FINITE DIFFERENCE METHOD FOR ULTRA PARABOLIC EQUATIONS

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APPROVAL PAGE

This is to certify that I have read this thesis written by Serhat YILMAZ and that in my opinion it is fully adequate, in scope and quality, as a thesis for the degree of Doctor of Philosophy in Mathematics.

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ABSTRACT

The stable difference schemes for the approximate solution of the initial boundary value problem for ultra parabolic equations are presented. Stability of these difference schemes are established. Theoretical results are supported by the result of numerical examples.

Keywords: Ultra-Parabolic Equations, Difference Schemes, Stability Estimates, Matlab Implementation, Numerical Solutions.

ULTRA PARABOLIK DENKLEMLER İÇİN SONLU FARK METODU

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ÖZ

Başlangıç değer ultra-parabolik problemlerin yaklaşık çözümleri için, kararlı fark şemaları elde edilmiştir. Bu fark şemaları için elde edilen teorik bulgular sayısal örneklerle desteklenmiştir.

Anahtar Kelimeler: Ultra-Parabolik Denklemler, Fark Şemaları, Kararlılık, Matlap Uygulamaları, Sayısal Çözümler.

To my family

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INTRODUCTION

Mathematical models which are formulated in terms of ultra parabolic equations are of great importance in various problems for instance in the probability theory, in the theory of Brownian motion and in the theory of boundary layers (Kolmogorov, 1933), (Kolmogorov, 1934), (Genčev, 1963). The existence of mild solutions, as well as other properties, for various classes of ultra-parabolic problem arising from age-structured models with diffusion have been investigated by a number of authors (Kunisch et al., 1985), (Deng and Hallam, 2006), (Dyson et al., 2007), (Magal and Ruan, 2007).

Moreover, in (Di Blasio, 2009) the author studies the well-posedness and the regularity of solutions for a class of ultra parabolic equations of the form

$$\begin{cases} u_t(t,a,x) + u_a(t,a,x) = -d(t,a,x) + \Delta u(t,a,x), \quad t>0, \quad a>0, \\ x \in \Omega \subset R^n, \\ \\ u(t,0,x) = b(t,x), \quad t>0, \quad x \in \Omega, \\ \\ u(t,a,x) = 0, \quad t>0, \quad a>0, \quad x \in \partial\Omega, \\ \\ u(0,a,x) = u_0(a,x), \quad a>0, \quad x \in \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset R^n$ is an open and bounded set with regular boundary $\partial\Omega$ and Δ denote the Laplace operator in Ω. Problems of this kind occur in the study of the dynamics of a population subject to birth, death and diffusion, in a given domain Ω. The

death term d(t,a,x) and the birth process b(t,x) are assumed to be of the form

$$d(t,a,x) = \mu(t,a,x)u(t,a,x),$$

$$b(t,x) = \int_0^\infty \beta(t,\alpha,x)u(t,\alpha,x)d\alpha.$$

In order to obtain the results, the author reduces the problem (1.1) to the relative parabolic one using the characteristic coordinates on the (t,a)-plane. Here, note that ultra-parabolic equations describe as well the processes where in some direction the effect of diffusion is negligible as compared to convection. Later (Di Blasio and Lorenzi, 2011) deals with an identification problem for a class of semi-linear population dynamics models structured by age and with spatial diffusion.

In (Akrivis et al., 1994), Akrivis, Crouzeix, Thomée analyze semi-discrete as well as fully discrete finite element approximations to the solution of the following initial boundary value problem for a model ultra-parabolic equation

$$\begin{cases} u_t + \lambda u_s(t,a,x) - \Delta u = 0, \quad \Omega \times (0,S) \times (0,T), \\ u = 0, \quad \partial\Omega \times [0,S] \times [0,T], \\ u(t,0,x) = v^0, \quad \overline{\Omega} \times [0,T], \\ u(t,a,0) = w^0, \quad \overline{\Omega} \times [0,S], \end{cases} \tag{1.2}$$

where S, T > 0, and $\Omega \subset R^d$ is a bounded domain with smooth boundary $\partial\Omega$, and λ is a positive smooth function on $\overline{\Omega} \times [0,S] \times [0,T]$. They show an optimal order estimate with respect to the norm

$$\|u\|_\Gamma^2 = \int_\Gamma \|u\|^2 \, d\sigma \le C \int_0^T \left\|v^0(t,\cdot,\cdot)\right\|^2 dt + \int_0^S \left\|w^0(\cdot,s,\cdot)\right\|^2 ds\,),$$

where Γ is a piecewise smooth curve contained in $[0,S] \times [0,T]$ and connecting the intervals $[0,S]$ with [0, T] on the s and x axes. Note that a corresponding maximum-norm error estimate holds when it is independent of x. Then they consider a fully discrete implicit backward Euler method, using backward difference quotients to approximate the derivatives with respect to both s and t in (1.2) and show fully

discrete analogues of the above error estimates in the semi-discrete case is considered. Finally, they discuss a second order fully discrete scheme based on a variant of the box scheme for the hyperbolic part of the equation and show a corresponding optimal order error estimate. Very recently, numerical solutions of ultra-parabolic problems are investigated in (Ashyralyev and Yilmaz, 2011), (Ashyralyev and Yilmaz, 2012a), (Ashyralyev and Yilmaz, 2012b), (Ashyralyev and Yilmaz, 2012c).

In the present work we consider the well-posedness of initial value problems for equations of ultra-parabolic type. The main goal of this work is to construct and investigate the difference schemes for initial value problems for ultra-parabolic equations.

It is known that initial value problems for ultra-parabolic equation can be solved by Fourier series method, by Laplace transform method and by Fourier transform method. Now, let us illustrate these three different analytical methods by examples.

Example 1.1. First let us consider the simple initial boundary value problem for ultra parabolic equation

$$\begin{cases} \frac{\partial u}{\partial t} + \frac{\partial u}{\partial s} - \frac{\partial^2 u}{\partial x^2} + 2u = e^{-(s+t)} \sin x, \ 0 < t, s < \infty, \ 0 < x < \pi, \\ u(0, s, x) = e^{-s} \sin x, \ 0 \leq x \leq \pi, 0 \leq s < \infty, \\ u(t, 0, x) = e^{-t} \sin x, \ 0 \leq x \leq \pi, 0 \leq t < \infty, \\ u(t, s, 0) = u(t, s, \pi) = 0, 0 \leq t, s < \infty. \end{cases} \tag{1.3}$$

For the solution of the problem (1.3), we use the method of separation of variables or so called Fourier series method. In order to solve the problem we need to separate $u(t, x)$ into two parts

$$u(t, s, x) = v(t, s, x) + w(t, s, x),$$

where

$$\begin{cases} \frac{\partial v}{\partial t} + \frac{\partial v}{\partial s} - \frac{\partial^2 v}{\partial x^2} + 2v = 0,\ 0 < t, s < \infty,\ 0 < x < \pi, \\ v(0, s, x) = e^{-s} \sin x,\ 0 \leq x \leq \pi, 0 \leq s < \infty, \\ v(t, 0, x) = e^{-t} \sin x,\ 0 \leq x \leq \pi, 0 \leq t < \infty, \\ v(t, s, 0) = v(t, s, \pi) = 0, 0 \leq t, s < \infty \end{cases} \tag{1.4}$$

and

$$\begin{cases} \frac{\partial w}{\partial t} + \frac{\partial w}{\partial s} - \frac{\partial^2 w}{\partial x^2} + 2w = e^{-(s+t)} \sin x,\ 0 < t, s < \infty,\ 0 < x < \pi, \\ w(0, s, x) = 0,\ 0 \leq x \leq \pi, 0 \leq s < \infty, \\ w(t, 0, x) = 0,\ 0 \leq x \leq \pi, 0 \leq t < \infty, \\ w(t, s, 0) = w(t, s, \pi) = 0, 0 \leq t, s < \infty. \end{cases} \tag{1.5}$$

First we will obtain the solution of the problem (1.4) by the method of separation of variables. To do this, a solution of the form

$$v(t, s, x) = T(t, s)X(x) \neq 0$$

is suggested. Taking the partial derivative and substituting the result in (1.4), we obtain

$$\frac{T_t(t, s) + T_s(t, s)}{T(t, s)} - \frac{X''(x) - 2X(x)}{X(x)} = 0$$

or

$$\frac{T_t(t, s) + T_s(t, s)}{T(t, s)} = \frac{X''(x) - 2X(x)}{X(x)} = -(\lambda + 2).$$

Since

$$X''(x) = \lambda X(x), X(0) = X(\pi) = 0,$$

we have that

$$X_k(x) = \sin \sqrt{-\lambda_k} x, \lambda_k = -k^2, k = 1, 2, \cdots.$$

Now to get $T(t,s)$ we can write

$$\frac{T_t(t,s)+T_s(t,s)}{T(t,s)}=-(k^2+2).$$

Since

$$\frac{\partial(t+s)}{\partial t}=\frac{\partial(t+s)}{\partial s}=1,$$

we have that

$$\frac{2T_{t+s}(t,s)}{T(t,s)}=-(k^2+2).$$

So the solution for $T(t,s)$ can be obtained by using Cauchy formula

$$T_k(t,s)=e^{-\frac{1}{2}(k^2+2)(t+s)}T_k(0,0)\sin kx.$$

By using superposition principle, we get

$$v(t,s,x)=\sum_{k=1}^{\infty}v_k(t,s,x)=\sum_{k=1}^{\infty}e^{-\frac{1}{2}(k^2+2)(t+s)}T_k(0,0)\sin kx.$$

Using the initial conditions, we get

$$v(0,0,x)=\sum_{k=1}^{\infty}T_k(0,0)\sin kx=\sin x.$$

From it follows that $T_k(0,0)=0$, $k\neq 1$ and $T_1(0,0)=1$. Hence

$$v\left(t,s,x\right)\equiv e^{\frac{3}{2}(t+s)}\sin x.$$

Second we will obtain the solution for (1.5). Let

$$w\left(t,s,x\right)=\sum_{k=1}^{\infty}A_k\left(t,s\right)\sin kx.$$

Putting into the equation and using the initial conditions, we get

$$\sum_{k=1}^{\infty}[\left(A_k\left(t,s\right)\right)_t+\left(A_k\left(t,s\right)\right)_s]\sin kx$$

$$=\sum_{k=1}^{\infty}[-k^2A_k\left(t,s\right)sinkx-2A_k\left(t,s\right)\sin kx]+e^{(t+s)}\sin x$$

and

$$A_k\left(0,0\right)=0 \text{ for } k=1,2,\cdots,$$

which implies

$$\frac{\partial A_k(t,s)}{\partial(t)}+\frac{\partial A_k(t,s)}{\partial(s)}+(k^2+2)A_k(t,s)=0, A_k(t,s)=0, k\neq 1$$

and

$$2\frac{\partial A_1(t,s)}{\partial(t+s)}+3A_1(t,s)=e^{-(t+s)}, A_1(0,0)=0.$$

Using the Cauchy formula, we get

$$A_1(t,s)=e^{(t+s)}-e^{\frac{3}{2}(t+s)}$$

and

$$w\,(t,s,x)=(e^{(t+s)}-e^{\frac{3}{2}(t+s)})sinx.$$

So,

$$u\,(t,s,x)=e^{(t+s)}sinx.$$

Note that using the same manner one can obtain the solution of the following boundary value problem for the multidimensional ultra-parabolic equation

$$\begin{cases} \frac{\partial u}{\partial s}+\frac{\partial u}{\partial t}+\sum\limits_{|r|=2m} a_r \frac{\partial^{|\tau|}u}{\partial x_1^{r1}\ldots\partial x_n^{rn}}+\delta u(t,s,x)=f(t,s,x), \\ x=(x_1,\ldots,x_n)\in\overline{\Omega}, 0<t,s<T, |r|=r_1+\cdots+r_n, \\ u(t,0,x)=\varphi(t,x), 0\leq t\leq T,\ x\in\overline{\Omega}, \\ u(0,s,x)=\psi(s,x), 0\leq s\leq T,\ x\in\overline{\Omega}, \\ u(t,s,x)=0, 0\leq t,s\leq T,\ x\in\partial\Omega, \end{cases}$$

where $\delta>0$ and f(t,s,x) $(t,s\in[0,T],\ x\in\overline{\Omega})$, $\varphi(t,x)$ $(t\in[0,T], x\in\overline{\Omega})$, $\psi(s,x)$ $(s\in[0,T], x\in\overline{\Omega})$ are given smooth functions. Here Ω is the unit open cube in the n-dimensional Euclidean space $\mathbb{R}^n$ $(0<x_k<1, 1\leq k\leq n)$ with boundary $\partial\Omega$, $\overline{\Omega}=\Omega\cup\partial\Omega$.

Example 1.2. Another example of initial boundary value problem for ultra parabolic equation is given below. It can be solved by Laplace transformation method (in x).

$$\begin{cases} \frac{\partial u}{\partial t} + \frac{\partial u}{\partial s} - \frac{\partial^2 u}{\partial x^2} + u = -2e^{-(s+t+x)},\ 0 < t, s < \infty,\ 0 < x < \infty, \\ u(0,s,x) = e^{-(s+x)},\ 0 \leq x < \infty, 0 \leq s < \infty, \\ u(t,0,x) = e^{-(t+x)},\ 0 \leq x < \infty, 0 \leq t < \infty, \\ u(t,s,0) = u(t,s,\pi) = e^{-(t+s)}, 0 \leq t, s < \infty. \end{cases} \tag{1.6}$$

Let denote $\mathbf{L}\{u(t,s,x)\} = U(t,s,\xi)$. Taking the Laplace transform of both sides of the differential equation we can write that

$$\mathbf{L}\{u_t(t,s,x)\} + \mathbf{L}\{u_s(t,s,x)\} - \mathbf{L}\{u_{xx}(t,s,x)\} + \mathbf{L}\{u(t,s,x)\} = \mathbf{L}\left\{\left(-2e^{-(t+s+x)}\right)\right\},$$

then we have

$$U_t(t,s,\xi) + U_s(t,s,\xi) = (\xi^2 - 1)U(t,s,\xi) - \xi e^{-(t+s)} + e^{-(t+s)} + \mathbf{L}\left\{\left(-2e^{-(t+s+x)}\right)\right\}.$$

So our problem becomes

$$U_t(t,s,\xi) + U_s(t,s,\xi) + (1-\xi^2)U(t,s,\xi) = -\frac{1+\xi^2}{1+\xi}e^{-(t+s)}$$

or

$$2U_{t+s}(t,s,\xi) + (1-\xi^2)U(t,s,\xi) = -\frac{1+\xi^2}{1+\xi}e^{-(t+s)}.$$

Applying the Cauchy formula, we get

$$U(t,s,\xi) = \frac{1}{1+\xi}e^{-(t+s)}.$$

Hence taking the inverse of Laplace transform, we get

$$u(t,s,x) = \mathbf{L}^{-1}\left\{\frac{1}{1+\xi}e^{-(t+s)}\right\} = e^{-(t+s+x)}.$$

Note that using the same manner one can obtain the solution of the following

boundary value problem for the multidimensional ultra-parabolic equation

$$\begin{cases} \frac{\partial u}{\partial s}+\frac{\partial u}{\partial t}+\sum\limits_{r=1}^{n} a_r \frac{\partial^2 u}{\partial x_r^2}+\delta u(t,s,x)=f(t,s,x), \\ x=(x_1,\ldots,x_n)\in\Omega^+, 0\le t,s\le T, \\ u(t,0,x)=\varphi(t,x), 0\le t\le T,\ x\in\overline{\Omega}^+, \\ u(0,s,x)=\psi(s,x), 0\le s\le T,\ x\in\overline{\Omega}^+, \\ u_{x_r}(t,s,x)=0, r=1,...,n, u(t,s,x)=0, 0\le t,s\le T,\ x\in\partial\Omega^+, \end{cases}$$

where $\delta>0$ and f(t,s,x) $(t,s\in[0,T],\ x\in\Omega^+)$, $\varphi(t,x)$ $(t\in[0,T], x\in\overline{\Omega}^+)$, $\psi(s,x)$ $(s\in[0,T], x\in\overline{\Omega}^+)$ are given smooth functions. Here Ω^+ is the unit open set in the n-dimensional Euclidean space $\mathbb{R}^n$ $(0<x_k<\infty, 1\le k\le n)$ with boundary $\partial\Omega^+$, $\overline{\Omega}^+=\Omega^+\cup\partial\Omega^+$.

Example 1.3. The last example is an initial boundary value problem for ultra parabolic equation solved by using Fourier transform method.

$$\begin{cases} \frac{\partial u}{\partial t}+\frac{\partial u}{\partial s}-\frac{\partial^2 u}{\partial x^2}+u=(1-4x^2)e^{-(s+t+x^2)},\ 0<t,s<\infty,\ -\infty<x<\infty, \\ u(0,s,x)=e^{-(s+x^2)},\ -\infty<x<\infty, 0\le s<\infty, \\ u(t,0,x)=e^{-(t+x^2)},\ -\infty<x<\infty, 0<\infty. \end{cases} \tag{1.7}$$

Let denote $\mathbf{F}\{u(t,s,x)\}=U(t,s,\xi)$. By taking the Fourier transform of both sides, we obtain

$$\mathbf{F}\{u_t(t,s,x)\}+\mathbf{F}\{u_s(t,s,x)\}-\mathbf{F}\{u_{xx}(t,s,x)\}+\mathbf{F}\{u(t,s,x)\}$$

$$=\mathbf{F}\left\{(1-4x^2)e^{-(t+s+x^2)}\right\}.$$

Then we have

$$U_t(t,s,\xi)+U_s(t,s,\xi)+(1+\xi^2)U(t,s,\xi)=\mathbf{F}\left\{(1-4x^2)e^{-(t+s+x^2)}\right\},$$

it implies

$$2U_{t+s}(t,s,\xi)+(1+\xi^2)U(t,s,\xi)=e^{-(t+s)}\left(\mathbf{F}\left\{(e^{-x^2})'\right\}-\mathbf{F}\left\{(e^{-x^2}\right\}\right)$$

or

$$U_{t+s}(t,s,\xi)+\frac{1+\xi^2}{2}U(t,s,\xi)=\frac{e^{-(t+s)}}{2}(\xi^2-1)\mathbf{F}\left\{e^{-x^2}\right\}.$$

Applying the Cauchy formula, we get

$$U(t,s,\xi)=\mathbf{F}\left\{e^{-x^2}\right\}e^{-(t+s)}.$$

Hence taking the inverse of Fourier transform, we get

$$u(t,s,x)=e^{-x^2}e^{-(t+s)}=e^{-(t+s+x^2)}.$$

Note that using the same manner one obtains the solution of the following initial value problem for the multidimensional ultra-parabolic equation

$$\begin{cases} \frac{\partial u}{\partial s}+\frac{\partial u}{\partial t}+\sum\limits_{|r|=2m}a_r\frac{\partial^{|\tau|}u}{\partial x_1^{r1}\ldots\partial x_n^{rn}}+\delta u(t,s,x)=f(t,s,x), \\ x=(x_1,\ldots,x_n)\in\mathbb{R}^n, 0<t,s<T, |r|=r_1+\cdots+r_n, \\ u(t,0,x)=\varphi(t,x), 0\le t\le T,\ x\in\mathbb{R}^n, \\ u(0,s,x)=\psi(s,x), 0\le s\le T,\ x\in\mathbb{R}^n, \end{cases}$$

where $\delta>0$ and f(t,s,x) $(t,s\in[0,T],\ x\in\mathbb{R}^n)$, $\varphi(t,x)$ $(t\in[0,T], x\in\mathbb{R}^n)$, $\psi(s,x)$ $(s\in[0,T], x\in\mathbb{R}^n)$ are given smooth functions.

However, the Fourier transform method can be used only in the case when it has constant coefficients. It is well-known that the most useful method for solving partial differential equations with dependent coefficients in x and in the space variables is difference method.

It is known that various initial boundary value problem for the ultra-parabolic equations can be reduced to the boundary value problem

$$\begin{cases} \frac{\partial u(t,s)}{\partial t}+\frac{\partial u(t,s)}{\partial s}+Au(t,s)=f(t,s), \quad 0<t,s<T, \\ u(0,s)=\psi(s), \quad 0\le s\le T, \\ u(t,0)=\varphi(t), \quad 0\le t\le T \end{cases} \tag{1.8}$$

for differential equation in a Banach space E with a strongly positive operator A.

In the present work, the well-posedness of the initial boundary problem (1.8) is considered. The difference schemes of the first and second orders of accuracy of these porblem are presented. Under the applicability conditions, well-posedness of these difference schemes are investigated.

Let us briefly describe the contest of the various sections. It consist of six chapters and a conclusion.

First chapter is the introduction.

Second chapter proves that following theorems on the stability analysis of ultra- parabolic problems.

Theorem 1.1. For the solution of problem (1.8), the following stability inequality holds

$$\max_{0\leq t,s\leq T} \| u(t,s) \|_E \leq M \left[\max_{0\leq t\leq T} \| \varphi(t) \|_E + \max_{0\leq s\leq T} \| \psi(s) \|_E + \max_{0\leq s,t\leq T} \| f(t,s) \|_E \right],$$

here M is independent of φ (t), ψ (s) and f(t,s).

Theorem 1.2. For the solution of problem (1.8), the following coercive stability inequality holds

$$\max_{0\leq t,s\leq T} \| u_t(t,s) + u_s(t,s) \|_{E_\alpha} + \max_{0\leq t,s\leq T} \| Au(t,s) \|_{E_\alpha}$$

$$\leq M \left[\max_{0\leq t\leq T} \| A\varphi \|_{E_\alpha} + \max_{0\leq s\leq T} \| A\psi(s) \|_{E_\alpha} + \frac{1}{1-\alpha} \max_{0\leq t,s\leq T} \| f(t,s) \|_{E_\alpha} \right],$$

where M is independent of φ (t), ψ (s) and f(t,s). Here, $\| v \|_{E_\alpha}$ denotes norm of the Banach space of E_α consist of those $v \in E$ for which the norm

$$\| v \|_{E_\alpha} = \sup_{\lambda>0} \lambda^{1-\alpha} \| Ae^{-\lambda A} v \|_E + \|v\|_E$$

is finite and $\alpha \in (0,1)$.

Theorem 1.3. For the solution of problem (1.8), the following coercive stability inequality holds

$$\max_{0<s,t<T} \| u_s(t,s) \|_{E_\alpha} + \max_{0<s,t<T} \| u_t(t,s) \|_{E_\alpha} \leq M \left[\max_{0<t<T} \| A\varphi(t) \|_{E_\alpha} \right.$$

$$+ \max_{0<t<T} \| \varphi'(t) \|_{E_\alpha} + \max_{0<s,t<T} \| f(t,s \|_{E_\alpha} + \max_{0<s,t<T} \| f_s(t,s) \|_{E_\alpha}$$

$$\left. + \max_{0<t<T} \| A\psi(t) \|_{E_\alpha} + \max_{0<t<T} \| \psi'(t) \|_{E_\alpha} + \max_{0<s,t<T} \| f_t(t,s) \|_{E_\alpha} \right],$$

where M is independent of φ (t), ψ (s) and f(t,s).

Three applications of these theorems are given. First, the mixed boundary value problem for a ultra-parabolic equation

$$\begin{cases} \frac{\partial u}{\partial s} + \frac{\partial u}{\partial t} - a(x)\frac{\partial^2 u}{\partial x^2} + \delta u = f(t,s,x), 0 < t,s < T, 0 < x < 1, \\ u(t,0,x) = \varphi(t,x),\ 0 \leq t \leq T, 0 \leq x \leq 1, \\ u(0,s,x) = \psi(s,x),\ 0 \leq s \leq T, 0 \leq x \leq 1, \\ u(t,s,0) = u(t,s,1),\ \ u_x(t,s,0) = u_x(t,s,1),\ \ 0 \leq t,s \leq T, \end{cases} \tag{1.9}$$

is considered. Here $a(x)$, $\varphi(t,x), \psi(s,x)$ and $f(t,s,x)$ are given sufficiently smooth functions and $a(x) > 0,\ \delta > 0$ is a sufficiently large number.

Theorem 1.4. For the solution of the mixed problem (1.9) the following stability inequality holds

$$\max_{0\leq t,s\leq T} \| u(t,s) \|_{C^\beta[0,1]}$$

$$\leq M \left[\max_{0\leq t\leq T} \| \varphi(t) \|_{C^\beta[0,1]} + \max_{0\leq s\leq T} \| \psi(s) \|_{C^\beta[0,1]} + \max_{0\leq s,t\leq T} \| \ f(t,s) \|_{C^\beta[0,1]} \right],$$

here M is independent of $\beta, \varphi(t), \psi(s)$ and $f(t,s)$, where $0 \leq \beta \leq 1$.

Theorem 1.5. Let $\alpha \in (0, \frac{1}{2})$. Then for the solution of the mixed problem (1.9) the following coercive stability inequality hold

$$\max_{0\leq t,s\leq T} \| u_t(t,s) + u_s(t,s) \|_{C^{2\alpha}[0,1]} + \max_{0\leq t,s\leq T} \| u(t,s) \|_{C^{2+2\alpha}[0,1]}$$

$$\leq M(\alpha) \left[\max_{0\leq t\leq T} \| \varphi(t) \|_{C^{2+2\alpha}[0,1]} \right.$$

$$+\max_{0\le s\le T}\parallel \psi(s)\parallel_{C^{2+2\alpha}[0,1]}+\max_{0\le s,t\le T}\parallel\ f(t,s)\parallel_{C^{2\alpha}[0,1]}\Bigg],$$

where $M(\alpha)$ is independent of $\varphi(t),\psi(s)$ and $f(t,s)$.

Theorem 1.6. Let $\alpha\in(0,\frac{1}{2})$. Then for the solution of the mixed problem (1.9) the following coercive stability inequality hold

$$\max_{0\le t,s\le T}\parallel u_t(t,s)\parallel_{C^{2\alpha}[0,1]}+\max_{0\le t,s\le T}\parallel u_s(t,s)\parallel_{C^{2\alpha}[0,1]}$$

$$\le M(\alpha)\Bigg[\max_{0\le t\le T}\parallel \varphi(t)\parallel_{C^{2+2\alpha}[0,1]}+\max_{0\le t\le T}\parallel \varphi^{'}(t)\parallel_{C^{2\alpha}[0,1]}+\max_{0\le s\le T}\parallel \psi(s)\parallel_{C^{2+2\alpha}[0,1]}$$

$$+\max_{0\le s\le T}\parallel \psi^{'}(s)\parallel_{C^{2\alpha}[0,1]}+\max_{0\le s,t\le T}\parallel\ f_t(t,s)\parallel_{C^{2\alpha}[0,1]}+\max_{0\le s,t\le T}\parallel\ f_s(t,s)\parallel_{C^{2\alpha}[0,1]}\Bigg],$$

where $M(\alpha)$ is independent of $\varphi(t),\psi(s)$ and $f(t,s)$.

Second, let Ω be the unit open cube in the n-dimensional Euclidean space $\mathbb{R}^n\ \ (0<x_k<1,1\le k\le n)$ with boundary $S,\ \overline{\Omega}=\Omega\cup S$. In $[0,T]\times[0,T]\times\overline{\Omega}$ we consider the mixed boundary value problem for the multidimensional ultra-parabolic equation

$$\begin{cases}\frac{\partial u(t,s,x)}{\partial s}+\frac{\partial u(t,s,x)}{\partial t}-\sum\limits_{r=1}^{n}\alpha_r(x)\frac{\partial^2 u(t,s,x)}{\partial x_r^2}+\delta u(t,s,x)=f(t,s,x),\\ x=(x_1,\ldots,x_n)\in\overline{\Omega},0\le t,s\le T,\\ u(t,0,x)=\varphi(t,x),0\le t\le T,\ x\in\overline{\Omega},\\ u(0,s,x)=\psi(s,x),0\le s\le T,\ x\in\overline{\Omega},\\ u(t,s,x)=0,0\le t,s\le T,\ x\in S,\end{cases}\tag{1.10}$$

where $\alpha_r(x),\ f(t,s,x),\varphi(t,x),\psi(s,x)(t,s\in[0,T],\ x\in\overline{\Omega})$ are given smooth functions and $\alpha_r(x)>0,\ \delta>0$ is a sufficiently large number.

Theorem 1.7. For the solution of the mixed boundary value problem (1.10) the following stability inequality is valid

$$\max_{0\le t,s\le T}\parallel u(t,s)\parallel_{C^{\beta}_{01}(\overline{\Omega})}\le$$

$$M\Bigg[\max_{0\le t\le T}\parallel \varphi(t)\parallel_{C^{\beta}_{01}(\overline{\Omega})}+\max_{0\le s\le T}\parallel \psi(s)\parallel_{C^{\beta}_{01}(\overline{\Omega})}+\max_{0\le s,t\le T}\parallel\ f(t,s)\parallel_{C^{\beta}_{01}(\overline{\Omega})}\Bigg],$$

where M is independent of $\beta, \varphi(t,x), \psi(s,x)$ and $f(t,s,x)$. Here $\beta = (\beta_1, \ldots, \beta_n), \beta_k \in [0,1]$.

Third, we consider the initial-value problem on the range $\{0 \le t,s \le T,\ x \in \mathbb{R}^n\}$ for the 2m-th order multidimensional parabolic equation

$$\begin{cases} \frac{\partial u}{\partial s} + \frac{\partial u}{\partial t} + \sum\limits_{|r|=2m} a_r(x) \frac{\partial^{|\tau|} u}{\partial x_1^{r_1} \ldots \partial x_n^{r_n}} + \delta u(t,s,x) = f(t,s,x), \\ 0 < t,s < T,\ x, r \in \mathbb{R}^n, |r| = r_1 + \cdots + r_n, \\ u(t,0,x) = \varphi(t,x), 0 \le t \le T,\ x \in \mathbb{R}^n, \\ u(0,s,x) = \psi(s,x), 0 \le s \le T,\ x \in \mathbb{R}^n, \end{cases} \tag{1.11}$$

where $\alpha_r(x)$, $f(t,s,x), \varphi(t,x)$ and $\psi(s,x)$are given sufficiently smooth functions and $\delta > 0$ is the sufficiently large number .

Theorem 1.8. Let $\alpha \in (0, \frac{1}{2m})$. Then for the solution of the mixed boundary value problem (1.11) the following stability inequality is valid

$$\max_{0\le t,s\le T} \| u(t,s) \|_{C^{2m\alpha}(R^n)}$$

$$\le M \left[\max_{0<t<T} \| \varphi(t) \|_{C^{2m\alpha}(R^n)} \right.$$

$$\left. + \max_{0<t<T} \| \psi(s) \|_{C^{2m\alpha}(R^n)} + \max_{0<s,t<T} \| f(t,s) \|_{C^{2m\alpha}(R^n)} \right],$$

where M is independent of $\alpha, \varphi(t,x), \psi(s,x)$ and $f(t,s,x)$.

Theorem 1.9. Let $\alpha \in (0, \frac{1}{2m})$. Then for the solution of the mixed problem (1.11) the following coercive stability inequalities hold

$$\max_{0\le t,s\le T} \| u_t(t,s) + u_s(t,s) \|_{C^{2m\alpha}(R^n)} + \sum_{|r|=2m} \max_{0\le t,s\le T} \| a_r(x) \frac{\partial^{|r|} u(t,s)}{\partial x_1^{r_1} \ldots \partial x_n^{r_n}} \|_{C^{2m\alpha}(R^n)}$$

$$\le M(\alpha) \left[\sum_{|r|=2m} \max_{0\le t\le T} \| a_r(x) \frac{\partial^{|r|} \varphi(t)}{\partial x_1^{r_1} \ldots \partial x_n^{r_n}} \|_{C^{2m\alpha}(R^n)} \right.$$

$$\left. + \sum_{|r|=2m} \max_{0\le s\le T} \| a_r(x) \frac{\partial^{|r|} \psi(s)}{\partial x_1^{r_1} \ldots \partial x_n^{r_n}} \|_{C^{2m\alpha}(R^n)} \right.$$

$$+\max_{0\le s,t\le T}\| \ f(t,s)\ \|_{C^{2m\alpha}(R^n)}\Bigg],$$

where $M(\alpha)$ is independent of $\varphi(t,x), \psi(s,x)$ and $f(t,s,x)$.

Theorem 1.10. Let $\alpha \in (0, \frac{1}{2m})$. Then for the solution of the mixed problem (1.11) the following coercive stability inequality hold

$$\max_{0\le t,s\le T}\| \ u_t(t,s)\ \|_{C^{2m\alpha}(R^n)} + \max_{0\le t,s\le T}\| \ u_s(t,s)\ \|_{C^{2m\alpha}(R^n)}$$

$$\le M(\alpha)\Bigg[\sum_{|r|=2m}\max_{0\le t\le T}\| \ a_r(x)\frac{\partial^{|r|}\varphi(t)}{\partial x_1^{r_1}\dots\partial x_n^{r_n}}\ \|_{C^{2m\alpha}(R^n)}$$

$$+\max_{0\le t\le T}\| \ \varphi^{'}(t)\ \|_{C^{2m\alpha}(R^n)} + \sum_{|r|=2m}\max_{0\le s\le T}\| \ a_r(x)\frac{\partial^{|r|}\psi(s)}{\partial x_1^{r_1}\dots\partial x_n^{r_n}}\ \|_{C^{2m\alpha}(R^n)}$$

$$+\max_{0\le s\le T}\| \ \psi^{'}(s)\ \|_{C^{2m\alpha}(R^n)} + \max_{0\le s,t\le T}\| \ f_t(t,s)\ \|_{C^{2m\alpha}(R^n)}$$

$$+\max_{0\le s,t\le T}\| \ f_s(t,s)\ \|_{C^{2m\alpha}(R^n)}\Bigg],$$

where $M(\alpha)$ and $M_1(\alpha)$ are independent of $\varphi(t,x), \psi(s,x)$ and $f(t,s,x)$.

