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DIFFERENTIAL TRANSFORM METHOD
AND ITS APPLICATIONS

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APPROVAL PAGE

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DIFFERENTIAL TRANSFORM METHOD AND ITS APPLICATIONS

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ABSTRACT

The differential transform method is one of the useful and powerful methods which can be easily applied to boundary valuable and initial condition problems,many linear and nonlinear problems.In our study,some numerical examples are given to demonstrate the difference of the Differential transform method to other usual techniques.Some algebraic computations were solved by using Maple computer algebra systems.

Keywords:Differential transform method,boundary valuable and initial condition problems,Maple

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ÖZ

Diferansiyel dönüşüm yöntemi,sınır değer ve başlangıç değer şartı problemlerine,birçok doğrusal ve doğrusal olmayan problemlere kolaylıkla uygulanan kullanışlı ve güçlü yöntemlerden biridir.Çalışmalarımızda bazı sayısal örnekler diferansiyel dönüşüm yönteminin diğer sıradan yöntemlerden farkını göstermek amacıyla verildi.Bazı sayısal hesaplamalar Maple bilgisayar cebir sistemi kullanılarak çözüldü.

Anahtar Kelimeler: Diferansiyel dönüşüm yöntemi,Sınır değer ve başlangıç değer problemi,Maple.

To my parents

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CHAPTER I

LITERATURE SURVEY ON DIFFERENTIAL TRANSFORM METHOD

2.1 INTRODUCTION

The Differential Transform Method (DTM) has been initially introduced by Zhou [1] to solve linear and nonlinear initial value problems in electric circuit analysis. This method develops a solution in the form of a polynomial that is based on Taylor series, but it is totally different from the traditional higher order Taylor series method. The DTM is a kind of an alternative solution process for obtaining Taylor series solution of the differential equations. According to the method, the transform of a function at k integer value is represented by k^{th} order derivative of the function at a specific point, such as x_0. The differential transform method is mainly used for solution of differential equations, ordinary or partial, and their systems. Also, the solution of integral or integro-differential equations along with their systems can be solved by using differential transform method. There are many problems solved by differential transform method. As an example, the solution related to the eigenvalue problems was examined by Hassan [2] using one-dimensional differential transform method. Also, many researchers were worked on differential transform to solve differential equations [3-4], partial differential equations of physics [5-6], and the systems of either ordinary or partial differential equations [7]. It is introduced the application of concept of the differential transform method to the fixed grid size for using approximate solution of linear and nonlinear initial value problems by Jang et. al. [3]. In some areas of application such mechanics of solid and vibration, the differential transform method is used to solve vibration on beam by Catal [5] whose work it is examined the vibration of a beam on an elastic solid.

Examples are given for different types of boundary conditions and results are compared with some closeform solutions. The highly motivated problems of heat conductions were examined by using differential transform method [6]. Since the idea can be applied for the problem that has more than one variable, then the system of differential equations can be solved by using differential transform method [7-9]. Specifically, linear and nonlinear systems of partial differential equations have been widely investigated and the solutions of sample problems have been compared with the close form solutions by Ayaz [7]. Some problems which contain the derivative of unknown function, such as integro-differential equations or finite difference equations, can be solved by the method differential transform method [10]. Some modification on the differential transform method has been done by Zhou *et.al.* DTM is generalized to solve differential-difference equations and is combined with a Pade technique in order to extend the convergence rate of presented series solutions. Generalized Differential Transform Method (GDTM) is also applied to discrete KdV and mKdV equations and acquired successful results [11].The solution of integral equations and integro-differential equations by differential transform method have examined by Tari and Shahmorad [12]. They used DTM to solve two dimensional linear and non-linear Volterra integro-differential equations of the second kind. The results of their solutions were compared with the closeform solutions. The solution nonlinear heat transfer problem for specific heat coefficient is examined by Yaghoobi and Torabi [13] in which the comparisons of the results obtained by DTM are done by other semi-analytic method such as variational iteration method, homotopy perturbation method, perturbation method and the exact solutions, if it is possible.In the second chapter of this study, it is examined all theorems and their proofs related to the operations and fundamental functions used in differential transform method. Some of these theorems are proved step by step to explain the method to the reader. Some basic fundamental transforms are shown in Appendix as a table. In the third chapter, some examples related to the theorems are given and the results obtained by the method differential transform compared with the close form solutions. Finally, in the last chapter, some conclusions related to the thesis are given.

CHAPTER II

ONE DIMENSIONAL DIFFERENTIAL TRANSFORM METHOD

In this chapter, it will be analyzed some theorems related to the differential transform of additions, multiplications and some fundamental functions, particularly, single-valued functions, and their operations. In the research of differential transform method, there are different types of approaches to the properties of the transform related to the structure of the problem. In the following part of the thesis, it will be shown the proofs of the theorems mostly used in the solution of the partial and ordinary differential equations and integral equations. Also, the differential transform of some fundamental functions are shown.

Definition

The differential transform of function $f(x)$ at $x = x_0$ is defined as

$$\mathcal{D}_T\{f(x)\} = F(k) = \frac{1}{k!}\left[\frac{d^k f(x)}{dx^k}\right]_{x=x_0} \tag{2.1}$$

where k is an integer and $F(k)$ is the transformed function. However, the inverse differential transform of $F(k)$ can be represented as

$$\mathcal{D}_T^{-1}\{F(k)\} = f(x) = \sum_{k=0}^{\infty} x^k F(k) = \sum_{k=0}^{\infty} \frac{x^k}{k!}\left[\frac{d^k f(x)}{dx^k}\right]_{x=x_0} \tag{2.2}$$

which implies that the concept of differential transform is derived from Taylor series expansion, but the method does not evaluate the derivatives symbolically.

Theorem 2.1

Assuming that c is a constant and $g(x)$ is an analytic function then the differential transform of

$f(x)=cg(x)$ can be shown as $F(k)=cG(k)$.

Proof 2.1

Since c is a constant and $g(x)$ is an analytic function then

$$F(k)=\frac{1}{k!}\left[\frac{d^k f(x)}{dx^k}\right]_{x=x_0}$$

We put $f(x)=cg(x)$ into the definition of transform then

$$F(k)=\frac{1}{k!}\left[\frac{d^k cg(x)}{dx^k}\right]_{x=x_0}$$

and using the linearity properties of derivative operator it can be easily written as

$$F(k)=\frac{1}{k!}\left[\frac{d^k cg(x)}{dx^k}\right]_{x=0}=c\frac{1}{k!}\left[\frac{d^k g(x)}{dx^k}\right]_{x=x_0}=cG(k).$$

Theorem 2.2

Assuming that $g(x)$ and $h(x)$ are analytic functions then the differential transform of

$$f(x)=g(x)\pm h(x)$$

can be shown as

$$F(k)=G(k)\pm H(k).$$

Proof 2.2

Directly, substituting $g(x)\pm h(x)$ into the definition of differential transform, it can be obtained as

$$F(k)=\frac{1}{k!}\left[\frac{d^k g(x)\pm h(x)}{dx^k}\right]_{x=x_0}=\frac{1}{k!}\left[\frac{d^k g(x)}{dx^k}\right]_{x=x_0}\pm\frac{1}{k!}\left[\frac{d^k h(x)}{dx^k}\right]_{x=x_0}.$$

$$=G(k)\pm H(k)$$

Theorem 2.3

Assuming that $f(x)$ and $g(x)$ are analytic functions then the differential transform of the derivative,

$$f(x)=\frac{d}{dx}g(x),$$

can be represented as

$$F(k)=(k+1)G(k+1).$$

Proof 2.3

Let us apply the definition of differential transform to the function $\frac{d}{dx}g(x)$ as follows:

$$F(k)=\frac{1}{k!}\left[\frac{d^k}{dx^k}\frac{dg(x)}{dx}\right]_{x=x_0}=\frac{(k+1)}{(k+1)!}\left[\frac{d^{k+1}g(x)}{dx^{k+1}}\right]_{x=x_0}$$

$$=(k+1)\underbrace{\frac{1}{(k+1)!}\left[\frac{d^{k+1}g(x)}{dx^{k+1}}\right]_{x=x_0}}_{G(k+1)}$$

$$=(k+1)G(k+1).$$

Theorem 2.4

Assuming that $f(x)$ and $g(x)$ are analytic functions then the differential transform of n^{th} order derivative,

