial_\mu\right)\phi(x)\\ &= -i\left(\omega^{\mu\nu}x_\nu\partial_\mu + \xi^\mu\partial_\mu\right)\phi(x)\\ &= -i\left(\omega^{\mu\nu}x_\nu + \xi^\mu\right)\partial_\mu\phi(x).\end{aligned} \tag{4.67}$$

If we denote $\varepsilon^\mu = -i(\omega^{\mu\nu}x_\nu + \xi^\mu)$, we can obtain

$$\begin{aligned}\delta(\xi,\omega)\,\phi(x) &= \varepsilon^\mu\partial_\mu\phi(x)\\ &= \mathscr{L}_\varepsilon\phi(x)\\ \delta(\varepsilon)\,\phi(x) &= \delta_{\mathrm{gct}}\phi(x).\end{aligned} \tag{4.68}$$

This shows us that the generalized coordinate transformations of a curved spacetime are equivalent to the gauged Poincaré transformations [45].

4.4.4 Pure Poincaré gauge theory

For the reasons of calculations of the future works in this thesis, we change our convention a little bit as in [13], and we will drop the complex number i. In this new convention, the Poincaré algebra is given as

$$\begin{aligned}[J_{ab},J_{cd}] &= 4\eta_{[a[c}J_{d]b]},\\ [J_{ab},P_c] &= 2\eta_{c[b}P_{a]},\\ [P_a,P_b] &= 0.\end{aligned} \tag{4.69}$$

Since the sum of 1-forms are equal to another 1-form, one can give the definition of the Lie algebra valued 1-form for Poincaré algebra as

$$\begin{aligned}A_\mu &= e_\mu + \omega_\mu\\ &= e_\mu{}^a P_a + \frac{1}{2}\omega_\mu{}^{ab}J_{ab}\end{aligned} \tag{4.70}$$

As calculated in Chapter 2, the gauge theoretic curvature 2-form is given by

$$R = \mathrm{d}A + \frac{1}{2}[A,A] \tag{4.71}$$

Using **(4.70)** and **(4.71)**, one can find the curvature 2-form for Poincaré gauge theory as

$$\begin{aligned}R &= \frac{1}{2}\left(\partial_\mu A_\nu - \partial_\nu A_\mu + \left[A_\mu,A_\nu\right]\right)\mathrm{d}x^\mu\,\mathrm{d}x^\nu\\ &= \frac{1}{2}\left(\partial_\mu(e_\nu+\omega_\nu) - \partial_\nu(e_\mu+\omega_\mu) + \left[(e_\mu+\omega_\mu),(e_\nu+\omega_\nu)\right]\right)\mathrm{d}x^\mu\,\mathrm{d}x^\nu\,.\end{aligned} \tag{4.72}$$

Rewriting the 1-forms as in **(4.70)**, and using the commutators given in **(4.69)**, we can obtain the curvature 2-form as

$$R_{\mu\nu}(A) = R_{\mu\nu}{}^{a}(e)P_a + \frac{1}{2}R_{\mu\nu}{}^{ab}(\omega)J_{ab} \tag{4.73}$$

where

$$\begin{aligned} R_{\mu\nu}{}^{a}(e) &= 2\left(\partial_{[\mu}e_{\nu]}{}^{a} - \omega_{[\mu}{}^{a}{}_{b}e_{\nu]}{}^{b}\right) \\ R_{\mu\nu}{}^{ab}(\omega) &= 2\left(\partial_{[\mu}\omega_{\nu]}{}^{ab} - \omega_{[\mu}{}^{a}{}_{|c|}\omega_{\nu]}{}^{cb}\right) \end{aligned} \tag{4.74}$$

We can also give the arbitrary symmetry transformations as

$$\begin{aligned} \delta(\varepsilon) &= \varepsilon^{A}T_A \\ T_a\phi^b &= -f_{ac}{}^{b}\phi^c \\ \delta(\varepsilon)A_\mu{}^{a} &= \partial_\mu\varepsilon^a + \varepsilon^c A_\mu{}^{b}f_{bc}{}^{a} \\ D_\mu\phi^a &= \left(\partial_\mu - \delta\left(A_\mu\right)\right)\phi^a \\ &= \partial_\mu\phi^a - A_\mu{}^{b}T_b\phi^a \\ \delta(\varepsilon)R_{\mu\nu}{}^{a} &= \varepsilon^c R_{\mu\nu}{}^{b}f_{bc}{}^{a} \end{aligned} \tag{4.75}$$

Using these rules, we are allowed to calculate the symmetry variation of e and ω

$$\begin{aligned} \delta(\xi)\,e_\mu{}^{a} &= \partial_\mu\xi^a - \xi^b\omega_\mu{}^{a}{}_{b} + \lambda^{ab}e_{\mu b} \\ \delta(\lambda)\,\omega_\mu{}^{ab} &= \partial_\mu\lambda^{ab} + 2\lambda^{c[a}\omega_\mu{}^{b]}{}_{c} \end{aligned} \tag{4.76}$$

where we use **(4.53)** as the structure constant, and since A^{ab} is antisymmetric, this simplifies as

$$A^{bc}f_{a[bc]}{}^{d} = A^{bc}\eta_{ab}\delta^{d}_{c}. \tag{4.77}$$

4.4.5 The soldering equation

So far, the variations act only on the internal space. However, we need the variations that act on spacetime for a gravity theory. Thus, we need to obtain a procedure to combine these two, priori different transformations in some way.

Let us first define another transformation, which is called "covariant general coordinate transformation" (cgct) as

$$\delta_{\text{cgct}}(\xi) = \delta_{\text{gct}}(\xi) - \delta\left(\xi^\mu A_\mu\right) \tag{4.78}$$

Acting on a gauge field, we obtain

$$\begin{aligned}
\delta_{\mathrm{cgct}}(\xi)A_\mu{}^a &= \delta_{\mathrm{gct}}(\xi)A_\mu{}^a - \delta(\xi^\sigma A_\sigma)A_\mu{}^a \\
&= \xi^\nu \partial_\nu A_\mu{}^a + A_\nu{}^a \partial_\mu \xi^\nu - \partial_\mu(\xi^\nu A_\nu{}^a) - \xi^\nu A_\nu{}^c A_\mu{}^b f_{bc}{}^a \\
&= \xi^\nu \partial_\nu A_\mu{}^a + A_\nu{}^a \partial_\mu \xi^\nu - \xi^\nu \partial_\mu A_\nu{}^a - A_\nu{}^a \partial_\mu \xi^\nu - \xi^\nu A_\nu{}^c A_\mu{}^b f_{bc}{}^a \\
&= \xi^\nu \partial_\nu A_\mu{}^a - \xi^\nu \partial_\mu A_\nu{}^a - \xi^\nu A_\nu{}^c A_\mu{}^b f_{bc}{}^a \\
&= -\xi^\nu \left(\partial_{[\mu} A_{\nu]}{}^a + A_\mu{}^b A_\nu{}^c f_{bc}{}^a \right) = -\xi^\nu R_{\mu\nu}{}^a(A)
\end{aligned} \tag{4.79}$$

Thus we have

$$\delta_{\mathrm{cgct}}(\xi)e_\mu{}^a = -\xi^\nu R_{\mu\nu}{}^a(e) \tag{4.80}$$

If we expand the LHS for the Poincaré algebra, we find that

$$\delta_{\mathrm{cgct}}(\xi)e_\mu{}^a = \delta_{\mathrm{gct}}e_\mu{}^a - \delta_P(\xi^\lambda e_\lambda)e_\mu{}^a - \delta_J(\xi^\lambda \omega_\lambda)e_\mu{}^a \tag{4.81}$$

and we obtain

$$-\xi^\nu R_{\mu\nu}{}^a(e) = \delta_{\mathrm{gct}}e_\mu{}^a - \delta_P(\xi^\lambda e_\lambda)e_\mu{}^a - \delta_J(\xi^\lambda \omega_\lambda)e_\mu{}^a \tag{4.82}$$

Let us deform the original Poincaré algebra by imposing the constraint[6]

$$R_{\mu\nu}{}^a(e) = 0 \tag{4.83}$$

thus we find that

$$\begin{aligned}
0 &= \delta_{\mathrm{gct}}e_\mu{}^a - \delta_P(\xi^\lambda e_\lambda)e_\mu{}^a - \delta_J(\xi^\lambda \omega_\lambda)e_\mu{}^a \\
\delta_P(\xi^\lambda e_\lambda)e_\mu{}^a &= \delta_{\mathrm{gct}}e_\mu{}^a - \delta_J(\xi^\lambda \omega_\lambda)e_\mu{}^a \\
\delta_P(\xi)e_\mu{}^a &= \delta_{\mathrm{gct}}e_\mu{}^a - \delta_J(\xi^\lambda \omega_\lambda)e_\mu{}^a
\end{aligned} \tag{4.84}$$

This result provides the connection between the translational symmetries of Poincaré algebra and general coordinate transformation, so one can transform these into each other [19].

4.4.6 The last step: action

In the previous chapter, we have defined all the objects we need for a gravity theory, such as the metric and the connection between the metric and the vielbein or the covariant derivatives, etc. In this subsection, we only give the action.

[6]Note that this constraint corresponds to the torsionless condition, i.e. $\Gamma^\lambda{}_{[\mu\nu]} = 0$.

