# Solution of the Dirac Equation in the Non-Asymptotically Flat Geometries

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# ABSTRACT

This thesis is devoted to the solution of the Dirac equation in the two non-asymptotically flat geometries; the Bertotti-Robinson (BR) geometry and the near horizon (NH) geometry of an extreme Kerr black hole (BH), respectively. While the Dirac equation with charge coupling is considered in the uniform electromagnetic field space of BR, in the NH geometry, the test Dirac i.e., Chandrasekhar-Dirac (CD) equation is used. The methods of separation of variables and decoupling are achieved for each geometry. In the BR geometry, the solution of angular equation is given both in terms of Jacobi polynomials and of spin-weighted spheroidal harmonics. The separated axial equation is reduced to a rare Riccati type of differential equation. In the neutrino (massless and chargeless particles) case, it is shown that the axial equation takes the from of a hypergeometric differential equation. Unlike the BR geometry, the angular equation depends on the mass, while the axial equation turns out to be independent of the mass and is exactly solved in the NH geometry. The angular equation, in the neutrino case, reduces to a confluent Heun equation. In general for a massive case, the angular equation is expressible at best, as a set of coupled first order differential equations apt for numerical investigation. For each geometry, the axial potentials corresponding to the associated Schrödinger-type wave equations and their conserved currents are found. The thesis concludes with a verification of the absence of superradiance for Dirac particles in the NH geometry, a result which is well known within the context of general Kerr background.

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# Chapter 1

# Introduction

The Dirac equation is a relativistic quantum mechanical wave equation invented by Paul Dirac in 1928. It provides a description of spin$-\frac{1}{2}$ particles (fermions), such as the electron, proton, and neutron that is fully consistent with the principles of quantum mechanics and largely consistent with the theory of general relativity. It also accounts in a natural way for the nature of particle spin and the existence of antiparticles.

Since the Dirac equation was originally invented to describe the charged spin$-\frac{1}{2}$ particles, we will generally speak of "charged Dirac particles" in the following thesis. If chrage is present in spacetime (charged spacetime background), charge-charge interaction will be important for a charged Dirac particle. Naturally, the Dirac equation describing the behavior of the charged particle will be modified. The Dirac equation applies to uncharged spin$-\frac{1}{2}$ point particles too, such as the neutron (or "test Dirac particle") and the neutrino (Dirac particle which is massless and chargeless).

Pioneering work in separating the Dirac equation in the spacetime around a Kerr BH into axial (radial) and angular equations was done by Chandrasekhar in 1976 [1]. After this work, the study of spin$-\frac{1}{2}$ particles on type-D spacetimes has thus far attracted much interest and a large number of results have accumulated in the literature [2-8] (and references cited therein). The main reason for this is that all well known black holes (BHs) are in this category and their better understanding involves detailed analysis of various physical fields in their vicinity. Dirac particles constitute one such potential candidate whose interaction and behavior around BHs may reveal information of much significance. After Chandrasekhar, progress was made to analyzing the Dirac equation in non-asymptotically flat spacetimes. Very recently, the Dirac equation in a non-asymptotically flat BH solution, the Kerr-NUT geometry, was studied [9].

Studies of the Dirac equation in spacetimes other than BHs have also started to arouse interest for various reasons. From this token, we cite studies for a test Dirac particle in the Bell-Szekeres (which is equivalent to BR geometry) , Robertson-Walker and BR (due to cosmological constant) geometries [10-12].

In this thesis, we firstly study the Dirac equation with charge coupling in the BR geometry [13-15]. The BR geometry is a non-asymptotically flat solution of the Einstein-Maxwell equations representing a geometry of space filled by a uniform electromagnetic radiation. Using some special choices of coordinate transformation it can be made to look like either a pure electric or pure magnetic field. Here, we shall consider a BR geometry having a pure electric field. From a geometrical point of view, the BR geometry represents a throat region of the extreme Reissner-Nordström BH. Until now, the BR geometry gained much recognition in connection with extremal BHs, colliding electromagnetic waves, higher dimensions, holography and the brane world [16-23]. Also in the literature, many applications can be found on the BR geometry [24 and references therein].

Secondly, we consider the CD equation in the NH geometry of an extreme Kerr BH. This geometry represents the most important region of the outerworld prior to the horizon of a Kerr BH enhanced with the extremality condition. Extremal BHs are believed to have connection with the ground states of quantum gravity. This alone justifies, in spite of the absence of backreaction effects, the importance of spin$-\frac{1}{2}$ particles in such backgrounds. Recently, the behavior of massless scalar fields in the extreme Kerr throat have been considered, and it is found that certain modes with large azimuthal quantum number exhibit superradiance [25]. This implies that the geodesics near the horizon can escape to infinity carrying away energy-momentum more than the amount that infalls. Our solution enables us to investigate a similar phenomenon with Dirac fields which turns out to be negative as far as superradiance is concerned. This result is in accordance with the treatment of the Dirac equation in the general background in the absence of an exact solution [2]. The analysis of fermions in Kerr-Newman background and absence of superradiance was shown first by Lee [3]. Our aim is to revisit this matter – not just separating equations and deducing results on general grounds – so as to obtain and employ exact solutions. The advantage of confining ourselves to the NH alone is that it allows us to express solutions with the known polynomials. In the general, for the Kerr family of BHs even this much remains a problem beyond technical reach. Meanwhile, it is important to note that in contrast to the general Kerr metric, the throat metric is not asymptotically flat. This difference naturally shows itself in the potentials too. In other words, the extreme Kerr throat metric represents such a local region that the geometry of interest is different than the geometry of the general Kerr metric. Hence, we can mentally say that the behavior of particles in the two geometries will not be the same. There seems to be no such study in the literature, so we think it is useful to explore it.

This thesis is organized as follows. This first chapter introduces the historical developments of the Dirac equation, the features of the BR and NH geometries and the difference of our study from the others. Chapter two is devoted to the mathematical formulation of the Dirac equation with charge coupling and consequently its test form, the CD equation, by using the null tetrad formalism i.e. the Newman-Penrose (NP) formalism [26]. The third chapter introduces the spacetime of the BR geometry. We separate the Dirac equation, with charge coupling, into axial and angular equations. The possible solutions of the angular equation are given. Both in general and neutrino cases, we focus on the reduction and solution of the axial part of the Dirac equation. We also study the behavior of the potentials and the large distance solutions of the axial equation. We conclude the chapter by presenting the conserved currents of the Dirac equation. In chapter four, we shall review the NH geometry of an extreme Kerr BH. We employ the method of separation of variables of the CD equation. We show that the axial equation is analytically solvable. The massless and massive cases of the angular equation are studied. The reduction of the axial equation to one-dimensional Schrödinger-type wave equations with their conserved currents and the analysis of the superradiance are all included in this chapter. The thesis ends with conclusions and discussion in chapter five.

# Chapter 2

# Formulation of Dirac Equation

## 2.1 Introduction

In General Relativity, many tools have been developed to study the Dirac equation in curved spacetimes and two of them are the NP and spinor formalisms. In this chapter we briefly review the Dirac equation, mainly following Chandrasekhar's monograph [2].

## 2.2 Dirac Equation

As is well known, the most useful way of writing the Dirac equation is in the framework of the spinor formalism[1]. In the relativistic theory of spin$-\frac{1}{2}$ particles, the wave function is represented by a pair of spinors, $\phi_B$ and $\chi_{B'}$; and in the spinor formalism, the Dirac equation for a charged Dirac particle in a curved spacetime background with a background electromagnetic field has the form [4,27],

$$\left(\nabla_B{}^{B'} + iq\mathcal{A}_B{}^{B'}\right)\chi_{B'} = -i\frac{\mu}{\sqrt{2}}\phi_B$$

[1] One can use Chandrasekhar's monograph [2] to get a brief account of spinor analysis and the spinorial basis of the NP formalism. In the spinor formalism, while all lowercase letters denote the spacetime indices, the spinor indices are denoted with capital latin letters, with primed ones indicating that complex conjugate transformations are to be applied. Furthermore, the Levi-Civita symbols ($\varepsilon' s$) are used to lower and raise spinor indices.

$$\left(\nabla^{B}{}_{B'} + iq\mathcal{A}^{B}{}_{B'}\right)\phi_B = i\frac{\mu}{\sqrt{2}}\chi_{B'} \tag{2.1}$$

where $\mathcal{A}_{AB'}$ is the vector potential of the background electromagnetic field, $\chi_{B'}$ and $\phi_B$ are the pair of spinors and $\nabla_{AB'}$ is the standard operator of the covariant derivative of spinor fields [9]. $\mu = \sqrt{2}\mu^*$ and $q$ are the mass and the charge of the charged Dirac particle, respectively. In the NP formalism, using the spin coefficients, Dirac equations have the form

$$\left(\overline{\delta} + iq\mathcal{A}_{10'} + \pi - \alpha\right)\phi_0 - \left(D + iq\mathcal{A}_{00'} + \varepsilon - \rho\right)\phi_1 = i\mu^*\chi_{0'}$$

$$\left(\Delta + iq\mathcal{A}_{11'} + \mu - \gamma\right)\phi_0 - \left(\delta + iq\mathcal{A}_{01'} + \beta - \tau\right)\phi_1 = i\mu^*\chi_{1'}$$

$$\left(D + iq\mathcal{A}_{00'} + \overline{\varepsilon} - \overline{\rho}\right)\chi_{1'} - \left(\delta + iq\mathcal{A}_{01'} + \overline{\pi} - \overline{\alpha}\right)\chi_{0'} = i\mu^*\phi_0$$

$$\left(\overline{\delta} + iq\mathcal{A}_{10'} + \overline{\beta} - \overline{\tau}\right)\chi_{1'} - \left(\Delta + iq\mathcal{A}_{11'} + \overline{\mu} - \overline{\gamma}\right)\chi_{0'} = i\mu^*\phi_1 \tag{2.2}$$

where

$$D = l^j\nabla_j \qquad \Delta = n^j\nabla_j \qquad \delta = m^j\nabla_j \tag{2.3}$$

are the directional derivatives for NP tetrads.

