MAMEDOV TYPE M-SINGULAR INTEGRAL OPERATORS AND THEIR CONVERGENCE PROPERTIES

by

AYŞENUR YAVUZ

THESIS SUBMITTED TO
THE GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES
OF
THE ABANT İZZET BAYSAL UNIVERSITY
IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE
DEGREE OF
MASTER OF SCIENCE
IN
THE DEPARTMENT OF MATHEMATICS

FEBRUARY 2014

Approval of the Graduate School of Natural and Applied Sciences

_____

Prof. Dr. Yaşar DÜRÜST

Director

I certify that this thesis satisfies all the requirements as a thesis for the degree of Master of Science.

_____

Prof. Dr. Zafer ERCAN

Head of Department

This is to certify that we have read this thesis and that in our opinion it is fully adequate, in scope and quality, as a thesis for the degree of Master of Science.

_____

Assist. Prof. Dr. İsmail Uğur TİRYAKİ

Supervisor

Examining Committee Members

Assoc. Prof. Dr. Ayşegül ERENÇİN (AİBÜ) _____

Assist. Prof. Dr. İsmail Uğur TİRYAKİ (AİBÜ) _____

Assist. Prof. Dr. Ayşe Çiçek GÖZÜTOK (AİBÜ) _____

ABSTRACT

MAMEDOV TYPE M-SINGULAR INTEGRAL OPERATORS AND THEIR CONVERGENCE PROPERTIES

YAVUZ, Ayşenur

M.Sc., Department of Mathematics

Supervisor: Assist. Prof. Dr. İsmail Uğur TİRYAKİ

February 2014, 76 pages

This thesis is a survey on some approximation properties of Mamedov type m-singular integral operators in the approximation theory. One of the fundamental problems of analysis is to approximate a given function f in some sense or other by functions having certain properties, and generally, by functions which have 'better' properties than f. It is to be expected that the better behaved functions are to be constructed from the given f by some smoothing operation on f itself.

This thesis consists of five chapters. The first chapter is devoted to the introduction. The second chapter contains concepts, definitions and also some important theorems, which we need further studies and calculations. In the third chapter, we give some definitions and properties about the modulus of continuity and we give informations Approximation problem. In the fourth chapter, we study m-singular integral operators and m-th modulus of smoothness, we prove theorems about the rate of convergence of m-singular integral operator on (p, m)-Lebesgue point and (μ, p, m)-Lebesgue point.

Finally in the last chapter we study Mellin type nonlinear m-singular integral operators and its pointwise convergence.

Keywords: Approximate identity, Modulus of continuity, Singular integral operator, m-singular Integral, m-Modulus of smoothness.

ÖZET

MAMEDOV TİPLİ M-SİNGÜLER İNTEGRAL OPERATÖRLER VE ONLARIN YAKINSAKLIK ÖZELLİKLERİ

YAVUZ, Ayşenur

Yüksek lisans, Matematik Bölümü

Tez Yöneticisi: Yard.Doç. Dr. İsmail Uğur TİRYAKİ

February 2014, 76 sayfa

Bu tez, Mamedov tipli m-singüler integral operatörlerin yaklaşım özellikleri üzerine yapılan bir çalışmadır. Analizin temel problemlerinden biri f gibi bazı kötü özelliklere sahip bir fonksiyona, daha iyi özellikleri olan başka bir fonksiyonla yaklaşmaktır.

Bu tez beş bölümden oluşmaktadır. Birinci kısım giriş bölümüne ayrılmıştır. İkinci bölüm çalışmaya kaynak oluşturacak kavramlar, tanımlar ve ayrıca ileriki çalışmalar ve hesaplamalar için gerekli olan bazı teoremler içermektedir. Üçüncü bölümde pozitif çekirdekli operatorlerle ilgili bazı bilgiler, süreklilik modülü ile ilgili tanımlar ve bu tanımlara ait bazı özellikler ve yaklaşım problemi hakkında bilgi verilmiştir. Dördüncü bölümde m-singular integral kavramı tanıtılarak m-yinci düzgünlük modülü tanımı ve bu tanıma dair bazı teoremler ispatlanmıştır. Bu bölümün sonunda ise (p, m)-Lebesgue ve (μ, p, m)-Lebesgue noktaları tanımları yapılmış ve bu noktalardaki m-singüler integralin yaklaşım teoremleri ispatlanmıştır.

Son bölümde m-singular integral yapısını Mellin tipli nonlinear m-singular in-

tegral operatörde incelenmiştir. Çarpımsal grup üzerinde Lebesgue noktası tanımı verilip noktasal yakınsaklığı incelenmiştir.

Anahtar Kelimeler: Pozitif çekirdekli integral operatörler, Süreklilik modülü, Lebesgue noktası, m-singüler integral, m-yinci düzgünlük modülü.

To my family, my friends and my supervisor...

ACKNOWLEDGEMENTS

I would like to express sincere appreciation to my supervisor, Assist. Prof. Dr. İsmail Uğur Tiryaki for his motivation, helpful discussions, encouragement, patience and constant guidance during this work. I would also like to thank Assoc. Prof. Dr Harun Karslı for his suggestions.

I am grateful to express my appreciation to my family, relatives and friends for their continuous helps.

TABLE OF CONTENTS

NOTATIONS

$\omega_{\mathbb{C}[a,b]}(f;\delta)$	$The\ \ modulus\ \ of\ \ continuity\ \ of\ \ f\ defined\ on\ [a,b]$
$\omega_{L_p}(f;\delta)$	$The\ \ modulus\ \ of\ \ continuity\ \ of\ \ f\ \in L_p$
$\omega_m(f;\delta)_{L_p}$	$The\ \ modulus\ \ of\ \ smoothness\ \ of\ \ f\ \in L_p$
$\omega_m(f;\delta)_{C[a,b]}$	$The\ \ modulus\ \ of\ \ smoothness\ \ of\ \ f\ defined\ on\ [a,b]$
$L_1(-\infty,+\infty)$	$Functions\ defined\ on\ (-\infty,+\infty)\ and\ integrable$
$L_p(-\infty,+\infty)$	$Functions\ defined\ on\ (-\infty,+\infty)\ and\ integrable\ to\ the\ p-th\ power$
$\|.\|_{L_p}$	$The\ norm\ on\ the\ L_p\ space$
$\|.\|_{L_1}$	$The\ norm\ on\ the\ L_1\ space$
$(F_n)(x)$	$Fejer\ kernel$
$(P_r)(\theta)$	$Poisson\ kernel$
$(A_\varepsilon)(x)$	$Abel-Poisson\ kernel$
$(W_n)(x)$	$Gauss-Weierstrass\ kernel$
$(L_n^m f)(x)$	$Mamedov\ Type\ m-singular\ integral\ operator$
$\triangle_t^m f(x)$	$m-th\ finite\ difference\ of\ f(x)$
(O,o)	$Landou\ symbols$

CHAPTER 1

INTRODUCTION

One of the fundamental problems of analysis is to approximate a given function f in some sense or other by functions having certain properties and generally, by functions which have "better" properties than f. Approximation Theory is that area of analysis which, at its cone, is concerned with the ability to approximate functions by simpler and more easily calculated functions.

This theory takes its origin from at 19th century. At the beginning of the century the functions were viewed by the formulas, such as series, or as a solutions of differential equations. However largely as a consequence of the claims Fourier and result of Dirichlet and the modern concept of a function by its properties were introduced.

The birth of Approximation Theory becomes an unavailable development after a new definitions of the functions. It is in the theory of Fourier series that we find some of the first result of the approximation theory. This include conditions as a functions that ensure the pointwise convergence and uniform convergence of its Fourier Series.

The first question we ask in Approximation Theory concerns the possibility of approximation. Is the given family of functions from which we plan to approximate dense in the set of functions we wish to approximate? The Weierstrass Approximation Theorems spawned numerous generalization which were applied to other families of functions. They also led to the development of two general methods for determining the density. These are Stone-Weierstrass (S.W.) Theorem generalizing the Weierstrass Theorem to the normed space. Especially in

S.W. Theorem the normed space must be compact. A different and more modern approach to density theorem is due to Functional Analysis.

Suppose that a given function f can be shown that the limit of set of functions $\varphi_n(x)$ at the point which has good properties at x_0. Namely,

$$\lim_{n\to\infty} \varphi_n(x_0) = f(x_0)$$

be as described above. In this case the series of $\varphi_n(x)$ approximates or converges to f.

The second fundamental problem of the approaching theorem is finding the rate of approach to the problem.

$$\lim_{n\to\infty} \varphi_n(x_0) = f(x_0)$$

while $n \to \infty$, then the series $(\alpha_n) = (f(x_0) - \varphi_n(x_0))$ is a zero sequence. If we find an another zero sequence β_n with $\alpha_n = o(\beta_n)$, then one has

$$f(x_0) - \varphi_n(x_0) = o(\beta_n).$$

This Landau representation shows that the $f(x_0) - \varphi_n(x_0)$ is approaches to zero more rapidly than β_n, when $n \to \infty$.

In calculus we consider the real line $\mathbb{R}$ and real-valued functions on $\mathbb{R}$ (or on a subset of $\mathbb{R}$). Obviously, any such function is a mapping of its domain into $\mathbb{R}$. In functional analysis we consider more genaral spaces, such as metric spaces and normed spaces, and mappings of these spaces. In the case of vector spaces and, in particular, normed spaces, a mapping is called an operator.

Throughout this thesis we will consider the m-singular integral operator. Sup-

pose that the m-th finite difference of $f(x)$ with the step t

$$\triangle_t^m f(x) = \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} f(x+kt) \quad , \quad m \in \mathbb{N}.$$

In 1963 Mamedov defined a generalization of the linear integral operators by using m-th finite difference operator as

$$(L_n^m f)(x) = \int_{-\infty}^{\infty} \left[\sum_{k=1}^{m} (-1)^{k-1} \binom{m}{k} f(x+kt) \right] K_n(t)\, dt, \qquad n = 1, 2, ...$$

where $m \geq 1$, called Mamedov type m-singular integral operators.

CHAPTER 2

PRELIMINARIES

In this chapter, we give some necessary definitions and theorems, which is used in this thesis.

2.1 Some Definitions

Definition 2.1. If A is a subset of the topological space X and if x is a point of X, we say that x is a *limit point* (or *accumulation point*) of A if every neighborhood of x intersects A in some point other than x itself. Said differently, x is a limit point of A if it belongs to the closure of $A - \{x\}$. The point x may lie in A or not; for this definition it does not matter.

Definition 2.2. We say that $f(x)$ approaches the limit L as x approaches a, and we write

$$\lim_{x \to a} f(x) = L$$

if the following condition is satisfied:
For every number $\varepsilon > 0$ there exists a number $\delta > 0$, depending on ε, such that

$$0 < |x - a| < \delta \quad \textit{implies} \quad |f(x) - L| < \varepsilon.$$

Definition 2.3. Suppose $\{f_n\}$, $n = 1, 2, 3, ..,$ is a sequence of functions defined on set E, and suppose that the sequence of numbers $\{f_n(x)\}$ converges for every $x \in E$. We shall say that $\{f_n\}$ converges to f *pointwise* on E, that is,

$$\lim_{n \to \infty} f_n(x) = f(x) \qquad , \ (x \in E).$$

Definition 2.4. We say that a sequence of functions $\{f_n\}$, $n = 1, 2, 3, ..,$ converges $uniformly$ on E to a function f if for every $\varepsilon > 0$ there is an integer N such that $n \geq N$ implies

$$|f_n(x) - f(x)| \leq \varepsilon$$

for all $x \in E$.

Definition 2.5. The sequence f_n uniformly converge to f on $A \Leftrightarrow \forall \varepsilon > 0\ \exists n_0$ such that $n \geq n_0$ and $\forall x \in A$, $|f_n(x) - f(x)| < \varepsilon$.

Definition 2.6. $(C[a, b])$. As a set X we take the set of all real- valued functions $x, y, \cdots$ which are functions of an independent real variable t and are defined and continuous on a given closed interval $J = [a, b]$. Choosing the metric defined by

$$d(x, y) = \max_{t \in J} |x(t) - y(t)|,$$

where max denotes the $maximum$, a metric space is denoted by $C[a, b]$.

Definition 2.7. A map $T : X \to Y$ between two vector spaces is called an operator.

Definition 2.8. A linear operator T is an operator such that

- The domain $D(T)$ of T is a vector space and the range $R(T)$ lies in a vector space over the same field,
- for each $x, y \in D(T)$ and scalar α,

$$T(x + y) = Tx + Ty$$

and

$$T(\alpha x) = \alpha Tx.$$

Definition 2.9. (Integral Operator). We can define an integral operator $T : C[0,1] \to C[0,1]$ by

$$y = Tx, \qquad where \qquad y(t) = \int_0^1 x(t)k(t,\tau)d\tau.$$

Here k is a given function which is called the kernel of T and is assumed to be continuous on the closed square $G = J \times J$ in the $t\tau - plane$, where $J = [0,1]$. This operator is linear.

