TURKISH REPUBLIC
ERCIYES UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCE
DEPARTMENT OF MATHEMATICS

FIXED POINT THEOREMS AND SOME APPLICATIONS

Prepared by
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Supervisor
Prof. Dr. Osman MUCUK

M. Sc. Thesis

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KAYSERİ

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HANAN ABDULLAH IDAN AL-JANABI

SUITABILITY FOR GUIDE

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Hanan Abdullah Idan Al-JANABI

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ABSTRACT

Fixed point theory is an exciting branch of mathematics. It is a mixture of analysis, topology and geometry. Fixed point techniques have been applied in fields such as biology, chemistry, economics, engineering and physics. It has very fruitful applications in control theory, game theory, category theory, functional equations, integral equations, mathematical physics, mathematical chemistry, functional analysis and many other areas. Thus the study of the fixed point theory has been researched extensively.
In this thesis some results aboult fixed point theorems in different metrix spaces are stated. The thesis consists of three chapters: A general basic definitions and preliminaries related to metric fixed point theory are introduced in Chapter One.
Chapter Two consists of two sections: in Section 1 we extend and improve the results of [1,2,3,4,5,6], relate to approximate fixed point, coincidence point and approximate best proximity points are studied. In Section 2, common fixed point theorems for the maps for hybrid integral type contraction conditions are established which modifies the results of [7].

Chapter Three consists of two sections, The concepts of compatible maps and weakly compatible maps in G-metric spaces are introduced in Section 1. In Section 2, weakly compatible mappings and some common fixed point theorems on fuzzy metric spaces have been stated.

In Chapter Four our common fixed point theorems for weakly compatible mappings in fuzzy metric spaces have been presented, in addition to some common fixed theorems for *R*-weakly commuting mappings and some of their variants in fuzzy metric spaces are given.

Key words: Metric Space, fuzzy metric space, contraction mappings, compatible contraction mappings, fixed point theorem.

SABİT NOKTA TEOREMLERİ VE BAZI UYGULAMALAR

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ÖZET

Sabit nokta teoremi analiz, topoloji ve geometrinin karışımı olan ilginç matematik dalıdır. Sabit nokta teknikleri biyoloji, kimya, ekonomi, mühendislik ve fizik gibi bilim dallarına uygulanmıştır. Sabit nokta teoremi kontrol teorisi, oyun teorisi, kategori teorisi, fonksiyonel denklemler, integral denklemleri, matematiksel fizik, matematiksel kimya, fonksiyonel analiz ve diğer bazı alanlarda güzel uygulamalara sahiptir. Bu yüzden bu teori yoğun olarak araştırılmaktadır.

Bu tezde farklı metrik uzaylarda sabit nokta theoremi ile ilgili bazı sonuçlar verilmiştir.
Tez üç bölümden oluşmaktadır: Metrik uzaylarda sabit nokta teoremi ile ilgili genel temel tanımlar ve bazı önbilgiler Bölüm 1 de tanıtılmıştır.
İki kesimden oluşan 2. Bölüm'ün birinci kesiminde [1,2,3,4,5,6] daki sonuçlar incelenmiş, ilgili sabit nokta teoremleriyle ilişkilendirilmiş ve yaklaşık yakınlık noktası çalışılmış, ikinci kesiminde ise fonksiyonların hybrid integral tipi daralma şartları için genel sabit nokta teoremleri verilmiş ve [7] nin sonuçları gözden geçirilmiştir.
Bölüm 3 ise iki kesimden oluşur. Birinci kesimde G-metrik uzaylarda uyumlu ve zayıf uyumlu dönüşüm kavramları tanıtılmıştır. İkinci kesimde ise fuzzy metrik uzaylarda zayıf uyumlu dönüşümler ve bazı sabit nokta teoremleri verilmiştir.
Bölüm 4 de fuzzy metrik uzaylarda zayıf uyumlu dönüşümler için genel sabit nokta teoremleri sunulmuş ve ayrıca fuzzy metrik uzaylarda R-zayıf değişimli dönüşümler ve varyantları için bazı sabit nokta teoremleri verilmiştir.

Anahtar Kelimeler: Metrik uzay, fuzzy metrik uzay, büzülme dönüşümlşeri, uyumlu büzülme dönüşümleri, sabit nokta teoremi

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INTRODUCTION

Fixed point theory is an exciting branch of mathematics. It is a mixture of analysis, topology and geometry. Fixed point theory is one of the most dynamic research areas in nonlinear analysis.

It has a wide range of applications in fields such as economics, computer science and many others. The most important and core result in this direction was given by Polish mathematician Stefan Banach [8] in 1922, popularly known as Banach contraction principle. This principle not only guarantees the existence and uniqueness of the fixed points of self-mappings but also gives a procedure to obtain the fixed points of these mappings. This technique is used as a powerful tool to solve various problems in different branches of Mathematics such as integral equations, ordinary differential equations, partial differential equations, dynamic programming and game theory.

Motivated by the Banach contraction principle, several researchers (see, e.g., Banach [9], Chatterjea [10], Ciric [11], Edelstein [12], Hardy and Rogers [13], Kannan [14], Rakotch [15], Reich [16], Sehgal [17], Singh [18] and Zamfirescu [19]) have introduced various other types of contractive conditions and have established number of related fixed point theorems. In 1977, Rhoades [20] investigated various contractive mappings that have appeared in the literature in the last twenty years and has analyzed the inter relationships among them. The generalization of the existing idea of Banach is a very important aspect for the advancement in the field of fixed point theory.

The space X is said to have the fixed point property if for any continuous function $F: X \to X$, there exists $x \in X$ such that $F(x) = x$, that is, a point which remains invariant under the mapping F. Geometric representation of fixed point is diagonal line. We can say identity mappings have all its points as fixed point.

During the sixties, the notion of 2-metric spaces was introduced by Gahler [21, 22, 23], as a generalization of usual notion of metric space (X, d). Many applications of fixed point theory in 2-metric spaces can be found. A new structure of a generalized metric space called D-metric space was introduced by Dhage [24], on the lines of ordinary metric spaces. Generally the usual ordinary metric is called the distance function. D-metric is called the diameter function of the points of spaces. Mustafa and Sims [25] introduced a new structure of generalized metric space, which is called G-metric space as generalization of metric space (X, d), to develop and introduce a new fixed point theory for various mappings in this new structure.

Motivated by G-metric spaces and b-metric spaces, we define the concept of generalized b-metric spaces and then prove some basic approximate fixed point results in generalized metric spaces. Further, we establish some results in concerning approximate fixed points, approximate common fixed points, approximate coincidence points, coincidence points and endpoints of multi-valued contractions.

Fixed point theory was extended to multi-valued mappings in 1941 with the fixed point theorems of Nadler [26]. If f is a multi-valued map from X to a collection of non empty subsets of X, then a point $x \in X$ is a fixed point of f if $x \in f(x)$. Fixed point theory for multi-valued maps find special significance in the theory of inclusions, differential equations for multi-valued operators, game theory, fractal theory, discrete dynamics and other areas of mathematical sciences.

Thus a huge development is reported in the study of fixed point theory of single valued, multi-valued and hybrid maps in different directions. This in turn enhances the applications of fixed point theorems to diverse disciplines of mathematics, statistics, engineering and economics. In the present work, our attempt is to state improvements and generalizations of several recent results for application purpose.

Main Objectives

The main objective of the work reported in the thesis are:

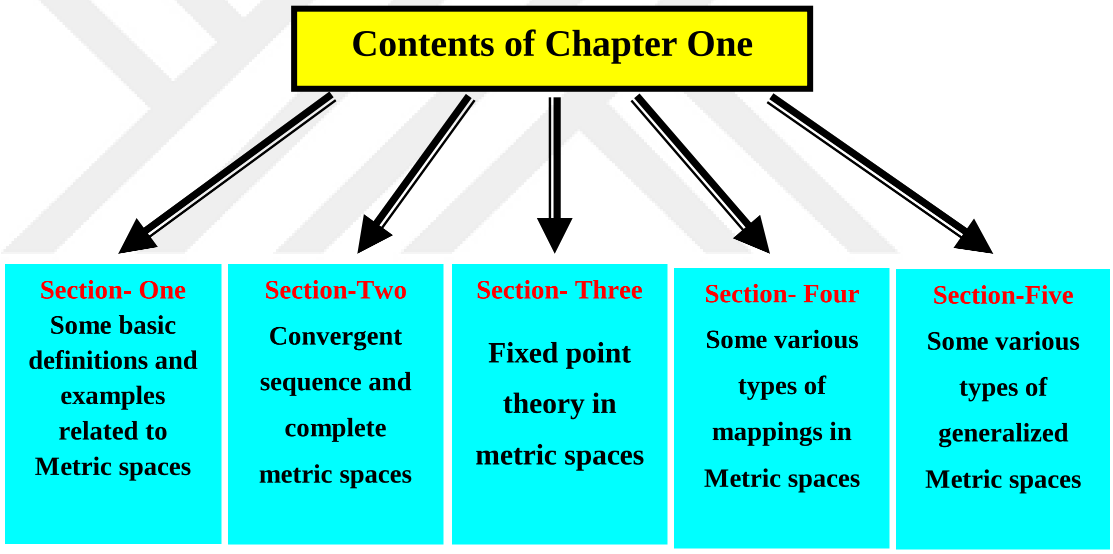
- Get some approximate fixed point results and explore them for applications,
- Establish some common fixed point theorems and their applications,

- Generalizing the results using compatible mappings, R-weakly commuting mappings and their variants in different spaces.
- Weakly compatible mappings and some common fixed point theorems on fuzzy metric spaces.
- Study and extend the fixed point results for iterated function and multifunction systems.

CHAPTER ONE

BASIC CONCEPTS

The first chapter is introductory in nature, provides the development of the subject, necessary background to the rest of the chapters. This thesis consists of five sections stated as follow.



1.1 Some Basic Definitions and Examples Related to Metric Spaces

Metric spaces which plays an important role in the study of topology and functional analysis has been introduced by Frechet [27] in 1906. However, definition given by the German mathematician, Hausdroff [28] in 1914 is commonly used and is stated as follows:

Definition.1.1.1 Let X be a non-empty set. A metric on X is a real valued map d: $X \times X \to \mathbb{R}$ which satisfies the following conditions for all $x, y, z \in X$,

i) $d(x, y) \geq 0$;

ii) $d(x, y) = 0$ iff $x = y$;

iii) $d(x, y) = d(y, x)$;

iv) $d(x, z) \leq d(x, y) + d(y, z)$. (Triangle inequality)

A metric space is a non-empty set X equipped with a metric d on X and is denoted by (X, d). Geometrically, $d(x, y)$ represents distance between two points x and y on the real line.

Examples.1.1.2

i) The set $\mathbb{R}$ of real numbers with the distance function $d(x, y) = |x - y|$ for all $x, y \in \mathbb{R}$ is known as usual metric space.

ii) The plane with the supremum or maximum metric space

$$d((x_1, y_1), (x_2, y_2)) = max\{ |x_1 - x_2|, |y_2 - y_1| \}$$

is often called the infinity metric d_∞.

iii) Take X to be any set. The discrete metric on X is given by: $d(x, y) = 0$ if $x = y$ and $d(x, y) = 1$ otherwise.

Definition.1.1.3 A norm on a linear space X is a map $\|.\| : X \rightarrow \mathbb{R}$ with the following properties:

i) $\|\boldsymbol{x}\| \geq 0$ for all $x \in X$ (non-negative)

ii) $\|\boldsymbol{\lambda x}\| = |\lambda|\|\boldsymbol{x}\|$ for all $x \in$ X and for all $\lambda \in \mathbb{R}$;

iii) $\|\boldsymbol{x} + \boldsymbol{y}\| \leq \|\boldsymbol{x}\| + \|\boldsymbol{y}\|$ for all $x, y \in$ X;

iv) $\|\boldsymbol{x}\| = \mathbf{0} \Rightarrow x = 0$.

A normed linear space $(X, \|.\|)$ is a linear space X equipped with a norm $\|.\|$.

Remark.1.1.4 Every normed linear space is a metric space under the metric $d(x, y) = \|x - y\|$.

Examples.1.1.5

i) If $X = \mathbb{R}$, then $\|x\| = |x|$ is a norm.

ii) Let $X = \mathbb{C}$ for z in $\mathbb{C}$ the modulus of z, $|z| = (|Re\, z|^2 + |Im\, z|^2)^{1/2}$ is a norm for the complex numbers.

1.2. Convergent Sequence and Complete Metric Spaces

Definition.1.2.1 A sequence is a function whose domain is the set of natural numbers or a subset of the natural numbers. We usually use the symbol $\{x_n\}$ to represent a sequence, where n is a natural number and x_n is the value of the function on n.

Remark.1.2.2 A sequence may be finite or infinite. If a sequence is finite, we sometimes write $\{x_1, x_2, x_3, x_4, \ldots\ldots, x_n\}$to represent the sequence. If a sequence is infinite, we write $\{x_1, x_2, x_3, x_4, \ldots\ldots\}$ or $\{x_n\}_{n=1}^{\infty}$.

Examples.1.2.3 The followings are examples of sequences.

i) $\{x_n\} = (n)_{n=1}^{\infty}$ .We obtain the sequence of positive integers: $\{x_n\}_{n=1}^{\infty}$ = {1, 2, 3 , 4 …..}.

ii) $\{x_n\} = (x^2 - 3)_{n=1}^{\infty}$. We obtain the following sequence $\{x_n\}_{n=1}^{\infty}$ = {-2, 1, 6, 13, 22, 33, 46, …..}.

iii) $\{x_n\} = \left(\frac{n}{n+1}\right)_{n=1}^{\infty}$. We obtain the following sequence $\{x_n\}_{n=1}^{\infty}$ = $\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \ldots..\}$.

Definition.1.2.4 A sequence $\{x_n\}$ in a metric space *(X, d)* is called **Cauchy sequence** if $\lim_{n,m\to\infty} d(x_n, x_m) = 0$ for all natural number m,n.

Example.1.2.5 Let $X = (0, 1]$ with usual metric and $\{x_n\} = \frac{1}{n}$ be a sequence in *X*. then $\{x_n\}$ is Cauchy sequence.

Definition.1.2.6 A sequence in $\{x_n\}$ of a metric space *(X, d)* is called **Convergent-Sequence** to a point $x \in X$ if $\lim_{n\to\infty} d(x_n, x) = 0.$

Example.1.2.7 *Let (X, d)* be a usual metric space and let $\{x_n\} = \frac{1}{n}$ be a sequence. Given $\in > 0$, we choose a natural number m such that: $m > \frac{1}{\in} \Rightarrow \frac{1}{m} < \in$, therefore $for\ all\ n \geq m, \frac{1}{n} \leq \frac{1}{m} < \in \Rightarrow \left|\frac{1}{n} - 0\right| = \frac{1}{n} < \in, \forall\, n \geq m \Rightarrow \{x_n\}$ converge to 0 in *X*.

Definition.1.2.8 A metric space *(X, d)* is called **complete** if each Cauchy sequence in *X* is convergent to a point of *X*.

Example.1.2.9 Let *X* be a non empty set. For all $x, y \in X$, d: $X \times X \to \mathbb{R}$ is defined by

$$d(x,y) = \begin{cases} 0, & when \quad x = y \\ 1, & when \quad x \neq y \end{cases}$$

Then *(X, d)* is a metric space and called the discrete metric space. As every Cauchy sequence in X is convergent, therefore *(X, d)* a complete metric space.

Remark.1.2.10 Every convergent sequence in a metric-space is a Cauchy sequence but the converse is not true. For instance, Euclidean n-space with the Euclidean distance is complete metric space whereas the set of rational numbers with metric $d(x, y) = |x - y|$ is not a complete metric space.

1.3 Fixed Point Theory

Definition.1.3.1 Let *X* be a non-empty set and $\Gamma: X \to X$ be a map. A point $x \in X$ is called **fixed point** of Γ if x remains invariant under Γ i.e., $\Gamma(x) = x$. In graphical terms, a fixed point means the point $(x, \Gamma(x))$ is on the line $y = x$, or in other words, the graph of Γ has a point in common with line $y = x$.