In spinor analysis, any vector $X_{AB'} = \sigma^j_{AB'}X_j$ ; $A, B = 0, 1$ . Here, the Pauli matrices[2] are defined as

$$\sigma^j_{AB'} = \begin{pmatrix} l^j & m^j \\ \overline{m}^j & n^j \end{pmatrix} \tag{2.4}$$

Representing the Dirac spinors by

[2]Due to the factor $\sqrt{2}$ in the definition of the mass,$\mu$, the Pauli matrices differ from their customary definitions as defined in [2] by the lack of the normalization factor $\frac{1}{\sqrt{2}}$.

$$F_1 = -\phi_1, \quad F_2 = \phi_0, \quad G_1 = \chi_{0'}, \quad G_2 = \chi_{1'} \tag{2.5}$$

we obtain [9,27],

$$\left[D + iql^j\mathcal{A}_j + \varepsilon - \rho\right] F_1 + \left[\overline{\delta} + iq\overline{m}^j\mathcal{A}_j + \pi - \alpha\right] F_2 = i\mu^* G_1$$

$$\left[\delta + iqm^j\mathcal{A}_j \;\; + \beta - \tau\right] F_1 + \left[\Delta + iqn^j\mathcal{A}_j \;\; + \mu - \gamma\right] F_2 = i\mu^* G_2$$

$$\left[D + iql^j\mathcal{A}_j + \overline{\varepsilon} - \overline{\rho}\right] G_2 - \left[\delta + iqm^j\mathcal{A}_j + \overline{\pi} - \overline{\alpha}\right] G_1 = i\mu^* F_2$$

$$\left[\Delta + iqn^j\mathcal{A}_j \;\; + \overline{\mu} - \overline{\gamma}\right] G_1 - \left[\overline{\delta} + iq\overline{m}^j\mathcal{A}_j + \overline{\beta} - \overline{\tau}\right] G_2 = i\mu^* F_1 \tag{2.6}$$

These are the Dirac equations with charge coupling in the NP formalism. For the case of the test Dirac particles, these equations reduce to the well known CD equations [2]

$$\left[D + \varepsilon - \rho\right] F_1 + \left[\overline{\delta} + \pi - \alpha\right] F_2 = i\mu^* G_1$$

$$\left[\delta + \beta - \tau\right] F_1 + \left[\Delta + \mu - \gamma\right] F_2 = i\mu^* G_2$$

$$\left[D + \overline{\varepsilon} - \overline{\rho}\right] G_2 - \left[\delta + \overline{\pi} - \overline{\alpha}\right] G_1 = i\mu^* F_2$$

$$\left[\Delta + \overline{\mu} - \overline{\gamma}\right] G_1 - \left[\overline{\delta} + \overline{\beta} - \overline{\tau}\right] G_2 = i\mu^* F_1 \tag{2.7}$$

# Chapter 3

# Solution of the Dirac Equation in the BR Geometry

## 3.1 Introduction

The aim of the present chapter[1] is to study the Dirac equation with charge coupling, in the BR spacetime. It is an Einstein-Maxwell solution which resembles the geometry of $AdS_2 \times S^2$. The anti-de Sitter (AdS) structure and its correspondence with conformal field theory has been a fashionable topic in recent years. This spacetime has the topology of $R^2 \times S^2$ and the underlying group structure of $SL(2, R) \times SO(3)$. It is conformally flat, but asymptotically non-flat.

In the next section, we first represent the line element of the BR geometry and discuss the geodesics of radially moving particles. Then, we separate the Dirac equation with charge coupling on the BR geometry. In Sec. 3.2, we give explicitly the solutions of the angular part. In Sec. 3.3, we show that the axial part of the Dirac equation reduces to a Riccati type differential equation. Furthermore, the exact solution of the axial part is found in the case of a neutrino. Next, we study the behavior of the potentials and the asymptotic solutions of the axial equation. Finally, in Sec. 3.4, we present the conserved currents of the Dirac equation.

[1] The material of this chapter was published in [28]

## 3.2 BR Geometry and Separation of Dirac Equation

Our line-element describing BR spacetime is given by

$$dS^2 = \left(1 + z^2\right) dt^2 - \frac{dz^2}{1 + z^2} - \left(d\theta^2 + \sin^2\theta d\varphi^2\right) \tag{3.1}$$

The transformation which yields the standard BR line-element [14-15] as a solution of Einstein-Maxwell equation, is

$$z = \frac{1}{2r}\left(\bar{t}^2 - r^2 + 1\right) \tag{3.2}$$

and

$$t = \tan^{-1}\left[\frac{1}{2\bar{t}}\left(\bar{t}^2 - r^2 - 1\right)\right] \tag{3.3}$$

Since we are mostly interested in the geodesics of the axially moving particles, it suffices to consider the geodesics with constant $\theta$ and $\phi$. The effective Lagrangian is then

$$L_{eff} = \frac{1}{2}\left[\left(1 + z^2\right)\dot{t}^2 - \frac{\dot{z}^2}{\left(1 + z^2\right)}\right] \tag{3.4}$$

where a dot over a quantity denotes a derivative with respect to an affine parameter, $\tau$. Since $t$ is a cyclic coordinate, $P_t$ (canonical momentum) will be a constant of motion.

$$P_t = \left(1 + z^2\right)\dot{t} \tag{3.5}$$

Together with the 4-velocity identity $u_\mu u^\mu = \epsilon$, where $\epsilon = 1, 0, -1$ for timelike,null and spacelike geodesics, respectively, it yields the axial differential equation

$$\dot{z}^2 + \epsilon z^2 = P_t^2 - \epsilon \tag{3.6}$$

This equation can easily be integrated

$$\epsilon = 1 : \qquad z = \pm\sqrt{P_t^2 + 1}\,\sin(\tau - \tau_0)$$

$$\epsilon = 0 : \qquad z = \pm\left|P_t\right|(\tau - \tau_0) \tag{3.7}$$

$$\epsilon = -1 : \qquad z = \pm\sqrt{P_t^2 - 1}\,\sinh(\tau - \tau_0)$$

The solution of the spacelike case tells us that spatial infinity is at infinite spatial distance; i.e. $\tau \to \infty$ as $z \to \infty$ . Massive particles have $\epsilon = 1$ (timelike) and will recollapse after a time $\tau - \tau_0 = \pi$. However, photons will reach spatial infinity $z \to \infty$ .

In terms of the coordinate time $t$, from Eq.(3.5), one can find for light

$$t = t_0 + \tan^{-1}\left[\left|P_t\right|(\tau - \tau_0)\right] \tag{3.8}$$

Since $t - t_0 < \frac{\pi}{2}$, photons reach spatial infinity within finite coordinate time. As we follow a photon, it will reach infinity within a finite coordinate time but infinite affine parameter. This shows that the BR solution is geodesically complete, and has a timelike boundary.

The basis vectors of the null tetrad in terms of elements of the BR geometry are chosen as

$$l_j = \frac{1}{\sqrt{2}}\left[\sqrt{1+z^2}, -\frac{1}{\sqrt{1+z^2}}, 0, 0\right]$$

$$n_j = \frac{1}{\sqrt{2}}\left[\sqrt{1+z^2}, \frac{1}{\sqrt{1+z^2}}, 0, 0\right]$$

$$m_j = \frac{1}{\sqrt{2}}\left[0, 0, 1, i\sin\theta\right]$$

$$\overline{m}_j = \frac{1}{\sqrt{2}}\left[0, 0, 1, -i\sin\theta\right] \tag{3.9}$$

so that the corresponding nonzero spin coefficients are

$$\epsilon = \gamma = \frac{z}{2\sqrt{2(1+z^2)}}, \qquad \alpha = -\beta = \frac{\cot\theta}{2\sqrt{2}} \tag{3.10}$$

Its uniform electromagnetic character can be deduced from the only non-vanishing Maxwell scalar,

$$\Phi_{11} = \frac{1}{2} \tag{3.11}$$

The form of the Dirac equations suggests that we introduce [2,28],

$$F_1 = f_1(z)\, A_1(\theta)\, e^{i(kt+m\varphi)}$$

$$G_1 = g_1(z)\, A_2(\theta)\, e^{i(kt+m\varphi)}$$

$$F_2 = f_2(z)\, A_3(\theta)\, e^{i(kt+m\varphi)}$$

$$G_2 = g_2(z)\, A_4(\theta)\, e^{i(kt+m\varphi)} \tag{3.12}$$

where $k$ is the frequency of the incoming wave corresponding to the Dirac particle – it is assumed to be a positive and real constant due to its relation with the energy of the incoming wave – and $m$ is the azimuthal quantum number. With this choice one can see that the only difference among the spinors is in the axial and angular dependence.

Throughout the chapter, we use the simplest case (scalar potential) for the 4-vector potential so

$$\mathcal{A}_j = \left[\sqrt{2}z, 0, 0, 0\right] \tag{3.13}$$

Substituting Eq.(3.12) into Eq.(2.6) and using spin coefficients (3.10), we can rewrite the Dirac equation as

$$\frac{1}{f_2}\widetilde{Z}f_1 - i\mu\frac{g_1}{f_2}\frac{A_2}{A_1} - \frac{(LA_3)}{A_1} = 0$$

$$-\frac{1}{f_1}\overline{\widetilde{Z}}f_2 - i\mu\frac{g_2}{f_1}\frac{A_4}{A_3} - \frac{(L^+A_1)}{A_3} = 0$$

$$\frac{1}{g_1}\widetilde{Z}g_2 - i\mu\frac{f_2}{g_1}\frac{A_3}{A_4} + \frac{(L^+A_2)}{A_4} = 0$$

$$-\frac{1}{g_2}\overline{\widetilde{Z}}g_1 + i\mu\frac{f_1}{g_2}\frac{A_1}{A_2} - \frac{(LA_4)}{A_2} = 0 \tag{3.14}$$

where

$$\widetilde{Z} = \sqrt{1+z^2}\partial_z + \frac{1}{2\sqrt{1+z^2}}\left[z + 2i\left(k + q^*z\right)\right]$$

$$\overline{\widetilde{Z}} = \sqrt{1+z^2}\partial_z + \frac{1}{2\sqrt{1+z^2}}\left[z - 2i\left(k + q^*z\right)\right]$$

$$L = \partial_\theta + \frac{m}{\sin\theta} + \frac{\cot\theta}{2}$$

$$L^\dagger = \partial_\theta - \frac{m}{\sin\theta} + \frac{\cot\theta}{2} \tag{3.15}$$

and

$$q^* = \sqrt{2}q \tag{3.16}$$

It is obvious that $L$ and $L^\dagger$ are purely angular operators.

Since $\{f_1, f_2, g_1, g_2\}$ and $\{A_1, A_2, A_3, A_4\}$ are functions of two different variables, we introduce a separation constant $\lambda$. Next, to write Eq.(3.14) in compact form, one can assume that

$$f_1 = g_2, \quad f_2 = g_1, \quad A_1 = A_2, \quad A_3 = A_4 \tag{3.17}$$

Thus, we get two sets of equations

$$\overline{\widetilde{Z}} g_1 = -(\lambda + i\mu)\, g_2 \tag{3.18}$$

$$\widetilde{Z} g_2 = -(\lambda - i\mu)\, g_1 \tag{3.19}$$

and

$$L^{\dagger} A_1 = \lambda A_3 \tag{3.20}$$

$$L A_3 = -\lambda A_1 \tag{3.21}$$

In principle, one should solve the angular equation with the eigenvalue $\lambda$ which is the separation constant of the complete Dirac equation. Then inserting the value of $\lambda$ into the axial equation, one should solve the axial equation.