Definition 2.10. (Modulus of Continuity) Let $f \in C[a,b]$ and $\delta > 0$ be given. Then modulus of continuity of f is given by;

$$\omega(f;\delta) = \sup_{|t-x|\leq\delta \ , \ t,x\in[a,b]} |f(t) - f(x)|$$

Definition 2.11. ($L_1[a,b]$ Space). $L_1[a,b]$ is the set of functions which are Lebesgue integrable to the first power of the period $[a,b]$. For $f \in L_1[a,b]$

$$||f||_1 = \int_a^b |f(t)|dt.$$

(Butzer and Nessel, 1971)

Definition 2.12. ($L_p[a,b]$ Space). $L_p[a,b]$ is the set of functions which are Lebesgue integrable to the p^{th} power of $[a,b]$. For $f \in L_p[a,b]$

$$||f||_p = \left\{ \int_a^b |f(t)|^p dt \right\}^{1/p}.$$

(Butzer and Nessel, 1971)

Definition 2.13. ($L_p(-\infty,\infty)$ Space). L_p is the set of functions which are Lebesgue integrable to the p^{th} power over $\mathbb{R}$, for $1 \leq p < \infty$, and essentially

bounded (bounded almost everywhere) on $\mathbb{R}$ if $p = \infty$. For $f \in L_p(-\infty, \infty)$

$$||f||_p = \left\{ \int_{-\infty}^{\infty} |f(t)|^p dt \right\}^{\frac{1}{p}},$$

if $1 \leq p < \infty$ and in case $p = \infty$

$$||f||_\infty = ess \sup_{x \in \mathbb{R}} |f(t)|.$$

(Butzer and Nessel, 1971)

Definition 2.14. (Hölder Inequality). Let $f \in L_p$, $1 \leq p \leq \infty$, and $g \in L_q$. Then $fg \in L_1$ and $||fg||_1 \leq ||f||_p ||g||_q$. Namely;

$$\left| \int fg \right| \leq \left(\int |f|^p \right)^{1/p} \cdot \left(\int |g|^q \right)^{1/q}.$$

Definition 2.15. (Minkowsky Inequality). Let $f, g \in X(\mathbb{R})$. Then $(f + g) \in X(\mathbb{R})$ and

$$||f + g||_{X(\mathbb{R})} \leq ||f||_{X(\mathbb{R})} + ||g||_{X(\mathbb{R})}.$$

If p is such that $1 \leq p \leq \infty$, the conjugate number q is defined through $\frac{1}{p} + \frac{1}{q} = 1$ in case $1 < p < \infty$; $q = \infty$ if, $p = 1$, and $q = 1$ if $p = \infty$.

Definition 2.16. (Generalized Minkowsky Inequality). Let $f(x, y)$ be defined and measurable on $\mathbb{R}^2$. If $||f(x, y)||_{X(\mathbb{R})} \in L_1$, then

$$\left|\left| \int_{-\infty}^{\infty} f(x, y) dy \right|\right|_{X(\mathbb{R})} \leq \int_{-\infty}^{\infty} ||f(x, y)||_{X(\mathbb{R})} dy.$$

Definition 2.17. ($X(\mathbb{R})$ Space). $X(\mathbb{R})$ always denotes one of the spaces C or L_p, $1 \leq p < \infty$. For $f, g \in X(\mathbb{R})$ we write $f(x) = g(x)$ (a.e.) if equality holds for all $x \in X(\mathbb{R})$ in case $X(\mathbb{R}) = C$, and almost everywhere in case $X(\mathbb{R}) = L_p$, $1 \leq p < \infty$, i.e., if $||f - g|| = 0$. In this event we also write $f = g$ in $X(\mathbb{R})$.

Definition 2.18. (Lipschitz condition) A finite function $f(x)$ defined on $[a,b]$ is said to satisfy a Lipschitz condition if there exists a constant K such that for any two points x and y in $[a,b]$,

$$|f(x) - f(y)| \leq K|x-y|$$

holds true.

(Natanson, Volume I, 1964)

Definition 2.19. A finite function $f(x)$ is defined on $[a,b]$, is said to satisfy a Hölder condition if there exists a constant K such that for any two points x and y in $[a,b]$,

$$|f(x) - f(y)| \leq K.|x-y|^{\alpha}$$

where $0 < \alpha \leq 1$. Then the functions satisfying Hölder condition define the following set of functions ;

$$Lip_K\alpha = \{f : [a,b] \to \mathbb{R}, |f(x) - f(y)| \leq K.|x-y|^{\alpha}\},$$

for all x,$y \in [a,b]$ and $0 < \alpha \leq 1$.

CHAPTER 3

SINGULAR INTEGRAL OPERATORS

In this chapter we will define singular integral and its properties.Also we will give some examples of singular integral .

Definition 3.1. Let X and Y be function space and L be a rule

$$\begin{array}{rcl} L & : & X \to Y \\ & & f \to (Lf) \end{array},$$

where (Lf) is defined as

$$(Lf)(x) = g(x).$$

Let X be a function space and $f_{1,}f_2 \in X$, a_1 and a_2 are scalars, if

$$(La_1f_1 + a_2f_2)(x) = a_1(Lf_1)(x) + a_2(Lf_2)(x)$$

is satisfied that L is called linear operator. If there exists a positive function on Y for all positive function on X, then L is called positive linear integral operator. Linear integral operator is defined as

$$(Lf)(x) = \int_a^b f(t)K(x,t)dt.$$

Let X be function space and f be a function on the same space, Λ is the set of indices λ_0 is a limit point of this set.

We consider the integral

$$(L_\lambda f)(x) = \int_a^b f(t)K_\lambda(x,t)dt \tag{3.1}$$

for $\lambda \in \Lambda$ on $x \in (a,b)$. By substituting f to different function from X space, find new $f(x)$ functions. Therefore equation (3.1) defines an integral operator. Interval (a,b) can be bounded or unbounded. $K_\lambda(x;t)$ is a function that it is defined on $(a,b) \times (a,b)$ and properties are defined before.

This function is kernel of operator. If $K_\lambda(x;t)$ is a positive function, then operator (3.1) is called positive integral operator.

If $t = t_0$ point

$$\lim_{\lambda \to \lambda_0} K_\lambda(t_0;x) = \infty$$

yields that (3.1) is called singular integral family.

When Λ is natural number set and $\lambda_0 = \infty$, (3.1) is called the sequence of singular integral operator.

Definition 3.2. (Kernel) A kernel is a function $\phi_n(t,x)$ which $(n = 1,2,3,...)$ defined in the square $(a \leq t \leq b, a < x < b)$ and such that;

$$\lim_{n\to\infty} \int_\beta^\theta \phi_n(t,x)\,dt = 1$$

provided where, $a \leq \beta < x < \theta \leq b$. It is self-evident that $\phi_n(t,x)$ is assumed differentiable with respect to t for every fixed x.

And also note that; an integral of the form;

$$f_n(x) = \int_a^b \phi_n(t,x)\, f(t)\, dt$$

where $\phi_n(t,x)$ is a kernel, is called a *singular integral*.

(Natanson Volume II, 1960)

Definition 3.3. Let $f \in X_{2\pi}$ and $\{K_\lambda(x)\}$ be a kernel. Then we call an expression of the form

$$I_\lambda(f;x) = (f * K_\lambda)(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-u)\,K_\lambda(u)\,du$$

a (periodic) singular integral (or convolution integral). We say that the singular integral is positive, continuous if the corresponding kernel is positive, continuous.

(Butzer and Nessel 1971)

In general, a singular integral may be written in the form

$$\Phi_n(x) = \int_a^b f(t)\,K_n(x,t)\,dt$$

where $K_n(x,t)$ is the "kernel", defined for $a \le x \le b$, $a \le t \le b$, which has the property that for functions $f(x)$ of a certain class and in a certain sense, $\Phi_n(x)$ converges to $f(x)$ as $n \to \infty$.

There are lots of singular integral which has positive kernel. Some of them will be given as example in the next part.

First example of singular integral is Fejér intgeral. Now we will show some properties of Fejér integral.

Fejér Integral

Let f be a 2π periodic function. the equation

$$S_n f(x) = \frac{a_0}{2} + \sum_{k=1}^{n}(a_k \cos kx + b_k \sin kx)$$

is partial sum of Fourier Series.

And using Fourier coefficients as

$$\begin{aligned} a_k &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\cos ktdt \quad , \quad k=0,1,2... \\ b_k &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\sin ktdt \quad , \quad k=0,1,2... \end{aligned}$$

we find

$$S_n f(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\frac{\sin\left(n+\frac{1}{2}\right)(t-x)}{2\sin\frac{t-x}{2}}dt \quad , \quad n=0,1,2...$$

Fejér find the Cesàro mean of the sum as

$$\sigma_n f(x) = \frac{S_0 f(x) + S_1 f(x) + ... - S_{n-1} f(x)}{n}.$$

This mean equal to the integral

$$\sigma_n f(x) = \int_{-\pi}^{\pi} \frac{1}{2n\pi}\left[\frac{\sin\frac{n}{2}(t-x)}{\sin\frac{1}{2}(t-x)}\right]^2 dt.$$

This integral is Fejér integral.

Then

$$F_n(t) = \frac{1}{2n\pi}.\left[\frac{\sin\frac{n}{2}t}{\sin\frac{1}{2}t}\right]^2$$

is Fejér Kernel. Fejér Kernel is nonnegative even function.

Properties of Fejér Kernel

i) $F_n(t) > 0$

ii) $F_n(t) = F_n(-t)$

iii) $\int_{-\pi}^{+\pi} Fn(t)dt = 1$

iv) where $t \neq 0$

$$\begin{aligned}
\lim_{n\to+\infty} F_n(t) &= \lim_{n\to+\infty} \frac{1}{n+1}\left[\frac{\sin\left[n+1\right]\frac{t}{2}}{\sin\frac{t}{2}}\right]^2 \\
&= \frac{1}{\sin^2\frac{t}{2}} \lim_{n\to+\infty} \frac{1}{n+1}\left[\sin\left[n+1\right]\frac{t}{2}\right]^2 \\
&\leq \frac{1}{\sin^2\frac{t}{2}} \lim_{n\to+\infty} \frac{1}{n+1} \\
&= 0
\end{aligned}$$

As a result

$$\lim_{n\to+\infty} F_n(t) = 0$$

Let $t = 0$

$$\lim_{n\to+\infty} F_n(0) = \lim_{n\to+\infty}\left[\lim_{t\to 0} F_n(t)\right]$$

$$\lim_{t\to 0} \frac{1}{n+1}\left[\frac{\sin\left[n+1\right]\frac{t}{2}}{\sin\frac{t}{2}}\right]^2 = \lim_{t\to 0} \frac{1}{n+1}\left[\frac{\sin\left[n+1\right]\frac{t}{2}}{\sin\frac{t}{2}}\right]^2 \cdot \left[\frac{\left[n+1\right]\frac{t}{2}}{\left[n+1\right]\frac{t}{2}}\right]^2.$$

By using

$$\lim_{t\to 0} \frac{\sin\frac{t}{2}}{\frac{t}{2}} = 1 \Rightarrow \lim_{t\to 0}\left[\frac{\sin\frac{t}{2}}{\frac{t}{2}}\right]^2 = 1$$

and

$$\lim_{t\to 0} \frac{\sin\left[n+1\right]}{n+1} = 1 \Rightarrow \lim_{t\to 0}\left[\frac{\sin\left[n+1\right]}{n+1}\right]^2 = 1.$$

Also

$$\lim_{t\to 0} \frac{\frac{t}{2}}{\sin\frac{t}{2}} = 1 \Rightarrow \lim_{t\to 0}\left[\frac{\frac{t}{2}}{\sin\frac{t}{2}}\right]^2 = 1$$

$$\begin{aligned}
\lim_{t\to 0} \frac{1}{n+1}\left[\frac{\sin\left[n+1\right]\frac{t}{2}}{\sin\frac{t}{2}}\right]^2 \cdot \left[\frac{\left[n+1\right]\frac{t}{2}}{\left[n+1\right]\frac{t}{2}}\right]^2 &= \lim_{t\to 0}\left[n+1\right] \\
&= n+1.
\end{aligned}$$

We get

$$\lim_{N\to+\infty} F_n(0) = +\infty$$

v)For each fixed $0 < \delta < \pi$

$$\lim_{n\to+\infty} \int_{\delta\leq x\leq\pi} F_n(x)dx = 0.$$

For each $\delta \leq x \leq \pi$ we have

$$1 \leq \frac{1}{\sin^2 \frac{x}{2}} \leq \frac{1}{\sin^2 \frac{\delta}{2}}$$

$$0 \leq F_n(x) \leq \frac{1}{n+1}\frac{1}{\sin^2 \frac{\delta}{2}}$$

We taking integral right hand side

$$\begin{aligned}\lim_{n\to+\infty} \int_{\delta\leq x\leq\pi} F_n(x)dx &\leq \lim_{n\to+\infty} \int_{\delta\leq x\leq\pi} \frac{1}{n+1}\frac{1}{\sin^2 \frac{\delta}{2}}dx \\ &= \lim_{n\to+\infty} \frac{1}{n+1}\frac{1}{\sin^2 \frac{\delta}{2}}.2(\pi-\delta) \\ &= 0\end{aligned}$$

As a result

$$\lim_{n\to+\infty} \int_{\delta\leq x\leq\pi} F_n(x)dx = 0.$$

Poisson Integral

The polar coordinate form of the Laplace Operator is given by

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

and similarly

$$\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

is called Laplace Equation for $x = r\cos\theta$, $y = r\sin\theta$. By using this equation we write

$$u(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\frac{1-r^2}{1-2r\cos(t-\theta)+r^2}dt \quad , 0 < r < 1 \tag{3.2}$$

This is the solution of Dirichlet problem on unit circle. The equation (3.2) is called Poisson integral.