First, let us calculate the metric determinant in terms of vielbeine

$$\det\left(g_{\mu\nu}\right) = \det\left(\eta_{ab} e^{a}{}_{\mu} e^{b}{}_{\nu}\right) = -\det\left(e^{a}{}_{\mu}\right)^{2} \equiv -e^{2}, \tag{4.85}$$

thus we have

$$\sqrt{-\det\left(g_{\mu\nu}\right)} = \sqrt{e^{2}} = e \tag{4.86}$$

Since $R_{\mu\nu}{}^{a}(e) = 0$, using **(4.73)**, we can find that only curvature term is $R_{\mu\nu}{}^{ab}$. Contracting with vielbeine, one can see that the Ricci scalar is given by

$$R(\omega) = e^{\mu}_{a} e^{\nu}{}_{b} R_{\mu\nu}{}^{ab}(\omega). \tag{4.87}$$

As the result, the action becomes

$$S = \int \mathrm{d}^{d}x\, e R(\omega). \tag{4.88}$$

Note that this action is called the “tetradic Palatini action” [45].

5. BI-GRAVITY THEORIES AND SCALING LIMIT

In this chapter, we will discuss general aspects with some examples of scaling limit and the relation between bi-gravity theories [18,23,48–52] and massive gravity [21,22,53–66]. Since Einstein's General Relativity is also able to be understood as a field theory which describes the interactions of massless spin-2 particles i.e. graviton. Having this perspective leads to considering the case that GR might be generalized as allowing small graviton mass. Surely, the starting point is constructed by Fierz and Pauli Theory [60] however this theory is not the easiest theory that anyone might want to extend as a well-defined interacting theory. First of all, one of the fundamental issues is the existence of an unphysical situation such as negative energy scalar modes as known as Boulware-Deser(BD) ghost or instability [58]. Even though we will not focus on the BD ghost, we will understand the progress of achievement as we move further.

5.1 First Attempt: Non-Linear Massive Gravity Model

As a starting point, we will consider two general metrics $g_{\mu\nu}$ and $f_{\mu\nu}$ and the structure that they contribute. Then we will investigate this whole structure by imposing a few conditions. So let us begin with the massive action of [23]:

$$S = M_p^2 \int d^4x\sqrt{-g}\left[R + 2m^2\sum_{n=0}^{4}\left(\beta_n e_n(\sqrt{g^{-1}f}\right)\right] \tag{5.1}$$

where the square root of a matrix is defined as $\left(\sqrt{g^{-1}f}\sqrt{g^{-1}f}\right)^{\mu}{}_{\lambda} = g^{\mu\nu}f_{\nu\lambda}$. And, $e_n(\sqrt{g^{-1}f})$ are called elementary symmetric polynomials with the eigenvalues λ_n of the matrix $\sqrt{g^{-1}f}$. If we are interested in the 4×4 matrix:

$$\begin{aligned}
e_0(\sqrt{g^{-1}f}) &= 1\\
e_1(\sqrt{g^{-1}f}) &= \lambda_1+\lambda_2+\lambda_3+\lambda_4\\
e_2(\sqrt{g^{-1}f}) &= \lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_1\lambda_4+\lambda_2\lambda_3+\lambda_2\lambda_4+\lambda_3\lambda_4\\
e_3(\sqrt{g^{-1}f}) &= \lambda_1\lambda_2\lambda_3+\lambda_1\lambda_2\lambda_4+\lambda_1\lambda_3\lambda_4+\lambda_2\lambda_3\lambda_4\\
e_4(\sqrt{g^{-1}f}) &= \lambda_1\lambda_2\lambda_3\lambda_4 = \det\sqrt{g^{-1}f}
\end{aligned} \tag{5.2}$$

Let us focus on the last term on the potential **(5.1)**:

$$\sqrt{-\det g}\beta_4 e_4(\sqrt{g^{-1}f}) = \beta_4\sqrt{-\det g}\det\sqrt{g^{-1}f} = \beta_4\sqrt{-\det f} \tag{5.3}$$

As you may see easily, it is independent of $g_{\mu\nu}$ so this term does not contribute to the e.o.m of the $g_{\mu\nu}$. Therefore, when the reference metric $f_{\mu\nu}$ is assumed as non-dynamical, we are able to drop this term. The remaining four β_n describe combinations of the fundamental parameters such as the mass of graviton, the cosmological constant, and two free parameters.

From now on, $f_{\mu\nu}$ will be considered as dynamical. Therefore, since the potential term in **(5.1)** includes both $g_{\mu\nu}$ and $f_{\mu\nu}$, it will act as a potential for both $g_{\mu\nu}$ and $f_{\mu\nu}$. Of course in that case we keep the **(5.3)** term. As a result, we totally have 5 free parameters which include cosmological constant and graviton mass for both $g_{\mu\nu}$ and $f_{\mu\nu}$.

5.1.1 View as bi-metric gravity

Let us go further one step and add a kinetic term for $f_{\mu\nu}$ into **(5.1)**. However, we have to add such a kinetic term that should be compatible with the potential term with the aim of constructing a well-defined ghost-free gravity theory. For now, this mass term might seem like it does not have the same form for both metrics. Even though this is problematic, we might write a kinetic term for $f_{\mu\nu}$ that is not an Einstein-Hilbert kinetic term. In general, we can add $f_{\mu\nu}$ kinetic terms which are constructed by using both $g_{\mu\nu}$ and $f_{\mu\nu}$. As a starting point, we will try to add only one term. Also in order to understand what the structure of this new term should be, let us examine the mass term

$$\begin{aligned} \text{Since} \quad & \sqrt{-\det g} = \sqrt{-\det f}\sqrt{\det f^{-1}g} \\ \text{We have} \quad & \sqrt{-\det g}\sum_{n=0}^{4}\beta_n e_n(\sqrt{g^{-1}f}) = \sqrt{-\det f}e_4(\sqrt{f^{-1}g})\sum_{n=0}^{4}\beta_n e_n(\sqrt{g^{-1}f}) \end{aligned} \tag{5.4}$$

Now let us try to whether we can express $e_n(\sqrt{g^{-1}f})$ by relating the $f^{-1}g$. One can easily see that $e_n(\sqrt{f^{-1}g})$ can be expressed via the eigenvalues of $g^{-1}f$ as

$$\begin{aligned}
e_0(\sqrt{f^{-1}g}) &= 1 \\
e_1(\sqrt{f^{-1}g}) &= \frac{1}{\lambda_1}+\frac{1}{\lambda_2}+\frac{1}{\lambda_3}+\frac{1}{\lambda_4} \\
e_2(\sqrt{f^{-1}g}) &= \frac{1}{\lambda_1\lambda_2}+\frac{1}{\lambda_1\lambda_3}+\frac{1}{\lambda_1\lambda_4}+\frac{1}{\lambda_2\lambda_3}+\frac{1}{\lambda_2\lambda_4}+\frac{1}{\lambda_3\lambda_4} \\
e_3(\sqrt{f^{-1}g}) &= \frac{1}{\lambda_1\lambda_2\lambda_3}+\frac{1}{\lambda_1\lambda_2\lambda_4}+\frac{1}{\lambda_1\lambda_3\lambda_4}+\frac{1}{\lambda_2\lambda_3\lambda_4} \\
e_4(\sqrt{f^{-1}g}) &= \frac{1}{\lambda_1\lambda_2\lambda_3\lambda_4}
\end{aligned} \tag{5.5}$$

Note that there is a relation between **(5.2)** and **(5.5)** such that

$$e_n(\sqrt{g^{-1}f}) = \frac{e_{4-n}(\sqrt{f^{-1}g})}{e_4(\sqrt{f^{-1}g})} \tag{5.6}$$

Then the mass term **(5.4)** can be written as

$$\sqrt{-\det g}\sum_{n=0}^{4}\beta_n e_n(\sqrt{g^{-1}f}) = \sqrt{-\det f}\sum_{n=0}^{4}\beta_n e_{4-n}(\sqrt{f^{-1}g}) \tag{5.7}$$

At this moment, it is clear that mass term indeed has the same ghost-free structure for both $g_{\mu\nu}$ and $f_{\mu\nu}$ with different coefficients. Therefore, we understand that a proper kinetic term for $f_{\mu\nu}$ is the Einstein-Hilbert kinetic term. Adding this term leads to a theory that is expected to become ghost-free for both $g_{\mu\nu}$ and $f_{\mu\nu}$

$$\begin{aligned}
S =& M_g^2\int d^4x\sqrt{-g}R^{(g)} + M_f^2\int d^4x\sqrt{-f}R^{(f)} \\
&+2m^2M_{eff}^2\int d^4x\sqrt{-g}\sum_{n=0}^{4}\beta_n e_n(\sqrt{-g^{-1}f})
\end{aligned} \tag{5.8}$$

where $R^{(g)}$ and $R^{(f)}$ denotes Ricci scalars for $g_{\mu\nu}$ and $f_{\mu\nu}$, respectively. Here, we have two different Planck masses for both metrics. Also, we defined a new "Effective Planck Mass"

$$M_{eff}^2 = \left(\frac{1}{M_g^2}+\frac{1}{M_f^2}\right)^{-1} \tag{5.9}$$

If we wonder whether there is another kinetic term for $f_{\mu\nu}$, it is possible that there exist other terms or consistent mixing terms that can be introduced between $g_{\mu\nu}$ and $f_{\mu\nu}$. Thanks to the theorem given in [52], we know that such a term must vanish in the limit $m \to 0$.