$$f(x)=\frac{d^n g(x)}{dx^n},$$

can be represented as

$$F(k)=\frac{(k+n)!}{k!}G(k+n).$$

Proof 2.4

Starting with the definition of differential transform method

$$F(k)=\frac{1}{k!}\left[\frac{d^k}{dx^k}\frac{d^n g(x)}{dx^n}\right]_{x=x_0}=\frac{1}{k!}\left[\frac{d^{k+n} g(x)}{dx^{k+n}}\right]_{x=x_0}$$

and using the identity

$$\frac{(k+1)(k+2)\ldots(k+n)}{(k+n)!}=\frac{1}{k!},$$

it can be written that

$$F(k)=\frac{(k+1)(k+2)\cdots(k+n)}{(k+n)!}\left[\frac{d^{k+n} g(x)}{dx^{k+n}}\right]_{x=x_0},$$

$$=(k+1)(k+2)\cdots(k+n)\frac{1}{(k+n)!}\left[\frac{d^{k+n} g(x)}{dx^{k+n}}\right]_{x=x_0},$$

$$=\frac{(k+n)!}{k!}\underbrace{\frac{1}{(k+n)!}\left[\frac{d^{k+n} g(x)}{dx^{k+n}}\right]_{x=x_0}}_{G(k+n)},$$

$$= \frac{(k+n)!}{k!} G(k+n)$$

Theorem 2.5

Assuming that $f(x)$, $g(x)$ and $h(x)$ are analytic functions then the differential transform of the product of two functions,

$$f(x) = g(x)h(x),$$

can be represented as

$$F(k) = \sum_{r=0}^{k} G(r)H(k-r).$$

Proof 2.5

Starting with the k^{th} order derivative of the product of two functions like

$$\frac{d^k\left(g(x)h(x)\right)}{dx^k} = h(x)\frac{d^k}{dx^k}g(x) + \binom{k}{1}\frac{d}{dx}h(x)\frac{d^{k-1}}{dx^{k-1}}g(x) + \binom{k}{2}\frac{d^2}{dx^2}h(x)\frac{d^{k-2}}{dx^{k-2}}g(x) + \ldots$$

$$\ldots + \binom{k}{k-1}\frac{d^{k-1}}{dx^{k-1}}h(x)\frac{d}{dx}g(x) + \left(\frac{d^k}{dx^k}h(x)\right)g(x), \tag{2.3}$$

the differential transform of the k^{th} order derivative of the product of the two functions can be obtained as follows:

$$F(k) = \frac{1}{k!}\left[\frac{d^k\left(g(x)h(x)\right)}{dx^k}\right]_{x=x_0} = \frac{1}{k!}\left\{\left[h(x)\frac{d^k}{dx^k}g(x) + \binom{k}{1}\frac{d}{dx}h(x)\frac{d^{k-1}}{dx^{k-1}}g(x)\right]_{x=x_0}\right.$$

$$\left. + \left[\binom{k}{2}\frac{d^2}{dx^2}h(x)\frac{d^{k-2}}{dx^{k-2}}g(x) + \ldots + \binom{k}{k-1}\frac{d^{k-1}}{dx^{k-1}}h(x)\frac{d}{dx}g(x) + g(x)\frac{d^k}{dx^k}h(x)\right]_{x=x_0}\right\}.$$

The last expression can be rewritten from the last term again as

$$= \frac{1}{k!}\left[h(x)\frac{d^k g(x)}{dx^k}\right]_{x=x_0} + \frac{1}{(k-1)!}\left[\frac{d^{k-1}g(x)}{dx^{k-1}}\frac{dh(x)}{dx}\right]_{x=x_0}$$

$$+ \frac{1}{2(k-2)!}\left[\frac{d^{k-2}g(x)}{dx^{k-2}}\frac{d^2h(x)}{dx^2}\right]_{x=x_0}$$

$$+ \cdots + \frac{1}{(k-1)!}\left[\frac{dg(x)}{dx}\frac{d^{k-1}h(x)}{dx^{k-1}}\right]_{x=x_0} + \frac{1}{k!}\left[\frac{d^k h(x)}{dx^k}g(x)\right]_{x=x_0}.$$

As a result, it can be written that,

$$F(k) = G(k)H(0) + G(k-1)H(1) + G(k-2)H(2),$$

$$+ \cdots + G(1)H(k-1) + G(0)H(k)$$

$$F(k) = \sum_{r=0}^{k} G(r)H(k-r).$$

Theorem 2.6

Assuming that $f(x)$ and $g_1(x)$, $g_2(x)$, $g_3(x)$ are analytic functions then the differential transform of the product of three functions,

$$f(x) = g_1(x)g_2(x)g_3(x),$$

can be shown as

$$F(k) = \sum_{k_2=0}^{k_3} \sum_{k_1=0}^{k_2} G_1(k_1)G_2(k_2-k_1)G_3(k-k_2).$$

Proof 2.6

$$F(0) = \frac{1}{0!}\left[g_1(x)g_2(x)g_3(x)\right]_{x=x_0} = G_1(0)G_2(0)G_3(0),$$

$$F(1)=\frac{1}{1!}\frac{d}{dx}\left[g_1(x)g_2(x)g_3(x)\right]_{x=x_0},$$

$$=\left[g_1'(x)g_2(x)g_3(x)\right]_{x=x_0}+\left[g_1(x)g_2'(x)g_3(x)\right]_{x=x_0},$$

$$+\left[g_1(x)g_2(x)g_3'(x)\right]_{x=x_0}$$

$$F(2)=\frac{1}{2!}\frac{d^2}{dx^2}\left[g_1(x)g_2(x)g_3(x)\right]_{x=x_0},$$

$$=\frac{1}{2!}\left[g_1''(x)g_2(x)g_3(x)+g_1'(x)g_2'(x)g_3(x)+g_1'(x)g_2(x)g_3'(x)\right]_{x=x_0},$$

$$+\frac{1}{2!}\left[g_1'(x)g_2'(x)g_3(x)+g_1(x)g_2''(x)g_3(x)+g_1(x)g_2'(x)g_3'(x)\right]_{x=x_0},$$

$$+\frac{1}{2!}\left[g_1'(x)g_2(x)g_3'(x)+g_1(x)g_2'(x)g_3'(x)+g_1(x)g_2(x)g_3''(x)\right]_{x=x_0},$$

$$=G_1(2)G_2(0)G_3(0)+G_1(1)G_2(1)G_3(0)+G_1(1)G_2(0)G_3(1),$$

$$+G_1(1)G_2(1)G_3(0)+G_1(0)G_2(2)G_3(0)+G_1(0)G_2(1)G_3(1),$$

$$+G_1(1)G_2(0)G_3(1)+G_1(0)G_2(1)G_3(1)+G_1(0)G_2(0)G_3(2).$$

Finally, it can be obtained that

$$F(k)=\sum_{k_2=0}^{k_3}\sum_{k_1=0}^{k_2}G_1(k_1)G_2(k_2-k_1)G_3(k-k_2).$$

Theorem 2.7

Assuming that $f(x)$ and $g_1(x)$, $g_2(x)$, …, $g_n(x)$ are analytic functions then the differential transform of the product of *n* analytic functions,

$$f(x)=g_1(x)g_2(x)\cdots g_{n-1}(x)g_n(x),$$

can be shown as

$$F(k)=\sum_{k_{n-1}=0}^{k}\sum_{k_{n-2}=0}^{k_{n-1}}\cdots\sum_{k_2=0}^{k_3}\sum_{k_1=0}^{k_2}G_1(k_1)\times G_2(k_2-k_1)\times\cdots\times G_{n-1}(k_{n-1}-k_{n-2})\times G_n(k-k_{n-1})$$