Let

$$P_r(\theta) = \frac{1}{2\pi}\frac{1-r^2}{1-2r\cos(\theta)+r^2}.$$

The above function $P_r(\theta)$ is known as Poisson Kernel.

Properties of Poisson Kernel

i) $P_r(\theta) \geq 0$ is nonnegative

ii) $P_r(\theta) = P_r(-\theta)$ is even as a function of θ

iii)

$$\int_0^{2\pi} P_r(\theta)d\theta = \int_0^{2\pi} \frac{1-r^2}{1-2r\cos\theta+r^2}d\theta = 1$$

Indeed; if we choose $u(x) = 1$then we reach the result

iv) Where $\theta \neq 0$

$$\lim_{r\to 1} P_r(\theta) = \lim_{r\to 1}\frac{1}{2\pi}\frac{1-r^2}{1-2r\cos\theta+r^2} = 0$$

And where $\theta = 0$

$$\lim_{r\to 1} P_r(\theta) = \lim_{r\to 1}\frac{1}{2\pi}\frac{1-r^2}{1-2r\cos\theta+r^2} = +\infty$$

Abel-Poisson Integral

Laplace Operator is given as

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

and equation

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

is called Laplace equation. Dirichlet problem is finding the u function which satisfied Laplace equation and satisfied $u(x,0) = f(x)$.

The function

$$f_\varepsilon(x) = \frac{\varepsilon}{\pi} \int_{-\infty}^{+\infty} f(t).\frac{1}{\varepsilon^2 + (t-x)^2} dt,\ \varepsilon > 0 \tag{3.3}$$

is solution of the dirichlet problem on the upper half plane.

Let

$$A_\varepsilon(x) = \frac{\varepsilon}{\pi}.\frac{1}{\varepsilon^2 + x^2}$$

$A_\varepsilon(x)$ is known as the Abel-Poisson kernel and (3.3) is called Abel-Poisson integral.

Properties of Abel-Poisson Kernel

i) $A_\varepsilon(x)$ is nonnegative

ii) $A_\varepsilon(x) = A_\varepsilon(-x)$ is even kernel

iii)

$$\int_{-\infty}^{+\infty} A_\varepsilon(x)dx = 1$$

iv)where $x \neq 0$

$$\lim_{\varepsilon \to 0} A_\varepsilon(x) = \lim_{\varepsilon \to 0} \frac{\varepsilon}{\pi}.\frac{1}{\varepsilon^2 + x^2} = 0$$

where x=0

$$\lim_{\varepsilon \to 0} A_\varepsilon(0) = \lim_{\varepsilon \to 0} \frac{\varepsilon}{\pi}.\frac{1}{\varepsilon^2} = +\infty.$$

Gauss-Weierstrass Integral

Let f be a given function

$$u(x,0) = f(x)$$

holds as initial condition

$$u(x,t) = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} f(\xi) e^{-\frac{(\xi - x)^2}{4t}} d\xi.$$

Also satisfy heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}.$$

If we write $\lambda = \frac{1}{2}\sqrt{t}$ and $u(x,t) = u_\lambda(x)$.

Then one has

$$u_\lambda(x) = \frac{\lambda}{\pi} \int_{-\infty}^{\infty} f(\xi) e^{-\lambda^2(\xi - x)^2} d\xi \ , \ \lambda > 0.$$

denote this integral is called Gauss-Weierstrass integral.

$$w_\lambda(x) = \frac{\lambda}{\pi} e^{-\lambda^2 x^2}$$

is known as Gauss-Weierstrass kernel. It is clear that Gauss-Weierstrass kernel is nonnegative even function.

3.1 Approximation Problem

Generally we investigate the approximation problem to a function f by functions which have better properties than f .

Suppose that a given function f can be shown that the limit of sequence $\varphi_n(x)$ at the point x_0, Namely

$$\lim_{n\to+\infty} \varphi_n(x_0) = f(x_0) \equiv \lim_{n\to+\infty} [\varphi_n(x_0) - f(x_0)] = 0$$

This convergence is the first problem in analysis.The second problem is to find the rate of the convergence.

Let us consider $\alpha_n = \varphi_n(x_0) - f(x_0)$. Then $\alpha_n \to 0$ so it is a zero sequence.

If we find a new zero sequence (β_n) and

$$\lim_{n\to\infty} \frac{\alpha_n}{\beta_n} = 0 \quad \Rightarrow \quad 0 \le \alpha_n \le \beta_n$$

In this case the rate of the convergence of α_n is greater than the rate of convergence β_n

$$\lim_{n\to\infty} \frac{\alpha_n}{\beta_n} = a \quad (a \in \mathbb{R})$$

In this case they have same rate.

$$\lim_{n\to\infty} \frac{\alpha_n}{\beta_n} = 0 \Rightarrow 0 \le \beta_n \le \alpha_n$$

(β_n) is faster than (α_n) .

Definition 3.4. Let

$$\lim_{n\to+\infty} \|f_n(x) - f(x)\|_x = K \qquad \|f_n(x) - f(x)\|_x = O(\alpha_n)$$

then the rate of convergence of $f_n(x)$ to $f(x)$ in the norm is equal to the rate of convergence of α_n to zero

If

$$\lim_{n\to+\infty} \frac{\|f_n(x) - f(x)\|_x}{\alpha_n} = 0,$$

then one can say that $\|f_n(x) - f(x)\|$ is faster than α_n and we write

$$\|f_n(x) - f(x)\| = o(\alpha_n).$$

If

$$\lim_{n\to+\infty} \frac{\|f_n(x) - f(x)\|_x}{\alpha_n} = +\infty,$$

we write

$$\alpha_n = o\left(\|f_n(x) - f(x)\|_x\right)$$

This kind of representations or the symbols (o, O) are called Landau symbols.

In general we can define the comparin zero sequence by the following method.

Such kind of zero sequence can be considered as " Modulus of Continuity ".

3.2 Modulus of Continuity

Definition 3.5. Let $[a, b]$ be a bounded or unbounded interval and f be a bounded function on this interval where $\delta > 0$

$$w_{\mathbb{C}[a,b]}(f;\delta) = \sup_{|t|\leq\delta} |f(x+t) - f(x)|$$

is modulus of continuity of function f. $w_{\mathbb{C}[a,b]}$ is a nonnegative and monotone increasing function with respect to δ.

Also f is continuous function and $[a,b]$ is a closed interval

$$\lim_{\delta \to 0^+} w_{\mathbb{C}[a,b]}(f;\delta) = 0 \tag{3.4}$$

really f is continuous on $[a,b]$ then f is also uniformly continuous on this interval. That is $\forall \varepsilon > 0$ and $\forall x,t \in [a,b]$ such that where $|x-t| < \eta$

$$|f(x) - f(t)| < \varepsilon \tag{3.5}$$

this yields.

If we choose $\delta > 0$ and $\delta < \eta$, then equation (3.5) again holds. And this shows that equation (3.4) is true.

Equality (3.4) is the most important property of modulus of continuity at $\mathbb{C}[a,b]$.

Definition 3.6. For $f \in X_{2\pi}$ the modulus of continuity is defined by for $\delta > 0$ by ;

$$w(f;\delta) = \sup_{|h| \le \delta} \|f(.,h) - f(.)\|_{X_{2\pi}}$$

If $X_{2\pi} = \mathbb{C}[-\pi,\pi]$, then

$$w(f;\delta) = \sup_{|t| \le \delta} \left[\max_{x \in [-\pi,\pi]} |f(x+h) - f(x)| \right] = 2\delta$$

Definition 3.7. The modulus of continuity of $f \in L_p$ is shown $w_{L_p}(f;\delta)$ and $\delta > 0$ defined as

$$w_{L_p}(f;\delta) = \sup_{|h| \le \delta} \left(\int_{-\infty}^{+\infty} |f(x+h) - f(x)|^p \, dx \right)^{\frac{1}{p}} \qquad , 1 \le p < \infty \tag{3.6}$$

w_{L_p} is a monotone increasing with respect to δ and nonnegative function. Also it holds for $f \in L_p(-\infty, +\infty)$

$$\lim_{\delta \to 0} w_{L_p}(f;\delta) = 0 \tag{3.7}$$

Theorem 3.1. *Let* $f \in L_p$

i) $w_{L_p}(f; a\delta) \leq a w_{L_p}(f;\delta)$ where $a \in \mathbb{N}$

ii) Let $\lambda > 0$ be an arbitrary number $w_{L_p}(f;\lambda\delta) \leq (\lambda+1)\, w_{L_p}(f;\delta)$

Proof.

i)Point out the formulation of modulus of continuity (3.6)

$$w_{L_p}(f; a\delta) = \sup_{|h| \leq a\delta} \left(\int_{-\infty}^{+\infty} |f(x+h) - f(x)|^p \, dx \right)^{\frac{1}{p}}$$

let $h = ay$

$$\begin{aligned} w_{L_p}(f; a\delta) &= \sup_{|y| \leq \delta} \left(\int_{-\infty}^{+\infty} |f(x+ay) - f(x)|^p \, dx \right)^{\frac{1}{p}} \\ &= \sup_{|y| \leq \delta} \left(\int_{-\infty}^{+\infty} |f(x+ay) - f(x+(a-1)y) + f(x+(a-1)y) \right. \\ &\quad -f(x+(a-2)y) + f(x+(a-2)y) \\ &\quad \left. + \cdots + f(x+y) - f(x)|^p \, dx \right)^{\frac{1}{p}} \end{aligned}$$

That is

$$\begin{aligned} w_{L_p}(f;a\delta) &= \sup_{|y|\leq\delta}\left(\int_{-\infty}^{+\infty}\left|\sum_{k=1}^{m} f(x+ky)-f(x+(k-1)y)\right|^p dx\right)^{\frac{1}{p}} \\ &\leq \sup_{|y|\leq\delta}\left(\int_{-\infty}^{+\infty}\sum_{k=1}^{m}|f(x+ky)-f(x+(k-1)y)|^p\,dx\right)^{\frac{1}{p}} \end{aligned}$$

if we say $x+(k-1)y=z$, then

$$w_{L_p}(f;a\delta)\leq\sup_{|y|\leq\delta}\left(\int_{-\infty}^{+\infty}\sum_{k=1}^{m}|f(z+y)-f(z)|^p\,dz\right)^{\frac{1}{p}}$$

this proves (i).

ii) If we use the property of $\lambda < [|\lambda|]+1$ in addition, w_{L_p} is a monotone increasing function. then

$$w(f;\lambda\delta)\leq w(f;([|\lambda|]+1)\,\delta)$$

holds.

use $[|\lambda|]+1$ is an integer and $[|\lambda|]<\lambda$

$$\begin{aligned} w(f;\lambda\delta) &\leq w(f;([|\lambda|]+1)\,\delta) \\ &\leq ([|\lambda|]+1)\,w(f;\delta) \\ &\leq (\lambda+1)\,w(f;\delta). \end{aligned}$$

Lemma: Let $f\in X_{2\pi}$

i) $w(f;\delta)$ is monotone increasing with respect to $\delta\,(\delta\geq 0)$

ii) $w(f;\lambda\delta)\leq(1+\lambda)\,w(f;\delta)$ for each $\lambda>0$

iii)

$$\lim_{\delta \to 0^+} w\left(f;\delta\right) = 0.$$

Theorem 3.2. *Let L_n be an operator and L_n be defined as*

$$L_n f(x) = \int_{-\infty}^{+\infty} f(x-t) K_n(t) dt$$

and for all n, $K_n(t) \geq 0$.

$$\int_{-\infty}^{+\infty} K_n(t) dt = 1$$

where $f \in L_p$ satisfies

$$w_{L_p}(L_n f;\delta) \leq w_{L_p}(f;\delta).$$

Proof.If $f \in L_p$, $K_n \in L_1$then $L_n f \in L_p$ .if we use generalized Minkowsky inequality

$$\begin{aligned}
w_{L_p}(L_n f;\delta) &= \sup_{|h|\leq\delta} \left(\int_{-\infty}^{+\infty} \left| L_n f(x+h) - L_n f(x) \right|^p dx \right)^{\frac{1}{p}} \\
&= \sup_{|h|\leq\delta} \left(\int_{-\infty}^{+\infty} \left| \int_{-\infty}^{+\infty} f\left((x+h)-t\right) K_n(t) dt - \int_{-\infty}^{+\infty} f\left(x-t\right) K_n(t) dt \right|^p dx \right)^{\frac{1}{p}} \\
&= \sup_{|h|\leq\delta} \left(\int_{-\infty}^{+\infty} \left| \int_{-\infty}^{+\infty} \left[f\left((x+h)-t\right) - f\left(x-t\right) \right] K_n(t) dt \right|^p dx \right)^{\frac{1}{p}}
\end{aligned}$$

using by generalized Minkowsky inequality

$$\sup_{|h|\leq\delta} \left(\int_{-\infty}^{+\infty} \left| \int_{-\infty}^{+\infty} \left[f\left((x+h)-t\right) - f\left(x-t\right) \right] K_n(t) dt \right|^p dx \right)^{\frac{1}{p}}$$

$$\begin{aligned}
&\leq \sup_{|h|\leq\delta} \int_{-\infty}^{+\infty} \left(\int_{-\infty}^{+\infty} |f((x+h)-t) - f(x-t)|^p (K_n(t))^p \, dx \right)^{\frac{1}{p}} dt \\
&= \sup_{|h|\leq\delta} \int_{-\infty}^{+\infty} \left(\int_{-\infty}^{+\infty} |f((x+h)-t) - f(x-t)|^p \, dx \right)^{\frac{1}{p}} K_n(t) dt
\end{aligned}$$

$$\begin{aligned}
&= \sup_{|h|\leq\delta} \int_{-\infty}^{+\infty} \left(\int_{-\infty}^{+\infty} |f(x+h) - f(x)|^p \, dx \right)^{\frac{1}{p}} K_n(t) dt \\
&= w_{L_p}(f;\delta).
\end{aligned}$$

This completes the proof.