## 3.3 Solution of the Angular Dirac Equation

The solutions of the eigenvalue problem for $A_1$ and $A_3$ arising from Eqs.(3.20), (3.21) can be treated in three cases.

When eigenvalue $\lambda$ is different from zero, according to the value chosen for $m$, the solutions are found in terms of Jacobi polynomials [11, 29] and spin-weighted spheroidal harmonics [30–33].

*I. Solution of the angular equations in terms of Jacobi polynomials.*

i) if $m = 1, 2, 3, 4...$

$$A_1 = c_1 \left(\sin\frac{\theta}{2}\right)^{m-\frac{1}{2}} \left(\cos\frac{\theta}{2}\right)^{m+\frac{1}{2}} J_{l-m}\left(2m+1, m+\frac{1}{2}; \frac{1}{2}(1-\cos\theta)\right)$$

$$A_3 = c_2 \left(\sin\frac{\theta}{2}\right)^{m+\frac{1}{2}} \left(\cos\frac{\theta}{2}\right)^{m-\frac{1}{2}} J_{l-m}\left(2m+1, m+\frac{3}{2}; \frac{1}{2}(1-\cos\theta)\right) \quad (3.22)$$

ii) if $m = -1, -2, -3, -4...$

$$A_1 = c_3 \left(\sin\frac{\theta}{2}\right)^{-m+\frac{1}{2}} \left(\cos\frac{\theta}{2}\right)^{-m-\frac{1}{2}} J_{l+m}\left(-2m+1, -m+\frac{3}{2}; \frac{1}{2}(1-\cos\theta)\right)$$

$$A_3 = c_4 \left(\sin\frac{\theta}{2}\right)^{-m-\frac{1}{2}} \left(\cos\frac{\theta}{2}\right)^{-m+\frac{1}{2}} J_{l+m}\left(-2m+1, -m+\frac{1}{2}; \frac{1}{2}(1-\cos\theta)\right) \quad (3.23)$$

where $J_{l\pm m}$ represents the Jacobi polynomails.

For the above subcases the eigenvalue solutions are,

$$\lambda^2 = \left(l+\frac{1}{2}\right)^2 \qquad (l = |m|, |m|+1, |m|+2, .....) \quad (3.24)$$

iii) if $m = 0$

$$A_1 = c_5 \left(\sin\frac{\theta}{2}\right)^{\frac{1}{2}} \left(\cos\frac{\theta}{2}\right)^{\frac{1}{2}} J_n\left(2, \frac{3}{2}; \frac{1}{2}(1-\cos\theta)\right)$$

$$A_3 = c_6 \left(\sin\frac{\theta}{2}\right)^{\frac{1}{2}} \left(\cos\frac{\theta}{2}\right)^{\frac{1}{2}} J_n\left(2, \frac{3}{2}; \frac{1}{2}(1-\cos\theta)\right) \quad (3.25)$$

where

$$\lambda^2 = (n+1)^2 \qquad (n = 0, 1, 2, 3, ...) \quad (3.26)$$

*II. When the eigenvalue $\lambda$ is zero with $0 < \theta < \pi$, we get the following solutions*

$$A_1 = c_7 (1-\cos\theta)^m (\sin\theta)^{-m-\frac{1}{2}}$$

$$A_3 = c_8\left(1-\cos\theta\right)^{-m}\left(\sin\theta\right)^{m-\frac{1}{2}} \tag{3.27}$$

where $c_1, c_2, c_3, c_4, c_5, c_6, c_7$ and $c_8$ are arbitrary constants.

*III. Solution of the angular equation in terms of spin-weighted spheroidal harmonics.*

It was found earlier that in the spherical case the laddering operators ($L$ and $L^{\dagger}$) [30] govern the spin-weighted spheroidal harmonics as

$$\left(\partial_\theta - \frac{m}{\sin\theta} - s\cot\theta\right)\left({}_sY_{l'}^{m}(\theta)\right) = -\sqrt{\left(l'-s\right)\left(l'+s+1\right)}\,{}_{s+1}Y_{l'}^{m}\left(\theta\right)$$

$$\left(\partial_\theta + \frac{m}{\sin\theta} + s\cot\theta\right)\left({}_sY_{l'}^{m}(\theta)\right) = \sqrt{\left(l'+s\right)\left(l'-s+1\right)}\,{}_{s-1}Y_{l'}^{m}\left(\theta\right) \tag{3.28}$$

where the eigenfunctions ${}_sY_{l'}^{m}(\theta)$, called the spin-weighted spheroidal harmonics, are complete and orthogonal and have an explicit form

$${}_sY_{l'}^{m}(\theta,\varphi) = e^{im\varphi}\sqrt{\frac{2l'+1}{4\pi}\frac{(l'+m)!\,(l'-m)!}{(l'+s)!\,(l'-s)!}}\left(\sin\frac{\theta}{2}\right)^{2l'}$$

$$\times\sum_{r=-l'}^{l'}(-1)^{l'+m-r}\binom{l'-s}{r-s}\binom{l'+s}{r-m}\left(\cot\frac{\theta}{2}\right)^{2r-m-s} \tag{3.29}$$

and $s$ is identified as the spin-weight with $l' = |s|, |s|+1, |s|+2....$ and $-l' < m < +l'$. Comparison with Eqs.(3.20), (3.21) leads us to identify

$$A_1 = {}_{-\frac{1}{2}}Y_{l'}^{m}; \qquad A_3 = {}_{\frac{1}{2}}Y_{l'}^{m} \tag{3.30}$$

and the eigenvalue is

$$\lambda = -\left(l' + \frac{1}{2}\right) \tag{3.31}$$

## 3.4 Reduction of the Axial Dirac Equation Into a Riccati-Type Differential Equation

On the right-hand sides of Eqs.(3.18), (3.19), we have similar factors which can be eliminated by some appropriate substitutions. Now changing first the independent variable to

$$z = \sinh x \tag{3.32}$$

and then changing the dependent variables as

$$g_1 = fe^{-K} + ige^{-K}$$

$$g_2 = fe^{-K} - ige^{-K} \tag{3.33}$$

where

$$K = \mu x + \frac{1}{2}\ln(\cosh x) \tag{3.34}$$

We obtain [2], by substituting these changes into Eqs.(3.18), (3.19),

$$f_{,x} + (\lambda - \mu) f = -(\mu + X) g \tag{3.35}$$

$$g_{,x} - (\lambda + \mu) g = -(\mu - X) f \tag{3.36}$$

in which

[2] Throughout this chapter, a subscript of a quantity, after the comma, denotes a derivative with respect to it.

$$X = \frac{k + q^* \sinh x}{\cosh x} \tag{3.37}$$

We shall decouple Eqs.(3.35), (3.36) to a form of first order differential equation. First, replacing $f$ and $g$ by

$$f = e^{2\rho} \cos \psi$$

$$g = e^{2\rho} \sin \psi \tag{3.38}$$

we obtain the decoupled equation

$$\psi_{,x} - \lambda \sin 2\psi + \mu \cos 2\psi = X \tag{3.39}$$

This can equivalently be written in the form

$$\psi_{,x} - a \sin(2\psi - \alpha) = X \tag{3.40}$$

where

$$a = \sqrt{\mu^2 + \lambda^2} \qquad \text{and} \qquad \alpha = \tan^{-1} \frac{\mu}{\lambda} \tag{3.41}$$

Now letting

$$\psi = \frac{2t + \alpha + \frac{\pi}{2}}{2} \tag{3.42}$$

we find that with the substitution of Eq.(3.42), Eq.(3.40) reduces to

$$t_{,x} - a \cos 2t = X \tag{3.43}$$

Furthermore, by the substitution

$$t = \tan^{-1} y \tag{3.44}$$

the equation becomes

$$y_{,x} + (a - X)\, y^2 - (a + X) = 0 \tag{3.45}$$

One can easily see that Eq.(3.43) is reduced to a standard form of Riccati equation. Unfortunately, Eq.(3.45) is such a rare Riccati type that its solution (if there is any at all) is unknown. Essentially, if one finds a solution for Eq.(3.43) or Eq.(3.45), Dirac equation will be determined completely.

## 3.5 Solution of the Axial Equation in the Neutrino Case

The equations appropriate to two-component neutrinos (satisfying the Weyl form of the massless Dirac equation) can be obtained by simply setting $\mu = q = 0$ in Eqs.(3.18), (3.19). Starting from the new definitions of the functions with the same substitution (3.32)

$$g_1 = \frac{\widetilde{a}}{\sqrt{\cosh x}} + i\frac{\widetilde{b}}{\sqrt{\cosh x}}$$

$$g_2 = \frac{\widetilde{a}}{\sqrt{\cosh x}} - i\frac{\widetilde{b}}{\sqrt{\cosh x}} \tag{3.46}$$

Eqs. (3.18), (3.19) take the simple form

$$\widetilde{a}_{,x} + \lambda\widetilde{a} = -\frac{k}{\cosh x}\widetilde{b}$$

$$\widetilde{b}_{,x} - \lambda\widetilde{b} = \frac{k}{\cosh x}\widetilde{a} \tag{3.47}$$

Also by letting

$$\upsilon = 2\tan^{-1} e^{x} \tag{3.48}$$

the foregoing equations can be brought to the forms

$$\widetilde{a}_{,\upsilon} + \frac{\lambda}{\sin \upsilon}\widetilde{a} = -k\widetilde{b} \tag{3.49}$$

$$\widetilde{b}_{,\upsilon} - \frac{\lambda}{\sin \upsilon}\widetilde{b} = k\widetilde{a} \tag{3.50}$$

which allow us to obtain a pair of the one-dimensional Schrödinger-type wave equations with non-trivial potentials [2]

$$\widetilde{a}_{,\upsilon\upsilon} + k^2\widetilde{a} = V_1\widetilde{a} \tag{3.51}$$

$$\widetilde{b}_{,\upsilon\upsilon} + k^2\widetilde{b} = V_2\widetilde{b} \tag{3.52}$$

where the potentials of the BR geometry for the incidence of neutrino waves

$$V_1 = \lambda \cosh x \left(\lambda \cosh x - \sinh x\right)$$

$$V_2 = \lambda \cosh x \left(\lambda \cosh x + \sinh x\right) \tag{3.53}$$

As can be seen from Fig.3.1, the potentials have a structure similar to an infinite potential well; a fundamental problem in standard quantum mechanics. So we deduce that the waves should vanish at the boundaries ($x \to \pm\infty$) of the geometry. Meanwhile, the reason of why the potentials do not die off while $x \to \pm\infty$ is the non-asymptotic structure of the BR spacetime.