Third chapter studies the stability of first order of accuracy Rothe difference scheme

$$\begin{cases} \frac{u_{k,m}-u_{k-1,m}}{\tau}+\frac{u_{k-1,m}-u_{k-1,m-1}}{\tau}+Au_{k,m}=f_{k,m}\ , \\ f_{k,m}=f(t_k,s_m),\ t_k=k\tau,\ s_m=m\tau,\ 1\le k,m\le N,\ N\tau=1, \\ u_{0,m}=\psi_m,\ 0\le m\le N,\ u_{k,0}=\varphi_k,\ 0\le k\le N \end{cases} \tag{1.12}$$

for the approximate solution of initial boundary value problem (1.8). The following theorems on stability estimates for the solution of difference schemes (1.12) are established.

Theorem 1.11. Let τ be a sufficiently small number. Then for the solution of (1.12), we have the following stability inequality

$$\max_{1\le k,m\le N}\|u_{k,m}\|_E$$
$$\le M\left(\max_{0\le m\le N}\|\psi_m\|_E+\max_{0\le k\le N}\|\varphi_k\|_E+\max_{1\le k,m\le N}\|f_{k,m}\|_E\right),$$

where M is independent of ψ_m, φ_k, and $f_{k,m}$.

Theorem 1.12. Let τ be a sufficiently small number and $\psi_m, \varphi_k \in D(A)$. Then for the solution of (1.12), we have the following almost coercive inequality

$$\max_{1\le k,m\le N}\|\tfrac{u_{k,m}-u_{k-1,m-1}}{\tau}\|_E + \max_{1\le k,m\le N}\|Au_{k,m}\|_E$$

$$\le M\left(\max_{0\le m\le N}\|A\psi_m\|_E + \max_{0\le k\le N}\|A\varphi_k\|_E\right.$$

$$\left. + \min\left\{\ln\tfrac{1}{\tau},\ 1+|\ln\|A\|_{E\to E}|\right\}\max_{1\le k,m\le N}\|f_{k,m}\|\right),$$

where M is independent of τ, ψ_m, φ_k, and $f_{k,m}$.

Theorem 1.13. Let τ be a sufficiently small number and $\psi_m, \varphi_k \in D(A)$. Then for the solution of (1.12), we have the following almost coercive inequality

$$\max_{1\le k,m\le N}\|\tfrac{u_{k,m}-u_{k-1,m}}{\tau}\|_E + \max_{1\le k,m\le N}\|\tfrac{u_{k-1,m}-u_{k-1,m-1}}{\tau}\|_E$$

$$\le M\left(\max_{0\le m\le N}\|A\psi_m\|_E + \max_{1\le k,m\le N}\|\tfrac{\psi_m-\psi_{m-1}}{\tau}\|_E + \max_{1\le k,m\le N}\|\tfrac{f_{k,m}-f_{k,m-1}}{\tau}\|_E\right.$$

$$+\max_{0\le k\le N}\|A\varphi_k\|_E + \max_{1\le k,m\le N}\|\tfrac{\varphi_k-\varphi_{k-1}}{\tau}\|_E$$

$$\left.+\max_{1\le k,m\le N}\|\tfrac{f_{k,m}-f_{k-1,m}}{\tau}\|_E + \max_{1\le k,m\le N}\|f_{k,m}\|_E\right),$$

where M is independent of τ, ψ_m, φ_k, and $f_{k,m}$.

Theorem 1.14. Let τ be a sufficiently small number and $\psi_m, \varphi_k \in D(A)$. Then for the solution of (1.12), we have the following coercive inequality

$$\max_{1\le k,m\le N}\|\tfrac{u_{k,m}-u_{k-1,m-1}}{\tau}\|_{E_\alpha} + \max_{1\le k,m\le N}\|Au_{k,m}\|_{E_\alpha}$$

$$\le M\left(\max_{0\le m\le N}\|A\psi_m\|_{E_\alpha} + \max_{0\le k\le N}\|A\varphi_k\|_{E_\alpha} + \tfrac{1}{\alpha(1-\alpha)}\max_{1\le k,m\le N}\|f_{k,m}\|_{E_\alpha}\right),$$

where M is independent of τ, ψ_m, φ_k, and $f_{k,m}$.

Theorem 1.15. Let τ be a sufficiently small number and $\psi_m, \varphi_k \in D(A)$.

Then for the solution of (1.12), we have the following coercive inequality

$$\max_{1\le k,m\le N}\|\tfrac{u_{k,m}-u_{k-1,m}}{\tau}\|_{E_\alpha}+\max_{1\le k,m\le N}\|\tfrac{u_{k-1,m}-u_{k-1,m-1}}{\tau}\|_{E_\alpha}$$

$$\le M\left(\max_{0\le m\le N}\|A\psi_m\|_{E_\alpha}+\max_{1\le k,m\le N}\|\tfrac{\psi_m-\psi_{m-1}}{\tau}\|_{E_\alpha}\right.$$

$$+\max_{0\le k\le N}\|A\varphi_k\|_{E_\alpha}+\max_{1\le k,m\le N}\|\tfrac{\varphi_k-\varphi_{k-1}}{\tau}\|_{E_\alpha}\ +\max_{1\le k,m\le N}\|f_{k,m}\|_{E_\alpha}\ ,$$

$$\left.+\max_{1\le k,m\le N}\|\tfrac{f_{k,m}-f_{k,m-1}}{\tau}\|_{E_\alpha}+\max_{1\le k,m\le N}\|\tfrac{f_{k,m}-f_{k-1,m}}{\tau}\|_{E_\alpha}\right),$$

where M is independent of τ, ψ_m, φ_k, and $f_{k,m}$.

We consider three applications of these theorems. First, the mixed boundary value problem for one dimensional ultra-parabolic equation (1.9) is considered. The discretization of problem (1.9) is carried out in two steps. In the first step, let us define the grid sets

$$[0,1]_h=\{x=x_n: x_n=nh,\ 0\le n\le M,\ Mh=1\}.$$

We introduce the Banach spaces $C_h=C_h[0,1]_h, C_h^\beta=C^\beta[0,1]_h$ of grid functions $\varphi^h(x)=\{\varphi(h_1m_1,\cdots,h_nm_n)\}$ defined on $[0,1]_h$, equipped with the norms

$$\|\ \varphi^h\ \|_{C[0,1]_h}=\max_{x\in[0,1]_h}|\varphi^h(x)|,$$

$$\|\ \varphi^h\ \|^\beta_{C[0,1]_h}=\|\ \varphi^h\ \|_{C[0,1]_h}+\sup_{0\le x<x+h\le 1}\frac{|v^h(x+h)-v^h(x)|}{h^\beta}.$$

To the differential operator A generated by problem (1.9) we assign the difference operator A_h^x by the formula

$$A_h^xu^h(x)=\left\{-a(x)\frac{u_{n+1}-2u_n+u_{n-1}}{h^2}\right\}_1^M+\delta u^h(x)$$

acting in the space of grid functions $u^h(x)=\{u_n\}_0^M$ satisfying conditions $u_0^h=u_M^h$, $u_1^h-u_0^h=u_M^h-u_{M-1}^h$.

With the help of A_h^x we arrive at the initial boundary-value problem

$$\begin{cases} \frac{\partial u^h(t,s,x)}{\partial t} + \frac{\partial u^h(t,s,x)}{\partial s} + A_h^x u^h(t,s,x) = f^h(t,s,x), \\ \\ 0 < t, s < 1,\ x \in [0,l]_h, \\ \\ u^h(0,s,x) = \psi^h(s,x),\ 0 \le s \le 1, u^h(t,0,x) = \varphi^h(t,x), \\ \\ 0 \le t \le 1,\ x \in [0,l]_h, \end{cases} \tag{1.13}$$

for an infinite system of ordinary differential equations. In the second step, we replace problem (1.13) by difference scheme (1.12)

$$\begin{cases} \frac{u_{k,m}^h - u_{k-1,m-1}^h}{\tau} + A_h^x u_{k,m}^h = f_{k,m}^h(x), x \in [0,l]_h, \\ \\ f_{k,m}^h(x) = f^h(t_k, s_m, x), t_k = k\tau, s_m = m\tau, \\ \\ 1 \le k, m \le N,\ x \in [0,l]_h, \\ \\ u_{0,m}^h = \psi_m^h, 0 \le m \le N, u_{k,0}^h = \varphi_k^h, 0 \le k \le N. \end{cases} \tag{1.14}$$

Theorem 1.16. For the solution of difference scheme (1.14), we have the following stability inequality

$$\max_{1\le k,m\le N} \|u_{k,m}^h\|_{C[0,1]_h} \le M \left(\max_{0\le m\le N} \|\psi_m^h\|_{C[0,1]_h} + \max_{0\le k\le N} \|\varphi_k^h\|_{C[0,1]_h} \right.$$

$$\left. + \max_{1\le k,m\le N} \|f_{k,m}^h\|_{C[0,1]_h} \right),$$

where M is independent of τ, ψ_m^h, φ_k^h, and $f_{k,m}^h$.

Theorem 1.17. For the solution of difference scheme (1.14), we have the

following coercive stability inequalities

$$\max_{1\le k,m\le N}\|\tfrac{u^h_{k,m}-u^h_{k-1,m-1}}{\tau}\|_{C[0,1]_h}+\max_{1\le k,m\le N}\|h^{-2}\left\{u^{n+1}_{k,m}-2u^n_{k,m}+u^{n-1}_{k,m}\right\}^N_1\|_{C[0,1]_h}$$

$$\le M\left(\max_{0\le m\le N}\|h^{-2}\left\{\psi^{n+1}_m-2\psi^n_m+\psi^{n-1}_m\right\}^N_1\|_{C[0,1]_h}\right.$$

$$+\max_{0\le k\le N}\|h^{-2}\left\{\varphi^{n+1}_k-2\varphi^n_k+\varphi^{n-1}_k\right\}^N_1\|_{C[0,1]_h}$$

$$\left.+\ln\tfrac{1}{\tau+|h|}\max_{1\le k,m\le N}\|f^h_{k,m}\|_{C[0,1]_h}\right).$$

where M is independent of τ, ψ^h_m, φ^h_k, and $f^h_{k,m}$.

Theorem 1.18. For the solution of difference scheme (1.14), we have the following coercive stability inequalities

$$\max_{1\le k,m\le N}\|\tfrac{u^h_{k,m}-u^h_{k-1,m}}{\tau}\|_{C[0,1]_h}+\max_{1\le k,m\le N}\|\tfrac{u^h_{k-1,m}-u^h_{k-1,m-1}}{\tau}\|_{C[0,1]_h}$$

$$\le M\left(\max_{0\le m\le N}\|h^{-2}\left\{\psi^{n+1}_m-2\psi^n_m+\psi^{n-1}_m\right\}^N_1\|_{C[0,1]_h}+\max_{1\le k,m\le N}\left\|f^h_{k,m}\right\|\right.$$

$$+\max_{1\le m\le N}\|\tfrac{\psi^h_m-\psi^h_{m-1}}{\tau}\|_{C[0,1]_h}+\max_{1\le k,m\le N}\|\tfrac{f^h_{k,m}-f^h_{k-1,m}}{\tau}\|_{C[0,1]_h}$$

$$+\max_{0\le k\le N}\|h^{-2}\left\{\varphi^{n+1}_k-2\varphi^n_k+\varphi^{n-1}_k\right\}^N_1\|_{C[0,1]_h}$$

$$\left.+\max_{1\le k,m\le N}\|\tfrac{f^h_{k,m}-f^h_{m-1}}{\tau}\|_{C[0,1]_h}+\max_{0\le k\le N}\|\tfrac{\varphi^h_k-\varphi^h_{k-1}}{\tau}\|_{C[0,1]_h}\right)$$

where M is independent of τ, ψ^h_m, φ^h_k, and $f^h_{k,m}$.

Secondly, the mixed boundary value problem for the multidimensional ultra-parabolic equation (1.10) is considered. The discretization of problem (1.10) is carried out in two steps. In the first step, let us define the grid sets

$$\widetilde{\Omega}_h=\{x=x_r=(h_1r_1,\cdots,h_nr_n),r=(r_1,\cdots,r_n),$$

$$0\le r_j\le K_j,h_jK_j=L,j=1,\cdots,n\},$$

$$\Omega_h=\widetilde{\Omega}_h\cap\Omega,S_h=\widetilde{\Omega}_h\cap S.$$

We introduce the Banach spaces $C_h = C_h(\widetilde{\Omega}_h), C_h^\beta = C_{01}^\beta(\widetilde{\Omega}_h)$ of grid functions $\varphi^h(x) = \{\varphi(h_1 m_1, \cdots, h_n m_n)\}$ defined on $\widetilde{\Omega}_h$, equipped with the norms

$$\| \varphi^h \|_{C(\overline{\Omega}_h)} = \max_{x \in \overline{\Omega}_h} |\varphi^h(x)|,$$

$$\| \varphi^h \|_{C_{01}^\beta(\overline{\Omega}_h)} = \| \varphi^h \|_{C(\overline{\Omega_h})}$$

$$+ \sup_{\substack{0 \le x_k < x_k + h_k \le 1 \\ 1 \le k \le n}} |\varphi^h(x_1, \cdots, x_n) - \varphi^h(x_1 + h_1, \cdots, x_n + h_n)|$$

$$\times \prod_{k=1}^{n} h_k^{-\beta_k} x_k^{\beta_k} (1 - x_k - h_k)^{\beta_k}.$$

To the differential operator A generated by problem (1.10) we assign the difference operator A_h^x by the formula

$$A_h^x u_x^h = -\sum_{r=1}^{n} a_r(x) (u_{\underline{\ }}^h)_{x_r, j_r}$$

acting in the space of grid functions $u^h(x)$, satisfying the condition $u^h(x) = 0$ for all $x \in S_h$.

With the help of A_h^x we arrive at the initial boundary-value problem

$$\begin{cases} \frac{\partial u^h(t,s,x)}{\partial t} + \frac{\partial u^h(t,s,x)}{\partial s} + A_h^x u^h(t,s,x) = f^h(t,s,x), \\ 0 < t, s < 1, \ x \in \Omega_h, \\ u^h(0,s,x) = \psi^h(s,x), \ 0 \le s \le 1, u^h(t,0,x) = \varphi^h(t,x), \\ 0 \le t \le 1, \ x \in \widetilde{\Omega}_h \end{cases} \tag{1.15}$$

for an infinite system of ordinary differential equations.

In the second step, we replace problem (1.15) by difference scheme (1.15)

$$\begin{cases} \frac{u_{k,m}^h - u_{k-1,m-1}^h}{\tau} + A_h^x u_{k,m}^h = f_{k,m}^h(x), x \in \Omega_h, \\ f_{k,m}^h(x) = f^h(t_k, s_{m,}, x), t_k = k\tau, s_m = m\tau, \ 1 \le k, m \le N, \ x \in \widetilde{\Omega}_h, \ , \\ u_{0,m}^h = \psi_m^h, 0 \le m \le N, u_{k,0}^h = \varphi_k^h, 0 \le k \le N \end{cases} \tag{1.16}$$

Theorem 1.19. For the solution of difference scheme (1.16), we have the following stability inequality

$$\max_{1\le k,m\le N}\|u^h_{k,m}\|_{C(\widetilde{\Omega}_h)}$$

$$\le M\left(\max_{0\le m\le N}\|\psi^h_m\|_{C(\widetilde{\Omega}_h)}+\max_{0\le k\le N}\|\varphi^h_k\|_{C(\widetilde{\Omega}_h)}+\max_{1\le k,m\le N}\|f^h_{k,m}\|_{C(\widetilde{\Omega}_h)}\right),$$

where M is independent of τ, ψ^h_m, φ^h_k, and $f^h_{k,m}$.

Theorem 1.20. For the solution of difference scheme (1.16), we have the following coercive stability inequalities

$$\max_{1\le k,m\le N}\|\tfrac{u^h_{k,m}-u^h_{k-1,m-1}}{\tau}\|_{C(\widetilde{\Omega}_h)}+\max_{1\le k,m\le r}\sum_{r=1}^{n}\|u^h_{(\bar{x}_r x_r,\, j_r)\; k,m}\|_{C(\widetilde{\Omega}_h)}$$

$$\le M\ln\tfrac{1}{|h|}\left(\max_{1\le m\le N}\sum_{r=1}^{n}\|\psi^h_{(\bar{x}_r x_r,\, j_r)\; m}\|_{C(\widetilde{\Omega}_h)}+\max_{0\le k\le N}\sum_{r=1}^{n}\|\varphi^h_{(\bar{x}_r x_r,\, j_r)\; k}\|_{C(\widetilde{\Omega}_h)}\right)$$

$$+C\ln\tfrac{1}{\tau+|h|}\max_{1\le k,\, m\le N}\|f^{\,h}_{k,m}\|_{C(\widetilde{\Omega}_h)}.$$

where M is independent of τ, ψ^h_m, φ^h_k, and $f^h_{k,m}$.

Theorem 1.21. For the solution of difference scheme (1.16), we have the following coercive stability inequalities

$$\max_{1\le k,m\le N}\|\tfrac{u^h_{k,m}-u^h_{k-1,m}}{\tau}\|_{C(\widetilde{\Omega}_h)}+\max_{1\le k,m\le N}\|\tfrac{u^h_{k-1,m}-u^h_{k-1,m-1}}{\tau}\|_{C(\widetilde{\Omega}_h)}$$

$$\le M\left(\max_{1\le m\le N}\sum_{r=1}^{n}\|\psi^h_{(\bar{x}_r x_r,\, j_r)\; m}\|_{C(\widetilde{\Omega}_h)}+\max_{1\le k,m\le N}\left\|f^h_{k,m}\right\|_{C(\widetilde{\Omega}_h)}\right.$$

$$+\max_{1\le m\le N}\|\tfrac{\psi^h_m-\psi^h_{m-1}}{\tau}\|_{C(\widetilde{\Omega}_h)}+\max_{1\le k,m\le N}\|\tfrac{f^h_{k,m}-f^h_{k-1,m}}{\tau}\|_{C(\widetilde{\Omega}_h)}$$

$$+\max_{0\le k\le N}\sum_{r=1}^{n}\|\varphi^h_{(\bar{x}_r x_r,\, j_r)\; k}\|_{C(\widetilde{\Omega}_h)}$$

$$\left.+\max_{1\le k,m\le N}\|\tfrac{f^h_{k,m}-f^h_{m-1}}{\tau}\|_{C(\widetilde{\Omega}_h)}+\max_{0\le k\le N}\|\tfrac{\varphi^h_k-\varphi^h_{k-1}}{\tau}\|_{C(\widetilde{\Omega}_h)}\right)$$

where M is independent of τ, ψ^h_m, φ^h_k, and $f^h_{k,m}$.

Third, the mixed boundary value problem for the multidimensional ultra-parabolic equation (1.11) is considered. The discretization of problem (1.11) is

carried out in two steps. In the first step let us give the difference operator A_h^x by the formula

$$A_h^x u_x^h = \sum_{2m \le |r| \le S} b_r^x D_h^r u_x^h + \delta u_x^h. \tag{1.17}$$

The coefficients are chosen in such a way that the operator A_h^x approximates in a specified way the operator

$$\sum_{|r|=2m} a_r(x) \frac{\partial^{|r|}}{\partial x_1^{r_1} ... \partial x_n^{r_n}} + \delta.$$

We shall assume that for $|\xi_k h| \le \pi$ the symbol $A(\xi h, h)$ of the operator $A_h^x - \delta$ satisfies the inequalities

$$(-1)^m A^x(\xi h, h) \ge M_1 |\xi|^{2m}, |\arg A^x(\xi h, h)| \le \phi < \phi_0 < \frac{\pi}{2}.$$

With the help of A_h^x we arrive at the boundary value problem

$$\begin{cases} \frac{\partial u^h(t,s,x)}{\partial t} + \frac{\partial u^h(t,s,x)}{\partial s} + A_h^x u^h(y,x) = f^h(t,s,x), \\ 0 < t, s < 1, x \in \mathbb{R}_h^n \\ u^h(t,0,x) = \varphi^h(t,x), 0 \le t \le 1, u^h(0,s,x) = \psi^h(s,x), \\ 0 \le s \le 1, x \in \mathbb{R}_h^n \end{cases} \tag{1.18}$$

for an infinite system of ordinary differential equations.

In the second step we replace problem (1.18) by the difference scheme

$$\begin{cases} \frac{u_{k,m}^h - u_{k-1,m-1}^h}{\tau} + A_h^x u_{k,m}^h = f_{k,m}^h(x), x \in \mathbb{R}_h^n, \\ f_{k,m}^h(x) = f^h(t_k, s_m, x), t_k = k\tau, s_m = m\tau, \ 1 \le k, m \le N, \ x \in \mathbb{R}_h^n, \ , \\ u_{0,m}^h = \psi_m^h, 0 \le m \le N, u_{k,0}^h = \varphi_k^h, 0 \le k \le N \end{cases} \tag{1.19}$$

Fourth chapter studies the stability of second order of accuracy difference

schemes

$$\begin{cases} \frac{u_{k,m}-u_{k-1,m}}{\tau}+\frac{u_{k-1,m}-u_{k-1,m-1}}{\tau}+\frac{1}{2}A(u_{k,m}+u_{k-1,m-1})=f_{k,m}\ , \\ f_{k,m}=f(t_k-\frac{\tau}{2},s_m-\frac{\tau}{2}),t_k=k\tau,\ s_m=m\tau,\ 1\le k,m\le N,\ N\tau=1,\ \ , \\ u_{0,m}=\psi_m,\ 0\le m\le N,\ u_{k,0}=\varphi_k,\ 0\le k\le N \end{cases} \tag{1.20}$$

for the approximate solution of initial boundary value problem(1.8). The following theorems on stability estimates for the solution of difference scheme (1.20) is established.

Theorem 1.22. Let τ be a sufficiently small number. Then for the solution of (1.20), we have the following stability inequality

$$\max_{1\le k,m\le N}\|u_{k,m}\|_E$$

$$\le M\left(\max_{0\le m\le N}\|\psi_m\|_E+\max_{0\le k\le N}\|\varphi_k\|_E+\max_{1\le k,m\le N}\|f_{k,m}\|_E\right),$$

where M is independent of ψ_m, φ_k, and $f_{k,m}$.

Theorem 1.23. Let τ be a sufficiently small number and $\psi_m,\varphi_k\in D(A)$. Then for the solution of (1.20), we have the following almost coercive inequality

$$\max_{1\le k,m\le N}\|\tfrac{u_{k,m}+u_{k-1,m-1}}{2}\|_E+\max_{1\le k,m\le N}\|\tfrac{1}{2}A(u_{k,m}+u_{k-1,m-1})\|_E$$

$$\le M\left(\max_{0\le m\le N}\|A\psi_m\|_E+\max_{0\le k\le N}\|A\varphi_k\|_E\right.$$

$$\left.+\min\left\{\ln\tfrac{1}{\tau},\ 1+|\ln\|A\|_{E\to E}|\right\}\max_{1\le k,m\le N}\|f_{k,m}\|\right),$$

where M is independent of τ, ψ_m, φ_k, and $f_{k,m}$.

Theorem 1.24. Let τ be a sufficiently small number and $\psi_m,\varphi_k\in D(A)$.

Then for the solution of (1.20), we have the following almost coercive inequality

$$\max_{1\le k,m\le N}\|\tfrac{u_{k,m}-u_{k-1,m}}{\tau}\|_E+\max_{1\le k,m\le N}\|\tfrac{u_{k-1,m}-u_{k-1,m-1}}{\tau}\|_E$$

$$\le M\left(\max_{0\le m\le N}\|A\psi_m\|_E+\max_{1\le k,m\le N}\|\tfrac{\psi_m-\psi_{m-1}}{\tau}\|_E\ +\max_{1\le k,m\le N}\|\tfrac{f_{k,m}-f_{k,m-1}}{\tau}\|_E\right.$$

$$+\max_{0\le k\le N}\|A\varphi_k\|_E+\max_{1\le k,m\le N}\|\tfrac{\varphi_k-\varphi_{k-1}}{\tau}\|_E$$

$$\left.+\max_{1\le k,m\le N}\|\tfrac{f_{k,m}-f_{k-1,m}}{\tau}\|_E+\max_{1\le k,m\le N}\|f_{k,m}\|_E\right),$$

where M is independent of τ, ψ_m, φ_k, and $f_{k,m}$.

Theorem 1.25. Let τ be a sufficiently small number and $\psi_m,\varphi_k\in D(A)$. Then for the solution of (1.20), we have the following coercive inequality

$$\max_{1\le k,m\le N}\|\frac{u_{k,m}+u_{k-1,m-1}}{2}\|_{E_\alpha}+\max_{1\le k,m\le N}\|\frac{1}{2}A(u_{k,m}+u_{k-1,m-1})\|_{E_\alpha}$$

$$\le M\left(\max_{0\le m\le N}\|A\psi_m\|_{E_\alpha}+\max_{0\le k\le N}\|A\varphi_k\|_{E_\alpha}+\ \frac{1}{\alpha(1-\alpha)}\max_{1\le k,m\le N}\|f_{k,m}\|_{E_\alpha}\right),$$

where M is independent of τ, ψ_m, φ_k, and $f_{k,m}$.

Theorem 1.26. Let τ be a sufficiently small number and $\psi_m,\varphi_k\in D(A)$. Then for the solution of (1.20), we have the following coercive inequality

$$\max_{1\le k,m\le N}\|\tfrac{u_{k,m}-u_{k-1,m}}{\tau}\|_{E_\alpha}+\max_{1\le k,m\le N}\|\tfrac{u_{k-1,m}-u_{k-1,m-1}}{\tau}\|_{E_\alpha}$$

$$\le M\left(\max_{0\le m\le N}\|A\psi_m\|_{E_\alpha}+\max_{1\le k,m\le N}\|\tfrac{\psi_m-\psi_{m-1}}{\tau}\|_{E_\alpha}\ +\max_{1\le k,m\le N}\|\tfrac{f_{k,m}-f_{k,m-1}}{\tau}\|_{E_\alpha}\right.$$

$$+\max_{0\le k\le N}\|A\varphi_k\|_{E_\alpha}+\max_{1\le k,m\le N}\|\tfrac{\varphi_k-\varphi_{k-1}}{\tau}\|_{E_\alpha}$$

$$\left.+\max_{1\le k,m\le N}\|\tfrac{f_{k,m}-f_{k-1,m}}{\tau}\|_{E_\alpha}+\max_{1\le k,m\le N}\|f_{k,m}\|_{E_\alpha}\right),$$

where M is independent of τ, ψ_m, φ_k, and $f_{k,m}$.

We consider three applications of these theorems. First, the mixed boundary value problem for one dimensional ultra-parabolic equation (1.9) is considered. We

replace problem (1.13) by difference scheme (1.20)

$$\begin{cases} \frac{u^h_{k,m}-u^h_{k-1,m-1}}{\tau}+\frac{1}{2}A^x_h(u^h_{k,m}+u^h_{k-1,m-1})=f^h_{k,m}(x), x\in[0,l]_h, \\ f^h_{k,m}(x)=f^h(t_k-\frac{\tau}{2},s_m-\frac{\tau}{2},x), t_k=k\tau, s_m=m\tau, \\ 1\le k,m\le N,\ x\in[0,l]_h, \\ u^h_{0,m}=\psi^h_m, 0\le m\le N, u^h_{k,0}=\varphi^h_k, 0\le k\le N. \end{cases} \tag{1.21}$$

Theorem 1.27. For the solution of difference scheme (1.21), we have the following stability inequality

$$\max_{1\le k,m\le N}\|u^h_{k,m}\|_{C[0,1]_h}\le M\left(\max_{0\le m\le N}\|\psi^h_m\|_{C[0,1]_h}+\max_{0\le k\le N}\|\varphi^h_k\|_{C[0,1]_h}\right.$$

$$\left.+\max_{1\le k,m\le N}\|f^h_{k,m}\|_{C[0,1]_h}\right),$$

where M is independent of τ, ψ^h_m, φ^h_k, and $f^h_{k,m}$.

Theorem 1.28. For the solution of difference scheme (1.21), we have the following coercive stability inequalities

$$\max_{1\le k,m\le N}\|\tfrac{u^h_{k,m}-u^h_{k-1,m-1}}{\tau}\|_{C[0,1]_h}$$

$$+\max_{1\le k,m\le N}\|h^{-2}\left[\left\{u^{n+1}_{k,m}-2u^n_{k,m}+u^{n-1}_{k,m}\right\}^N_1\right.$$

$$\left.+\left\{u^{n+1}_{k-1,m-1}-2u^n_{k-1,m-1}+u^{n-1}_{k-1,m-1}\right\}\right]^N_1\|_{C[0,1]_h}$$

$$\le M\left(\max_{0\le m\le N}\|h^{-2}\left\{\psi^{n+1}_m-2\psi^n_m+\psi^{n-1}_m\right\}^N_1\|_{C[0,1]_h}\right.$$

$$+\max_{0\le k\le N}\|h^{-2}\left\{\varphi^{n+1}_k-2\varphi^n_k+\varphi^{n-1}_k\right\}^N_1\|_{C[0,1]_h}$$

$$\left.+\ln\tfrac{1}{\tau+|h|}\max_{1\le k,m\le N}\|f^h_{k,m}\|_{C[0,1]_h}\right).$$

where M is independent of τ, ψ^h_m, φ^h_k, and $f^h_{k,m}$.

Theorem 1.29. For the solution of difference scheme (1.21), we have the following coercive stability inequalities

$$\max_{1\le k,m\le N}\|\tfrac{u^h_{k,m}-u^h_{k-1,m}}{\tau}\|_{C[0,1]_h}+\max_{1\le k,m\le N}\|\tfrac{u^h_{k-1,m}-u^h_{k-1,m-1}}{\tau}\|_{C[0,1]_h}$$

$$\le M\left(\max_{0\le m\le N}\|h^{-2}\left\{\psi^{n+1}_m-2\psi^n_m+\psi^{n-1}_m\right\}_1^N\|_{C[0,1]_h}+\max_{1\le k,m\le N}\left\|f^h_{k,m}\right\|\right.$$

$$+\max_{1\le m\le N}\|\tfrac{\psi^h_m-\psi^h_{m-1}}{\tau}\|_{C[0,1]_h}+\max_{1\le k,m\le N}\|\tfrac{f^h_{k,m}-f^h_{k-1,m}}{\tau}\|_{C[0,1]_h}$$

$$+\max_{0\le k\le N}\|h^{-2}\left\{\varphi^{n+1}_k-2\varphi^n_k+\varphi^{n-1}_k\right\}_1^N\|_{C[0,1]_h}$$

$$\left.+\max_{1\le k,m\le N}\|\tfrac{f^h_{k,m}-f^h_{m-1}}{\tau}\|_{C[0,1]_h}+\max_{0\le k\le N}\|\tfrac{\varphi^h_k-\varphi^h_{k-1}}{\tau}\|_{C[0,1]_h}\right)$$

where M is independent of τ, ψ^h_m, φ^h_k, and $f^h_{k,m}$.

Second, the mixed boundary value problem (1.10) is considered. We replace problem (1.15) by difference scheme (1.20)

$$\begin{cases}\frac{u^h_{k,m}-u^h_{k-1,m-1}}{\tau}+\frac{1}{2}A^x_h(u^h_{k,m}+u^h_{k-1,m-1})=f^h_{k,m}(x),x\in\Omega_h,\\ f^h_{k,m}(x)=f^h(t_k-\frac{\tau}{2},s_m-\frac{\tau}{2},x),t_k=k\tau,s_m=m\tau,\\ 1\le k,m\le N,\ x\in\widetilde{\Omega}_h,\\ u^h_{0,m}=\psi^h_m,0\le m\le N,u^h_{k,0}=\varphi^h_k,0\le k\le N\end{cases},\tag{1.22}$$

Theorem 1.30. For the solution of difference scheme (1.22), we have the following stability inequality

$$\max_{1\le k,m\le N}\|u^h_{k,m}\|_{C(\widetilde{\Omega}_h)}$$

$$\le M\left(\max_{0\le m\le N}\|\psi^h_m\|_{C(\widetilde{\Omega}_h)}+\max_{0\le k\le N}\|\varphi^h_k\|_{C(\widetilde{\Omega}_h)}+\max_{1\le k,m\le N}\|f^h_{k,m}\|_{C(\widetilde{\Omega}_h)}\right),$$

where M is independent of τ, ψ^h_m, φ^h_k, and $f^h_{k,m}$.

Theorem 1.31. For the solution of difference scheme (1.22), we have the

following coercive stability inequalities

$$\max_{1\le k,m\le N}\|\tfrac{u^h_{k,m}-u^h_{k-1,m-1}}{\tau}\|_{C(\widetilde{\Omega}_h)}$$

$$+\max_{1\le k,m\le r}\sum_{r=1}^{n}\tfrac{1}{2}\|u^h_{(\bar{x}_r x_r,\, j_r)\ k,m}+u^h_{(\bar{x}_r x_r,\, j_r)\ k-1,m-1}\|_{C(\widetilde{\Omega}_h)}$$

$$\le M\ln\tfrac{1}{|h|}\left(\max_{1\le m\le N}\sum_{r=1}^{n}\|\psi^h_{(\bar{x}_r x_r,\, j_r)\ m}\|_{C(\widetilde{\Omega}_h)}+\max_{0\le k\le N}\sum_{r=1}^{n}\|\varphi^h_{(\bar{x}_r x_r,\, j_r)\ k}\|_{C(\widetilde{\Omega}_h)}\right)$$

$$+M\ln\tfrac{1}{\tau+|h|}\max_{1\le k,\, m\le N}\|f^{\,h}_{k,m}\|_{C(\widetilde{\Omega}_h)}.$$

where M is independent of τ, ψ^h_m, φ^h_k, and $f^h_{k,m}$.