We can define the modulus of continuity for $f \in X(\mathbb{R})$ as follows

$$w(f;\delta) = \sup_{|h|\leq\delta} \left[\sup_{x\in\mathbb{R}} |f(x+h) - f(x)| \right]$$

or

$$w(f;\delta) = \sup_{|h|\leq\delta} \left[\int_{-\infty}^{+\infty} |f(x+h) - f(x)| \, dx \right].$$

Definition 3.8. Let ρ be a parameter ranging over some set A which is either an interval (a,b) , $-\infty < a < b < +\infty$ or the set $\mathbb{N}$,and ρ_0 be one of the point (a,b) or $+\infty$

A set of functions $\{K_\rho(x)\}$ will be called a periodic kernel.

if $K_\rho \in L^1_{2\pi}$ for each $\rho \in A$ and

$$\int_{-\pi}^{+\pi} K_\rho(x) dx = 2\pi$$

We call the kernel $\{K_\rho(x)\}$ bounded if $K_\rho \in L^\infty_{2\pi}$ continuous if $K_\rho \in \mathbb{C}_{2\pi}$ is absolutely continuous if $K_\rho \in \mathbb{C}_{2\pi}$ for each $\rho \in A$.

A kernel is said to be even if $K_\rho(x) = K_\rho(-x)$, positive $K_\rho(x) \geq 0$ for each $\rho \in A$.

Instead of this condition

$$\int_{-\pi}^{+\pi} K_\rho(x)dx = 2\pi$$

One can use the following general condition

$$\lim_{\rho\to\rho_0} \int_{-\pi}^{+\pi} K_\rho(x)dx = 2\pi$$

Definition 3.9. A kernel $\{K_\rho(x)\}$ is called an (periodic) Approximate identity if with some constant $M > 0$, $\|K_\rho\|_1 \leq M \quad (\rho \in A)$

$$\lim_{\rho\to\rho_0} \int_{-\pi}^{+\pi} |K_\rho(u)|\, du = 0 \tag{3.8}$$

We call an approximate identity even, positive, bounded or continuous if the kernel is even, bounded, continuous or positive.

Instead of (3.8), we can consider the following condition

$$\lim_{\rho\to\rho_0} \left[\sup_{\delta\leq|u|\leq\pi} |K_\rho(u)| \right] = 0$$

If the kernel is positive, then we can assume that $M = 1$.

$M = 1$ is not special choice. This is a choice without loos of the generality.

If $M \neq 1$, then we can normalize it.

Theorem 3.3. *Let $f \in (L_{2\pi}$ or $\mathbb{C}_{2\pi})$ and $\{K_\rho(x)\}$ be a kernel, then we call the expression of the form*

$$(I_g f)(x) = (f * K_\rho)(x) = \frac{1}{2\pi} \int_{-\pi}^{+\pi} f(x-u) K_\rho(u) du$$

a (periodic) Singular Integral (or convolution integral).

Theorem 3.4. *If the kernel function of convolution integral is an approximate identity,*

$$\|(I_g f)(x)\|_{X_{2\pi}} \leq \|K_\rho\|_1 . \|f\|_{X_{2\pi}}$$

for every $f \in X_{2\pi} = (L_{2\pi}, \mathbb{C}_{2\pi})$.

Proof. Let $X_{2\pi} = L_{2\pi}$

$$\begin{aligned} \frac{1}{2\pi} \int_{-\pi}^{+\pi} \left| \int_{-\pi}^{+\pi} f(x-u) K_\rho(u) du \right| dx &\leq \frac{1}{2\pi} \int_{-\pi}^{+\pi} \int_{-\pi}^{+\pi} |f(x-u)| \, |K_\rho(u)| \, du dx \\ &= \frac{1}{2\pi} \int_{-\pi}^{+\pi} |K_\rho(u)| \int_{-\pi}^{+\pi} |f(x-u)| \, dx du \\ &\leq \|K_\rho\|_{L_1} . \|f\|_{L_{2\pi}} . \end{aligned}$$

Let $X_{2\pi} = \mathbb{C}_{2\pi}$

$$\begin{aligned} \max_{x \in [-\pi,\pi]} \left| \int_{-\pi}^{+\pi} f(x-u) K_\rho(u) du \right| &\leq \max_{x \in [-\pi,\pi]} \int_{-\pi}^{+\pi} |f(x-u)| \, |K_\rho(u)| \, du \\ &\leq \max_{x \in [-\pi,\pi]} \int_{-\pi}^{+\pi} \max_{x \in [-\pi,\pi]} |f(x-u)| \, du \\ &\leq \|K_\rho\|_{L_1} . \|f\|_{\mathbb{C}_{2\pi}} \end{aligned}$$

Theorem 3.5. *If the kernel of the convolution (singular) integral is an approximate identity, then for every $f \in X_{2\pi}$*

$$\lim_{\rho \to \rho_0} \|I_\rho f - f\|_{X_{2\pi}} = 0.$$

Proof.

$$\begin{aligned}
(I_\rho f)(x) - f(x) &= \frac{1}{2\pi} \int_{-\pi}^{+\pi} f(x-u) K_\rho(u) du - f(x) \\
&= \frac{1}{2\pi} \int_{-\pi}^{+\pi} f(x-u) K_\rho(u) du - f(x) \frac{1}{2\pi} \int_{-\pi}^{+\pi} K_\rho(u) du \\
&\quad + f(x) \frac{1}{2\pi} \int_{-\pi}^{+\pi} K_\rho(u) du - f(x)
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2\pi} \int_{-\pi}^{+\pi} [f(x-u) - f(x)] K_\rho(u) du \\
&\quad + f(x) \left[\frac{1}{2\pi} \int_{-\pi}^{+\pi} K_\rho(u) du - 1 \right]
\end{aligned}$$

$$\begin{aligned}
\|I_\rho f - f\|_{X_{2\pi}} &= \int_{-\pi}^{+\pi} \left| \frac{1}{2\pi} \int_{-\pi}^{+\pi} [f(x-u) - f(x)] K_\rho(u) du \right. \\
&\quad \left. + f(x) \left[\frac{1}{2\pi} \int_{-\pi}^{+\pi} K_\rho(u) du - 1 \right] \right| dx
\end{aligned}$$

$$\begin{aligned}\|I_\rho f - f\|_{X_{2\pi}} &\leq \frac{1}{2\pi}\int_{-\pi}^{+\pi}\int_{-\pi}^{+\pi} |f(x-u)-f(x)|\,|K_\rho(u)|\,dudx \\ &\quad + \int_{-\pi}^{+\pi} |f(x)| \left|\frac{1}{2\pi}\int_{-\pi}^{+\pi} K_\rho(u)du - 1\right| dx \\ &= I_1 + I_2\end{aligned}$$

We see that $I_2 \to 0$ as $\rho \to \rho_0$, because $K\rho$ is approximate identity

$$I_2 = \int_{-\pi}^{+\pi} |f(x)| \left|\frac{1}{2\pi}\int_{-\pi}^{+\pi} K_\rho(u)du - 1\right| dx \to 0$$

as $\rho \to \rho_0$

$$\begin{aligned}I_1 &= \frac{1}{2\pi}\int_{-\pi}^{+\pi}\int_{-\pi}^{+\pi} |f(x-u)-f(x)|\,|K_\rho(u)|\,dxdu \\ &= \frac{1}{2\pi}\int_{-\pi}^{+\pi} |K_\rho(u)| \left[\int_{-\pi}^{+\pi} |f(x-u)-f(x)|\,dx\right] du \\ &= \frac{1}{2\pi}\left(\int_{-\pi}^{-\delta} + \int_{+\delta}^{+\pi}\right) |K_\rho(u)| \left[\int_{-\pi}^{+\pi} |f(x-u)-f(x)|\,dx\right] du \\ &\quad + \frac{1}{2\pi}\int_{-\delta}^{+\delta} |K_\rho(u)| \left[\int_{-\pi}^{+\pi} |f(x-u)-f(x)|\,dx\right] du\end{aligned}$$

$$\leq 2M.\frac{1}{2\pi}\left(\int_{\delta\leq|u|\leq\pi} |K_\rho(u)|\,du\right) + \frac{1}{2\pi}\int_{-\delta}^{+\delta} |K_\rho(u)| \left[\int_{-\pi}^{+\pi} |f(x-u)-f(x)|\,dx\right] du$$

By using K_ρ is approximate identity

$$\int_{\delta\leq|u|\leq\pi} |K_\rho(u)|\,du = 0$$

and using the continuity of f at the point x.

One has $w(f,\delta)$ is modulus of continuity

$$|f(x-u)-f(x)|<\varepsilon.$$

We get

$$2M.\frac{1}{2\pi}\left(\int\limits_{\delta\leq|u|\leq\pi}|K_\rho(u)|\,du\right)+\frac{1}{2\pi}\int\limits_{-\delta}^{+\delta}|K_\rho(u)|\left[\int\limits_{-\pi}^{+\pi}|f(x-u)-f(x)|\,dx\right]du\to 0$$

as $\rho\to\rho_0$ then finally

$$I_1\to 0$$

Then

$$\lim_{\rho\to\rho_0}\|I_\rho f-f\|_{X_{2\pi}}=0$$

The proof is completed.

Note.Given $f\in\mathbb{C}_{2\pi}$, a convolution integral of type

$$(f*K_\rho)(x)=\int\limits_{-\pi}^{+\pi}f(x-u)K_\rho(u)du$$

will be called a singular integral. If the kernel functions satisfy the conditions of the approximate identity, they usually have the familiar bell-shaped graph. The area under the curve $y=K_\rho(u)$ is equal to 2π. The peak at $u=0$ becomes higher and narrower in such a way taht the curve near $u=0$ ones out equal to 2π.

$$\int\limits_{-\delta}^{+\delta}f(x-u)K_\rho(u)du\approx f(x)\int\limits_{-\delta}^{+\delta}K_\rho(u)du=f(x)$$

Since $f(x-u)$ is then near to $f(x)$ (f being continuous), $(f*K_\rho)(x)$ is roughly

equal to

$$f(x)\int_{-\delta}^{+\delta} K_\rho(u)du$$

which tends to $f(x)$ as $\rho \to \rho_0$.

Definition 3.10. Let ρ be a positive parameter tending to infinity. A set of functions $\{K_\rho(x)\}$ will be called a kernel or a kernel on the real line if $K_\rho(x) \in L_1$ for each $\rho > 0$ and

$$\int_{-\infty}^{+\infty} K_\rho(x)dx = \sqrt{2\pi}$$

A kernel $\{K_\rho(x)\}$ will be said to be real, bounded, continuous, or absolutely continuous. If $K_\rho(x)$ is real, bounded, continuous, or absolutely continuous function of x for each $\rho > 0$.

Definition 3.11. For $f \in \mathbb{R}$ the convolution integral

$$(I_\rho f)(x) = \int_{-\infty}^{+\infty} f(x-u)K_\rho(u)du$$

defines a singular integral generated by the kernel $\{K_\rho(x)\}$. The singular integral is said to be positive or continuous if the kernel is positive or continuous.

Definition 3.12. A kernel $\{K_\rho(x)\}$ is called approximate identity if there is a constant $M > 0$ with

$$\|K_\rho\|_1 \leq M$$

and

$$\lim_{\rho\to\infty} \int_{0<\delta\leq|u|} |K_\rho(u)|\, du = 0$$

we can also write the last condition as;

$$\lim_{\rho\to\infty} \left[\sup_{0<\delta\leq|u|} |K_\rho(u)|\right] = 0.$$

Proposition. Let $f \in \mathbb{R}$ and $\{K_\rho(x)\}$ be a kernel for each $\rho > 0$, $(I_\rho f)(x) \in X(\mathbb{R})$

and

$$\|I_\rho f\|_{X(\mathbb{R})} \leq \|K_\rho f\|_1 \|f\|_{X(\mathbb{R})}$$

hence

$$I_\rho : X(\mathbb{R}) \to X(\mathbb{R}).$$

Theorem 3.6. *If $\{K_\rho(x)\}$ is an approximate identity, then*

$$\lim_{\rho\to\infty} \|I_\rho f - f\|_{X(\mathbb{R})} = 0$$

holds true for every $f \in \mathbb{R}$.