5.2 An Improved Approach: Zwei-Dreibein Gravity Model

This section is about a three-dimensional case. Many of the problems were resolved in three-dimensional spin-2 theory thanks to "New Massive Gravity" (NMG) [57]. Even though this theory is constructed in a general covariant way with the 4^{th} order in derivatives when it is first formulated, linearization shows that this is indeed a free field theory that describes the three-dimensional analog of the Fierz-Pauli (FP) massive spin-2 theory. Furthermore, the non-linear structure is constructed in a specific way to eliminate the Boulware-Deser ghost.

Also, for the four-dimensional case, there has been an attempt to build a non-linear extension of the FP theory in a ghost-free way. This theory is known as the "dRGT Model" thanks to de Rham, Gabadadze, and Tolley [56]. Surely, there exist undesired sides of the dRGT Model. The main one is the fact that we have a fixed background metric while we already have a dynamical metric. Therefore, it does not yield the general covariance of GR. Of course, general covariance is not completely lost. We can easily bring it to the stage with an alternative "bi-metric" gravity model, as we did in the previous section. However, the cost of implementing this structure is the introduction of the massless graviton, again [23]. After that, one can obtain the dRGT Model as a truncation when one of the metrics is considered as non-dynamical. This theory becomes simpler when it is formulated in terms of two vierbein i.e. "zwei-vierbein", instead of two metrics. In this formulation, one can easily see that it is ghost free. Let us have a look at these "Zwei-Dreibein Gravity" (ZDG) fields. These fields are *Lorentz vector valued* 1-forms i.e. e^a_I; $\quad I=1,2 \quad \& \quad a=0,1,2$, and *Lorentz vector valued* connection one-forms i.e. w^a_I. With the help of these fields, one is allowed to build curvature 2-forms and torsion as we did in **(3.28)** and **(3.29)**:

$$R^a_I = d\omega^a_I + \frac{1}{2}\varepsilon^{abc}\omega_{Ib}\omega_{Ic}, \quad T^a_I = de^a_I + \varepsilon^{abc}\omega_{Ib}e_{Ic} \tag{5.10}$$

where we did not present the forms explicitly. Let us present a few new parameters, for our later purposes, such as a sign parameter i.e. $\sigma = \pm 1$, two independent positive mass parameters, M_I, and

$$M_{12} = \frac{\sigma M_1 M_2}{\sigma M_1 + M_2} \tag{5.11}$$

This quantity is non-negative for $\sigma = 1$ and finite for any finite M_I. Note that if $\sigma = -1$, we consider the class $M_1 \neq M_2$. As a next step, let us consider the Lagrangian of ZDG:

$$\mathscr{L}_{ZDG} = \mathscr{L}_1 + \mathscr{L}_2 + \mathscr{L}_{12} \tag{5.12}$$

Here we have $\mathscr{L}_1$ and $\mathscr{L}_2$ which are Einstein-Cartan (EC) Lagrangians:

$$\begin{aligned} \mathscr{L}_1 &= -\sigma M_1 e_{1a} R_1^a - \frac{m^2}{6} M_1 \alpha_1 \varepsilon_{abc} e_1^a e_1^b e_1^c \\ \mathscr{L}_2 &= -M_2 e_{2a} R_2^a - \frac{m^2}{6} M_2 \alpha_2 \varepsilon_{abc} e_2^a e_2^b e_2^c \end{aligned} \tag{5.13}$$

where we interpret m as the mass parameter. Also, we have dimensionless α_I parameters. The last term in **(5.12)** is responsible for the coupling of the first two terms as

$$\mathscr{L}_{12} = \frac{m^2}{2} M_{12} \varepsilon_{abc} \left(\beta_1 e_1^a e_1^b e_2^c + \beta_2 e_1^a e_2^b e_2^c \right) \tag{5.14}$$

Here β_I are our new parameters which are obviously dimensionless. As a result, there are five parameters in the ZDG which are α_I, β_I, and M_1/M_2. If we drop $\mathscr{L}_{12}$ term from **(5.12)**, all we have is just the summation of two EC Lagrangians which means that the new action yields the diffeomorphism and local Lorentz gauge invariance independently for the two sets that contributed to the **(5.13)**.

The next step is expanding e_I^a around a fixed $\bar{e}^a$ which yields maximally symmetric background with cosmological constant Λ also having the same logic for ω_I^a:

$$\begin{aligned} e_1^a &= \bar{e}^a + \kappa h_1^a, \qquad & \omega_1^a &= \bar{\omega}^a + \kappa v_1^a \\ e_2^a &= \gamma(\bar{e}^a + \kappa h_2^a), & \omega_2^a &= \bar{\omega}^a + \kappa v_2^a \end{aligned} \tag{5.15}$$

where γ is another constant. κ is, in principle, an infinitesimal parameter. Since we are about the gather the motivation of the "scaling limit", we may consider it as a parameter of expansion. After putting these expanded 1-forms and spin connection into the **(5.12)**, Λ is fixed thanks to the cancellation of the linear terms in modified **(5.12)**

$$\frac{\Lambda}{m^2} = -\gamma^2 \alpha_2 + \frac{M_{12}}{M_2} (\beta_1 + 2\gamma\beta_2) \tag{5.16}$$

which resulted in a quadratic equation on γ:

$$[\alpha_2 (\sigma M_1 + M_2) + \beta_2 M_2] \gamma^2 + 2 (M_2 \beta_1 - \sigma M_1 \beta_2) \gamma - \sigma [\alpha_1 (\sigma M_1 + M_2) + \beta_1 M_1] = 0 \tag{5.17}$$

The quadratic terms in the expansion can be diagonalized by introducing

$$M_{crit} = \sigma M_1 + \gamma M_2 \neq 0 \tag{5.18}$$

Let us introduce the *diagonal* one-form fields as

$$\begin{aligned} h_+^a &= \frac{\sigma M_1 h_1^a + \gamma M_2 h_2^a}{M_{crit}}, \quad h_-^a = h_1^a - h_2^a \\ v_+^a &= \frac{\sigma M_1 v_1^a + \gamma M_2 v_2^a}{M_{crit}}, \quad v_-^a = v_1^a - v_2^a \end{aligned} \tag{5.19}$$

Putting **(5.19)** and **(5.18)** into the **(5.12)**, we obtain $\mathscr{O}\left(\kappa^2\right)$ Lagrangian in the diagonalized form as it has been promised:

$$\begin{aligned} \mathscr{L}^{(2)} =& \mathscr{L}_+^{(2)} + \mathscr{L}_-^{(2)} \\ \text{where} \quad \mathscr{L}_+^{(2)} =& -M_{crit}\left[h_{+a}\bar{D}v_+^a + \frac{\varepsilon_{abc}}{2}\bar{e}^a v_+^b v_+^c - \frac{\Lambda}{2}\varepsilon_{abc}\bar{e}^a h_+^b h_+^c\right] \\ \text{and} \quad \mathscr{L}_-^{(2)} =& -\frac{\sigma\gamma M_1 M_2}{M_{crit}}\left[h_{-a}\bar{D}v_-^a + \frac{\varepsilon_{abc}}{2}\bar{e}^a v_-^b v_-^c + \left(\mathscr{M}^2 - \Lambda\right)\frac{\varepsilon_{abc}}{2}\bar{e}^a h_-^b h_-^c\right] \\ \text{where} \quad \mathscr{M}^2 =& \frac{m^2 M_{crit}}{\sigma M_1 + M_2}\left(\beta_1 + \gamma\beta_2\right) \\ \text{and} \quad \bar{D} \quad & \text{is a covariant exterior derivative which only depends on } \bar{e}^a \ \& \ \bar{\omega}^a. \end{aligned} \tag{5.20}$$

The $v_\pm^a$ can be eliminated by their e.o.m since they are auxiliary. By eliminating v_+^a, one can see that $\mathscr{L}_+^{(2)}$ is just a second-order approximation to the **(4.10)** in AdS background also it appears that there is no such mode that propagates. On the other hand, if v_-^a are eliminated one can also understand that $\mathscr{L}_-^{(2)}$ is a multiplicity of Fierz-Pauli Lagrangian, again in the AdS background, with FP mass $\mathscr{M}$. Note that Λ has a contribution towards the mass term unless $\mathscr{M}^2 = \Lambda$. That specific case is called the "partially massless" case. The benefit of this case is that the linearized theory gains another gauge invariance. Of course, we do not have such a relationship when $\Lambda < 0$ since $\mathscr{M}^2 < 0$ means that we have a tachyon with spin-2.