Theorem 1.32. For the solution of difference scheme (1.22), we have the following coercive stability inequalities

$$\max_{1\le k,m\le N}\|\tfrac{u^h_{k,m}-u^h_{k-1,m}}{\tau}\|_{C(\widetilde{\Omega}_h)}+\max_{1\le k,m\le N}\|\tfrac{u^h_{k-1,m}-u^h_{k-1,m-1}}{\tau}\|_{C(\widetilde{\Omega}_h)}$$

$$\le M\left(\max_{1\le m\le N}\sum_{r=1}^{n}\|\psi^h_{(\bar{x}_r x_r,\, j_r)\ m}\|_{C(\widetilde{\Omega}_h)}+\max_{1\le k,m\le N}\left\|f^h_{k,m}\right\|_{C(\widetilde{\Omega}_h)}\right.$$

$$+\max_{1\le m\le N}\|\tfrac{\psi^h_m-\psi^h_{m-1}}{\tau}\|_{C(\widetilde{\Omega}_h)}+\max_{1\le k,m\le N}\|\tfrac{f^h_{k,m}-f^h_{k-1,m}}{\tau}\|_{C(\widetilde{\Omega}_h)}$$

$$+\max_{0\le k\le N}\sum_{r=1}^{n}\|\varphi^h_{(\bar{x}_r x_r,\, j_r)\ k}\|_{C(\widetilde{\Omega}_h)}$$

$$\left.+\max_{1\le k,m\le N}\|\tfrac{f^h_{k,m}-f^h_{k-1,m}}{\tau}\|_{C(\widetilde{\Omega}_h)}+\max_{0\le k\le N}\|\tfrac{\varphi^h_k-\varphi^h_{k-1}}{\tau}\|_{C(\widetilde{\Omega}_h)}\right)$$

where M is independent of τ, ψ^h_m, φ^h_k, and $f^h_{k,m}$.

Moreover, we consider the r-modified Crank-Nicholson difference scheme

$$\begin{cases} \frac{u_{k,m}-u_{k-1,m-1}}{\tau} + Au_{k,m} = f_{k,m}\ ,\ \ 1 \le k \le r \ \ or \ \ 1 \le m \le r, \\ \\ \frac{u_{k,m}-u_{k-1,m-1}}{\tau} + \frac{A}{2}(u_{k,m} + u_{k-1,m-1)} = f_{k,m}\ ,\ \ r+1 \le k,m \le N, \\ \\ f_{k,m} = f(t_k, s_m),\ t_k = k\tau,\ \ s_m = m\tau,\ \ 1 \le k,m \le N,\ \ N\tau = 1, \\ \\ u_{0,m} = \psi_m, \psi_m = \psi(s_m),\ \ 0 \le m \le N, \\ \\ u_{k,0} = \varphi_k, \varphi_k = \varphi(t_k),\ \ 0 \le k \le N \end{cases} \tag{1.23}$$

for the approximate solution of initial boundary value problem (1.8).

The following theorems on stability estimates for the solution of difference scheme (1.23) are established.

Theorem 1.33. Let τ be a sufficiently small number. Then for the solution of problem (1.23), we have the following stability inequalities

$$\max_{1\le k,m\le N} \|u_{k,m}\|_E \le M \left(\max_{0\le m\le N} \|\psi_m\|_E + \max_{0\le k\le N} \|\varphi_k\|_E + \max_{1\le k,m\le N} \|f_{k,m}\|_E \right),$$

where M is independent of τ, ψ_m, φ_k, and $f_{k,m}$.

Theorem 1.34. Let τ be a sufficiently small number and $\psi_m, \varphi_k \in D(A)$. Then for the solution of (1.23), we have the following almost coercive inequality

$$\begin{aligned} &\max_{1\le k,m\le N} \|\tfrac{u_{k,m}-u_{k-1,m-1}}{\tau}\|_E + \max_{1\le k,m\le r} \|Au_{k,m}\|_E \\ &+ \max_{r+1\le k,m\le N} \|\tfrac{A}{2}(u_{k,m} + u_{k-1,m-1})\|_E \\ &\le M \left(\max_{0\le m\le N} \|A\psi_m\|_E + \max_{0\le k\le N} \|A\varphi_k\|_E \right. \\ &\left. + \min\left\{\ln \tfrac{1}{\tau},\ 1 + |\ln \|A\|_{E\to E}|\right\} \max_{1\le k,m\le N} \|f_{k,m}\| \right), \end{aligned}$$

where M is independent of τ, ψ_m, φ_k, and $f_{k,m}$.

Theorem 1.35. Let τ be a sufficiently small number and $\psi_m, \varphi_k \in D(A)$. Then for the solution of (1.23), we have the following almost coercive inequality

$$\max_{1\le k,m\le N}\|\tfrac{u_{k,m}-u_{k-1,m}}{\tau}\|_E+\max_{1\le k,m\le N}\|\tfrac{u_{k-1,m}-u_{k-1,m-1}}{\tau}\|_E$$

$$\le M\left(\max_{0\le m\le N}\|A\psi_m\|_E+\max_{1\le k,m\le N}\|\tfrac{\psi_m-\psi_{m-1}}{\tau}\|_E\ +\max_{1\le k,m\le N}\|\tfrac{f_{k,m}-f_{k,m-1}}{\tau}\|_E\right.$$

$$+\max_{0\le k\le N}\|A\varphi_k\|_E+\max_{1\le k,m\le N}\|\tfrac{\varphi_k-\varphi_{k-1}}{\tau}\|_E$$

$$\left.+\max_{1\le k,m\le N}\|\tfrac{f_{k,m}-f_{k-1,m}}{\tau}\|_E+\max_{1\le k,m\le N}\|f_{k,m}\|_E\right),$$

where M is independent of τ, ψ_m, φ_k, and $f_{k,m}$.

Theorem 1.36. Let τ be a sufficiently small number and $\psi_m,\varphi_k\in D(A)$. Then for the solution of (1.23), we have the following coercive inequality

$$\max_{1\le k,m\le N}\|\tfrac{u_{k,m}-u_{k-1,m-1}}{\tau}\|_{E_\alpha}+\max_{1\le k,m\le r}\|Au_{k,m}\|_{E_\alpha}+$$

$$\max_{r+1\le k,m\le N}\|\tfrac{A}{2}(u_{k,m}+u_{k-1,m-1})\|_{E_\alpha}$$

$$\le M\left(\max_{0\le m\le N}\|A\psi_m\|_{E_\alpha}+\max_{0\le k\le N}\|A\varphi_k\|_{E_\alpha}+\ \tfrac{1}{\alpha(1-\alpha)}\max_{1\le k,m\le N}\|f_{k,m}\|_{E_\alpha}\right),$$

where M is independent of τ, ψ_m, φ_k, and $f_{k,m}$.

Theorem 1.37. Let τ be a sufficiently small number and $\psi_m,\varphi_k\in D(A)$. Then for the solution of (1.23), we have the following coercive inequality

$$\max_{1\le k,m\le N}\|\tfrac{u_{k,m}-u_{k-1,m}}{\tau}\|_{E_\alpha}+\max_{1\le k,m\le N}\|\tfrac{u_{k-1,m}-u_{k-1,m-1}}{\tau}\|_{E_\alpha}$$

$$\le M\left(\max_{0\le m\le N}\|A\psi_m\|_{E_\alpha}+\max_{1\le k,m\le N}\|\tfrac{\psi_m-\psi_{m-1}}{\tau}\|_{E_\alpha}\ +\max_{1\le k,m\le N}\|\tfrac{f_{k,m}-f_{k,m-1}}{\tau}\|_{E_\alpha}\right.$$

$$+\max_{0\le k\le N}\|A\varphi_k\|_{E_\alpha}+\max_{1\le k,m\le N}\|\tfrac{\varphi_k-\varphi_{k-1}}{\tau}\|_{E_\alpha}$$

$$\left.+\max_{1\le k,m\le N}\|\tfrac{f_{k,m}-f_{k-1,m}}{\tau}\|_{E_\alpha}+\max_{1\le k,m\le N}\|f_{k,m}\|_{E_\alpha}\right),$$

where M is independent of τ, ψ_m, φ_k, and $f_{k,m}$.

First, we consider the application of these theorems to problem (1.9). We replace problem (1.13) by difference scheme (1.23)

$$\begin{cases} \frac{u^h_{k,m}-u^h_{k-1,m-1}}{\tau} + A^x_h u^h_{k,m} = f^h_{k,m}(x), x \in [0,l]_h, 1 \le k,m \le r, \\ \\ \frac{u^h_{k,m}-u^h_{k-1,m-1}}{\tau} + \frac{1}{2}A^x_h(u_{k,m}+u_{k-1,m-1}) = f^h_{k,m}(x), x \in [0,l]_h, \\ \\ r+1 \le k,m \le N, \\ \\ f^h_{k,m}(x) = f^h(t_k - \frac{\tau}{2}, s_m - \frac{\tau}{2}, x), t_k = k\tau, s_m = m\tau, \\ \\ 1 \le k,m \le N,\ x \in [0,l]_h, \\ \\ u^h_{0,m} = \psi^h_m, 0 \le m \le N, u^h_{k,0} = \varphi^h_k, 0 \le k \le N. \end{cases} \tag{1.24}$$

Theorem 1.38. For the solution of difference scheme (1.24), we have the following stability inequality

$$\max_{1\le k,m\le N} \|u^h_{k,m}\|_{C[0,1]_h} \le M \left(\max_{0\le m\le N} \|\psi^h_m\|_{C[0,1]_h} + \max_{0\le k\le N} \|\varphi^h_k\|_{C[0,1]_h} \right.$$

$$\left. + \max_{1\le k,m\le N} \|f^h_{k,m}\|_{C[0,1]_h} \right),$$

where M is independent of τ, ψ^h_m, φ^h_k, and $f^h_{k,m}$.

Theorem 1.39. For the solution of difference scheme (1.24), we have the

following coercive stability inequalities

$$\max_{1\le k,m\le N}\|\tfrac{u^h_{k,m}-u^h_{k-1,m-1}}{\tau}\|_{C[0,1]_h}$$

$$+\max_{1\le k,m\le N}\|h^{-2}\left\{u^{n+1}_{k,m}-2u^n_{k,m}+u^{n-1}_{k,m}\right\}^N_1\|_{C[0,1]_h}$$

$$+\tfrac{1}{2}\max_{1\le k,m\le N}\|h^{-2}\Big[\left\{u^{n+1}_{k,m}-2u^n_{k,m}+u^{n-1}_{k,m}\right\}^N_1$$

$$+\left\{u^{n+1}_{k-1,m-1}-2u^n_{k-1,m-1}+u^{n-1}_{k-1,m-1}\right\}\Big]^N_1\|_{C[0,1]_h}$$

$$\le M\left(ln\tfrac{1}{|h|}\left(\max_{0\le m\le N}\|h^{-2}\left\{\psi^{n+1}_m-2\psi^n_m+\psi^{n-1}_m\right\}^N_1\|_{C[0,1]_h}\right.\right.$$

$$\left.+\max_{0\le k\le N}\|h^{-2}\left\{\varphi^{n+1}_k-2\varphi^n_k+\varphi^{n-1}_k\right\}^N_1\|_{C[0,1]_h}\right)$$

$$\left.+\ln\tfrac{1}{\tau+|h|}\max_{1\le k,m\le N}\|f^h_{k,m}\|_{C[0,1]_h}\right).$$

where M is independent of τ, ψ^h_m, φ^h_k, and $f^h_{k,m}$.

Theorem 1.40. For the solution of difference scheme (1.24), we have the following coercive stability inequalities

$$\max_{1\le k,m\le N}\|\tfrac{u^h_{k,m}-u^h_{k-1,m}}{\tau}\|_{C[0,1]_h}+\max_{1\le k,m\le N}\|\tfrac{u^h_{k-1,m}-u^h_{k-1,m-1}}{\tau}\|_{C[0,1]_h}$$

$$\le M\left(\max_{0\le m\le N}\|h^{-2}\left\{\psi^{n+1}_m-2\psi^n_m+\psi^{n-1}_m\right\}^N_1\|_{C[0,1]_h}+\max_{1\le k,m\le N}\left\|f^h_{k,m}\right\|\right.$$

$$+\max_{1\le m\le N}\|\tfrac{\psi^h_m-\psi^h_{m-1}}{\tau}\|_{C[0,1]_h}+\max_{1\le k,m\le N}\|\tfrac{f^h_{k,m}-f^h_{k-1,m}}{\tau}\|_{C[0,1]_h}$$

$$+\max_{0\le k\le N}\|h^{-2}\left\{\varphi^{n+1}_k-2\varphi^n_k+\varphi^{n-1}_k\right\}^N_1\|_{C[0,1]_h}$$

$$\left.+\max_{1\le k,m\le N}\|\tfrac{f^h_{k,m}-f^h_{m-1}}{\tau}\|_{C[0,1]_h}+\max_{0\le k\le N}\|\tfrac{\varphi^h_k-\varphi^h_{k-1}}{\tau}\|_{C[0,1]_h}\right)$$

where M is independent of τ, ψ^h_m, φ^h_k, and $f^h_{k,m}$.

Secondly, the mixed boundary value problem for the multidimensional ultra-parabolic equation (1.10) is considered. We replace problem (1.15) by difference

scheme (1.23)

$$\begin{cases} \frac{u^h_{k,m}-u^h_{k-1,m-1}}{\tau}+A^x_h u^h_{k,m}=f^h_{k,m}(x), x\in\Omega_h, 1\le k,m\le r \\ \\ \frac{u^h_{k,m}-u^h_{k-1,m-1}}{\tau}+\frac{1}{2}A^x_h(u_{k,m}+u_{k-1,m-1})=f^h_{k,m}(x), \\ \\ x\in\Omega_h, r+1\le k,m\le N \\ \\ f^h_{k,m}(x)=f^h(t_k-\frac{\tau}{2},s_m-\frac{\tau}{2},x), t_k=k\tau, s_m=m\tau, \\ \\ 1\le k,m\le N,\ x\in\widetilde{\Omega}_h, \\ \\ u^h_{0,m}=\psi^h_m, 0\le m\le N, u^h_{k,0}=\varphi^h_k, 0\le k\le N \end{cases} \tag{1.25}$$

Theorem 1.41. For the solution of difference scheme (1.25), we have the following stability inequality

$$\max_{1\le k,m\le N}\|u^h_{k,m}\|_{C(\widetilde{\Omega}_h)}$$

$$\le M\left(\max_{0\le m\le N}\|\psi^h_m\|_{C(\widetilde{\Omega}_h)}+\max_{0\le k\le N}\|\varphi^h_k\|_{C(\widetilde{\Omega}_h)}+\max_{1\le k,m\le N}\|f^h_{k,m}\|_{C(\widetilde{\Omega}_h)}\right),$$

where M is independent of τ, ψ^h_m, φ^h_k, and $f^h_{k,m}$.

Theorem 1.42. For the solution of difference scheme (1.25), we have the following coercive stability inequalities

$$\max_{1\le k,m\le N}\|\frac{u^h_{k,m}-u^h_{k-1,m-1}}{\tau}\|_{C(\widetilde{\Omega}_h)}+\max_{1\le k,m\le r}\sum_{r=1}^{n}\|u^h_{(\bar{x}_r x_r,\ j_r)\ k,m}\|_{C(\widetilde{\Omega}_h)}$$

$$+\frac{1}{2}\max_{1\le k,m\le r}\sum_{r=1}^{n}\|u^h_{(\bar{x}_r x_r,\ j_r)\ k,m}+u^h_{(\bar{x}_r x_r,\ j_r)\ k-1,m-1}\|_{C(\widetilde{\Omega}_h)}$$

$$\le M\ln\frac{1}{|h|}\left(\max_{1\le m\le N}\sum_{r=1}^{n}\|\psi^h_{(\bar{x}_r x_r,\ j_r)\ m}\|_{C(\widetilde{\Omega}_h)}+\max_{0\le k\le N}\sum_{r=1}^{n}\|\varphi^h_{(\bar{x}_r x_r,\ j_r)\ k}\|_{C(\widetilde{\Omega}_h)}\right)$$

$$+M\ln\frac{1}{\tau+|h|}\max_{1\le k,\ m\le N}\|f^{\ h}_{k,m}\|_{C(\widetilde{\Omega}_h)}.$$

where M is independent of τ, ψ^h_m, φ^h_k, and $f^h_{k,m}$.

Theorem 1.43. For the solution of difference scheme (1.25), we have the following coercive stability inequalities

$$\max_{1\le k,m\le N}\Big\|\frac{u^h_{k,m}-u^h_{k-1,m}}{\tau}\Big\|_{C(\widetilde{\Omega}_h)}+\max_{1\le k,m\le N}\Big\|\frac{u^h_{k-1,m}-u^h_{k-1,m-1}}{\tau}\Big\|_{C(\widetilde{\Omega}_h)}$$

$$\le M\left(\max_{1\le m\le N}\sum_{r=1}^{n}\|\psi^h_{(\bar{x}_r x_r,\, j_r)\; m}\|_{C(\widetilde{\Omega}_h)}+\max_{1\le k,m\le N}\left\|f^h_{k,m}\right\|_{C(\widetilde{\Omega}_h)}\right.$$

$$+\max_{1\le m\le N}\Big\|\frac{\psi^h_m-\psi^h_{m-1}}{\tau}\Big\|_{C(\widetilde{\Omega}_h)}+\max_{1\le k,m\le N}\Big\|\frac{f^h_{k,m}-f^h_{k-1,m}}{\tau}\Big\|_{C(\widetilde{\Omega}_h)}$$

$$+\max_{0\le k\le N}\sum_{r=1}^{n}\|\varphi^h_{(\bar{x}_r x_r,\, j_r)\; k}\|_{C(\widetilde{\Omega}_h)}$$

$$\left.+\max_{1\le k,m\le N}\Big\|\frac{f^h_{k,m}-f^h_{m-1}}{\tau}\Big\|_{C(\widetilde{\Omega}_h)}+\max_{0\le k\le N}\Big\|\frac{\varphi^h_k-\varphi^h_{k-1}}{\tau}\Big\|_{C(\widetilde{\Omega}_h)}\right)$$

where M is independent of τ, ψ^h_m, φ^h_k, and $f^h_{k,m}$.

Fifth chapter presents numerical examples supporting the theoretical statements obtained in the previous chapters.

Sixth chapter is the programming for given applications.

CHAPTER 2

WELL POSEDNESS OF ULTRA-PARABOLIC DIFFERENTIAL EQUATIONS

We consider initial value problem for ultra-parabolic equations

$$\begin{cases} \frac{\partial u(t,s)}{\partial t}+\frac{\partial u(t,s)}{\partial s} + Au(t,s) = f(t,s), \quad 0 < t,s < T, \\ u(0,s) = \psi(s), \quad 0 \le s \le T, \\ u(t,0) = \varphi(t), \quad 0 \le t \le T \end{cases} \tag{2.1}$$

in a Banach space E with a strongly positive operator A. Here, f(t,s), $\psi(s)$, $\varphi(t)$ are given smooth fuctions.

Recall that the operator A is said to be strongly positive if its spectrum $\sigma(A)$ lies in the interior of the sector of angle ϕ, $0 < 2\phi < \pi$, symmetric with respect to the real axis, and if on the edges of this sector, $S_1(\phi) = \{\rho e^{i\phi} : 0 \le \rho < \infty\}$ and $S_2(\phi) = \{\rho e^{-i\phi} : 0 \le \rho < \infty\}$,

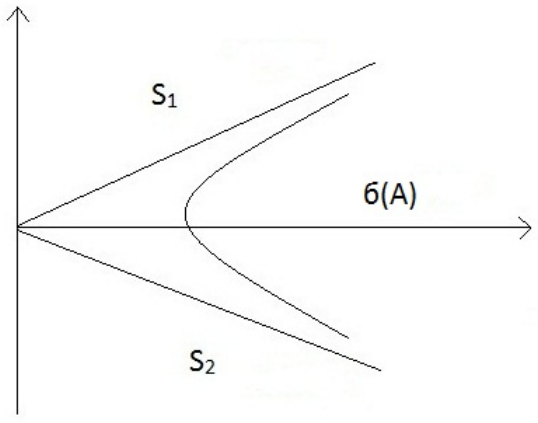

Figure 2.1 Spectrum of A

and outside it the resolvent $(\lambda - A)^{-1}$ is subject to the bound

$$\|(\lambda - A)^{-1}\|_{E\to E} \leq \frac{M(\phi)}{1+|\lambda|} \ .$$

A function u(t,s) is called a solution of problem (2.1) if the following conditions are satisfied:

1) u(t,s) is continuously differentiable with respect to t and s on the segment $[0, T]$. The partial derivative at the endpoints of the segment are understood as the appropriate unilateral derivatives.

2) The element u(t, s) belongs to D(A) for all t,s$\in$ $[0, T]$, and the function Au(t,s) is continuous on $[0, T] \times [0, T]$.

3) u(t,s) satisfies the equation and initial conditions (2.1).

A solution of problem (2.1) defined in this manner will from now on be referred to as a solution of problem (2.1) in the space $C(E) = C([0, T], [0, T], E)$. Here $C(E)$ stands for the Banach spaces of all continuous functions v satisfying a Hölder condition for which the following norm is finite

$$\| v \|_{C(E)} = \max_{0\leq t,s\leq T} \ \| v(t, s) \|_E \ .$$

Then, following theorems on well-posedness of problem (2.1) are established.

Theorem 2.1. For the solution of problem (2.1), the following stability inequality holds

$$\max_{0\leq s,t\leq T} \| u(t, s) \|_E \leq M \left[\max_{0\leq t\leq T} \| \varphi(t) \|_E + \max_{0\leq s\leq T} \| \psi(s) \|_E + \max_{0\leq s,t\leq T} \| \ f(t, s) \|_E \right],$$

here M is independent of φ (t), ψ (s) and f(t,s).

Proof. Problem (2.1) has an unique solution (Lorenz, 2012), represented by following formulas

$$u\,(t, s) = e^{-sA}\varphi(t - s) + \textstyle\int_0^s e^{-(s-\xi)A} f(t - s + \xi, \xi) d\xi, s \leq t,$$

$$u\,(t, s) = e^{-tA}\psi(s - t) + \textstyle\int_0^s e^{-(t-\xi)A} f(\xi, s - t + \xi) d\xi, t \leq s.$$

Using the triangle inequality, we can write

$$\| u(t,s) \|_E$$

$$\leq \| e^{-sA} \|_{E\to E} \| \varphi(t-s) \|_E + \int_0^s \| e^{-(s-\xi)A} \|_{E\to E} \| f(t-s+\xi,\xi) \|_E \, d\xi \quad (2.2)$$

$$\leq M_1 \left[\max_{0\leq t\leq T} \| \varphi(t) \|_E + \max_{0\leq t,s\leq T} \| f(t,s) \|_E \right], \quad s \leq t,$$

$$\| u(t,s) \|_E$$

$$\leq \| e^{-tA} \|_{E\to E} \| \psi(s-t) \|_E + \int_0^t \| e^{-(t-\xi)A} \|_{E\to E} \| f(\xi,s-t+\xi) \|_E \, d\xi \quad (2.3)$$

$$\leq M_2 \left[\max_{0\leq s\leq T} \| \psi(s) \|_E + \max_{0\leq t,s\leq T} \| f(t,s) \|_E \right], \quad t \leq s.$$

Combining (2.2) and (2.3), we get

$$\| u(t,s) \|_E$$

$$\leq M \left[\max_{0\leq t\leq T} \| \varphi(t) \|_E + \max_{0\leq t\leq T} \| \psi(s) \|_E + T \max_{0\leq s,t\leq T} \| f(t,s) \|_E \right].$$

Theorem 2.2. For the solution of problem (2.1), the following coercive stability inequality holds

$$\max_{0\leq t,s\leq T} \| u_t(t,s) + u_s(t,s) \|_{E_\alpha} + \max_{0\leq t,s\leq T} \| Au(t,s) \|_{E_\alpha}$$

$$\leq M \left[\max_{0\leq t\leq T} \| A\varphi \|_{E_\alpha} + \max_{0\leq s\leq T} \| A\psi(s) \|_{E_\alpha} + \frac{1}{1-\alpha} \max_{0\leq t,s\leq T} \| f(t,s) \|_{E_\alpha} \right],$$

where M is independent of φ (t), ψ (s) and f(t,s). Here, $\| v \|_{E_\alpha}$ denotes norm of the Banach space of E_α consist of those $v \in E$ for which the norm

$$\| v \|_{E_\alpha} = \sup_{\lambda>0} \lambda^{1-\alpha} \| Ae^{-\lambda A} v \|_E + \| v \|_E$$

is finite and $\alpha \in (0,1)$.

Proof. Using the identities

$$\lambda^{1-\alpha}Ae^{-\lambda A}Au\,(t,s)$$

$$=\lambda^{1-\alpha}A^2e^{-(\lambda+s)A}\varphi(t-s)+\lambda^{1-\alpha}\int_0^s A^2e^{-(\lambda+s-\xi)A}f(t-s+\xi,\xi)d\xi, s\le t,$$

$$\lambda^{1-\alpha}Ae^{-\lambda A}Au\,(t,s)$$

$$=\lambda^{1-\alpha}A^2e^{-(\lambda+t)A}\psi(s-t)+\lambda^{1-\alpha}\int_0^t A^2e^{-(\lambda+t-\xi)A}f(\xi,s-t+\xi)d\xi, t\le s,$$

the triangle inequality, the definition of the norm in the space E_α and the following estimate (Ashyralyev and Sobolevskii, 1994)

$$\parallel Ae^{-tA}\parallel_{E\to E}\le\frac{M}{t}, \tag{2.4}$$

we obtain

$$\lambda^{1-\alpha}\parallel Ae^{-\lambda A}Au\,(t,s)\parallel_E\le\frac{\lambda^{1-\alpha}}{(\lambda+s)^{1-\alpha}}(\lambda+s)^{1-\alpha}\parallel Ae^{-(\lambda+s)A}A\varphi(t-s)\parallel_E$$

$$+\lambda^{1-\alpha}\int_0^s\parallel Ae^{-\frac{\lambda+s-\xi}{2}A}\parallel_{E\to E}\parallel Ae^{-\frac{\lambda+s-\xi}{2}A}f(t-s+\xi,\xi)\parallel_E \tag{2.5}$$

$$\le\max_{0\le t\le T}\parallel A\varphi(t)\parallel_{E_\alpha}+\frac{M}{1-\alpha}\max_{0\le s,t\le T}\parallel f(t,s\parallel_{E_\alpha}, s\le t,$$

for any $\lambda>0$. In a similar manner one can establish the estimate

$$\lambda^{1-\alpha}\parallel Ae^{-\lambda A}Au\,(t,s)\parallel_E$$

$$\le\max_{0\le t\le T}\parallel A\psi(s)\parallel_{E_\alpha}+\frac{N}{1-\alpha}\max_{0\le s,t\le T}\parallel f(t,s\parallel_{E_\alpha}, t\le s, \tag{2.6}$$

combining estimates (2.5) and (2.6), we have

$$\max_{0\le t,s\le T}\parallel Au(t,s)\parallel_{E_\alpha}$$

$$\le M_1\left[\max_{0\le t\le T}\parallel A\varphi(t)\parallel_{E_\alpha}+\max_{0\le s\le T}\parallel A\psi(s)\parallel_{E_\alpha}+\frac{1}{1-\alpha}\max\parallel f(t,s)\parallel_{E_\alpha}\right]. \tag{2.7}$$

Further, we have

$$\max_{0\le t,s\le T}\|u_s\,(t,s)+u_t\,(t,s)\|_{E_\alpha}\le\max_{0\le t,s\le T}\|Au(t,s)\|_{E_\alpha}+\max_{0\le t,s\le T}\|f(t,s)\|_{E_\alpha}. \tag{2.8}$$

From this estimate and estimate (2.7), it follows that

$$\max_{0\leq t,s\leq T}\|u_s(t,s)+u_t(t,s)\|_{E_\alpha}$$

$$\leq M_1\left[\max_{0\leq t\leq T}\|A\varphi(t)\|_{E_\alpha}+\max_{0\leq s\leq T}\|A\psi(s)\|_{E_\alpha}+\tfrac{1}{1-\alpha}\max\|f(t,s)\|_{E_\alpha}\right]$$

$$+\max_{0\leq t\leq T}\|f(t,s)\|_{E_\alpha}$$

$$\leq M_2\,[\max_{0\leq t\leq T}\|A\varphi(t)\|_{E_\alpha}+\max_{0\leq s\leq T}\|A\psi(s)\|_{E_\alpha}+\tfrac{1}{1-\alpha}\max\|f(t,s)\|_{E_\alpha}].$$

Combining this estimate and estimate (2.7), we have

$$\max_{0\leq t,s\leq T}\|u_t(t,s)+u_s(t,s)\|_{E_\alpha}+\max_{0\leq t,s\leq T}\|Au(t,s)\|_{E_\alpha}$$

$$\leq M_3\left[\max_{0\leq t\leq T}\|A\varphi\|_{E_\alpha}+\max_{0\leq s\leq T}\|A\psi(s)\|_{E_\alpha}+\tfrac{1}{1-\alpha}\max_{0\leq t,s\leq T}\|f(t,s)\|_{E_\alpha}\right].$$

Theorem 2.3. For the solution of problem (2.1), the following coercive stability inequality holds

$$\max_{0\leq s,t\leq T}\|u_s(t,s)\|_{E_\alpha}+\max_{0\leq s,t\leq T}\|u_t(t,s)\|_{E_\alpha}\leq M\left[\max_{0\leq t\leq T}\|A\varphi(t)\|_{E_\alpha}\right.$$

$$+\max_{0\leq t\leq T}\|\varphi'(t)\|_{E_\alpha}+\max_{0\leq s,t\leq T}\|f(t,s\|_{E_\alpha}+\max_{0\leq s,t\leq T}\|f_s(t,s)\|_{E_\alpha}$$

$$\left.+\max_{0\leq t\leq T}\|A\psi(t)\|_{E_\alpha}+\max_{0\leq t\leq T}\|\psi'(t)\|_{E_\alpha}+\max_{0\leq s,t\leq T}\|f_t(t,s)\|_{E_\alpha}\right],$$

where M is independent of φ (t), ψ (s) and f(t,s).

Proof. We have the formula

$$u_t(t,s)=e^{-sA}\varphi'(t-s)+\int_0^s e^{-(s-\xi)A}f_{t-s+\xi}(t-s+\xi,\xi)d\xi,\ \ s\leq t. \tag{2.9}$$

Using the triangle inequality, the estimate (2.4) and the definition of the norm in

the space E_α, we get

$$\lambda^{1-\alpha} \parallel Ae^{-\lambda A}u_t\,(t,s) \parallel_E$$

$$= \lambda^{1-\alpha} \parallel Ae^{-(\lambda+s)A}\varphi^{'}(t-s) \parallel_E + \int_0^s \lambda^{1-\alpha} \parallel Ae^{-(\lambda+s-\xi)A} f_t(t-s+\xi,\xi) \parallel d\xi$$

$$\leq \frac{\lambda^{1-\alpha}}{(\lambda+s)^{1-\alpha}} \max_{0\leq t\leq T} \parallel \varphi^{'}(t) \parallel_{E_\alpha} + \int_0^s \frac{\lambda^{1-\alpha}}{(\lambda+s-\xi)^{1-\alpha}} d\xi \max_{0\leq s,t\leq T} \parallel f_t(t,s \parallel_{E_\alpha}$$

$$\leq M_1 \left[\max_{0\leq t\leq T} \parallel A\varphi^{'}(t) \parallel_{E_\alpha} + \max_{0\leq s,t\leq T} \parallel f_t(t,s \parallel_{E_\alpha} \right], s \leq t \ , \tag{2.10}$$

and in a similar manner using the following formulas

$$u_s\,(t,s) = Ae^{-sA}\varphi(t-s) - e^{-sA}\varphi^{'}(t-s) - \int_0^s e^{-(s-\xi)A} f_{t-s+\xi}(t-s+\xi,\xi)d\xi$$

$$+ \int_0^s Ae^{-(s-\xi)A} f(t-s+\xi,\xi)d\xi + f(t,s),\ s \leq t, \tag{2.11}$$

$$u_t\,(t,s) = Ae^{-tA}\psi(s-t) - e^{-tA}\psi^{'}(s-t) - \int_0^t e^{-(t-\xi)A} f_{s-t+\xi}(\xi,s-t+\xi)d\xi$$

$$+ \int_0^t Ae^{-(t-\xi)A} f(\xi,s-t+\xi)d\xi + f(t,s),\ t \leq s \ , \tag{2.12}$$

$$u_s\,(t,s) = e^{-tA}\psi^{'}(s-t) + \int_0^t e^{-(t-\xi)A} f_{s-t+\xi}(\xi,s-t+\xi)d\xi,\ t \leq s \ ,$$

one can establish following estimates

$$\max_{0\leq s,t\leq T} \parallel u_s(t,s) \parallel_{E_\alpha}$$

$$\leq M_2 \left[\max_{0\leq t\leq T} \parallel A\varphi(t) \parallel_{E_\alpha} + \max_{0\leq t\leq T} \parallel \varphi^{'}(t) \parallel_{E_\alpha} \right. \tag{2.13}$$

$$\left. + \max_{0\leq s,t\leq T} \parallel f_s(t,s) \parallel_{E_\alpha} + \max_{0\leq s,t\leq T} \parallel f(t,s) \parallel_{E_\alpha} \right], s \leq t \ ,$$

$$\max_{0\le s,t\le T} \| u_t(t,s) \|_{E_\alpha}$$

$$\le M_3 \left[\max_{0\le t\le T} \| A\psi(t) \|_{E_\alpha} + \max_{0\le t\le T} \| \psi'(t) \|_{E_\alpha} \right. \tag{2.14}$$

$$\left. + \max_{0\le s,t\le T} \| f_t(t,s) \|_{E_\alpha} + \max_{0\le s,t\le T} \| f(t,s) \|_{E_\alpha} \right], t \le s \ ,$$

$$\max_{0\le s,t\le T} \| u_s(t,s) \|_{E_\alpha}$$

$$\le M_4 \left[\max_{0\le t\le T} \| \psi'(t) \|_{E_\alpha} + \max_{0\le s,t\le T} \| f_s(t,s \|_{E_\alpha} \right], t \le s \ . \tag{2.15}$$

Combining estimates (2.12), (2.15), (2.16) and (2.17), we have

$$\max_{0\le s,t\le T} \| u_s(t,s) \|_{E_\alpha} + \max_{0\le s,t\le T} \| u_t(t,s) \|_{E_\alpha}$$

$$\le M \left[\max_{0\le t\le T} \| A\varphi(t) \|_{E_\alpha} + \max_{0\le t\le T} \| A\varphi'(t) \|_{E_\alpha} \right.$$

$$+ \max_{0\le s,t\le T} \| f(t,s \|_{E_\alpha} + \max_{0\le s,t\le T} \| f_s(t,s) \|_{E_\alpha} + \max_{0\le t\le T} \| A\psi(t) \|_{E_\alpha} \tag{2.16}$$

$$+ \max_{0\le t\le T} \| A\psi'(t) \|_{E_\alpha} + \max_{0<t<T} \| \psi'(t) \|_{E_\alpha}$$

$$\left. + \max_{0\le t\le T} \| \varphi'(t) \|_{E_\alpha} + \max_{0\le s,t\le T} \| f_t(t,s) \|_{E_\alpha} \right].$$

So, this is the end of proof of the Theorem 2.3.