Proof. Let $X(\mathbb{R}) = L_1(\mathbb{R})$ we start with

$$\begin{aligned}(I_\rho f)(x) - f(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x-u)K_\rho(u)du - \frac{1}{\sqrt{2\pi}} f(x) \int_{-\infty}^{+\infty} K_\rho(u)du \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} [f(x-u) - f(x)] K_\rho(u)du\end{aligned}$$

$$\|I_\rho f - f\|_1 = \int_{-\infty}^{+\infty} \left| \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} [f(x-u) - f(x)] K_\rho(u)du \right| dx$$

$$\|I_\rho f - f\|_1 \leq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |f(x-u) - f(x)| \, |K_\rho(u)| \, dudx$$

Applying the generalized Minkowsky inequality to the righthand-side of the last

inequality, it yields

$$\begin{aligned}
& \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |f(x-u)-f(x)|\,|K_\rho(u)|\,dudx \\
\leq\ & \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |f(x-u)-f(x)|\,|K_\rho(u)|\,dxdu \\
=\ & \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} |K_\rho(u)| \left(\int_{-\infty}^{+\infty} |f(x-u)-f(x)|\,dx \right) du
\end{aligned}$$

$$\begin{aligned}
=\ & \frac{1}{\sqrt{2\pi}} \int_{\delta \leq |u|} |K_\rho(u)| \left(\int_{-\infty}^{+\infty} |f(x-u)-f(x)|\,dx \right) du \\
& + \frac{1}{\sqrt{2\pi}} \int_{-\delta}^{+\delta} |K_\rho(u)| \left(\int_{-\infty}^{+\infty} |f(x-u)-f(x)|\,dx \right) du \\
=\ & I_1 + I_2
\end{aligned}$$

since $f \in L_1(\mathbb{R})$ then there exists a number K with $\|f\|_1 \leq K$.

Hence

$$0 \leq I_1 \leq 2K.\frac{1}{\sqrt{2\pi}} \int_{\delta \leq |u|} |K_\rho(u)|\,du$$

So

$$\lim_{\rho \to \infty} I_1 = 0$$

$$\lim_{\rho \to \infty} I_2 = 0$$

because of modulus of continuity.

3.3 Characteristic Points for $f \in L_1$and $f \in L_p$

Let $f \in L_1$, we define

$$F(x) = \int_0^x f(t)dt.$$

Then

$$\lim_{h\to 0} \frac{F(x+h) - F(x)}{h} = f(x).$$

This implies that

$$\begin{aligned}
\lim_{h\to 0} \frac{1}{h} \int_x^{x+h} f(t)dt &= f(x) \equiv \lim_{h\to 0} \frac{1}{h} \int_x^{x+h} f(t)dt - f(x).1 \\
&= \lim_{h\to 0} \left[\frac{1}{h} \int_x^{x+h} f(t)dt - f(x).\frac{1}{h} \int_x^{x+h} dt \right] \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\lim_{h\to 0} \frac{1}{h} \int_x^{x+h} [f(t) - f(x)]\, dt &= 0 \\
\lim_{h\to 0} \frac{1}{h} \int_0^{h} [f(x+t) - f(x)]\, dt &= 0
\end{aligned}$$

A point $x \in \mathbb{R}$ is called a D-point of the function $f \in L_1(\mathbb{R})$ (D stands for differentiability).

If $f \in L_1(\mathbb{R})$ and

$$\lim_{h\to 0} \frac{1}{h} \int_0^{h} |f(x+t) - f(x)|\, dt = 0$$

then the point x is called the Lebesgue point (L-point) of $f \in L_1(\mathbb{R})$.

Now we will show that almost all points of $f \in L_1(\mathbb{R})$ are the L-point of f

We set

$$\varphi(x;h) = \frac{1}{h}\int_0^h |f(x+u) - f(x)|\, du$$

Since $f \in L_1, \varphi(x;h) \in L_1$, Indeed;

$$\begin{aligned} \|\varphi\|_{L_1} &= \int_{-\infty}^{+\infty} \left| \frac{1}{h}\int_0^h |f(x+u) - f(x)|\, du \right| dx \\ &= \frac{1}{h}\int_0^h \left(\int_{-\infty}^{+\infty} |f(x+u) - f(x)|\, dx \right) du. \end{aligned}$$

Since

$$\frac{1}{h}\int_0^h \left(\int_{-\infty}^{+\infty} |f(x+u) - f(x)|\, dx \right) du \leq 2\|f\|_{L_1} < +\infty$$

one has

$$\begin{aligned} \|\varphi\|_{L_1} &\leq \frac{1}{h}\int_0^h \left(\sup_{|u|\leq} \int_{-\infty}^{+\infty} |f(x+u) - f(x)|\, dx \right) du \\ &= w_{L_1}(f;\delta) \end{aligned}$$

$$\|\varphi\|_{L_1} = w_{L_1}(f;\delta)$$

$$\lim_{\delta\to 0^+} w(f;\delta) = 0 \Rightarrow \lim_{h\to 0} \|\varphi(x;h)\|_{L_1} = 0.$$

This shows that for almost all x points

$$\lim_{h\to 0} \varphi(x;h) = 0$$

Almost all points of $f \in L_1(\mathbb{R})$ are the L-points of f.

Theorem 3.7. *Let $f \in L_1(-\infty, +\infty)$ and $K_n(t)$ be a monotone decreasing and nonnegative even kernel on $[0, \infty)$ and $\int_{-\infty}^{+\infty} K_n(t)dt = 1$. Also for arbitrary $\alpha > 0$, $\forall \delta > 0$ and $\Delta_n = \int_{-\infty}^{+\infty} tK_n(t)dt, K_n(\delta) = o\left(\Delta_n^{\alpha}\right), \int_{\delta}^{+\infty} K_n(t)dt = o\left(\Delta_n^{\alpha}\right)$. And then at the point x which satisfies*

$$\int_0^h |f(x+t) - f(x)|\, dt = o\left(h^{\alpha+1}\right)$$

we have

$$|(L_n f)(x) - f(x)| = o(\Delta_n^{\alpha})$$

for singular integral operatorL_n

Proof.$\forall \varepsilon > 0,\ \exists \delta > 0$ such that $h < \delta$

$$\int_0^h |f(x+t) - f(x)|\, dt \leq \varepsilon h^{\alpha+1}$$

Let define a function F

$$F(t) = \int_0^t |f(x+\xi) - f(x)|\, d\xi$$

where $t < \alpha$,

$$F(t) < \varepsilon t^{\alpha+1}$$

$$\begin{aligned} \int_0^{\infty} |f(x+t) - f(x)|\, K_n(t)dt &= \left(\int_0^{\delta} + \int_{\delta}^{\infty}\right) |f(x+t) - f(x)|\, K_n(t)dt \\ &= I_1 + I_2 \end{aligned}$$

$$I_1 = \int_0^{\delta} K_n(t)dF(t)$$

$$I_2 = \int_{\delta}^{\infty} |f(x+t) - f(x)| \, K_n(t) dt$$

Apply to integration by parts

$$\begin{aligned} I_1 &= K_n(t)F(t) \mid_0^{\delta} - \int_0^{\delta} F(t) dK_n(t) \\ &= K_n(\delta)F(\delta) - \int_0^{\delta} F(t) dK_n(t) \\ &< \varepsilon \delta^{\alpha+1} K_n(\delta) + \varepsilon \int_0^{\delta} t^{\alpha+1} d(-K_n(t)) \end{aligned}$$

Again using integration by parts

$$I_1 < \varepsilon \delta^{\alpha+1} K_n(\delta) + \varepsilon \left(-t^{\alpha+1} K_n(t) \mid_0^t + \int_0^{\delta} K_n(t)(\alpha+1) t^{\alpha} dt \right)$$

$$I_1 < \varepsilon(\alpha+1) \left(\int_0^{\delta} K_n(t) t^{\alpha} dt \right)$$

Let $p = \frac{1}{\alpha}$ and use Hölder's inequality

$$\begin{aligned} \int_0^{\delta} K_n(t) t^{\alpha} dt &= \int_0^{\delta} [K_n(t)]^{1-\alpha+\alpha} \, t^{\alpha} dt \\ &= \int_0^{\delta} (K_n(t))^{1-\alpha} \, (K_n(t)t)^{\alpha} \, dt \\ &\leq \left(\int_0^{\delta} \left[(K_n(t))^{1-\alpha} \right]^{\frac{1}{1-\alpha}} dt \right)^{1-\alpha} \left(\int_0^{\delta} \left[(tK_n(t))^{\alpha} \right]^{\frac{1}{\alpha}} dt \right)^{\alpha} \end{aligned}$$

$$= \left(\int_0^{\delta} K_n(t)dt\right)^{1-\alpha} \left(\int_0^{\delta} tK_n(t)dt\right)^{\alpha}$$
$$< \left(\int_{-\infty}^{+\infty} K_n(t)dt\right)^{1-\alpha} \left(\int_{-\infty}^{+\infty} tK_n(t)dt\right)^{\alpha}$$

Therefore

$$I_1 < \varepsilon(\alpha+1)\Delta_n^{\alpha}. \tag{3.9}$$

The other part,

$$\begin{aligned} I_2 &\leq \int_{\delta}^{+\infty} |f(x+t)|\, K_n(t)dt + \int_{\delta}^{+\infty} |f(x)|\, K_n(t)dt \\ &= \int_{\delta}^{+\infty} |f(x+t)|\, K_n(t)dt + |f(x)| \int_{\delta}^{+\infty} K_n(t)dt \\ &\leq K_n(\delta) \int_{-\infty}^{+\infty} |f(x)|\, dt + |f(x)| \int_{\delta}^{+\infty} K_n(t)dt \end{aligned}$$

$$= K_n(\delta)\, \|f\|_{L_1} + |f(x)| \int_{\delta}^{+\infty} K_n(t)dt$$

$$I_2 \leq K_n(\delta)\, \|f\|_{L_1} + |f(x)| \int_{\delta}^{+\infty} K_n(t)dt \tag{3.10}$$

by the equations (3.9) and (3.10)

$$\int_{\delta}^{\infty} |f(x+t) - f(x)|\, K_n(t)dt < \varepsilon(\alpha+1)\Delta_n^{\alpha} + K_n(\delta)\, \|f\|_{L_1} + |f(x)| \int_{\delta}^{+\infty} K_n(t)dt$$

Every terms of inequality are divided by Δ_n^α

$$\frac{|(L_n f)(x) - f(x)|}{\Delta_n^\alpha} < \varepsilon(\alpha+1) + \frac{K_n(\delta)\|f\|_{L_1}}{\Delta_n^\alpha} + \frac{|f(x)| \int_\delta^{+\infty} K_n(t)dt}{\Delta_n^\alpha}$$

This equation is true for $\forall \varepsilon > 0$

Therefore

$$\lim_{n\to\infty} \frac{|(L_n f)(x) - f(x)|}{\Delta_n^\alpha} = 0.$$

Definition 3.13. If $f \in L_p$ and the equation

$$\lim_{h\to 0} \left(\frac{1}{h} \int_0^h |f(x+t) - f(x)|^p \, dt \right)^{\frac{1}{p}} = 0$$

is true for x point, then x point is called p-Lebesgue point of f.

Let $f \in L_p(-\infty, +\infty)$, almost all x points are p-Lebesgue point of f

$$\Psi_p(x;h) = \left(\frac{1}{h} \int_0^h |f(x+t) - f(x)|^p \, dt \right)^{\frac{1}{p}}$$

$\Psi_p(x;h)$ is nonnegative function and

$$(\Psi_p(x;h))^p = \frac{1}{h} \int_0^h |f(x+t) - f(x)|^p \, dt$$

L_p-norm of function $\Psi_p(x;h)$ is

$$\begin{aligned}\left\|\Psi_p^p(x;h)\right\|_p &= \left[\int_{-\infty}^{+\infty}\frac{1}{h^p}\left(\int_0^h |f(x+t)-f(x)|^p\,dt\right)^p dx\right]^{\frac{1}{p}} \\ &= \frac{1}{h}\left[\int_{-\infty}^{+\infty}\left(\int_0^h |f(x+t)-f(x)|^p\,dt\right)^p dx\right]^{\frac{1}{p}}\end{aligned}$$

by using generalized Minkowsky inequality

$$\left\|\Psi_p^p(x;h)\right\|_p \leq \frac{1}{h}\int_0^h\left(\int_{-\infty}^{+\infty}|f(x+t)-f(x)|^p\,dx\right)^{\frac{1}{p}} dt$$

And then

$$\sup_{t\leq h}\left(\int_{-\infty}^{+\infty}|f(x+t)-f(x)|^p\,dx\right)^{\frac{1}{p}} = w_{L_p}(f;h)$$

$$0 \leq \left\|\Psi_p^p(x;h)\right\|_p \leq w_{L_p}(f;h)$$

$$\lim_{h\to 0} w_{L_p}(f;h) = 0 \Rightarrow \lim_{h\to 0}\left\|\Psi_p^p(x;h)\right\|_p = 0$$

for almost all x

$$\lim_{h\to 0}\Psi_p^p(x;h) = 0$$

That is almost all x points is p-Lebesgue points of f.

CHAPTER 4

M -SINGULAR OPERATORS AND THEIR APPROXIMATION PROPERTIES

4.1 m-Singular Integral

Let

$$\triangle_t^m f(x) = \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} f(x+kt) \qquad m \in \mathbb{N},$$

is the m-th finite difference of $f(x)$ with the step t. In 1963 Mamedov defined a generalization of the linear integral operators by using m-th finite difference operator as

$$(L_n^m f)(x) = \int_{-\infty}^{\infty} \left[\sum_{k=1}^{m} (-1)^{k-1} \binom{m}{k} f(x+kt) \right] K_n(t)\, dt, \qquad n = 1, 2, ... \quad (4.1)$$

where $m \geq 1$, called m-singular integral operators.