As a final examination, let us look at how ZDG unifies earlier 3D massive gravity theories. In order to understand its connection with the three-dimensional dRGT model, set σ=1 and let

$$e_2^a = \bar{e}_2^a + \lambda h_2^a, \quad \omega_2^a = \bar{\omega}_2^a + \lambda v_2^a \tag{5.21}$$

where $\bar{e}_2^a$ is "reference dreibein", $\bar{\omega}_2^a$ corresponds to spin connection with zero torsion also λ is the "expansion parameter". When the limit $\lambda \to 0$ is taken while keeping

$\lambda^2 M_2^2 = M_1 = 2M_p$ fixed, **(5.12)** can be written as

$$\mathscr{L} = 2M_p \mathscr{L}_0(h_2, v_2) + 2M_p \mathscr{L}_{dRGT}(e_1, \omega_1) \tag{5.22}$$

Here the first term corresponds to second-order approximation to the three-dimensional **(4.20)** which only includes the reference background for the fluctuations (h_2^a, v_2^a). Since it has EC gauge invariances at a linearized level, this term provides no modes that propagate. On the other hand, when we look at the second term, $(e_1^a, \omega_1^a) \to (e^a, \omega^a)$ leads to

$$\mathscr{L}_{dRGT} = -e_a R^a - \alpha_1 \frac{m^2}{6} \varepsilon^{abc} e_a e_b e_c + \frac{m^2}{2} \varepsilon^{abc} \left(\beta_1 e_a e_b \bar{e}_c + \beta_2 e_a \bar{e}_b \bar{e}_c\right) \tag{5.23}$$

But this is nothing but the dreibein formulation of three-dimensional dRGT.

In order to reveal the connection between ZDG and NMG, let us set $\sigma = -1$ and

$$e_2^a = e_1^a + \frac{\lambda}{m^2} f^a, \quad \omega_2^a = \omega_1^a - \lambda h^a \tag{5.24}$$

As a next step, let us consider the following parameterization

$$\begin{aligned} M_1(\lambda) =& 2\left(1 + \frac{1}{\lambda}\right) M_p, \quad \alpha_1(\lambda) = -\frac{\lambda_0}{m^2}\lambda + \frac{1}{\lambda}, \quad \beta_1(\lambda) = 0 \\ M_2(\lambda) =& \frac{2}{\lambda} M_p, \qquad\qquad \alpha_2(\lambda) = 2\left(1 + \frac{1}{\lambda}\right), \quad \beta_2(\lambda) = 1 \end{aligned} \tag{5.25}$$

After putting these into the Lagrangian, if we take the limit $\lambda \to 0$ while keeping the Planck Mass, M_p, fixed. We obtain the first-order formulation of the limit examined in [17] which requires the $\sigma = -1$. As a result this limit in $\mathscr{L}_{ZDG}$ exists as well

$$\lim_{\lambda \to 0} \mathscr{L}_{ZDG} = \mathscr{L}$$

$$\text{where} \quad \mathscr{L} = 2M_p \left[e_a R^a + \frac{\lambda_0}{6} \varepsilon^{abc} e_a e_b e_c + h_a T^a - \frac{1}{m^2} \left(f_a R^a + \frac{\varepsilon^{abc}}{2} e_a f_b f_c \right) \right] \tag{5.26}$$

This Lagrangian is known as the "Chern-Simons-like" Lagrangian for NMG [64]. As you may observe, the second dreibein just became an auxiliary field. Therefore, when we eliminate it, we see the higher-derivative action for NMG.

5.3 Scaling Limits

In this section, we are going to unify some of the massive gravity models. In order to understand the structure that enables the unification let us refresh our knowledge about massive gravity theories in a quick review.

By modifying the **(4.10)**, some models were introduced in order to add higher curvature terms to the action, such as NMG [21]. Also, there is a well-established, i.e. ghost-free, construction of a nonlinear extension of the Fierz-Pauli mass term. That is the de Rham-Gabadadze-Tolley model [56]. We will explore and understand how we can represent these models in the light of scaling limits of the ghost-free bi-gravity models of [23]. After that, with the help of this knowledge, we will construct extensions of both types of massive gravity models. Since we will start with a ghost-free theory, the outcomes are guaranteed to not include any ghosts or instabilities.

One way to extend this theory is to have a holographic c-function [61–63] since we want to understand the allowed structure of the higher curvature terms which will be added into the action. This is completely enough at the cubic level however at the higher order, it is not enough and degeneracies occur.

5.3.1 A smooth introduction to bi-gravity models

Let us start with the ghost(instability)-free d-dimensional bi-gravity models. As we will demonstrate, with the help of the generalization of the allowed mass-terms given in [56], the models given in [23] can be seen as their specific limits & choices. As the name suggests, bi-gravity models describe two independent metrics which may or may not interact via a potential that depends on some specific polynomials as we will see. Also, they yield one propagating massless & one massive spin-2 modes. The mentioned specific polynomials are characteristic polynomials and for the sake of completeness, let us start by introducing them. They are related to a matrix M as

$$\mathscr{P}^{(N)} \equiv \delta^{\alpha_1\cdots\alpha_n}_{[\beta_1\cdots\beta_n]} M^{\beta_1}_{\alpha_1}\cdots M^{\beta_n}_{\alpha_n} \tag{5.27}$$

where

$$\delta^{\alpha_1\cdots\alpha_n}_{[\beta_1\cdots\beta_n]} = \frac{1}{n!}\left[\delta^{\alpha_1}_{\beta_1}\cdots\delta^{\alpha_n}_{\beta_n} + \text{permutations}\right] \tag{5.28}$$

The first few polynomials are:

$$\begin{aligned} &\mathscr{P}^0(M^\beta_\alpha) = 1, \quad \mathscr{P}^1(M^\beta_\alpha) = [M], \quad \mathscr{P}^2(M^\beta_\alpha) = \frac{1}{2}\left([M]^2 - \left[M^2\right]\right) \\ &\text{and,} \quad \mathscr{P}^3(M^\beta_\alpha) = \frac{1}{6}\left([M]^3 - 3[M]\left[M^2\right] + 2\left[M^3\right]\right) \end{aligned} \tag{5.29}$$

Here $[M] = M^\alpha_\alpha$, $\left[M^2\right] = M_{\alpha\beta}M^{\alpha\beta}$ and $\left[M^3\right] = M_{\alpha\beta}M^{\beta\gamma}M^\alpha_\gamma$. Understanding any theory is established by examining the Lagrangian as the easiest way. In order to

do so let us introduce bi-gravity theories. They are described two dynamical metrics g_{ab} and f_{ab} with action [23]:

$$S=\int d^dx\left[\frac{M_a^{d-2}}{2}\sqrt{-g}R[g]+\frac{M_b^{d-2}}{2}\sqrt{-f}R[f]+\bar{M}^{d-2}m^2\sqrt{-g}U(g,f)\right] \tag{5.30}$$

where $\bar{M}^{d-2}=M_a^{d-2}+M_b^{d-2}$. Here the potential term can be expressed as

$$U(g,f)=\sum_{n=0}^{d}\alpha_n\mathscr{P}^{(n)}(\mathscr{K}),\quad \text{with}\quad \mathscr{K}_b^a(g,f)\equiv\delta_b^a-\sqrt{g^{ac}f_{cb}} \tag{5.31}$$

where α_n are free parameters. To be more precise, $\mathscr{K}_{ab}=g_{ac}\mathscr{K}_b^c$ is a tensor given in [56] which satisfy

$$2\mathscr{K}_b^a-\mathscr{K}_c^a\mathscr{K}_b^c=\delta_b^a-g^{ac}f_{cb} \tag{5.32}$$

As you may see from **(5.30)** when one takes a scaling (decoupling) limit i.e. $M_b\to\infty$ while keeping M_a fixed, dynamics of the metric decouples if $f_{\mu\nu}=\bar{f}_{\mu\nu}+M_b^{d-2}\delta f_{\mu\nu}$. Here $\bar{f}_{\mu\nu}$ is a fixed background metric. Then, effective theory for $g_{\mu\nu}$ becomes a massive gravity theory with the massless mode hidden by $\delta f_{\mu\nu}$ decoupled. As a special case, $\bar{f}_{\mu\nu}=\eta_{\mu\nu}$, we got the dRGT.

Even though we did not fix the coefficients α_n, we can choose $\alpha_1=0$ via conformally rescaling f_{ab} and adding a multiple of the identity matrix to $\mathscr{K}$ at the same time and $\alpha_2=1$ via rescaling the mass. If one chooses the coefficients as $\alpha_n=\alpha^n$, potential term becomes:

$$U(g,f)=\det(\delta_b^a+\alpha\mathscr{K}_b^a) \tag{5.33}$$

Therefore, the general choices of parameters can be understood as a deformation of this determinant. Then it is straightforward to realize that the action is symmetric under the interchange $g_{ab}\to f_{ab}$. Keep in mind that in this case, one has to redefine all the constants α_n in the action [49]. This comes from the fact that

$$\sqrt{-g}\det(\delta_b^a+\alpha\mathscr{K}_b^a)=\alpha^d\sqrt{-f}\det\left(\delta_b^a+\alpha^{-1}\tilde{\mathscr{K}}_b^a\right) \tag{5.34}$$

where $\tilde{\mathscr{K}}_b^a=\delta_b^a-\sqrt{f^{ac}g_{cb}}$. Despite the fact that [23] considered bi-gravity actions containing solely **(4.10)** term, in higher dimensions, we can generalize this. The leading way to the generalization goes through considering both these two metrics theory can be described via Lovelock gravity [53]:

$$S_L=\int d^dx\left[\frac{M_a^{d-2}}{2}\sqrt{-g}\sum c_g^{(K)}\mathscr{L}^{(K)}[g]+\frac{M_b^{d-2}}{2}\sqrt{-f}\sum c_f^{(K)}\mathscr{L}^{(K)}[f]+\bar{M}^{d-2}m^2\sqrt{-g}U(g,f)\right] \tag{5.35}$$

Here $\mathscr{L}^{(K)}$ correspond to "Lovelock invariants". The benefit of the "Lovelock Bi-Gravity" models is that they are the most general non-chiral theories of a single massless field coupled to a single massive field in any dimension. In the appendix, you may find a very quick review of Lovelock gravity.