Example.1.3.2

- **i)** A mapping $\Gamma: \mathbb{R} \to \mathbb{R}$ defined by $\Gamma(x) = x^2$, for all $x \in \mathbb{R}$ has two fixed points 0 and 1.
- **ii)** A mapping $\Gamma: \mathbb{R} \to \mathbb{R}$ defined by $\Gamma(x) = x$ (identity-map), for all $x \in \mathbb{R}$ has infinitely many fixed points. i.e., every point of $\mathbb{R}$ is a fixed point of Γ.
- **iii)** A translation mapping $\Gamma: \mathbb{R} \to \mathbb{R}$ defined by $\Gamma(x) = x + 1$, for all $x \in \mathbb{R}$ has no fixed point.

Remark.1.3.3 Example.1.3.2 refer that a mapping may have a unique fixed point, more than one fixed point or infinitely many fixed points and even it may not have any fixed point.

Definition.1.3.4 Let (X, d) be a metric space. A map $\Gamma: X \to X$ is called

- i) **Contraction mapping** if there exists a real number q, $0 \leq q < 1$ such that $d(\Gamma(x), \Gamma(y)) < q\, d(x, y)$ for all $x, y \in X$.
- ii) **Non-expansive mapping** if $d(\Gamma(x), \Gamma(y)) < d(x, y)$ for all $x, y \in X$.
- iii) **Contractive mapping** if $d(\Gamma(x), \Gamma(y)) < d(x, y)$ for all $x, y \in X$ and $x \neq y$.

Remark.1.3.5 Every contraction map is contractive map but the converse is not true.

Remark.1.3.6 Every contractive map is non-expansive map but the converse is not true.

Example.1.3.7

- i) If $\Gamma(x) = x^2$, $0 \leq x \leq \frac{1}{3}$, then Γ is contraction mapping on $\left[0, \frac{1}{3}\right]$ with usual metric space.

ii) The mapping I: $X \to X$, is non-expansive but not contractive since for all $x, y \in X$, $d(I(x), I(y)) = d(x, y)$.

iii) Let $X = [1,\infty)$ be a complete metric space equipped with the metric of absolute value. Define $\Gamma: X \to X$, by $\Gamma(x) = \frac{x^2+1}{x}$, for all $x \in X$. Then Γ is a contractive mapping, but Γ is not a contraction.

Theorem.1.3.8 ([8] **Banach fixed point theorem** or **Banach contraction principle)** Let (X , d) be a complete metric space and Γ be a contraction mapping on X. Then Γ has a unique fixed point in X, i.e. every contraction mapping on a complete metric space has a unique fixed point.

Remark.1.3.9 Banach contraction principle has many applications but it suffers from one drawback, i.e., it requires the mapping to be continuous at all points of its domain.

In 2002, Dhage [49] obtained a fixed point result for a single self -mapping satisfying an analogue of Banach contraction principle for an integral n type inequality.

Recently, many researchers (see, e.g. [30], [31], [32], [33], [34]**)** have obtained common fixed point results on metric spaces.

1.4. Some Various Types Of Mappings in Metric Spaces

In this section we recall some important types of mappings in metric spaces:

Definition.1.4.1 Let Γ, $P: (X ,d) \to (X ,d)$ be a self-maps of a metric space (X ,d). Then Γ and P said to be **commuting** if $\Gamma(P(x)) = P(\Gamma(x))\ \forall\, x \in X$ [35].

Example.1.4.2 Let $X = [0, 1]$. Define self-maps Γ and P on X by $\Gamma(x) = 2x$ and $P(x) = 3x$, for all $x \in X$. Then Γ and P are commuting maps.

Definition.1.4.3 Let $\Gamma, P: (X,d) \to (X,d)$ be self-maps of a metric space (X,d). Then Γ and P said to be **weakly commuting** if $d(\Gamma(P(x)), P(\Gamma(x))) \leq d(\Gamma(x), P(x))$, for all $x \in X$ **[36].**

Remark.1.4.4 Every pair of commuting self-maps is weakly commuting but the converse is not true. See the following example:

Example.1.4.5 Let $X = [0, 1]$, and (X, d) be the usual metric space. Define self-maps Γ, $P: X \to X$ by $\Gamma(x) = \frac{1}{2}x$ and $P(x) = \frac{x}{x+2}$, $\forall x \in X$. Then for any $x \in X$, we get:

$$d(\Gamma(P(x)), P(\Gamma(x))) \leq d(\Gamma(x), P(x))$$

This implies that f and g are weakly commuting on X but are not commuting on X, since $\Gamma(P(x)) \neq P(\Gamma(x))$ for any non-zero $x \in X$ [36].

Remark.1.4.6 In **[37]** Jungck defined the notion of compatible maps which is more general than that of commuting and weakly commuting maps.

Definition.1.4.7 Let $\Gamma, P: (X,d) \to (X,d)$ be self-maps of a metric space (X,d). Then Γ and P said to be **compatible mappings** if

$$\lim_{n\to\infty} d\{\Gamma(P(x_n)), P(\Gamma(x_n))\} = 0$$

whenever a sequence $\{x_n\}$ sequence in X such that

$$\lim_{n\to\infty} \Gamma(x_n) = \lim_{n\to\infty} P(x_n) = r$$

for some $r \in X$.

Remark.1.4.8 Every weakly commuting map is compatible map, but the converse is not true. See the following example:

Example.1.4.9 Let $X = [0, 1]$ be endowed with usual metric d and let Γ, $P: (X,d) \to (X,d)$ be maps where $\Gamma(x) = x^3$ and $P(x) = 2x^3$. Then $\Gamma(P(x)) \neq P(\Gamma(x))$. So Γ and P are not commuting on X and

$$|\Gamma(P(x)) - P(\Gamma(x))\,| < |\Gamma(x) - P(x)\,|$$

Thus, Γ and P are not weakly commuting on X. Moreover, for any sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} \Gamma(x_n) = \lim_{n\to\infty} P(x_n) = u$ for some $u \in X$. Then

$$\lim_{n\to\infty} d(\Gamma(P(x_n)), P(\Gamma(x_n))) = \lim_{n\to\infty} |\Gamma(P(x)) - P(\Gamma(x))\,| = 0.$$

So Γ and P are compatible [37].

In 1994, Pant [38] introduced the concept of r -weakly commuting maps as follow.

Definition.1.4.10 Let Γ, P: $(X,d) \to (X,d)$ be self-maps of a metric space (X,d), then Γ and P are said to be r **-weakly commuting** if there exists some real number $r > 0$ such that

$$d(\Gamma(P(x)), P(\Gamma(x))) \leq r\, d\,(\Gamma(x), P(x)),\ \forall\, x \in X.$$

Definition.1.4.11 Let Γ, P: $(X,d) \to (X,d)$ be self-maps of a metric space (X,d). Then Γ and P is said to be **point-wise r -weakly commuting** if for a given $x \in X$, $\exists$ $r > 0$ such that

$$d(\Gamma(P(x)), P(\Gamma(x))) \leq r\, d\,(\Gamma(x), P(x)),\ \forall\, x \in X.$$

Remark.1.4.12

i) Every $\mathbb{R}$ -weakly commuting pair is weakly commuting when $r = 1$.

ii) Every weak commutativity is r -weak commutativity, but the converse is not true [38].

Definition.1.4.13 Let Γ, P: $(X,d) \to (X,d)$ be self-maps of a metric space (X,d). Then Γ and P are said to be **non-compatible** if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} \Gamma(x_n) = \lim_{n\to\infty} P(x_n) = r$$

for some $r \in X$ but $\lim_{n\to\infty} d(\Gamma(P(x)), P(\Gamma(x)))$ is either non-zero or does not exist **[39].**

In 1998, Jungck and Rhoades [40] introduced the notion of weakly compatible mappings which is more general than that of compatibility as follows:

Definition.1.4.14 Let Γ, $P: (X, d) \to (X, d)$ be a self-maps of a metric space (X, d). Then Γ and P are said to be **weakly compatible maps** if the mappings commute at all of their coincidence points, i.e. $\Gamma(x) = P(x)$ for some $x \in X$ implies $\Gamma(P(x)) = P(\Gamma(x))$.

Remark.1.4.15 Compatible maps are weakly compatible but the converse is not true.

Example.1.4.16 Let $X = [2, 20]$, and (X, d) be the usual metric space and let Γ, $P: (X, d) \to (X, d)$ be defined as:

$$\Gamma(x) = \begin{cases} 2, & x = 2 \quad or \quad x > 5 \\ 6, & 2 < x \leq 5 \end{cases} \quad \text{and} \quad P(x) = \begin{cases} 2, & x = 2 \\ 12, & 2 < x \leq 5 \\ x - 3, & x > 5 \end{cases}$$

Let $\{x_n\} = \left(5 + \frac{1}{n}\right)$. Clearly, f and g are non-compatible and weakly compatible at $x = 2$.

Definition.1.4.17 Let Γ, $P: (X, d) \to (X, d)$ be a self-maps of a metric space (X, d). Then Γ and P are said to be **occasionally weakly compatible (O-W-C)** if there exists a point $x \in X$ which is a coincidence point of Γ and P at which Γ and P commute [41].

Remark.1.4.18 The weakly compatible self-maps form a proper subclass of **(O-W-C)** self-Mappings. See the following example.

Example.1.4.19 Let $X = [0, \infty)$ and (X, d) be the usual metric space and let Γ, $P: (X, d) \to (X, d)$ be defined by

$$\Gamma(x) = 2x \text{ and } P(x) = x^2, \ \forall\, x \in X.$$

Then , $\Gamma(x) = P(x)$ at points 0 and 2 but $\Gamma(P(0)) = P(\Gamma(0))$ and $\Gamma(P(2)) \neq P(\Gamma(2))$. Therefore maps Γ and P are **(O-W-C)** but not weakly-compatible [41].

Aamri and Moutawakil **[42]** generalized the concept of non -compatible mappings by introducing a new property called property (E-A). This property is defined as follow.
Definition.1.4.20 Let Γ, $P: (X ,d)\rightarrow (X ,d)$ be self-maps of a metric space (X ,d). Then Γ and P are said to be **property (E-A),** if there exists a sequence $\{x_n\}$ in X such that:

$$\lim_{n\to\infty} \Gamma(x_n)= \lim_{n\to\infty} P(x_n) = r \text{ for some } r \in X.$$

Example.1.4.21 Let $X = [0, \infty)$, and (X ,d) be the usual metric space and let Γ, $P: (X ,d)\rightarrow (X ,d)$ defined by

$$\Gamma(x) = \frac{x}{4} \text{ and } P(x) = \frac{3x}{4},\ \forall\, x \in X.$$

Consider the sequence $\{x_n\} = \left(\frac{1}{n}\right)$, obviously $\lim_{n\to\infty} \Gamma(x_n)= \lim_{n\to\infty} P(x_n) = 0 \in X$. Then Γ and P satisfy property **(E-A).**

In 2005, Liu et al. **[7]** introduced the notion of common property (E-A) which contains the property (E-A) for pairs of self-mappings.

Definition.1.4.22 Γ, $P: (X ,d)\rightarrow (X ,d)$ and T, $F: (X ,d)\rightarrow (X ,d)$ are said to satisfy the **common property (E-A)** if there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} \Gamma(x_n)= \lim_{n\to\infty} P(x_n) = \lim_{n\to\infty} T(y_n)= \lim_{n\to\infty} F(y_n) = r \text{ for some } r \in X.$$

In 2011, Sintunavarat and Kumam **[43]** introduced a new property called "**common limit in the range of P property**".

Definition.1.4.23 Let Γ, $P: (X ,d)\rightarrow (X ,d)$ be self-maps of a metric space (X ,d). Then Γ and P are said to be satisfy the **common limit in the range of *P* property** if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} \Gamma(x_n)= \lim_{n\to\infty} P(x_n)= P(r) \text{ for some } r \in X.$$

Example.1.4.24 Let $X = [0, \infty)$ and (X ,d) be the usual metric space and let Γ, $P: (X ,d)\rightarrow (X ,d)$ defined by $\Gamma(x) = x+1$ and $P(x) = 2x$, $\forall\, x \in X$. Consider the sequence $\{x_n\} = \left(1+\frac{1}{n}\right)$. Since $\lim_{n\to\infty} \Gamma(x_n) = \lim_{n\to\infty} P(x_n) = P(1) = 2$, therefore, Γ and P satisfy the **common limit in the range of *P* property** [43].

In 2012, Imdad et al. [44] defined the common limit in the range of P and F property and in that way extending the notion of (**common limit in the range of *P* property**) as below.

Definition.1.4.25 Let X be a non-empty set *and* Γ , $P:$ $(X ,d)\to (X ,d.)$ Then T, F: $(X ,d)\to (X ,d)$ are said to satisfy the **(common limit in the range of P & F property)** if there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} \Gamma(x_n) = \lim_{n\to\infty} P(x_n) = \lim_{n\to\infty} T(y_n) = \lim_{n\to\infty} F(y_n) = r, \text{ where } r \in P(X) \cap F(X).$$

Chauhan et al**.** [45] defined the notion of "**The joint common limit in the range of *P* & *F* property**" for two pairs of self-Mappings as follow.

Definition.1.4.26 Let X be a non-empty set *and* Γ , $P:$ $(X ,d)\to (X ,d)$. Then T, F: $(X ,d)\to (X ,d)$ are said to satisfy the **(The joint common limit in the range of *P* and *F* property)** if there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} \Gamma(x_n) = \lim_{n\to\infty} P(x_n) = \lim_{n\to\infty} T(y_n) = \lim_{n\to\infty} F(y_n) = P(r) = F(r) \text{ where } r \in X.$$

Example.1.4.27 Let X = [-1, 1], and (X ,d) be the usual metric space and let Γ, P, T, F: $(X ,d)\to (X ,d)$ be defined by $\Gamma(x) = \frac{x}{3}$ and $P(x) = x$, & $T(x) = -\frac{x}{3}$ and $F(x) = -x$, $\forall x \in X$. Consider the sequences $\{x_n\} = \left(\frac{1}{n}\right)$ and $\{y_n\} = \left(-\frac{1}{n}\right)$ in X, such that

$$\lim_{n\to\infty} \Gamma(x_n) = \lim_{n\to\infty} P(x_n) = \lim_{n\to\infty} T(y_n) = \lim_{n\to\infty} F(y_n) = P(0) = F(0).$$

This explains that the pairs (Γ, P) and (T, F) satisfy the (The joint common limit in the range of P & F property). Since where, $0 \in P(X) \cap F(X)$ so both the pairs (Γ, P) and (T, F) satisfy the (The joint common limit in the range of P & F property).

In 1999, Pant [46] introduced a new continuity condition, known as reciprocal continuity as follows:

Definition.1.4.28 Let Γ, P: $(X,d) \to (X,d)$ be self-maps of a metric space (X,d). Then Γ and P are said to be **reciprocally continuous** if

$$\lim_{n\to\infty} \Gamma(P(x_n)) = \Gamma(r) \text{ and } \lim_{n\to\infty} P(\Gamma(x_n)) = P(r)$$

when $\{x_n\}$ in X is a sequence such that $\lim_{n\to\infty} \Gamma(x_n) = \lim_{n\to\infty} P(x_n) = r$ for some $r \in X$.

Remark.1.4.29 If Γ and P are both continuous, then Γ and P are reciprocally continuous but the converse is not true. See the following example.

Example.1.4.30 Let $X = [2, 20]$, and (X, d) be the usual metric space and let Γ, P: $(X,d) \to (X,d)$ such that:

$$\Gamma(x) = \begin{cases} 2, & x = 2 \\ 3, & x > 2 \end{cases} \quad \text{and} \quad P(x) = \begin{cases} 2, & x = 2 \\ 6, & x > 2 \end{cases}$$

Then Γ and P are reciprocally continuous-maps for the sequence $\{x_n\} = \{2\}$ but are not continuous at $x = 2$ [46].