Next, by introducing the following scalings

$$\widetilde{a} = \widetilde{c}\left(\frac{\sin \upsilon}{1 + \cos \upsilon}\right)^{-\lambda}$$

$$\widetilde{b} = \widetilde{d}\left(\frac{\sin \upsilon}{1 + \cos \upsilon}\right)^{\lambda} \tag{3.54}$$

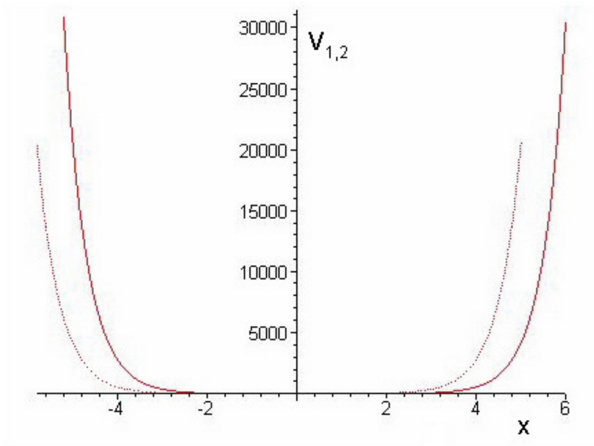


Figure 3-1: The potential barriers, $V_1$ (the full-line curves) and $V_2$ (the dotted curves), of a BR spacetime for the incidence of neutrino waves with $\lambda = \frac{3}{2}$ and $k = 1$.

we obtain, after some reductions,

$$\widetilde{c} = \frac{\widetilde{d}_{,\upsilon}}{k}\left(\frac{1-e^{\upsilon}}{1+e^{\upsilon}}\right)^{2\lambda} \tag{3.55}$$

$$\widetilde{d} = -\frac{\widetilde{c}_{,\upsilon}}{k}\left(\frac{1-e^{\upsilon}}{1+e^{\upsilon}}\right)^{-2\lambda} \tag{3.56}$$

If we decouple Eqs.(3.55), (3.56) for $\widetilde{c}$ and $\widetilde{d}$ , respectively, we are finally left with the following second order differential equations

$$\widetilde{c}_{,\upsilon\upsilon} - \frac{2\lambda}{\sin\upsilon}\widetilde{c}_{,\upsilon} + k^2\widetilde{c} = 0 \tag{3.57}$$

$$\widetilde{d}_{,\upsilon\upsilon} + \frac{2\lambda}{\sin\upsilon}\widetilde{d}_{,\upsilon} + k^2\widetilde{d} = 0 \tag{3.58}$$

Now defining a new variable $\upsilon = \cos^{-1}(1-2\eta)$ , the foregoing equations reduce to the hypergeometric differential equations [34, 29]

$$\left(\eta^2 - \eta\right)\widetilde{c}_{,\eta\eta} + \left(\eta + \lambda - \frac{1}{2}\right)\widetilde{c}_{,\eta} - k^2\widetilde{c} = 0 \tag{3.59}$$

$$\left(\eta^2-\eta\right)\widetilde{d}_{,\eta\eta}+\left(\eta-\lambda-\frac{1}{2}\right)\widetilde{d}_{,\eta}-k^2\widetilde{d}=0 \tag{3.60}$$

which admit the following physically acceptable solutions (namely solutions generating a $g_1$ (or $g_2$) solution which vanishes for $\eta=0$ and $\eta=1$) in the range $0\leq\eta\leq1$

$$\widetilde{c}=c_9\eta^{\left(\frac{1}{2}+\lambda\right)}F\left(\frac{1}{2}+\lambda+k,\frac{1}{2}+\lambda-k;\frac{3}{2}+\lambda;\eta\right) \tag{3.61}$$

$$\widetilde{d}=c_{10}\eta^{\left(\frac{1}{2}-\lambda\right)}F\left(\frac{1}{2}+k-\lambda,\frac{1}{2}-\lambda-k;\frac{3}{2}-\lambda;\eta\right) \tag{3.62}$$

where $c_9$ and $c_{10}$ are the integration constants.

## 3.6 Large Distance Solution of the Axial Equation

With our customary choice of $x=\sinh^{-1}z$ as the independent variable and applying the substitutions

$$g_1=\frac{P_1}{\sqrt{\cosh x}}$$

$$g_2=\frac{P_2}{\sqrt{\cosh x}} \tag{3.63}$$

to Eqs.(3.18), (3.19), the equations take the form

$$P_{1,x}-iXP_1=-\left(\lambda-i\mu\right)P_2 \tag{3.64}$$

$$P_{2,x}+iXP_2=-\left(\lambda+i\mu\right)P_1 \tag{3.65}$$

Recalling that

$$\lambda \pm i\mu = e^{\pm i\alpha} a \tag{3.66}$$

We can write Eqs.(3.64), (3.65) in the alternative form

$$P_{1,r} - i\left(k + q^* \sinh x\right) P_1 = -a \cosh x e^{i\alpha} P_2 \tag{3.67}$$

$$P_{2,r} + i\left(k + q^* \sinh x\right) P_2 = -a \cosh x e^{-i\alpha} P_1 \tag{3.68}$$

where the new independent variable is defined as

$$r = 2\tan^{-1}(e^x) \tag{3.69}$$

We can eliminate the exponential factors on the right-hand sides of Eqs.(3.67), (3.68) by the substitutions

$$P_1 = \Psi_- e^{\frac{i\alpha}{2}}$$

$$P_2 = \Psi_+ e^{-\frac{i\alpha}{2}} \tag{3.70}$$

We are then left with

$$\Psi_{\pm,r} \pm ik(1 + \frac{q^*}{k}\sinh x)\Psi_+ = -a\cosh x \Psi_- \tag{3.71}$$

We can obtain the same factor in front of the first term on the left hand-sides as there is in front of the second by changing the independent variable to

$$r^* = r + \frac{q^*}{k}\ln\left(\cosh x\right) \tag{3.72}$$

in which case

$$dr^* = \left(1 + \frac{q^*}{k}\sinh x\right) dr \tag{3.73}$$

Thus, we obtain the equations

$$\Psi_{\pm,r^*} \pm ik\Psi_+ = W\Psi_- \tag{3.74}$$

where

$$W = \frac{-a\cosh x}{1 + \frac{q^*}{k}\sinh x} \tag{3.75}$$

Now letting

$$Z_\pm = \Psi_+ \pm \Psi_- \tag{3.76}$$

we can combine Eqs.(3.74), (3.76) to give

$$Z_{\pm,r^*} \mp WZ_+ = -ikZ_- \tag{3.77}$$

From the above set of equations, we readily obtain a pair of one-dimensional Schrödinger-type wave equations,

$$Z_{\pm,r^*r^*} + k^2 Z_\pm = V_\pm Z_\pm \tag{3.78}$$

with

$$V_\pm = W^2 \pm W_{,r^*} = \frac{\sqrt{a}\cosh x}{\left(1 + \frac{q^*}{k}\sinh x\right)^2}\left[\sqrt{a}\cosh x \pm \frac{\frac{q^*}{k} - \sinh x}{1 + \frac{q^*}{k}\sinh x}\right] \tag{3.79}$$

where $V_+$ and $V_-$ are the potentials felt by a Dirac particle.

As seen in Fig.3.2, both potentials have a singularity at a definite value of $x$, $x = x^*$

$$x^* = -\sinh^{-1}\left(\frac{k}{q^*}\right) \tag{3.80}$$

It is now more apparent that in contrast to a test particle, charge is a strong

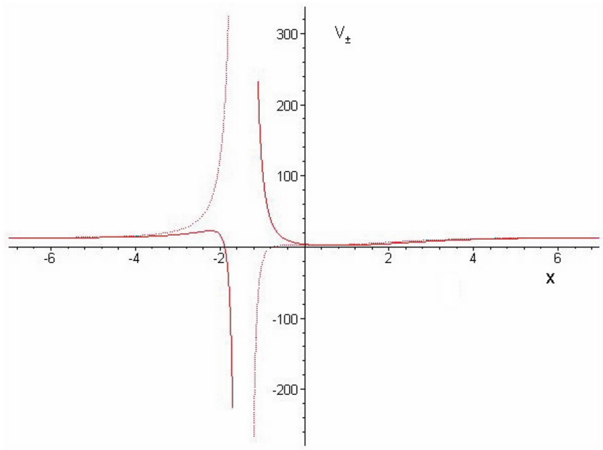


Figure 3-2: The potentials, $V_+$ (the full-line curves) and $V_-$ (the dotted curves) of a BR spacetime for the waves of charged Dirac particle with $\lambda = \frac{3}{2}$, $k = 1$, $\mu = 1$ and $q^* = \frac{1}{2}$ .

coupling parameter. Because its existence in Eq.(3.79) gives rise to shift the location of divergence of the potentials from $\pm\infty$ to $x^*$. In addition to this, it causes to fall down the potentials to negative infinity, which is unphysical. However, in general physics we are familiar with such problems about the charge. For instance, the Coulomb singularity at the origin. Chandrasekhar also encountered with the same problem, i.e. falling down of the potential to negative infinity, while he was studying electromagnetic waves in Kerr geometry [2]. On the other hand, for neutrinos the mentioned problem is removed since the neutrinos have no charge. It is easy to check that in the limit of neutrino case $V_\pm$ reduces to $V_1$ and $V_2$, respectively.

### 3.6.1 Large Distance Solution

It is observed that potentials (3.79) are involved for an analytic solution. They become appropriate, however, for large distance treatment of the exact solution. One can easily check that while $x \to \pm\infty$ , $V_\pm \to \frac{k^2 a}{q^2}$. This means the potentials have similar behavior for independent waves while $x \to \pm\infty$.

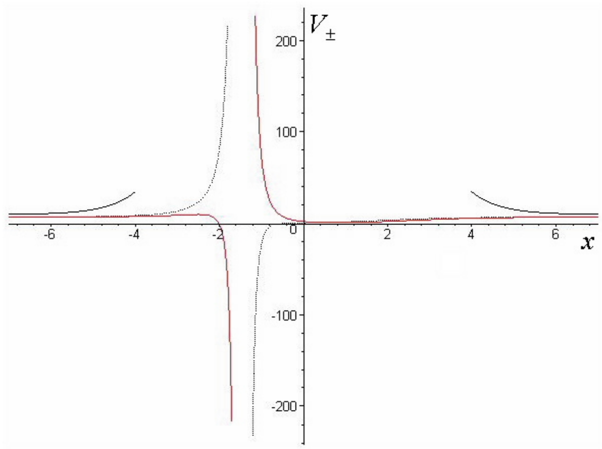


Figure 3-3: Large distance behavior of $Z_+$ and $Z_-$ for case $(i)$. Behavior can be seen by the damped curves over the potential while $x \to \pm\infty$ . Physical parameters are $a_o = b_o = 1$, $\lambda = \frac{3}{2}$,$k = 1$, and $\mu = 1$.