Proof 2.7

$$F(0)=\frac{1}{0!}\left[g_1(x)g_2(x)\cdots g_{n-1}(x)g_n(x)\right]_{x=x_0}=G_1(0)G_2(0)\cdots G_{n-1}(0)G_n(0),$$

$$F(1)=\frac{1}{1!}\frac{d}{dx}\left[g_1(x)g_2(x)\cdots g_{n-1}(x)g_n(x)\right]_{x=x_0},$$

$$=\left[g_1'(x)g_2(x)\cdots g_{n-1}(x)g_n(x)\right]_{x=x_0}+\left[g_1(x)g_2'(x)\cdots g_{n-1}(x)g_n(x)\right]_{x=x_0},$$

$$+\cdots+\left[g_1(x)g_2(x)\cdots g_{n-1}'(x)g_n(x)\right]_{x=x_0}+\left[g_1(x)g_2(x)\cdots g_{n-1}(x)g_n'(x)\right]_{x=x_0},$$

$$F(1)=G_1(1)G_2(0)\cdots G_{n-1}(0)G_n(0)+G_1(0)G_2(1)\cdots G_{n-1}(0)G_n(0)+\cdots$$

$$+G_1(0)G_2(0)\cdots G_{n-1}(1)G_n(0)+G_1(0)G_2(0)\cdots G_{n-1}(0)G_n(1),$$

$$F(2)=G_1(1)G_2(1)G_3(0)\cdots G_n(0)+G_1(1)G_2(0)G_3(1)\cdots G_n(0)+\cdots$$

$$+G_1(1)G_2(0)G_3(0)\cdots G_n(1)$$

$$+G_1(0)G_2(1)G_3(1)\cdots G_n(0)+G_1(0)G_2(1)G_3(0)G_4(1)\cdots G_n(0)$$

$$+G_1(0)G_2(1)G_3(0)G_4(0)\cdots G_n(1)$$

$$+\cdots+G_1(0)G_2(0)G_3(0)\cdots G_n(2).$$

In general, it can be represented as

$$F(k)=\sum_{k_{n-1}=0}^{k}\sum_{k_{n-2}=0}^{k_{n-1}}\cdots\sum_{k_2=0}^{k_3}\sum_{k_1=0}^{k_2}G_1(k_1)\times G_2(k_2-k_1)\times\cdots\times G_{n-1}(k_{n-1}-k_{n-2})\times G_n(k-k_{n-1})$$

Theorem 2.8

Let $f(x)$ be an analytic function then the differential transform of power function,

$$f(x)=x^m,\quad m\in\mathbb{R},$$

can be shown as

$$F(k)=\delta(k-m)=\begin{cases}1, & k=m\\ 0, & \text{otherwise}\end{cases}.$$

Proof 2.8

First of all, in terms of the sign of m, let us examine the derivative of $f(x)$ as follows:

Case 1

If $(k<m)$,

$$\left[\frac{d^k}{dx^k}x^m\right]_{x=x_0}=\left[m(m-1)\ldots(m-k+1)x^{m-k}\right]_{x=x_0}=0$$

then

$$F(k)=\left[\frac{d^k}{dx^k}x^m\right]_{x=x_0}=\frac{0}{k!}=0.$$

Case 2

If $(k=m)$,

$$\left[\frac{d^m}{dx^m}x^m\right]_{x=x_0}=\left[m(m-1)\ldots(m-m+1)x^{m-m}\right]_{x=x_0}=m!$$

then

$$F(m)=\frac{1}{m!}\left[\frac{d^m}{dx^m}x^m\right]_{x=x_0}=\frac{1}{m!}m!=1.$$

Case 3

If $(k>m)$,

$$\left[\frac{d^k}{dx^k}x^m\right]_{x=x_0}=\left[m(m-1)\ldots(m-m+1)0\right]_{x=x_0}=0$$

then

$$F(k)=\frac{1}{k!}\left[\frac{d^k}{dx^k}x^m\right]_{x=x_0}=\frac{0}{k!}=0.$$

Finally, the differential transform of a polynomial type power function has a value only in the case of $k=m,\ m\in\mathbb{R}$, such as

$$F(k)=\delta(k-m)=\begin{cases}1, & k=m\\ 0, & \text{otherwise}\end{cases}.$$

Table 1.1 Some examples related to differential transform of power functions.

$f(x)$	$F(k)$
$f(x)=27$	$F(k)=27\delta(k) \Rightarrow F(0)=27$
$f(x)=17x$	$F(k)=17\delta(k-1) \Rightarrow F(1)=17$
$f(x)=6x^2$	$F(k)=6\delta(k-2) \Rightarrow F(2)=6$
$f(x)=3x^5$	$F(k)=3\delta(k-5) \Rightarrow F(5)=3$
$f(x)=2x^8+13x^{10}$	$F(k)=2\delta(k-8)+13\delta(k-10)$

Theorem 2.9

Assuming that $f(x)$ is analytic function and $\lambda \in \mathbb{R}$ then the differential transform of an exponential function,

$$f(x)=a^{\lambda x},$$

can be shown as

$$F(k)=\frac{\lambda^k(\ln a)^k}{k!}.$$

Proof 2.9

From the definition of differential transform, it can be written as

$$F(k)=\frac{1}{k!}\left[\frac{d^k f(x)}{dx^k}\right]_{x=0}=\frac{1}{k!}\left[\frac{d^k a^{\lambda x}}{dx^k}\right]_{x=0}$$

then for $k=0$,

$$F(0)=\frac{1}{0!}\left[a^{\lambda x}\right]_{x=0}=\frac{1}{0!}=1,$$

for $k=1$,

$$F(1)=\frac{1}{1!}\left[\frac{d}{dx}a^{\lambda x}\right]_{x=0}=\frac{1}{1!}\left[\lambda \ln a a^{\lambda x}\right]_{x=0}=\frac{1}{1!}\lambda \ln a\,,$$

for $k=2$,

$$F(2)=\frac{1}{2!}\left[\frac{d^2}{dx^2}a^{\lambda x}\right]_{x=0}=\frac{1}{2!}\left[\lambda^2(\ln a)^2 a^{\lambda x}\right]_{x=0}=\frac{1}{2!}\lambda^2(\ln a)^2,$$

for $k=3$,

$$F(3)=\frac{1}{3!}\left[\frac{d^3}{dx^3}a^{\lambda x}\right]_{x=0}=\frac{1}{3!}\left[\lambda^3(\ln a)^3 a^{\lambda x}\right]_{x=0}=\frac{1}{3!}\lambda^3(\ln a)^3,$$

$$\vdots$$

Finally, it can be summarize the transformed function as

$$F(k)=\frac{\lambda^k(\ln a)^k}{k!}=\frac{1}{k!}\lambda^k(\ln a)^k.$$

Theorem 2.10

Assuming that $f(x)$ is analytic function and $\lambda\in\mathbb{R}$ then the differential transform of a natural exponential function,

$$f(x)=e^{\lambda x},$$

can be shown as

$$F(k)=\frac{\lambda^k}{k!}.$$

Proof 2.10

If it is substituted e instead of a in Theorem 2.9, it can be easily obtained the differential transform of a natural exponential function as follows:

$$F(k)=\frac{1}{k!}\left[\lambda^k(\ln a)^k\right]_{x=0}=\frac{\lambda^k(\ln e)^k}{k!}=\frac{\lambda^k}{k!}.$$

Theorem 2.11

Assuming that $a,b\in\mathbb{R}$ then the differential transform of a trigonometric function $f(x)=\sin(ax+b)$ can be given as

$$F(k)=\mathcal{D}_T\{f(x)\}=\frac{a^k}{k!}\sin\left(\frac{\pi}{2}k+b\right).$$

Proof 2.11

By substituting the function into the definition of the differential transformation, it can be obtained as

$$F(k)=\frac{1}{k!}\left[\frac{d^k}{dx^k}\sin(ax+b)\right]_{x=0},$$

For $k=0$,

$$F(0)=\frac{1}{0!}\left[\sin(ax+b)\right]_{x=0}=\frac{1}{0!}\sin b,$$

For $k=1$,

$$F(1)=\frac{1}{1!}\left[\frac{d}{dx}\sin(ax+b)\right]_{x=0}=\frac{1}{1!}\left[a\cos(ax+b)\right]_{x=0}=a\cos(b)=\frac{1}{1!}a\sin\left(\frac{\pi}{2}+b\right),$$

For $k=2$,

$$F(2)=\frac{1}{2!}\left[\frac{d^2}{dx^2}\sin(ax+b)\right]_{x=0}=\frac{1}{2!}\left[-a^2\sin(ax+b)\right]_{x=0},$$

$$=-\frac{1}{2!}a^2\sin(b)=\frac{1}{2!}a^2\sin\left(\frac{2\pi}{2}+b\right),$$

For $k=3$,

$$F(3)=\frac{1}{3!}\left[\frac{d^3}{dx^3}\sin(ax+b)\right]_{x=0}=\frac{1}{3!}\left[-a^3\cos(ax+b)\right]_{x=0},$$

$$=-\frac{1}{3!}a^3\cos(b)=\frac{1}{3!}a^3\sin\left(\frac{3\pi}{2}+b\right).$$

Finally, it will be obtained the following result:

$$F(k)=\frac{a^k}{k!}\sin\left(\frac{\pi}{2}k+b\right).$$

Theorem 2.12

Assuming that $a,b\in\mathbb{R}$ then the differential transform of a trigonometric function

$$f(x)=\cos(ax+b)$$

can be given as

$$F(k)=\mathcal{D}_T\left\{f(x)\right\}=\frac{a^k}{k!}\cos\left(\frac{\pi}{2}k+b\right).$$