Theorem 4.1. *Let $f \in \mathbb{C}(-\infty, +\infty)$ and $K_n(t)$ is the kernel of m-singular integral operator. Then,$L_n^m f$ define an operator from $\mathbb{C}(-\infty, +\infty)$ to $\mathbb{C}(-\infty, +\infty)$ and for every $n \in \mathbb{N}$*

$$\|L_n^m f\|_{\mathbb{C}(-\infty,+\infty)} \leq \|f\|_{\mathbb{C}(-\infty,+\infty)},$$

holds true.

Proof . We will find the norm of $(L_n^m f)(x)$. When $f \in \mathbb{C}(-\infty, +\infty)$

$$\begin{aligned}(L_n^m f)(x) &= \int_{-\infty}^{\infty} \left[\sum_{k=1}^{m} (-1)^{k-1} \binom{m}{k} f(x+kt)\right] K_n(t)\, dt \\ |(L_n^m f)(x)| &= \left|\int_{-\infty}^{\infty} \left[\sum_{k=1}^{m} (-1)^{k-1} \binom{m}{k} f(x+kt)\right] K_n(t)\, dt\right|\end{aligned}$$

$$\begin{aligned}&\leq \int_{-\infty}^{\infty} \sum_{k=1}^{m} \left|(-1)^{k-1}\right| \binom{m}{k} |f(x+kt)|\, K_n(t)\, dt \\ &\leq \int_{-\infty}^{\infty} \sum_{k=1}^{m} \binom{m}{k} |f(x+kt)|\, K_n(t)\, dt \\ &\leq (2^m - 1) \sup_{t\in(-\infty,+\infty)} |f(x+kt)| \int_{-\infty}^{+\infty} K_n(t)\, dt\end{aligned}$$

$$|(L_n^m f)(x)| \leq (2^m - 1) \|f\|_{\mathbb{C}(-\infty,+\infty)} .1$$

Taking the supremum of both sides of the last inequality over $x \in (-\infty, +\infty)$.

Then one has

$$\sup_{x\in(-\infty,+\infty)} |(L_n^m f)(x)| \leq (2^m - 1) \|f\|_{\mathbb{C}(-\infty,+\infty)} .$$

We get

$$\|L_n^m f\|_{\mathbb{C}(-\infty,+\infty)} \leq \|f\|_{\mathbb{C}(-\infty,+\infty)}$$

and so $L_n^m : \mathbb{C}(-\infty, +\infty) \to \mathbb{C}(-\infty, +\infty)$.

Theorem 4.2. *Let $f \in L_1(-\infty, +\infty)$ and $K_n(t)$ is the kernel of m-singular integral operator. Then, L_n^m defines an operator from $L_1(-\infty, +\infty)$ into $L_1(-\infty, +\infty)$*

and for every $n \in \mathbb{N}$

$$\left\| L_n^m f \right\|_{L_1(-\infty,+\infty)} \leq \left\| f \right\|_{L_1(-\infty,+\infty)},$$

holds true.

Proof. Now we use the definition of the norm of $L_n^m f$ on L_1

$$\begin{aligned}
\left\| L_n^m f \right\| &= \int_{-\infty}^{+\infty} \left| \int_{-\infty}^{\infty} \left[\sum_{k=1}^{m} (-1)^{k-1} \binom{m}{k} f(x+kt) \right] K_n(t)\, dt \right| dx \\
&\leq \int_{-\infty}^{+\infty} \int_{-\infty}^{\infty} \sum_{k=1}^{m} \left| (-1)^{k-1} \binom{m}{k} f(x+kt) \right| K_n(t)\, dt dx \\
&\leq \int_{-\infty}^{+\infty} \int_{-\infty}^{\infty} \sum_{k=1}^{m} \left| (-1)^{k-1} \right| \binom{m}{k} |f(x+kt)| K_n(t)\, dx dt
\end{aligned}$$

$$\begin{aligned}
&\leq \int_{-\infty}^{+\infty} K_n(t) \int_{-\infty}^{\infty} \sum_{k=1}^{m} \binom{m}{k} |f(x+kt)|\, dx dt \\
&\leq \int_{-\infty}^{+\infty} K_n(t) \left[\sum_{k=1}^{m} \binom{m}{k} \int_{-\infty}^{\infty} |f(x+kt)|\, dx \right] dt \\
&= (2^m - 1) \left\| f \right\|_{L_1} . \int_{-\infty}^{+\infty} K_n(t)\, dt \\
&= (2^m - 1) \left\| f \right\|_{L_1} .1
\end{aligned}$$

We get

$$\left\| L_n^m f \right\|_{L_1} \leq \left\| f \right\|_{L_1}$$

and so $L_n^m : L_1(-\infty,+\infty) \to L_1(-\infty,+\infty)$.

This completes the proof.

Theorem 4.3. *Let $f \in L_p(-\infty,+\infty)$ and $K_n(t)$ is the kernel of m-singular integral operator. Then, L_n^m defines an operator from $L_p(-\infty,+\infty)$ to $L_p(-\infty,+\infty)$ and for every $n \in \mathbb{N}$*

$$\|L_n^m f\|_{L_p(-\infty,+\infty)} \leq \|f\|_{L_p(-\infty,+\infty)},$$

holds true.

Proof . Now we use the definition of the norm of $L_n^m f$ on L_p

$$\begin{aligned}
\|L_n^m f\|_{L_p} &= \left(\int_{-\infty}^{+\infty} \left| \int_{-\infty}^{\infty} \left[\sum_{k=1}^{m} (-1)^{k-1} \binom{m}{k} f(x+kt) \right] K_n(t)\, dt \right|^p dx \right)^{\frac{1}{p}} \\
&\leq \left(\int_{-\infty}^{+\infty} \int_{-\infty}^{\infty} \left| \sum_{k=1}^{m} (-1)^{k-1} \binom{m}{k} f(x+kt) \right|^p K_n(t)^p\, dtdx \right)^{\frac{1}{p}}
\end{aligned}$$

Applying the Hölder-Minkowsky inequality, then we get

$$\begin{aligned}
\|L_n^m f\|_{L_p} &\leq \int_{-\infty}^{+\infty} \left(\int_{-\infty}^{\infty} \left| \sum_{k=1}^{m} (-1)^{k-1} \binom{m}{k} f(x+kt) \right|^p \right)^{\frac{1}{p}} K_n(t)\, dtdx \\
&\leq (2^m-1)^p . \|f\|_{L_p} .1.
\end{aligned}$$

We get

$$\|L_n^m f\|_{L_p} \leq \|f\|_{L_p}$$

and so $L_n^m : L_p(-\infty,+\infty) \to L_p(-\infty,+\infty)$.

This completes the proof.

There are lots of example of m-singular integral. Gauss-Weierstrass m-singular integral, Abel-Poisson m-singular integral are some of them.

4.2 Properties of m-Singular Integral

Proposition 4.1. . *Let* $\triangle_t^1 f(x) = f(x+t) - f(x)$ *and* $\triangle_t^m = \triangle_t^1 \triangle_t^{m-1} f(x)$. *Then*

$$\triangle_t^m f(x) = \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} f(x+kt)$$

Proof *We will use induction method for* m

i) If $m = 1$,

$$\triangle_t^1 f(x) = \sum_{k=0}^{1} (-1)^{1-k} \binom{1}{k} f(x+kt) = -f(x) + f(x+t)$$

is true.

ii) Assume that equation is true for m.

iii) Show that equation is true for $m+1$.

$$\begin{aligned}
\triangle_t^{m+1} f(x) &= \triangle_t^1 \triangle_t^m f(x) = \triangle_t^1 \left[\sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} f(x+kt) \right] \\
&= \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} f(x+(k+1)t) - \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} f(x+kt) \\
&= \sum_{k=1}^{m+1} (-1)^{m-k+1} \binom{m}{k-1} f(x+kt) - \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} f(x+kt)
\end{aligned}$$

$$\begin{aligned}
&= \sum_{k=1}^{m+1} (-1)^{m-k+1} \left[\binom{m+1}{k} - \binom{m}{k} \right] f(x+kt) - \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} f(x+kt) \\
&= \sum_{k=1}^{m+1} (-1)^{m-k+1} \binom{m+1}{k} f(x+kt) - \sum_{k=1}^{m+1} (-1)^{m-k+1} \binom{m}{k} f(x+kt) \\
&\quad - \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} f(x+kt)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=1}^{m+1} (-1)^{m-k+1} \binom{m+1}{k} f\left(x+kt\right) \\
&\quad - \left\{ \sum_{k=0}^{m} (-1)^{m-k+1} \binom{m}{k} f\left(x+kt\right) - (-1)^{m+1} \binom{m}{0} f(x) \right. \\
&\quad \left. + (-1)^{m-(m+1)+1} \binom{m}{m+1} f(x+(m+1)t) \right\} \\
&\quad - \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} f\left(x+kt\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{k=1}^{m+1} (-1)^{m-k+1} \binom{m+1}{k} f\left(x+kt\right) + (-1)^{m+1} \binom{m}{0} f(x) \\
&= \sum_{k=0}^{m+1} (-1)^{m-k+1} \binom{m+1}{k} f\left(x+kt\right).
\end{aligned}
$$

Therefore equation is true for $m+1$. Also for $\forall m \in \mathbb{N}$ is true.

Proposition 4.2. *.For all* $m \in \mathbb{N}$

i)

$$\sum_{k=1}^{m} (-1)^{k-1} \binom{m}{k} = 1$$

ii)

$$\sum_{k=0}^{m} \binom{m}{k} = 2^m.$$

Let we consider the equation (4.1) we can write new form of equation (4.1).

Assume that $\int_{-\infty}^{\infty} K_n(t)\,dt = 1$ and by using (proposition 4.1 and 4.2)

$$\begin{aligned}
(L_n^m f)(x) - f(x) &= \int_{-\infty}^{\infty} \left[\sum_{k=1}^{m} (-1)^{k-1} \binom{m}{k} f(x+kt)\right] K_n(t)\,dt - f(x) \\
&= \int_{-\infty}^{\infty} \left[\sum_{k=0}^{m} (-1)^{k-1} \binom{m}{k} f(x+kt)\right] K_n(t)\,dt \\
&= (-1)^{m+1-2k} \int_{-\infty}^{\infty} \left[\sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} f(x+kt)\right] K_n(t)\,dt \\
&= (-1)^{m+1-2k} \int_{-\infty}^{\infty} \left[\Delta_t^m f(x)\right] K_n(t)\,dt
\end{aligned}$$

it yields. Then we will use this new form of equation.

Definition 4.1.

$$w_m(f;\delta)_{\mathbb{C}[a,b]} = \sup_{|t|<\delta} \left|\Delta_t^m f(x)\right|, \quad \delta > 0, \quad x \in [a,b]$$

is called m-modulus of smoothness of $f \in \mathbb{C}[a,b]$ function.

Definition 4.2.

$$w_m(f;\delta)_{L_p} = \sup_{|t|\leq\delta} \left[\int_{-\infty}^{+\infty} \left|\Delta_t^m f(x)\right|^p dx\right]^{\frac{1}{p}}$$

is called $m-$modulus of smoothness of $f \in L_p(-\infty,+\infty)$ function.

Proposition 4.3. *Let $f \in \mathbb{C}[a,b]$ for $\forall \delta > 0$*

i) $n \in \mathbb{N}, w_m(f;\delta)_{\mathbb{C}[a,b]} \leq n^m w_m(f;\delta)_{\mathbb{C}[a,b]}$

ii) $\lambda > 0, w_m(f;\lambda\delta)_{\mathbb{C}[a,b]} \leq (1+\lambda)^m w_m(f;\delta)_{\mathbb{C}[a,b]}$.

Proof:

i) Let

$$\Delta_{nt}^m f(x) = \sum_{k_1=0}^{n-1} \sum_{k_2=0}^{n-1} \ldots\ldots\ldots \sum_{km=0}^{n-1} \Delta_t^m f(x + k_1 t + \ldots\ldots\ldots + k_m t). \qquad (4.2)$$

If $m = 1$, we see that righthand-side is

$$\begin{aligned} \sum_{k_1=0}^{n-1} \Delta_t f(x + k_1 t) &= [f(x+t) - f(x)] + + \ldots\ldots\ldots + [f(x+nt) - f(x+nt-1)] \\ &= f(x+nt) - f(x) \\ &= \Delta_{nt} f(x) \end{aligned}$$

suppose equation (4.2) is true for some m. Then

$$\Delta_{nt}^{m+1} f(x) = \sum_{k_1=0}^{n-1} \sum_{k_2=0}^{n-1} \ldots\ldots\ldots \sum_{km=0}^{n-1} \Delta_{nt}(\Delta_t^m f(x + k_1 t + \ldots\ldots\ldots + k_m t)) \ .$$

Using the identity (4.2) for $m = 1$, which is the case we just proved, we see that (4.2) is true for $m + 1$ as well. Thus (4.2) is proved by induction on m. Part (i) follows easily from (4.2)

if we use

$$\Delta_{nt}^m f(x) = \sum_{k_1=0}^{n-1} \sum_{k_2=0}^{n-1} \ldots\ldots\ldots \sum_{km=0}^{n-1} \Delta_t^m f(x + k_1 t + \ldots\ldots\ldots + k_m t)$$

then

$$\begin{aligned} w_m(f;\delta) &\leq \sum_{k_1=0}^{n-1} \sum_{k_2=0}^{n-1} \ldots\ldots\ldots \sum_{km=0}^{n-1} \Delta_t^m f(x + k_1 t + \ldots\ldots\ldots + k_m t) + \varepsilon \\ &\leq n^m w_m(f;\delta) + \varepsilon. \end{aligned}$$

So (i) is proved.