At large distances ($x \to \pm\infty$), Eq.(3.78) take the following form

$$\frac{d^2 Z_\pm}{dx^2} + (k^2 - \mu^2 - \lambda^2) Z_\pm = 0 \tag{3.81}$$

It is evident from the above equation that at large distances the charge loses its effect on the behavior of the waves. The solution for $Z_\pm$ can be treated in two cases.

*i.* for $k^2 \leq \mu^2 + \lambda^2$

$$Z_- = a_0 e^{-\omega_0 |x|} \tag{3.82}$$

$$Z_+ = b_0 e^{-\omega_0 |x|} \tag{3.83}$$

where $\omega_0 = \sqrt{\mu^2 + \lambda^2 - k^2}$.

It is obvious from Fig.3.3 that the large distance behaviors of $Z_+$ and $Z_-$ for case $(i)$ represent the wave which is in the case of damping oscillation.

*ii.* for $k^2 > \mu^2 + \lambda^2$

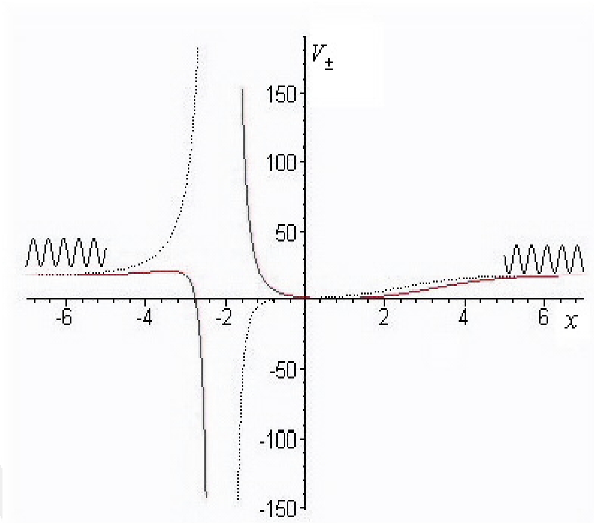


Figure 3-4: Large distance behavior of $Z_+$ and $Z_-$ for case $(ii)$. Behavior can be seen by the underdamped curves over the potential while $x \to \pm\infty$ . Physical parameters are $a_o = b_o = 1$, $\lambda = \frac{3}{2}$,$k = 1$, and $\mu = 1$.

$$Z_- = a_0' \cos \omega_0' x \tag{3.84}$$

$$Z_+ = b_0' \cos \omega_0' x \tag{3.85}$$

where $\omega_0' = \sqrt{k^2 - \mu^2 - \lambda^2}$.

It is clear that behaviors of $Z_+$ and $Z_-$ for case $(ii)$ represent that the wave is sinusoidal at the large distance, as shown in Fig.3.4.

On the other hand, in the neutrino case, since we have infinite walls at $x \to \pm\infty$, $Z_+$ and $Z_-$ vanish at the walls i.e. the waves of the neutrinos are totally reflected back by the walls.

## 3.7 The Conserved Current of the Dirac Equation

To find the conserved net current of particles, we shall use the Chandrasekhar's notation [2]. In this notation, the basic spinors are defined by $P^A$ and $\overline{Q}^{A'}$ and they , in the BR space-time, correspond to the following forms

$$\begin{aligned} P^0 &= F_1, \qquad P^1 = F_2 \\ \overline{Q}^{0'} &= -G_2, \quad \overline{Q}^{1'} = G_1 \end{aligned} \tag{3.86}$$

In determination of the conserved net current of particles, one needs to apply the conservation law for the particle number current. This is done by considering the surface $S$ of a sphere of time-independent radius $r$ and finding the integral of the particle number conservation equation

$$0 = \partial_t\left(\sqrt{-g}J^t\right) + \partial_i\left(\sqrt{-g}J^i\right) \tag{3.87}$$

over the surface $S$ within a time coordinate hypersurface. Thus

$$\begin{aligned} \frac{dN_S}{dt} &= \frac{d}{dt}\int_S \sqrt{-g}J^t d\theta d\varphi \\ &= -\int_S \sqrt{-g}J^r d\theta d\varphi \end{aligned} \tag{3.88}$$

This gives the rate of change of the number $N_S$ of particles in this surface in terms of the flux entering the outer surface and exiting the inner surface.

Adapting the foregoing conserved net current of particles to the spin$-\frac{1}{2}$ particles in the BR geometry, we find

$$\frac{\partial N}{\partial t} = -\int_0^{2\pi}\int_0^{\pi} J^z\sqrt{-g}d\theta d\varphi \tag{3.89}$$

where $g$ is the determinant of the spacetime metric and the current is defined [2], in the standard spinor formalism,

$$\frac{1}{\sqrt{2}}J^z = \sigma^z_{AB'}\left(P^A\overline{P}^{B'} + Q^A\overline{Q}^{B'}\right) \tag{3.90}$$

Using the axial Eqs.(3.18), (3.19), namely,

$$\sqrt{1+z^2}\left[\left(\partial_z + \frac{z}{2\left(1+z^2\right)}\right) - \frac{i\left(k+qz\right)}{1+z^2}\right]g_1 = -\left(\lambda + i\mu\right)g_2 \tag{3.91}$$

$$\sqrt{1+z^2}\left[\left(\partial_z + \frac{z}{2\left(1+z^2\right)}\right) + \frac{i\left(k+qz\right)}{1+z^2}\right]g_2 = -\left(\lambda - i\mu\right)g_1 \tag{3.92}$$

we easily find that

$$\partial_z\left(\left|g_1\right|^2 - \left|g_2\right|^2\right) + \frac{z}{1+z^2}\left(\left|g_1\right|^2 - \left|g_2\right|^2\right) = 0 \tag{3.93}$$

Eq.(3.93) implies that

$$\left|g_1\right|^2 - \left|g_2\right|^2 = \frac{-\alpha}{\sqrt{1+z^2}} \tag{3.94}$$

where $\alpha$ is an integration constant.

Using the chosen basis vectors, the matrix for the axial part is written as

$$\sigma^z_{AB'} = \frac{1}{\sqrt{2}}\begin{pmatrix} \sqrt{1+z^2} & 0 \\ 0 & -\sqrt{1+z^2} \end{pmatrix} \tag{3.95}$$

Finally we can evaluate $J^z$ explicitly by the substitutions into

$$J^z = \sqrt{\left(1+z^2\right)}\left(\left|g_2\right|^2 - \left|g_1\right|^2\right)\left(\left|A_1\right|^2 + \left|A_3\right|^2\right) \tag{3.96}$$

One can normalize the angular functions $A_1$ and $A_3$ to unity to get the conserved net current of particles as

$$\frac{dN}{dt} = 4\pi\alpha \tag{3.97}$$

Namely, the rate $\left(\frac{\partial N}{\partial t}\right)_{in}$ at which particles are moving *in* through a sector of the BR geometry per unit time is constant. In other words, Eq.(3.97) guarantees the constancy of the Wronskian , which should be constant due to the continuity equation. In order to show that the Wronskian is constant, in accordance with Eqs.(3.63), (3.70), (3.76) we successively find

$$\begin{aligned}
\frac{1}{4\pi}\frac{dN}{dt} &= \sqrt{(1+z^2)}\left(|g_2|^2 - |g_1|^2\right) = |\Psi_+|^2 - |\Psi_-|^2 \\
&= \frac{1}{4}\left[|Z_+ + Z_-|^2 - |Z_+ - Z_-|^2\right] \\
&= Z_+^{(\text{real})} Z_-^{(\text{real})} + Z_+^{(im)} Z_-^{(im)} \\
&= -\frac{1}{k}\left[Z_+^{(\text{real})}\left(Z_{+,r^*}^{(im)} - W Z_+^{(im)}\right) - Z_+^{(im)}\left(Z_{+,r^*}^{(\text{real})} - W Z_+^{(\text{real})}\right)\right] \\
&= -\frac{1}{k}\left[Z_+^{(\text{real})} Z_{+,r^*}^{(im)} - Z_+^{(im)} Z_{+,r^*}^{(\text{real})}\right] \\
&= \frac{1}{2ik}\left[Z_+ Z_{+,r^*} - Z_+ Z_{+,r^*}\right] \\
&= \frac{\left[Z_+, \overline{Z}_+\right]_{r^*}}{2ik}
\end{aligned} \tag{3.98}$$

Since $\frac{dN}{dt}$ retains the same constant value for $-\infty < z < \infty$ , the constancy of the Wronskian, $\left[Z_+, \overline{Z}_+\right]_{r^*}$ , over the same range of $z$ follows. The constancy of $\left[Z_-, \overline{Z}_-\right]_{r^*}$ similarly follows.

# Chapter 4

# Solution of the Dirac Equation in the NH Geometry of an Extreme Kerr BH

## 4.1 Introduction

The NH geometry of an extreme Kerr BH is already known as a throat geometry. It is a completely regular vacuum solution with an enhanced symmetry group $SL(2,R) \times U(1)$ [25,35]. It was shown in [25] that it partly preserves such typical $AdS$ features as confinement of timelike geodesics and discrete energy spectrum of a minimally coupled test scalar field. However, the presence of $\theta-$dependent factors violates this simple picture: timelike geodesics with sufficiently large angular momentum, just like radial null geodesics in structure of $AdS$, can escape to infinity in finite coordinate time but infinite affine parameter, and the Klein-Gordon spectrum also contains a continuous part with an associated superradiance phenomenon. Essentially, in many aspects, this solution shares the common features with the $AdS_2 \times S_2$ geometry arising in the near horizon limit of extreme Reissner-Nordström BH.

In order to separate equations we employ the well-known method due to Chandrasekhar so that we study CD equations. We separate the $y$ (axial) and $\theta$ (angular) dependence in such a way that the resulting axial equation remains independent of mass. This leads us to an exact solution irrespective of mass. The angular equation on the other hand depends strictly on the mass. For the massless case (which we refer to as neutrino or Weyl equation) the angular equation reduces to a confluent Heun equation [36]. When the mass is nonzero, however, we cannot identify our equations but instead we express them as a set of linear equations suitable for numerical analysis.

The organization of the chapter[1] is as follows: in Sec. 4.2 we review the NH geometry of an extreme Kerr BH and separation of variables of the CD equation. Solution of the axial equation follows in Sec. 4.3. The massless and massive cases are discussed in Secs.4.4 and 4.5, respectively. The reduction of our equations into one-dimensional Schrödinger-type equations with their conserved currents and superradiance are all included in the final section, Sec. 4.6.