Proof 2.12

By substituting the function into the definition of the differential transformation, it can be obtained as

$$F(k)=\frac{1}{k!}\left[\frac{d^k}{dx^k}\cos(ax+b)\right]_{x=0},$$

For $k=0$,

$$F(0)=\frac{1}{0!}\left[\cos(ax+b)\right]_{x=0}=\frac{1}{0!}\cos b,$$

For $k=1$,

$$F(1)=\frac{1}{1!}\left[\frac{d}{dx}\cos(ax+b)\right]_{x=0}=\frac{1}{1!}\left[-a\sin(ax+b)\right]_{x=0},$$

$$=-\frac{1}{1!}a\sin b=\frac{1}{1!}a\cos\left(\frac{\pi}{2}+b\right),$$

For $k=2$,

$$F(2)=\frac{1}{2!}\left[\frac{d^2}{dx^2}\cos(ax+b)\right]_{x=0}=\frac{1}{2!}\left[-a^2\cos(ax+b)\right]_{x=0},$$

$$=-\frac{1}{2!}a^2\cos b=\frac{1}{2!}a^2\cos\left(\frac{2\pi}{2}+b\right),$$

For $k=3$,

$$F(3)=\frac{1}{3!}\left[\frac{d^3}{dx^3}\cos(ax+b)\right]_{x=0}=\frac{1}{3!}\left[a^3\sin(ax+b)\right]_{x=0},$$

$$=\frac{1}{3!}a^3\sin b=\frac{1}{3!}a^3\cos\left(\frac{3\pi}{2}+\mathrm{b}\right).$$

Finally, it will be obtained the following result:

$$F(k)=\frac{a^k}{k!}\cos\left(\frac{\pi}{2}k+b\right).$$

Theorem 2.13

$$f(x)=\sinh(\lambda x) \quad \Rightarrow \quad \mathcal{D}_T\left\{\sinh(\lambda x)\right\}=\begin{cases} 0, & \text{for even } k \\ \dfrac{\lambda^k}{k!}, & \text{for odd } k \end{cases}$$

Proof 2.13

The function can be written as

$$f(x)=\sinh(\lambda x)=\frac{e^{\lambda x}-e^{-\lambda x}}{2}=\frac{e^{\lambda x}}{2}-\frac{e^{-\lambda x}}{2}$$

From the linearity of the differential transformation function

$$\mathcal{D}_T\left\{\sinh(\lambda x)\right\}=\frac{1}{2}\left(\frac{\lambda^k}{k!}-\frac{(-\lambda)^k}{k!}\right)$$

If k is odd;

$$\mathcal{D}_T\left\{\sinh(\lambda x)\right\}=\frac{1}{2}\left(\frac{\lambda^k}{k!}-\frac{(-\lambda)^k}{k!}\right)=\frac{\lambda^k}{k!}$$

If k is even number;

$$\mathcal{D}_T\left\{\sinh(\lambda x)\right\}=\frac{1}{2}\left(\frac{\lambda^k}{k!}-\frac{(-\lambda)^k}{k!}\right)=0$$

Theorem 2.14

$$f(x)=\cosh(\lambda x)\Rightarrow \mathcal{D}_T\left\{\cosh(\lambda x)\right\}=\begin{cases} \dfrac{\lambda^k}{k!}, & \text{for even } k \\ 0, & \text{for odd } k \end{cases}$$

Proof 2.14

$$f(x)=\sinh(\lambda x)=\frac{e^{\lambda x}+e^{-\lambda x}}{2}=\frac{e^{\lambda x}}{2}+\frac{e^{-\lambda x}}{2}$$

From the linearity of the differential transformation function

$$\mathcal{D}_T\left\{\cosh(\lambda x)\right\}=\frac{1}{2}\left(\frac{\lambda^k}{k!}+\frac{(-\lambda)^k}{k!}\right)$$

If k is even number;

$$\mathcal{D}_T\left\{\cosh(\lambda x)\right\}=\frac{1}{2}\left(\frac{\lambda^k}{k!}+\frac{(-\lambda)^k}{k!}\right)=0$$

If k is odd;

$$\mathcal{D}_T\left\{\cosh(\lambda x)\right\}=\frac{1}{2}\left(\frac{\lambda^k}{k!}-\frac{(-\lambda)^k}{k!}\right)=\frac{\lambda^k}{k!}$$

Theorem 2.15

Using the definition (2), let us define $G(k)$ as a differential transform of $g(x)$. Then, the differential transform of the integral containing $g(x)$,

$$f(x)=\int_{x_0}^{x} g(t)\,dt,$$

can be expressed as

$$\mathcal{D}_T\left\{f(x)\right\}=\frac{G(k-1)}{k}.$$

Proof 2.15

By using definition (2) the transform of an integral can be found as follows:

$$f(x)=\int_{x_0}^{x}\sum_{k=0}^{\infty}G(k)(t-x_0)^k\,dt\ ,$$

$$=\sum_{k=0}^{\infty}\int_{x_0}^{x}G(k)(t-x_0)^k\,dt=\sum_{k=0}^{\infty}\left[\frac{G(k)}{k+1}(t-x_0)^{k+1}\Big|_{x_0}^{x}\right],$$

$$=\sum_{k=0}^{\infty}\frac{G(k)}{k+1}(x-x_0)^{k+1}.$$

By using the index from $k=1$ instead of $k=0$ we can obtain $f(x)$ as follows:

$$f(x)=\sum_{k=0}^{\infty}\frac{G(k)}{k+1}(x-x_0)^{k+1}.$$

By using definition (2), we get $F(k)=\dfrac{G(k-1)}{k}$, where $k\geq 1$ and $F(0)=0$.

Theorem 2.16

If a given function is a n-folded integrals like

$$f(x)=\int_{x_0}^{x}\int_{x_0}^{x_{n-1}}\cdots\int_{x_0}^{x_3}\int_{x_0}^{x_2}\int_{x_0}^{x_1}g(t)\,dtdx_1dx_2\ldots dx_{n-2}dx_{n-1}$$

then

$$\mathcal{D}_T\left\{f(x)\right\}=F(k)=\frac{(k-n)!G(k-n)}{k}$$

where $k\geq n$.

Proof 2.16

By using the definition of differential transform given in

$$f(x)=\int_{x_0}^{x}\int_{x_0}^{x_{N-1}}\dots\int_{x_0}^{x_3}\int_{x_0}^{x_2}\int_{x_0}^{x_1}\sum_{k=0}^{\infty}G(k)(t-x_0)^k\,dtdx_1dx_2dx_3\dots dx_{n-1}$$

$$=\sum_{k=0}^{\infty}\int_{x_0}^{x}\int_{x_0}^{x_{N-1}}\dots\int_{x_0}^{x_3}\int_{x_0}^{x_2}\int_{x_0}^{x_1}G(k)(t-x_0)^k\,dtdx_1dx_2dx_3\dots dx_{n-1}$$

$$=\sum_{k=0}^{\infty}\int_{x_0}^{x}\int_{x_0}^{x_{N-1}}\dots\int_{x_0}^{x_3}\int_{x_0}^{x_2}\frac{G(k)}{k+1}(x_1-x_0)^{k+1}$$

$$=\sum_{k=0}^{\infty}\int_{x_0}^{x}\int_{x_0}^{x_{N-1}}\dots\int_{x_0}^{x_3}\frac{G(k)}{(k+1)(k+2)}(x_2-x_0)^{k+2}\,dx_2dx_3\dots dx_{n-1}$$

$$=\sum_{k=0}^{\infty}\frac{G(k)(x-x_0)^{k+n}}{(k+1)(k+2)\dots(k+n)}$$

By starting the index of the series from $k=n$ instead of $k=0$

$$f(x)=\sum_{k=n}^{\infty}\frac{G(k-n)}{(k+1-n)(k+2-n)\dots k}(x-x_0)^k,$$

$$=\sum_{k=n}^{\infty}\frac{(k-n)!}{k}G(k)(x-x_0)^{k+n},$$

$$\mathcal{D}_T\{f(x)\}=F(k)=\frac{(k-n)!G(k-n)}{k}.$$

Theorem 2.17

$$f(x)=g(x)\int_{x_0}^{x}h(t)dt\quad\Rightarrow\quad F(k)=\sum_{k_1=1}^{k}\frac{1}{k_1}G(k-k_1)H(k_1-1)$$

where $k\geq 1$.