5.3.2 Another way to NMG i.e. scaling limit

Let us take the scaling limit of the **(5.30)** which the Planck masses go to infinity. While taking this limit, let us take one of the masses to negative infinity. We are aware that we are explicitly breaking the unitarity of the theory for $d > 3$ but not for the special case $d = 3$. Let us consider $f_{ab} = g_{ab} + \lambda q_{ab}$ and look for what will go on when the limit $\lambda \to 0$ combined with $M_a^{d-2} + M_b^{d-2} = M_P^{d-2}$ and $\lambda M_b^{d-2} = A$ kept fixed. We can rescale q_{ab} so that $A = M_P^{d-2}$. In this specific limit, the kinetic terms turn out to be

$$\begin{aligned}\mathscr{L}_{kinetic} &= \left[\frac{M_a^{d-2}+M_b^{d-2}}{2}\sqrt{-g}R[g] - \frac{M_b^{d-2}}{2}\lambda\sqrt{-g}q^{ab}G_{ab} + \mathscr{O}(\lambda)\right] \\ &\to \frac{M_P^{d-2}}{2}\sqrt{-g}\left(R[g] - q^{ab}G_{ab} + \mathscr{O}(\lambda)\right)\end{aligned} \tag{5.36}$$

The next is the potential term. But first, keep in mind that in this specific limit

$$\mathscr{K}_{ab} = -\frac{1}{2}\lambda q_{ab} + \mathscr{O}(\lambda^2) \tag{5.37}$$

As a result

$$\mathscr{P}^{(n)}(\mathscr{K}) = \left(-\frac{\lambda}{2}\right)^n \mathscr{P}^{(n)}(q) \tag{5.38}$$

Furthermore, one can also take the scaling limit as

$$\bar{M}^{d-2}m^2\alpha_n\left(\frac{-\lambda}{2}\right)^n \to \frac{M_P^{d-2}}{2}\beta_n \tag{5.39}$$

where $\beta_0 = -2\Lambda$ i.e. the cosmological constant. When we gathered all these limit results, the action turns out to be

$$S = \frac{M_P^{d-2}}{2}\int d^dx\sqrt{-g}\left[R + 2\Lambda - q^{ab}G_{ab} + \sum_{n=2}^{d}\beta_n\mathscr{P}^n(q)\right] \tag{5.40}$$

If one set $\beta_{n>2} = 0$, **(5.40)** turns out to be nothing but the auxiliary field formulation of NMG. As a result, it is shown that NMG appears as a specific case of the bimetric gravity theory.

One can easily follow the same footprints for bimetric theories of Lovelock gravity. As a result, these models are identified via action:

$$S = \frac{M_P^{d-2}}{2} \int d^d x \left[\left(\sum_K a_K \mathscr{L}^{(K)} \right) + q^{ab} \left(\sum_K b_K G_{ab}^{(K)} \right) + \sum_{n=2}^{d} \beta_n \mathscr{P}^n(q) \right] \qquad \textbf{(5.41)}$$

where the G_{ab} are Lovelock tensors. Again, their definition can be found in the appendix. Since these theories are generalizations of the previous ones, they yield a ghost field. Thankfully, it has been shown in [17], with some special choices, that the ghost can be eliminated.

For general β_n, it is possible to have higher order curvature terms as much as anyone wants since they are created by integrating out the auxiliary field. Of course, with specific choices, one can write **(5.41)** in a closed form. However, this is the point where we conclude our chapter.

6. CARROLL GRAVITY

In Chapter 4, we obtained the GR by gauging the Poincaré algebra. Thus the question is obtaining Carrollian Gravity by gauging the algebra of the underlying symmetries. In this chapter, we will show how can be obtained ultra-relativistic gravity[1]. This chapter of the thesis is based on the [1]. So let us begin with Lie Algebra Expansion.

6.1 Lie Algebra Expansion

Lie algebra expansion is a method that focuses on a core Lie algebra $\mathfrak{g}$. Furthermore, that process will provide us new and higher dimensional algebras as long as $\mathfrak{g}$ can be written as a direct sum of two subspaces V_0 and V_1 such that they satisfy the relations:

$$[V_0, V_0] \subset V_0, \quad [V_0, V_1] \subset V_1, \quad [V_1, V_1] \subset V_0 \tag{6.1}$$

Based on these relations, V_0 represents the even class of generators while V_1 represents the odd class. The direct sum structure of the core Lie algebra suggests that we may also assign a gauge field to each of the generators

$$A_\mu = A^i_\mu X_i + A^\alpha_\mu Y_\alpha \tag{6.2}$$

where X^i represent the even subset of the generators while Y^α represent the odd ones. As a next step, we expand the gauge fields with the expansion parameter λ with respect to the class that they belong to [1]:

$$A^i = \sum_{n=0}^{N_0} \lambda^{2n} A^i_{(2n)}, \quad A^\alpha = \sum_{n=0}^{N_1} \lambda^{(2n+1)} A^\alpha_{(2n+1)} \tag{6.3}$$

Here, the sum can be extended to infinity to produce an infinite dimensional Lie algebra. The consistent truncation at order $g = (N_0, N_1)$ requires that either $N_0 = N_1$ or $N_0 = N_1 + 1$ is satisfied. With this expansion in hand, one can start with the Maurer-Cartan equations of the core Lie algebra $\mathfrak{g}$, expand the gauge fields with

[1] Throughout this thesis, we will use the ultra-relativistic gravity≡ Carroll gravity

respect to **(6.3)**, and obtain the structure constants of the expanded algebra from the expanded Maurer-Cartan equations at each order. Equivalently, based on their even/odd character, we may expand the generators X^i, Y^α with $X^i \in V_0$ and $Y^\alpha \in V_1$ as

$$X_i^{(2n)} = \lambda^{2n} \otimes X_i, \quad Y_\alpha^{(2n+1)} = \lambda^{2n+1} \otimes Y_\alpha \tag{6.4}$$

Also using the commutation relations of the core algebra

$$\left[X_i, X_j\right] = f_{ij}{}^k X_k, \quad [X_i, Y_\alpha] = f_{i\alpha}{}^\beta Y_\beta, \quad \left[Y_\alpha, Y_\beta\right] = f_{\alpha\beta}{}^i X_i \tag{6.5}$$

Now we are able to give the commutation relations for the expanded algebra:

$$\begin{aligned} \left[X_i^{(2m)}, X_j^{(2n)}\right] =& f_{ij}{}^k X_k^{(2m+2n)}, \quad \left[X_i^{(2m)}, X_\alpha^{(2n+1)}\right] = f_{i\alpha}{}^\beta Y_\beta^{(2m+2n+1)} \\ \text{and} \quad & \left[Y_\alpha^{(2m+1)}, Y_\beta^{(2n+1)}\right] = f_{\alpha\beta}{}^i X_i^{(2m+2n+2)} \end{aligned} \tag{6.6}$$

It is immediately seen that Jacobi identities for algebras are closed at each order without referring to any higher order. As a result, the group theoretical curvatures of the resulting truncated algebras satisfy the Bianchi identity without referring to higher-order terms. Now let us focus on the space-time split Poincaré algebra and its ultra-relativistic Lie algebra expansion. The d-dimensional Poincaré algebra consists of translations, P_A, and Lorentz transformations, M_{AB}, with non-vanishing commutation relations:

$$[M_{AB}, P_C] = 2\eta_{C[B}P_{A]}, \qquad [M_{AB}, M_{CD}] = 4\eta_{[A[C}M_{D]B]} \tag{6.7}$$

The space-time decomposition can be achieved by decomposing the d-dimensional index A as $A = (0, a)$ which leads to

$$M_{AB} = \{M_{0a} \equiv G_a, J_{ab}\}, \qquad P_A = \{P_0 = H, P_a\} \tag{6.8}$$

In this case, the Poincaré algebra decomposes as

$$\begin{aligned} &[G_a, P_b] = \delta_{ab} H, && [G_a, H] = P_a, && [J_{ab}, P_c] = \delta_{bc}P_a - \delta_{ac}P_b \\ &[J_{ab}, G_c] = \delta_{bc}G_a - \delta_{ac}G_b, && [J_{ab}, J_{cd}] = 4\delta_{[a[c}J_{d]b]}, && [G_a, G_b] = J_{ab} \end{aligned} \tag{6.9}$$

Since we obtained the spacetime decomposed Poincaré algebra, we can go further and discuss ultra-relativistic Lie algebra expansions and action principles.