In 2011, Pant et al. [47] generalized the concept of reciprocal continuity by introducing the notion of weak reciprocal continuity.

Definition.1.4.31 Let Γ, P: $(X,d) \to (X,d)$ be self-maps of a metric space (X,d). Then Γ and P said to be **weakly reciprocally continuous** if

$$\lim_{n\to\infty} \Gamma(P(x_n)) = \Gamma(r) \text{ or } \lim_{n\to\infty} P(\Gamma(x_n)) = P(r)$$

whenever the sequences $\{x_n\}$ in X such that $\lim_{n\to\infty} \Gamma(x_n) = \lim_{n\to\infty} P(x_n) = r$ for some $r \in X$.

Example.1.4.32 Let $X = [2, 20]$, and (X, d) be the usual metric space and let Γ, P: $(X,d) \to (X,d)$ defined as

$$\Gamma(x) = \begin{cases} 2, & x = 2 \quad or \quad x > 5 \\ 6, & 2 < x \le 5 \end{cases} \quad \text{and} \quad P(x) = \begin{cases} 2, & x = 2 \\ 12, & 2 < x \le 5 \\ \frac{x+1}{3}, & x > 5 \end{cases}$$

Clearly, mappings Γ and P are weakly reciprocal continuous maps but are not reciprocal continuous [47].

1.5 Some Various Types Of Generlized Metric Spaces

In sixties, Gahler [23, 21] introduced the concept of 2-metric space and proved that a 2-metric is a generalization of usual notion of a metric. But different authors proved that there is no relation between these two functions. For instance, Ha et al**.** [48] showed that a 2-metric need not be a continuous function of its variables, whereas an ordinary metric is continuous function of its variables. These considerations let Dhage [29], in 1992, to introduce a new class of generalized metric called as D-metric. Dhage claimed in his doctoral research work that D-metric provides a generalization of ordinary metric functions. Further, in 2004, Mustafa and Sims **[49]** demonstrated that most of the claims concerning the fundamental topological properties of D-metric spaces are incorrect and proved that a D-metric need not be a continuous function of its variables. These considerations let researchers to seek a more appropriate notion of generalized metric space. In 2006, Mustafa and Sims **[25]** introduced the concept of G-metric spaces to overcome fundamental flaws in Dhage"s theory of generalized metric spaces.

Definition 1.5.1 Let X be a non-empty set and G: $X\times X\times X\rightarrow[0, \infty)$ a mapping which satisfy the followings.

i) G (x, y, z) = 0 if $x = y = z$,

ii) G (x, x, y) > 0; whenever x ≠ y, for all $x, y \in X$,

iii) G(x, x, y) ≤ G(x, y, z); whenever y ≠ z, for all $x, y, z \in X$,

iv) G (x, y, z) = G (x, z, y) = G (y, x, z) = … (symmetric in all three variables),

v) G (x, y, z) ≤ G (x, b, b) + G (b, y, z) for all $x, y, z, b \in X$.

Then, the function G is called a generalized metric (G-Metric) on X and (X, G) namely a generalized metric space (G-Metric-Space).

Example.1.5.2 For any metric space define $\mathbf{G_r}$ and $\mathbf{G_s}$ on X as following:

a) $\mathbf{G_r}$ (x, y, z) = d(x, y) + d(y, z) + d(x, z).

b) $\mathbf{G_s}$ (x, y, z) = Max{d(x, y) , d(y, z) , d(x, z)}.

For all $x, y, z \in X$. Then $(X, \mathbf{G_r})$ and $(X, \mathbf{G_s})$ are G-Metric-Spaces.

Definition 1.5.3 Let X be a non-empty set. A function, $D: X\times X\times X \rightarrow \mathbb{R}$ is called a D-metric on X if it satisfies the following conditions:

i) D (x, y, z) ≥ 0 for all $x, y, z \in X$ and equality holds $\Leftrightarrow$ $x = y = z$,

ii) D (x, y, z) = D (x, z, y) = D (y, x, z) = D (z, x, y) = D (y, z, x) = D (z, y, x);

iii) D (x, y, z) $\leq$ D (x, y, a) + D (x, a, z) + D (a, y, z) for all $x, y, z, a \in X$",

Then (X, D) is called a D-metric space.

Example.1.5.4 Let $\mathbb{R}$ denote the set of all real numbers and define a function D on $\mathbb{R}\times\mathbb{R}\times\mathbb{R}$ by D (x, y, z) = Max {|x-y |, {|y-z |, {|z-x |} For all x, y, z $\in \mathbb{R}$. Then $(\mathbb{R}, D)$ is a D-metric space.

Definition 1.5.5 Let X be a non-empty set. A map, $d: X\times X\times X \rightarrow \mathbb{R}$ is called a 2-metric on X if it satisfies the following conditions.

i) d (x, y, z) = 0 only if at least two of x, y, z are equal.

ii) d (x, y, z) = d(p (x, z, y)) for all $x, y, z \in X$ and permutation p(x y, z) of x, y, z.

iii) $d (x, y, z) \leq d (x, y, a) + d (x, a, z) + d (a, y, z)$ for all $x, y, z, a \in X$,

Then (X, d) is called a 2-metric space.

Example.1.5.6 Let a map $d\colon \to [0, \infty)$ be defined by:

d (x, y, z) = min {|x-y |, {|y-z |, {|z-x |} for all $x, y, z \in R$. Then d is a 2-metric space on $\mathbb{R}$.

Definition 1.5.7 Let X be a non-empty set. A map $d\colon X \times X \to [0, \infty)$ is called a b-metric on X if the following conditions are holding:

i) $d(x, y) = 0 \Leftrightarrow x = y$,

ii) $d(x, y) = d(y, x)$;

iii) $d(x, y) \leq r\,[d(x, z) + d(z, y)]$ *for all* $x, y, z \in X$,

The pair (X, d) is called a b-Metric Space.

Example.1.5.8 Let $X = \{0, 1, 2\}$ and let

$$d(x, y) = \begin{cases} 2, & x = y = 0 \\ \frac{1}{2}, & otherwise \end{cases}$$

Then (X, d) is a b-metric space with the constant $r = 2$.

CHAPTER TWO

THIS CHAPTER CONSISTS OF TWO SECTIONS ARE SHOWING AS FOLLOWING

2.1. Some Conceptes Of Approximate Fixed Point Results And Discover Them For Applications

In this section, the concepts of approximate fixed points are discussed for various cases. In many situations of practical utility, the mapping under consideration may not have an exact fixed point due to some tight restriction on the space or the map, or an approximate fixed point is more than enough, an approximate solution plays an important role in such situations. This leads to the necessity of a proper theory regarding approximate fixed point. Tijs et al [1] obtained an approximate fixed point result in Banach spaces which is further extended for several types of operators on metric spaces by Berinde [2]. The most general operator considered in that was the weak contraction which has been introduced in [51]. In this section, we obtain some basic approximate fixed point results in generalized metric spaces. Further, we establish some existence results concerning approximate fixed points, approximate common fixed points, approximate coincidence points, coincidence points and endpoints of multi-valued contractions. We also develop quantitative estimates of the sets of approximate fixed points and approximate endpoints in generalized metric spaces.

We first obtain the following basic results in b-metric spaces:

Theorem 2.1.1: Let (X, d) be a b-metric space and $F: (X, d) \to (X, d)$ satisfy $d(Fx, Fy) \leq \mu \in (0,1)$. Then F has approximate fixed point property, i.e., for every $\alpha > 0$, $\mathrm{Fix}_{\alpha}(F) \neq \emptyset$, where $\mathrm{Fix}_{\alpha}(F)$ is the set of all approximate fixed points of F.

Theorem 2.1.2: Let *(X, d)* be a b-metric space and *F: (X, d) → (X, d)* satisfy

$$d\,(Fx, Fy) \leq \mu \in (0,1).$$

Then for every $\alpha > 0$ the diameter of $\mathrm{Fix}_{\alpha}(F)$ is not larger than $b\alpha(\frac{1+b}{1-\mu\, b^{2}})$.

Similar results are derived for the maps satisfying quasi contraction and weak contraction conditions. These theorems extend the various results given by Tijs et al [1], Berinde [2], Pacurar, M. and Pacurar, R. [3] Singh, and Prasad, B. [4]. In addition, we obtain some approximate fixed and coincidence point results for multi-valued maps satisfying different contraction conditions. Following is one of the results:

Theorem 2.1.3: Let *(X, d)* be a b-metric space and *F: X → Cl(X)* (*Cl(X) is the closure of X*)satisfy multi-valued contraction, i.e., there exists a number $\mu \in (0,1)$ such that

$$H(Fx, Fy) \leq \mu d(x,y) \; for\ all\ x, y \in X.$$

Then F has a fixed point provided, either *(X, d)* is compact and the function $\Gamma(x) = d(x, Fx)$ is lower semi-Continuous or F is closed and compact.

The following theorem ensures the existence of approximate coincidence point of two maps and includes the result of Hussain et al [5].

Theorem 2.1.4: Let *(X, d)* be a b-metric space and *F: X → Cl(X)* satisfy multi-valued T-Contraction, i.e., there exists a number $\mu \in (0,1)$ such that

$$H(Fx, Fy) \leq \mu d(Tx, Ty) \; for\ all\ x, y \in X.$$

Then F has the approximate coincidence point property, provided every Fx is T-invariant. Additional, if *(X, d)* is compact and the function $\Gamma(x) = d(Tx, Fx)$ is lower semi-Continuous then F and T have a coincidence point.

Next we extend the results of Hussain et al [5] and Amini-Harandi [6], regarding the existence of unique endpoint.

Theorem 2.1.5: Let *(X, d)* be a complete b-metric space and T*: X → Cl(X)* be a continuous single-valued mapping such that $rd(x,y) \leq d(Tx,Tx)$, where $r > 0$ is a constant. Let *F: X → Cl(X)* be lower semi-Continuous and satisfy multi-valued T-Contraction. Then *F* and *T* have a unique endpoint if and only if I and T have the approximate end point property. Similar results are obtained for other multi-valued contractions as well.

In [52] the concept of approximate best proximity pair is defined and some existence theorems for approximate best proximity pair are proved. Inspired from them, we obtain following existence results of approximate best proximity pair:

Theorem 2.1.6: Let *W* and *K* be non-empty subsets of a b-metric space *X* and let the mapping *F:* $W \cup K \rightarrow W \cup K$ satisfy $\text{Ғ}(W) \subseteq K$ and $\text{Ғ}(K) \subseteq W$ and $\lim_{n\to\infty} d(\text{Ғ}^n x, \text{Ғ}^{n+1} x) = d(\mathrm{W}, K)$ for some $x \in \mathrm{W} \cup K$. Then the pair *(*W*, K)* is an approximate best proximity pair.

Theorem 2.1.7: Let *W* and *K* be non-empty subsets of a b-metric space *X* and let the mapping *F:* $\mathrm{W} \cup K \rightarrow \mathrm{W} \cup K$ satisfies $\text{Ғ}(\mathrm{W}) \subseteq K$, $\text{Ғ}(K) \subseteq \mathrm{W}$ and

$\Psi\big(d(\text{Ғ}x, \text{Ғ}y)\big) \leq \Psi(d(x,y)) - \phi(d(x,y)) + \phi(d(\mathrm{W},K)), \forall\, x \in \mathrm{W}$ *and* $y \in K$ *where* $\phi, \Psi \in \Phi$, which is the class of the mappings $\phi, \Psi : [0, \infty) \longrightarrow [0, \infty)$ satisfying:

i) ϕ *and* Ψ are continuous and monotone non-decreasing.

ii) $\phi(r) = 0 \Leftrightarrow r = 0,\ \Psi(r) = 0 \Leftrightarrow r = 0.$

Then the pair *(W, K)* is an approximate best proximity pair.

Theorem 2.1.8: Let *W* and *K* be non-empty subsets of a b-metric space *X* and let the mapping *F:* $\mathrm{W} \cup K \rightarrow \mathrm{W} \cup K$ satisfies $\text{Ғ}(\mathrm{W}) \subseteq K$ and $\text{Ғ}(K) \subseteq \mathrm{W}$ and

$[d(\text{Ғ}x, \text{Ғ}y)]^3 \leq s_1\, d(x,y)[d(y,\ \text{Ғ}y)]^2 + s_2\, d(x, \text{Ғ}x)\ [d(\text{Ғ}x,\ \text{Ғ}y)]^2 + s_3\, d(y, \text{Ғ}^2 x)[d(\text{Ғ}x,\ \text{Ғ}^2 x)]^2 + s_4\, d(\mathrm{W},K)[d(y,\ \text{Ғ}^2 x)]^2$, $\forall\, x,y \in \mathrm{W} \cup K$ and $s_i \in \mathbb{R}$ *such that* $\sum_{i=1}^{4} s_i < 1$. Then the pair *(W, K)* is an approximate best proximity pair.

As an application of Theorem **2.1.7** we obtain an existence result for Hammerstein integral equation as follows:

Theorem 2.1.9: Let $F(r,\tau)$ satisfy $\left\{\int_a^b F^2(r,\tau)d\tau\right\}^{\frac{1}{2}} \leq Q < \infty$, where Q is a positive constant and $\mathrm{P}(r) \in C[a,b]$. Also let $H(\tau,x)$ satisfy a uniform Lipschitz condition with respect to its second argument, i. e., $\|H(\tau,x) - H(\tau,y)\| \leq q\|x-y\|$ for every $\tau \in [a,b]$, $q > 0$ and $x,y \in \mathbb{R}$. additional, presume that $Qq(b-a)^{\frac{1}{2}} < 1$. Then $\exists$ a solution $\Gamma(\mathrm{r}) \in \mathrm{C}[\mathrm{a},\mathrm{b}]$ and $\alpha > 0$ such that $\left|\Gamma(r) - \mathrm{P}(r) - \int_a^b F(r,\tau)H(\tau,x(\tau))d\tau\right| < \alpha$.

2.2. Some Common Fixed Point Theorems And Their Applications

The aim of this section is to get some coincidence point and common fixed point theorems. The existence and approximation results for common fixed points of families of mappings have been studied by various authors (see, for example [53, 54, 55]), mostly by relaxing the contraction condition of the map or sometimes by relaxing the condition on the space or both. First we obtain common fixed point theorems for hybrid pair of maps satisfying an integral inequality and common property **(E. A.)** in b-metric spaces, which modifies the results of Liu et al [7].

Theorem 2.2.1: Let *(X, d)* be a complete b-metric space and let Γ, $P: X \to X$ and I, $G: X \to CK(X)$ such that

a) $IX \subset PX$ and $\subset GX \subset \Gamma X$,

b) the pairs *(I, Γ)* and *(G, P)* satisfy the common property **(E.A),**

c) For all $x,y \in X$,

$$\int_0^{H(Ix,Gy)} \emptyset(r)\,\mathrm{dr} \leq \mathrm{h}\left\{\int_0^{\mathrm{Q}(\mathrm{x},\ \mathrm{y})} \emptyset(\mathrm{r})\,\mathrm{dr}\right\}$$

where $\emptyset: \mathbb{R}^+ \to \mathbb{R}^+$ is Lebesgue integrable mapping which is summable, non-negative and $\int_0^\alpha \emptyset(\mathrm{r})\mathrm{dr} > 0$, for all $\alpha > 0$. Also let

Q(x, y) = max$\{d(\Gamma x, Py), d(Ix, \Gamma x), d(Gy, Py), [d(Ix, Py) + d(Gy, \Gamma x)]\backslash 2\}$ with

$hs < 1, \lambda s < 1,$ where $\lambda = max\left\{h, \frac{hs}{2-hs}\right\}$. If ΓX and PX are closed subspace of X then

1) Γ and I have a coincidence point.

2) P and G have a coincidence point.

3) Γ and I have a common fixed point provided that Γ is $I-$ weakly commuting at u and $\Gamma\Gamma u = \Gamma u\ for\ u \in C(\Gamma, I)$; C(Γ, I) = { x: x is a coincidence point of Γ and I}.