Proof 2.17

By utilizing the definition of transform:

$$F(0)=\frac{1}{0!}\left[g(x)\int_{x_0}^{x}h(t)\,dt\right]_{x=x_0}=0,$$

$$F(1)=\frac{1}{1!}\frac{d}{dx}\left[g(x)\int_{x_0}^{x}h(t)\,dt\right]_{x=x_0},$$

$$=\left[g'(x)\int_{x_0}^{x}h(t)\,dt+g(x)h(x)\right]_{x=x_0}=G(0)H(0)$$

$$F(2)=\frac{1}{2!}\frac{d^2}{dx^2}\left[g(x)\int_{x_0}^{x}h(t)\,dt\right]_{x=x_0},$$

$$=\frac{1}{2!}\left[g''(x)\int_{x_0}^{x}h(t)\,dt+2g'(x)h(x)+g(x)h'(x)\right]_{x=x_0},$$

$$=G(1)H(0)+\frac{1}{2}G(0)H(1).$$

$$F(3)=\frac{1}{3!}\frac{d^3}{dx^3}\left[g(x)\int_{x_0}^{x}h(t)\,dt\right]_{x=x_0},$$

$$=\frac{1}{3!}\left[g'''(x)\int_{x_0}^{x}h(t)\,dt+3g''(x)h(x)+3g'(x)h(x)+g(x)h''(x)\right]_{x=x_0},$$

$$=G(2)H(0)+\frac{1}{2}G(1)H(1)+\frac{1}{3}G(0)H(2).$$

In general, we have

$$f(x)=g(x)\int_{x_0}^{x}h(t)\,dt \quad\Rightarrow\quad F(k)=\sum_{k_1=1}^{k}\frac{1}{k_1}G(k-k_1)H(k_1-1).$$

where $k\geq 1$.

Theorem 2.18

$$f(x)=\int_{x_0}^{x} g_1(t)g_2(t)dt \quad \Rightarrow \quad F(k)=\frac{1}{k}\sum_{k_1=0}^{k-1} G_1(k_1)G_2(k-k_1-1)$$

where $k \geq 1$.

Proof 2.18

By using the definition of the transform:

$$F(0)=\frac{1}{0!}\left[\int_{x_0}^{x} g_1(t)g_2(t)dt\right]_{x=x_0}=0,$$

$$F(1)=\frac{1}{1!}\frac{d}{dx}\left[\int_{x_0}^{x} g_1(t)g_2(t)dt\right]_{x=x_0}=\left[g_1(x)g_2(x)\right]_{x=x_0}=G_1(0)G_2(0),$$

$$F(2)=\frac{1}{2!}\frac{d^2}{dx^2}\left[\int_{x_0}^{x} g_1(t)g_2(t)dt\right]_{x=x_0}$$

$$=\left[g_1'(x)g_2(x)+g_1(x)g_2'(x)\right]$$

$$=\frac{1}{2}\left[G_1(1)G_2(0)+G_1(0)G_2(1)\right],$$

$$F(3)=\frac{1}{3!}\frac{d^3}{dx^3}\left[\int_{x_0}^{x} g_1(t)g_2(t)dt\right]_{x=x_0}$$

$$=\frac{1}{3!}\left[g_1''(x)g_2(x)+2g_1'(x)g_2'(x)+g_1(x)g_2''(x)\right]$$

$$=\frac{1}{3!}\left[G_1(2)G_2(0)+G_1(1)G_2(1)+G_1(0)G_2(2)\right],$$

$$F(4)=\frac{1}{4!}\left[G_1(3)G_2(0)+G_1(2)G_2(1)+G_1(1)G_2(2)+G_1(0)G_2(3)\right].$$

In general, we have

$$f(x)=\int_{x_0}^{x} g_1(t)g_2(t)dt \quad \Rightarrow \quad F(k)=\frac{1}{k}\sum_{k_1=0}^{k-1} G_1(k_1)G_2(k-k_1-1)$$

where $k \geq 1$.

Theorem 2.19

The differential transform of the integral of products such as

$$f(x)=\int_{x_0}^{x} g_1(t)g_2(t)\cdots g_{n-1}(t)g_n(t)dt$$

can be represented by

$$F(k)=\frac{1}{k}\sum_{k_{n-1}=0}^{k-1}\sum_{k_{n-2}=0}^{k_{n-1}}\cdots\sum_{k_2=0}^{k_3}\sum_{k_1=0}^{k_2} G_1(k_1)G_2(k_2-k_1)\cdots G_{n-1}(k_{n-1}-k_{n-2})G_n(k-k_{n-1}-1)$$

Proof 2.19

By using the definition of the differential transform

$$F(0)=\frac{1}{0!}\left[\int_{x_0}^{x} g_1(t)g_2(t)\cdots g_{n-1}(t)g_n(t)dt\right]_{x=x_0}=0,$$

$$F(1)=\frac{1}{1!}\frac{d}{dx}\left[\int_{x_0}^{x} g_1(t)g_2(t)\cdots g_{n-1}(t)g_n(t)dt\right]_{x=x_0},$$

$$=\left[g_1(x)g_2(x)\cdots g_{n-1}(t)g_n(t)\right]_{x=x_0},$$

$$F(1)=G_1(0)G_2(0)\cdots G_{n-1}(0)G_n(0),$$

$$F(2)=\frac{1}{2!}\frac{d^2}{dx^2}\left[\int_{x_0}^{x} g_1(t)g_2(t)\cdots g_{n-1}(t)g_n(t)\right]_{x=x_0},$$

$$= \frac{1}{2}\begin{bmatrix} g_1'(x)g_2(x)\cdots g_{n-1}(x)g_n(x) + g_1(x)g_2'(x)\cdots g_{n-1}(x)g_n(x) + \cdots \\ + g_1(x)g_2(x)\cdots g_{n-1}'(x)g_n(x) + g_1(x)g_2(x)\cdots g_{n-1}(x)g_n'(x) \end{bmatrix},$$

$$F(2) = \frac{1}{2}\begin{bmatrix} G_1(1)G_2(0)\cdots G_{n-1}(0)G_n(0) + G_1(0)G_2(1)\cdots G_{n-1}(0)G_n(0) + \cdots \\ + G_1(0)G_2(0)\cdots G_{n-1}(1)G_n(0) + G_1(0)G_2(0)\cdots G_{n-1}(0)G_n(1) \end{bmatrix}$$

In general,

$$F(k) = \frac{1}{k}\sum_{k_{n-1}=0}^{k-1}\sum_{k_{n-2}=0}^{k_{n-1}}\cdots\sum_{k_2=0}^{k_3}\sum_{k_1=0}^{k_2} G_1(k_1)G_2(k_2-k_1)\cdots G_{n-1}(k_{n-1}-k_{n-2})G_n(k-k_{n-1}-1).$$

CHAPTER III

Example 1

Solve the basic first order linear differential equation

$$y' - 3y = e^{2x}$$

subject to the initial condition $y(0) = 1$ by using differential transform method.

Solution 1

The differential transform of $y'(x)$ can be shown as

$$\mathcal{D}_T\{y'\} = (k+1)Y(k+1).$$

If we put into the equations, it will be obtained

$$(k+1)Y(k+1) - 3Y(k) = \frac{2^k}{k!},$$

$$Y(k+1) = \frac{3}{k+1}Y(k) + \frac{2^k}{(k+1)k!},$$

$$Y(0) = 1,$$

which is a recurrence relation for the transformed solution $Y(k)$. Then, using the recurrence,

$$k = 0 \Rightarrow \quad Y(1) = \frac{3}{1}Y(0) + \frac{2^0}{1 \cdot 0!} = 4,$$

$$k = 1 \Rightarrow \quad Y(2) = \frac{3}{2}Y(1) + \frac{2^1}{2 \cdot 1!} = 7,$$

$$k=2 \;\Rightarrow\; Y(3)=\frac{3}{3}Y(2)+\frac{2^2}{3\cdot 2!}=\frac{23}{3}$$

$$k=3 \;\Rightarrow\; Y(4)=\frac{3}{4}Y(3)+\frac{2^3}{4\cdot 3!}=\frac{73}{12}$$

$$k=4 \;\Rightarrow\; Y(5)=\frac{3}{5}Y(4)+\frac{2^4}{5\cdot 4!}=\frac{227}{60},$$

$$\vdots$$

Using inverse differential transform method, it can be obtained the solution of differential equation as

$$y(x)=\sum_{k=0}^{\infty}Y(k)x^k=Y(0)+Y(1)x+Y(2)x^2+Y(3)x^3+Y(4)x^4+Y(5)x^5+\cdots$$

$$y(x)=\sum_{k=0}^{\infty}Y(k)x^k=1+4x+7x^2+\frac{23}{3}x^3+\frac{73}{12}x^4+\frac{227}{60}x^5+\cdots$$