6.2 Ultra-Relativistic Algebras and Actions

The ultra-relativistic higher-dimensional algebras are achieved by the following choice for the generators

$$V_0 = \{J_{ab}, P_a\}, \qquad V_1 = \{H, G_a\} \tag{6.10}$$

With this choice of generators, we can follow the prescription that we presented in **(6.6)**. Thus the Lie algebra expansion of the spacetime decomposed Poincaré algebra is given by

$$\begin{aligned}
\left[G_a^{(2m+1)}, P_b^{(2n)}\right] &= \delta_{ab} H^{(2m+2n+1)}, & \left[G_a^{(2m+1)}, H^{(2n+1)}\right] &= P_a^{(2m+2n+2)}, \\
\left[J_{ab}^{(2m)}, P_c^{(2n)}\right] &= 2\delta_{c[b} P_{a]}^{(2m+2n)}, & \left[J_{ab}^{(2m)}, G_c^{(2n+1)}\right] &= 2\delta_{c[b} G_{a]}^{(2m+2n+1)}, \\
\left[J_{ab}^{(2m)}, J_{cd}^{(2n)}\right] &= 4\delta_{[a[c} J_{d]b]}^{2m+2n}, & \left[G_a^{(2m+1)}, G_b^{(2n+1)}\right] &= J_{ab}^{(2m+2n+2)}
\end{aligned} \tag{6.11}$$

The simplest truncation of this infinite-dimensional algebra is known as the Carroll algebra

$$\begin{aligned}
[C_a, P_b] &= \delta_{ab} H, & [J_{ab}, P_c] &= \delta_{bc} P_a - \delta_{ac} P_b, \\
[J_{ab}, C_c] &= \delta_{bc} C_a - \delta_{ac} C_b, & [J_{ab}, J_{cd}] &= 4\delta_{[a[c} J_{d]b]}
\end{aligned} \tag{6.12}$$

where we relabeled the generators as $P_a^{(0)} = P_a$, $J_{ab}^{(0)} = J_{ab}$, $H^{(1)} = H$ and we set $G_a^{(1)} = C_a$ to denote that it is the generator of the Carrollian boosts. One might wonder what would happen if we go just a little bit further i.e. put $J_{ab}^{(2)}$ and $P_a^{(2)}$ into the algebra. However, according to the consistent truncation conditions, we are not able to do that; we can go one further step and construct as we will discuss momentarily.

For the construction of an action principle, we first need to space-time decompose gauge fields of the Poincaré algebra, namely vielbein ($E_\mu{}^A$) and the spin connection ($\Omega_\mu{}^{AB}$):

$$E^A = \{E^0 = T, E^a\}, \qquad \Omega^{AB} = \{\Omega^{0a} = \Omega^a, \Omega^{ab}\} \tag{6.13}$$

and their expansion is given by

$$\begin{aligned}
T &= \sum_{n=0}^{N_1} \lambda^{2n+1} \tau_{(2n+1)}, & E^a &= \sum_{n=0}^{N_0} \lambda^{2n} e^a_{(2n)}, \\
\Omega^{ab} &= \sum_{n=0}^{N_0} \lambda^{2n} \omega^{ab}_{(2n)}, & \Omega^a &= \sum_{n=0}^{N_1} \lambda^{2n+1} \omega^a_{(2n+1)}
\end{aligned} \tag{6.14}$$

These expressions can be used in the space-time decomposed Einstein-Hilbert action in the first-order formulation to generate invariant non-relativistic gravity models. To perform the expansion explicitly, let us focus on three and four dimensions, however, our arguments are dimension independent. In four dimensions, the space-time decomposed Lagrangian is given by

$$\mathscr{L}_{EH} = \varepsilon_{ABCD} R^{AB} \wedge E^C \wedge E^D = 2\varepsilon_{abc} \left(R(\Omega^a) \wedge E^b \wedge E^c - R\left(\Omega^{ab}\right) \wedge E^c \wedge T \right) \quad \textbf{(6.15)}$$

Here R^{AB} refers to the group-theoretical curvature of the spin connection. $R(\Omega^a)$ and $R\left(\Omega^{ab}\right)$ refer to its spacetime decomposition with respect to **(6.13)**. While these curvatures can easily be seen from the Poincaré algebra, it is useful to present their expression explicitly for future purposes

$$R(\Omega^a) = d\Omega^a + \Omega^{ab} \wedge \Omega_b, \quad \textbf{(6.16)}$$

Consequently, the spacetime decomposed d=4 Einstein-Hilbert action is expanded as

$$\begin{aligned} \mathscr{L} = & -2\varepsilon_{abc} \sum_{m=0}^{N_0} \sum_{n=0}^{N_0} \sum_{l=0}^{N_1} \lambda^{2m+2n+2l+1} R\left(\omega^{ab}_{(2m)}\right) \wedge e^c_{(2n)} \wedge \tau_{(2l+1)} \\ & + 2\varepsilon_{abc} \sum_{m=0}^{N_1} \sum_{n=0}^{N_0} \sum_{l=0}^{N_0} \lambda^{2m+2n+2l+1} R\left(\omega^{a}_{(2m+1)}\right) \wedge e^b_{(2n)} \wedge e^c_{(2l)} \end{aligned} \quad \textbf{(6.17)}$$

Note that this case is much simpler than the non-relativistic version of it because all terms in the Lagrangian are expanded to the same λ-order. Therefore, it is guaranteed to have all possible contributions to any given λ-order as long as we have the equal number of even and odd generators. At the lowest order i.e. $\mathscr{O}(\lambda)$ we have the Carrollian gravity

$$\mathscr{L} = -2\varepsilon_{abc} R^{ab}(\omega) \wedge e^c \wedge \tau + 2\varepsilon_{abc} R^a(\omega) \wedge e^b \wedge e^c \quad \textbf{(6.18)}$$

where we set

$$\tau_{(1)} = \tau, \quad e^a_{(0)} = e^a, \quad \omega^{ab}_{(0)} = \omega^{ab}, \quad \omega^a_{(1)} = \omega^a \quad \textbf{(6.19)}$$

For the next-to-leading order Lagrangian i.e. $\mathscr{O}\left(\lambda^3\right)$

$$\begin{aligned} \mathscr{L} = & -2\varepsilon_{abc} R^{ab}(s) \wedge e^a \wedge \tau - 2\varepsilon_{abc} R^{ab}(\omega) \wedge t^c \wedge \tau - 2\varepsilon_{abc} R^{ab}(\omega) \wedge e^c \wedge m \\ & + 2\varepsilon_{abc} R^a(b) \wedge e^b \wedge e^c + 4\varepsilon_{abc} R^a(\omega) \wedge e^b \wedge t^c \end{aligned} \quad \textbf{(6.20)}$$

where we set

$$\begin{aligned} & \tau_{(1)} = \tau, \quad \tau_{(3)} = m, \quad \omega^{ab}_{(0)} = \omega^{ab}, \quad \omega^{ab}_{(2)} = s^{ab}, \\ & e^a_{(0)} = e^a, \quad e^a_{(2)} = t^a, \quad \omega^a_{(1)} = \omega^a, \quad \omega^a_{(3)} = b^a \end{aligned} \quad \textbf{(6.21)}$$

And the group theoretical curvatures read

$$\begin{aligned} R^{ab}(\omega) =& d\omega^{ab} - \omega^{ac} \wedge \omega^{b}{}_{c}, & R^{a}(\omega) =& d\omega^{a} + \omega^{ab} \wedge \omega_{b}, \\ R^{ab}(s) =& ds^{ab} - 2\omega^{ac} \wedge s^{b}{}_{c} - \omega^{a} \wedge \omega^{b}, & R^{a}(b) =& db^{a} + \omega^{ab} \wedge b_{b} + s^{ab} \wedge \omega_{b} \end{aligned} \tag{6.22}$$

This Lagrangian is invariant under the following extension of the Carroll algebra

$$\begin{aligned} [C_a, P_b] =& \delta_{ab} H, & [J_{ab}, P_c] =& \delta_{bc} P_a - \delta_{ac} P_b, & [J_{ab}, C_c] =& \delta_{bc} C_a - \delta_{ac} C_b, \\ [J_{ab}, J_{cd}] =& 4\delta_{[a[c} J_{d]b]}, & [B_a, P_b] =& \delta_{ab} M, & [C_a, T_b] =& \delta_{ab} M, \\ [C_a, H] =& T_a, & [J_{ab}, T_c] =& \delta_{bc} T_a - \delta_{ac} T_b, & [S_{ab}, P_c] =& \delta_{bc} T_a - \delta_{ac} T_b, \\ [J_{ab}, S_{cd}] =& 4\delta_{[a[c} S_{d]b]}, & [C_a, C_b] =& S_{ab}, & & \\ [J_{ab}, B_c] =& \delta_{bc} B_a - \delta_{ac} B_b, & [S_{ab}, C_c] =& \delta_{bc} B_a - \delta_{ac} B_b & & \end{aligned} \tag{6.23}$$

where we labeled $H^{(3)} = M$, $C_a^{(3)} = B_a$, $P_a^{(2)} = T_a$ and $J_{ab}^{(2)}$=S_{ab}. From now on, we will refer to this model as the Beyond-Carrollian Gravity. Note that all terms in the spacetime decomposed three-dimensional Einstein-Hilbert Lagrangian are still in the same order in the expansion. It is the unique structure of the Carrollian that you are allowed to write only odd-order Lagrangian. Also since it comes always in the same order you do not require to add off-diagonal $U(1) \times U(1)$ Chern-Simons gauge terms because we already have *exact* matching terms.