4) P and G have a common fixed point provided that P is $G-$weakly commuting at v and $PPv = Pv\ for\ v \in C(P, G)$.

5) Γ, P, I And G have a common fixed point provided in part-**(3)** and **(4)** are true.

As well as we get a common fixed point result for four self mappings in ***b*-metric** space.

Theorem 2.2.2: Let *(X, d)* be a complete b-metric space and let W, $K, S,$ Ƒ*:* $X \rightarrow X$ such that

a) $WX \subset$ ƑX and $KX \subset SX$,

b) The pairs *(W, S)* and *(K, Ƒ)* are weakly compatible,

c) For all $x, y \in X$, $\int_0^{d(Wx,Ky)} \emptyset(r)\,\mathrm{dr} \le \mathrm{h}\left\{\int_0^{\mathrm{Q(x,\ y)}} \emptyset(\mathrm{r})\,\mathrm{dr}\right\}$ where $\emptyset$*:* $\mathbb{R}^+ \rightarrow \mathbb{R}^+$is Lebesgue integrable mapping which is summable, non-negative and $\int_0^{\alpha} \emptyset(\mathrm{r})\mathrm{dr} > 0,$ For all $\alpha > 0$ and

Q(x, y) = max$\{d(Sx,$ Ƒ$y), d(Sx, Wx), d($Ƒ$y, Ky), [d(Sx, Ky) + d($Ƒ$y, Wx)]\backslash 2\}$ with

$hs < 1, \lambda s < 1,$ where $\lambda = max\left\{h, \frac{hs}{2-hs}\right\}$. If ƑX or S$X$ are closed, then W, K, S and Ƒ have a common fixed point.

As an application of Theorem 2.2.2**,** we get an existence and unique result for the functional equations arising in dynamic programming. Let X and Y be Banach spaces and $S \subset X$ the state space, $D \subset Y$ a decision space, $R = (-\infty, \infty)$ and K(S) the set of all bounded real valued mappings. Then the functional equations arising in dynamic programming are given as follows:

$$\Gamma_i\,(x) = \sup_{y \in D} H_i(x, y, \Gamma_i\,(Ƒ(x, y))), x \in S, \qquad (1)$$

$$P_i\,(x) = \sup_{y \in D} I_i(x, y, P_i\,(Ƒ(x, y))), x \in S, \qquad (2)$$

where Ƒ*:* $S \times D \to S$ and $H_i\,, I_i : S \times D\ \times \mathbb{R} \to \mathbb{R}$, *I =1, 2.*

Theorem 2.2.3: Let the following conditions be satisfied:

a) $H_i\ and\ \ I_i$ are bounded for *i = 1, 2.*

b) $\int_0^{|H_1(x,y,q(r)) - H_2\,(x,y,k(r))|} \emptyset(r)\mathrm{dr} \leq \mathrm{h}\left\{\int_0^{\mathrm{Q(q,k)}} \emptyset(\mathrm{r})\,\mathrm{dr}\right\}$, where

Q (q, k) = Max
$\{|Ƒ_2 q(r) - Ƒ_2 k(r)|\,, |Ƒ_1 q(r) - W_1 q(r)|\,, |Ƒ_2 k(r) - W_2 k(r)|\,, \frac{1}{2}[|Ƒ_1 q(r) - W_2 k(r)| + |Ƒ_2 k(r) - W_2 q(r)|]\}.$

For all $(x, y) \in S \times D, q, k \in K(S)$ and $r \in S. W_i$ and $Ƒ_i$ (I = 1, 2) are defined by:

a) $W_i\, q(x) = \sup_{y \in D} H_i(x, y, q(Ƒ(x, y))), for\ all\ x \in S$ and $q \in K(S);\ i =\ 1, 2.$

b) $Ƒ_i\, k(x) = \sup_{y \in D} I_i(x, y, k(Ƒ(x, y))), for\ all\ x \in S$ and $k \in K(S);\ i =\ 1, 2.$

c) For any sequence *{k_n}* ⊂ K(S) and $k \in$ K(S) with $\lim_{n \to \infty} \sup_{x \in S} |k_n(x) - k(x)| = 0$, there exists $q_i \in K(S)$ such that $k = Ƒ_i q_i$ for *i =1 or i = 2.*

d) For any $q \in$ K(S), there exist k_1 and $k_2 \in$ K(S), such that:

$$W_1\, q(x) = \text{Ƒ}_2 k_1(x)\ and\ W_2 q(x) = \text{Ƒ}_1 k_2(x),\ x \in S.$$

e) For any $q \in$ K(S), with $W_i\, q = \text{Ƒ}_i q\ (i = 1, 2)$, we have $\text{Ƒ}_i\, W_i\, q = W_i\, \text{Ƒ}_i q$.

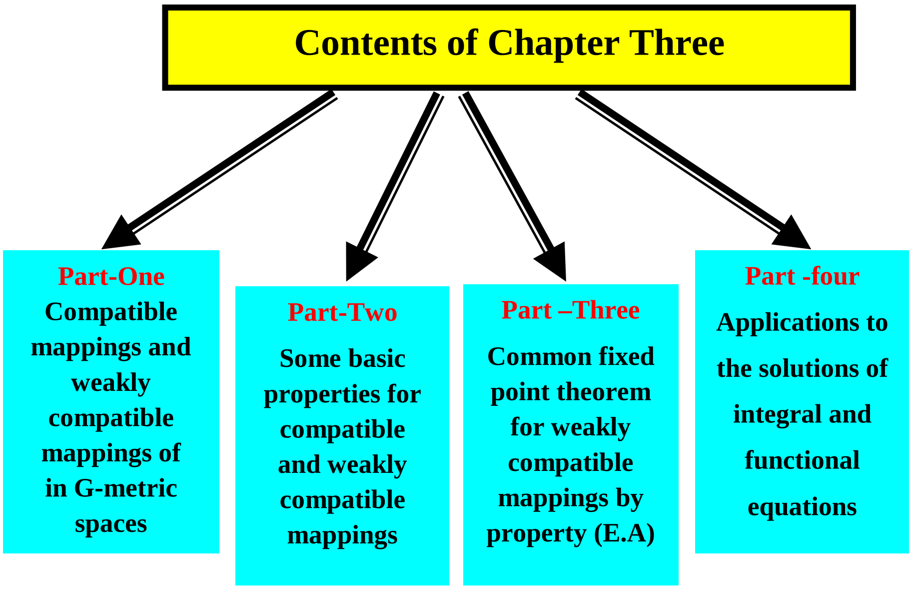
Then the system of functional equations **(1)** and **(2)** has a unique common solution in B(S).

CHAPTER-THREE

THIS CHAPTER CONSISTS OF TWO SECTIONS ARE SHOWING AS FOLLOWING

3.1. Common Fixed Point Theorems And Some Results For Compatible Mappings And Weakly Compatible Mappings in G-Metric Spaces

In this section some results for variants of compatible mappings along with property (E.A) in G-metric spaces have been introduced. Some properties related to these mappings have also been proved. This section consists of four parts. In part one, the concepts of compatible mappings and weakly compatible mappings in G-metric space have been discussed. Part two deals with some properties related to compatible mappings and weakly compatible mappings in G-metric spaces. In part three, a common fixed point theorem for weakly compatible mappings has been proved by using the notion of the property (E.A). Finally, in part four, applications to the solution of integral equation and the bounded solution of a functional equation arising in dynamic programming have been given.



3.2. Concepts of compatible and weakly compatible mappings in G-Metric Spaces

In this part, some basic notions of compatible mappings and weakly compatible mappings in G-metric spaces are introduced as below:

Definition 3.2.1 Let *(X, G)* and *(X*, G*)* be G-metric spaces and $\Gamma: (X, G) \to (X^*, G^*)$ be a mapping. Then Γ is said to be G-continuous at a point $u \in S \Leftrightarrow$ for a given $\alpha > 0$, $\exists\ \delta > 0$, such that $(u, x, y) < \delta \Rightarrow G^*(\Gamma u, \Gamma x, \Gamma y) < \alpha$, for all $x, y \in X$. A mapping Γ is G-continuous on $X \Leftrightarrow$ it is G-continuous at every $u \in X$ [25].

Definition 3.2.2 A pair (Γ, P) of self-mappings of a G-metric space *(X, G)* is said to be compatible mappings if

$$\lim_{n\to\infty} G(\Gamma(P(x_n)), P(\Gamma(x_n)), P(\Gamma(x_n))) = 0, \text{ or} \lim_{n\to\infty} G(P(\Gamma(x_n)), \Gamma(P(xn)), \Gamma(P(x_n))) = 0$$

whenever a sequence $\{x_n\}$ is in X such that $\lim_{n\to\infty} \Gamma(x_n) = \lim_{n\to\infty} P(x_n) = r$ for some $u \in X$.

Definition 3.2.3 A pair (Γ, P) of self-mappings of a G-metric space *(X, G)* is said to be non-compatible mappings if there exists at least one sequence $\{x_n\}$ in X such that

$\lim_{n\to\infty} \Gamma(x_n) = \lim_{n\to\infty} P(x_n) = u \in X$, but either $\lim_{n\to\infty} G(\Gamma(P(x_n)), P(\Gamma(x_n)), P(\Gamma(x_n))) \neq 0$, and $\lim_{n\to\infty} G(P(\Gamma(x_n)), \Gamma(P(xn)), \Gamma(P(x_n))) \neq 0$, or non-exists.

Definition 3.2.4 A pair (Γ, P) of self-mappings of a G-metric space *(X, G)* is said to be compatible mappings of type **(A)** if

$$\lim_{n\to\infty} (P(\Gamma(x_n)), \Gamma(\Gamma(x_n)), \Gamma(\Gamma(x_n))) = 0 \text{ and } \lim_{n\to\infty} G(\Gamma(P(x_n)), P(P(x_n)), P(P(x_n))) = 0$$

whenever a sequence $\{x_n\}$ is in X such that $\lim_{n\to\infty} \Gamma(x_n) = \lim_{n\to\infty} P(x_n) = u$ for some $u \in X$.

Definition 3.2.5 A pair (Γ, P) of self-mappings of a G-metric space *(X, G)* is said to be weakly compatible mappings of type (A) if

$$\lim_{n\to\infty} G(P(\Gamma(x_n)), \Gamma(\Gamma(x_n)), \Gamma(\Gamma(x_n))) \leq \lim_{n\to\infty} G(\Gamma(P(x_n)), \Gamma(\Gamma(x_n)), \Gamma(\Gamma(x_n))) \text{ and}$$

$$\lim_{n\to\infty} G(\Gamma(P(x_n)), P(P(x_n)), P(P(x_n))) \leq \lim_{n\to\infty} G(P(\Gamma(x_n)), P(P(x_n)), P(P(x_n)))$$

Whenever a sequence $\{x_n\}$ is in X such that $\lim_{n\to\infty} \Gamma(x_n) = \lim_{n\to\infty} P(x_n) = u$ for some $u \in X$.

3.3. Some Basic Properties For Compatible And Weakly Compatible Mappings

In this part, some fundamental properties concerning of compatible mappings and weakly compatible mappings in G-Metric spaces have been studied. The following proposition shows that under certain conditions, compatible mappings and compatible-mappings of type (A) are equivalent:

Proposition 3.3.1 Let Γ and P be G-Continuous mappings from a G-Metric space (X, G) into itself. Then Γ and P are compatible if and only if they are compatible of type-(A).

Proof Assumes Γ and P are compatible mappings. Let $\{x_n\}$ be a sequence in X such that:

$\lim_{n\to\infty} \Gamma(x_n) = \lim_{n\to\infty} P(x_n) = u$ for some $u \in X$. So, by definition **(1-5-1-part *v*)** we have:

$G(P(\Gamma(x_n)), \Gamma(\Gamma(x_n)), \Gamma(\Gamma(x_n))) \leq G(P(\Gamma(x_n)), \Gamma(P(x_n)), \Gamma(P(x_n)))+ G(\Gamma(P(x_n)), \Gamma(\Gamma(x_n)), \Gamma(\Gamma(x_n)))$. Since Γ and P are compatible and Γ is G-Continuous, we get:

$$\lim_{n\to\infty} G(P(\Gamma(x_n)), \Gamma(\Gamma(x_n)), \Gamma(\Gamma(x_n))) = 0,$$

Similarly, as P is G-Continuous, we get:

$$G(\Gamma(P(x_n)), P(P(x_n)), P(P(x_n))) = 0.$$

Therefore, Γ and P are compatible of type (A).

(Conversely), Let Γ and P are compatible of type **(A).**

Let $\{x_n\}$ be a sequence in X such that: $\lim_{n\to\infty} \Gamma(x_n) = \lim_{n\to\infty} P(x_n) = u$ for some $u \in X$.

Since P is G-Continuous, therefore, $\lim_{n\to\infty} P(\Gamma(x_n)) = \lim_{n\to\infty} P(P(x_n)) = P(u)$. By definition **(1-5-1-part *v*)** we have :

$G(\Gamma(P(x_n)), P(\Gamma(x_n)), P(\Gamma(x_n))) \leq G(\Gamma(P(x_n)), P(P(x_n)), P(P(x_n))) + G(P(P(x_n)), P(\Gamma(x_n)), P(\Gamma(x_n)))$. Since Γ and P are compatible-mappings of type (A) and g is G-Continuous, we have: $G(\Gamma(P(x_n)), P(\Gamma(x_n)), P(\Gamma(x_n))) = 0$. Similarly, as Γ is G-Continuous, we have:

$G(P(\Gamma(x_n)), \Gamma(P(x_n)), \Gamma(P(x_n))) = 0$. Therefore Γ and P are compatible.

Now we recall the following propositions which explains that under certain conditions, compatible mappings of type (A) and weakly compatible mappings of type (A) are equivalent:

Proposition 3.3.2 Let Γ and P be two compatible mappings of type (A) from a G-Metric space (X, G) into G-Metric space (X, G). Then Γ and P are weakly compatible mappings of type (A).

Proof- Since the pair { Γ and P} is compatible of type (A), then we get:

$$0 = \lim_{n\to\infty} G(P(\Gamma(x_n)), \Gamma(\Gamma(x_n)), \Gamma(\Gamma(x_n))) \leq \lim_{n\to\infty} G(\Gamma(P(x_n)), \Gamma(\Gamma(x_n)), \Gamma(\Gamma(x_n)))$$

And $0 = \lim_{n\to\infty} G(\Gamma(P(x_n)), P(P(x_n)), P(P(x_n))) \leq \lim_{n\to\infty} G(P(\Gamma(x_n)), P(P(x_n)), P(P(x_n)))$,

Whenever $\{x_n\}$ be a sequence in X such that: $\lim_{n\to\infty} \Gamma(x_n) = \lim_{n\to\infty} P(x_n) = u$ for some $u \in X$.

Therefore, Γ and P are weakly compatible of type (A).

Proposition 3.3.3 Let Γ and P be G-Continuous-mappings of a G-Metric space (X, G) into itself. If Γ and P are weakly compatible of type **(A),** then they are compatible of type (A).

Proof- let $\{x_n\}$ be a sequence in X such that: $\lim_{n\to\infty} \Gamma(x_n) = \lim_{n\to\infty} P(x_n) = u$ for some $u \in X$.

As Γ and P are G-Continuous mappings, therefore,

$\lim_{n\to\infty} \Gamma(\Gamma(x_n)) = \lim_{n\to\infty} \Gamma(P(x_n)) = \Gamma(u)$ and $\lim_{n\to\infty} P(\Gamma(x_n)) = \lim_{n\to\infty} P(P(x_n)) = P(u)$.