$$y(x)=(2\cdot 1-1)+(2\cdot 3-2)x+\left(2\cdot\frac{9}{2}-\frac{4}{2}\right)x^2+\left(2\cdot\frac{27}{6}-\frac{8}{6}\right)x^3$$

$$+\left(2\cdot\frac{81}{24}-\frac{16}{24}\right)x^4+\left(2\cdot\frac{243}{120}-\frac{32}{120}\right)x^5+\cdots$$

$$y(x)=2\left(1+\frac{3x}{1!}+\frac{(3x)^2}{2!}+\frac{(3x)^3}{3!}+\frac{(3x)^4}{4!}+\frac{(3x)^5}{5!}+\cdots\right)$$

$$-\left[1+2x+\frac{(2x)^2}{2!}+\frac{(2x)^3}{3!}+\frac{(2x)^4}{4!}+\frac{(2x)^5}{5!}+\cdots\right]$$

$$y(x)=2e^{3x}-e^{2x}.$$

Example 2

Find the solution of the nonlinear Volterra integral equation

$$u(x)+\int_0^x \left(u^2(t)+u(t)\right)dt=\frac{3}{2}-\frac{1}{2}e^{(-2x)}.$$

Solution 2

Using the definition of the differential transform along with Theorem 2.15 and Theorem 2.18, the given Volterra integral equation can be turned into a recurrence equation as

$$U(k)+\sum_{l=0}^{k-1}U(l)\frac{U(k-l-1)}{k}+\frac{U(k-1)}{k}=\frac{3}{2}\delta(k)-\frac{(-2)^k}{k!},\quad k\geq 1,\quad U(0)=1,$$

Consequently, for each k, the value of the transform function $U(k)$ can be obtained as

$$U(1)=-1,\ U(2)=\frac{1}{2!}$$

$$U(3)=-\frac{1}{3!},\ U(4)=\frac{1}{4!}$$

$$U(5)=-\frac{1}{5!},\ U(6)=\frac{1}{6!}$$

Finally using the inverse differential transform, the solution of the integral equation can be found as below:

$$u(x)=1-x+\frac{1}{2!}x^2-\frac{1}{3!}x^3+\frac{1}{4!}x^4-\frac{1}{5!}x^5+\frac{1}{6!}x^6.$$

When the result is examined carefully, it can be realized that it is the Taylor series expansion of e^{-x} about $x=0$.

Example 3

Find the solution of the linear Volterra integral equation

$$u(x)=1-x-\frac{x^2}{2}+\int_0^x (x-t)u(t)dt, \qquad 0<x<1.$$

Solution 3

By applying the differential transform to both sides of the linear Volterra integral equation along with Theorem 2.8 and Theorem 2.15, it can be obtained the following recurrence equation for $k \geq 1$:

$$U(k)=\delta(k)-\delta(k-1)-\frac{\delta(k-2)}{2}+\sum_{l=0}^{k-1}\delta(l-1)\frac{U(k-l-1)}{k-l}-\sum_{l=0}^{k-1}\delta(l-l)\frac{U(k-l-1)}{k}$$

where $u(0)=1$. Consequently, we find

$$U(1)=-1, \qquad U(2)=0,$$

$$U(3)=-\frac{1}{3!}, \qquad U(4)=0,$$

$$U(5)=-\frac{1}{5!}, \qquad U(6)=0,$$

$$U(7)=-\frac{1}{7!}, \qquad U(8)=0.$$

Finally, using these values in the inverse differential transform, it can be obtained the solution of the linear integral equation as

$$u(x)=1-x-\frac{1}{3!}-\frac{1}{5!}-\frac{1}{7!}-\ldots=1-\sinh(x)$$

which is the exact solution of the linear integral equation.

Example 4

Find the solution of the system of the non-homogeneous differential equations:

$$\frac{dx(t)}{dt} = z(t) - \cos t\,,$$

$$\frac{dy(t)}{dt} = z(t) - e^{t}\,,$$

$$\frac{dz(t)}{dt} = x(t) - y(t)$$

subject to the initial conditions $x(0)=1, \quad y(0)=0, \quad z(0)=2\,.$

Solution 4

Applying the definition of the differential transform to the system of differential equations, it can be obtained the transformed system of recurrence equations such as

$$(k+1)\,X(k+1) = Z(k) - \frac{1}{k!}\cos\left(\frac{\pi k}{2}\right),$$

$$(k+1)Y(k+1) = Z(k) - \frac{1}{k!},$$

$$(k+1)Z(k+1) = X(k) - Y(k),$$

and rearranging the system of recurrence equations it can be obtained the following system like

$$X(k+1) = \frac{1}{k+1}\left[Z(k) - \frac{1}{k!}\cos\left(\frac{\pi k}{2}\right)\right],$$

$$Y(k+1) = \frac{1}{k+1}\left[Z(k) - \frac{1}{k!}\right],$$

$$Z(k+1)=\frac{1}{k+1}\left[X(k)-Y(k)\right]$$

with initial conditions

$$X(0)=1,\qquad Y(0)=0,\qquad Z(0)=2.$$

Then, the first 6 iterations are given as follows:

for $k=0$, $X(1)=1$, $Y(1)=1$, $Z(1)=1$,

for $k=1$, $X(2)=\frac{1}{2}$, $Y(2)=0$, $Z(2)=0$,

for $k=2$, $X(3)=\frac{1}{6}$, $Y(3)=-\frac{1}{6}$, $Z(3)=\frac{1}{6}$,

for $k=3$, $X(4)=\frac{1}{24}$, $Y(4)=0$, $Z(4)=\frac{1}{12}$,

for $k=4$, $X(5)=\frac{1}{120}$, $Y(5)=\frac{1}{120}$, $Z(5)=\frac{1}{120}$,

for $k=5$, $X(6)=\frac{1}{720}$, $Y(6)=0$, $Z(6)=0$,

for $k=6$, $X(7)=\frac{1}{5040}$, $Y(7)=-\frac{1}{5040}$, $Z(7)=\frac{1}{5040}$.

Finally, using the inverse differential transform, the solution of the system of differential equations can be found as follows:

$$x(t)=\sum_{k=0}^{6}X(k)t^k=1+t+\frac{1}{2}t^2+\frac{1}{6}t^3+\frac{1}{24}t^4+\frac{1}{120}t^5+\frac{1}{720}t^6+\frac{1}{5040}t^7,$$

$$y(t)=\sum_{k=0}^{6}Y(k)t^k=t+\frac{1}{6}t^3+\frac{1}{120}t^5+\frac{1}{5040}t^7,$$

$$z(t)=\sum_{k=0}^{6}Z(k)t^k=2+t+\frac{1}{6}t^3+\frac{1}{12}t^4+\frac{1}{120}t^5+\frac{1}{5040}t^7.$$

The exact solution of the system of differential equations can be found by Laplace transform method as

$$x(t)=e^t,\qquad y(t)=\sin t,\qquad z(t)=e^t+\cos t$$

which is exactly same as the iterative solution.

Example 5

The system of non-linear Volterra integral equation that was studied by Goghary et. al.[2] is considered. Let us show the solution by differential transform method.

$$y_1(t)=t^2-\frac{1}{5}t^5-\frac{1}{10}t^{10}+\int_0^t\left[{y_1}^2(s)+{y_2}^3(s)\right]ds,$$

$$y_2(t)=t^3+\int_0^t\left[{y_1}^3(s)-{y_2}^2(s)\right]ds.$$

Solution 5

Applying the differential transform method to both sides of the integral equations in the system, it can be obtained as follows:

$$Y_1(k)=\delta(k-2)-\frac{\delta(k-5)}{5}-\frac{\delta(k-10)}{10}+\frac{1}{k}\sum_{k_1=0}^{k-1}Y_1(k_1)Y_1(k-k_1-1)$$

$$+\frac{1}{k}\sum_{k_2=0}^{k-1}\sum_{k_1=0}^{k_2}Y_2(k_1)Y_2(k_2-k_1)Y_2(k-k_2-1),$$

$$Y_2(k)=\delta(k-3)+\frac{1}{k}\sum_{k_2=0}^{k-1}\sum_{k_1=0}^{k_2}Y_1(k_1)Y_1(k_2-k_1)Y_1(k-k_2-1)$$

$$-\frac{1}{k}\sum_{k_1=0}^{k-1}Y_2(k_1)Y_2(k-k_1-1).$$

When the iteration is done the result can be represented as

$$y_1(t)=t^2, \qquad y_2(t)=t-1$$

Example 6

Consider the following non-linear equation system of Volterra integral equations of the second kind