## 4.2 Extreme Kerr Throat Geometry and Separation of Dirac Equation on It

The extreme Kerr metric in the Boyer-Lindquist coordinates is given by

$$ds^2 = e^{2\nu} d\widetilde{t}^2 - e^{2\psi}\left(d\widetilde{\phi} + \omega d\widetilde{t}\right)^2 - \rho^2\left(\frac{d\widetilde{r}^2}{\widetilde{\Delta}} + d\theta^2\right) \tag{4.1}$$

where

$$e^{2\nu} = \frac{\widetilde{\Delta}\rho^2}{\left(\widetilde{r}^2 + M^2\right)^2 - \widetilde{\Delta} M^2 \sin^2\theta}$$

$$e^{2(\nu+\psi)} = \widetilde{\Delta}\sin^2\theta$$

[1]The material of this chapter was published in [37].

$$\widetilde{\Delta} = (\widetilde{r} - M)^2$$

$$\omega = \frac{2M^2\widetilde{r}e^{2\nu}}{\widetilde{\Delta}\rho^2}$$

$$\rho^2 = \widetilde{r}^2 + M^2\cos^2\theta \tag{4.2}$$

In the extreme case, both the total mass $M$ and the rotation parameter $a$ become identical so that the angular momentum $J = M^2$ and the extremal horizon corresponds to $\widetilde{r} = M$. The area of the horizon is $A = 8\pi J$.

To describe the near horizon (or throat) limit of the extreme Kerr metric, due originally to Bardeen and Horowitz [25], one can set

$$\widetilde{r} = M + \lambda r$$

$$\widetilde{t} = \frac{t'}{\lambda}$$

$$\widetilde{\phi} = \phi - \frac{t'}{2\lambda M} \tag{4.3}$$

and takes the limit $\lambda \to 0$. In these new coordinates, the throat metric is obtained as

$$ds^2 = F\left[\frac{r^2}{r_0^2}dt'^2 - \frac{r_0^2}{r^2}dr^2 - r_0^2 d\theta^2\right] - \frac{r_0^2\sin^2\theta}{F}\left(d\phi + \frac{r}{r_0^2}dt'\right)^2 \tag{4.4}$$

where

$$F = \frac{1+\cos^2\theta}{2}$$

$$r_0^2 = 2M^2 \tag{4.5}$$

We set for simplicity, $r_0^2 = 1$ . This throat spacetime is no longer asymptotically flat.

Finally, passing to more general coordinates,

$$y = \frac{1}{2r}\left[r^2\left(1 + t'^2\right) - 1\right]$$

$$\cot t = \frac{1}{2t'r}\left[r^2\left(1 - t'^2\right) + 1\right]$$

$$\phi = \varphi + \ln\left|\frac{\cos t + y\sin t}{1 + t'r}\right| \tag{4.6}$$

we can write the throat metric (4.4) as

$$ds^2 = F\left[\left(1 + y^2\right)dt^2 - \frac{dy^2}{1 + y^2} - d\theta^2\right] - \frac{\sin^2\theta}{F}\left(d\varphi + ydt\right)^2 \tag{4.7}$$

The metric functions in Eq.(4.7) depend only on the variable $\theta$ thus as expected in the search for a solution of the Dirac equation in the angular equation forms the crux of the problem. The coordinates $-\infty < t < \infty$ , $-\infty < y < \infty$ cover the entire, singularity free spacetime. The Killing vector $\frac{\partial}{\partial t}$ is not timelike everywhere; it admits a region (for $\sin^2\theta > 0.536$) in which it becomes spacelike. Therefore by a coordinate transformation this particular region is transformable into the spacetime of colliding plane waves [37]. Recently, it has also been shown that the metric (4.7) can be obtained as a solution to dilaton-axion gravity which is similar to the rotating BR spacetime [35].

The singularity free character can best be seen by checking the Weyl scalar $\Psi_2$ and the Kretschmann scalar:

$$\Psi_2 = \frac{2}{\left(1 + \cos^2\theta\right)^3}\left[3\cos^2\theta - 1 + i\cos\theta\left(\cos^2\theta - 3\right)\right] \tag{4.8}$$

$$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = \frac{192\sin^2\theta}{(1+\cos^2\theta)^6}\left[\left(1+\cos^2\theta\right)^2 - 16\cos^2\theta\right] \tag{4.9}$$

When the backreaction of the spin$-\frac{1}{2}$ test particles on the background geometry is neglected, the Dirac field equation is given by the CD equations [2] on a fixed spacetime (4.7).

We choose a complex null tetrad $\{l, n, m, \overline{m}\}$ that satisfies the orthogonality conditions, $l.n = -m.\overline{m} = 1$. We note that, throughout the paper, a bar over a quantity denotes complex conjugation. Thus the covariant one-forms can be written as

$$\sqrt{2}l = \sqrt{F(1+y^2)}dt - \frac{\sqrt{F}}{\sqrt{1+y^2}}dy$$

$$\sqrt{2}n = \sqrt{F(1+y^2)}dt + \frac{\sqrt{F}}{\sqrt{1+y^2}}dy$$

$$\sqrt{2}m = \frac{iy\sin\theta}{\sqrt{F}}dt + \sqrt{F}d\theta + \frac{i\sin\theta}{\sqrt{F}}d\varphi$$

$$\sqrt{2}\overline{m} = -\frac{iy\sin\theta}{\sqrt{F}}dt + \sqrt{F}d\theta - \frac{i\sin\theta}{\sqrt{F}}d\varphi \tag{4.10}$$

and their corresponding directional derivatives are

$$\sqrt{2}D = \frac{1}{\sqrt{F(1+y^2)}}\partial_t + \frac{\sqrt{1+y^2}}{\sqrt{F}}\partial_y$$

$$\sqrt{2}\Delta = \frac{1}{\sqrt{F(1+y^2)}}\partial_t - \frac{\sqrt{1+y^2}}{\sqrt{F}}\partial_y$$

$$\sqrt{2}\delta = -\frac{1}{\sqrt{F}}\partial_\theta - \frac{i\sqrt{F}}{\sin\theta}\partial_\varphi$$

$$\sqrt{2}\overline{\delta} = -\frac{1}{\sqrt{F}}\partial_\theta + \frac{i\sqrt{F}}{\sin\theta}\partial_\varphi \tag{4.11}$$

One can determine the nonzero NP complex spin coefficients [11] in the above null-tetrad as

$$\pi = -\tau = \frac{\sin\theta\,(\cos\theta - i)}{(2F)^{\frac{3}{2}}}$$

$$\varepsilon = \gamma = \frac{y}{2\sqrt{2F(1+y^2)}}$$

$$\alpha = -\beta = \frac{2\cot\theta - i\sin\theta}{2\,(2F)^{\frac{3}{2}}} \tag{4.12}$$

Due to the form of the CD equation (2.7), similar to the case of BR, we introduce [2,28]

$$F_1 = f_1\,(y)\,A_1\,(\theta)\,e^{i(\sigma t + m\varphi)}$$

$$G_1 = g_1\,(y)\,A_2\,(\theta)\,e^{i(\sigma t + m\varphi)}$$

$$F_2 = f_2\,(y)\,A_3\,(\theta)\,e^{i(\sigma t + m\varphi)}$$

$$G_2 = g_2\,(y)\,A_4\,(\theta)\,e^{i(\sigma t + m\varphi)} \tag{4.13}$$

where $\sigma$ is the frequency of the corresponding Compton wave of the Dirac particle and $m$ is the azimuthal quantum number of the wave. Our convention is that $\sigma$ is always positive.

Inserting for the appropriate spin coefficients (4.12) with the spinors (4.13) into the four coupled CD equation (2.7), we obtain

$$\frac{\left(\hat{H}f_1\right)}{f_2} = i\mu^* \frac{g_1}{f_2}\frac{A_2}{A_1}\sqrt{F} - \frac{(ŁA_3)}{A_1}$$

$$\frac{\left(\overline{\hat{H}} f_2\right)}{f_1} = -i\mu^* \frac{g_2}{f_1}\frac{A_4}{A_3}\sqrt{F} - \frac{\left(\text{Ł}^\dagger A_1\right)}{A_3}$$

$$\frac{\left(\hat{H} g_2\right)}{g_1} = i\mu^* \frac{f_2}{g_1}\frac{A_3}{A_4}\sqrt{F} - \frac{\left(\pounds^\dagger A_2\right)}{A_4}$$

$$-\frac{\left(\overline{\hat{H}} g_1\right)}{g_2} = i\mu^* \frac{f_1}{g_2}\frac{A_1}{A_2}\sqrt{F} - \frac{\left(\pounds A_4\right)}{A_2} \tag{4.14}$$

where the axial and the angular operators are

$$\hat{H} = \sqrt{1+y^2}\partial_y + \frac{1}{2\sqrt{1+y^2}}\left[y + 2i\sigma\right]$$

$$\overline{\hat{H}} = \sqrt{1+y^2}\partial_y + \frac{1}{2\sqrt{1+y^2}}\left[y - 2i\sigma\right] \tag{4.15}$$

and

$$\text{Ł} = \partial_\theta + \frac{mF}{\sin\theta} + \frac{1}{2F}\left(\frac{\cos^3\theta}{\sin\theta} + \frac{i\sin\theta}{2}\right)$$

$$\text{Ł}^\dagger = \partial_\theta - \frac{mF}{\sin\theta} + \frac{1}{2F}\left(\frac{\cos^3\theta}{\sin\theta} + \frac{i\sin\theta}{2}\right)$$

$$\pounds = \partial_\theta + \frac{mF}{\sin\theta} + \frac{1}{2F}\left(\frac{\cos^3\theta}{\sin\theta} - \frac{i\sin\theta}{2}\right)$$

$$\pounds^\dagger = \partial_\theta - \frac{mF}{\sin\theta} + \frac{1}{2F}\left(\frac{\cos^3\theta}{\sin\theta} - \frac{i\sin\theta}{2}\right) \tag{4.16}$$

respectively. One can easily see that Ł= $\overline{\pounds}$ and $\text{Ł}^\dagger = \overline{\pounds^\dagger}$.

Further, choosing $f_1 = g_2$, $f_2 = g_1$, $A_1 = \overline{A_2}$ and $A_3 = \overline{A_4}$ and introducing the separation constant as $i\lambda$, where $\lambda$ is a real constant, we can separate Dirac equation (4.14) into axial and angular parts

$$\overline{\hat{H}}g_1 = -i\lambda g_2 \tag{4.17}$$

$$\hat{H}g_2 = i\lambda g_1 \tag{4.18}$$

and

$$\text{Ł}A_3 + i\mu^* A_2\sqrt{F} = i\lambda A_1 \tag{4.19}$$

$$\text{Ł}^\dagger A_1 + i\mu^* A_4\sqrt{F} = i\lambda A_3 \tag{4.20}$$

## 4.3 Solution of the Axial Equation

If we decouple the axial equations (4.17), (4.18) in Eq.(4.17) to get $g_1$, we obtain

$$\hat{H}(\overline{\hat{H}}g_1) = \lambda^2 g_1 \tag{4.21}$$

Similarly one can decouple the axial equations in Eq.(4.17) for $g_2$. The explicit form of Eq.(4.21) can be obtained as

$$\left(1+y^2\right)g_1''(y) + 2yg_1'(y) + \frac{1}{1+y^2}\left(\frac{1}{2}+\sigma^2+\frac{y^2}{4}-\lambda^2\left(1+y^2\right)+i\sigma y\right)g_1(y) = 0 \tag{4.22}$$

( Throughout this chapter, a prime denotes a derivative with respect to its argument.)