As well, Γ and P are weakly compatible of type (A), thus we get:

$\lim_{n\to\infty} G(P(\Gamma(x_n)), \Gamma(\Gamma(x_n)), \Gamma(\Gamma(x_n))) \leq \lim_{n\to\infty} G(\Gamma(P(x_n)), \Gamma(\Gamma(x_n)), \Gamma(\Gamma(x_n))) = G(\Gamma(u), \Gamma(u), \Gamma(u)) = 0$ and

$\lim_{n\to\infty} G(\Gamma(P(x_n)), P(P(x_n)), P(P(x_n))) \leq \lim_{n\to\infty} G(P(\Gamma(x_n)), P(P(x_n)), P(P(x_n))) = G(P(u), P(u), P(u)) = 0$. This implies that,

$\lim_{n\to\infty} G(P(\Gamma(x_n)), \Gamma(\Gamma(x_n)), \Gamma(\Gamma(x_n))) = 0$ *and* $\lim_{n\to\infty} G(\Gamma(P(x_n)), P(P(x_n)), P(P(x_n))) = 0$,

Whenever $\{x_n\}$ be a sequence in X such that: $\lim_{n\to\infty} \Gamma(x_n) = \lim_{n\to\infty} P(x_n) = u$ for some $u \in X$

Therefore, Γ and P compatible of type (A).

The following proposition directly consequence of Propositions 3.3.1, 3.3.2 and 3.3.3.

Proposition 3.3.4 Let Γ and P be G-continuous mappings from a G-metric space (X, G) into itself. Then:

i) The mappings Γ and P are compatible of type (A) $\Leftrightarrow$ they are weakly compatible of type (A).

ii) The mappings Γ and P are compatible $\Leftrightarrow$ they are weakly compatible of type (A).

Proposition 3.3.5 Let Γ and P be weakly compatible mappings of type (A) from a G-metric space (X, G) into itself. If $\Gamma(u) = P(u)$ for some $u \in X$, then

$$\Gamma(P(u)) = \Gamma(\Gamma(u)) = P(P(u)) = P(\Gamma(u)).$$

Proof Let $\{x_n\}$ be a sequence in X defined by $\{x_n\} = (u)$, n =1, 2, 3,.. and $\Gamma(u) = P(u)$. Then we have: $\lim_{n\to\infty} \Gamma(x_n) = \lim_{n\to\infty} P(x_n) = \Gamma(u) = P(u)$. Since, Γ and P be weakly compatible mappings of type (A), we get

$$G(\Gamma(P(u)), P(P(u)), P(P(u))) = \lim_{n\to\infty} G(\Gamma(P(x_n)), P(P(x_n)), P(P(x_n)))$$

$$\leq \lim_{n\to\infty} G(P(\Gamma(x_n)), P(P(x_n)), P(P(x_n)))$$

$$= G(P(\Gamma(u)), P(P(u)), P(P(u)))$$

$$= G(P(P(u)), P(P(u)), P(P(u))) = 0.$$

Thus, we get: $\Gamma(P(u)) = P(P(u))$. Hence $\Gamma(P(u)) = \Gamma(\Gamma(u)) = P(P(u)) = P(\Gamma(u))$.

Proposition 3.3.6 Let Γ and P be weakly compatible mappings of type (A) from a G-metric space (X, G) into itself and assume $\lim_{n\to\infty} \Gamma(x_n) = \lim_{n\to\infty} P(x_n) = u$ for some $u \in X$. Then we get the following:

1) $\lim_{n\to\infty} P(\Gamma(x_n)) = \Gamma(u)$ if Γ is G-continuous at $u \in X$,

2) $\lim_{n\to\infty} \Gamma(P(x_n)) = P(u)$ if P is G-continuous at $u \in X$,

3) $\Gamma(P(u)) = P(P(u))$ and $\Gamma(u) = P(u)$ if Γ and P are G-continuous at $u \in X$.

Proof- Assumes $\lim_{n\to\infty} \Gamma(x_n) = \lim_{n\to\infty} P(x_n) = u$ for some $u \in X$. since, Γ is G-continuous, we get:

$\lim_{n\to\infty} \Gamma(\Gamma(x_n)) = \lim_{n\to\infty} \Gamma(P(x_n)) = \Gamma(u)$. By definition **(1-5-1-part *v*)** we get:

$G(P(\Gamma(x_n)), \Gamma(u), \Gamma(u)) \leq G(P(\Gamma(x_n)), \Gamma(\Gamma(x_n)), \Gamma(\Gamma(x_n))) + G(\Gamma(\Gamma(x_n)), \Gamma(u), \Gamma(u))$.
When limit $n \to \infty$, we get:

$\lim_{n\to\infty} \mathrm{G}\,(P(\Gamma(x_n)), \Gamma(u), \Gamma(u))$

$\leq \lim_{n\to\infty} G(P(\Gamma(x_n)), \Gamma(\Gamma(x_n)), \Gamma(\Gamma(x_n))) + \lim_{n\to\infty} \mathrm{G}\,(\Gamma(\Gamma(x_n)), \Gamma(u), \Gamma(u))$

$$\leq \lim_{n\to\infty} G(\Gamma(P(x_n)), \Gamma(\Gamma(x_n)), \Gamma(\Gamma(x_n))) + G(\Gamma(u), \Gamma(u), \Gamma(u))$$

$$= G(\Gamma(u), \Gamma(u), \Gamma(u)) + G(\Gamma(u), \Gamma(u), \Gamma(u)) = 0 + 0 = 0.$$

This mean: $\lim_{n\to\infty} \mathrm{G}\,(P(\Gamma(x_n)), \Gamma(u), \Gamma(u)) = 0 \Rightarrow \lim_{n\to\infty} P(\Gamma(x_n)) = \Gamma(u)$. This completes the proof of part **(1).**

The proof of part **(2)** is similar to the proof of part **(1)** thus omitted. Now prove part **(3).** Since P is G-continuous at u, we have: $\lim_{n\to\infty} \Gamma(x_n) = \lim_{n\to\infty} P(x_n) = u. \Rightarrow \lim_{n\to\infty} P(\Gamma(x_n)) = \lim_{n\to\infty} P(P(x_n)) = P(u)$. Since P is G-continuous at u, then by using of part (1)

$\lim_{n\to\infty} P(\Gamma(x_n)) = \Gamma(u)$. Hence, by uniqueness of the limit, we get $\Gamma(u) = P(u)$ and by Proposition 3.3.5, we get

$$\Gamma(P(u)) = \Gamma(\Gamma(u)) = P(P(u)) = P(\Gamma(u)).$$

3.4. Some Common Fixed Point Theorems For Weakly Compatible Mappings By The Notion Of (E.A)

A common fixed point theorem for weakly compatible mappings has been proved in this part by using the notion of the property **(E.A).**

Let Φ denote the set of all continuous mappings $\phi : [0, \infty) \longrightarrow [0, \infty)$ such that:

1) ϕ is non-decreasing;

2) $\lim_{n\to\infty} \phi^n(r) = 0$, for all r $\in [0, \infty)$.

If $\phi \in \Phi$, then it is easy to show that $\phi(0) = 0$ and $\phi(r) < r$, $\forall$ $r \in (0, \infty)$.{see-**(**55)}.

Now we introduce the following theorem which is the main result of this chapter:

Theorem 3.4.1 Let (X, G) be a G-metric space and let $W, K, S, Ғ$: $(X, G) \rightarrow (X, G)$ such that :

a) $WX \subseteq ҒX$ and $KX \subseteq SX$;

b) One of the pairs (W, S) or $(K, Ғ)$ satisfies property **(E.A);**

c) For all $x, y \in X$, $\phi \in \Phi$, such that:

   $G\,(W(x),\ K(y),\ K(y)) \leq \phi[Max\{G(S(x), Ғ(y), Ғ(y)), G(S(x), K(y), K(y)), G(Ғ(y), K(y), K(y))\}];$

d) One of WX, $ҒX$ and KX OR SX is a complete subspace of X.

Then the pairs (W, S) and (K, $Ғ$) have a coincidence point. Additional, if (W,S) and (K, $Ғ$) are weakly compatible, then W, K, S *and* $Ғ$ have a unique common fixed point in X.

Proof Assume that the pair (K, $Ғ$) satisfies the property **(E. A).** Then there exists $\{x_n\}$ in X such that $\lim_{n\to\infty} K(x_n) = \lim_{n\to\infty} Ғ(x_n) = p$ for some $p \in X$. Since $KX \subseteq SX$, there exists a sequence $\{y_n\}$ in X such that $K(x_n) = S(x_n)$. Thus $\lim_{n\to\infty} S(y_n) = p$. Now, we prove that $\lim_{n\to\infty} W(y_n) = p$.

From part (c), we get:

$G\ (W(y_n),\ K(x_n),\ K(x_n)) \leq \phi\ [Max\{G(S(y_n),\ Ғ(x_n),\ Ғ(x_n)),\ G\ (S(y_n),\ K(x_n),\ K(x_n)),\ G\ (Ғ(x_n),\ K(x_n),\ K(x_n))\}]$, when the limit as $n \rightarrow \infty$ and using the fact that $\phi(t)$ is continuous at $t = 0$, we get:

$\lim_{n\to\infty} G\ (W(y_n),\ p,\ p) \leq \phi\ [Max\{G(p,\ p,\ p),\ G(p,\ p,\ p),\ G(p,\ p,\ p)\}] = \phi\ [max(0,\ 0,\ 0)] = \phi\ (0)=0,$ and thus, $\lim_{n\to\infty} W(y_n) = p.$ so, we have $\lim_{n\to\infty} W(y_n) = \lim_{n\to\infty} S(y_n) = \lim_{n\to\infty} K(y_n) = \lim_{n\to\infty} Ғ(y_n) = p.$

Assume that SX is a complete subspace of X. Then $p = S(u)$ for some $u \in X$. Therefore, we have

$\lim_{n\to\infty} W(y_n) = \lim_{n\to\infty} S(y_n) = \lim_{n\to\infty} K(x_n) = \lim_{n\to\infty} Ғ(x_n) = p = S(u)$. Now we prove that , $W(u) = S(u) = p$. from part (c) we have:

$G\ (W(u),\ K(x_n),\ K(x_n)) \leq \phi\ [Max\ (G\ (S(u)\ ,Ғ(x_n),\ Ғ(x_n)),\ G(Su,\ K(x_n),\ K(x_n)),\ G\ (Ғ(x_n), K(x_n), K(x_n)))].$

When the limit as $n \to \infty$ and using the property of ϕ , we get:

$G\ (W(u),\ S(u),\ S(u)) \leq \phi\ [Max\{G(p,\ p,\ p),\ G(p,\ p,\ p),\ G(p,\ p,\ p)\}] = \phi\ [Max(0,\ 0,\ 0)] = \phi\ (0)=0,$

$W(u) = S(u)$. Therefore, u is a coincidence point for the pair (W, S).

The weak compatibility of W and S $\Rightarrow WS(u) = SW(u)$ and hence

$WW(u) = WS(u) = SW\ (u) = SS(u)$. Since, $WX \subseteq ҒX,$ there exists $v \in X$ such that $W(u) = Ғ(v)$. We prove that:

$Ғ(v) = K(v)$ or $W(u) = K(v)$. Suppose not, from (c) and using the fact that $\phi(r) < r,$ we get:

$G\ (W(u), K(v), K(v)) \leq \phi\ [max\{G(S(u),\ Ғ(v),\ Ғ(v)),\ G\ (S(u),\ K(v),\ K(v)),\ G\ (Ғ(v),\ K(v),\ K(v))\}]$

$= \phi\ [Max\ \{0, G\ (W(u), K(v), K(v)), G\ (W(u), K(v), K(v))\}]$

$= \phi\ [\ G\ (W(u), K(v), K(v))] < G\ (W(u), K(v), K(v)),$

Which is a contradiction to the hypothesis that $W(u) \neq K(v)$. Thus $W(u) = K(v) = Ғ(v)$. So, the pair $(K, Ғ)$ has also a coincidence point. The weak compatibility of K and $Ғ$ implies that:

$KҒ(v) = ҒK(v) = ҒҒ(v) = KK(v)$. Finally, we show that $W(u)$ is a common fixed point of W and S. Presumes not, then from (c) and by using the property of ϕ, we get:

$$G(WW(u), W(u), W(u)) = G(WW(u), K(v), K(v))$$

$$\leq \phi\,[Max\{G(SW(u), Ғ(v), Ғ(v)), G(SW(u), K(v), K(v)), G(Ғ(v), K(v), K(v))\}]$$

$$= \phi\,[Max\{G(WW(u), W(u), W(u)), G(WW(u), W(u), W(u)), G(W(u), W(u), W(u))\}]$$

$$= \phi\,[Max\{G(WW(u), W(u), W(u)), G(WW(u), W(u), W(u)), 0\}]$$

$$= \phi\,[G(WW(u), W(u), W(u))] < G(WW(u), W(u), W(u)),$$

Which gives a contradiction. Therefore, $W(u) = WW(u) = SW(u)$ is a common fixed point of W and S.

Similarly, we can prove that $K(v)$ is a common fixed point of K and Ғ. Since $W(u) = K(v)$, we deduce that $W(u)$ is a common fixed point of W, K, S and $Ғ$.

For uniqueness, let $x^{\wedge}$ and $y^{\wedge}$, with $x^{\wedge} \neq y^{\wedge}$ are two common fixed points of W, K, S and $Ғ$. Then, from (c), and using the property of ϕ, we get:

$$G(x^{\wedge}, y^{\wedge}, y^{\wedge}) = G(W(x^{\wedge}), K(y^{\wedge}), K(y^{\wedge}))$$

$$\leq \phi\,[Max\{G(S(x^{\wedge}), Ғ(y^{\wedge}), Ғ(y^{\wedge})), G(S(x^{\wedge}), K(y^{\wedge}), K(y^{\wedge})), G(Ғ(y^{\wedge}), K(y^{\wedge}), K(y^{\wedge}))\}]$$

$$= \phi\,[Max\{G(x^{\wedge}, y^{\wedge}, y^{\wedge}), G(x^{\wedge}, y^{\wedge}, y^{\wedge}), G(y^{\wedge}, y^{\wedge}, y^{\wedge})\}]$$

$$= \phi\,[G(x^{\wedge}, y^{\wedge}, y^{\wedge})] < G(x^{\wedge}, y^{\wedge}, y^{\wedge}),$$

which is a contradiction and so there must be $x^{\wedge} = y^{\wedge}$. Therefore W, K, S and $Ғ$ have a unique common fixed point. Similarly we can get the same conclusion in case one of WX, $ҒX$ OR KX is a complete subspace of X instead of SX, and in case the pair (W, S) satisfies the property **(E.A)** in place of pair (K,F). This completes the proof of the theorem.

By choosing S = F in Theorem 3.4.1**,** the following result involving three self-mappings is concluded:

Corollary 3.4.2 Let *(X, G)* be a G-metric space and let W, K, S*: (X, G) → (X, G)* be three mappings such that :

a) $WX \subseteq SX$ and $KX \subseteq SX$;

b) One of the pairs *(W, S)* or *(K, S*) satisfies property **(E.A);**

c) For all $x, y \in X$, $\phi \in \Phi$, such that:

G (W(x) K(y) K(y)) $\leq \phi$*[Max {G(S(x), S(y), S(y)),G(S(x), K(y), K(y)), G(S(y), K(y), K(y))}];*

d) One of $WX\ \ KX\ OR\ SX$ is a complete subspace of *X*.

Then the pairs (W, S) and (K, *S*) have a coincidence point. Additional, if (W, S) and (K, *S*) are weakly compatible. Then *W*, *K*, and *S* have a unique common fixed point in X.

The following example shows the validity of Corollary 3.4.2.

Example 3.4.3 Let $X = [0, 2]$ and a mapping, G: $X^3 \to [0, \infty)$ be defined by:

$G(x, y, z) = Max\{|x - y|, |x - z|, |z - x|\}$, For all $x, y, z \in X$. Define W, K, S*: (X, G) → (X, G)* by *W(x) =1*, *K(x) =2-x* and *S(x) = x*, for all $x \in$ X and $\phi : [0, \infty) \longrightarrow [0, \infty)$ by $\phi(r) = \frac{r}{2}$, $for\ all\ r \geq 0$.