$$f_1(t)=\sin t-t+\int_0^t\left(f_1{}^2(s)+f_2{}^2(s)\right)ds,$$

$$f_2(t)=\cos t-\frac{1}{2}\sin^2 t+\int_0^t f_1(s)\,f_2(s)\,ds\,.$$

Solution 6

Applying the differential transform method to both sides of the integral equations in the system, it can be obtained as follows:

$$F_1(k)=\frac{1}{k!}\sin\frac{k\pi}{2}-\delta(k-1)+\frac{1}{k}\sum_{k_1=0}^{k-1}F_1(k_1)F_1(k-k_1-1)+\frac{1}{k}\sum_{k_1=0}^{k-1}F_2(k_1)F_2(k-k_1-1),$$

$$F_2(k)=\frac{1}{k!}\cos\frac{k\pi}{2}-\frac{1}{2}\sum_{k_1=0}^{k}\frac{1}{k!}\sin\left(\frac{k_1\pi}{2}\right)\frac{1}{(k-k_1)!}\sin\left(\frac{(k-k_1)\pi}{2}\right)+\frac{1}{k}\sum_{k_1=0}^{k-1}F_1(k_1)F_2(k-k_1-1)$$

According to these equations, the coefficients from first couple iterations are as follows:

$$F_1(0)=0, \qquad F_1(1)=1, \qquad F_1(2)=0, \qquad F_1(3)=-\frac{1}{6},\ldots$$

$$F_2(0)=1, \qquad F_2(1)=-\frac{1}{2}, \qquad F_2(2)=0, \qquad F_2(3)=\frac{1}{4},\ldots$$

It can be shown that

$$f_1(t)=\sum_{k=0}^{\infty}F_1(k)t^k=\sin t, \qquad f_2(t)=\sum_{k=0}^{\infty}F_2(k)t^k=\cos t.$$

Example 7

Find the solution of the following linear system of Volterra integral equations of the second kind

$$f_1(t)+\int_0^t e^{-(s-t)}f_1(s)ds+\int_0^t \cos(s-t)\,f_2(s)ds=\cosh t+t\sin t,$$

$$f_2(t)+\int_0^t e^{s+t}f_1(s)ds+\int_0^t t\cos s\,f_2(s)ds=2\sin t+t\sin^2 t+te^t.$$

Solution 7

One can readily find the differential transform of, as follows

$$F_1(k)=-\sum_{k_1=0}^{k}\frac{1}{k_1}\frac{1}{(k-k_1)!}F_1(k_1-1)-\frac{1}{k}\sum_{k_1=0}^{k-1}\frac{(-1)^{k_1}}{k_1!}F_1(k-k_1-1)$$

$$-\sum_{k_2=1}^{k}\sum_{k_1=1}^{k_2}\frac{1}{k_2}\left(\frac{1}{(k-1)!}\cos\frac{(k_1-1)\pi}{2}\right)F_2(k_2-k_1)\frac{1}{(k-k_2)!}\cos\frac{(k-k_2)\pi}{2}$$

$$-\sum_{k_2=1}^{k}\sum_{k_1=1}^{k_2}\frac{1}{k_2}\left(\frac{1}{(k-1)!}\sin\frac{(k_1-1)\pi}{2}\right)F_2(k_2-k_1)\frac{1}{(k-k_2)!}\sin\frac{(k-k_2)\pi}{2}$$

$$+\frac{1+(-1)^k}{2k!}+\sum_{k_1=0}^{k}\delta(k_1-1)\frac{1}{(k-k_1)!}\sin\left(\frac{\pi(k-k_1)}{2}\right),$$

$$F_2(k)=-\sum_{k_1=1}^{k}\frac{1}{k_1}\frac{1}{(k-k_1)!}F_1(k_1-1)-\frac{1}{k}\sum_{k_1=0}^{k-1}\frac{(-1)^{k_1}}{k_1!}F_1(k-k_1-1)$$

$$-\sum_{k_2=1}^{k}\sum_{k_1=1}^{k_2}\frac{1}{k_2}\left(\frac{1}{(k-1)!}\cos\frac{(k_1-1)\pi}{2}\right)F_2(k_2-k_1)\times\delta(k-k_2-1)+\frac{2}{k!}\sin\left(\frac{k\pi}{2}\right)$$

$$+\sum_{k_2=0}^{k}\sum_{k_1=0}^{k_2}\delta(k_1-1)\frac{1}{(k_2-k_1)!}\sin\left(\frac{\pi(k_2-k_1)}{2}\right)\times\frac{1}{(k-k_1)!}\sin\left(\frac{\pi(k-k_1)}{2}\right)$$

$$+\sum_{k_1=0}^{k}\delta(k_1-1)\frac{1}{(k-k_1)!}.$$

Consequently, the result of the iterations can be obtained as

$$F_1(0)=1,\qquad F_1(1)=-1,\qquad F_1(2)=\frac{1}{2},\qquad F_1(3)=-\frac{1}{6},\ldots$$

$$F_2(0)=0,\qquad F_2(1)=2,\qquad F_2(2)=0,\qquad F_2(3)=-\frac{1}{3},\ldots$$

Then, applying the inverse differential transform method it can be found the result as

$$f_1(t)=\sum_{k=0}^{\infty}F_1(k)t^k=e^{-t},$$

$$f_2(t)=\sum_{k=0}^{\infty}F_2(k)t^k=2\sin t$$

The results are same as the closed form solution.

Example 8

Let us solve the following equation

$$\frac{dy}{dt}=-y^2(t)+1$$

subject to the initial condition $y(0)=0$.

Solution 8

The exact solution of the differential equation is given by

$$y(t)=\frac{e^{2t}-1}{e^{2t}+1}.$$

The Taylor series expansion of $y(t)$ about $t=0$ is

$$y(t)=t-\frac{1}{3}t^3+\frac{2}{15}t^5-\frac{17}{315}t^7+\frac{62}{2835}t^9-\ldots.$$

Now, let us apply the differential transform to the differential equation as follows:

$$(k+1)Y(k+1)=-\sum_{r=0}^{k}Y(r)Y(k-r)+\delta(k)$$

with the initial condition $Y(0)=0$. Using the recurrence equation, the values of transformed function $Y(k)$ can be found as follows:

$$Y(1)=1,$$

$$Y(2)=0,$$

$$Y(3)=-\frac{1}{3},$$

$$Y(4)=0,$$

$$Y(5)=\frac{2}{15}.$$

Finally, using the inverse differential transform the solution can be easily written in the series form like

$$y(t)=\sum_{k=0}^{\infty}Y(k)t^k=t-\frac{1}{3}t^3+\frac{2}{15}t^5-\frac{17}{315}t^7+\frac{62}{2835}t^9-\cdots=\frac{e^{2t}-1}{e^{2t}+1}$$

that is the same as the exact solution.

Example 9

First, let us examine the singular two-point BVP (Cui,Geng [15])

$$y''(x)+\frac{1}{x}y'(x)+y(x)=4-9x+x^2-x^3,\quad 0<x\le 1,$$

subject to the boundary conditions $y(0)=0$ and $y(1)=0$.

Solution 9

The exact solution of the differential equation is

$$y(x)=x^2-x^3.$$

Now, let us try to find the same solution by using the differential transform method. First, as follows:

$$\sum_{l=0}^{k}\delta(l-1)(k-l+1)(k-l+2)Y(k-l+2)+(k+1)Y(k+1)$$

$$+\sum_{l=0}^{k}\delta(l-1)Y(k-l)-4\delta(k-1)+9\delta(k-2)-\delta(k-3)+\delta(k-4)=0$$

subject to the initial condition $Y(0)=0$ and $y(1)=\sum_{k=0}^{N}Y(k)1^k=\sum_{k=0}^{N}Y(k)=0$.

For $N=6$, the following system for $k=0,1,2,\ldots$ is obtained:

$$Y(1)=0,$$

$$4Y(2)+Y(0)=4,$$

$$9Y(3)+Y(1)=-9,$$

$$16Y(4)+Y(2)=1,$$

$$25Y(5)+Y(3)=-1,$$

$$36Y(6)+Y(4)=0.$$

Solving the above system and using the inverse differential transform, we get the following series solution which is exactly same as the close form solution, respectively,

$$y(t)=\sum_{k=0}^{6}Y(k)t^k=0+0\cdot t+1\cdot t^2-1\cdot t^3+0\cdot t^4+0\cdot t^5+0\cdot t^6,$$

$$y(x)=x^2-x^3.$$

Example 10

Consider the following singular two-point BVP (Junfeng [2])

$$y''(x)+\frac{1}{x}y'(x)+y(x)=\frac{5}{4}+\frac{x^2}{16},\qquad 0<x\leq 1,$$

subject to the boundary conditions $y'(0)=0$ and $y(1)=\frac{17}{16}$.