Finally, we consider the applications of these results to the ultra-parabolic equations. First, we consider the mixed boundary value problem for ultra-parabolic equation

$$\begin{cases} \frac{\partial u}{\partial s} + \frac{\partial u}{\partial t} - a(x)\frac{\partial^2 u}{\partial x^2} + \delta u = f(t,s,x), 0 < t,s < T, 0 < x < 1, \\ u(t,0,x) = \varphi(t,x), \ 0 \le t \le T, 0 \le x \le 1, \\ u(0,s,x) = \psi(s,x), \ 0 \le s \le T, 0 \le x \le 1, \\ u(t,s,0) = u(t,s,1), \quad u_x(t,s,0) = u_x(t,s,1), \quad 0 \le t,s \le T, \end{cases} \tag{2.17}$$

where $a(x)$, $\varphi(t,x), \psi(s,x)$ and $f(t,s,x)$ are given sufficiently smooth functions and $a(x) > 0,\ \delta > 0$ is a sufficiently large number.

We introduce the Banach spaces $C^{\beta}[0,1] \quad (0 < \beta < 1)$ of all continuous functions $\varphi(x)$ satisfying a Hölder condition for which the following norms are finite

$$\| \varphi \|_{C^{\beta}[0,1]} = \| \varphi \|_{C[0,1]} + \sup_{0 \le x < x+\tau \le 1} \frac{|\varphi(x+\tau) - \varphi(x)|}{\tau^{\beta}},$$

where $C[0,1]$ is the space of the all continuous functions $\varphi(x)$ defined on [0,1] with the usual norm

$$\| \varphi \|_{C[0,1]} = \max_{0 \le x \le 1} |\varphi(x)|.$$

It is known that the differential expression

$$A^x v = -a(x)v''(x) + \delta v(x) \tag{2.18}$$

define a positive operator A^x acting in $C^{\beta}[0,1]$ with domain $C^{\beta+2}[0,1]$ and satisfying the conditions $v(0) = v(1),\ v_x(0) = v_x(1)$.

Therefore, we can replace the mixed problem (2.19) by the abstract initial-value problem (2.1). Using the results of Theorem 2.1, Theorem 2.2, Theorem 2.3, we can obtain that

Theorem 2.4. For the solution of the mixed problem (2.19) the following stability inequality holds

$$\max_{0 \le t,s \le T} \| u(t,s) \|_{C^{\beta}[0,1]}$$

$$\le M \left[\max_{0 \le t \le T} \| \varphi(t) \|_{C^{\beta}[0,1]} + \max_{0 \le s \le T} \| \psi(s) \|_{C^{\beta}[0,1]} + \max_{0 \le s,t \le T} \| f(t,s) \|_{C^{\beta}[0,1]} \right],$$

here M is independent of $\beta, \varphi(t), \psi(s)$ and $f(t,s)$, where $0 \le \beta \le 1$.

The proof of Theorem 2.4 is based on the abstract Theorem 2.1, the positivity of operator A^x defined (2.6) in $C^{\beta}[0,1]$ for any $0 \le \beta \le 1$.

Theorem 2.5. Let $\alpha \in (0, \frac{1}{2})$. Then for the solution of the mixed problem (2.19) the following coercive stability inequalities hold

$$\max_{0 \le t,s \le T} \| u_t(t,s) + u_s(t,s) \|_{C^{2\alpha}[0,1]} + \max_{0 \le t,s \le T} \| u(t,s) \|_{C^{2+2\alpha}[0,1]}$$

$$\le M(\alpha) \left[\max_{0 \le t \le T} \| \varphi(t) \|_{C^{2+2\alpha}[0,1]} \right.$$

$$+\max_{0\leq s\leq T}\parallel \psi(s)\parallel_{C^{2+2\alpha}[0,1]}+\max_{0\leq s,t\leq T}\parallel\ f(t,s)\parallel_{C^{2\alpha}[0,1]}\Bigg],$$

where $M(\alpha)$ is independent of $\varphi(t), \psi(s)$ and $f(t,s)$.

Theorem 2.6. Let $\alpha \in (0, \frac{1}{2})$. Then for the solution of the mixed problem (2.19) the following coercive stability inequalities hold

$$\max_{0\leq t,s\leq T}\parallel u_t(t,s)\parallel_{C^{2\alpha}[0,1]}+\max_{0\leq t,s\leq T}\parallel u_s(t,s)\parallel_{C^{2\alpha}[0,1]}$$

$$\leq M(\alpha)\Bigg[\max_{0\leq t\leq T}\parallel \varphi(t)\parallel_{C^{2+2\alpha}[0,1]}+\max_{0\leq t\leq T}\parallel \varphi^{'}(t)\parallel_{C^{2\alpha}[0,1]}+\max_{0\leq s\leq T}\parallel \psi(s)\parallel_{C^{2+2\alpha}[0,1]}$$

$$+\max_{0\leq s\leq T}\parallel \psi^{'}(s)\parallel_{C^{2\alpha}[0,1]}+\max_{0\leq s,t\leq T}\parallel\ f_t(t,s)\parallel_{C^{2\alpha}[0,1]}+\max_{0\leq s,t\leq T}\parallel\ f_s(t,s)\parallel_{C^{2\alpha}[0,1]}\Bigg],$$

where $M(\alpha)$ is independent of $\varphi(t), \psi(s)$ and $f(t,s)$.

The proof of Theorem 2.5 and 2.6 is based on the abstract Theorem 2.2, the positivity of operator A^x defined (2.19) in $C^{\beta}[0,1]$ and on the following theorem on the structure of the fractional space $\mathrm{E}_{\alpha}(A^x, C[0,1])$.

Theorem 2.7. $\mathrm{E}_{\alpha}(A^x, C[0,1]) = C^{2\alpha}[0,1]$ for all $0 < \alpha < \frac{1}{2}$.

Second, let Ω be the unit open cube in the n-dimensional Euclidean space $\mathbb{R}^n$ $(0 < x_k < 1, 1 \leq k \leq n)$ with boundary S, $\overline{\Omega} = \Omega \cup S$. In $[0,T]\times[0,T]\times\overline{\Omega}$ we consider the mixed boundary value problem for the multidimensional ultra-parabolic equation

$$\begin{cases} \frac{\partial u(t,s,x)}{\partial s}+\frac{\partial u(t,s,x)}{\partial t}-\sum\limits_{r=1}^{n}\alpha_r(x)\frac{\partial^2 u(t,s,x)}{\partial x_r^2}+\delta u(t,s,x)=f(t,s,x), \\ x=(x_1,\ldots,x_n)\in\overline{\Omega}, 0\leq t,s\leq T, \\ u(t,0,x)=\varphi(t,x), 0\leq t\leq T,\ x\in\overline{\Omega}, \\ u(0,s,x)=\psi(s,x), 0\leq s\leq T,\ x\in\overline{\Omega}, \\ u(t,s,x)=0, 0\leq t,s\leq T,\ x\in S, \end{cases} \tag{2.19}$$

where $\alpha_r(x),\ f(t,s,x), \varphi(t,x), \psi(s,x) (t,s\in[0,T],\ x\in\overline{\Omega})$ are given smooth functions and $\alpha_r(x) > 0,\ \delta > 0$ is a sufficiently large number.

We introduce the Banach spaces $C_{01}^{\beta}(\overline{\Omega})$ $(\beta = (\beta_1,\ldots,\beta_n),\ 0 < x_k < 1, k = 1,\ldots,n)$ of all continuous functions satisfying a Hölder condition with the indicator

$\beta = (\beta_1, \ldots, \beta_n), \beta_k \in (0,1), 1 \le k \le n$ and with weight $x_k^{\beta_k}(1-x_k-h_k)^{\beta_k}, 0 \le x_k < x_k + h_k \le 1, 1 \le k \le n$ which equipped with the norm

$$\| f \|_{C_{01}^{\beta}(\overline{\Omega})} = \| f \|_{C(\overline{\Omega})}$$

$$+ \sup_{0 \le x_k < x_k + h_k \le 1, 1 \le k \le n} |f(x_1, \ldots, x_n) - f(x_1 + h_1, \ldots, x_n + h_n)|$$

$$\prod_{k=1}^{n} h_k^{-\beta_k} x_k^{\beta_k} (1 - x_k - h_k)^{\beta_k},$$

where $C(\overline{\Omega})$ is the space of the all continuous functions defined on $\overline{\Omega}$, equipped with the norm

$$\| f \|_{C(\overline{\Omega})} = \max_{x \in \overline{\Omega}} |f(x)|.$$

It is known that the differential expression

$$A^x v = -\sum_{r=1}^{n} \alpha_r(x) \frac{\partial^2 v(x)}{\partial x^2} + \delta v(x) \tag{2.20}$$

defines a positive operator A^x acting on $C_{01}^{\beta}(\overline{\Omega})$ with domain $D(A^x) \subset C_{01}^{2+\beta}(\overline{\Omega})$ and satisfying the condition $v = 0$ on S.

Therefore, we can replace the mixed problem (2.21) by the abstract initial-value problem (2.1). Using the result of Theorem 2.1, we can obtain that

Theorem 2.8. For the solution of the mixed boundary value problem (2.21) the following stability inequality is valid

$$\max_{0 \le t,s \le T} \| u(t,s) \|_{C_{01}^{\beta}(\overline{\Omega})} \le$$

$$M \left[\max_{0 \le t \le T} \| \varphi(t) \|_{C_{01}^{\beta}(\overline{\Omega})} + \max_{0 \le s \le T} \| \psi(s) \|_{C_{01}^{\beta}(\overline{\Omega})} + \max_{0 \le s,t \le T} \| f(t,s) \|_{C_{01}^{\beta}(\overline{\Omega})} \right],$$

where M is independent of $\beta, \varphi(t,x), \psi(s,x)$ and $f(t,s,x)$. Here $\beta = (\beta_1, \ldots, \beta_n), \beta_k \in [0,1]$.

Proof. The proof of Theorem 2.8 is based on the abstract Theorem 2.1, the positivity of operator A^x defined (2.21) in $C_{01}^{\beta}(\overline{\Omega})$ for any $0 \le \beta \le 1$.

Third, we consider the initial-value problem on the range $\{0 \le t, s \le T,\ x \in$

$\mathbb{R}^n\}$ for the 2m-th order multidimensional parabolic equation

$$\begin{cases} \frac{\partial u}{\partial s}+\frac{\partial u}{\partial t}+\sum\limits_{|r|=2m} a_r(x)\frac{\partial^{|\tau|}u}{\partial x_1^{r_1}...\partial x_n^{r_n}}+\delta u(t,s,x)=f(t,s,x), \\ 0<t,s<T,\ x,r\in\mathbb{R}^n, |r|=r_1+\cdots+r_n, \\ u(t,0,x)=\varphi(t,x), 0\le t\le T,\ x\in\mathbb{R}^n, \\ u(0,s,x)=\psi(s,x), 0\le s\le T,\ x\in\mathbb{R}^n, \end{cases} \tag{2.21}$$

where $\alpha_r(x)$, $f(t,s,x), \varphi(t,x)$ and $\psi(s,x)$are given sufficiently smooth functions and $\delta>0$ is the sufficiently large number .

Let us consider a differential operator with constant coefficients of the form

$$B=\sum_{|r|=2m} b_r\frac{\partial^{r_1+...+r_n}}{\partial_{x_1^{r_1}}...\partial_{x_n^{r_n}}},$$

acting on functions defined on the entire space R^n. Here $r\in R^n$ is a vector with nonnegative integer components, $|r|=r_1+...+r_n$. If $\varphi(y)\,(y=(y_1,...y_n)\in R^n)$ is an infinitely differentiable function that decays at infinity together with all its derivatives, then by means of the Fourier transformation one establishes the equality

$$F\left(B_\varphi\right)(\xi)=B(\xi)F(\varphi)(\xi).$$

Here the Fourier transform operator is defined by the rule

$$F(\varphi)(\xi)=(2\pi)^{-n/2}\int\limits_{R^n}\exp\{-i(y,\xi)\}\varphi(y)\,dy,$$

$$(y,\xi)=y_1\xi_1+...+y_n\xi_n.$$

The function $B(\xi)$ is called the symbol of the operator B and is given by

$$B(\xi)=\sum_{|r|=2m} b_r(i\xi_1)^{r_1}...(i\xi_n)^{r_n}.$$

We will assume that the symbol

$$B^x(\xi)=\sum_{|r|=2m} a_r(x)(i\xi_1)^{r_1}...(i\xi_n)^{r_n}, \xi=(\xi_1,\cdots,\xi_n)\in R^n$$

of the differential operator of the form

$$B^x = \sum_{|r|=2m} a_r(x) \frac{\partial^{|r|}}{\partial x_1^{r_1} \dots \partial x_n^{r_n}} \tag{2.22}$$

acting on functions defined on the space $\mathbb{R}^n$, satisfies the inequalities

$$0 < M_1 |\xi|^{2m} \le (-1)^m B^x(\xi) \le M_2 |\xi|^{2m} < \infty$$

for $\xi \neq 0$. The problem (2.23) has a unique smooth solution. This allows us to reduce the problem (2.23) to the abstract initial-value problem (2.1) in a Banach space $E = C^{\mu}(R^n)$ of all continuous bounded functions defined on $\mathbb{R}^n$satisfying a Hölder condition with the indicator $\mu \in (0,1)$ with a strongly positive operator $A^x = B^x + \delta I$ defined by (2.24).

Theorem 2.9. Let $\alpha \in (0, \frac{1}{2m})$. Then for the solution of the mixed boundary value problem (2.23) the following stability inequality is valid

$$\max_{0\le t,s\le T} \| u(t,s) \|_{C^{2m\alpha}(R^n)}$$

$$\le M \left[\max_{0\le t\le T} \| \varphi(t) \|_{C^{2m\alpha}(R^n)} \right.$$

$$\left. + \max_{0\le t\le T} \| \psi(s) \|_{C^{2m\alpha}(R^n)} + \max_{0\le s,t\le T} \| f(t,s) \|_{C^{2m\alpha}(R^n)} \right],$$

where M is independent of $\alpha, \varphi(t,x), \psi(s,x)$ and $f(t,s,x)$.

The proof of Theorem 2.9 is based on the abstract Theorem 2.1, the positivity of operator A^x defined (2.24) on $\mathbb{R}^n$.

Theorem 2.10. Let $\alpha \in (0, \frac{1}{2m})$. Then for the solution of the mixed problem (2.23) the following coercive stability inequalities hold

$$\max_{0\le t,s\le T} \| u_t(t,s) + u_s(t,s) \|_{C^{2m\alpha}(R^n)} + \sum_{|r|=2m} \max_{0\le t,s\le T} \| a_r(x) \frac{\partial^{|r|} u(t,s)}{\partial x_1^{r_1} \dots \partial x_n^{r_n}} \|_{C^{2m\alpha}(R^n)}$$

$$\le M(\alpha) \left[\sum_{|r|=2m} \max_{0\le t\le T} \| a_r(x) \frac{\partial^{|r|} \varphi(t)}{\partial x_1^{r_1} \dots \partial x_n^{r_n}} \|_{C^{2m\alpha}(R^n)} \right.$$

$$+ \sum_{|r|=2m} \max_{0\le s\le T} \| a_r(x) \frac{\partial^{|r|} \psi(s)}{\partial x_1^{r_1} \dots \partial x_n^{r_n}} \|_{C^{2m\alpha}(R^n)}$$

$$\left. + \max_{0\le s,t\le T} \| f(t,s) \|_{C^{2m\alpha}(R^n)} \right],$$

where $M(\alpha)$ is independent of $\varphi(t,x), \psi(s,x)$ and $f(t,s,x)$.

Theorem 2.11. Let $\alpha \in (0, \frac{1}{2m})$. Then for the solution of the mixed problem (2.22) the following coercive stability inequality hold

$$\max_{0\le t,s\le T} \| u_t(t,s) \|_{C^{2m\alpha}(R^n)} + \max_{0\le t,s\le T} \| u_s(t,s) \|_{C^{2m\alpha}(R^n)}$$

$$\le M(\alpha) \left[\sum_{|r|=2m} \max_{0\le t\le T} \| a_r(x) \frac{\partial^{|r|}\varphi(t)}{\partial x_1^{r_1} \dots \partial x_n^{r_n}} \|_{C^{2m\alpha}(R^n)} \right.$$

$$+ \max_{0\le t\le T} \| \varphi'(t) \|_{C^{2m\alpha}(R^n)} + \sum_{|r|=2m} \max_{0\le s\le T} \| a_r(x) \frac{\partial^{|r|}\psi(s)}{\partial x_1^{r_1} \dots \partial x_n^{r_n}} \|_{C^{2m\alpha}(R^n)}$$

$$+ \max_{0\le s\le T} \| \psi'(s) \|_{C^{2m\alpha}(R^n)} + \max_{0\le s,t\le T} \| f_t(t,s) \|_{C^{2m\alpha}(R^n)}$$

$$\left. + \max_{0\le s,t\le T} \| f_s(t,s) \|_{C^{2m\alpha}(R^n)} \right],$$

where $M(\alpha)$ and $M_1(\alpha)$ are independent of $\varphi(t,x), \psi(s,x)$ and $f(t,s,x)$.

The proofs of Theorem 2.10 and Theorem 2.11 are based on the abstract Theorems 2.2, the positivity of operator A^x defined by(no) in $C^{2m\alpha}(R^n)$ and on the theorem on the coercivity of elliptic operator A^x in $C^{\beta}(R^n)$ and on the theorem on the structure of the fractional spaces $E_\alpha(A^x, C(R^n))$.

Theorem 2.12. Let $\alpha \in (0, \frac{1}{2m})$. Then for the solution of the elliptic equation

$$\sum_{|r|=2m} a_r(x) \frac{\partial^{|r|}u}{\partial x_1^{r_1} \dots \partial x_n^{r_n}} + \delta u = F(x), x \in \mathbb{R}^n$$

the following coercive inequality holds

$$\sum_{|r|=2m} \max_{0\le t,s\le T} \| a_r(x) \frac{\partial^{|r|}u}{\partial x_1^{r_1} \dots \partial x_n^{r_n}} \|_{C^{2m\alpha}(R^n)} \le M(\alpha) \| F \|_{C^{2m\alpha}(R^n)},$$

here M(α) is independent of F.

Theorem 2.13. $E_\alpha(A^x, C(R^n)) = C^{2m\alpha}(R^n)$ for all $0 < \alpha < \frac{1}{2m}$.

CHAPTER 3

WELL POSEDNESS OF FIRST ORDER OF ACCURACY ROTHE DIFFERENCE SCHEME

Let us associate the initial boundary value problem for ultra parabolic equations(2.1) with the corresponding first order of accuracy difference scheme

$$\begin{cases} \frac{u_{k,m}-u_{k-1,m}}{\tau} + \frac{u_{k-1,m}-u_{k-1,m-1}}{\tau} + Au_{k,m} = f_{k,m} \ , \\ f_{k,m} = f(t_k, s_m),\ t_k = k\tau,\ s_m = m\tau,\ 1 \le k, m \le N,\ N\tau = 1, \\ u_{0,m} = \psi_m,\ 0 \le m \le N,\ u_{k,0} = \varphi_k,\ 0 \le k \le N. \end{cases} \tag{3.1}$$

Then, the following main theorems on stability estimates for the solution of difference schemes (3.1) are established.

Theorem 3.1. Let τ be a sufficiently small number. Then for the solution of (3.1), we have the following stability inequality

$$\begin{aligned} &\max_{1\le k,m\le N} \|u_{k,m}\|_E \\ &\le M \left(\max_{0\le m\le N} \|\psi_m\|_E + \max_{0\le k\le N} \|\varphi_k\|_E + \max_{1\le k,m\le N} \|f_{k,m}\|_E \right), \end{aligned}$$

where M is independent of ψ_m, φ_k, and $f_{k,m}$.

Proof. Using (3.1), we get

$$\frac{u_{k,m} - u_{k-1,m-1}}{\tau} + Au_{k,m} = f_{k,m}.$$

From that it follows

$$u_{k,m} = Ru_{k-1,m-1} + \tau R f_{k,m}, \tag{3.2}$$

where $R = (I + \tau A)^{-1}$. By the mathematical induction, we will prove that

$$u_{k,m} = R^n u_{k-n,m-n} + \sum_{j=1}^{n} \tau R^{n-j+1} f_{k-n+j,m-n+j} \tag{3.3}$$

is true for all positive integers n. It is obvious that for $n = 1, 2$ formula (3.3) is true. Assume that for $n = r$

$$u_{k,m} = R^r u_{k-r,m-r} + \sum_{j=1}^{r} \tau R^{r-j+1} f_{k-r+j,m-r+j} \tag{3.4}$$

is true. In formula (3.2), replacing k and m with $k - r$ and $m - r$, respectively, we have

$$u_{k-r,m-r} = R u_{k-r-1,m-r-1} + \tau R f_{k-r,m-r}. \tag{3.5}$$

Then, using (3.4) and (3.5), we get

$$u_{k,m} = R^{r+1} u_{k-r-1,m-r-1} + \tau R^{r+1} f_{k-r,m-r} + \sum_{j=1}^{r} \tau R^{r-j+1} f_{k-r+j,m-r+j}.$$

From that it follows

$$u_{k,m} = R^{r+1} u_{k-r-1,m-r-1} + \sum_{j=1}^{r+1} \tau R^{r-j+2} f_{k-r-1+j,m-r-1+j},$$

is true for $n = r + 1$. So, formula (3.3) is proved. For $m > k$, replacing n with k in formula (3.3), we obtain that

$$u_{k,m} = R^k \psi_{m-k} + \sum_{j=1}^{k} \tau R^{k-j+1} f_{j,m-k+j}. \tag{3.6}$$

Using estimate (Ashyralyev and Sobolevskii, 1994)

$$||R^k||_{E\to E} \le M \tag{3.7}$$

and triangle inequality, we get

$$\|u_{k,m}\|_E \le ||R^k||_{E\to E} \|\psi_{m-k}\|_E + \sum_{j=1}^{k} \tau \|R^{k-j+1}\|_{E\to E} \|f_{j,m-k+j}\|_E$$

$$\le M \left[\max_{0\le m\le N} \|\psi_m\| + \max_{1\le j,m\le N} \|f_{j,m}\| \right]$$

for any k and m. For $k > m$, replacing n with m in formula (3.3) we get

$$u_{k,m} = R^m \varphi_{k-m} + \sum_{j=1}^{m} \tau R^{m-j+1} f_{k-m+j,j}. \tag{3.8}$$

From estimate (3.7) and triangle inequality, it follows that

$$\|u_{k,m}\|_E \le ||R^m||_{E\to E}\|\varphi_{k-m}\|_E + \sum_{j=1}^{m}\tau\|R^{m-j+1}\|_{E\to E}\|f_{k-m+j,j})\|_E$$

$$\le M\left[\max_{0\le k\le N}\|\varphi_k\| + \max_{1\le j,m\le N}\|f_{j,m}\|\right]$$

for any k and m. Thus, Theorem 3.1 is proved.

Theorem 3.2. Let τ be a sufficiently small number and $\psi_m, \varphi_k \in D(A)$. Then for the solution of (3.1), we have the following almost coercive inequality

$$\max_{1\le k,m\le N}\|\tfrac{u_{k,m}-u_{k-1,m-1}}{\tau}\|_E + \max_{1\le k,m\le N}\|Au_{k,m}\|_E$$

$$\le M\left(\max_{0\le m\le N}\|A\psi_m\|_E + \max_{0\le k\le N}\|A\varphi_k\|_E\right.$$

$$\left. + \min\left\{\ln\tfrac{1}{\tau},\ 1+|\ln\|A\|_{E\to E}|\right\}\max_{1\le k,m\le N}\|f_{k,m}\|\right),$$

where M is independent of τ, ψ_m, φ_k, and $f_{k,m}$.

Proof. Using the identities(3.6) and (3.8), we get

$$u_{k,m} = R^k\psi_{m-k} + V_{k,m},\ k \le m,$$

$$u_{k,m} = R^m\varphi_{k-m} + V_{k,m},\ m \le k, \tag{3.9}$$

where

$$V_{k,m} = \sum_{j=1}^{k}\tau R^{k-j+1}f_{j,m-k+j}, k \le m,$$

$$V_{k,m} = \sum_{j=1}^{m}\tau R^{m-j+1}f_{k-m+j,j}, m \le k. \tag{3.10}$$

Let us estimate the norm of each term. Using the estimate (3.7), we get

$$||AR^k\psi_{m-k}||_E$$

$$\le ||R^k||_{E\to E}||A\psi_{m-k}||_E \le M\max_{0\le m\le N}\|\ A\psi_m\ \|_E,\ k \le m,$$

$$||AR^k\varphi_{k-m}||_E$$

$$\le ||R^m||_{E\to E}||A\varphi_{k-m}||_E \le M\max_{0\le k\le N}\|\ A\varphi_k\ \|_E,\ m \le k. \tag{3.11}$$

To estimate $V_{k,m}$ we use the following estimate (Ashyralyev and Sobolevskii, 1994)

$$||AR^k||_{E\to E} \le \frac{M}{k\tau} \tag{3.12}$$

and we get

$$\|AV_{k,m}\|_E \le \sum_{j=1}^{k} \tau \|AR^{k-j+1}\|_{E\to E} \|f_{j,m-k+j}\|_E$$

$$\le M \sum_{j=1}^{k} \tfrac{1}{k+1-j} \max_{1\le j,m\le N} \|f_{j,m}\|_E,\ k \le m,$$

$$\|AV_{k,m}\|_E \le \sum_{j=1}^{m} \tau \|AR^{m-j+1}\|_{E\to E} \|f_{k-m+j,j}\|_E$$

$$\le M \sum_{j=1}^{m} \tfrac{1}{m+1-j} \max_{1\le j,m\le N} \|f_{j,m}\|_E,\ m \le k.$$

Since

$$\sum_{j=1}^{k} \frac{1}{k+1-j} \le \int_1^k \frac{ds}{k+1-s} = lnk, k > 1,$$

we have

$$\|AV_{k,m}\|_E \le Mlnk \max_{1\le j,m\le N} \|f_{j,m}\|_E,$$

hence

$$\max_{1\le k,m\le N} \|AV_{k,m}\|_E \le Mln(\frac{1}{\tau}) \max_{1\le j,m\le N} \|f_{j,m}\|_E. \tag{3.13}$$

Further, we have

$$\|AV_{k,m}\|_E \le \sum_{j=1}^{k} \tau \|AR^{k-j+1}\|_{E\to E} \|f_{j,m-k+j}\|_E$$

$$\le \sum_{j=1}^{k} \tau \|AR^{k-j+1}\|_{E\to E} \max_{1\le j,m\le N} \|f_{j,m}\|_E\ ,k \le m,$$

$$\|AV_{k,m}\|_E \le \sum_{j=1}^{m} \tau \|AR^{m-j+1}\|_{E\to E} \|f_{k-m+j,j}\|_E$$

$$\le \sum_{j=1}^{m} \tau \|AR^{m-j+1}\|_{E\to E} \max_{1\le j,m\le N} \|f_{j,m}\|_E\ ,m \le k.$$

It remains to estimate the quantity

$$J_k = \sum_{j=1}^{k} \tau \|AR^{k-j+1}\|_{E\to E} = \sum_{j=1}^{k} \tau \|AR^s\|_{E\to E}.$$

$$J_m = \sum_{j=1}^{m} \tau \|AR^{m-j+1}\|_{E\to E} = \sum_{j=1}^{m} \tau \|AR^s\|_{E\to E}.$$

From the last identities it is clear that it suffices to estimate J_N. Using estimates (3.7) and (3.12), we obtain

$$\|AR^s\|_{E\to E} \le Mmin\{\tfrac{1}{s\tau}, \|A\|_{E\to E}\}.$$

If $\|A\|_{E\to E} > N$, then

$$J_N \le M \sum_{j=1}^{N} \tfrac{\tau}{s\tau} \le M \int_{\|A\|_{E\to E}}^{1} \tfrac{ds}{s} \le M \mid ln\|A\|_{E\to E} \mid .$$

If $\|A\|_{E\to E} \le 1$, then

$$J_N \le M \sum_{j=1}^{N} \|A\|_{E\to E}\tau \le M\|A\|_{E\to E} \le M.$$

Finally, if $\|A\|_{E\to E} \le N$, then

$$J_N \le M \left\{ \sum_{j=1}^{\left[N\|A\|^{-1}_{E\to E}\right]} \|A\|_{E\to E}\tau + \sum_{\left[N\|A\|^{-1}_{E\to E}\right]+1}^{N} \tfrac{\tau}{s\tau} \right\}$$

$$\le M \left(1 + \int_{\|A\|^{-1}_{E\to E}}^{1} \tfrac{ds}{s}\right) = M\left(1 + \ln \|A\|_{E\to E}\right).$$

Thus, in all three cases we have the estimate

$$J_N \le M(1 + ln\|A\|_{E\to E}), \tag{3.14}$$

which yields

$$\max_{1\le k,m\le N} \|AV_{k,m}\|_E \le M\left[1 + ln\|A\|_{E\to E}\right] \max_{1\le j,m\le N} \|f_{j,m}\|_E. \tag{3.15}$$

Combining the estimates (3.11),(3.13) and (3.15) we obtain the estimate

$$\max_{1\le k,m\le r} \|Au_{k,m}\|_E$$

$$\le M \left(\max_{0\le m\le N} \|A\psi_m\|_E + \max_{0\le k\le N} \|A\varphi_k\|_E \right.$$

$$\left. + \min\left\{\ln \tfrac{1}{\tau},\ 1 + |\ln \|A\|_{E\to E}|\right\} \max_{1\le k,m\le N} \|f_{k,m}\| \right).$$

Theorem 3.2 is proved.

Theorem 3.3. Let τ be a sufficiently small number and $\psi_m, \varphi_k \in D(A)$. Then for the solution of (3.1), we have the following almost coercive inequality

$$\max_{1\le k,m\le N}\left\|\frac{u_{k,m}-u_{k-1,m}}{\tau}\right\|_E + \max_{1\le k,m\le N}\left\|\frac{u_{k-1,m}-u_{k-1,m-1}}{\tau}\right\|_E$$

$$\le M\left(\max_{0\le m\le N}\|A\psi_m\|_E + \max_{1\le k,m\le N}\left\|\frac{\psi_m-\psi_{m-1}}{\tau}\right\|_E + \max_{1\le k,m\le N}\left\|\frac{f_{k,m}-f_{k,m-1}}{\tau}\right\|_E\right.$$

$$+\max_{0\le k\le N}\|A\varphi_k\|_E + \max_{1\le k,m\le N}\left\|\frac{\varphi_k-\varphi_{k-1}}{\tau}\right\|_E$$

$$\left.+\max_{1\le k,m\le N}\left\|\frac{f_{k,m}-f_{k-1,m}}{\tau}\right\|_E + \max_{1\le k,m\le N}\|f_{k,m}\|_E\right),$$

where M is independent of τ, ψ_m, φ_k, and $f_{k,m}$.