6.3 Scaling Limit of Multimetric Gravity

In this section, we introduce an ultra-relativistic scaling limit for multimetric gravity models based on the Lie algebra expansion that we discussed in the previous section. The fundamental idea to have a well-defined scaling limit is the necessity of using multiple copies of Einstein-Hilbert action to get rid of divergent lower-order Lagrangian(s).

The main difference between the non-relativistic and ultra-relativistic expansion is the change of the character of the generator of time translations H, and spatial translations P_a. In the ultra-relativistic expansion of the Poincaré algebra, H is expanded in odd powers of λ while P_a is expanded in even powers. This is also reflected in the expansion of the corresponding gauge fields as you see from **(6.14)**. As a result, the d-dimensional Einstein-Hilbert action captures all $\mathscr{O}\left(\lambda^{2N+1}\right)$ terms, giving rise to an invariant Lagrangian as long as $N_0 = N_1$. This statement is true for d = 3 as well, hence

we do not need to present a separate treatment for $d = 3$ but can provide a representative example for d = 4 which can be generalized to arbitrary dimensions.

In four dimensions, the structure of the space-time decomposed Einstein-Hilbert action **(6.15)** implies that the ultra-relativistic expansion with **(6.14)** yields the following structure

$$\mathscr{L}_{EH} = \lambda M^2 \mathscr{L}_1 + \lambda^3 M^2 \mathscr{L}_3 + \lambda^5 M^2 \mathscr{L}_5 + \cdots \tag{6.24}$$

here M^2 is a mass parameter. The lowest order Lagrangian can be isolated by the lowest order expressions for the relativistic fields **(6.19)** which can be followed by the rescaling of the mass parameter $M^2 \to M^2/\lambda$ and the scaling limit $\lambda \to 0$. We need to find an α expansion in order to isolate the higher-order Lagrangians. We start with non-interacting $N+1$-metric theory

$$\mathscr{L} = \sum_{i=1}^{N+1} \left[2M_i^2 \varepsilon_{abc} \left(R(\Omega_i^a) \wedge E_i^b \wedge E_i^c - R(\Omega_i^{ab}) \wedge E_i^c \wedge T_i \right) \right] \tag{6.25}$$

where $i = 1, \cdots N+1$ label the set of fields $\{T_i, E_i^a, \Omega_i^a, \Omega_i^{ab}\}$. Let us keep the expansion of the first set of fields $\{T_1, E_1^a, \Omega_1^a, \Omega_1^{ab}\}$ the same as **(6.14)**

$$\begin{aligned} T_1 &= \sum_{n=0}^{N_1} \lambda^{2n+1} \tau_{(2n+1)}, \quad & E_1^a &= \sum_{n=0}^{N_0} \lambda^{2n} e^a_{(2n)}, \\ \Omega_1^{ab} &= \sum_{n=0}^{N_0} \lambda^{2n} \omega^{ab}_{(2n)}, & \Omega_1^a &= \sum_{n=0}^{N_1} \lambda^{2n+1} \omega^a_{(2n+1)} \end{aligned} \tag{6.26}$$

while for $i > 1$, we include additional dimensionless parameters $\alpha_{(i)}$

$$\begin{aligned} T_{(i)} &= \sum_{n=0}^{N_1} \lambda^{2n+1} \alpha_{(i)}^{2n+1} \tau_{(2n+1)}, \quad & E^a_{(i)} &= \sum_{n=0}^{N_0} \lambda^{2n} \alpha_{(i)}^{2n} e^a_{(2n)}, \\ \Omega^{ab}_{(i)} &= \sum_{n=0}^{N_0} \lambda^{2n} \alpha_{(i)}^{2n} \omega^{ab}_{(2n)}, & \Omega^a_{(i)} &= \sum_{n=0}^{N_1} \lambda^{2n+1} \alpha_{(i)}^{2n+1} \omega^a_{(2n+1)} \end{aligned} \tag{6.27}$$

where $\alpha_i \neq 1$ and $\alpha_i \neq \alpha_j$ for $i \neq j$. This yields the following expansion of $\mathscr{L}_i$ that is the Einstein-Hilbert action for i-th set of fields $\{T_i, E_i^a, \Omega_i^a, \Omega_i^{ab}\}$

$$\mathscr{L}_{EH,i} = \lambda \alpha_i M_i^2 \mathscr{L}_1 + \lambda^3 \alpha_i^3 M_i^2 \mathscr{L}_3 + \lambda^5 \alpha_i^5 M^2 \mathscr{L}_5 + \cdots \tag{6.28}$$

Keeping the first set of fields with standard expansion while using the α expansion for $i > 1$, the Einstein-Hilbert action turns out to be

$$\mathscr{L}_{EH} = \sum_{n=0}^{N} \lambda^{2n+1} \left(M_1^2 + \sum_{i=2}^{N+1} \alpha_i^{2n+1} M_i^2 \right) \mathscr{L}_{2n+1} + \mathscr{O}\left(\lambda^{2N+3}\right) \tag{6.29}$$

To single out the order $2N+1$ Lagrangian, one needs to solve N number of algebraic equations that determines the values of α_i for $i = 2, 3 \cdots, N+1$

$$0 = M_1^2 + \sum_{i=2}^{N+1} \alpha_i^{2n+1} M_i^2, \quad \text{for} \quad n = 0, 1, \cdots, N-1 \tag{6.30}$$

These values can finally be used in the coefficient of the order $2N+1$ Lagrangian along with the scaling of the mass parameters $M_i \to M_i/\lambda^{2N+1}$. Finally, performing the limit $\lambda \to 0$ gets rid of all higher terms and yields the desired ultra-relativistic model. Let us provide some examples

- **Carroll Gravity**: The Carroll gravity appears as $\mathscr{L}_1$ in the ultra-relativistic expansion of Einstein-Hilbert action **(6.24)**. It is, thus, sufficient to consider the Poincaré algebra, a single set of its gauge fields, and the Einstein-Hilbert action. Using the lowest-order definitions for the relativistic fields

$$T = \lambda \tau, \quad E^a = e^a, \quad \Omega^{ab} = \omega^{ab}, \quad \Omega^a = \lambda \omega^a \tag{6.31}$$

  We obtain the Carroll gravity

$$\mathscr{L}_1 = -2\varepsilon_{abc} R^{ab}(\omega) \wedge e^c \wedge \tau + 2\varepsilon_{abc} R^a(\omega) \wedge e^b \wedge e^c \tag{6.32}$$

  by rescaling the mass parameter with $M^2 \to M^2/\lambda$ and taking the scaling limit $\lambda \to 0$.

- **Beyond the Carroll Gravity**: The next order ultra-relativistic model appears at the $\mathscr{O}(\lambda^3)$ in the Lie algebra expansion. Thus, we shall consider two copies of the Einstein- Hilbert action. We keep the same form as the Lie algebra expansion for the first set of fields:

$$\begin{aligned} T =& \lambda \tau + \lambda^3 m, \qquad E_1^a = e^a + \lambda^2 t^a, \\ \Omega_1^a =& \lambda \omega^a + \lambda^3 b^a, \quad \Omega_1^{ab} = \omega^{ab} + \lambda^2 s^{ab}. \end{aligned} \tag{6.33}$$

  For the second copy, we use the free parameter α_2 as

$$\begin{aligned} T_2 =& \lambda \tau + \lambda^3 \alpha_2^3 m, \qquad E_2^a = \alpha_2 e^a + \lambda^2 \alpha_2^2 t^a, \\ \Omega_2^{ab} =& \omega^{ab} + \lambda^2 \alpha_2^2 s^{ab}, \qquad \Omega_2^a = \lambda \alpha_2 \omega^a + \lambda^3 \alpha_2^3 b^a \end{aligned} \tag{6.34}$$

  The combination of these two Lagrangian with mass parameters $M_{1,2}$ give the result

$$\mathscr{L} = \lambda M^2 \left(1 + \frac{a_2}{2}\right) \mathscr{L}_1 + \lambda^3 M^2 \left(1 + \frac{a_2^3}{2}\right) \mathscr{L}_3 + \mathscr{O}\left(\lambda^5\right) \tag{6.35}$$

where we made the choice as $M_1^2 = 2M_2^2 = M^2$ for simplicity and $\mathscr{L}_1$ refers to the **(6.32)**. The term $\mathscr{L}_3$ is called *Beyond the Carroll Gravity*

$$\begin{aligned}\mathscr{L}_3 =& -2\varepsilon_{abc}R^{ab}(s)\wedge e^c\wedge\tau - 2\varepsilon_{abc}R^{ab}(\omega)\wedge t^c\wedge\tau - 2\varepsilon_{abc}R^{ab}(\omega)\wedge e^c\wedge m\\ &+2\varepsilon_{abc}R^{a}(b)\wedge e^b\wedge e^c + 4\varepsilon_{abc}R^{a}(\omega)\wedge e^b\wedge t^c\end{aligned} \tag{6.36}$$

We can fix α_2 as $\alpha_2 = -2$ which cancels the coefficient of the Carroll gravity and fixed the coefficient of $\mathscr{L}_3$ to be -3. Then, we scale the mass parameter as

$$M^2 = \frac{1}{3\lambda^3}m^2 \tag{6.37}$$

with the scaling limit $\lambda \to 0$. After these steps, we recover the beyond the Carrol gravity model up to an overall minus sign.