Solution 10

The exact solution of the differential equation is obtained as

$$y(x)=1+\frac{x^2}{16}.$$

The differential transform of the differential equation is given as

$$\sum_{l=0}^{k}\delta(l-1)(k-l+1)(k-l+2)Y(k-l+2)+(k+1)Y(k+1)$$

$$+\sum_{l=0}^{k}\delta(l-1)Y(k-l)-\frac{5}{4}\delta(k-1)-\frac{1}{16}\delta(k-3)=0$$

with the transformed boundary conditions

$$(k+1)Y(k+1)=0, \qquad \sum_{k=0}^{N}Y(k)=\frac{17}{16}.$$

For $N=6$, the following system for $k=0,1,2,\ldots$ is obtained:

$$Y(1)=0,$$

$$4Y(2)+Y(0)=\frac{5}{4},$$

$$9Y(3)+Y(1)=0,$$

$$16Y(4)+Y(2)=\frac{1}{16},$$

$$25Y(5)+Y(3)=0,$$

$$36Y(6)+Y(4)=0.$$

Solving the above system and using the inverse differential transform, we get the following series solution which is exactly same as the close form solution, respectively,

$$y(t)=\sum_{k=0}^{6}Y(k)t^k=1+0\cdot t+\frac{1}{16}\cdot t^2-0\cdot t^3+0\cdot t^4+0\cdot t^5+0\cdot t^6,$$

$$y(x)=1+\frac{x^2}{16}.$$

Example 11

Consider the following system of differential equations of order two

$$y''(x)+y_1(x)-y''(x)-4y_2(x)=0,$$

$$y'(x)+y_2'(x)=\cos(x)+2\cos(2x)$$

with the conditions

$$y_1(0)=0, \qquad y_1'(0)=1,$$

$$y_2(0)=0, \qquad y_2'(0)=2.$$

Solution 11

The exact solution of the system is

$$y_1(x)=\sin(x),$$

$$y_2(x)=\sin(2x).$$

Now, let us apply the differential transform to the system of differential equation. The new transformed system of recurrence equations can be obtained as follows:

$$(k+2)(k+1)Y_1(k+2)+Y_1(k)-(k+2)(k+1)Y_2(k+2)-4Y_2(k)=0,$$

$$(k+1)Y_1(k+1)+(k+1)Y_2(k+1)=\frac{1}{k!}\cos\left(\frac{k\pi}{2}\right)+\frac{2^{k+1}}{k!}\cos\left(\frac{k\pi}{2}\right)$$

with initial conditions

$$Y_1(0)=0, \qquad (k+1)Y_1(k+1)=1 \;\Rightarrow\; Y_1(1)=1 \text{ for } k=0$$

$$Y_2(0)=0, \qquad (k+1)Y_2(k+1)=2 \;\Rightarrow\; Y_2(1)=2 \text{ for } k=0.$$

Then, using the recurrence equation it can be obtained the following values

$$Y_1(2)=0, \qquad Y_2(2)=0,$$

$$Y_1(3)=-\frac{1}{3!}, \qquad Y_2(3)=-\frac{8}{3!},$$

$$Y_1(4)=0, \qquad Y_2(4)=0,$$

$$Y_1(5)=\frac{1}{5!}, \qquad Y_2(5)=\frac{32}{5!},$$

$$\vdots$$

Finally, the solution can be expressed as

$$y_1(x)=x-\frac{1}{3!}x^3+\frac{1}{5!}x^5+\ldots=\sin(x),$$

$$y_2(x)=2x-\frac{(2x)^3}{3!}+\frac{(2x)^5}{5!}+\ldots=\sin(2x).$$

CHAPTER IV

CONCLUSION

In this work, differential transformation method was successfully applied to find numerical solutions for linear and nonlinear system of ordinary differential equations. It is observed that DTM is an powerful tool for the solution of system of ordinary differential equations Linear and Nonlinear Volterra integral equation, non-homogeneous differential system, Linear and nonlinear Volterra integral equation system, initial condition problem, boundary valuable problem, differential equation system problem and Homogeneous reaction-diffusion singular perturbation problems were solved by DTM. In this paper, some theorems are introduced with their proofs and some algebraic computations are performed by using MAPLE 13 symbolic programming software.

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APPENDIX A

Original Function	*Transformed Function*
1) $f(x)=cg(x)$	$F(k)=cG(k).$
2) $f(x)=g(x)\pm h(x)$	$F(k)=G(k)\pm H(k)$
3) $f(x)=\dfrac{dg(x)}{dx}$	$F(k)=(k+1)G(k+1)$
4) $f(x)=\dfrac{d^n g(x)}{dx^n}$	$F(k)=\dfrac{(k+n)!}{k!}G(k+n)$
5) $f(x)=g(x)h(x)$	$F(k)=\sum_{r=0}^{k}G(r)H(k-r)$
6) $f(x)=g_1(x)g_2(x)g_3(x)$	$F(k)=\sum_{k_2=0}^{k_3}\sum_{k_1=0}^{k_2}G_1(k_1)G_2(k_2-k_1)G_3(k-k_2)$
7) $f(x)=g_1(x)g_2(x)\cdots g_{n-1}(x)g_n(x)$	$F(k)=\sum_{k_{n-1}=0}^{k}\sum_{k_{n-2}=0}^{k_{n-1}}\cdots\sum_{k_2=0}^{k_3}\sum_{k_1=0}^{k_2}G_1(k_1)G_2(k_2-k_1)\cdots G_{n-1}(k_{n-1}-k_{n-2})\times G_n(k-k_{n-1})$
8) $f(x)=x^m$	$F(k)=\delta(k-m)=\begin{cases}1, & k=m\\ 0, & \text{otherwise}\end{cases}$
9) $f(x)=a^{\lambda x}$	$F(k)=\dfrac{\lambda^k(\ln a)^k}{k!}$
10) $f(x)=e^{\lambda x}$	$F(k)=\dfrac{\lambda^k}{k!}$
11) $f(x)=\sin(ax+b)$	$F(k)=\dfrac{a^k}{k!}\sin\left(\dfrac{\pi k}{2}+b\right)$

Original Function	***Transformed Function***
12) $f(x)=\cos(ax+b)$	$F(k)=\frac{a^k}{k!}\cos\left(\frac{\pi k}{2}+b\right)$
13) $f(x)=\int_{x_0}^{x} g(t)\,dt$	$F(k)=\frac{G(k-1)}{k}$
14) $f(x)=\sinh(\lambda x)$	$F(k)=\begin{cases}0, & \text{for even } k\\ \frac{\lambda^k}{k!}, & \text{for odd } k\end{cases}$
15) $f(x)=\cosh(\lambda x)$	$F(k)=\begin{cases}\frac{\lambda^k}{k!}, & \text{for even } k\\ 0, & \text{for odd } k\end{cases}$
16) $f(x)=\int_{x_0}^{x}\int_{x_0}^{x_{n-1}}\dots\int_{x_0}^{x_3}\int_{x_0}^{x_2}\int_{x_0}^{x_1} g(t)\,dt dx_1 dx_2\dots dx_{n-2}dx_{n-1}$	$F(k)=\frac{(k-n)!G(k-n)}{k}$
17) $f(x)=g(x)\int_{x_0}^{x} h(t)\,dt$	$F(k)=\sum_{k_1=1}^{k}\frac{1}{k_1}G(k-k_1)H(k_1-1),$ $where\, k\geq 1$
18) $f(x)=\int_{x_0}^{x} g_1(t)g_2(t)\,dt$	$F(k)=\frac{1}{k}\sum_{k_1=0}^{k-1}G_1(k_1)G_2(k-k_1-1),$ $where\, k\geq 1$
19) $f(x)=\int_{x_0}^{x} g_1(t)g_2(t)\cdots g_{n-1}(t)g_n(t)\,dt$	$F(k)=\frac{1}{k}\sum_{k_{n-1}=0}^{k-1}\sum_{k_{n-2}=0}^{k_{n-1}}\cdots\sum_{k_2=0}^{k_3}\sum_{k_1=0}^{k_2}G_1(k_1)$ $G_2(k_2-k_1)\cdots G_{n-1}(k_{n-1}-k_{n-2})$ $G_n(k-k_{n-1}-1)$