Obviously, the hypotheses (a) and (d) of the Corollary 3.4.2 hold trivially. Furthermore, the pair (W, S) satisfies the property **(E.A)**. Here we illustrate only that the hypothesis (c) holds. In fact, for all $\forall\, x, y \in$ X we get:

$G\big(W(x), K(y), K(y)\big) = G(1, 2 - y, 2 - y) = |1 - y|$,

$G\big(S(x), S(y), S(y)\big) = G(x, y, y) = |x - y|,$

$GG\big(S(x), K(y), K(y)\big) = G(x, 2 - y, 2 - y) = |2 - x - y|,$

$\big(S(x), K(y), K(y)\big) = G(y, 2 - y, 2 - y) = 2|1 - y|$ And consequently

$G\big(W(x), K(y), K(y)\big) = |1 - y| = \frac{1}{2}\, 2|1 - y| = \phi\Big(G\big(S(x), K(y), K(y)\big)\Big)$

$\leq \phi$*[Max {G (S(x), S(y), S(y)),G (S(x), K(y), K(y)), G (S(y), K(y), K(y))}]*.

Then, by the Corollary 3.4.2, the pairs *(W, S)* and *(K, S)* have a coincidence point, that is, $x = 1$. Furthermore, if *(W, S)* and *(K, S)* are weakly compatible, then $x = 1$ is the unique common fixed point of *W*, *K* and *S* in *X*.

3.5. Some Applications To The Solutions Of Integral And Functional Equations

Two useful applications of Corollary 3.4.2 are presented in this section. Firstly, the solution of an integral equation is obtained by applying the Corollary 3.4.2.

Let $\mho = [0, 1]$ and *C(℧)* be the space of all real valued continuous mappings defined on ℧. Clearly, this space endowed with the G-Metric given by:

$G(x, y, z) = \sup_{r\in\mho}|x(r) - y(r)| + \sup_{r\in\mho}|y(r) - z(r)| + \sup_{r\in\mho}|z(r) - x(r)| \quad \forall\ x, y, z \in X \in C(\mho)$, is a G-Complete metric space.

Assume:- $P: \mho \times \mathbb{R} \rightarrow \mathbb{R}$ and $h: \mho \times \mho \times \mathbb{R} \rightarrow \mathbb{R}$ be two continuous functions. Consider an integral equation of the following kind: $P\big(r, x(r)\big) = \int_{\mho} h(r, s, x(s))$, for all $s, r \in \mho$…(3)

Theorem 3.5.1 Let there exists $H: \mho \times \mathbb{R} \rightarrow [0, \infty)$ such that:

1) $H\big(s, v(r)\big) \leq \int_{\mho} h(r, s, u(s))\, ds \leq P\big(s, v(r)\big),\ \forall\, s, r \in \mho$;

2) $P\big(s, v(r)\big) - H(s, v(r)) \leq k\big|p\big(s, v(r)\big) - v(r)\big|$ Where, $k \in (0, 1)$.

Then the integral equation **(3)** has a unique solution in *C (℧).*

Proof First define $W(x)(r) = \int_{\mho} h(r, s, x(s))ds \ and \ K(x)(r) = p(r, x(r))$. Then now,

$$G(W(x), k(x), k(x)) = 2 \sup_{r\in\mho} |(W(x)(r)) - (K(y)(r))|$$

$$= 2 \sup_{r\in\mho} \left| p(r, y(r)) - \int_{\mho} h(r, s, x(s))ds \right|$$

$$\le 2 \sup_{r\in\mho} |p(r, y(r)) - H(r, \ y(r)|$$

$$\le 2k \sup_{r\in\mho} |p(r, y(r)) - \ y(r)| = kG(y, K(y), K(y)).$$

Thus, all the hypotheses of the Corollary 3.4.2 are satisfied by choosing $S = T_{C(\mho)}$ and $\phi(t) = kt$, for all $t \ \ge 0, k \ \in (0, 1)$. Therefore, there is a unique solution of the integral equation (3) in *C(℧).* Additional, by using the Corollary 3.4.2, the existence and uniqueness of the bounded solution of a functional equation arising in dynamic programming has been presented. Here, let *U* and *V* are Banach-Spaces, $A \subseteq U$ is a state space and $D \subseteq U$DV is a decision space.

It is well known that the dynamic programming provides useful tools for mathematical optimization and computer programming as well (see, **[56, 57, 58]).** In particular, the problem of dynamic programming related to multistage process reduces to the problem of solving the functional equation:

$$\mathbb{Q}(x) = \sup_{y\in D}\{\Gamma(x, y) + B(x, y, \mathbb{Q}(\tau(x, y)))\}, \ x \ \in A, \tag{4}$$

where, τ: $A \times$ D $\rightarrow$ A, Γ: $A \times$ D $\rightarrow \mathbb{R}$ and B: $A \times$ D $\times \mathbb{R} \rightarrow \mathbb{R}$. Let *K (A)* denote the space of all bounded real-valued functions on A. Clearly, this space endowed with the G-metric given by:

$$G(q, k, p) = max\{\sup_{x\in A} |q(x) - k(x)|, \sup_{x\in A} K(X) - p(x), \sup_{x\in A} |p(x) - q(x)|\}$$ For all $q, k, p \in K(A)$. is a G-Complete metric space.

Theorem 3.5.2 Let $B\colon A \times D \times \mathbb{R} \to \mathbb{R}$ and $\Gamma\colon A \times D \to \mathbb{R}$ be two bounded mappings and let W: K(A) → K(A) be defined by:

$$W\big(q(x)\big) = \sup_{y \in D}\{\Gamma(x,y) + B(x,y,q(\tau(x,y)))\}, \forall\, q \in K(A) \text{ and } x \in A. \qquad (5)$$

Presume the following condition holds:

$$|\mathrm{B}(\mathrm{x},\mathrm{y},\mathrm{q}_1(\tau(\mathrm{x},\mathrm{y}))) - \mathrm{B}(\mathrm{x},\mathrm{y},\mathrm{q}_2(\tau(\mathrm{x},\mathrm{y})))| \le \phi(|\mathrm{q}_1(\mathrm{x}) - \mathrm{q}_2(\mathrm{x})|),$$

Where $q_1, q_2 \in K(A)$ and $x \in A, y \in D$ and $\phi \in \Phi$. Then functional equation (4) has a unique bounded solution.

Proof We can see that *(K(A), G)* is a complete G-metric space. **.** Let α be an arbitrary positive number, $q_1, q_2 \in K(A)$ and $x \in A$. Then there exist $y_1, y_2 \in D$, such that:

$$W\big(q_1(x)\big) < \Gamma(x,y_1) + B\{x,y_1,q_1(\tau(x,y_1))\} + \alpha, \qquad (6)$$

$$W\big(q_2(x)\big) < \Gamma(x,y_2) + B\{x,y_2,q_2(\tau(x,y_2))\} + \alpha, \qquad (7)$$

$$W\big(q_1(x)\big) \ge \Gamma(x,y_2) + B\{x,y_2,q_1(\tau(x,y_2))\}, \qquad (8)$$

$$W\big(q_2(x)\big) \ge \Gamma(x,y_1) + B\{x,y_1,q_2(\tau(x,y_1))\}, \qquad (9)$$

Then, from **(6)** and **(9),** it follows simply that:

$$\begin{aligned} W\big(q_1(x)\big) - W\big(q_2(x)\big) &< B\{x,y_1,q_1(\tau(x,y_1))\} - B\{x,y_1,q_2(\tau(x,y_1))\} + \alpha \\ &\le |B\{x,y_1,q_1(\tau(x,y_1))\} - B\{x,y_1,q_2(\tau(x,y_1))\}| + \alpha \\ &\le \phi(|q_1(x) - q_2(x)|) + \alpha. \end{aligned}$$

Hence, we get:

$$W\big(q_1(x)\big) - W\big(q_2(x)\big) < \phi(|q_1(x) - q_2(x)|) + \alpha. \qquad (10)$$

By same method, from **(7)** and **(8),** we get:

$$W\big(q_2(x)\big) - W\big(q_1(x)\big) < \phi(|q_1(x) - q_2(x)|) + \alpha. \qquad (11)$$

Therefore, from **(10)** and **(11)**, we have:

$$\left|W\left(q_1(x)\right) - W\left(q_2(x)\right)\right| < \phi(|q_1(x) - q_2(x)|) + \alpha, \tag{12}$$

This implies,

$$G\{W(q_1(x)), W(q_2(x)), W(q_2(x))\} < \phi\{G(q_1(x), q_2(x), q_2(x))\} + \alpha.$$

The above inequality is right for all $x \in A$. Since $\alpha > 0$ is arbitrary, we get:

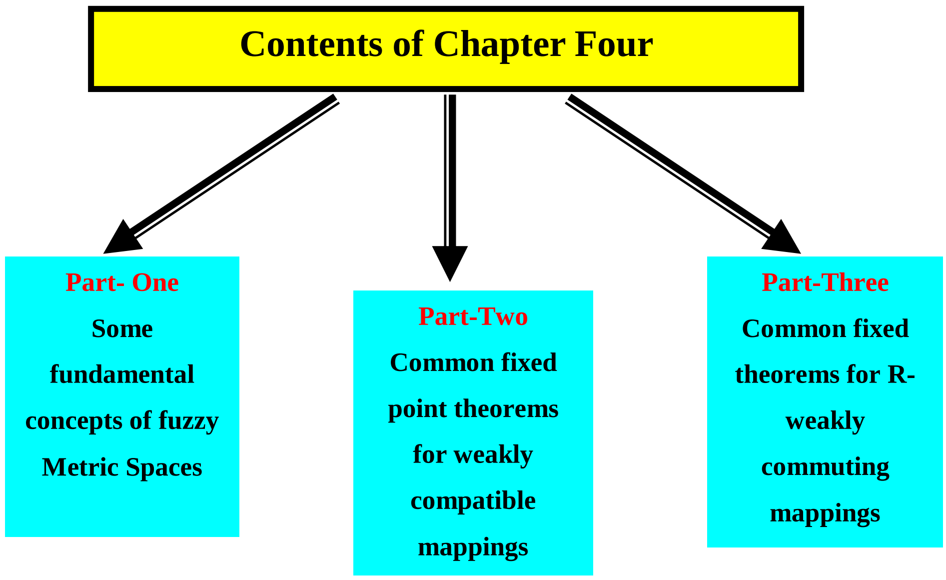
$$G\{W(q_1(x)), W(q_2(x)), W(q_2(x))\} \leq \phi\{G(q_1(x), q_2(x), q_2(x))\}.$$

Thus, all the hypothesis of the Corollary 3.4.2 are satisfied with $W = K$ and $S = T_{K(A)}$ (the identity mapping on *K(A))*. Therefore, the functional equation (4) has a unique bounded solution.

CHAPTER FOUR

WEAKLY COMPATIBLE MAPPINGS AND SOME COMMON FIXED POINT THEOREMS ON FUZZY METRIC SPACES

Some common fixed point theorems for weakly compatible mappings on fuzzy metric spaces have been proved in this chapter by making use of some implicit relations. This section has been divided into three parts. In part one we introduced some basic concepts of fuzzy metric spaces which we need them in this section. Some common fixed point theorems for weakly compatible mappings in fuzzy metric spaces by using common limit in the range property have been discussed in part two. As well in part three we presented some common fixed theorems for R-weakly commuting mappings and some of their variants in fuzzy metric spaces.



4.1. Some Basic Concets Of Fuzzy Metric Spaces

Now we recall some fundamental concepts which need then in this section:

The authors in **[59]** introduced fuzzy metric spaces by generalizing the concept of probabilistic metric spaces **[60]** to fuzzy situations using continuous r-norm in the following way:

Definition 4.1.1 **[59]** A binary operation $\otimes$: $[0, \infty] \times [0, \infty] \to [0, \infty]$ is continuous r-norm if $\otimes$ satisfies the following conditions for all $a, b, c, d \in [0, \infty]$.

1) $\otimes$ is commutative and associative;
2) $\otimes$ is continuous;
3) $a \otimes 1 = a;$
4) $a \otimes b \leq c \otimes d$, whenever, $a \leq c$ *and* $b \leq d$.

Definition 4.1.2 The 3-tuple $(X, M, \otimes)$ is said to be a fuzzy metric space if X is an arbitrary set, $\otimes$ is a continuous r-norm and M is a fuzzy set on $X^2 \times [0, \infty)$ satisfies the following conditions for all $x, y, z \in X$ and $r, s > 0$,

1) $M(x, y, 0) = 0$,
2) $M(x, y, r) = 1 \;\forall\; x = y$;
3) $M(x, y, r) = M(y, x, r)$;
4) $M(x, y, r) \otimes M(y, z, s) \leq M(x, z, r + s)$,
5) $M(x, y, *) : [0, \infty) \to [0, \infty]$ is left continuous.

Mishra et al**. [61]** In 1994, introduced the notion of compatible mappings in fuzzy metric spaces similar to the notion of compatible mappings in metric spaces as follows:

Definition 4.1.3 Let Γ, P: $(X, M, \otimes) \to (X, M, \otimes)$ be self-mappings of a fuzzy metric $(X, M, \otimes)$ into itself. Then Γ and P are said to be compatible if

$$\lim_{n\to\infty} M\big(\Gamma\big(P(x_n)\big), P\big(\Gamma(x_n)\big), r\big) = 1$$

For the sequences $\{x_n\}$ in X such that: $\lim_{n\to\infty} \Gamma(x_n) = \lim_{n\to\infty} P(x_n) = r$ for some $r \in X$.

In addition, a pair (Γ, P) of self -mappings of a fuzzy metric space $(X, M, \otimes)$ is said to be non-compatible if there exists at least one sequence $\{x_n\}$ in X such that:

$\lim_{n\to\infty} \Gamma(x_n) = \lim_{n\to\infty} P(x_n) = r$ for some $r \in X$ but $\lim_{n\to\infty} M\big(\Gamma\big(P(x_n)\big), P\big(\Gamma(x_n)\big), r\big) \neq 1$, or non-existent for at least one $r > 0$.

Pant **[39]** in 1994, introduced the notion of $\mathbb{R}$-weakly commuting mappings in metric spaces. Afterward, Vasuki **[62]** introduced the notion of $\mathbb{R}$-weakly commuting mappings in fuzzy metric spaces as:

Definition 4.1.4 Let Γ, P: $(X, M, \otimes) \to (X, M, \otimes)$ be self-mappings of a fuzzy metric space $(X, M, \otimes)$ into itself. Then Γ and P are said to be R-weakly commuting if there exists some $R > 0$ such that

$$M\big(\Gamma\big(P(x)\big), P\big(\Gamma(x)\big), r\big) \geq M\left\{\Gamma(x), P(x), \frac{r}{R}\right\} \text{ for all } x \in X \text{ and } r > 0\,.$$

In 1997, Pathak et al. **[63]** improved the notion of R-weakly commuting mappings in metric spaces by introducing the notions of $\mathbb{R}$-weakly commutativity of type (W_{P}) and $\mathbb{R}$-weakly commutativity of type (W_Γ). Imdad and Ali **[64]** in 2008 generalized the notion of R-weakly commutativity of type $(W_{\mathrm{P}})$and $\mathbb{R}$-weakly commutativity of type (W_Γ) in fuzzy metric with inspiration from Pathak et al. **[63].** Further, they have also introduced the notion of R-weakly commuting mappings of type (P) in fuzzy metric spaces.