We end this section with a brief discussion on how to obtain the ultra-relativistic higher-order algebras by contracting the multiple copies of the Poincaré algebra. The ultra-relativistic algebras that admit an invariant action formulation arises from the Lie algebra expansion for $N_0 = N_1$. This implies that for an algebra of order (N_1, N_1), one has the same number of generators as (N_1+1)-copies of Poincaré algebra. Furthermore, the expansion does not change the structure constant of the smaller core algebra, indicating that we can combine the generators of multiple copies of the Poincaré algebra in a linearly independent way to exhibit the generators of the larger ultra-relativistic algebras. Based on the α-expansion of the relativistic fields, we introduce the following expressions for the ultra-relativistic generators in terms of the generators of the Poincaré algebra

$$\begin{aligned} J_{ab}^{(2n)} =& \lambda^{2n}\bigoplus_{i=1}^{N_0}\alpha_i^{2n}J_{ab}^i, \qquad H^{(2n+1)} = \lambda^{2n+1}\bigoplus_{i=1}^{N_1}\alpha_i^{2n+1}H^i,\\ P_a^{(2n)} =& \lambda^{2n}\bigoplus_{i=1}^{N_0}\alpha_i^{2n}P_a^i, \qquad G_a^{(2n+1)} = \lambda^{2n+1}\bigoplus_{i=1}^{N_1}\alpha_i^{2n+1}G_a^i \end{aligned} \tag{6.38}$$

where $\alpha_i \neq 1$ and $\alpha_i \neq \alpha_j$ for $i \neq j$. Notice that in this case, these α parameters are free and not necessary to be fixed. This direct sum trivially satisfies the algebra **(6.11)**. These relations can be inverted to express the relativistic generators in terms of the ultra-relativistic ones. As a reminder, the truncation condition is $N_0 = N_1$.

7. CONCLUSION

In this thesis, we studied the general aspects of ultra-relativistic gravity as a gauge theory, and we presented an action principle in the first-order formalism for Carroll gravity. Throughout this thesis, we also showed that this theory is invariant under the symmetry transformations obtained from the Carroll group and algebra.

We first studied the General Relativity as a gauge theory. This showed us that to obtain a gravity theory as a gauge theory. Then, we examined the specific massive and bi-metric gravity models in order to understand the relation between these structures and the usage of bi-metric gravity effectively. Also, we showed that some of these theories are able to be represented as a specific choice or limit of a more general theory. This methodology showed us the effectiveness of bi-metric gravity theories. Via gathering all these pieces of information, we constructed the Carroll Algebra and Beyond the Carroll Algebra corresponding to Carroll Gravity and Beyond the Carroll Gravity, respectively. Without a doubt, Lie Algebra expansion is a powerful tool to obtain gravity theories when it is combined with the scaling limit.

As a result, we showed that the ultra-relativistic theories (in any desired order) can be obtained from a contraction of a multi-metric theory and we shortly discussed the possibility of the existence of another expansion with more freedom for ultra-relativistic theories.

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APPENDICES

APPENDIX A.1 : Identities for metric determinants and its derivatives
APPENDIX A.2 : Variation of metric and its determinant
APPENDIX B.1 : Lovelock Theories of Gravity

APPENDIX A.1: Identities for metric determinants and its derivatives

Let A be any invertible matrix, then generally

$$\delta(\ln(|\det A|)) = \mathrm{Tr}(A^{-1} \times \delta A) \tag{A.1.1}$$

where δ denotes any variation. Using this relation, we can calculate that

$$\begin{aligned} \partial_\mu \ln(g) &= \mathrm{Tr}(g^{\rho\sigma}\partial_\mu g_{\alpha\beta}) \\ \frac{1}{g}\partial_\mu g &= g^{\rho\sigma}\partial_\mu g_{\rho\sigma}. \end{aligned} \tag{A.1.2}$$

Thus we have

$$\begin{aligned} \partial_\mu \ln\left(\sqrt{-g}\right) &= \frac{1}{\sqrt{-g}}\partial_\mu\sqrt{-g} = \frac{1}{2}\frac{1}{-g}\partial_\mu(-g) \\ &= \frac{1}{2}\frac{1}{g}\partial_\mu g = \frac{1}{2}g^{\rho\sigma}\partial_\mu g_{\rho\sigma}. \end{aligned} \tag{A.1.3}$$

APPENDIX A.2: Variation of metric and its determinant

Let us calculate the variation of the metric using the fact that $g_{\mu\rho}g^{\rho\nu} = \delta^\nu_\mu$ as

$$\delta\left(g_{\mu\rho}g^{\rho\nu}\right) = g_{\mu\rho}\,\delta(g^{\rho\nu}) + \delta\left(g_{\mu\rho}\right)g^{\rho\nu} = 0. \tag{A.2.1}$$

Thus we have

$$\delta\left(g_{\mu\rho}\right)g^{\rho\sigma} = -g_{\mu\rho}\,\delta(g^{\rho\sigma}), \tag{A.2.2}$$

and contracting with $g_{\sigma\nu}$, we obtain

$$\delta\left(g_{\mu\nu}\right) = -g_{\mu\rho}g_{\sigma\nu}\,\delta(g^{\rho\sigma}). \tag{A.2.3}$$

We can also calculate the variation of the determinant of the metric using (A.1.1) as

$$\delta\left(\sqrt{-g}\right) = \frac{1}{2}\sqrt{-g}g^{\mu\nu}\,\delta\left(g_{\mu\nu}\right). \tag{A.2.4}$$

Substituting (A.2.3),

$$\begin{aligned} \delta\left(\sqrt{-g}\right) &= -\frac{1}{2}\sqrt{-g}g^{\mu\nu}g_{\mu\rho}g_{\sigma\nu}\,\delta(g^{\rho\sigma}) \\ &= -\frac{1}{2}\sqrt{-g}g_{\rho\sigma}\,\delta(g^{\rho\sigma}). \end{aligned} \tag{A.2.5}$$

APPENDIX B.1: Lovelock Theories of Gravity

The N-th order Lovelock invariants are given by

$$\mathscr{L}^{(N)} \equiv \delta^{a_1 b_1 \cdots a_N b_N}_{[c_1 d_1 \cdots c_N d_N]} R_{a_1 b_1}{}^{c_1 d_1} \cdots R_{a_N b_N}{}^{c_N d_N} \tag{B.1.1}$$

Similarly, the N-th order Lovelock tensor is given by

$$\left(G^{(N)}\right)^{f}_{e} \equiv -\frac{1}{2}\delta^{f a_1 b_1 \cdots a_N b_N}_{[e c_1 d_1 \cdots c_N d_N]} R_{a_1 b_1}{}^{c_1 d_1} \cdots R_{a_N b_N}{}^{c_N d_N} \tag{B.1.2}$$

The Lovelock tensor is nothing but the equation of motion following from a lagrangian given by the corresponding Lovelock invariant, which guarantees that its a covariantly conserved tensor i.e

$$\nabla_a \left(G^{(N)}\right)^{ab} = 0 \tag{B.1.3}$$

The first few of the **(B.1.1)** & **(B.1.2)**:

$$\begin{aligned} \mathscr{L}^{(0)} =& 1, \quad \mathscr{L}^{(1)} = R, \quad \mathscr{L}^{(2)} = R_{abcd}R^{abcd} - 4R_{ab}R^{ab} + R^2 \\ G^{(0)}_{ab} =& -\frac{g_{ab}}{2}, \quad G^{(1)}_{ab} = R_{ab} - \frac{g_{ab}}{2}R \\ G^{(2)}_{ab} =& 2R_{ab}R - 4R_{ac}R^{c}_{b} - 4R_{acbd}R^{cd} + 2R_{acde}R^{cde}_{b} - \frac{g_{ab}}{2}\mathscr{L}^{(2)} \end{aligned} \tag{B.1.4}$$

Keep in mind that

$$\begin{aligned} G^{(N)}_{ab} &= \frac{1}{2}\left(\frac{\partial \mathscr{L}^{(N)}}{\partial g^{ab}} - g_{ab}\mathscr{L}^{(N)}\right) \\ \text{Therefore,} \quad g^{ab}G^{(N)}_{ab} &= \left(\frac{2N-d}{2}\right)\mathscr{L}^{(N)} \end{aligned} \tag{B.1.5}$$

Here while taking the partial derivative with respect to the metric, we kept the Riemann tensor R_{abcd} fixed. Also **(B.1.5)** tells us that whenever $d = 2K$, $G^{(K)}_{ab}$ becomes traceless.

CURRICULUM VITAE

Name Surname : Ertuğrul Ekiz

EDUCATION :

- **B.Sc.** : 2017-2021, Istanbul Technical University, Faculty of Science and Letters, Department of Physics Engineering

PROFESSIONAL EXPERIENCE AND REWARDS:

- 2022 Research Assistant at Istanbul Technical University

PUBLICATIONS, PRESENTATIONS AND PATENTS ON THE THESIS:

- **Ekiz, E., Kasikci, O., Ozkan, M., Senisik, C. B., Zorba, U.** 2022. Non-relativistic and ultra-relativistic scaling limits of multimetric gravity, *Journal of High Energy Physics,* 10, 151.