From the properties of supremum $\forall \varepsilon > 0$, $\exists t \in [a, b,]$ $\ni |t| \leq \delta$.

$$w_m(f; \delta) \leq |\Delta_{nt}^m f(x)| + \varepsilon$$

ii) $w_m(f; \delta)_{\mathbb{C}[a,b]}$ is monotone increasing with respect to δ and using by $\lambda \leq [\lambda]+1$

$$w_m(f; \delta) \leq w_m(f; ([\lambda] + 1)\delta)_{\mathbb{C}[a,b]}$$

is true.

$([\lambda] + 1)$ is an integer and from (i)

$$w_m(f; \lambda\delta) \leq w_m(f; ([\lambda] + 1)\delta) \leq ([\lambda] + 1)^m w_m(f; \delta) \leq (\lambda + 1)^m w_m(f; \delta)$$

Therefore

$$w_m(f; \lambda\delta)_{\mathbb{C}[a,b]} \leq (\lambda + 1)^m w_m(f; \delta)_{\mathbb{C}[a,b]}.$$

This proposition is true for $m-$modulus of smoothness of $f \in L_p$ functions.

Theorem 4.4. *Let $f \in L_p(-\infty, \infty)$. Then*

$$\lim_{\delta \to 0} w_m(f; \delta)_{L_p} = 0$$

holds true.

Proof.

From m-th difference

$$\Delta_t^{m+1} f(x) = \Delta_t^m f(x + t) - \Delta_t^m f(x)$$

equality exist and

$$\begin{aligned} \left|\Delta_t^{m+1} f(x)\right| &\leq |\Delta_t^m f(x+t)| + |\Delta_t^m f(x)| \\ \sup_{|t|\leq\delta} \left(\int_{-\infty}^{+\infty} \left|\Delta_t^{m+1} f(x)\right|^p dx \right)^{\frac{1}{p}} &\leq \sup_{|t|\leq\delta} \left(\int_{-\infty}^{+\infty} |\Delta_t^m f(x+t)|^p dx \right)^{\frac{1}{p}} + \sup_{|t|\leq\delta} \left(\int_{-\infty}^{+\infty} |\Delta_t^m f(x)|^p dx \right)^{\frac{1}{p}} \end{aligned}$$

and then

$$\sup_{|t|\leq\delta} \left(\int_{-\infty}^{+\infty} \left|\Delta_t^{m+1} f(x)\right|^p dx \right)^{\frac{1}{p}} \leq 2 \sup_{|t|\leq\delta} \left(\int_{-\infty}^{+\infty} |\Delta_t^m f(x)|^p dx \right)^{\frac{1}{p}}$$

That is

$$w_{m+1}(f;\delta)_{L_p} \leq 2w_m(f;\delta)L_p$$

continuing like this

$$w_{m+1}(f;\delta)_{L_p} \leq 2w_m(f;\delta)L_p \leq 2^2 w_{m-1}(f;\delta)_{L_p} \leq \ldots\ldots\ldots \leq 2^m w_1(f;\delta)_{L_p}$$

yields.

On the other hand 1-th modulus of smoothness is equal to modulus of continuity

$$w_{m+1}(f;\delta)_{L_p} \leq 2^m w(f;\delta)_{L_p}$$

from equation (3.7)

$$\lim_{\delta\to 0} w_m(f;\delta)_{L_p} = 0$$

The proof is completed.If f is uniformly continuous , then theorem is true for $f \in \mathbb{C}[a,b]$.

Theorem 4.5. *Let $K_n(t)$ be a nonnegative and even kernel ,satisfying*

$$\int_{-\infty}^{+\infty} K_n(t)dt = 1$$

and $f \in \mathbb{C}[a,b]$, $(-\infty \leq a < b \leq +\infty)$ *such that*

$$\delta_n^m = \int_{-\infty}^{+\infty} t^m K_n(t)dt$$

for L_n^m

$$\|L_n^m f - f\|_{\mathbb{C}[a,b]} \leq 3.2^m w_m(f;\delta_n)$$

inequality holds.

Proof. If we consider Definition 4.1, then

$$\begin{aligned} \|L_n^m f - f\| &= \left| \int_{-\infty}^{+\infty} \left[\Delta_t^m f(x)\right] K_n(t)\, dt \right| \\ &\leq \int_{-\infty}^{+\infty} \left|\Delta_t^m f(x)\right| K_n(t)\, dt. \end{aligned} \tag{4.3}$$

From Proposition 4.3 (ii)

$$w_m\left(f; \frac{|t|}{\delta_n}.\delta_n\right) \leq \left(1 + \frac{|t|}{\delta_n}\right)^m w_m(f;\delta_n)$$

We will use the property on equation (4.3)

$$\begin{aligned} \|(L_n^m f)(x) - f(x)\| &\leq \int_{-\infty}^{+\infty} \left(1 + \frac{|t|}{\delta_n}\right)^m w_m(f;\delta_n) K_n(t)\, dt \\ &= w_m(f;\delta_n) \int_{-\infty}^{+\infty} \left(1 + \frac{|t|}{\delta_n}\right)^m K_n(t)\, dt \end{aligned}$$

$$
\begin{aligned}
&\leq 2^m w_m(f;\delta_n) \int_{-\infty}^{+\infty} \left(1+\frac{|t|}{\delta_n}\right)^m K_n(t)\,dt \\
&= 2^m w_m(f;\delta_n) \left\{ \int_{-\infty}^{+\infty} K_n(t)\,dt + \frac{1}{\delta_n^m} \int_{-\infty}^{+\infty} |t|^m K_n(t)\,dt \right\} \\
&= 2^m w_m(f;\delta_n) \left\{ 1 + \frac{2}{\delta_n^m} \int_{0}^{+\infty} |t|^m K_n(t)\,dt \right\}
\end{aligned}
$$

$$
\begin{aligned}
&= 2^m w_m(f;\delta_n) + \frac{2^{m+1}}{\delta_n^m} w_m(f;\delta_n) \int_{0}^{+\infty} t^m K_n(t)\,dt \\
&= 3.2^m w_m(f;\delta_n).
\end{aligned}
$$

By taking supremum of two side the equation

$$
\sup_{x\in[a,b]} \left|\left(L_n^m f\right)(x) - f(x)\right| = \sup_{x\in[a,b]} 3.2^m w_m(f;\delta_n)
$$

We get

$$
\left\|L_n^m f - f\right\|_{\mathbb{C}[a,b]} \leq 3.2^m w_m(f;\delta_n)_{\mathbb{C}[a,b]}.
$$

Theorem 4.6. *Let $K_n(t)$ be a positive kernel with*

$$
\int_{-\infty}^{+\infty} K_n(t)dt = 1.
$$

If $f \in L_p$, then

$$
w_m(L_n^m f;\delta)_{L_p} \leq w_m(f;\delta)_{L_p}.
$$

holds true.

Proof. From the definition of m-modulus of smoothness

$$w_m(L_n^m f;\delta)_{L_p} = \sup_{|h|\leq\delta}\left(\int_{-\infty}^{+\infty}\left|\Delta_h^m\left(L_n f\right)(x)\right|^p dx\right)^{\frac{1}{p}}$$

And using the equation

$$\Delta_h^m\left(L_n f\right)(x) = \sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}\left(L_n f\right)(x+kh)$$

We get

$$\begin{aligned}
\left(L_n f\right)(x+kh) &= \int_{-\infty}^{+\infty}\left[\sum_{k=0}^{m}(-1)^{k-1}\binom{m}{k}f\left(x+kh+kt\right)\right]K_n(t)dt \\
\Delta_h^m\left(L_n f\right)(x) &= \sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}\int_{-\infty}^{+\infty}\left[\sum_{k=0}^{m}(-1)^{k-1}\binom{m}{k}f\left(x+(t+h)\,k\right)\right]K_n(t)dt
\end{aligned}$$

$$\begin{aligned}
&= \sum_{k=0}^{m}(-1)^{k-1}\binom{m}{k}\int_{-\infty}^{+\infty}\left[\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}f\left(x+(t+h)\,k\right)\right]K_n(t)dt \\
&= 1.\int_{-\infty}^{+\infty}\left[\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}f\left(x+(t+h)\,k\right)\right]K_n(t)dt
\end{aligned}$$

If we write the equation on the equation

$$\begin{aligned}
w_m(L_n f;\delta)_{L_p} &= \sup_{|h|\leq\delta}\left(\int_{-\infty}^{+\infty}\left|\int_{-\infty}^{\infty}\sum_{k=0}^{m}\binom{m}{k}f\left(x+(t+h)k\right)K_n\left(t\right)dt\right|^p dx\right)^{\frac{1}{p}} \\
&\leq \sup_{|h|\leq\delta}\int_{-\infty}^{+\infty}\left(\int_{-\infty}^{\infty}\left|\sum_{k=0}^{m}\binom{m}{k}f\left(x+(t+h)k\right)\right|^p K_n\left(t\right)^p dx\right)^{\frac{1}{p}}dt
\end{aligned}$$

$$= \sup_{|h|\leq\delta} \int_{-\infty}^{+\infty} \left(\int_{-\infty}^{\infty} \left| \sum_{k=0}^{m} \binom{m}{k} f\left(x+(t+h)k\right) \right|^p dx \right)^{\frac{1}{p}} K_n(t)\, dt$$

$$= \sup_{|h|\leq\delta} \int_{-\infty}^{+\infty} \left(\int_{-\infty}^{\infty} \left| \sum_{k=0}^{m} \binom{m}{k} f\left(x+(t+h)k\right) \right|^p dx \right)^{\frac{1}{p}}$$

Let $x+k=u$

$$\begin{aligned} w_m(L_n f;\delta)_{L_p} &\leq \sup_{|h|\leq\delta} \int_{-\infty}^{+\infty} \left(\int_{-\infty}^{\infty} \left| \sum_{k=0}^{m} \binom{m}{k} f(u+hk) \right|^p du \right)^{\frac{1}{p}} \\ &= w_m(f;\delta). \end{aligned}$$

This completes te proof.

Definition 4.3. Let $f \in L_p(-\infty,+\infty)$ $(p \geq 1)$

$$\lim_{h\to 0} \frac{1}{h^m} \left(\int_0^h \left| \left[\Delta_t^m + \Delta_{-t}^m\right] f(x) \right|^p dt \right)^{\frac{1}{p}} = 0$$

x point is called (p,m)-Lebesgue point of f function.

Theorem 4.7. *Let $f \in L_p(-\infty,+\infty)$, $K_n(t)$ be nonnegative even, monotone decreasing kernel on $[0,\infty)$ and $\int_{-\infty}^{+\infty} K_n(t)dt = 1$. Also , arbitrary $\alpha > 0$*

$\Delta_n = \int_0^{\infty} t^{N\alpha} K_n(t)dt$, $K_n(\delta) = o(\Delta_n)$, $\int_{\delta}^{\infty} K_n(t)dt = o(\Delta_n)$ for
$N_\alpha = m+\alpha-1$, Then x points which

$$\lim_{h\to 0} \frac{1}{h^{m+\alpha}} \int_0^h \left| \left[\Delta_t^m + \Delta_{-t}^m\right] f(x) \right|^p dt = 0$$

holds, we have

$$\lim_{n\to\infty} \frac{|(L_n^m f)(x) - f(x)|}{\Delta_n} = 0$$

for operator L_n^m.