Proof. Using the identity(3.6), we get

$$\frac{u_{k,m}-u_{k-1,m}}{\tau} = R^{k-1}\frac{\psi_{m-k}-\psi_{m-k+1}}{\tau}$$

$$+\frac{R^k-R^{k-1}}{\tau}\psi_{m-k} + \tau AR f_{k,m}$$

$$+\sum_{j=1}^{k-1}\tau\left\{R^{k-j}\frac{f_{j,m-k+j}-f_{j,m-k+1+j}}{\tau}\right.$$

$$+\frac{R^{k-j+1}-R^{k-j}}{\tau}f_{j,m-k+j}\ , k\le m.$$

Using triangle inequality, we get

$$\left\|\frac{u_{k,m}-u_{k-1,m}}{\tau}\right\|_E \le \left\|R^{k-1}\right\|_{E\to E}\left\|\frac{\psi_{m-k}-\psi_{m-k+1}}{\tau}\right\|_E$$

$$+\left\|R^k\right\|_{E\to E}\|A\psi_{m-k}\|_E + \|\tau AR\|_{E\to E}\|f_{k,m}\|_E$$

$$+\sum_{j=1}^{k-1}\tau\left\|R^{k-j}\right\|_{E\to E}\left\|\frac{f_{j,m-k+j}-f_{j,m-k+1+j}}{\tau}\right\|_E \qquad (3.16)$$

$$+\left\|\sum_{j=1}^{k-1}\tau AR^{k-j+1}f_{j,m-k+j}\right\|_E,\ k\le m.$$

Here,

$$\sum_{j=1}^{k-1} \tau A R^{k-j+1} f_{j,m-k+j} = \sum_{j=1}^{k-1} (I-R) R^{k-j} f_{j,m-k+j}$$

$$= \sum_{j=1}^{k-1} R^{k-j} f_{j,m-k+j} - \sum_{j=1}^{k-1} R^{k-j+1} f_{j,m-k+j}$$

$$= \sum_{j=1}^{k-1} R^{k-j} f_{j,m-k+j} - R^k f_{1,m-k+1} - \sum_{j=1}^{k-2} R^{k-j} f_{j+1,m-k+j+1}$$

$$= R f_{k-1,m-1} + R^k f_{1,m-k+1} + \sum_{j=2}^{k-2} \tau R^{k-j} \tfrac{f_{j,m-k+j} - f_{j+1,m-k+j+1}}{\tau}.$$

Therefore,

$$\left\| \sum_{j=1}^{k-1} \tau A R^{k-j+1} f_{j,m-k+j} \right\|_E \leq \|R\|_{E\to E} \|f_{k-1,m-1}\|_E + \left\|R^k\right\|_{E\to E} \|f_{1,m-k+1}\|_E$$

$$+ \sum_{j=1}^{k-2} \tau \left\|R^{k-j}\right\|_{E\to E} \left\| \tfrac{f_{j+1,m-k+j+1} - f_{j,m-k+j}}{\tau} \right\|_E. \tag{3.17}$$

Combining estimates (3.7), (3.16) and (3.17), we get

$$\max_{1\leq k,m\leq N} \left\| \tfrac{u_{k,m} - u_{k-1,m}}{\tau} \right\|_E \leq M \left[\max_{1\leq m\leq N} \left\| \tfrac{\psi_m - \psi_{m-1}}{\tau} \right\|_E \right.$$

$$\left. + \max_{1\leq m\leq N} \|A\psi_m\|_E + \max_{1\leq k,m\leq N} \|f_{k,m}\|_E + \max_{1\leq k,m\leq N} \left\| \tfrac{f_{k,m} - f_{k-1,m-1}}{\tau} \right\|_E \right]. \tag{3.18}$$

Using the identity (3.8), we get

$$\tfrac{u_{k,m} - u_{k-1,m}}{\tau} = R^m \tfrac{\varphi_{k-m} - \varphi_{k-1-m}}{\tau}$$

$$+ \sum_{j=1}^{m} \tau R^{m-j+1} \tfrac{f_{k-m+j,j} - f_{k-1+m+j,j}}{\tau}, \ m \leq k.$$

Using the triangle inequality, we get

$$\left\| \tfrac{u_{k,m} - u_{k-1,m}}{\tau} \right\|_E \leq \|R^m\|_{E\to E} \left\| \tfrac{\varphi_{k-m} - \varphi_{k-1-m}}{\tau} \right\|_E$$

$$+ \sum_{j=1}^{m} \tau \|R^{m-j+1}\|_{E\to E} \left\| \tfrac{f_{k-m+j,j} - f_{k-1+m+j,j}}{\tau} \right\|_E, \ m \leq k.$$

Applying the estimate (3.7), we get

$$\begin{aligned}&\max_{1\le k,m\le N}\left\|\tfrac{u_{k,m}-u_{k-1,m}}{\tau}\right\|_E\\&\le M\left[\max_{1\le k\le N}\left\|\tfrac{\varphi_k-\varphi_{k-1}}{\tau}\right\|_E+\max_{1\le k,m\le N}\left\|\tfrac{f_{k,m}-f_{k-1,m}}{\tau}\right\|_E\right],\ m\le k.\end{aligned}\tag{3.19}$$

Using the identity (3.6), we get

$$\begin{aligned}&\tfrac{u_{k-1,m}-u_{k-1,m-1}}{\tau}=R^k\tfrac{\psi_{m-k-1}-\psi_{m-k-2}}{\tau}\\&+\sum_{j=1}^{k}\tau R^{k-j+1}\tfrac{f_{j,m-k+j}-f_{j,m-1-k+j}}{\tau},\ k\le m.\end{aligned}$$

Using the triangle inequality,we get

$$\begin{aligned}&\left\|\tfrac{u_{k-1,m}-u_{k-1,m-1}}{\tau}\right\|_E\le\left\|R^k\right\|_{E\to E}\left\|\tfrac{\psi_{m-k-1}-\psi_{m-k-2}}{\tau}\right\|_E\\&+\sum_{j=1}^{k}\tau\left\|R^{k-j+1}\right\|_{E\to E}\left\|\tfrac{f_{j,m-k+j}-f_{j,m-1-k+j}}{\tau}\right\|_E,\ k\le m.\end{aligned}$$

Applying the estimate (3.7), we get

$$\begin{aligned}&\max_{1\le k,m\le N}\left\|\tfrac{u_{k-1,m}-u_{k-1,m-1}}{\tau}\right\|_E\\&\le M\left[\max_{1\le m\le N}\left\|\tfrac{\psi_m-\psi_{m-1}}{\tau}\right\|_E+\max_{1\le k,m\le N}\left\|\tfrac{f_{k,m}-f_{k,m-1}}{\tau}\right\|_E\right],\ m\le k.\end{aligned}\tag{3.20}$$

Using the identity (3.8), we get

$$\begin{aligned}&\tfrac{u_{k-1,m}-u_{k-1,m-1}}{\tau}=R^{m-1}\tfrac{\varphi_{k-m}-\varphi_{k-m+1}}{\tau}\\&=\tfrac{R^m-R^{m-1}}{\tau}\varphi_{k-m}+\tau ARf_{k,m}\\&+\sum_{j=1}^{m-1}\tau\Big\{R^{m-j}\tfrac{f_{k-m+j,j}-f_{k-m+1+j,j}}{\tau}\\&+\tfrac{R^{m-j+1}-R^{m-j}}{\tau}f_{k-m+j,j}\Big\},\ m\le k.\end{aligned}$$

Using the triangle inequality, we get

$$\left\|\frac{u_{k-1,m}-u_{k-1,m-1}}{\tau}\right\|_E \le \|R^{m-1}\|_{E\to E}\left\|\frac{\varphi_{k-m}-\varphi_{k-m+1}}{\tau}\right\|_E$$

$$+\|R^m\|_{E\to E}\|A\varphi_{k-m}\|_E+\|\tau AR\|_{E\to E}\|f_{k,m}\|_E$$

$$+\sum_{j=1}^{m-1}\tau\|R^{m-j}\|_{E\to E}\left\|\frac{f_{k-m+j,j}-f_{k-m+1+j,j}}{\tau}\right\|_E \tag{3.21}$$

$$+\sum_{j=1}^{m-1}\|\tau AR^{m-j+1}f_{k-m+j,j}\|_E,\ m\le k$$

Here,

$$\sum_{j=1}^{m-1}\tau AR^{m-j+1}f_{k-m+j,j}=\sum_{j=1}^{m-1}(I-R)R^{m-j}f_{k-m+j,j}$$

$$=\sum_{j=1}^{m-1}R^{m-j}f_{k-m+j,j}-\sum_{j=1}^{m-1}R^{m-j+1}f_{k-m+j,j}$$

$$=\sum_{j=1}^{m-1}R^{m-j}f_{k-m+j,j}-R^m f_{k-m+1,1}-\sum_{j=1}^{m-2}R^{m-j}f_{k-m+j+1,j+1}$$

$$=Rf_{k-1,m-1}+R^m f_{k-m+1,1}+\sum_{j=1}^{m-2}\tau R^{m-j}\frac{f_{k-m+j,j}-f_{k-m+j+1,j+1}}{\tau}.$$

Therefore,

$$\left\|\sum_{j=1}^{m-1}\tau AR^{m-j+1}f_{k-m+j,j}\right\|_E \le \|R\|_{E\to E}\|f_{k-1,m-1}\|_E+\|R^m\|_{E\to E}\|f_{k-m+1,1}\|_E$$

$$+\sum_{j=1}^{m-2}\tau\|R^{m-j}\|_{E\to E}\left\|\frac{f_{k-m+j,j+1}-f_{k-m+j,j}}{\tau}\right\|_E. \tag{3.22}$$

Combining estimates (3.7), (3.21) and (3.22), we get

$$\max_{1\le k,m\le N}\left\|\frac{u_{k-1,m}-u_{k-1,m-1}}{\tau}\right\|_E \le M\left[\max_{1\le m\le N}\left\|\frac{\varphi_k-\varphi_{k-1}}{\tau}\right\|_E\right. \tag{3.23}$$

$$\left.+\max_{1\le k\le N}\|A\varphi_k\|_E+\max_{1\le k,m\le N}\|f_{k,m}\|_E+\max_{1\le k,m\le N}\left\|\frac{f_{k,m}-f_{k-1,m-1}}{\tau}\right\|_E\right]$$

From the estimates (3.18), (3.19), (3.20) and (3.23) we obtain the statement of the Theorem 3.3.

Theorem 3.4. Let τ be a sufficiently small number and $\psi_m, \varphi_k \in D(A)$. Then for the solution of (3.1), we have the following coercive inequality

$$\max_{1\le k,m\le N}\|\tfrac{u_{k,m}-u_{k-1,m-1}}{\tau}\|_{E_\alpha}+\max_{1\le k,m\le N}\|Au_{k,m}\|_{E_\alpha}$$

$$\le M\left(\max_{0\le m\le N}\|A\psi_m\|_{E_\alpha}+\max_{0\le k\le N}\|A\varphi_k\|_{E_\alpha}+\tfrac{1}{\alpha(1-\alpha)}\max_{1\le k,m\le N}\|f_{k,m}\|_{E_\alpha}\right),$$

where M is independent of τ, ψ_m, φ_k, and $f_{k,m}$.

Proof. It is very well known that, from the fact that the operators R, exp{-λA} and A commute, it follows that (Ashyralyev and Sobolevskii, 1994)

$$\|R\|_{E_\alpha\to E_\alpha}\le\|R\|_{E\to E}. \tag{3.24}$$

Using the identity(3.9), the definition of norm of the space E_α and estimate(3.24), we obtain

$$\left\|R^kA\psi_{m-k}\right\|_{E_\alpha}\le\left\|R^k\right\|_{E\to E}\|A\psi_{m-k}\|_{E_\alpha}\le M\|A\psi_{m-k}\|_{E_\alpha},\ k\le m. \tag{3.25}$$

To estimate V_k in the norm of E_α we use the following Cauchy-Riesz representation formula for the operator $A(\lambda+A)^{-1}\sum_{j=1}^{k}A\tau R^{k-j+1}$:

$$\lambda^\alpha A(\lambda+A)^{-1}V_k$$

$$=\tfrac{1}{2\pi i}\int_G\tfrac{z\lambda^\alpha}{(1+z)^{k-j+1}}\tfrac{\tau}{\lambda\tau+z}A(z-\tau A)^{-1}f_{j,m-k+j}\ ,$$

where $G=\left\{\rho e^{\pm i\theta},0\le\rho<\infty,0\le\theta\le\tfrac{\pi}{2}\right\}$. Then we can write

$$V_k=\frac{1}{2\pi i}\int_G\sum_{j=1}^{k}\frac{z\lambda^\alpha}{(1+z)^{k-j+1}}\frac{\tau}{\lambda\tau+z}A(z-\tau A)^{-1}f_{j,m-k+j}\tau.$$

Since $z=\rho e^{\pm i\theta}$, with $|\theta|\le\tfrac{\pi}{2}$, from the strong positivity of A it follows that

$$\left|(\tfrac{z}{\tau})^\alpha\right|\left\|A(\tfrac{z}{\tau}-A)^{-1}f_{j,m-k+j}\right\|_E\le M(\tfrac{\rho}{\tau})^\alpha\left\|A(\tfrac{\rho}{\tau}+A)^{-1}f_{j,m-k+j}\right\|_E,$$

$$\tfrac{1}{|\lambda\tau+z|}\le\tfrac{M}{\lambda\tau+\rho}.$$

Hence,

$$\|\lambda^\alpha A(\lambda+A)^{-1}V_k\|_E$$

$$\le M\int_0^\infty\sum_{j=1}^{k}\frac{\rho^{1-\alpha}}{(1+2\rho\cos\theta+\rho^2)^{\frac{k-j+1}{2}}}\frac{(\tau\lambda)^\alpha}{\lambda\tau+\rho}d\rho\left\|f_{j,m-k+j}\right\|_{E_\alpha}.$$

Summing the geometric progression and using the estimate (3.7), we get

$$\|\lambda^{\alpha}A(\lambda+A)^{-1}V_k\|_E$$

$$\leq M\int_0^{\infty}\sum_{j=1}^{k}\frac{\rho^{1-\alpha}}{(1+2\rho\cos\theta+\rho^2)^{\frac{1}{2}}}[1-\frac{1}{(1+2\rho\cos\theta+\rho^2)^{\frac{1}{2}}}]\frac{(\tau\lambda)^{\alpha}}{\lambda\tau+\rho}d\rho\,\|f_{j,m-k+j}\|_{E_\alpha} \tag{3.26}$$

$$\leq \frac{M_1}{\cos\theta}\int_0^{\infty}\frac{(\tau\lambda)^{\alpha}}{\rho^{\alpha}(\lambda\tau+\rho)}d\rho\,\|f_{j,m-k+j}\|_{E_\alpha}\leq\frac{M(\theta)}{\alpha(1-\alpha)}\,\|f_{j,m-k+j}\|_{E_\alpha}.$$

Finally, using the triangle inequality and the last estimate and the estimates (3.25), (3.26), we obtain the following estimate

$$\max_{1\leq k,m\leq N}\|\tfrac{u_{k,m}-u_{k-1,m-1}}{\tau}\|_{E_\alpha}+\max_{1\leq k,m\leq N}\|Au_{k,m}\|_{E_\alpha}$$

$$\leq M\left(\max_{0\leq m\leq N}\|A\psi_m\|_{E_\alpha}+\tfrac{1}{\alpha(1-\alpha)}\max_{1\leq k,m\leq N}\|f_{k,m}\|_{E_\alpha}\right).$$

In a similar manner one can show that

$$\max_{1\leq k,m\leq N}\|\tfrac{u_{k,m}-u_{k-1,m-1}}{\tau}\|_{E_\alpha}+\max_{1\leq k,m\leq N}\|Au_{k,m}\|_{E_\alpha}$$

$$\leq M\left(+\max_{0\leq k\leq N}\|A\varphi_k\|_{E_\alpha}+\tfrac{1}{\alpha(1-\alpha)}\max_{1\leq k,m\leq N}\|f_{k,m}\|_{E_\alpha}\right).$$

Theorem 3.4 is proved.

Theorem 3.5. Let τ be a sufficiently small number and $\psi_m,\varphi_k\in D(A)$. Then for the solution of (3.1), we have the following coercive inequality

$$\max_{1\leq k,m\leq N}\|\tfrac{u_{k,m}-u_{k-1,m}}{\tau}\|_{E_\alpha}+\max_{1\leq k,m\leq N}\|\tfrac{u_{k-1,m}-u_{k-1,m-1}}{\tau}\|_{E_\alpha}$$

$$\leq M\left(\max_{0\leq m\leq N}\|A\psi_m\|_{E_\alpha}+\max_{1\leq k,m\leq N}\|\tfrac{\psi_m-\psi_{m-1}}{\tau}\|_{E_\alpha}\right.$$

$$+\max_{0\leq k\leq N}\|A\varphi_k\|_{E_\alpha}+\max_{1\leq k,m\leq N}\|\tfrac{\varphi_k-\varphi_{k-1}}{\tau}\|_{E_\alpha}+\max_{1\leq k,m\leq N}\|f_{k,m}\|_{E_\alpha},$$

$$\left.+\max_{1\leq k,m\leq N}\|\tfrac{f_{k,m}-f_{k,m-1}}{\tau}\|_{E_\alpha}+\max_{1\leq k,m\leq N}\|\tfrac{f_{k,m}-f_{k-1,m}}{\tau}\|_{E_\alpha}\right),$$

where M is independent of τ, ψ_m, φ_k, and $f_{k,m}$.

Proof. Using the identity (3.6), we get

$$\begin{aligned}
&\tfrac{u_{k,m}-u_{k-1,m}}{\tau} = R^{k-1}\tfrac{\psi_{m-k}-\psi_{m-k+1}}{\tau} \\
&+\tfrac{R^k-R^{k-1}}{\tau}\psi_{m-k} + \tau AR f_{k,m} \\
&+\sum_{j=1}^{k-1}\tau\Big\{R^{k-j}\tfrac{f_{j,m-k+j}-f_{j,m-k+1+j}}{\tau} \\
&+\tfrac{R^{k-j+1}-R^{k-j}}{\tau}f_{j,m-k+j}\Big\}, k \le m.
\end{aligned}$$

Using the triangle inequality, we get

$$\begin{aligned}
&\left\|\tfrac{u_{k,m}-u_{k-1,m}}{\tau}\right\|_{E_\alpha} \le \left\|R^{k-1}\right\|_{E_\alpha\to E_\alpha}\left\|\tfrac{\psi_{m-k}-\psi_{m-k+1}}{\tau}\right\|_{E_\alpha} \\
&+\left\|R^k\right\|_{E_\alpha\to E_\alpha}\|A\psi_{m-k}\|_{E_\alpha} + \|\tau AR\|_{E_\alpha\to E_\alpha}\|f_{k,m}\|_{E_\alpha} \\
&+\sum_{j=1}^{k-1}\tau\left\|R^{k-j}\right\|_{E_\alpha\to E_\alpha}\left\|\tfrac{f_{j,m-k+j}-f_{j,m-k+1+j}}{\tau}\right\|_{E_\alpha} \\
&+\left\|\sum_{j=1}^{k-1}\tau AR^{k-j+1}f_{j,m-k+j}\right\|_{E_\alpha}, \ k \le m.
\end{aligned} \tag{3.27}$$

Here,

$$\begin{aligned}
&\sum_{j=1}^{k-1}\tau AR^{k-j+1}f_{j,m-k+j} = \sum_{j=1}^{k-1}(I-R)R^{k-j}f_{j,m-k+j} \\
&= \sum_{j=1}^{k-1}R^{k-j}f_{j,m-k+j} - \sum_{j=1}^{k-1}R^{k-j+1}f_{j,m-k+j} \\
&= \sum_{j=1}^{k-1}R^{k-j}f_{j,m-k+j} - R^k f_{1,m-k+1} - \sum_{j=1}^{k-2}R^{k-j}f_{j+1,m-k+j+1} \\
&= Rf_{k-1,m-1} + R^k f_{1,m-k+1} + \sum_{j=2}^{k-2}\tau R^{k-j}\tfrac{f_{j,m-k+j}-f_{j+1,m-k+j+1}}{\tau}.
\end{aligned}$$

Therefore,

$$\left\|\sum_{j=1}^{k-1}\tau AR^{k-j+1}f_{j,m-k+j}\right\|_{E_\alpha}\leq\|R\|_{E\to E}\|f_{k-1,m-1}\|_{E_\alpha}+\left\|R^k\right\|_{E\to E}\|f_{1,m-k+1}\|_{E_\alpha}$$

$$+\sum_{j=1}^{k-2}\tau\left\|R^{k-j}\right\|_{E_\alpha\to E_\alpha}\left\|\tfrac{f_{j+1,m-k+j+1}-f_{j,m-k+j}}{\tau}\right\|_{E_\alpha}. \tag{3.28}$$

Combining estimates (3.7), (3.27) and (3.28), we get

$$\max_{1\leq k,m\leq N}\left\|\tfrac{u_{k,m}-u_{k-1,m}}{\tau}\right\|_{E_\alpha}\leq M\left[\max_{1\leq m\leq N}\left\|\tfrac{\psi_m-\psi_{m-1}}{\tau}\right\|_{E_\alpha}\right. \tag{3.29}$$

$$\left.+\max_{1\leq m\leq N}\|A\psi_m\|_{E_\alpha}+\max_{1\leq k,m\leq N}\|f_{k,m}\|_{E_\alpha}+\max_{1\leq k,m\leq N}\left\|\tfrac{f_{k,m}-f_{k-1,m-1}}{\tau}\right\|_{E_\alpha}\right].$$

Using the identity (3.8), we get

$$\tfrac{u_{k,m}-u_{k-1,m}}{\tau}=R^m\tfrac{\varphi_{k-m}-\varphi_{k-1-m}}{\tau}$$

$$+\sum_{j=1}^{m}\tau R^{m-j+1}\tfrac{f_{k-m+j,j}-f_{k-1+m+j,j}}{\tau},\ m\leq k.$$

Using the triangle inequality, we get

$$\left\|\tfrac{u_{k,m}-u_{k-1,m}}{\tau}\right\|_{E_\alpha}\leq\|R^m\|_{E_\alpha\to E_\alpha}\left\|\tfrac{\varphi_{k-m}-\varphi_{k-1-m}}{\tau}\right\|_{E_\alpha}$$

$$+\sum_{j=1}^{m}\tau\|R^{m-j+1}\|_{E_\alpha\to E_\alpha}\left\|\tfrac{f_{k-m+j,j}-f_{k-1+m+j,j}}{\tau}\right\|_{E_\alpha},\ m\leq k.$$

Applying the estimate (3.7), we get

$$\max_{1\leq k,m\leq N}\left\|\tfrac{u_{k,m}-u_{k-1,m}}{\tau}\right\|_{E_\alpha}$$

$$\leq M\left[\max_{1\leq k\leq N}\left\|\tfrac{\varphi_k-\varphi_{k-1}}{\tau}\right\|_{E_\alpha}+\max_{1\leq k,m\leq N}\left\|\tfrac{f_{k,m}-f_{k-1,m}}{\tau}\right\|_{E_\alpha}\right],\ m\leq k. \tag{3.30}$$

Using the identity (3.6), we get

$$\tfrac{u_{k-1,m}-u_{k-1,m-1}}{\tau}=R^k\tfrac{\psi_{m-k-1}-\psi_{m-k-2}}{\tau}$$

$$+\sum_{j=1}^{k}\tau R^{k-j+1}\tfrac{f_{j,m-k+j}-f_{j,m-1-k+j}}{\tau},\ k\leq m.$$

Using the triangle inequality,we get

$$\left\| \frac{u_{k-1,m}-u_{k-1,m-1}}{\tau} \right\|_{E_\alpha} \leq \left\| R^k \right\|_{E_\alpha \to E_\alpha} \left\| \frac{\psi_{m-k-1}-\psi_{m-k-2}}{\tau} \right\|_{E_\alpha}$$

$$+ \sum_{j=1}^{k} \tau \left\| R^{k-j+1} \right\|_{E_\alpha \to E_\alpha} \left\| \frac{f_{j,m-k+j}-f_{j,m-1-k+j}}{\tau} \right\|_{E_\alpha}, \ k \leq m.$$

Applying the estimate (3.7), we get

$$\begin{aligned} &\max_{1\leq k,m\leq N} \left\| \frac{u_{k-1,m}-u_{k-1,m-1}}{\tau} \right\|_{E_\alpha} \\ &\leq M \left[\max_{1\leq m\leq N} \left\| \frac{\psi_m-\psi_{m-1}}{\tau} \right\|_{E_\alpha} + \max_{1\leq k,m\leq N} \left\| \frac{f_{k,m}-f_{k,m-1}}{\tau} \right\|_{E_\alpha} \right], \ m \leq k. \end{aligned} \tag{3.31}$$

Using the identity (3.8), we get

$$\begin{aligned} \frac{u_{k-1,m}-u_{k-1,m-1}}{\tau} &= R^{m-1} \frac{\varphi_{k-m}-\varphi_{k-m+1}}{\tau} \\ &+ \frac{R^m-R^{m-1}}{\tau} \varphi_{k-m} + \tau A R f_{k,m} \\ &+ \sum_{j=1}^{m-1} \tau \left\{ R^{m-j} \frac{f_{k-m+j,j}-f_{k-m+1+j,j}}{\tau} \right. \\ &\left. + \frac{R^{m-j+1}-R^{m-j}}{\tau} f_{k-m+j,j} \right\}, \ m \leq k \end{aligned}$$

Using the triangle inequality, we get

$$\begin{aligned} \left\| \frac{u_{k-1,m}-u_{k-1,m-1}}{\tau} \right\|_{E_\alpha} &\leq \|R^{m-1}\|_{E_\alpha \to E_\alpha} \left\| \frac{\varphi_{k-m}-\varphi_{k-m+1}}{\tau} \right\|_{E_\alpha} \\ &+ \|R^m\|_{E_\alpha \to E_\alpha} \|A\varphi_{k-m}\|_{E_\alpha} + \|\tau AR\|_{E\to E} \|f_{k,m}\|_{E_\alpha} \\ &+ \sum_{j=1}^{m-1} \tau \|R^{m-j}\|_{E_\alpha \to E_\alpha} \left\| \frac{f_{k-m+j,j}-f_{k-m+1+j,j}}{\tau} \right\|_{E_\alpha} \\ &+ \sum_{j=1}^{m-1} \|\tau A R^{m-j+1} f_{k-m+j,j}\|_{E_\alpha}, \ m \leq k. \end{aligned} \tag{3.32}$$

Here,

$$\sum_{j=1}^{m-1} \tau A R^{m-j+1} f_{k-m+j,j} = \sum_{j=1}^{m-1} (I-R) R^{m-j} f_{k-m+j,j}$$

$$= \sum_{j=1}^{m-1} R^{m-j} f_{k-m+j,j} - \sum_{j=1}^{m-1} R^{m-j+1} f_{k-m+j,j}$$

$$= \sum_{j=1}^{m-1} R^{m-j} f_{k-m+j,j} - R^m f_{k-m+1,1} - \sum_{j=1}^{m-2} R^{m-j} f_{k-m+j+1,j+1}$$

$$= R f_{k-1,m-1} + R^m f_{k-m+1,1} + \sum_{j=1}^{m-2} \tau R^{m-j} \tfrac{f_{k-m+j,j} - f_{k-m+j+1,j+1}}{\tau}.$$

Therefore,

$$\begin{aligned} \left\| \sum_{j=1}^{m-1} \tau A R^{m-j+1} f_{k-m+j,j} \right\|_{E_\alpha} &\le \|R\|_{E_\alpha \to E_\alpha} \|f_{k-1,m-1}\|_{E_\alpha} \\ &+ \|R^m\|_{E_\alpha \to E_\alpha} \|f_{k-m+1,1}\|_{E_\alpha} \\ &+ \sum_{j=1}^{m-2} \tau \|R^{m-j}\|_{E_\alpha \to E_\alpha} \left\| \tfrac{f_{k-m+j,j+1} - f_{k-m+j,j}}{\tau} \right\|_{E_\alpha}. \end{aligned} \tag{3.33}$$

Combining estimates (3.7), (3.32) and (3.33), we get

$$\begin{aligned} \max_{1\le k,m\le N} \left\| \tfrac{u_{k-1,m} - u_{k-1,m-1}}{\tau} \right\|_{E_\alpha} &\le M \left[\max_{1\le m\le N} \left\| \tfrac{\varphi_k - \varphi_{k-1}}{\tau} \right\|_{E_\alpha} \right. \\ &\left. + \max_{1\le k\le N} \|A\varphi_k\|_{E_\alpha} + \max_{1\le k,m\le N} \|f_{k,m}\|_{E_\alpha} + \max_{1\le k,m\le N} \left\| \tfrac{f_{k,m} - f_{k-1,m-1}}{\tau} \right\|_{E_\alpha} \right]. \end{aligned} \tag{3.34}$$

From the estimates (3.29), (3.30), (3.31) and(3.34) we obtain the statement of the Theorem 3.5.

Now, the applications of Theorem 3.1, Theorem 3.2, Theorem 3.3, Theorem 3.4 and Theorem 3.5 will be given. First, the mixed boundary value problem for one dimensional ultra-parabolic equation (2.19) is considered. The discretization of problem (2.19) is carried out in two steps. In the first step, let us define the grid sets

$$[0,1]_h = \{x = x_n : x_n = nh,\ 0 \le n \le M,\ Mh = 1\}.$$

We introduce the Banach spaces $C_h = C_h[0,1]_h, C_h^\beta = C^\beta[0,1]_h$ of grid functions $\varphi^h(x) = \{\varphi(h_1 m_1, \cdots, h_n m_n)\}$ defined on $[0,1]_h$, equipped with the norms

$$\| \varphi^h \|_{C[0,1]_h} = \max_{x \in [0,1]_h} |\varphi^h(x)|,$$

$$\| \varphi^h \|^\beta_{C[0,1]_h} = \| \varphi^h \|_{C[0,1]_h} + \sup_{0 \le x < x+h \le 1} \frac{|v^h(x+h) - v^h(x)|}{h^\beta}.$$

To the differential operator A generated by problem (2.19) we assign the difference operator A_h^x by the formula

$$A_h^x u^h(x) = \left\{ -a(x) \frac{u_{n+1} - 2u_n + u_{n-1}}{h^2} \right\}_1^M + \delta u^h(x)$$

acting in the space of grid functions $u^h(x) = \{u_n\}_0^M$ satisfying conditions $u_0^h = u_M^h$, $u_1^h - u_0^h = u_M^h - u_{M-1}^h$.

With the help of A_h^x we arrive at the initial boundary-value problem

$$\begin{cases} \frac{\partial u^h(t,s,x)}{\partial t} + \frac{\partial u^h(t,s,x)}{\partial s} + A_h^x u^h(t,s,x) = f^h(t,s,x), \\ \\ 0 < t, s < 1, \ x \in [0,l]_h, \\ \\ u^h(0,s,x) = \psi^h(s,x), \ 0 \le s \le 1, u^h(t,0,x) = \varphi^h(t,x), \\ \\ 0 \le t \le 1, \ x \in [0,l]_h, \end{cases} \tag{3.35}$$

for an infinite system of ordinary differential equations. In the second step, we replace problem (3.35) by difference scheme (3.1)

$$\begin{cases} \frac{u_{k,m}^h - u_{k-1,m-1}^h}{\tau} + A_h^x u_{k,m}^h = f_{k,m}^h(x), x \in [0,l]_h, \\ \\ f_{k,m}^h(x) = f^h(t_k, s_m, x), t_k = k\tau, s_m = m\tau, \\ \\ 1 \le k, m \le N, \ x \in [0,l]_h, \\ \\ u_{0,m}^h = \psi_m^h, 0 \le m \le N, u_{k,0}^h = \varphi_k^h, 0 \le k \le N. \end{cases} \tag{3.36}$$

It is known that A_h^x is a positive operator in $C[0,l]_h$ and $C^\beta[0,l]_h$ (*see*[19]). Let us give a number of the corollary of Theorems (3.1), (3.2), (3.3), (3.4) and (3.5)

Theorem 3.6. For the solution of difference scheme (3.36), we have the following stability inequality

$$\max_{1\le k,m\le N}\|u^h_{k,m}\|_{C[0,1]_h}\le M\left(\max_{0\le m\le N}\|\psi^h_m\|_{C[0,1]_h}+\max_{0\le k\le N}\|\varphi^h_k\|_{C[0,1]_h}\right.$$

$$\left.+\max_{1\le k,m\le N}\|f^h_{k,m}\|_{C[0,1]_h}\right),$$

where M is independent of τ, ψ^h_m, φ^h_k, and $f^h_{k,m}$.

The proof of Theorem 3.6 is based on the Theorem 3.1 and the positivity of operator A^x_h .

Theorem 3.7. For the solution of difference scheme (3.36), we have the following coercive stability inequalities

$$\max_{1\le k,m\le N}\|\tfrac{u^h_{k,m}-u^h_{k-1,m-1}}{\tau}\|_{C[0,1]_h}+\max_{1\le k,m\le N}\|h^{-2}\left\{u^{n+1}_{k,m}-2u^n_{k,m}+u^{n-1}_{k,m}\right\}^N_1\|_{C[0,1]_h}$$

$$\le M\left(\max_{0\le m\le N}\|h^{-2}\left\{\psi^{n+1}_m-2\psi^n_m+\psi^{n-1}_m\right\}^N_1\|_{C[0,1]_h}\right.$$

$$+\max_{0\le k\le N}\|h^{-2}\left\{\varphi^{n+1}_k-2\varphi^n_k+\varphi^{n-1}_k\right\}^N_1\|_{C[0,1]_h}$$

$$\left.+\ln\tfrac{1}{\tau+|h|}\max_{1\le k,m\le N}\|f^h_{k,m}\|_{C[0,1]_h}\right),$$

where M is independent of τ, ψ^h_m, φ^h_k, and $f^h_{k,m}$.

The proof of Theorem 3.7 is based on the Theorem 3.2 and the positivity of operator A^x_h .

Theorem 3.8. For the solution of difference scheme (3.36), we have the

following coercive stability inequalities

$$\max_{1\le k,m\le N}\|\tfrac{u^h_{k,m}-u^h_{k-1,m}}{\tau}\|_{C[0,1]_h}+\max_{1\le k,m\le N}\|\tfrac{u^h_{k-1,m}-u^h_{k-1,m-1}}{\tau}\|_{C[0,1]_h}$$

$$\le M\left(\max_{0\le m\le N}\|h^{-2}\left\{\psi^{n+1}_m-2\psi^n_m+\psi^{n-1}_m\right\}_1^N\|_{C[0,1]_h}+\max_{1\le k,m\le N}\left\|f^h_{k,m}\right\|\right.$$

$$+\max_{1\le m\le N}\|\tfrac{\psi^h_m-\psi^h_{m-1}}{\tau}\|_{C[0,1]_h}+\max_{1\le k,m\le N}\|\tfrac{f^h_{k,m}-f^h_{k-1,m}}{\tau}\|_{C[0,1]_h}$$

$$+\max_{0\le k\le N}\|h^{-2}\left\{\varphi^{n+1}_k-2\varphi^n_k+\varphi^{n-1}_k\right\}_1^N\|_{C[0,1]_h}$$

$$\left.+\max_{1\le k,m\le N}\|\tfrac{f^h_{k,m}-f^h_{m-1}}{\tau}\|_{C[0,1]_h}+\max_{0\le k\le N}\|\tfrac{\varphi^h_k-\varphi^h_{k-1}}{\tau}\|_{C[0,1]_h}\right),$$

where M is independent of τ, ψ^h_m, φ^h_k, and $f^h_{k,m}$.