Definition. 4.1.5 Let Γ, P: $(X, M, \otimes) \to (X, M, \otimes)$ be self-mappings of a fuzzy metric $(X, M, \otimes)$ into itself. Then Γ and P is said to be **R-weakly commuting of type** $(\boldsymbol{W_\Gamma})$ if there exists some $\mathbb{R} > 0$ such that

$$M\big(P(P(x)),\Gamma(P(x)),r\big) \geq M\left\{P(x),\Gamma(x),\frac{r}{R}\right\} \text{for all } x \in X \text{ and } r > 0\,.$$

Definition 4.1.6 Let Γ, P: *(X, M, ⊗) → (X, M, ⊗)* of self-mappings of a fuzzy metric (X, M, ⊗) into itself, then Γ and P is said to be **R-weakly commuting of type** (W_{P}) if there exists some $\mathbb{R} > 0$ such that

$$M\big(\Gamma(\Gamma(x)),\ P(\Gamma(x)),r\big) \geq M\left\{\Gamma(x),P(x),\frac{r}{\mathbb{R}}\right\} \ \forall\, x \in \mathrm{X} \text{ and } r > 0.$$

Definition 4.1.7 Let Γ, P: *(X, M, ⊗) → (X, M, ⊗)* be self-mappings of a fuzzy metric (X, M, ⊗) into itself. Then Γ and P is said to be **R-weakly commuting of type (*P*)** if there exists some $R > 0$ such that

$$M\big(\Gamma(\Gamma(x)),\ P(P(x)),r\big) \geq M\left\{\Gamma(x),P(x),\frac{r}{R}\right\} \ \forall\, x \in X \text{ and } r > 0.$$

Now we introduce a new property "common limit in the range property" for four self-mappings with an inspiration from Sintunavarat and Kumam **[50]**.

Definition 4.1.8 The pairs (W, S) and (K, Ƒ) on a fuzzy metric space *(X, M, ⊗)* are said to share the **common limit in the range of S property** if there exists two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} W(x_n) = \lim_{n\to\infty} S(x_n) = \lim_{n\to\infty} Ƒ(y_n) = \lim_{n\to\infty} K(y_n) = S(r)$$

for some $r \in X$ **[43].**

Proposition 4.1.9 Let $\{y_n\}$ be a sequence in a fuzzy metric space *(X, M, ⊗)*. If ∃ a constant $k \in (0, 1)$ such that: $\mathrm{M}(\mathrm{y_n},\mathrm{y_{n+1}},\mathrm{kr}) \geq \mathrm{M}(\mathrm{y_{n-1}},\mathrm{y_n},\mathrm{r})\ \forall\, n = 0, 1, 2, \dots, \mathrm{r} > 0$, then $\{y_n\}$ is a Cauchy sequence in X **[61].**

Proposition 4.1.10 Let *(X, M, ⊗)* be a fuzzy metric space. If ∃ a constant $k \in (0, 1)$ such that: $\mathrm{M}(\mathrm{x},\ \mathrm{y},\ \mathrm{kr}) \geq \mathrm{M}(\mathrm{x},\ \mathrm{y},\ \mathrm{r})$ for all $x, y \in X$, then $x = y$ **[61].**

In **[65]** Popa introduced the idea of implicit relations in metric spaces which are fruitful in unifying new contraction conditions and proved some fixed point results by using these relations. Singh and Jain **[66]** further extended the results of Popa **[65]** in fuzzy metric spaces.

Now recall the following implicit relation for the proof of our main result:

- **Implicit relation**: Let Ψ_4 be the set of all continuous mapping I: $[0,1]^4 \to \mathbb{R}$ satisfying the following conditions:

i) For each $u > 0,\ v \geq 0$ *with* $I(u,v,u,v) \geq 0$ *or* $I(u,v,v,u) \geq 0$, *we get* $u > v$;

ii) $I(u,u,1,1) < 0$ *for all* $u \in (0,1)$.

Proposition 4.1.11 Let *W*, *K*, *S* and *F* be four self-mappings of a fuzzy metric space *(X, M, ⊗)* satisfying the following conditions:

1) the pair *(W, S)* satisfies the common limit in the range of S property (or *(K, F)* satisfies the common limit in the range of *F* property);

2) for every $x, y \in X,\ I \in \Psi_4$ *and* $r > 0$,

$$I\{M(W(x),K(y),r), M(S(x),F(y),r), M(S(x),W(x),r), M(F(y),K(y),r)\} \geq 0;$$

3) $W(X) \subseteq F(X)$ *OR* $K(X) \subseteq S(X)$.

Then the pairs *(W, S)* and *(K, F)* share the common limit in the range of S property (or *F* property)

4.2. Some Common Fıxed Poınt Theorems For Weakly Compatıble Mappıngs In Fuzzy Metrıc Spaces

In this part we establish the following theorems concerning weakly compatible mappings by using common limit in the range property.

Theorem 4.2.1 Let *W*, *K*, *S* and *F* be four self-mappings of a fuzzy metric space *(X, M, ⊗)* satisfying the following conditions:

1) The pair (W, S) satisfies the common limit in the range of S property (or (K, Ғ) satisfies the common limit in the range of *F* property);

2) for every $x, y \in X,\ I \in \Psi_4\ and\ t > 0,$

$I\{M(W(x), K(y), t), M(S(x), Ғ(y), t), M(S(x), W(x), t), M(Ғ(y), K(y), t)\} \geq 0;$

3) $\mathrm{W(X)} \subseteq \mathrm{Ғ(X)}$ OR $\mathrm{K(X)} \subseteq \mathrm{S(X)}$.

Then the pairs *(W, S)* and *(K, F)* have a point of coincidence each. Moreover, *A, B, S* and *F* have a unique common fixed point provided that both the pairs *(W, S)* and *(K, F)* are weakly compatible.

Proof From Proposition 4.1.11 the pairs (W, S) and (*K*, Ғ) share the common limit in the range of S property, that is, there exists two sequences $\{x_n\}$ and $\{y_n\}$ such that:

$\lim_{n\to\infty} W(x_n) = \lim_{n\to\infty} S(x_n) = \lim_{n\to\infty} Ғ(y_n) = \lim_{n\to\infty} K(y_n) = S(r)$ for some $r \in X$.

First we state that W(r) = S(r). If not, then by (2), we have:

$$I\{M(W(r), K(y_n), t), M(S(r), Ғ(y_n), t), M(S(r), W(r), t), M(Ғ(y_n), K(y_n), t)\} \geq 0;$$

Now taking $n \to \infty$, we get:

$I\{M(W(r), S(r), t), M(S(r), S(r), t), M(S(r), W(r), r), M(S(r), S(r), t)\} \geq 0;$

$I\{M(W(r), S(r), t), M(S(r), W(r), t), 1\} \geq 0;$ By using (I_1), we have:

$\{M(S(r), W(r), t\ )\} > 1;$ which is a contradiction and therefore, $W(r) = S(r)$. In addition, since, $\mathrm{W(X)} \subseteq \mathrm{Ғ(X)}$, therefore, $\exists\ v \in X$, such that $W(r) = Ғ(v)$.

Now, $K(v) = Ғ(v)$. If not, then by the use of (2), we have:

$I\{M(W(r), K(v), t), M(S(r), Ғ(v), t), M(S(r), W(r), t), M(Ғ(v), K(v), t)\} \geq 0;$

$I\{M(F(v),K(v),t),M(F(v),F(v),t),M(F(v),F(v),t),M(F(v),K(v),t)\} \geq 0;$

$I\{M(F(v),K(v),t),1,1,M(F(v),K(v),t)\} \geq 0$; Again, by using ($I_1$), we have:

$\{M(F(v),K(v),t)\} > 1$; which is contradiction and therefore, *F(v) = K(v) = W(r) = S(r).*

Since the pairs *(W, S) and (K, F)* are weakly compatible and *F(v) = K(v) and W(r) = S(r)*. Therefore, *WS(r) = SW(r) = WW(r) = SS(r), KF(v) = FK(v) = FF(v) = KK(v).* Finally, we assert that WW(r) = W(r). Suppose not, then by (2), we get:

$$I\{M(WW(r),K(v),t),M(SW(r),F(v),t),M(SW(r),WW(r),t),M(F(v),K(v),t)\} \geq 0;$$

$$I\{M(WW(r),K(v),t),M(WW(r),K(v),t),M(WW(r),WW(r),t),M(K(v),K(v),t)\} \geq 0;$$

$I\{M(WW(r),W(r),t),M(WW(r),W(r),t),1,1\} \geq 0;$

which is a contradiction to **((I_2))** and therefore, *WW(r) = W(r) = SW(r)* Hence, *W(r)* is common fixed point of W and S. Similarly, one can easily establish that *KK(v) = K(v) = FK(v),* this mean that *K(v)* is common fixed point of K & *F*. Since *W(r) = K(v)* , therefore, *W(r)* is common fixed point of *W*, *S* ,*K* and *F*. The uniqueness of common fixed point is an easy consequence of inequality (2).

Corollary 4.2.2 Let *W* and *S be* two self-mappings of a fuzzy metric space *(X, M, ⊗)* satisfying the following conditions:

1) The pair *(W, S)* satisfies the common limit in the range of S property;

2) for every $x,y \in X,\ I \in \Psi_4$ and $t > 0$,

$I\{M(W(x),W(y),t),M(S(x),S(y),t),M(S(x),W(x),t),M(S(y),W(y),t)\} \geq 0;$

3) W(X) ⊆ F(X).

Then the pairs *W* and *S* have a point of coincidence each. Additionally, *A* and *S* have a unique common fixed point provided that both the pairs *W* and *S* are weakly compatible

Proof The proof follows from Theorem 4.2.1 by taking $W = B$ and $S = F$.

Corollary 4.2.3 Let *W*, *B*, *S* and *F* be four self-mappings of a fuzzy metric space *(X, M, ⊗)* satisfying the following conditions:-

1) The pairs *(W, S)* and *(B, F)* share the common limit in the range of *S* property (or *F* property);

2) for every $x, y \in X,\ I \in \Psi_4$ and $t > 0$,

$$I\{M(W(x), K(y), t), M(S(x), \mathrm{F}(y), t), M(S(x), W(x), t), M(\mathrm{F}(y), K(y), t)\} \geq 0;$$

3) $\mathrm{W(X)} \subseteq \mathrm{F(X)}$ OR $\mathrm{K(X)} \subseteq \mathrm{S(X)}$.

Then the pairs *(W, S)* and *(B, F)* have a point of coincidence each. Additionally, *A*, *K*, *S* and *F* have a unique common fixed point provided that both the pairs *(W, S)* and *(B, F)* are weakly compatible.

Proof The proof follows from Theorem 4.2.1 by using Proposition 4.1.11.

Theorem 4.2.4 Let *W*, *K*, *S* and *F* be four self-mappings of a fuzzy metric space *(X, M, ⊗)* satisfying the following conditions:

1) for every $x, y \in X,\ I \in \Psi_4$ and $t > 0$ $such\ that$:

$$I\{M(W(x), K(y), t), M(S(x), \mathrm{F}(y), t), M(S(x), W(x), t), M(\mathrm{F}(y), K(y), t)\} \geq 0;$$

2) $\mathrm{W(X)} \subseteq \mathrm{F(X)}$ OR $\mathrm{K(X)} \subseteq \mathrm{S(X)}$.

3) The pair *(W, S)* satisfies the property of *(E.A)* and S(X) is a closed sub-space of X (or (K, *F*) satisfying property *(E.A)* and *F(X)* is a closed sub-space of X).

Then the pairs *(W, S)* and *(K, F)* have a point of coincidence each. Additionally, *A, B, S* and *F* have a unique common fixed point provided that both the pairs *(W, S)* and *(K, F)* are weakly compatible.

Proof Suppose that the pair (W, S) satisfy property ***(E. A).*** Then there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} W(x_n) = \lim_{n\to\infty} S(x_n) = p$ for some $p \in X$. Since S (X) is a closed sub-space of X. Therefore, p = S(r), for some. $r \in X$. Hence, the pair *(W, S)* satisfies the common limit in the range of S property. Thus, by the use of Corollary 4.2.3 and Proposition 4.1.11, mappings W, K, S and *F* have a unique common fixed point.

Corollary 4.2.5 Let *W, B, S* and *F* be four self-mappings of a fuzzy metric space *(X, M,* ⊗*)* satisfying the following conditions:-

1) The pairs *(W, S)* and *(B, F)* satisfies common property *(E.A)* and S(X) is a closed sub-space of X,

2) for every $x, y \in X,\ I \in \Psi_4\ and\ t > 0$,

$I\{M(W(x), K(y), t), M(S(x), F(y), t), M(S(x), W(x), t), M(F(y), K(y), t)\} \geq 0$;

3) $W(X) \subseteq F(X)$ OR $K(X) \subseteq S(X)$.

Then the pairs *(W, S)* and *(B, F)* have a point of coincidence each. Additionally, *A, K, S* and *F* have a unique common fixed point provided that both the pairs *(W, S)* and *(B, F)* are weakly compatible.

Proof Since the pairs (W, S) and (K, *F*) satisfies common property *(E.A)*, therefore, there exists two sequences $\{x_n\}$ and $\{y_n\}$ in X such that:

$$\lim_{n\to\infty} W(x_n) = \lim_{n\to\infty} S(x_n) = \lim_{n\to\infty} F(y_n) = \lim_{n\to\infty} K(y_n) = p \text{ for some } p \in X.$$

It follows by the fact that *S(X)* is a closed subspace of *X* that p = S(r) for some $r \in X$ and thus the pairs (W, S) and (K, *F*) share the common limit in the range of *S* property. Therefore, by Theorem 4.2.1, mappings *W, K, S* and *F* have a unique common fixed point.

Remark 4.2.6 The result of Theorem 4.2.4, holds true even if the condition of property ***(E.A)*** is replaced by non-compatibility condition as shown in the following corollary:

Corollary 4.2.7 Let *W, B, S* and *F* be four self-mappings of a fuzzy metric space *(X, M, ⊗)* satisfying the following conditions:-

1) The pairs *(W, S)* and *(B, F*) are non-compatible mappings and *S(X)* is a closed sub-space of *X*.

2) for every $x, y \in X,\ I \in \Psi_4\ and\ t > 0$,

$$I\{M(W(x), K(y), t), M(S(x), \text{F}(y), t), M(S(x), W(x), t), M(\text{F}(y), K(y), t)\} \geq 0;$$

3) $\mathrm{W(X)} \subseteq \mathrm{F(X)}$ OR $\mathrm{K(X)} \subseteq \mathrm{S(X)}$.

Then the pairs *(W, S)* and *(B, F)* have a point of coincidence each. Additionally, *A, K, S* and *F* have a unique common fixed point provided that both the pairs *(W, S)* and *(K, F)* are weakly compatible.

Proof Since the pair *(W, S)* is non-compatible mappings, we get W and S satisfy property ***(E.A).*** Therefore, by Theorem 4.2.4, we get W, K, S and *F* have a unique common fixed point in X.

Remark 4.2.8 Although, property **(E.A)** (common property **(E.A)**) is an essential tool to claim the existence of common fixed points of some mappings. But, this property requires the condition of closedness of *S(X).*In Theorem 4.2.1, the condition of closedness of subspace *S(X)* has been weakened. Therefore, it is interesting to use common limit in the range of S property as another auxiliary tool to claim the existence of a common fixed point. As an application of Theorem 4.2.1, a common fixed point theorem for four finite families of mappings on fuzzy metric spaces has been established.