Thus the solutions of the decoupled equations for $g_1$, Eq.(4.22), and $g_2$ (not given here) can be found in terms of the associated Legendre functions as follows:

$$g_1(y) = c_1 P^{\widehat{\beta}}_{\lambda-\frac{1}{2}}(iy)$$

$$g_2(y) = c_2 P^{\widehat{\beta}}_{\lambda-\frac{1}{2}}(-iy) \tag{4.23}$$

where

$$\widehat{\beta} = \sqrt{\sigma^2 + \frac{\sigma}{2} + \frac{1}{4}} \tag{4.24}$$

and $c_1$, $c_2$ are complex constants.

Here, as a result of physical necessities, we considered only the first kind of associated Legendre functions. Although solutions (4.23) seem like complex solutions, it is possible to draw the real functions from the above associated Legendre functions. We may define

$$\lambda = \widetilde{m} + \frac{1}{2} \quad \text{with} \quad \widetilde{m} = 1, 2, 3.... \tag{4.25}$$

and

$$\sigma = \frac{1}{4}\left(\sqrt{16\tilde{n}^2 - 3} - 1\right) \tag{4.26}$$

so that

$$\widehat{\beta} = \tilde{n} \qquad \text{with} \qquad \tilde{n} = -\widetilde{m}, -\widetilde{m}+1, .....-1, 1, ......\widetilde{m}-1, \widetilde{m} \tag{4.27}$$

In order to get the real functions for solutions (4.21), the required condition is $\widetilde{m} - |\tilde{n}|$ =even number.

It is worth drawing attention to the following remarks:

*i)* In the case of $\lambda = 0$, Eqs.(4.17), (4.18) reduce to simple first order differential equations which admit the following solutions

$$g_1(y) = c_3 \left(1 + y^2\right)^{-\frac{1}{4}} e^{i\sigma \tan^{-1}(y)}$$

$$g_2(y) = c_4 \left(1 + y^2\right)^{-\frac{1}{4}} e^{-i\sigma \tan^{-1}(y)} \tag{4.28}$$

with $c_3, c_4$ complex constants.

These two solutions can be interpreted as representing ingoing and outgoing waves.

*ii)* In the case of $\lambda = \frac{1}{2}$, we obtain the following complex solutions from Eqs.(4.17), (4.18):

$$g_1(y) = c_5 \left(\frac{iy+1}{iy-1}\right)^{\frac{\widehat{\beta}}{2}} + c_6 \left(\frac{iy+1}{iy-1}\right)^{-\frac{\widehat{\beta}}{2}}$$

$$g_2(y) = c_7 \left(\frac{1-iy}{1+iy}\right)^{\frac{\widehat{\beta}}{2}} + c_8 \left(\frac{1-iy}{1+iy}\right)^{-\frac{\widehat{\beta}}{2}} \tag{4.29}$$

where again $c_j$ with $j = 5, 6, 7, 8$ are complex constants.

## 4.4 Reduction of the Angular Equation to Heun Equation: The Massless Case

In this section, we shall show that the angular equations (4.19), (4.20) for the neutrino particles can be decoupled to the confluent Heun equation. For this purpose, let us reconsider Eqs.(4.19),(4.20) in the explicit form for $\mu^* = 0$,

$$A_3'(\theta) + (K + G)A_3(\theta) = i\lambda A_1(\theta) \tag{4.30}$$

$$A_1'(\theta) + (K - G)A_1(\theta) = i\lambda A_3(\theta) \tag{4.31}$$

where

$$K = \frac{1}{2F}\left(\frac{\cos^3\theta}{\sin\theta} + \frac{i\sin\theta}{2}\right) \tag{4.32}$$

$$G = \frac{mF}{\sin\theta} \tag{4.33}$$

By introducing the scalings

$$A_1(\theta) = H_1(\theta)e^{-\int (K-G)d\theta} \tag{4.34}$$

$$A_3(\theta) = H_3(\theta)e^{-\int (K+G)d\theta} \tag{4.35}$$

we get

$$H_1'(\theta) = i\lambda H_3(\theta)e^{-\int 2Gd\theta} \tag{4.36}$$

$$H_3'(\theta) = i\lambda H_1(\theta)e^{\int 2Gd\theta} \tag{4.37}$$

If we decouple Eqs.(4.36), (4.37) in Eq.(4.36) for $H_1(\theta)$, we get

$$H_1''(\theta) + 2GH_1'(\theta) + \lambda^2 H_1(\theta) = 0 \tag{4.38}$$

In similar fashion, we find, for $H_3(\theta)$,

$$H_3''(\theta) - 2GH_3'(\theta) + \lambda^2 H_3(\theta) = 0 \tag{4.39}$$

Introducing a new variable $\theta = \cos^{-1}(1-2z)$, Eqs.(4.38), (4.39) turn out to be

$$H_1''(z) + \left(-2m + \frac{\frac{1}{2}+m}{z} + \frac{\frac{1}{2}-m}{z-1}\right)H_1'(z) - \frac{\lambda^2}{z(z-1)}H_1(z) = 0 \tag{4.40}$$

$$H_3''(z) + \left(-2m + \frac{m-\frac{1}{2}}{z} + \frac{m+\frac{1}{2}}{z-1}\right)H_3'(z) - \frac{\lambda^2}{z(z-1)}H_3(z) = 0 \tag{4.41}$$

Let us recall the general confluent form of Heun equation [36],

$$H''(z) + \left(A + \frac{B}{z} + \frac{C}{z-1}\right) H'(z) - \frac{QBz - h}{z(z-1)} H(z) = 0 \tag{4.42}$$

Drawing the similarities between Eq.(4.42) and Eqs.(4.40), (4.41), we observe the following correspondences:

*a)* For Eq.(4.40),

$$Q = 0,\ h = \lambda^2,\ A = -2m,\ B = \frac{1}{2} + m \text{ and } C = \frac{1}{2} - m \tag{4.43}$$

*b)* For Eq.(4.39),

$$Q = 0,\ h = \lambda^2,\ A = -2m,\ B = m - \frac{1}{2} \text{ and } C = m + \frac{1}{2} \tag{4.44}$$

The confluent Heun equation (4.42), with its accessory parameter $h$, has two regular singular points at $z = 0, 1$ with exponents $(0, 1-B)$ and $(0, 1-C)$, respectively, as well as an irregular singularity at the infinity. In the vicinity of the point $z = 0$, its power series can be written as

$$H(Q, A, B, C, h; z) = \sum_{j=0}^{\infty} W_j z^j \tag{4.45}$$

and the coefficient $W_j$ satisfies a three-term recurrence relation [36],

$$\begin{aligned} W_0 &= 1, \qquad W_1 = \frac{-h}{B} \\ (j+1)(j+B) W_{j+1} - A(j-1+Q) W_{j-1} &= [j(j-1-A+B+C) - h] W_j \end{aligned} \tag{4.46}$$

It is also possible to obtain the power series solution in the vicinity of the point $z = 1$ by a linear transformation interchanging the regular singular points $z = 0$ and $z = 1$. Namely, $z \to 1 - z$.

Expansion of solutions to the confluent Heun equation in terms of the hypergeometric and confluent hypergeometric function can be seen in [36]. In Ref. [36], it is also shown that the confluent Heun equation can be normalized to constitute a group of orthogonal complete functions and the confluent Heun equation also admits quasipolynomial solutions for particular values of the parameters.

Since $Q = 0$ in our case, it follows from the three-term recurrence relation that $H(Q, A, B, C, h; z)$ is a polynomial solution if $W_1(h) = 0$, where $W_1$ stands for a polynomial of degree 1 in $h$. Namely , there is only one eigenvalue $h_i$ for $h$ such that $W_1(h_i) = 0$ (i.e. $\lambda = 0$).

## 4.5 Reduction of the Angular Equation Into a Set of Linear First Order Differential Equations: The Case With Mass

To complete our analysis of the angular equation, we need to discuss the angular equation for Dirac particles with mass.

The angular equations (4.19), (4.20) can be rewritten in the forms

$$LA_3 + i\mu_p\sqrt{1+\cos^2\theta}\overline{A_1} = i\lambda A_1 \tag{4.47}$$

$$L^+A_1 + i\mu_p\sqrt{1+\cos^2\theta}\overline{A_3} = i\lambda A_3 \tag{4.48}$$

With substitutions

$$A_1(\theta) = (A_0(\theta) + iB_0(\theta))\, e^{\int \frac{\cos^3\theta}{2\sin\theta F}d\theta} \tag{4.49}$$

$$A_3(\theta) = (M_0(\theta) + iN_0(\theta))\, e^{\int \frac{\cos^3\theta}{2\sin\theta F}d\theta} \tag{4.50}$$

we can transform Eqs.(4.47), (4.48) into a set of first order differential equations

$$M_0'(\theta) + GM_0(\theta) - \frac{\sin\theta}{4F} N_0(\theta) = -\left(\lambda + \mu_p\sqrt{1+\cos^2\theta}\right) B_0(\theta)$$

$$N_0'(\theta) + GN_0(\theta) + \frac{\sin\theta}{4F} M_0(\theta) = \left(\lambda - \mu_p\sqrt{1+\cos^2\theta}\right) A_0(\theta)$$

$$A_0'(\theta) - GA_0(\theta) - \frac{\sin\theta}{4F} B_0(\theta) = -\left(\lambda + \mu_p\sqrt{1+\cos^2\theta}\right) N_0(\theta)$$

$$B_0'(\theta) - GB_0(\theta) + \frac{\sin\theta}{4F} A_0(\theta) = \left(\lambda - \mu_p\sqrt{1+\cos^2\theta}\right) M_0(\theta) \tag{4.51}$$

Introducing a new variable $x = \cos\theta$ and with the further substitutions

$$M_0(\theta) = \frac{1}{2}\left(m_0(\theta) + a_0(\theta)\right) \qquad N_0(\theta) = \frac{1}{2}\left(n_0(\theta) + b_0(\theta)\right)$$

$$A_0(\theta) = \frac{1}{2}\left(m_0(\theta) - a_0(\theta)\right) \qquad B_0(\theta) = \frac{1}{2}\left(n_0(\theta) - b_0(\theta)\right) \tag{4.52}$$

we may obtain the final form of the set as linear first order differential equations

$$m_0'(x) + \alpha_1 a_0(x) + (\alpha_2 - \alpha_3)\, n_0(x) = 0$$

$$a_0'(x) + \alpha_1 m_0(x) + (\alpha_4 + \alpha_3)\, b_0(x) = 0$$

$$n_0'(x) + \alpha_1 b_0(x) - (\alpha_2 + \alpha_3)\, m_0(x) = 0$$

$$b_0'(x) + \alpha_1 n_0(x) - (\alpha_4 - \alpha_3)\, a_0(x) = 0 \tag{4.53}$$

where

$$\alpha_1 = -\frac{m\,(1+x^2)}{2\,(1-x^2)} \qquad \alpha_2 = \frac{1}{2\,(1+x^2)} - \frac{\lambda}{\sqrt{1-x^2}}$$

$$\alpha_3 = \frac{\mu_p \sqrt{1+x^2}}{\sqrt{1-x^2}} \qquad \alpha_2 = \frac{1}{2\,(1+x^2)} + \frac{\lambda}{\sqrt{1-x^2}} \tag{4.54}$$

Although the system (4.53) does not seem solvable analytically, one may develop an appropriate numerical technique to study it. In the literature, there may exist such interesting systems which are more or less of this type.