The proof of Theorem 3.7 is based on the Theorem 3.3 and the positivity of operator A^x_h .

Secondly, the mixed boundary value problem for the multidimensional ultra-parabolic equation (2.21) is considered. The discretization of problem (2.21) is carried out in two steps. In the first step, let us define the grid sets

$$\widetilde{\Omega}_h=\{x=x_r=(h_1r_1,\cdots,h_nr_n),r=(r_1,\cdots,r_n),$$

$$0\le r_j\le N_j,N_jK_j=L,j=1,\cdots,n\},$$

$$\Omega_h=\widetilde{\Omega}_h\cap\Omega,S_h=\widetilde{\Omega}_h\cap S.$$

We introduce the Banach spaces $C_h=C_h(\widetilde{\Omega}_h),C^\beta_h=C^\beta_{01}(\widetilde{\Omega}_h)$ of grid functions

$\varphi^h(x) = \{\varphi(h_1 m_1, \cdots, h_n m_n)\}$ defined on $\widetilde{\Omega}_h$, equipped with the norms

$$\| \varphi^h \|_{C(\overline{\Omega}_h)} = \max_{x \in \overline{\Omega}_h} |\varphi^h(x)|,$$

$$\| \varphi^h \|_{C_{01}^{\beta}(\overline{\Omega}_h)} = \| \varphi^h \|_{C(\overline{\Omega_h})}$$

$$+ \sup_{\substack{0 \le x_k < x_k + h_k \le 1 \\ 1 \le k \le n}} |\varphi^h(x_1, \cdots, x_n) - \varphi^h(x_1 + h_1, \cdots, x_n + h_n)|$$

$$\times \prod_{k=1}^{n} h_k^{-\beta_k} x_k^{\beta_k} (1 - x_k - h_k)^{\beta_k}.$$

To the differential operator A generated by problem (2.20) we assign the difference operator A_h^x by the formula

$$A_h^x u_x^h = -\sum_{r=1}^{n} a_r(x) (u_{\underline{\ }_{x_r}}^h)_{x_r, j_r}$$

acting in the space of grid functions $u^h(x)$, satisfying the condition $u^h(x) = 0$ for all $x \in S_h$.

With the help of A_h^x we arrive at the initial boundary-value problem

$$\begin{cases} \frac{\partial u^h(t,s,x)}{\partial t} + \frac{\partial u^h(t,s,x)}{\partial s} + A_h^x u^h(t,s,x) = f^h(t,s,x), \\ 0 < t, s < 1, \ x \in \Omega_h, \\ u^h(0,s,x) = \psi^h(s,x), \ 0 \le s \le 1, u^h(t,0,x) = \varphi^h(t,x), \\ 0 \le t \le 1, \ x \in \widetilde{\Omega}_h \end{cases} \tag{3.37}$$

for an infinite system of ordinary differential equations.

In the second step, we replace problem (3.37) by difference scheme (3.1)

$$\begin{cases} \frac{u_{k,m}^h - u_{k-1,m-1}^h}{\tau} + A_h^x u_{k,m}^h = f_{k,m}^h(x), x \in \Omega_h, \\ f_{k,m}^h(x) = f^h(t_k, s_m, x), t_k = k\tau, s_m = m\tau, \ 1 \le k, m \le N, \ x \in \widetilde{\Omega}_h, \ , \\ u_{0,m}^h = \psi_m^h, 0 \le m \le N, u_{k,0}^h = \varphi_k^h, 0 \le k \le N \end{cases} \tag{3.38}$$

It is known that A_h^x is a positive operator in $C(\widetilde{\Omega}_h)$ and $C_{01}^{\beta}(\overline{\Omega}_h)$. Let us give another corollary of the Theorems (3.1), (3.2) and (3.3).

Theorem 3.9. For the solution of difference scheme (3.42), we have the following stability inequality

$$\max_{1\le k,m\le N}\|u_{k,m}^h\|_{C(\widetilde{\Omega}_h)}$$

$$\le M\left(\max_{0\le m\le N}\|\psi_m^h\|_{C(\widetilde{\Omega}_h)}+\max_{0\le k\le N}\|\varphi_k^h\|_{C(\widetilde{\Omega}_h)}+\max_{1\le k,m\le N}\|f_{k,m}^h\|_{C(\widetilde{\Omega}_h)}\right),$$

where M is independent of τ, ψ_m^h, φ_k^h, and $f_{k,m}^h$.

The proof of Theorem 3.9 is based on the Theorem 3.1 and the positivity of operator A_h^x .

Theorem 3.10. For the solution of difference scheme (3.36), we have the following coercive stability inequalities

$$\max_{1\le k,m\le N}\|\tfrac{u_{k,m}^h-u_{k-1,m-1}^h}{\tau}\|_{C(\widetilde{\Omega}_h)}+\max_{1\le k,m\le r}\sum_{r=1}^{n}\|u^h_{(\bar{x}_r x_r,\, j_r)\ k,m}\|_{C(\widetilde{\Omega}_h)}$$

$$\le M\ln\tfrac{1}{|h|}\left(\max_{1\le m\le N}\sum_{r=1}^{n}\|\psi^h_{(\bar{x}_r x_r,\, j_r)\ m}\|_{C(\widetilde{\Omega}_h)}+\max_{0\le k\le N}\sum_{r=1}^{n}\|\varphi^h_{(\bar{x}_r x_r,\, j_r)\ k}\|_{C(\widetilde{\Omega}_h)}\right)$$

$$+M\ln\tfrac{1}{\tau+|h|}\max_{1\le k,\ m\le N}\|f^{\ h}_{k,m}\|_{C(\widetilde{\Omega}_h)},$$

where M is independent of τ, ψ_m^h, φ_k^h, and $f_{k,m}^h$.

The proof of Theorem 3.10 is based on the Theorem 3.2 and the positivity of operator A_h^x .

Theorem 3.11. For the solution of difference scheme (3.36), we have the

following coercive stability inequalities

$$\max_{1\le k,m\le N}\Big\|\frac{u^h_{k,m}-u^h_{k-1,m}}{\tau}\Big\|_{C(\widetilde{\Omega}_h)}+\max_{1\le k,m\le N}\Big\|\frac{u^h_{k-1,m}-u^h_{k-1,m-1}}{\tau}\Big\|_{C(\widetilde{\Omega}_h)}$$

$$\le M\left(\max_{1\le m\le N}\sum_{r=1}^{n}\big\|\psi^h_{(\bar{x}_r x_r,\, j_r)\; m}\big\|_{C(\widetilde{\Omega}_h)}+\max_{1\le k,m\le N}\big\|f^h_{k,m}\big\|_{C(\widetilde{\Omega}_h)}\right.$$

$$+\max_{1\le m\le N}\Big\|\frac{\psi^h_m-\psi^h_{m-1}}{\tau}\Big\|_{C(\widetilde{\Omega}_h)}+\max_{1\le k,m\le N}\Big\|\frac{f^h_{k,m}-f^h_{k-1,m}}{\tau}\Big\|_{C(\widetilde{\Omega}_h)}$$

$$+\max_{0\le k\le N}\sum_{r=1}^{n}\big\|\varphi^h_{(\bar{x}_r x_r,\, j_r)\; k}\big\|_{C(\widetilde{\Omega}_h)}$$

$$\left.+\max_{1\le k,m\le N}\Big\|\frac{f^h_{k,m}-f^h_{m-1}}{\tau}\Big\|_{C(\widetilde{\Omega}_h)}+\max_{0\le k\le N}\Big\|\frac{\varphi^h_k-\varphi^h_{k-1}}{\tau}\Big\|_{C(\widetilde{\Omega}_h)}\right),$$

where M is independent of τ, ψ^h_m, φ^h_k, and $f^h_{k,m}$.

The proof of Theorem 3.11 is based on the Theorem 3.3 and the positivity of operator A^x_h .

Third, the mixed boundary value problem for the multidimensional ultra-parabolic equation (2.23) is considered. The discretization of problem (2.23) is carried out in two steps. In the first step let us give the difference operator A^x_h by the formula

$$A^x_h u^h_x=\sum_{2m\le|r|\le S} b^x_r D^r_h u^h_x+\delta u^h_x. \tag{3.39}$$

The coefficients are chosen in such a way that the operator A^x_h approximates in a specified way the operator

$$\sum_{|r|=2m} a_r(x)\frac{\partial^{|r|}}{\partial x_1^{r_1}...\partial x_n^{r_n}}+\delta.$$

We shall assume that for $|\xi_k h|\le\pi$ the symbol $A(\xi h,h)$ of the operator $A^x_h-\delta$ satisfies the inequalities

$$(-1)^m A^x(\xi h,h)\ge M_1|\xi|^{2m},\,|\arg A^x(\xi h,h)|\le\phi<\phi_0<\frac{\pi}{2}. \tag{3.40}$$

With the help of A_h^x we arrive at the boundary value problem

$$\begin{cases} \frac{\partial u^h(t,s,x)}{\partial t} + \frac{\partial u^h(t,s,x)}{\partial s} + A_h^x u^h(y,x) = f^h(t,s,x), \\ \\ 0 < t,s < 1, x \in \mathbb{R}_h^n \\ \\ u^h(t,0,x) = \varphi^h(t,x), 0 \le t \le 1, u^h(0,s,x) = \psi^h(s,x), \\ \\ 0 \le s \le 1, x \in \mathbb{R}_h^n \end{cases} \tag{3.41}$$

for an infinite system of ordinary differential equations.

In the second step we replace problem (3.41) by the difference scheme

$$\begin{cases} \frac{u_{k,m}^h - u_{k-1,m-1}^h}{\tau} + A_h^x u_{k,m}^h = f_{k,m}^h(x), x \in \mathbb{R}_h^n, \\ \\ f_{k,m}^h(x) = f^h(t_k, s_{m,}x), t_k = k\tau, s_m = m\tau,\ 1 \le k,m \le N,\ x \in \mathbb{R}_h^n, \\ \\ u_{0,m}^h = \psi_m^h, 0 \le m \le N, u_{k,0}^h = \varphi_k^h, 0 \le k \le N. \end{cases} \tag{3.42}$$

It is known that A_h^x is a positive operator in $C(\widetilde{\Omega}_h)$and $C_{01}^{\beta}(\overline{\Omega}_h)$.

CHAPTER 4

SECOND ORDER OF DIFFERENCE SCHEMES

of accuracy difference scheme

$$\begin{cases} \frac{u_{k,m}-u_{k-1,m}}{\tau}+\frac{u_{k-1,m}-u_{k-1,m-1}}{\tau}+\frac{1}{2}A(u_{k,m}+u_{k-1,m-1})=f_{k,m}\ , \\ \\ f_{k,m}=f(t_k-\frac{\tau}{2},s_m-\frac{\tau}{2}),t_k=k\tau,\ s_m=m\tau,\ 1\le k,m\le N,\ N\tau=1, \\ \\ u_{0,m}=\psi_m,\ 0\le m\le N,\ u_{k,0}=\varphi_k,\ 0\le k\le N \end{cases} \tag{4.1}$$

is constructed. The following main theorems on stability estimates for the solution of difference scheme (4.1) is established.

Theorem 4.1. Let τ be a sufficiently small number. Then for the solution of (4.1), we have the following stability inequality

$$\max_{1\le k,m\le N}\|u_{k,m}\|_E$$

$$\le M\left(\max_{0\le m\le N}\|\psi_m\|_E+\max_{0\le k\le N}\|\varphi_k\|_E+\max_{1\le k,m\le N}\|f_{k,m}\|_E\right),$$

where M is independent of ψ_m, φ_k, and $f_{k,m}$.

Proof. Using (4.1), we get

$$\frac{u_{k,m}-u_{k-1,m-1}}{\tau}+\frac{1}{2}A(u_{k,m}+u_{k-1,m-1})=f_{k,m}.$$

From that it follows

$$u_{k,m}=Bu_{k-1,m-1}+\tau Cf_{k,m}. \tag{4.2}$$

where B=$\left(I-\frac{\tau A}{2}\right)\left(I+\frac{\tau A}{2}\right)^{-1}$ and C=$\left(I+\frac{\tau A}{2}\right)^{-1}$. By the mathematical induction, we will prove that

$$u_{k,m}=B^nu_{k-n,m-n}+\sum_{j=1}^{n}\tau B^{n-j}Cf_{k-n+j,m-n+j} \tag{4.3}$$

is true for all positive integers n. It is obvious that for n=1,2 formula (4.3) is true. Assume that for n=s

$$u_{k,m} = B^s u_{k-s,m-s} + \sum_{j=1}^{s} \tau B^{r-j} C f_{k-s+j,m-s+j} \tag{4.4}$$

is true. In formula (4.2), replacing k and m with k-s and m-s, respectively, we have

$$u_{k-s,m-s} = B u_{k-s-1,m-s-1} + \tau C f_{k-s,m-s}. \tag{4.5}$$

Then, using (4.4) and (4.5), we get

$$u_{k,m} = B^{s+1} u_{k-s-1,m-s-1} + \tau R^s C f_{k-s,m-s} + \sum_{j=1}^{s} \tau R^{s-j} C f_{k-s+j,m-s+j}.$$

From that it follows

$$u_{k,m} = B^{s+1} u_{k-s-1,m-s-1} + \sum_{j=1}^{s+1} \tau B^{s-j+1} C f_{k-s-1+j,m-s-1+j},$$

is true for n=s+1. So, formula (4.3) is proved. For $m \leq k$, replacing n with k-r in formula (4.3), we obtain that

$$u_{k,m} = B^{k-r} u_{r,m-k+r} + \sum_{j=1}^{k-r} \tau B^{k-r-j} C f_{r+j,m-k+r+j}. \tag{4.6}$$

Then using (3.6), we obtain that

$$u_{k,m} = B^{k-r} R^r \psi_{m-k} + \sum_{j=1}^{r} \tau B^{k-r} R^{r-j+1} f_{j,m-r+j} + \sum_{j=r+1}^{k} \tau B^{k-j} C f_{j,m-k+j}. \tag{4.7}$$

Using estimate (see[18])

$$||B^{k-r} R^r||_{E\to E} \leq M \text{ and } ||B^{k-j} C||_{E\to E} \leq M \tag{4.8}$$

and triangle inequality, we get

$$\|u_{k,m}\|_E \leq ||B^{k-r} R^r||_{E\to E} \|\psi_{m-k}\|_E + \sum_{j=1}^{r} \tau \|R^{k-j+1}\|_{E\to E} \|f_{j,m-k+j}\|_E$$

$$+ \sum_{j=r+1}^{k} \tau \|B^{k-j} C\|_{E\to E} \|f_{j,m-k+j}\|_E \tag{4.9}$$

$$\leq M \left[\max_{0\leq m\leq N} \|\psi_m\|_E + \max_{1\leq j\leq N} \max_{1\leq m\leq N} \|f_{j,m}\|_E \right].$$

For $k \geq m$, replacing n with m-r in formula (4.3) we get

$$u_{k,m} = B^{k-r}u_{k-m+r,r} + \sum_{j=1}^{m-r} \tau B^{m-r-j}Cf_{k-m+r+j,r+j}.$$

Then using(3.8), we obtain that

$$u_{k,m} = B^{k-r}R^r\varphi_{k-m} + \sum_{j=1}^{r} \tau B^{k-r}R^{r-j+1}f_{k-m+j,j} + \sum_{j=r+1}^{k} \tau B^{m-j}Cf_{k-m+j,j}. \quad (4.10)$$

From estimate (4.8) and triangle inequality, it follows that

$$\|u_{k,m}\|_E \leq ||B^{k-r}R^r||_{E\to E}\|\varphi_{k-m}\|_E + \sum_{j=1}^{m} \tau\|R^{m-j+1}\|_{E\to E}\|f_{k-m+j,j})\|_E$$

$$+ \sum_{j=r+1}^{k} \tau\|B^{m-j}C\|_{E\to E}\|f_{k-m+j,j}\|_E \quad (4.11)$$

$$\leq M\left[\max_{0\leq k\leq N}\|\varphi_k\|_E + \max_{1\leq j\leq N}\max_{1\leq m\leq N}\|f_{j,m}\|_E\right].$$

Combining the estimates (4.9) and (4.11) we get the Theorem 4.1.

Theorem 4.2. Let τ be a sufficiently small number and $\psi_m, \varphi_k \in D(A)$. Then for the solution of (4.1), we have the following almost coercive inequality

$$\max_{1\leq k,m\leq N}\|\tfrac{u_{k,m}+u_{k-1,m-1}}{2}\|_E + \max_{1\leq k,m\leq N}\|\tfrac{1}{2}A(u_{k,m}+u_{k-1,m-1})\|_E$$

$$\leq M\left(\max_{0\leq m\leq N}\|A\psi_m\|_E + \max_{0\leq k\leq N}\|A\varphi_k\|_E\right.$$

$$\left. + \min\left\{\ln\tfrac{1}{\tau},\ 1+|\ln\|A\|_{E\to E}|\right\}\max_{1\leq k,m\leq N}\|f_{k,m}\|\right),$$

where M is independent of τ, ψ_m, φ_k, and $f_{k,m}$.

Proof. Using the identities(3.6) and (3.8), we get

$$\tfrac{A}{2}(u_{k,m}+u_{k-1,m-1}) = AR^k\psi_{m-k} + AW_{k,m},\ k \leq m,$$

$$\tfrac{A}{2}(u_{k,m}+u_{k-1,m-1}) = AR^m\varphi_{k-m} + AW_{k,m},\ m \leq k,$$

where

$$W_{k,m} = \sum_{j=2}^{k} \left[\tau\tfrac{1}{2}R^{k-j+1}(f_{j,m-k+j} + f_{j-1,m-k+j-1})\right]$$

$$+\tau\tfrac{1}{2}R^{k}f_{1,m-k+1},\ k \le m,$$

$$W_{k,m} = \sum_{j=2}^{m} \left[\tau\tfrac{1}{2}R^{m-j+1}(f_{k-m+j,j} + f_{k-m+j-1,j-1})\right] \tag{4.12}$$

$$+\tau\tfrac{1}{2}R^{m}f_{k-m+1,1},\ m \le k.$$

To estimate $W_{k,m}$ we use the estimate (3.12) and triangle inequality, we get

$$\|AW_{k,m}\|_E \le \tau\tfrac{1}{2}\left(\sum_{j=2}^{k} \|AR^{k-j+1}\|_{E\to E}\|f_{j,m-k+j}\|_E\right.$$

$$\left.+\sum_{j=2}^{k} \|AR^{k-j+1}\|_{E\to E}\|f_{j-1,m-k+j-1}\|_E + \|AR^{k}\|_{E\to E}\|f_{1,m-k+1}\|_E\right)$$

$$\le M_1 \sum_{j=1}^{k} \tfrac{1}{k+1-j} \max_{1\le j,m\le N}\|f_{j,m}\|_E,\ k \le m,$$

$$\|AW_{k,m}\|_E \le \tau\tfrac{1}{2}\left(\sum_{j=2}^{m} \|AR^{m-j+1}\|_{E\to E}\|f_{k-m+j,j}\|_E\right.$$

$$\left.+\sum_{j=2}^{m} \|AR^{m-j+1}\|_{E\to E}\|f_{k-m+j-1,j-1}\|_E + \|AR^{m}\|_{E\to E}\|f_{k-m+1,1}\|_E\right)$$

$$\le M_2 \sum_{j=1}^{m} \tfrac{1}{m+1-j} \max_{1\le j,m\le N}\|f_{j,m}\|_E,\ m \le k.$$

Since

$$\sum_{j=1}^{k} \frac{1}{k+1-j} \le \int_1^k \frac{ds}{k+1-s} = lnk, k > 1,$$

we have

$$\|AW_{k,m}\|_E \le Mlnk \max_{1\le j,m\le N}\|f_{j,m}\|_E,$$

hence

$$\max_{1\le k,m\le N}\|AW_{k,m}\|_E \le Mln(\frac{1}{\tau}) \max_{1\le k,m\le N}\|f_{k,m}\|_E. \tag{4.13}$$

Further, we have

$$\|AW_{k,m}\|_E \le \tau^{\frac{1}{2}} \left(\sum_{j=2}^{k} \|AR^{k-j+1}\|_{E\to E} \|f_{j,m-k+j}\|_E \right.$$

$$\left. + \sum_{j=2}^{k} \|AR^{k-j+1}\|_{E\to E} \|f_{j-1,m-k+j-1}\|_E + \|AR^k\|_{E\to E} \|f_{1,m-k+1}\|_E \right.$$

$$\le \sum_{j=1}^{k} \tau \|AR^{k-j+1}\|_{E\to E} \max_{1\le j,m\le N} \|f_{j,m}\|_E \ , \ k \le m,$$

$$\|AW_{k,m}\|_E \le \tau^{\frac{1}{2}} \left(\sum_{j=2}^{m} \|AR^{m-j+1}\|_{E\to E} \|f_{k-m+j,j}\|_E \right.$$

$$\left. + \sum_{j=2}^{m} \|AR^{m-j+1}\|_{E\to E} \|f_{k-m+j-1,j-1}\|_E + \|AR^m\|_{E\to E} \|f_{k-m+1,1}\|_E \right.$$

$$\le \sum_{j=1}^{m} \tau \|AR^{m-j+1}\|_{E\to E} \max_{1\le j,m\le N} \|f_{j,m}\|_E \ , \ m \le k.$$

Using the estimate (3.14) and triangle inequality, we get

$$\|AW_{k,m}\|_E \le M[1 + \ln ||A||_{E\to E}] \max_{1\le k,m\le N} \|f_{k,m}\|_E. \tag{4.14}$$

Combining estimates (3.11), (4.13) and (4.14) we obtain the Theorem 4.2.

Theorem 4.3. Let τ be a sufficiently small number and $\psi_m, \varphi_k \in D(A)$. Then for the solution of (4.1), we have the following almost coercive inequality

$$\max_{1\le k,m\le N} \|\tfrac{u_{k,m}-u_{k-1,m}}{\tau}\|_E + \max_{1\le k,m\le N} \|\tfrac{u_{k-1,m}-u_{k-1,m-1}}{\tau}\|_E$$

$$\le M \left(\max_{0\le m\le N} \|A\psi_m\|_E + \max_{1\le k,m\le N} \|\tfrac{\psi_m-\psi_{m-1}}{\tau}\|_E \ + \max_{1\le k,m\le N} \|\tfrac{f_{k,m}-f_{k,m-1}}{\tau}\|_E \right.$$

$$+ \max_{0\le k\le N} \|A\varphi_k\|_E + \max_{1\le k,m\le N} \|\tfrac{\varphi_k-\varphi_{k-1}}{\tau}\|_E$$

$$\left. + \max_{1\le k,m\le N} \|\tfrac{f_{k,m}-f_{k-1,m}}{\tau}\|_E + \max_{1\le k,m\le N} \|f_{k,m}\|_E \right),$$

where M is independent of τ, ψ_m, φ_k, and $f_{k,m}$.

Proof. The proof of Theorem 4.3 is based on the formula (4.2), the following formulas

$$u_{k,m} = B^k \psi_{m-k} + \sum_{j=1}^{k} \tau B^{k-j} C f_{j,m-k+j}), \quad m > k,$$

$$u_{k,m} = B^m \varphi_{k-m} + \sum_{j=1}^{m} \tau B^{m-j} C f_{k-m+j,j}), \ k > m$$

and the estimate (4.8) .

Theorem 4.4. Let τ be a sufficiently small number and $\psi_m, \varphi_k \in D(A)$. Then for the solution of (4.1), we have the following coercive inequality

$$\max_{1\le k,m\le N} \| \frac{u_{k,m} + u_{k-1,m-1}}{2} \|_{E_\alpha} + \max_{1\le k,m\le N} \| \frac{1}{2} A(u_{k,m} + u_{k-1,m-1}) \|_{E_\alpha}$$

$$\le M \left(\max_{0\le m\le N} \|A\psi_m\|_{E_\alpha} + \max_{0\le k\le N} \|A\varphi_k\|_{E_\alpha} + \frac{1}{\alpha(1-\alpha)} \max_{1\le k,m\le N} \|f_{k,m}\|_{E_\alpha} \right),$$

where M is independent of τ, ψ_m, φ_k, and $f_{k,m}$.

Proof. To estimate $W_{k,m}$ in the norm of E_α we use the estimate (3.26), (3.7) and triangle inequality, we get

$$\|W_{k,m}\|_E \le \frac{M(\theta)}{\alpha(1-\alpha)} \|f_{j,m-k+j}\|_{E_\alpha} . \tag{4.15}$$

Finally, using the triangle inequality,the last estimate and the estimate (3.11), we obtain the following estimate

$$\max_{1\le k,m\le N} \| \tfrac{u_{k,m} - u_{k-1,m-1}}{\tau} \|_{E_\alpha} + \max_{1\le k,m\le N} \|Au_{k,m}\|_{E_\alpha}$$

$$\le M \left(\max_{0\le m\le N} \|A\psi_m\|_{E_\alpha} + \tfrac{1}{\alpha(1-\alpha)} \max_{1\le k,m\le N} \|f_{k,m}\|_{E_\alpha} \right).$$

In a similar manner one can show that

$$\max_{1\le k,m\le N} \| \tfrac{u_{k,m} - u_{k-1,m-1}}{\tau} \|_{E_\alpha} + \max_{1\le k,m\le N} \|Au_{k,m}\|_{E_\alpha}$$

$$\le M \left(+ \max_{0\le k\le N} \|A\varphi_k\|_{E_\alpha} + \tfrac{1}{\alpha(1-\alpha)} \max_{1\le k,m\le N} \|f_{k,m}\|_{E_\alpha} \right).$$

Theorem 4.4 is proved.

Theorem 4.5. Let τ be a sufficiently small number and $\psi_m, \varphi_k \in D(A)$. Then for the solution of (4.1), we have the following coercive inequality

$$\max_{1\le k,m\le N}\|\tfrac{u_{k,m}-u_{k-1,m}}{\tau}\|_{E_\alpha}+\max_{1\le k,m\le N}\|\tfrac{u_{k-1,m}-u_{k-1,m-1}}{\tau}\|_{E_\alpha}$$

$$\le M\left(\max_{0\le m\le N}\|A\psi_m\|_{E_\alpha}+\max_{1\le k,m\le N}\|\tfrac{\psi_m-\psi_{m-1}}{\tau}\|_{E_\alpha}+\max_{1\le k,m\le N}\|\tfrac{f_{k,m}-f_{k,m-1}}{\tau}\|_{E_\alpha}\right.$$

$$+\max_{0\le k\le N}\|A\varphi_k\|_{E_\alpha}+\max_{1\le k,m\le N}\|\tfrac{\varphi_k-\varphi_{k-1}}{\tau}\|_{E_\alpha}$$

$$\left.+\max_{1\le k,m\le N}\|\tfrac{f_{k,m}-f_{k-1,m}}{\tau}\|_{E_\alpha}+\max_{1\le k,m\le N}\|f_{k,m}\|_{E_\alpha}\right),$$

where M is independent of τ, ψ_m, φ_k, and $f_{k,m}$.

Proof. The proof of Theorem 4.5 is based on the formula (4.2) and the estimate (4.8) .

Now, the applications of Theorem (4.1), Theorem (4.2), Theorem (4.3), Theorem (4.4) and Theorem (4.5) will be given. First, the mixed boundary value problem (2.19) is considered. We replace problem (3.35) by difference scheme (4.1)

$$\begin{cases}\frac{u^h_{k,m}-u^h_{k-1,m-1}}{\tau}+\frac{1}{2}A^x_h(u^h_{k,m}+u^h_{k-1,m-1})=f^h_{k,m}(x), x\in[0,l]_h\\ f^h_{k,m}(x)=f^h(t_k-\frac{\tau}{2},s_m-\frac{\tau}{2},x), t_k=k\tau, s_m=m\tau,\\ 1\le k,m\le N,\ x\in[0,l]_h,\\ u^h_{0,m}=\psi^h_m, 0\le m\le N, u^h_{k,0}=\varphi^h_k, 0\le k\le N.\end{cases}\tag{4.16}$$

Let us give a series of these abstract theorems.

Theorem 4.6. For the solution of difference scheme (4.16), we have the following stability inequality

$$\max_{1\le k,m\le N}\|u^h_{k,m}\|_{C[0,1]_h}\le M\left(\max_{0\le m\le N}\|\psi^h_m\|_{C[0,1]_h}+\max_{0\le k\le N}\|\varphi^h_k\|_{C[0,1]_h}\right.$$

$$\left.+\max_{1\le k,m\le N}\|f^h_{k,m}\|_{C[0,1]_h}\right),$$

where M is independent of τ, ψ^h_m, φ^h_k, and $f^h_{k,m}$.

The proof of Theorem 4.6 is based on the Theorem 4.1 and the positivity of operator A_h^x .

Theorem 4.7. For the solution of difference scheme (4.16), we have the following coercive stability inequalities

$$\max_{1\leq k,m\leq N}\|\tfrac{u_{k,m}^h-u_{k-1,m-1}^h}{\tau}\|_{C[0,1]_h}$$

$$+\max_{1\leq k,m\leq N}\|h^{-2}\left[\left\{u_{k,m}^{n+1}-2u_{k,m}^n+u_{k,m}^{n-1}\right\}_1^N\right.$$

$$\left.+\left\{u_{k-1,m-1}^{n+1}-2u_{k-1,m-1}^n+u_{k-1,m-1}^{n-1}\right\}\right]_1^N\|_{C[0,1]_h}$$

$$\leq M\left(\tfrac{1}{|h|}\left(\max_{0\leq m\leq N}\|h^{-2}\left\{\psi_m^{n+1}-2\psi_m^n+\psi_m^{n-1}\right\}_1^N\|_{C[0,1]_h}\right.\right.$$

$$\left.+\max_{0\leq k\leq N}\|h^{-2}\left\{\varphi_k^{n+1}-2\varphi_k^n+\varphi_k^{n-1}\right\}_1^N\|_{C[0,1]_h}\right)$$

$$\left.+\ln\tfrac{1}{\tau+|h|}\max_{1\leq k,m\leq N}\|f_{k,m}^h\|_{C[0,1]_h}\right),$$

where M is independent of τ, ψ_m^h, φ_k^h, and $f_{k,m}^h$.

The proof of Theorem 4.7 is based on the Theorem 4.2 and the positivity of operator A_h^x .

Theorem 4.8. For the solution of difference scheme (4.16), we have the following coercive stability inequalities

$$\max_{1\leq k,m\leq N}\|\tfrac{u_{k,m}^h-u_{k-1,m}^h}{\tau}\|_{C[0,1]_h}+\max_{1\leq k,m\leq N}\|\tfrac{u_{k-1,m}^h-u_{k-1,m-1}^h}{\tau}\|_{C[0,1]_h}$$

$$\leq M\left(\max_{0\leq m\leq N}\|h^{-2}\left\{\psi_m^{n+1}-2\psi_m^n+\psi_m^{n-1}\right\}_1^N\|_{C[0,1]_h}+\max_{1\leq k,m\leq N}\left\|f_{k,m}^h\right\|\right.$$

$$+\max_{1\leq m\leq N}\|\tfrac{\psi_m^h-\psi_{m-1}^h}{\tau}\|_{C[0,1]_h}+\max_{1\leq k,m\leq N}\|\tfrac{f_{k,m}^h-f_{k-1,m}^h}{\tau}\|_{C[0,1]_h}$$

$$+\max_{0\leq k\leq N}\|h^{-2}\left\{\varphi_k^{n+1}-2\varphi_k^n+\varphi_k^{n-1}\right\}_1^N\|_{C[0,1]_h}$$

$$\left.+\max_{1\leq k,m\leq N}\|\tfrac{f_{k,m}^h-f_{m-1}^h}{\tau}\|_{C[0,1]_h}+\max_{0\leq k\leq N}\|\tfrac{\varphi_k^h-\varphi_{k-1}^h}{\tau}\|_{C[0,1]_h}\right)$$

where M is independent of τ, ψ_m^h, φ_k^h, and $f_{k,m}^h$.

The proof of Theorem 4.8 is based on the Theorem 4.3 and the positivity of operator A_h^x .

Second, the mixed boundary value problem (2.21) is considered. We replace problem (3.37) by difference scheme (4.1)

$$\begin{cases} \frac{u_{k,m}^h - u_{k-1,m-1}^h}{\tau} + \frac{1}{2}A_h^x(u_{k,m}^h + u_{k-1,m-1}^h) = f_{k,m}^h(x), x \in \Omega_h, \\ \\ f_{k,m}^h(x) = f^h(t_k - \frac{\tau}{2}, s_m - \frac{\tau}{2}, x), t_k = k\tau, s_m = m\tau, \\ \\ 1 \le k, m \le N,\ x \in \widetilde{\Omega}_h, \\ \\ u_{0,m}^h = \psi_m^h, 0 \le m \le N, u_{k,0}^h = \varphi_k^h, 0 \le k \le N. \end{cases}, \tag{4.17}$$

Theorem 4.9. For the solution of difference scheme (4.17), we have the following stability inequality

$$\max_{1\le k,m\le N} \|u_{k,m}^h\|_{C(\widetilde{\Omega}_h)}$$

$$\le M\left(\max_{0\le m\le N}\|\psi_m^h\|_{C(\widetilde{\Omega}_h)} + \max_{0\le k\le N}\|\varphi_k^h\|_{C(\widetilde{\Omega}_h)} + \max_{1\le k,m\le N}\|f_{k,m}^h\|_{C(\widetilde{\Omega}_h)}\right),$$

where M is independent of τ, ψ_m^h, φ_k^h, and $f_{k,m}^h$.

The proof of Theorem 4.9 is based on the Theorem 4.1 and the positivity of operator A_h^x .