Theorem 4.2.9 Let $\{W_1, W_2, W_3, \ldots, Wm\}$, $\{K_1, K_2, K_3, \ldots Kn\}$, $\{S_1, S_2, S_3, \ldots Sp\}$, *and* $\{F_1, F_2, F_3, \ldots Fq\}$, be four finite collections of self-mappings of a fuzzy metric space

$(X, M, \otimes)$ such that: W = $W_1, W_2, W_3,\ldots, Wm,$ K = $K_1, K_2, K_3,\ldots, Kn,$ S = $S_1, S_2, S_3,\ldots, Sp,$ and $F = F_1, F_2, F_3,\ldots Fq,$ satisfying the following conditions:-

1) for every $x, y \in X,\ I \in \Psi_4$ and $t > 0$,

$I\{M(W(x), K(y), t), M(S(x), \mathrm{F}(y), t), M(S(x), W(x), t), M(\mathrm{F}(y), K(y), t)\} \geq 0;$

2) $\mathrm{W(X)} \subseteq \mathrm{F(X)}$ OR $\mathrm{K(X)} \subseteq \mathrm{S(X)}$

3) The pairs (W, S) OR (B, F) satisfies the common limit in the range of S property. Then:

i) The pairs (W, S) and (K, F) have a point of coincidence each.

ii) $W_{i,}, S_k, K_r$, and F_t, have a unique common fixed point provided that the pairs of collections $[\{W_{i,}\}, \{S_k\}]$ & $[\{K_r\}, \{F_t\}]$ commute pair-wise for every $i = 1, 2, 3,\ldots,m,\ k = 1, 2, 3,\ldots,p,\ r = 1, 2, 3,\ldots,n,$ and $t = 1, 2, 3,\ldots,q.$

Proof As the pairs of collections $[\{W_{i,}\}, \{S_k\}]$ and $[\{K_r\}, \{F_t\}]$ commute pair-wise, thus,

$$WS = (W_1W_2\ldots W_m)(S_1S_2\ldots S_p) = (W_1W_2\ldots W_{m-1})(W_m S_1S_2\ldots S_p)$$

$$= (W_1W_2\ldots W_{m-1})(S_1S_2\ldots S_pW_m) = (W_1W_2\ldots W_{m-2})(W_{m-1} S_1S_2\ldots S_pW_m)$$

$$= (W_1W_2\ldots W_{m-2})(S_1S_2\ldots S_p W_{m-1} W_m) = \ldots = W_1 (S_1S_2\ldots S_p W_2\ldots W_{m-1}W_m)$$

$$= (S_1S_2\ldots S_p)(W_1W_2\ldots W_m) = SW.$$

Similarly, one can prove that $KF = FK$. Hence, the pair (W, S) and (K, F) are weakly compatible. Now, using Theorem 4.2.1, we conclude that W, S, K and F have a unique common fixed point in X, say r. Now, we establish that r remains the fixed point of all the component mappings. For this consider:

$$W(W_i(r)) = (W_1W_2\ldots W_m) W_i(r) = (W_1W_2\ldots W_{m-1})(W_m W_i)(r)$$

$$= (W_1W_2\ldots W_{m-1})(W_i W_m)(r) = (W_1W_2\ldots W_{m-2})(W_{m-1} W_i W_m)(r)$$

$= (W_1W_2\ldots W_{m-2})\ (W_iW_{m-1}W_m)\ (r) = \ldots\ldots = W_1\ (W_iW_2\ldots W_{m-1}\ W_m)\ (r)$

$= (W_1W_i)\ (W_2W_3\ldots W_m)\ (r) = (W_iW_1)\ (W_2W_3\ldots W_m)\ (r)$

$= W_i\ (W_1W_2\ldots W_m)\ (r) = W_i\ W(r) = W_i\ (r)$ in the same way, one can establish that

$W\ (S_k(r)) = S_k\ (W(r)) = S_k(r)$ and $S\ (S_k(r)) = S_k\ (S(r)) = S_k(r)$,

$S\ (W_i(r)) = W_i\ (S(r)) = Wi(r)$ and $K(K_z(r)) = K_z\ (K(r)) = k_z\ (r)$,

$K\ (F_t\ (r)) = F_t\ (K(r)) = F_t\ (r)$ and $F(F_t\ (r)) = F_t\ (F(r)) = F_t\ (r)$,

As well as $F(K_z(r)) = K_z\ (F(r)) = K_z(r))$.

Which explain that (for all i, r, k and t) $W_i(r)$ and $S_k(r)$, are other fixed point of the pair (W, S) whereas $k_z\ (r)$, and $F_t\ (r)$, are other fixed points of the pair (K, F). Since W, K, S and F have a unique common fixed point, so, we get: $r = W_i(r) = S_k(r) = k_z\ (r) = F_t\ (r)$, for every $i = 1, 2, 3,\ldots,m$, $k = 1, 2, 3,\ldots,p$, $r = 1, 2, 3,\ldots,n$, and $t = 1, 2, 3,\ldots,q$. This shows that r is a unique common fixed point of $\{A_i\}_{i=1}^{m}$, $\{S_k\}_{k=1}^{p}$, $\{K_z\}_{z=1}^{n}$, and $\{F_t\}_{t=1}^{q}$. Hence, the evidence of the Theorem 4.2.9 is established.

4.3 Some Common Fixed Poınt Theorems For *R*-Weakly Commuting Mappings in Fuzzy Metric Spaces

In this part we introduce some theorems relating to common fixed theorems for weakly compatible and $\mathbb{R}$-weakly commuting mappings and some of their variants in fuzzy metric spaces:

Theorem 4.3.1 Let $(X, M, \otimes)$ be complete fuzzy metric space. Let Γ and P be weakly compatible self-mappings of X satisfying:

1) $Q(P(x), P(y), F(r)) \geq Q(\Gamma(x), \Gamma(y), r)$ *for all* x, y$\in X$, $r > 0$ and $0 < F < 1$;

2) $P(X) \subseteq \Gamma(X)$.

If one of P(X) OR Γ(X) is complete, then Γ and P have a unique common fixed point.

Proof Let $x_0 \in X$. Since $\mathrm{P(X)} \subseteq \Gamma(\mathrm{X})$, chooses $\mathrm{x_1} \in \mathrm{X}$ such that $\mathrm{P(x_0)} = \big(\Gamma(\mathrm{x_1})\big)$. In general, choose $x_{n+1} \in X$ such that : $\mathrm{y_n} = \Gamma(\mathrm{x_{n+1}}) = \big(\mathrm{P(x_n)}\big)$. then by (1), we have:

$Q(\Gamma(\mathrm{x_1}), \Gamma(\mathrm{x_{n+1}}), r) = Q(\mathrm{P(x_{n-1})}, \mathrm{P(x_n)}, r) \geq Q\left\{\Gamma(\mathrm{x_{n-1}}), \Gamma(\mathrm{x_n}), \frac{r}{F}\right\} =$
$Q\left\{\mathrm{P(x_{n-2})}, \mathrm{P(x_{n-1})}, \frac{\mathrm{r}}{\mathrm{F}}\right\} \geq \cdots \geq Q\left\{\Gamma(\mathrm{x_0}), \Gamma(\mathrm{x_1}), \frac{r}{F^n}\right\}$. Therefore, for any p, we get:

$$Q\big(\Gamma(\mathrm{x_n}), \Gamma\big(\mathrm{x_{n+p}}\big), r\big) \geq Q\left(\Gamma(\mathrm{x_n}), \Gamma(\mathrm{x_{n+1}}), \frac{r}{p}\right) \geq \cdots (p - time)..$$
$$\geq Q\left\{\Gamma\big(\mathrm{x_{n+p-1}}\big), \Gamma\big(\mathrm{x_{n+p}}\big), \frac{r}{p}\right\} \geq Q\left\{\Gamma(\mathrm{x_0}), \Gamma(\mathrm{x_1}), \frac{\mathrm{r}}{\mathrm{pF^n}}\right\} \geq \cdots (p - time)$$
$$\geq Q\left\{\Gamma(\mathrm{x_0}), \Gamma(\mathrm{x_1}), \frac{r}{pF^{n+p-1}}\right\}.$$

As $n \to \infty$, $\{\Gamma(\mathrm{x_n})\} = \{y_n\}$ is cauchy sequence and so, by completeness of X, $\{\Gamma(\mathrm{x_n})\}$, is convergent sequence. Therefore, $\lim_{n\to\infty} \Gamma(x_n) = \lim_{n\to\infty} P(x_n) = z$ for some $z \in X$.

Since, $\Gamma(\mathrm{x_n})$ is complete, so there exist a point $p \in X$ such that $\Gamma(\mathrm{p}) = \mathrm{z}$. now, from (1), we get

$Q(P(\mathrm{p}), P(\mathrm{x_n}), F(r)) \geq \mathrm{Q}(\Gamma(\mathrm{p}), \Gamma(\mathrm{x_n}), \mathrm{r})$. Taking the limit as $n \to \infty$, we have

$Q(P(\mathrm{p}), z, F(r)) \geq Q(\Gamma(\mathrm{p}), z, r) = Q(z, z, r) = 1 \implies P(\mathrm{p}) = \mathrm{z} = \Gamma(\mathrm{p})$

As Γ and P are weakly compatible, therefore, $\Gamma\big(P(p)\big) = P(\Gamma(p))$ this means $\Gamma(z) = P(z)$.

We now, show that ***z*** is fixed point of Γ and P. From (1) we have

$Q(P(\mathrm{z}), P(\mathrm{x_n}), F(r)) \geq Q(\Gamma(\mathrm{z}), \Gamma(\mathrm{x_n}), r)$. Taking the limit as $n \to \infty$ we get:

$Q(P(\mathrm{z}), z, F(r)) \geq Q(\Gamma(\mathrm{z}), z, r)$ & $Q(\mathrm{P(z)}, \mathrm{z}, \mathrm{F(r)}) \geq \mathrm{Q(P(z), z, r)}$. Therefore, by Proposition 4.1.10, we get: P(z) = z = Γ(z).

Hence, z is a common fixed point of Γ and P. For uniqueness, let w be another fixed point of Γ and P, then by (1),

$Q(P(\mathrm{z}), P(\mathrm{w}), F(r)) \geq Q(\Gamma(\mathrm{z}), \Gamma(\mathrm{w}), r)$ or $Q(z, w, F(r)) \geq Q(z, w, r)$.

Therefore, by Proposition 4.1.10, we get: $z = w$. Therefore, $\boldsymbol{z}$ is unique common fixed point of Γ and P.

Remark 4.3.2 The results to the above theorem hold true even if we replace complete fuzzy metric space with fuzzy metric space.

Theorem 4.3.3 Let *(X, M, ⊗)* be a fuzzy metric space. Let Γ and P be weakly compatible self-mappings of X satisfying the following conditions:

1) Q(P(x), P(y), F(r)) ≥ Q($\Gamma(x), \Gamma(y), r$) *for all* $x, y \in X$, $r > 0$ and $0 < F < 1$;

2) P(X) ⊆ Γ(X) .

If one of $P(X)$ OR Γ(X) is complete, then Γ and P have a unique common fixed point.

Proof By using the proof of Theorem 4.3.1 we conclude that $\{\Gamma(\mathrm{y_n})\} = \{\mathrm{y_n}\}$ is Cauchy sequence in X. Now,. Let Γ (X) be a complete subspace of X, and then the sub-sequence of $\{\mathrm{y_n}\}$ must get a limit in Γ (X). Let it be u and $u \in \Gamma^{-1}(v)$then Γ *(v).* = u for some $v \in X$ .As $\{\mathrm{y_n}\}$ is a Cauchy sequence containing a convergent subsequence, therefore, the sequence $\{\mathrm{y_n}\}$ also converges. . Now, from a part **(1)** we get:

$Q(P(\mathrm{v}), P(\mathrm{x_n}), F(r)) \geq Q(\Gamma(\mathrm{v}), \Gamma(\mathrm{x_n}), r)$. Taking the limit as $n \to \infty$ we get:

$Q(P(v), u, F(r)) \geq Q(\Gamma(v), u, r) = Q(u, u, r) = 1$.

This gives, P(v) = u = Γ(v) which shows that pair (Γ, P) has a point of coincidence. Since, Γ and P are weakly compatible, therefore, $\Gamma\big(P(v)\big) = P(\Gamma(v))$ this mean $\Gamma(u) = P(u)$. Now, we show that u is a fixed point of Γ and P. From part **(1)**, we have

$Q(P(\mathrm{u}), P(\mathrm{x_n}), F(r)) \geq Q(\Gamma(\mathrm{v}), \Gamma(\mathrm{x_n}), r)$. Taking the limit as $n \to \infty$ we get:

$Q(P(\mathrm{u}), u, F(r)) \geq Q(\Gamma(\mathrm{u}), u, r)$ and $Q(P(\mathrm{u}), u, F(r)) \geq Q(P(\mathrm{u}), u, r)$. By using proposition (5, 1, 10) we get P(u) = u = $\Gamma(\mathrm{u})$. Hence, u is a fixed point of Γ and P.

For uniqueness, let w be another fixed point of Γ & P, then by part (1), we get:

$Q(P(\mathrm{z}), P(\mathrm{w}), F(r)) \geq Q(\Gamma(\mathrm{z}), \Gamma(\mathrm{w}), r)$ and $Q(z, w, F(r)) \geq Q(z, w, r)$ By Proposition 4.1.10 we get z = w. Therefore, z is unique common fixed point of Γ and P.

Theorem 4.3.4 The Theorems 4.3.1 and 4.3.3 stay true if weakly compatible self-mappings are replaced by any one of the following self-mappings:

1) ℝ-weakly commuting;
2) ℝ-weakly commuting of type (A_Γ);
3) ℝ-weakly commuting of type (A_P);
4) ℝ-weakly commuting of type (P);
5) Weakly commuting property.

Proof With respect to **(1)** using the hypothesis of Theorems 4.3.1 and 4.3.3, the proof for the existence of coincidence point follows the same method. So, let x be any arbitrary coincidence point for the pair (Γ, P), then by using R-weak commutativity, one gets

$$Q(\Gamma(P(x)), P(\Gamma(x)), r)) \geq Q\left(\Gamma(x), P(x), \frac{r}{R}\right) = Q\left(\Gamma(x), \Gamma(x), \frac{r}{\mathbb{R}}\right) = 1. \text{ Where } \mathbb{R} > 0.$$

This gives, $\Gamma\big(P(x)\big)$ = $P\big(\Gamma(x)\big)$ therefore, the pair (Γ, P), is weakly compatible. Now for **(2)** in case (Γ, P), is a ℝ-weakly commuting pair of type (A_Γ); after that:

$$Q(P(P(x)), \Gamma(P(x)), r)) \geq Q\left(P(x), \Gamma(x), \frac{r}{\mathbb{R}}\right) = Q\left(\Gamma(x), \Gamma(x), \frac{r}{\mathbb{R}}\right) = 1.$$

Which gives, $P\big(P(x)\big)$ = $\Gamma(P(x))$. Moreover,

$$Q(\Gamma(P(x)), P(\Gamma(x)), r)) \geq Q\left(\Gamma(P(x)), P(P(x)), \frac{r}{2}\right) \otimes Q\left(P(P(x)), P(\Gamma(x)), \frac{r}{2}\right)$$
$$= 1 \otimes 1 = 1$$

Which gives, $\Gamma\big(P(x)\big) = P\big(\Gamma(x)\big)$. Therefore, the pair ($\Gamma$, P), is weakly compatible.

Now With respect to **(3)** in case (Γ, P), is a $\mathbb{R}$-weakly commuting pair of type (A_P); Therefore:

$$Q(\Gamma(\Gamma(x)), P(\Gamma(x)), r)) \geq Q\left(\Gamma(x), P(x), \frac{r}{\mathbb{R}}\right) = Q\left(\Gamma(x), \Gamma(x), \frac{r}{\mathbb{R}}\right) = 1.$$

Which gives, $P\big(\Gamma(x)\big) = \Gamma\big(\Gamma(x)\big)$. Also,

$$Q(\Gamma(P(x)), P(\Gamma(x)), r)) \geq Q\left(\Gamma(P(x)), \Gamma(\Gamma(x)), \frac{r}{2}\right) \otimes Q\left(\Gamma(\Gamma(x)), P(\Gamma(x)), \frac{r}{2}\right)$$
$$= 1 \otimes 1 = 1$$

Which gives, $\Gamma\big(P(x)\big) = P\big(\Gamma(x)\big)$. Therefore, the pair ($\Gamma$, P), is weakly compatible.

Similarly, if pair (Γ, P) is $\mathbb{R}$-weakly commuting of type (P) or weakly commuting property, then (Γ, P) furthermore commutes at their point of coincidence. Hence Γ and P have a unique common fixed point in all the five cases. This proves the declaration.

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Date of Birth and place: 7-9-1969 Bagdad-Iraq

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Education

Degree	Institute	Graduation Date
Master	Erciyes University , Turkey	2017-2018
Bachelor	Ibn Al-Haitham University, Bagdad	2010-2012
High School	Al NahDaa' School - Bagdad	1989-1990

Work Experience

Year	Institute	Duty
1998 – up to date	Ministry of Education	Mathematics teacher
	-Experience in teaching in secondary schools for 20 years. -Experience in high school administration for several years -Training courses for mathematics	

Language

Arabic ,English, Turkish