## 4.6 Reduction of Dirac Equation to a One-Dimensional Schrödinger-Type Equation With a Conserved Current

It is possible to get more compact forms the axial Eqs.(4.17),(4.18) by introducing the scalings

$$g_1(y) = Z_1(y)\left(1+y^2\right)^{-\frac{1}{4}} \tag{4.55}$$

$$g_2(y) = Z_2(y)\left(1+y^2\right)^{-\frac{1}{4}} \tag{4.56}$$

and applying the coordinate transformation $y = \tan u$ , the axial equations take the form

$$Z_1'(u) - i\sigma Z_1(u) = -i\lambda\Lambda Z_2(u) \tag{4.57}$$

$$Z_2'(u) + i\sigma Z_2(u) = i\lambda\Lambda Z_1(u) \tag{4.58}$$

where $\Lambda = \sqrt{1+y^2} \equiv \frac{1}{\cos u.}$.

Letting

$$Z_1(u) = \frac{iP_1(u) - P_2(u)}{2} \tag{4.59}$$

$$Z_2(u) = \frac{iP_1(u) + P_2(u)}{2} \tag{4.60}$$

we can combine Eqs.(4.57), (4.58) to give

$$P_1'(u) = -E_+ P_2(u) \tag{4.61}$$

$$P_2'(u) = E_- P_1(u) \tag{4.62}$$

where

$$E_+ = \sigma + \lambda\Lambda \tag{4.63}$$

$$E_- = \sigma - \lambda\Lambda \tag{4.64}$$

Decoupling is attained by introducing

$$P_1(u) = \sqrt{E_+}T(u) \tag{4.65}$$

$$P_2(u) = \sqrt{E_-}S(u) \tag{4.66}$$

where we obtain a pair of one-dimesional Schrödinger-type wave equations,

$$T''(u) + \sigma^2 T(u) = V_T T(u) \tag{4.67}$$

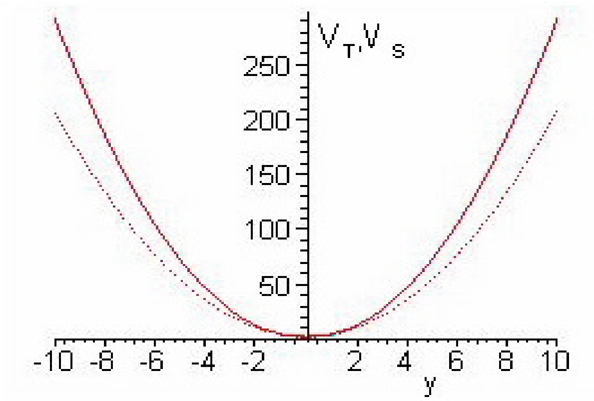


Figure 4-1: The potential barriers, $V_T$ (the full-line curves) and $V_S$ (the dotted curves) surrounding a NH geometry of Kerr BH for the incidence of neutrino waves with $\lambda = \frac{3}{2}$ and $\sigma = 0.65$.

$$S''(u) + \sigma^2 S(u) = V_S S(u) \tag{4.68}$$

with the potentials

$$V_T = \lambda^2 \Lambda^2 (1 + \frac{y^2}{4E_+^2}) - \frac{\lambda X}{2E_+^2} \left(\sigma \left(1 + 2y^2\right) - \lambda \Lambda^3\right) \tag{4.69}$$

$$V_S = \lambda^2 \Lambda^2 (1 + \frac{y^2}{4E_-^2}) + \frac{\lambda X}{2E_-^2} \left(\sigma \left(1 + 2y^2\right) + \lambda \Lambda^3\right) \tag{4.70}$$

One can easily observe that for $y \to \pm\infty$ the potentials diverge. This result stems from the fact that our space-time is not asymptotically flat.

To examine the existence of the superradiance, one may consider the conserved net current of Dirac particles [2] – in other words, the rate $\left(\frac{\partial N}{\partial t}\right)_{in}$ at which particles falling through the horizon per unit time, which must be negative for the superradiance to occur:

$$\left(\frac{\partial N}{\partial t}\right)_{in} = -\left(\int \sqrt{-g} J^y d\theta d\varphi\right)|_{Horizon} < 0 \tag{4.71}$$

where $g$ is the determinant of the spacetime metric and $J^y$ is the axial component of the neutrino particle current.

We recall from metric (4.7) that we have

$$\sqrt{-g} = F \sin\theta \tag{4.72}$$

It is clear from transformations (4.3), (4.6) that the horizon of metric (4.7) corresponds to $y \to (-\infty)$. In other words, integral (4.71) is taken over $y \to (-\infty)$.

Using the definition (3.90), we evaluate $J^y$ as

$$J^y = \sqrt{\frac{1+y^2}{F}} \left(|g_2|^2 - |g_1|^2\right) \left(|A_1|^2 + |A_3|^2\right) \tag{4.73}$$

Assuming that the angular functions $A_1(\theta)$ and $A_3(\theta)$ are normalized to unity, the integral in Eq.(4.71) yields

$$\int \sqrt{-g} J^y d\theta d\varphi = 3.246\pi \left(|Z_2|^2 - |Z_1|^2\right) \tag{4.74}$$

From Eqs.(4.59), (4.60), (4.61), (4.62), we successively find

$$\begin{aligned} |Z_2|^2 - |Z_1|^2 &= \frac{i}{2}\left(P_1\overline{P}_2 - P_2\overline{P}_1\right) \\ &= \frac{-i}{2E_+}\left[P_1\frac{d\overline{P}_1}{du} - \overline{P}_1\frac{dP_1}{du}\right] \\ &= \frac{-i}{2}\left[T,\overline{T}\right]_u \end{aligned} \tag{4.75}$$

where $\left[T,\overline{T}\right]_u$ is the Wronskian.

Therefore, in order to check the existence of superradiance, it will suffice to seek the solution for $T$ at the horizon.

The reality that the potentials $V_T$ and $V_S$ become infinite both at the horizon and $y \to \infty$, leads us to think the problem as a problem of particles in an infinite potential well. Since the particles are bound inside the well, the principal physical fact requires that the solutions of the wave equations (4.67), (4.68) must be identically zero at the walls (the horizon and $y \to \infty$). Clearly, the Wronskian vanishes at the horizon and it follows that the number of particles exiting the horizon per unit time is zero. Consequently, similar to the general Kerr background [2,3], there is also no superradiance in the extreme Kerr throat geometry.

# Chapter 5

# Conclusion

Our aim in this thesis was to go beyond separating the Dirac equation in the geometries of BR and the throat – namely, the extremal Kerr throat geometry – and to obtain exact solutions if possible. These targets have mostly been accomplished and they will definitely contribute to the wave mechanical aspects of spin$-\frac{1}{2}$ particles around these geometries (the regions which encompass these geometries).

In the BR geometry, the motivation for using the Dirac equation with charge coupling came originally from the attempt to show how charged spin$-\frac{1}{2}$ particles behave in a uniform electromagnetic field. Here, using the static gauge, we considered only the electric fields. The separation of the Dirac equation into angular and axial parts was shown. Due to the spherical symmetry, the presence of the charge does not affect the behavior of the incoming Dirac particle in the angular direction. The possible solutions of the angular part are given in terms of both Jacobi polynomials and spin-weighted spheroidal harmonics. However, the dominant effect grew up to be in the axial direction. It was shown that the axial Dirac equation can be reduced to a Riccati type of differential equation. In addition to this, for the neutrino case, the axial part was analytically solved. In order to see the nature of potentials, the axial part was redecoupled to a pair of one-dimensional Schrödinger-type wave equations. The behavior of potentials were studied graphically by considering the massive and neutrino cases keeping the other parameters unchanged. In the neutrino case, the potentials acted like an infinite potential well, which has boundaries at $x \rightarrow \pm\infty$. It was seen that the charge plays a crucial role in the behavior of the potentials. In particular, the existence of charge effects locations where the potentials diverge and as a result, changes the structures of the potentials. For a charged Dirac particle, the corresponding potential has a singularity at a definite location, $x^*$, of spacetime and it varies smoothly to the large distances. But the problem of using the charge coupling in the Dirac equation showed itself when the potentials fall dawn to negative infinity at $x^*$, which is a well known problem since the study of electromagnetic waves on Kerr geometry [2]. At large distances, we found that the waves exhibit two types of behavior: while $k^2 \leq \mu^2 + \lambda^2$, the waves are in the case of the damping oscillation and while $k^2 > \mu^2 + \lambda^2$, they become sinusoidal. Finally, we showed that the net current of the Dirac particles is conserved and Wronskian is constant in the BR spacetime.

In the general Kerr background the radial Dirac equation was the harder part to tackle compared with the angular part [2]. But in the present problem of the extremal Kerr throat we found the opposite is true: the axial part poses no more difficulty than the angular part does. Essentially, the axial part is independent of mass and admits solution in terms of the well-known associated Legendre functions. The difficulty in the angular part is overcome for the massless case and we attain an exact solution in terms of Heun polynomials. Inclusion of mass prevents this reduction and as a result we are unable to express the angular equation in terms of a set of known equations. This part of the problem can be handled numerically. Alternatively, the angular equation is cast into a pair of one-dimensional Schrödinger-type wave equations. Unlike the scalar field case Dirac fields exhibit no superradiance.

The charge coupling of a Dirac particle to an extremal Kerr-Newmann BH in its near horizon limit may reveal more information compared to the present case. Another interesting application for our study would be to find reflection and transmission coefficients of the incoming wave during its interaction with the potential. These are beyond the scope of this thesis and will be dealt within the future.

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