Theorem 4.10. For the solution of difference scheme (4.17), we have the following coercive stability inequalities

$$\max_{1\le k,m\le N} \|\tfrac{u_{k,m}^h - u_{k-1,m-1}^h}{\tau}\|_{C(\widetilde{\Omega}_h)}$$

$$+\tfrac{1}{2}\max_{1\le k,m\le r}\sum_{r=1}^{n}\|u^h_{(\bar{x}_r x_r,\, j_r)\ k,m} + u^h_{(\bar{x}_r x_r,\, j_r)\ k-1,m-1}\|_{C(\widetilde{\Omega}_h)}$$

$$\le M\ln\tfrac{1}{|h|}\left(\max_{1\le m\le N}\sum_{r=1}^{n}\|\psi^h_{(\bar{x}_r x_r,\, j_r)\ m}\|_{C(\widetilde{\Omega}_h)} + \max_{0\le k\le N}\sum_{r=1}^{n}\|\varphi^h_{(\bar{x}_r x_r,\, j_r)\ k}\|_{C(\widetilde{\Omega}_h)}\right)$$

$$+M\ln\tfrac{1}{\tau+|h|}\max_{1\le k,\, m\le N}\|f^{\,h}_{k,m}\|_{C(\widetilde{\Omega}_h)}.$$

where M is independent of τ, ψ_m^h, φ_k^h, and $f_{k,m}^h$.

The proof of Theorem 4.10 is based on the Theorem 4.2 and the positivity of operator A_h^x .

Theorem 4.11. For the solution of difference scheme (4.17), we have the following coercive stability inequalities

$$\max_{1\le k,m\le N}\|\tfrac{u_{k,m}^h-u_{k-1,m}^h}{\tau}\|_{C(\widetilde{\Omega}_h)}+\max_{1\le k,m\le N}\|\tfrac{u_{k-1,m}^h-u_{k-1,m-1}^h}{\tau}\|_{C(\widetilde{\Omega}_h)}$$

$$\le M\left(\max_{1\le m\le N}\sum_{r=1}^{n}\|\psi^h_{(\bar{x}_r x_r,\, j_r)\; m}\|_{C(\widetilde{\Omega}_h)}+\max_{1\le k,m\le N}\left\|f_{k,m}^h\right\|_{C(\widetilde{\Omega}_h)}\right.$$

$$+\max_{1\le m\le N}\|\tfrac{\psi_m^h-\psi_{m-1}^h}{\tau}\|_{C(\widetilde{\Omega}_h)}+\max_{1\le k,m\le N}\|\tfrac{f_{k,m}^h-f_{k-1,m}^h}{\tau}\|_{C(\widetilde{\Omega}_h)}$$

$$+\max_{0\le k\le N}\sum_{r=1}^{n}\|\varphi^h_{(\bar{x}_r x_r,\, j_r)\; k}\|_{C(\widetilde{\Omega}_h)}$$

$$\left.+\max_{1\le k,m\le N}\|\tfrac{f_{k,m}^h-f_{m-1}^h}{\tau}\|_{C(\widetilde{\Omega}_h)}+\max_{0\le k\le N}\|\tfrac{\varphi_k^h-\varphi_{k-1}^h}{\tau}\|_{C(\widetilde{\Omega}_h)}\right),$$

where M is independent of τ, ψ_m^h, φ_k^h, and $f_{k,m}^h$.

The proof of Theorem 4.11 is based on the Theorem 4.3 and the positivity of operator A_h^x .

4.1 R-MODIFIED CRANK-NICHOLSON DIFFERENCE SCHEME

Finally, we consider the r-modified Crank-Nicholson difference scheme

$$\begin{cases}\frac{u_{k,m}-u_{k-1,m-1}}{\tau}+Au_{k,m}=f_{k,m}\ ,\ \ 1\le k\le r\ \ or\ \ 1\le m\le r,\\ \\ \frac{u_{k,m}-u_{k-1,m-1}}{\tau}+\frac{A}{2}(u_{k,m}+u_{k-1,m-1})=f_{k,m}\ ,\ \ r+1\le k,m\le N,\\ \\ f_{k,m}=f(t_k,s_m),\ t_k=k\tau,\ \ s_m=m\tau,\ \ 1\le k,m\le N,\ \ N\tau=1,\\ \\ u_{0,m}=\psi_m,\psi_m=\psi(s_m),\ \ 0\le m\le N,\\ \\ u_{k,0}=\varphi_k,\varphi_k=\varphi(t_k),\ \ 0\le k\le N\end{cases}\tag{4.18}$$

for the approximate solution of (2.1). Note that the modified Crank-Nicholson difference schemes were firstly presented in (Ashyralyev, 1989) for the approximate solution of the Cauchy problem for abstract parabolic equations with non-smooth data

$$\mathrm{v}_t(t) + Av(t) = f(t),\ 0 < t < 1,\ v(0) = v_0 \tag{4.19}$$

in a Banach space with the strongly positive operator A. These difference schemes were constructed with combination of the Crank-Nicholson difference scheme with step τ and the Rothe scheme with step $2^{-1}\tau$ in the case with non-smooth data. By the way, r-modified Crank-Nicholson difference schemes for $r = 1, 2, 3$ have a smooth property as the Rothe difference scheme and the same order of approximation as the Crank-Nicholson difference schemes. Moreover, the modified Crank-Nicholson difference schemes for various evolution partial differential equations have been studied in papers (Ashyralyev and Sirma, 2009), (Ashyralyev, 2008), (Ashyralyev and Yildirim, 2011), (Ashyralyev, 2007) and (Ashyralyev et al., 2010).

The following stability estimates and almost coercive stability for the solution of difference schemes (4.18) are obtained.

Theorem 4.12. Let τ be a sufficiently small number. Then for the solution of problem (4.18), we have the following stability inequalities

$$\max_{1\le k,m\le N} \|u_{k,m}\|_E \le M \left(\max_{0\le m\le N} \|\psi_m\|_E + \max_{0\le k\le N} \|\varphi_k\|_E + \max_{1\le k,m\le N} \|f_{k,m}\|_E \right),$$

where M is independent of τ, ψ_m, φ_k, and $f_{k,m}$.

Proof. For $1 \le k, m \le r$, using the theorem (3.1) we obtain

$$\|u_{k,m}\|_E \le M \left[\max_{0\le k,m\le r} \|\psi_m\|_E + \max_{0\le k\le r} \|\varphi_k\|_E + \max_{1\le k,m\le r} \|f_{k,m}\|_E \right]. \tag{4.20}$$

For $r + 1 \le k, m \le N$, using the Theorem (4.1) we obtain

$$\|u_{k,m}\|_E \le M \left[\max_{r+1\le k,m\le N} \|\psi_m\|_E + \max_{r+1\le k\le N} \|\varphi_k\|_E + \max_{r+1\le k,m\le N} \|f_{k,m}\|_E \right]. \tag{4.21}$$

Combining the estimates (4.20) and (4.21) we get the statement of the Theorem 4.12.

Theorem 4.13. Let τ be a sufficiently small number and $\psi_m, \varphi_k \in D(A)$.

Then for the solution of (4.18), we have the following almost coercive inequality

$$\max_{1\le k,m\le N}\|\tfrac{u_{k,m}-u_{k-1,m-1}}{\tau}\|_E+\max_{1\le k,m\le r}\|Au_{k,m}\|_E$$

$$+\max_{r+1\le k,m\le N}\|\tfrac{A}{2}(u_{k,m}+u_{k-1,m-1})\|_E$$

$$\le M\left(\max_{0\le m\le N}\|A\psi_m\|_E+\max_{0\le k\le N}\|A\varphi_k\|_E\right.$$

$$\left.+\min\left\{\ln\tfrac{1}{\tau},\ 1+|\ln\|A\|_{E\to E}|\right\}\max_{1\le k,m\le N}\|f_{k,m}\|\right),$$

where M is independent of τ, ψ_m, $\varphi_{k,}$ and $f_{k,m}$.

The proof of the Theorem (4.13) is based on the Theorems (3.2) and (4.2)

Theorem 4.14. Let τ be a sufficiently small number and $\psi_m,\varphi_k\in D(A)$. Then for the solution of (4.18), we have the following almost coercive inequality

$$\max_{1\le k,m\le N}\|\tfrac{u_{k,m}-u_{k-1,m}}{\tau}\|_E+\max_{1\le k,m\le N}\|\tfrac{u_{k-1,m}-u_{k-1,m-1}}{\tau}\|_E$$

$$\le M\left(\max_{0\le m\le N}\|A\psi_m\|_E+\max_{1\le k,m\le N}\|\tfrac{\psi_m-\psi_{m-1}}{\tau}\|_E\ +\max_{1\le k,m\le N}\|\tfrac{f_{k,m}-f_{k,m-1}}{\tau}\|_E\right.$$

$$+\max_{0\le k\le N}\|A\varphi_k\|_E+\max_{1\le k,m\le N}\|\tfrac{\varphi_k-\varphi_{k-1}}{\tau}\|_E$$

$$\left.+\max_{1\le k,m\le N}\|\tfrac{f_{k,m}-f_{k-1,m}}{\tau}\|_E+\max_{1\le k,m\le N}\|f_{k,m}\|_E\right),$$

where M is independent of τ, ψ_m, $\varphi_{k,}$ and $f_{k,m}$.

The proof of the Theorem (4.14) is based on the Theorems (3.3) and (4.3)

Theorem 4.15. Let τ be a sufficiently small number and $\psi_m,\varphi_k\in D(A)$. Then for the solution of (4.18), we have the following coercive inequality

$$\max_{1\le k,m\le N}\|\tfrac{u_{k,m}-u_{k-1,m-1}}{\tau}\|_{E_\alpha}+\max_{1\le k,m\le r}\|Au_{k,m}\|_{E_\alpha}+$$

$$\max_{r+1\le k,m\le N}\|\tfrac{A}{2}(u_{k,m}+u_{k-1,m-1})\|_{E_\alpha}$$

$$\le M\left(\max_{0\le m\le N}\|A\psi_m\|_{E_\alpha}+\max_{0\le k\le N}\|A\varphi_k\|_{E_\alpha}+\ \tfrac{1}{\alpha(1-\alpha)}\max_{1\le k,m\le N}\|f_{k,m}\|_{E_\alpha}\right),$$

where M is independent of τ, ψ_m, φ_k, and $f_{k,m}$.

The proof of Theorem 4.15 is based on the Theorems (3.4) and (4.4).

Theorem 4.16. Let τ be a sufficiently small number and $\psi_m, \varphi_k \in D(A)$. Then for the solution of (4.18), we have the following coercive inequality

$$\max_{1\le k,m\le N}\left\|\frac{u_{k,m}-u_{k-1,m}}{\tau}\right\|_{E_\alpha}+\max_{1\le k,m\le N}\left\|\frac{u_{k-1,m}-u_{k-1,m-1}}{\tau}\right\|_{E_\alpha}$$

$$\le M\left(\max_{0\le m\le N}\|A\psi_m\|_{E_\alpha}+\max_{1\le k,m\le N}\left\|\frac{\psi_m-\psi_{m-1}}{\tau}\right\|_{E_\alpha}+\max_{1\le k,m\le N}\left\|\frac{f_{k,m}-f_{k,m-1}}{\tau}\right\|_{E_\alpha}\right.$$

$$+\max_{0\le k\le N}\|A\varphi_k\|_{E_\alpha}+\max_{1\le k,m\le N}\left\|\frac{\varphi_k-\varphi_{k-1}}{\tau}\right\|_{E_\alpha}$$

$$\left.+\max_{1\le k,m\le N}\left\|\frac{f_{k,m}-f_{k-1,m}}{\tau}\right\|_{E_\alpha}+\max_{1\le k,m\le N}\|f_{k,m}\|_{E_\alpha}\right),$$

where M is independent of τ, ψ_m, φ_k, and $f_{k,m}$.

The proof of Theorem 4.16 is based on the Theorems (3.5) and (4.5).

First, we consider the application of these theorems to problem (2.19). We replace problem (3.7) by difference scheme (4.18)

$$\begin{cases}\frac{u^h_{k,m}-u^h_{k-1,m-1}}{\tau}+A^x_h u^h_{k,m}=f^h_{k,m}(x), x\in[0,l]_h, 1\le k,m\le r,\\ \frac{u^h_{k,m}-u^h_{k-1,m-1}}{\tau}+\frac{1}{2}A^x_h(u_{k,m}+u_{k-1,m-1})=f^h_{k,m}(x), x\in[0,l]_h,\\ r+1\le k,m\le N,\\ f^h_{k,m}(x)=f^h(t_k-\frac{\tau}{2},s_m-\frac{\tau}{2},x), t_k=k\tau, s_m=m\tau,\\ 1\le k,m\le N,\ x\in[0,l]_h,\\ u^h_{0,m}=\psi^h_m, 0\le m\le N, u^h_{k,0}=\varphi^h_k, 0\le k\le N.\end{cases} \tag{4.22}$$

It is known that A^x_h is a positive operator in $C[0,l]_h$ and $C^\beta[0,l]_h$.

Theorem 4.17. For the solution of difference scheme (4.22), we have the

following stability inequality

$$\max_{1\le k,m\le N}\|u^h_{k,m}\|_{C[0,1]_h}\le M\left(\max_{0\le m\le N}\|\psi^h_m\|_{C[0,1]_h}+\max_{0\le k\le N}\|\varphi^h_k\|_{C[0,1]_h}\right.$$

$$\left.+\max_{1\le k,m\le N}\|f^h_{k,m}\|_{C[0,1]_h}\right),$$

where M is independent of τ, ψ^h_m, φ^h_k, and $f^h_{k,m}$.

The proof of Theorem 4.17 is based on the Theorem 4.11 and the positivity of operator A^x_h .

Theorem 4.18. For the solution of difference scheme (4.22), we have the following coercive stability inequalities

$$\max_{1\le k,m\le N}\|\tfrac{u^h_{k,m}-u^h_{k-1,m-1}}{\tau}\|_{C[0,1]_h}$$

$$+\max_{1\le k,m\le N}\|h^{-2}\left\{u^{n+1}_{k,m}-2u^n_{k,m}+u^{n-1}_{k,m}\right\}^N_1\|_{C[0,1]_h}$$

$$+\tfrac{1}{2}\max_{1\le k,m\le N}\|h^{-2}\left[\left\{u^{n+1}_{k,m}-2u^n_{k,m}+u^{n-1}_{k,m}\right\}^N_1\right.$$

$$\left.+\left\{u^{n+1}_{k-1,m-1}-2u^n_{k-1,m-1}+u^{n-1}_{k-1,m-1}\right\}^N_1\right]\|_{C[0,1]_h}$$

$$\le M\left(ln\tfrac{1}{|h|}\left(\max_{0\le m\le N}\|h^{-2}\left\{\psi^{n+1}_m-2\psi^n_m+\psi^{n-1}_m\right\}^N_1\|_{C[0,1]_h}\right.\right.$$

$$\left.+\max_{0\le k\le N}\|h^{-2}\left\{\varphi^{n+1}_k-2\varphi^n_k+\varphi^{n-1}_k\right\}^N_1\|_{C[0,1]_h}\right)$$

$$\left.+\ln\tfrac{1}{\tau+|h|}\max_{1\le k,m\le N}\|f^h_{k,m}\|_{C[0,1]_h}\right).$$

The proof of Theorem 4.18 is based on the Theorem 4.12 and the positivity of operator A^x_h .

Theorem 4.19. For the solution of difference scheme (4.22), we have the

following coercive stability inequalities

$$\max_{1\le k,m\le N}\|\tfrac{u^h_{k,m}-u^h_{k-1,m}}{\tau}\|_{C[0,1]_h}+\max_{1\le k,m\le N}\|\tfrac{u^h_{k-1,m}-u^h_{k-1,m-1}}{\tau}\|_{C[0,1]_h}$$

$$\le M\left(\max_{0\le m\le N}\|h^{-2}\left\{\psi^{n+1}_m-2\psi^n_m+\psi^{n-1}_m\right\}^N_1\|_{C[0,1]_h}+\max_{1\le k,m\le N}\left\|f^h_{k,m}\right\|\right.$$

$$+\max_{1\le m\le N}\|\tfrac{\psi^h_m-\psi^h_{m-1}}{\tau}\|_{C[0,1]_h}+\max_{1\le k,m\le N}\|\tfrac{f^h_{k,m}-f^h_{k-1,m}}{\tau}\|_{C[0,1]_h}$$

$$+\max_{0\le k\le N}\|h^{-2}\left\{\varphi^{n+1}_k-2\varphi^n_k+\varphi^{n-1}_k\right\}^N_1\|_{C[0,1]_h}$$

$$\left.+\max_{1\le k,m\le N}\|\tfrac{f^h_{k,m}-f^h_{m-1}}{\tau}\|_{C[0,1]_h}+\max_{0\le k\le N}\|\tfrac{\varphi^h_k-\varphi^h_{k-1}}{\tau}\|_{C[0,1]_h}\right)$$

where M is independent of τ, ψ^h_m, φ^h_k, and $f^h_{k,m}$.

The proof of Theorem 4.19 is based on the Theorem 4.13 and the positivity of operator A^x_h .

Secondly, the mixed boundary value problem for the multidimensional ultra-parabolic equation (2.21) is considered. We replace problem (3.9) by difference scheme (4.18)

$$\begin{cases}\frac{u^h_{k,m}-u^h_{k-1,m-1}}{\tau}+A^x_hu^h_{k,m}=f^h_{k,m}(x),x\in\Omega_h,1\le k,m\le r,\\ \frac{u^h_{k,m}-u^h_{k-1,m-1}}{\tau}+\frac{1}{2}A^x_h(u_{k,m}+u_{k-1,m-1})=f^h_{k,m}(x),\\ x\in\Omega_h,r+1\le k,m\le N\\ f^h_{k,m}(x)=f^h(t_k-\frac{\tau}{2},s_m-\frac{\tau}{2},x),t_k=k\tau,s_m=m\tau,\\ 1\le k,m\le N,\ x\in\widetilde{\Omega}_h,\\ u^h_{0,m}=\psi^h_m,0\le m\le N,u^h_{k,0}=\varphi^h_k,0\le k\le N.\end{cases}\tag{4.23}$$

It is known that A^x_h is a positive operator in $C(\widetilde{\Omega}_h)$ and $C^\beta_{01}(\overline{\Omega}_h)$.

Theorem 4.20. For the solution of difference scheme (4.23), we have the

following stability inequality

$$\max_{1\le k,m\le N}\|u^h_{k,m}\|_{C(\widetilde{\Omega}_h)}$$

$$\le M\left(\max_{0\le m\le N}\|\psi^h_m\|_{C(\widetilde{\Omega}_h)}+\max_{0\le k\le N}\|\varphi^h_k\|_{C(\widetilde{\Omega}_h)}+\max_{1\le k,m\le N}\|f^h_{k,m}\|_{C(\widetilde{\Omega}_h)}\right),$$

where M is independent of τ, ψ^h_m, φ^h_k, and $f^h_{k,m}$.

The proof of Theorem 4.20 is based on the Theorem 4.11 and the positivity of operator A^x_h .

Theorem 4.21. For the solution of difference scheme (4.23), we have the following coercive stability inequalities

$$\max_{1\le k,m\le N}\|\tfrac{u^h_{k,m}-u^h_{k-1,m-1}}{\tau}\|_{C(\widetilde{\Omega}_h)}+\max_{1\le k,m\le r}\sum_{r=1}^{n}\|u^h_{(\bar{x}_rx_r,\,j_r)\;k,m}\|_{C(\widetilde{\Omega}_h)}$$

$$+\max_{1\le k,m\le r}\sum_{r=1}^{n}\tfrac{1}{2}[\|u^h_{(\bar{x}_rx_r,\,j_r)\;k,m}+u^h_{(\bar{x}_rx_r,\,j_r)\;k-1,m-1}\|_{C(\widetilde{\Omega}_h)}]$$

$$\le M\ln\tfrac{1}{|h|}\left(\max_{1\le m\le N}\sum_{r=1}^{n}\|\psi^h_{(\bar{x}_rx_r,\,j_r)\;m}\|_{C(\widetilde{\Omega}_h)}+\max_{0\le k\le N}\sum_{r=1}^{n}\|\varphi^h_{(\bar{x}_rx_r,\,j_r)\;k}\|_{C(\widetilde{\Omega}_h)}\right)$$

$$+C\ln\tfrac{1}{\tau+|h|}\max_{1\le k,\,m\le N}\|f^{\,h}_{k,m}\|_{C(\widetilde{\Omega}_h)},$$

where M is independent of τ, ψ^h_m, φ^h_k, and $f^h_{k,m}$.

The proof of Theorem 4.21 is based on the Theorem 4.12 and the positivity of operator A^x_h .

Theorem 4.22. For the solution of difference scheme (4.23), we have the

following coercive stability inequalities

$$\max_{1\le k,m\le N}\|\tfrac{u^h_{k,m}-u^h_{k-1,m}}{\tau}\|_{C(\widetilde{\Omega}_h)}+\max_{1\le k,m\le N}\|\tfrac{u^h_{k-1,m}-u^h_{k-1,m-1}}{\tau}\|_{C(\widetilde{\Omega}_h)}$$

$$\le M\left(\max_{1\le m\le N}\sum_{r=1}^{n}\|\psi^h_{(\bar{x}_r x_r,\, j_r)\; m}\|_{C(\widetilde{\Omega}_h)}+\max_{1\le k,m\le N}\left\|f^h_{k,m}\right\|_{C(\widetilde{\Omega}_h)}\right.$$

$$+\max_{1\le m\le N}\|\tfrac{\psi^h_m-\psi^h_{m-1}}{\tau}\|_{C(\widetilde{\Omega}_h)}+\max_{1\le k,m\le N}\|\tfrac{f^h_{k,m}-f^h_{k-1,m}}{\tau}\|_{C(\widetilde{\Omega}_h)}$$

$$+\max_{0\le k\le N}\sum_{r=1}^{n}\|\varphi^h_{(\bar{x}_r x_r,\, j_r)\; k}\|_{C(\widetilde{\Omega}_h)}$$

$$\left.+\max_{1\le k,m\le N}\|\tfrac{f^h_{k,m}-f^h_{m-1}}{\tau}\|_{C(\widetilde{\Omega}_h)}+\max_{0\le k\le N}\|\tfrac{\varphi^h_k-\varphi^h_{k-1}}{\tau}\|_{C(\widetilde{\Omega}_h)}\right),$$

where M is independent of τ, ψ^h_m, φ^h_k, and $f^h_{k,m}$.

The proof of Theorem 4.22 is based on the Theorem 4.13 and the positivity of operator A^x_h .

CHAPTER 5

NUMERICAL RESULTS

We consider the initial boundary value problem for one dimensional ultra parabolic equation

$$\begin{cases} \frac{\partial u(s,t,x)}{\partial t} + \frac{\partial u(s,t,x)}{\partial s} - \frac{\partial^2 u(s,t,x)}{\partial x^2} + 2u(t,s,x) = f(t,s,x), \\ f(t,s,x) = e^{-(t+s)} \sin \pi x, 0 < s,t < 1,\ 0 < x < 1, \\ u(0,t,x) = e^{-t} \sin \pi x,\ 0 < t < 1,\ 0 < x < 1, \\ u(s,0,x) = e^{-s} \sin \pi x,\ 0 < s < 1,\ 0 < x < 1, \\ u(s,t,0) = u(s,t,\pi) = 0, \quad 0 < s,t < 1 \end{cases} \tag{5.1}$$

and ultra-parabolic equation with Neumann condition

$$\begin{cases} \frac{\partial u(s,t,x)}{\partial t} + \frac{\partial u(s,t,x)}{\partial s} - \frac{\partial^2 u(s,t,x)}{\partial x^2} + 2u(t,s,x) = f(t,s,x), \\ f(t,s,x) = e^{-(t+s)} \cos \pi x, 0 < s,t < 1,\ 0 < x < 1, \\ u(0,s,x) = e^{-s} \cos \pi x,\ 0 \le t \le 1,\ 0 \le x \le 1, \\ u(t,0,x) = e^{-t} \cos \pi x,\ 0 \le s \le 1,\ 0 \le x \le 1, \\ u_x(s,t,0) = u_x(s,t,\pi) = 0, \quad 0 \le s,t \le 1. \end{cases} \tag{5.2}$$

The exact solution of problem (5.1) is

$$u\,(t,s,x) = e^{-(t+s)} \sin \pi x.$$

and the exact solution of problem (5.2) is

$$u\left(t,s,x\right)=e^{-(t+s)}\cos\pi x.$$

In the present chapter for the approximate solutions of the problem (5.1) and (5.2), we will use the first and second orders of accuracy difference schemes. We have the second order difference equations with respect to n with matrix coefficients. To solve these difference equations, we have applied a procedure of modified Gauss elimination method for difference equations with respect to n with matrix coefficients. The results of numerical experiments permit us to show that the second order of accuracy difference scheme is more accurate compared with the first order of accuracy difference scheme.

5.1 THE FIRST ORDER OF ACCURACY DIFFERENCE SCHEME

For the approximate solution of the problem (5.1), applying the Rothe difference scheme (3.1), we obtain

$$\begin{cases} \frac{u_n^{k,m}-u_n^{k-1,m}}{\tau}-\frac{u_n^{k-1,m}-u_n^{k-1,m-1}}{\tau}-\frac{u_{n+1}^{k,m}-2u_n^{k,m}+u_{n-1}^{k,m}}{h^2}+2u_n^{k,m}=f_{k,m}^h, \\ f_{k,m}^h=\ f(t_k,s_m,x_n)=e^{-(t_k+s_m)}\sin x_n, 1\le k,m\le N,\ 1\le n\le M-1, \\ u_n^{0,m}=e^{-s_m}\sin x_n,\ 0\le m\le N,\ 0\le n\le M, \\ u_n^{k,0}=e^{-t_k}\sin x_n,\ 0\le k\le N,\ 0\le n\le M, \\ u_0^{k,m}=u_M^{k,m}=0,\ 0\le k,m\le N, \\ t_k=k\tau, s_m=m\tau,\ 1\le k,m\le N,\ N\tau=1, \\ x_n=nh,\ 1\le n\le M,\ Mh=\pi \end{cases} \tag{5.3}$$

for approximate solutions of initial boundary value problem (5.1). It can be written in the matrix form

$$\begin{cases} A\,u_{n+1} + B\,u_n + C\,u_{n-1} = \varphi_n, \quad 1 \le n \le M-1, \\ \\ u_0 = \vec{0}, \;\; u_M = \vec{0}. \end{cases} \tag{5.4}$$

Here

$$A = \begin{bmatrix} 0 & 0 & \cdots & \cdots & \cdots & 0 & 0 \\ 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & a & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & a & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & a \end{bmatrix}_{(N+1)^2 \times (N+1)^2},$$

$$B = \begin{bmatrix} 1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & b & 0 & \cdots & c & d & 0 & \cdots & 0 \\ 0 & 0 & b & 0 & \cdots & c & d & 0 & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots & b & 0 & \cdots & c & d \end{bmatrix}_{(N+1)^2 \times (N+1)^2},$$

$$C = \begin{bmatrix} 0 & 0 & \cdots & \cdots & \cdots & 0 & 0 \\ 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & a & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & a & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & a \end{bmatrix}_{(N+1)^2 \times (N+1)^2},$$

where

$$a = -\frac{1}{h^2}, b = -\frac{1}{\tau}, c = -\frac{1}{\tau}, d = \frac{1}{\tau} + \frac{1}{\tau} + \frac{2}{h^2} + 2,$$

$$\varphi_n = \begin{bmatrix} \varphi_n^{0,0} \\ \varphi_n^{1,0} \\ \vdots \\ \varphi_n^{N,0} \\ \varphi_n^{0,1} \\ \varphi_n^{1,1} \\ \vdots \\ \varphi_n^{N,1} \\ \vdots \\ \varphi_n^{0,N} \\ \varphi_n^{1,N} \\ \vdots \\ \varphi_n^{N,N} \end{bmatrix}_{(N+1)^2 \times 1} , \quad \varphi_n^{k,m} = f(t_k, s_m, x_n) = e^{-(t_k + s_m)} \sin x_n \ ,$$

and

$$u_n = \begin{bmatrix} u_n^{0,0} \\ u_n^{1,0} \\ \vdots \\ u_n^{N,0} \\ u_n^{0,1} \\ u_n^{1,1} \\ \vdots \\ u_n^{N,1} \\ \vdots \\ u_n^{0,N} \\ u_n^{1,N} \\ \vdots \\ u_n^{N,N} \end{bmatrix}_{(N+1)^2 \times 1} .$$

This type system was used by Samarskii and Nikolaev (Samarskii and Nikolaev, 1989) for difference equations. For the solution of matrix equation (5.4), we will use

the modified Gauss elimination method. We seek a solution of the matrix equation by the following form:

$$u_n = \alpha_{n+1} u_{n+1} + \beta_{n+1}, \quad n = M-1, \cdots, 2, 1, \tag{5.5}$$

where $u_M = \vec{0}$, $\alpha_j \quad (j = 1, \cdots, M-1)$ are $(N+1)^2 \times (N+1)$ ^{2}square matrices, β_j $(j = 1, \cdots, M-1)$ are $(N+1)^2 \times 1$ coloumn matrices, $\alpha_1,\ \beta_1$ are zero matrices and

$$\begin{aligned} &\alpha_{n+1} = -\left(B + C\alpha_n\right)^{-1} A, \\ &\beta_{n+1} = \left(B + C\alpha_n\right)^{-1} \left(\varphi_n - C\beta_n\right), n = 1, 2, 3, \cdots (M-1). \end{aligned} \tag{5.6}$$

For the approximate solution of the problem (5.2), again applying the Rothe difference scheme (3.1), we obtain

$$\begin{cases} \frac{u_n^{k,m} - u_n^{k-1,m}}{\tau} - \frac{u_n^{k-1,m} - u_n^{k-1,m-1}}{\tau} - \frac{u_{n+1}^{k,m} - 2u_n^{k,m} + u_{n-1}^{k,m}}{h^2} + 2u_n^{k,m} = f_{k,m}^h, \\ f_{k,m}^h = \ f(t_k, s_m, x_n) = e^{-(t_k + s_m)} \cos x_n, 1 \le k, m \le N,\ 1 \le n \le M-1, \\ u_n^{0,m} = e^{-s_m} \cos x_n,\ 0 \le m \le N,\ 0 \le n \le M, \\ u_n^{k,0} = e^{-t_k} \cos x_n,\ 0 \le k \le N,\ 0 \le n \le M, \\ u_0^{k,m} = u_1^{k,m},\ u_M^{k,m} = u_{M-1}^{k,m},\ 0 \le k, m \le N, \\ t_k = k\tau, s_m = m\tau,\ 0 \le k, m \le N,\ N\tau = 1, \\ x_n = nh,\ 0 \le n \le M,\ Mh = \pi \end{cases} \tag{5.7}$$

for approximate solutions of initial boundary value problem (5.9). It can be written

in the matrix form (5.4). Here

$$A = \begin{bmatrix} 0 & 0 & \cdots & \cdots & \cdots & 0 & 0 \\ 0 & 0 & \cdots & \cdots & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & a & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & a & \cdots & 0 \\ 0 & 0 & 0 & \cdots & \cdots & a & \cdots \\ 0 & 0 & 0 & 0 & \cdots & \cdots & a \end{bmatrix}_{(N+1)^2 \times (N+1)^2},$$

$$B = \begin{bmatrix} 1 & 0 & \cdots & \cdots & \cdots & 0 & 0 & 0 \\ 0 & 1 & \cdots & \cdots & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & b & \cdots & c & d & \cdots & \cdots & \cdots \\ 0 & 0 & b & \cdots & c & d & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & b & \cdots & c & d & 0 \\ 0 & 0 & 0 & \cdots & b & \cdots & c & d \end{bmatrix}_{(N+1)^2 \times (N+1)^2},$$

$$C = \begin{bmatrix} 0 & 0 & \cdots & \cdots & \cdots & 0 & 0 \\ 0 & 0 & \cdots & \cdots & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & a & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & a & \cdots & 0 \\ 0 & 0 & 0 & \cdots & \cdots & a & \cdots \\ 0 & 0 & 0 & 0 & \cdots & \cdots & a \end{bmatrix}_{(N+1)^2 \times (N+1)^2},$$

where

$$a = -\frac{1}{h^2}, \qquad b = -\frac{1}{\tau}, \quad c = -\frac{1}{\tau},$$
$$d = \frac{1}{\tau} + \frac{2}{h^2} + 2,$$

$$\varphi_n = \begin{bmatrix} \varphi_n^{0,0} \\ \varphi_n^{1,0} \\ \cdots \\ \varphi_n^{N,0} \\ \varphi_n^{0,1} \\ \varphi_n^{1,1} \\ \cdots \\ \varphi_n^{N,1} \\ \cdots \\ \varphi_n^{0,N} \\ \varphi_n^{1,N} \\ \cdots \\ \varphi_n^{N,N} \end{bmatrix}_{(N+1)^2 \times 1} , \quad \varphi_n^{k,m} = f(t_k, s_m, x_n) = e^{-(t_k + s_m)} \cos x_n \ ,$$

and

$$u_n = \begin{bmatrix} u_n^{0,0} \\ u_n^{1,0} \\ \cdots \\ u_n^{N,0} \\ u_n^{0,1} \\ u_n^{1,1} \\ \cdots \\ u_n^{N,1} \\ \cdots \\ u_n^{0,N} \\ u_n^{1,N} \\ \cdots \\ u_n^{N,N} \end{bmatrix}_{(N+1)^2 \times 1} .$$

We seek a solution of the matrix equation by the same algorithm (5.5) and (5.6).

5.2 THE SECOND ORDER OF ACCURACY DIFFERENCE SCHEME

Applying the Crank-Nicholson difference scheme (4.1), we obtain the difference scheme second order of accuracy in t and s and second order of accuracy in x

$$\begin{cases} \frac{u_n^{k,m}-u_n^{k-1,m}}{\tau} - \frac{u_n^{k-1,m}-u_n^{k-1,m-1}}{\tau} - \\ \frac{1}{2}\left[\frac{u_{n+1}^{k,m}-2u_n^{k,m}+u_{n-1}^{k,m}}{h^2} + 2u_n^{k,m} + \frac{u_{n+1}^{k-1,m-1}-2u_n^{k-1,m-1}+u_{n-1}^{k-1,m-1}}{h^2} + 2u_n^{k-1,m-1}\right] = f_{k,m}^h, \\ f_{k,m}^h = \ f(t_k, s_m, x_n) = e^{-(t_k+s_m-\tau)}\sin x_n, 1 \le k, m \le N, \ 1 \le n \le M-1, \\ u_n^{0,m} = e^{-s_m}\sin x_n, \ 0 \le m \le N, \ 0 \le n \le M, \\ u_n^{k,0} = e^{-t_k}\sin x_n, \ 0 \le k \le N, \ 0 \le n \le M, \\ u_0^{k,m} = u_M^{k,m} = 0, \ 0 \le k, m \le N, \\ t_k = k\tau, s_m = m\tau, \ 1 \le k, m \le N, \ N\tau = 1, \\ x_n = nh, \ 1 \le n \le M, \ Mh = \pi \end{cases} \tag{5.8}$$

for approximate solutions of initial boundary value problem (5.8). It can be written the matrix form (5.4). Here,

$$A = \begin{bmatrix} 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 \\ 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots \\ a & 0 & \cdots & a & 0 & \cdots & 0 \\ 0 & a & 0 & \cdots & a & 0 & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & a & 0 & \cdots & a \end{bmatrix}_{(N+1)^2\times(N+1)^2},$$

$$B = \begin{bmatrix} 1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ b & 0 & 0 & \cdots & 1 & c & 0 & \cdots & 0 \\ 0 & b & 0 & 0 & \cdots & 1 & c & 0 & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & b & 0 & 0 & \cdots & 1 & c \end{bmatrix}_{(N+1)^2 \times (N+1)^2},$$

$$C = \begin{bmatrix} 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 \\ 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots \\ a & 0 & \cdots & a & 0 & \cdots & 0 \\ 0 & a & 0 & \cdots & a & 0 & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & a & 0 & \cdots & a \end{bmatrix}_{(N+1)^2 \times (N+1)^2},$$

where

$$a = -\frac{1}{2h^2}, b = -\frac{1}{\tau} + \frac{1}{h^2} + 1, c = \frac{1}{\tau} + \frac{1}{h^2} + 1,$$

$$\varphi_n = \begin{bmatrix} \varphi_n^{0,0} \\ \varphi_n^{1,0} \\ \vdots \\ \varphi_n^{N,0} \\ \varphi_n^{0,1} \\ \varphi_n^{1,1} \\ \vdots \\ \varphi_n^{N,1} \\ \vdots \\ \varphi_n^{0,N} \\ \varphi_n^{1,N} \\ \vdots \\ \varphi_n^{N,N} \end{bmatrix}_{(N+1)^2 \times 1}, \quad \varphi_n^{k,m} = f(t_k - \tfrac{\tau}{2}, s_m - \tfrac{\tau}{2}, x_n) = e^{-(t_k + s_m)} \sin x_n.$$

We seek a solution of the matrix equation by the same algorithm (5.5) and (5.6).

Applying the modified Crank-Nicholson difference scheme (4.18), we obtain

$$
\begin{cases}
\frac{u_n^{k,m}-u_n^{k-1,m-1}}{\tau} - \frac{u_{n+1}^{k,m}-2u_n^{k,m}+u_{n-1}^{k,m}}{h^2} + 2u_n^{k,m} = f_{k,m}^h, 1 \le k,m \le r, \\
\\
\frac{u_n^{k,m}-u_n^{k-1,m-1}}{\tau} - \frac{u_{n+1}^{k,m}-2u_n^{k,m}+u_{n-1}^{k,m}}{h^2} + 2u_n^{k,m} \\
\\
-\frac{u_{n+1}^{k-1,m-1}-2u_n^{k-1,m-1}+u_{n-1}^{k-1,m-1}}{h^2} + 2u_n^{k-1,m-1} = f_{k,m}^h, r+1 \le k,m \le N, \\
\\
f_{k,m}^h = \ f(t_k - \frac{\tau}{2}, s_m - \frac{\tau}{2}, x_n) = e^{-(t_k+s_m-\tau)} \sin x_n, \\
\\
1 \le k,m \le N, \ 1 \le n \le M-1, \\
\\
u_n^{0,m} = e^{-s_m} \sin x_n, \ 0 \le m \le N, \ 0 \le n \le M, \\
\\
u_n^{k,0} = e^{-t_k} \sin x_n, \ 0 \le k \le N, \ 0 \le n \le M, \\
\\
u_0^{k,m} = u_M^{k,m} = 0, \ 0 \le k,m \le N, \\
\\
t_k = k\tau, s_m = m\tau, \ 1 \le k,m \le N, \ N\tau = 1, \\
\\
x_n = nh, \ 1 \le n \le M, \ Mh = \pi
\end{cases}
\tag{5.9}
$$

for approximate solutions of initial boundary value problem (5.1). It can be written in the matrix form (5.4). Here

$$A = \begin{bmatrix}
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & \cdots & 0 & a & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & \cdots & 0 & a & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & \cdots & \cdots & 0 & a & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & 0 & a & 0 & \cdots & \cdots & 0 & a & 0 & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & 0 & a & 0 & \cdots & \cdots & 0 & a & 0 & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & \cdots & 0 & a & 0 & \cdots & \cdots & 0 & a & 0 & \cdots & \cdots & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & a & 0 & \cdots & \cdots & 0 & a & 0 & \cdots \\
0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & a & 0 & \cdots & \cdots & 0 & a & 0 \\
0 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & a & 0 & \cdots & \cdots & 0 & a
\end{bmatrix},$$

$$B = \begin{bmatrix}
1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
b & 0 & \cdots & 0 & d & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & b & 0 & \cdots & 0 & d & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & b & 0 & \cdots & 0 & d & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & \cdots & 0 & c & 0 & \cdots & 0 & d & 0 & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & \cdots & 0 & c & 0 & \cdots & 0 & d & 0 & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & \cdots & \cdots & 0 & c & 0 & \cdots & 0 & d & 0 & \cdots & \cdots & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & c & 0 & \cdots & 0 & d & 0 & \cdots \\
0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & c & 0 & \cdots & 0 & d & 0 \\
0 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & c & 0 & \cdots & 0 & d
\end{bmatrix},$$

$$C = \begin{bmatrix}
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & \cdots & 0 & a & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & \cdots & 0 & a & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & \cdots & \cdots & 0 & a & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & 0 & a & 0 & \cdots & \cdots & 0 & a & 0 & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & 0 & a & 0 & \cdots & \cdots & 0 & a & 0 & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & \cdots & 0 & a & 0 & \cdots & \cdots & 0 & a & 0 & \cdots & \cdots & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & a & 0 & \cdots & \cdots & 0 & a & 0 & \cdots \\
0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & a & 0 & \cdots & \cdots & 0 & a & 0 \\
0 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & a & 0 & \cdots & \cdots & 0 & a
\end{bmatrix}$$

are $(N+1)^2 \times (N+1)^2$ square matrices, where

$$a = -\frac{1}{h^2}, b = -\frac{1}{\tau}, c = \frac{1}{\tau} + \frac{1}{h^2} + 1, d = \frac{1}{\tau} + \frac{2}{h^2} + 2,$$

for the solution of matrix equation (5.4). We seek a solution of the matrix equation by the same algorithm (5.5) and (5.6).

Applying again the modified Crank-Nicholson difference scheme (4.18), we obtain the difference scheme second order of accuracy in t and s and second order

of accuracy in x

$$
\begin{cases}
\frac{u_n^{k,m}-u_n^{k-1,m-1}}{\tau} - \frac{u_{n+1}^{k,m}-2u_n^{k,m}+u_{n-1}^{k,m}}{h^2} + 2u_n^{k,m} = f_{k,m}^h, 1 \le k,m \le r, \\
\\
\frac{u_n^{k,m}-u_n^{k-1,m-1}}{\tau} - \frac{u_{n+1}^{k,m}-2u_n^{k,m}+u_{n-1}^{k,m}}{h^2} + 2u_n^{k,m} \\
\\
-\frac{u_{n+1}^{k-1,m-1}-2u_n^{k-1,m-1}+u_{n-1}^{k-1,m-1}}{h^2} + 2u_n^{k-1,m-1} = f_{k,m}^h, r+1 \le k,m \le N, \\
\\
f_{k,m}^h = \ f(t_k - \frac{\tau}{2}, s_m - \frac{\tau}{2}, x_n) = e^{-(t_k+s_m-\tau)} \sin x_n, \\
\\
1 \le k,m \le N, \ 1 \le n \le M-1, \\
\\
u_n^{0,m} = e^{-s_m} \cos x_n, \ 0 \le m \le N, \ 0 \le n \le M, \\
\\
u_n^{k,0} = e^{-t_k} \cos x_n, \ 0 \le k \le N, \ 0 \le n \le M, \\
\\
u_0^{k,m} = u_1^{k,m}, u_M^{k,m} = u_{M-1}^{k,m} \ 0 \le k,m \le N, \\
\\
t_k = k\tau, s_m = m\tau, \ 1 \le k,m \le N, \ N\tau = 1, \\
\\
x_n = nh, \ 1 \le n \le M, \ Mh = \pi
\end{cases}
\tag{5.10}
$$

for approximate solutions of initial boundary value problem (5.2). It can be written in the matrix form (5.4). Here

$$
A = \begin{bmatrix}
0 & 0 & \cdots & \cdots & \cdots & 0 & 0 \\
0 & 0 & \cdots & \cdots & \cdots & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & a & \cdots & \cdots & \cdots \\
0 & 0 & 0 & \cdots & a & \cdots & 0 \\
0 & 0 & 0 & \cdots & \cdots & a & \cdots \\
0 & 0 & 0 & 0 & \cdots & \cdots & a
\end{bmatrix}_{(N+1)^2 \times (N+1)^2},
$$

$$B = \begin{bmatrix} 1 & 0 & \cdots & \cdots & \cdots & 0 & 0 & 0 \\ 0 & 1 & \cdots & \cdots & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & b & \cdots & c & d & \cdots & \cdots & \cdots \\ 0 & 0 & b & \cdots & c & d & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & b & \cdots & c & d & 0 \\ 0 & 0 & 0 & \cdots & b & \cdots & c & d \end{bmatrix}_{(N+1)^2 \times (N+1)^2},$$

$$C = \begin{bmatrix} 0 & 0 & \cdots & \cdots & \cdots & 0 & 0 \\ 0 & 0 & \cdots & \cdots & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & a & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & a & \cdots & 0 \\ 0 & 0 & 0 & \cdots & \cdots & a & \cdots \\ 0 & 0 & 0 & 0 & \cdots & \cdots & a \end{bmatrix}_{(N+1)^2 \times (N+1)^2},$$

where

$$a = \frac{-1}{2h^2}, b = \frac{-1}{\tau}, c = \frac{-1}{\tau} + \frac{-1}{h^2}, d = \frac{1}{\tau} + \frac{2}{h^2} + 2,$$

$$\varphi_n = \begin{bmatrix} \varphi_n^{0,0} \\ \varphi_n^{1,0} \\ \cdots \\ \varphi_n^{N,0} \\ \varphi_n^{0,1} \\ \varphi_n^{1,1} \\ \cdots \\ \varphi_n^{N,1} \\ \cdots \\ \varphi_n^{0,N} \\ \varphi_n^{1,N} \\ \cdots \\ \varphi_n^{N,N} \end{bmatrix}_{(N+1)^2 \times 1}, \quad \varphi_n^{k,m} = f(t_k, s_m, x_n) = e^{-(t_k + s_m)} \cos x_n \ ,$$

and

$$u_n = \begin{bmatrix} u_n^{0,0} \\ u_n^{1,0} \\ \cdots \\ u_n^{N,0} \\ u_n^{0,1} \\ u_n^{1,1} \\ \cdots \\ u_n^{N,1} \\ \cdots \\ u_n^{0,N} \\ u_n^{1,N} \\ \cdots \\ u_n^{N,N} \end{bmatrix}_{(N+1)^2 \times 1} .$$

5.3 ERROR ANALYSIS

Errors computed by

$$E_N^{K,M} = \max_{1 \le k,m \le N, 1 \le n \le M-1} \left| u(t_k, s_m, x_n) - u_n^{k,m} \right|$$

of the numerical solutions, where $u(t_k, s_m, x_n)$ represents the exact solution and $u_n^{k,m}$ represents the numerical solution at (t_k, s_m, x_n) and the results derived by difference schemes are given in following tables.

Table 5.1 Comparison of the errors of different schemes for (5.1), here N=M.

Difference schemes	$N = 10$	$N = 15$	$N = 20$	$N = 30$
Difference scheme (5.3)	0.02810	0.02093	0.01440	0.01002
Difference scheme (5.8)	0.00051	0.00022	0.00012	0.00005
Difference scheme (5.9),r=1	0.02063	0.01025	0.00616	0.00292
Difference scheme (5.9),r=2	0.03277	0.01752	0.01094	0.00541

Table 5.2 Comparison of the errors of different schemes for (5.2), here N=M.

Difference schemes	$N = 10$	$N = 15$	$N = 20$	$N = 30$
Difference scheme (5.7)	0.0646	0.0414	0.0363	0.0183
Difference scheme (5.11),r=0	0.0043	0.0022	0.0010	0.0003
Difference scheme (5.11),r=1	0.0574	0.0382	0.0251	0.0168

Thus, the results show that the second order of accuracy difference schemes are more accurate comparing with the first order of accuracy difference scheme. Moreover, modified Crank-Nicholson difference scheme is less accurate for greater r values.

CHAPTER 6

MATLAB PROGRAMMING

6.1 MATLAB IMPLEMENTATION OF THE FIRST ORDER OF ACCURACY DIFFERENCE SCHEMES

6.1.1 Matlab Implementation for (5.1)

function uparabolafirst(N,M)

close; close;

tau1=1/N; tau2=tau1;

h=pi/M; a=-1/h^2; $b = -1/(tau2)$;

c = -1/(tau1); d =-c+2+(2/(h^2));

N=N+1;

A=zeros(N*N,N*N);B=zeros(N*N,N*N);U=zeros(N*N,M+1);

alpha=zeros(N*N,N*N);betha=zeros(N*N,N*N);

for i=1:N A(i,i)=0;end;

for i=N+1:N*N A(i,i)=a; end;

for i=1:N-1 A(i*N+1,i*N+1)=0; end;

C=A; for i=1:N B(i,i)= 1 ; end;

for ii=1:1:N-1 for i=ii*N+2:ii*N+N ;B(i,i-N-1)=b;

B(1+ii*N,1+ii*N)=1;B(i,i-1)=0;B(i,i)=d; end; end;

Phi=zeros(N*N,M+1);

for n=1:1:M+1

for nn=1:N

Phi(nn,n) =(exp(-(nn-1)*tau1))* sin((n-1)*h);

end;

```
for i=1:N-1
Phi(N+1+(i-1)*N,n) = (exp((-i)*tau2))* sin((n-1)*h);
end; end;
for n=1:1:M+1
for ii=1:N-1
for nnn=ii*N+2:ii*N+N
Phi(nnn,n)=(exp(((nnn-ii*N-1)*-tau1)+ii*-tau2))* sin((n-1)*h);
end; end; end;
for j=1:1:M
alpha(:,:,j+1)=-inv(B+C*alpha(:,:,j))*A;
betha(:,j+1)=inv(B+C*alpha(:,:,j))*(Phi(:,j+1)-C*betha(:,j));
end;
for z = M:-1:2
U(:,z) = alpha(:,:,z)* U(:,z+1) + betha(:,z);
end;
es=zeros(N*N,M+1);
for n=1:1:M+1
for nn=1:N
es(nn,n) = (exp(-(nn-1)*tau1))* sin((n-1)*h);
end;
for i=1:N-1
es(N+1+(i-1)*N,n) = (exp((-i)*tau2))* sin((n-1)*h);
end; end
for n=1:1:M+1
for ii=1:N-1
for nnn=ii*N+2:ii*N+N
es(nnn,n)=(exp(((nnn-ii*N-1)*-tau1)+ii*-tau2))* sin((n-1)*h);
end; end; end;
figure ; m(1,1)=min(min(U))-0.01; m(2,2)=nan; surf(m); hold;
surf(es) ; rotate3d ;axis tight;
title('EXACT SOLUTION'); figure ;
```

```
m(1,1)=min(min(U))-0.01; m(2,2)=nan; surf(m); hold; surf(U) ;
rotate3d ;axis tight;
'EXACT SOLUTION OF THIS PROBLEM';
maxerror=max(max(abs(es-U)));
relativeerror=max(max(abs(es-U)))/max(max(abs(Phi)) );
cevap =[maxerror,relativeerror]
```

6.1.2 Matlab Implementation for (5.2)

```
function uparabolaneumann(N,M)
close; close;
tau1=1/N; tau2=tau1; h=pi/M; a=-1/h^2; b = -1/(tau2);
c = -1/(tau1); d = -c+2+(2/(h^2)); N = N + 1;
A=zeros(N*N,N*N);B=zeros(N*N,N*N);U=zeros(N*N,M+1);
alpha=eye(N*N,N*N);betha=zeros(N*N,N*N);
for i=1:N A(i,i)=0; end;
for i=N+1:N*N A(i,i)=a; end;
for i=1:N-1 A(i*N+1,i*N+1)=0; end; C=A;
for i=1:N B(i,i)= 1 ; end;
for ii=1:1:N-1
for i=ii*N+2:ii*N+N ; B(i,i-N-1)=b
;B(1+ii*N,1+ii*N)=1;B(i,i-1)=0;B(i,i)=d;
end; end;
B;
Phi=zeros(N*N,M+1);
for n=1:1:M+1
for nn=1:N
Phi(nn,n) =(exp(-(nn-1)*tau1))* cos((n-1)*h); end;
for i=1:N-1
Phi(N+1+(i-1)*N,n) = (exp((-i)*tau2))* cos((n-1)*h);
end; end
for n=1:1:M+1
```

```
for ii=1:N-1
for nnn=ii*N+2:ii*N+N
Phi(nnn,n)=(exp(((nnn-ii*N-1)*-tau1)+ii*-tau2))* cos((n-1)*h);
end; end; end;
for j=1:1:M;
alpha(:,:,j+1)=-inv(B+C*alpha(:,:,j))*A;
betha(:,j+1)=inv(B+C*alpha(:,:,j))*(Phi(:,j+1)-C*betha(:,j)); end;
for ii=1:N
for nnn=(ii-1)*N+1:ii*N
U(nnn,1)=exp(((nnn-(ii-1)*N-1)*-tau1)+(ii-1)*-tau2);
end; end;
alpha(:,:,M+1);
U(:,M+1)=inv(eye(N*N,N*N)-alpha(:,:,M+1))*betha(:,M+1);
for z = M:-1:1
U(:,z) = alpha(:,:,z)* U(:,z+1) + betha(:,z); end;
es=zeros(N*N,M+1);
for n=1:1:M+1
for nn=1:N es(nn,n) = (exp(-(nn-1)*tau1))* cos((n-1)*h); end;
for i=1:N-1
es(N+1+(i-1)*N,n) = (exp((-i)*tau2))* cos((n-1)*h);
end; end;
for n=1:1:M+1
for ii=1:N-1
for nnn=ii*N+2:ii*N+N
es(nnn,n)=(exp(((nnn-ii*N-1)*-tau1)+ii*-tau2))* cos((n-1)*h);
end; end; end;
figure ; m(1,1)=min(min(U))-0.01; m(2,2)=nan; surf(m); hold;
surf(es) ; rotate3d ;axis tight; title('EXACT SOLUTION');
figure ; m(1,1)=min(min(U))-0.01; m(2,2)=nan; surf(m); hold;
surf(U); rotate3d ;axis tight;
'EXACT SOLUTION OF THIS PROBLEM';
```

```
maxerror=max(max(abs(es-U)));
relativeerror=max(max(abs(es-U)))/max(max(abs(Phi)) );
cevap = [maxerror,relativeerror]
```

6.2 MATLAB IMPLEMENTATION OF CRANK-NICHOLSON DIFFERENCE SCHEME

6.2.1 Matlab Implementation for (5.1)

```
function uparabolacrank(N,M)
close; close;
tau1=1/N; tau2=tau1; h=pi/M; aa=-1/(2*h^2); a1 = 1/tau1; a2 = 1/tau2;
b =(1/2)*(a1-a2+1+(1/(h^2))); bb = -a1 + 1 + (1/(h^2));
c =(1/2)*(-a1+a2+1+(1/(h^2))); d = a1 + 1 + (1/(h^2));
N=N+1;
A=zeros(N*N,N*N);B=zeros(N*N,N*N);alpha=zeros(N*N,N*N);
betha=zeros(N*N,N*N);U=zeros(N*N,M+1);
for i=1:N ; A(i,i)= 0 ; end;
for ii=1:1:N-1 for i=ii*N+2:ii*N+N ; A(i,i-(N+1))=aa; A(i,i-N)=0;
A(i,i-1)=0;A(i,i)=aa; end; end;
for i=1:N-1 A(i*N+1,i*N+1)=0; end;
C=A;
for i=1:N ; B(i,i)= 1 ; end;
for ii=1:1:N-1
for i=ii*N+2:ii*N+N ;
B(i,i-(N+1))=bb; B(i,i-N)=0 ;B(1+ii*N,1+ii*N)=1;B(i,i-1)=0;B(i,i)=d;
end; end;
Phi=zeros(N*N,M+1);
for n=1:1:M+1
for nn=1:N
Phi(nn,n) = (exp(-(nn-1)*tau1))* sin((n-1)*h); end;
```

```
for i=1:N-1
Phi(N+1+(i-1)*N,n) = (exp((-i)*tau2))* sin((n-1)*h);
end; end;
for n=1:1:M+1
for ii=1:N-1
for nnn=ii*N+2:ii*N+N
Phi(nnn,n)=(exp(((nnn-ii*N-1)*-tau1+tau1/2)+ii*-tau2+tau2/2))* sin((n-1)*h);
end; end; end;
for j=1:1:M;
alpha(:,:,j+1)= -inv(B+C*alpha(:,:,j))*A;
betha(:,j+1)=inv(B+C*alpha(:,:,j))*(Phi(:,j+1)-C*betha(:,j));
end;
for z = M:-1:2
U(:,z) = alpha(:,:,z)* U(:,z+1 ) + betha(:,z); end;
'EXACT SOLUTION OF THIS PROBLEM';
es=zeros(N*N,M+1);
for n=1:1:M+1
for nn=1:N
es(nn,n) = (exp(-(nn-1)*tau1))* sin((n-1)*h);
end;
for i=1:N-1
es(N+1+(i-1)*N,n) = (exp((-i)*tau2))* sin((n-1)*h);
end; end
for n=1:1:M+1
for ii=1:N-1
for nnn=ii*N+2:ii*N+N
es(nnn,n)=(exp(((nnn-ii*N-1)*-tau1)+ii*-tau2))* sin((n-1)*h);
end; end; end;
figure ; m(1,1)=min(min(U))-0.01; m(2,2)=nan; surf(m); hold;
surf(es) ; rotate3d ;axis tight; title('EXACT SOLUTION');
figure ; m(1,1)=min(min(U))-0.01; m(2,2)=nan; surf(m); hold;
```

```
surf(U) ; rotate3d ;axis tight; title('FIRST ORDER');
maxes=max(max(es)) ;
maxerror=max(max(abs(es-U)));
relativeerror=max(max((abs(es-U))))/max(max(abs(U)) ); cevap =
[maxerror,relativeerror]
```

6.3 MATLAB IMPLEMENTATION OF MODIFIED CRANK-NICHOLSON DIFFERENCE SCHEME

6.3.1 Matlab Implementation for (5.1)

```
function uparabolarmod(N,M,k)
close; close;
tau1=1/N; tau2=tau1; h=pi/M; a=-1/(2*(h^2));
d = 1/(tau2)+2+(2/(h^2)); e = -1/(tau2) + 1 + (1/(h^2));
f=1/(tau2)+1+(1/(h^2));
N=N+1;
A=zeros(N*N,N*N);B=zeros(N*N,N*N);U=zeros(N*N,M+1);
alpha=zeros(N*N,N*N);betha=zeros(N*N,N*N);
for i=1:N A(i,i)= 0 ; end;
for ii=1:1:N-1
for i=ii*N+2:ii*N+k ;
A(i,i-N-1)=0;A(1+ii*N,1+ii*N)=0;A(i,i)=a*2;end;
for i=ii*N+k+1:ii*N+N ; A(i,i-N-1)=a ;A(1+ii*N,1+ii*N)=0;A(i,i)=a;
end; end;
C=A;
for i=1:N B(i,i)= 1 ; end; for ii=1:1:N-1
for i=ii*N+2:ii*N+k
B(i,i-N-1)=-1/(tau2) ;B(1+ii*N,1+ii*N)=1;B(i,i)=d; end;
for i=ii*N+k+1:ii*N+N ; B(i,i-N-1)=e ;B(1+ii*N,1+ii*N)=1;B(i,i)=f;
end; end;
```

```
Phi=zeros(N*N,M+1);
for n=1:1:M+1
for nn=1:N
Phi(nn,n) = (exp(-(nn-1)*tau1))* sin((n-1)*h); end;
for i=1:N-1
Phi(N+1+(i-1)*N,n) = (exp((-i)*tau2))* sin((n-1)*h);
end; end;
for n=1:1:M+1
for ii=1:N-1
for nnn=ii*N+2:ii*N+N
Phi(nnn,n)=(exp(((nnn-ii*N-1)*-tau1+tau1/2)+ii*-tau2+tau2/2))* sin((n-1)*h);
end; end; end;
for j=1:1:M;
alpha(:,:,j+1)= -inv(B+C*alpha(:,:,j))*A;
betha(:,j+1)=inv(B+C*alpha(:,:,j))*(Phi(:,j+1)- C*betha(:,j));
end;
for z = M:-1:2
U(:,z) = alpha(:,:,z)* U(:,z+1 ) + betha(:,z);
end;
'EXACT SOLUTION OF THIS PROBLEM';
es=zeros(N*N,M+1);
for n=1:1:M+1
for nn=1:N
es(nn,n) = (exp(-(nn-1)*tau1))* sin((n-1)*h);
end;
for i=1:N-1
es(N+1+(i-1)*N,n) = (exp((-i)*tau2))* sin((n-1)*h);
end; end;
for n=1:1:M+1
for ii=1:N-1
for nnn=ii*N+2:ii*N+N
```

```
es(nnn,n)=(exp(((nnn-ii*N-1)*-tau1)+ii*-tau2))* sin((n-1)*h);
end; end; end;
figure ; m(1,1)=min(min(U))-0.01; m(2,2)=nan; surf(m); hold;
surf(es) ; rotate3d ;axis tight; title('EXACT SOLUTION');
figure ; m(1,1)=min(min(U))-0.01; m(2,2)=nan; surf(m); hold;
surf(U); rotate3d ;axis tight; title('FIRST ORDER');
maxes=max(max(es)) ;
maxerror=max(max(abs(es-U)));
relativeerror=max(max((abs(es-U))))/max(max(abs(U)) );
cevap = [maxerror,relativeerror]
```

6.3.2 Matlab Implementation for (5.2)

```
function uparabolarmod3(N,M,k)
close; close;
tau1=1/N; tau2=tau1; h=pi/M; a=-1/(2*(h^2));
d = 1/(tau2)+9+(2/(h^2)); e = -1/(tau2) + 9/2 + (1/(h^2));
f=1/(tau2)+9/2+(1/(h^2));
N=N+1;
A=zeros(N*N,N*N);B=zeros(N*N,N*N);U=zeros(N*N,M+1);
alpha=zeros(N*N,N*N);betha=zeros(N*N,N*N);
for i=1:N A(i,i)= 0 ; end;
for ii=1:1:N-1
for i=ii*N+2:ii*N+k ; A(i,i-N-1)=0;
A(1+ii*N,1+ii*N)=0;A(i,i)=a*2; end;
for i=ii*N+k+1:ii*N+N ; A(i,i-N-1)=a ;A(1+ii*N,1+ii*N)=0;A(i,i)=a;
end; end;
C=A;
for i=1:N B(i,i)= 1 ; end; for ii=1:1:N-1
for i=ii*N+2:ii*N+k ; B(i,i-N-1)=-1/(tau2);
B(1+ii*N,1+ii*N)=1;B(i,i)=d; end;
for i=ii*N+k+1:ii*N+N ;
```

```
B(i,i-N-1)=e ;B(1+ii*N,1+ii*N)=1;B(i,i)=f;
end; end;
Phi=zeros(N*N,M+1);
for n=1:1:M+1
for nn=1:N
Phi(nn,n) =0; end;
for i=1:N-1
Phi(N+1+(i-1)*N,n) = 0;
end; end
for n=1:1:M+1
for ii=1:N-1
for nnn=ii*N+2:ii*N+N
Phi(nnn,n)=(sqrt((nnn-ii*N-1)*(tau1-tau1/2))
+sqrt(ii*(tau2-tau2/2)))*(3/2)*sqrt(((nnn-ii*N-1)
*(tau1-tau1/2))*(ii*(tau2-tau2/2)))*sin(3*(n-1)*h);
end; end; end;
for j=1:1:M;
alpha(:,:,j+1)= -inv(B+C*alpha(:,:,j))*A;
betha(:,j+1)=inv(B+C*alpha(:,:,j))*(Phi(:,j+1)- C*betha(:,j)); end
for z = M:-1:2
U(:,z) = alpha(:,:,z)* U(:,z+1 ) + betha(:,z); end;
'EXACT SOLUTION OF THIS PROBLEM';
es=zeros(N*N,M+1);
for n=1:1:M+1
for nn=1:N
es(nn,n) = 0; end;
for i=1:N-1 es(N+1+(i-1)*N,n) =0;
end; end
for n=1:1:M+1
for ii=1:N-1
for nnn=ii*N+2:ii*N+N
```

```
es(nnn,n)=(((nnn-ii*N-1)*(tau1-tau1/2))
*((ii)*(tau2-tau2/2)))^(3/2) * sin(3 * (n - 1) * h);
end; end; end;
figure ; m(1,1)=min(min(U))-0.01; m(2,2)=nan; surf(m); hold;
surf(es) ; rotate3d ;axis tight; title('EXACT SOLUTION');
figure ; m(1,1)=min(min(U))-0.01; m(2,2)=nan; surf(m); hold;
surf(U) ; rotate3d ;axis tight; title('FIRST ORDER');
maxes=max(max(es)) ;
maxerror=max(max(abs(es-U)));
relativeerror=max(max((abs(es-U))))/max(max(abs(U)) );
cevap = [maxerror,relativeerror]
```

CHAPTER 7

CONCLUSION

This thesis is devoted to study the well-posedness of ultra-parabolic equations. The following original results are obtained:

- The abstract theorem on the coercive stability estimate for the solution of Neumann initial boundary value differential problem for the ultra-parabolic equation in the Banach space is proved.
- The theorems on coercive stability estimates for the solution of initial boundary for the ultra-parabolic equation are presented.
- The first and second orders of accuracy difference schemes for approximate solutions of the initial boundary value problems for ultra-parabolic equations are presented.
- Theorems on the stability estimates, almost coercive stability estimates and coercive stability estimates for the solution of difference scheme for ultra-parabolic equations are proved.
- The Matlab implementation of the first and second orders difference schemes are presented.
- The theoretical statements for the solution of these difference schemes are supported by the results of numerical examples.

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APPENDIX A

DECLARATION STATEMENT FOR THE ORIGINALITY OF THE THESIS, FURTHER STUDIES AND PUBLICATIONS FROM THESIS WORK

A.1 DECLARATION STATEMENT FOR THE ORIGINALITY OF THE THESIS

I hereby declare that this thesis comprises my original work. No material in this thesis has been previously published and written by another person, except where due reference is made in the text of the thesis. I further declare that this thesis contains no material which has been submitted for a degree or diploma or other qualifications at any other university.

Signature:
Date: February 22, 2013

A.2 FURTHER STUDIES

Further studies related to this thesis work can be done in the following areas:

(a) Higher order accuracy difference schemes for approximate solutions of ultra-parabolic problems
(b) Applications of ultra parabolic equations.

A.3 PUBLICATIONS FROM THESIS WORK

Academic Journals

- Ashyralyev, A. and Yilmaz, S., "An Aproximation of Ultra-Parabolic Equations", *Abstr. Appl. Anal.*, Vol. 2012, pp. 1–14, 2012.
- Ashyralyev, A. and Yilmaz, S., "Modified Crank-Nicholson Difference Schemes for Ultra-Parabolic Equations", *Comput. Math. Appl*, Vol. 64, pp. 2756–2764, 2012.

Conference Proceedings

- Ashyralyev, A. and Yilmaz, S., "On the Numerical Solution of Ultra-Parabolic Equations with the Neumann Condition", *AIP Conference Proceedeings,* Gumushane-Turkey, Vol. 1470, pp. 240–243, October 2012.
- Ashyralyev, A. and Yilmaz, S., "Second Order of Accuracy Difference Schemes for Ultra-Parabolic Equations", *AIP Conference Proceedeings,,* Halkidiki-Greece, Vol. 1389, pp. 601–604, September 2011.

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- Ashyralyev, A. and Yilmaz, S., "Modified Crank-Nicholson Difference Schemes for Ultra-Parabolic Equations", *Comput. Math. Appl*, Vol. 64, pp. 2756–2764, 2012.

Conference Proceedings
- Ashyralyev, A. and Yilmaz, S., "Second Order of Accuracy Difference Schemes for Ultra-Parabolic Equations", *AIP Conference Proceedeings,*, Halkidiki-Greece, Vol. 1389, pp. 601–604, September 2011.
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TECHNICAL SKILLS
- Programming languages: C
- Analytical and statistics programs: Mathematica, MatLab
- Word processing and publication: Microsoft Word
- Spreadsheets and databases: Microsoft Excel
- Operating systems: Microsoft Windows 7
- Presentation: Microsoft PowerPoint