Proof. Since $K_n(t)$ be an even kernel

$$|(L_n^m f)(x) - f(x)| \leq \int_0^\infty \left|\left[\Delta_t^m + \Delta_{-t}^m\right] f(x)\right| K_n(t)\, dt$$

for $p > 1$ and $\frac{1}{p} + \frac{1}{p} = 1$, using the Hölder's inequality

$$\begin{aligned}
|(L_n^m f)(x) - f(x)| &\leq \int_0^\infty \left|\left[\Delta_t^m + \Delta_{-t}^m\right] f(x)\right| K_n(t)^{\frac{1}{p}} K_n(t)^{\frac{1}{p}} dt \\
&\leq \left(\int_0^\infty \left|\left[\Delta_t^m + \Delta_{-t}^m\right] f(x)\right|^p K_n(t)\, dt\right)^{\frac{1}{p}} \left(\int_0^\infty K_n(t) dt\right)^{\frac{1}{p}} \\
&= \frac{1}{2^{\frac{1}{p}}} \left(\int_0^\infty \left|\left[\Delta_t^m + \Delta_{-t}^m\right] f(x)\right|^p K_n(t)\, dt\right)^{\frac{1}{p}}
\end{aligned}$$

or

$$2^{\frac{p'}{p}} |(L_n^m f)(x) - f(x)| \leq \int_0^\infty \left|\left[\Delta_t^m + \Delta_{-t}^m\right] f(x)\right|^p K_n(t)\, dt$$

Let F be defined as

$$\begin{aligned}
F(t) &= \int_0^t \left|\left[\Delta_\zeta^m + \Delta_{-\zeta}^m\right] f(x)\right|^p d\zeta \\
dF(t) &= \left|\left[\Delta_t^m + \Delta_{-t}^m\right] f(x)\right|^p dt
\end{aligned}$$

From the hypothesis

$$\lim_{h\to 0} \frac{1}{h^{m+\alpha}} \int_0^h \left|\left[\Delta_t^m + \Delta_{-t}^m\right] f(x)\right|^p dt = 0.$$

In view of the definition of the function F, one has

$$\lim_{h\to 0}\frac{1}{h^{m+\alpha}}F(h)=0$$

If $\forall\varepsilon>0$, $\exists\delta>0\ \ni h<\delta$,then $F(h)<\varepsilon h^{m+\alpha}$

$$\begin{aligned}2^{\frac{p'}{p}}\left|(L_n^m f)(x)-f(x)\right| &\leq \left(\int_0^{\delta}+\int_{\delta}^{\infty}\right)\left|\left[\Delta_t^m+\Delta_{-t}^m\right]f(x)\right|^p K_n(t)\,dt\\ &= I_1+I_2\end{aligned}$$

$$I_1=\int_0^{\delta}K_n(t)\,dF(t)$$

By integration by parts

$$\begin{aligned}I_1 &= K_n(t)\,F(t)\mid_0^{\delta}+\int_0^{\delta}F(t)dK_n(t)\\ &= K_n(\delta)\,F(\delta)+\int_0^{\delta}F(t)d\left(-K_n(t)\right)\end{aligned}$$

and then

$$I_1\leq\varepsilon\delta^{m+\alpha}K_n(\delta)+\varepsilon\int_0^{\delta}t^{m+\alpha}d\left(-K_n(t)\right)$$

again using integration by parts

$$I_1\leq\varepsilon\delta^{m+\alpha}K_n(\delta)+\varepsilon\left(-t^{m+\alpha}K_n(t)\mid_0^{\delta}+\int_0^{\delta}K_n(t)(m+\alpha)t^{m+\alpha-1}dt\right)$$

so

$$I_1\leq\varepsilon(m+\alpha)\int_0^{\infty}K_n(t)t^{m+\alpha-1}dt=\varepsilon(m+\alpha)\Delta_n.$$

If we point out integral I_2

$$\left|\Delta_t^m f(x) + \Delta_{-t}^m f(x)\right|^p \leq 2^p \left(|\Delta_t^m f(x)|^p + \left|\Delta_{-t}^m f(x)\right|^p\right)$$

from the inequality

$$\begin{aligned} I_2 &= \int_\delta^\infty \left|\sum_{k=0}^m \binom{m}{k} \left[f\left(x+kt\right) + f\left(x-kt\right)\right]\right|^p K_n\left(t\right) dt \\ &= \int_\delta^\infty \left|\sum_{k=1}^m \binom{m}{k} \left[f\left(x+kt\right) + f\left(x-kt\right) - 2f(x)\right]\right|^p K_n\left(t\right) dt \end{aligned}$$

$$\begin{aligned} &\leq \int_\delta^\infty \left\{ (2^m-1)^p \left|\sum_{k=1}^m \binom{m}{k} f\left(x+kt\right)\right|^p \right. \\ &\quad \left. +(2^m-1)^p \left|\sum_{k=1}^m \binom{m}{k} f\left(x-kt\right)\right|^p + 2^p \left|f(x)\right|^p \right\} K_n\left(t\right) dt \\ &\leq M_1 K_n\left(\delta\right) \left\|f\right\|_p + M_2 \left|f(x)\right|^p \int_\delta^\infty K_n\left(t\right) dt \end{aligned}$$

where $M_1 = (2^m-1)^{p+1}$, and $M_2 = 2^p$

$$2^{\frac{p}{p}'} \left|(L_n^m f)(x) - f(x)\right| \leq \varepsilon(m+\alpha)\Delta_n + M_1 K_n\left(\delta\right) \left\|f\right\|_p + M_2 \left|f(x)\right|^p \int_\delta^\infty K_n\left(t\right) dt$$

If two sides of the inequality is divided by Δ_n and $2^{\frac{p}{p}'}$

$\varepsilon = \varepsilon(m+\alpha)$, $M_1' = \frac{M_1}{2^{\frac{p}{p}'}}$, $M_2' = \frac{M_2}{2^{\frac{p}{p}'}}$

$$\frac{\left|(L_n^m f)(x) - f(x)\right|}{\Delta_n} \leq \varepsilon + \frac{M_1' K_n\left(\delta\right) \left\|f\right\|_p}{\Delta_n} + \frac{M_2' \left|f(x)\right|^p \int_\delta^\infty K_n\left(t\right) dt}{\Delta_n}$$

$$|(L_n^m f)(x) - f(x)| = o(\Delta_n)$$

or

$$\lim_{n\to\infty} \frac{|(L_n^m f)(x) - f(x)|}{\Delta_n} = 0.$$

Example.

Let $K_n(t)$ be Fejer-type kernel and $\phi(t)$ be a nonnegative function

Also,

i) even function on $(-\infty, +\infty)$

ii) monotone decreasing on $[0, \infty)$

iii) $\int_{-\infty}^{+\infty} \phi(t)dt = 1$

iv) $q_{N_\alpha} = \int_o^{\infty} t^{N_\alpha}\phi(t)dt < \infty \quad$ (where $\; N_\alpha = m + \alpha - 1 \geq \alpha > 0$)

Theorem.4.7 holds true for Fejer-type kernel $K_n(t)$.

$K_n(t)$ is non-negative on $[0, \infty)$ and monotone decreasing even kernel because of properties of Fejer-type kernel.

Also

$$\int_{-\infty}^{+\infty} K_n(t)dt = \int_{-\infty}^{+\infty} n\phi(nt)dt = \int_{-\infty}^{+\infty} \phi(u)du = 1$$

In addition $\phi(t)$ is monotone decreasing function.

$$\begin{aligned} \int_{\frac{u}{2}}^{u} \phi(t)t^{N_\alpha}dt &\geq \phi(u)\int_{\frac{u}{2}}^{u} t^{N_\alpha}dt = \phi(u)\frac{t^{N_\alpha+1}}{N_\alpha+1}\,|_{\frac{u}{2}}^{u} \\ &= \phi(u)\frac{u^{N_\alpha+1}}{N_\alpha+1}.\frac{2^{N_\alpha+1}-1}{2^{N_\alpha+1}} \end{aligned}$$

$$u^{N_\alpha+1} \leq C\int_{\frac{u}{2}}^{u} \phi(t)t^{N_\alpha}dt$$

where

$$C = \frac{(N_\alpha+1)\,2^{N_\alpha+1}-1}{2^{N_\alpha+1}}.$$

Then

$$\begin{aligned} n^{N_\alpha+1}\phi(n\delta) &= \frac{1}{\delta^{N_\alpha+1}}\left(n\delta\right)^{N_\alpha+1}\phi(n\delta) \\ &\leq \frac{C}{\delta^{N_\alpha+1}}\int\limits_{\frac{n\delta}{2}}^{n\delta}\phi(t)t^{N_\alpha}dt \end{aligned}$$

On the other hand

$$\begin{aligned} \Delta_n &= \int\limits_0^\infty t^{N_\alpha}K_n(t)dt = \int\limits_{\frac{u}{2}}^{u} t^{N_\alpha}n\phi(nt)dt \\ &= \int\limits_0^\infty \frac{u^{N_\alpha}}{n^{N_\alpha}}\phi(u)du \\ &= \frac{1}{n^{N_\alpha}}\int\limits_0^\infty u^{N_\alpha}\phi(u)du = \frac{q_{N_\alpha}}{n^{N\alpha}} \end{aligned}$$

$$\begin{aligned} \frac{K_n(t)}{\Delta_n} &= \frac{n\phi(n\delta)}{\frac{q_{N_\alpha}}{n^{N\alpha}}} = \frac{n^{N\alpha+1}\phi(n\delta)}{q_{N_\alpha}} \\ &\leq \frac{C}{q_{N_\alpha}n^{N\alpha+1}}\int\limits_{n\frac{\delta}{2}}^{n\delta} t^{N_\alpha}\phi(t)dt. \end{aligned}$$

As $n \to \infty$,right side of the last inequality converges to 0

$$\lim_{n\to\infty}\frac{K_n(t)}{\Delta_n} = 0.$$

That is

$$K_n(\delta) = o(\Delta_n).$$

Now, we will show that

$$\int_{\delta}^{\infty} K_n(t)dt = o(\Delta_n)$$

$$\begin{aligned}\int_{\delta}^{\infty} K_n(t)dt &= \int_{\delta}^{\infty} n\phi(nt)dt = \int_{n\delta}^{\infty} \phi(t)dt \\ &= \int_{n\delta}^{\infty} \frac{t^{N_\alpha}\phi(t)}{t^{N_\alpha}}dt \\ &\leq \frac{1}{n^{N_\alpha}\delta^{N_\alpha}} \int_{n\delta}^{\infty} t^{N_\alpha}\phi(t)dt.\end{aligned}$$

As $n \to \infty \quad \int_{\delta}^{\infty} K_n(t)dt = o(\Delta_n)$

In conclusion,

Let $f \in L_p(-\infty, +\infty)$ and $K_n(t)$ be a Fejer-type kernel function. Assume that the nonnegative kernel function $\phi(t)$ satisfies (i)-(iv). For Fejer-type m-singular integral

$$\left(Q_n f\right)(x) = \int_{-\infty}^{+\infty} \left[\sum_{k=1}^{m} (-1)^{k-1} \binom{m}{k} f\left(x+kt\right)\right] n\phi(nt)dt$$

the following equality holds true

$$\left|\left(Q_n f\right)(x) - f(x)\right| = o(n^{-N\alpha}),$$

at the point x which satisfies

$$\lim_{h\to 0} \frac{1}{h^{m+\alpha}} \int_{0}^{h} \left|\left[\Delta_t^m + \Delta_{-t}^m\right] f(x)\right|^p dt = 0.$$

Indeed;

$$\begin{aligned}\Delta_n &= \int_0^\infty t^{N_\alpha} K_n(t)dt = \int_0^\infty t^{N_\alpha} n\phi(nt)dt \\ &= \int_0^\infty \frac{u^{N_\alpha}}{n^{N_\alpha}}\phi(u)du \\ &= \frac{1}{n^{N_\alpha}}\int_0^\infty u^{N_\alpha}\phi(u)du\end{aligned}$$

Integral on the right side of inequality is convergent by (iv)

$$\Delta_n = \frac{1}{n^{N_\alpha}}\int_0^\infty u^{N_\alpha}\phi(u)du \approx \frac{1}{n^{N_\alpha}} = n^{-N\alpha}.$$

If $K_n(t)$ considered as the Abel-Poisson Kernel, then

$$\Delta_n = \int_0^\infty t^{N_\alpha} A_n(t)dt = \frac{1}{n^{N_\alpha}}\int_0^\infty t^{N_\alpha}\frac{1}{\pi}.\frac{1}{1+t^2}dt$$

where $0 \leq N_\alpha < 1$, integral is convergent.

If $K_n(t)$ Gauss-Weierstrass Kernel, then

$$\Delta_n = \int_0^\infty t^{N_\alpha} W_n(t)dt = \frac{1}{n^{N_\alpha}}\int_0^\infty t^{N_\alpha}\frac{1}{\pi}.\frac{e^{-t^2}}{\sqrt{\pi}}dt$$

where $N_\alpha \geq 0$, integral is convergent.

Let $f \in L_p(-\infty, +\infty)$ and let x be a point satisfying

$$\lim_{h\to 0}\frac{1}{h^{m+\alpha}}\int_0^h \left|\left[\Delta_t^m + \Delta_{-t}^m\right] f(x)\right|^p dt = 0$$

Abel-Poisson m-singular integral defined as

$$(A_n f)(x) = \int_{-\infty}^{+\infty} \left[\sum_{k=1}^{m} (-1)^{k-1} \binom{m}{k} f(x+kt) \right] a_n(t) dt$$

where $0 \leq N_\alpha < 1$, the following equation holds true.

$$|(A_n f)(x) - f(x)| = o(n^{-N\alpha}),$$

that is

$$\lim_{n\to\infty} \frac{|(A_n f)(x) - f(x)|}{n^{N\alpha}} = 0.$$

$f \in L_p(-\infty, +\infty)$ and at the point x which satisfies

$$\lim_{h\to 0} \frac{1}{h^{m+\alpha}} \int_0^h \left| \left[\Delta_t^m + \Delta_{-t}^m \right] f(x) \right|^p dt = 0$$

$$(W_n f)(x) = \int_{-\infty}^{+\infty} \left[\sum_{k=1}^{m} (-1)^{k-1} \binom{m}{k} f(x+kt) \right] w_n(t) dt$$

for Gauss-Weierstrass m-singular integral where $N_\alpha > 0$, following equation holds true

$$|(W_n f)(x) - f(x)| = o(n^{-N\alpha}),$$

that is

$$\lim_{n\to\infty} \frac{|(W_n f)(x) - f(x)|}{n^{N\alpha}} = 0.$$

Let $f \in L_p(-\infty, +\infty)$, $\mu(h)$ absolutely continuous function

$$\lim_{h\to 0} \frac{1}{\mu(h)} \left(\int_0^h \left| \left[\Delta_t^m + \Delta_{-t}^m \right] f(x) \right|^p dt \right)^{\frac{1}{p}} = 0$$

x is called (μ, p, m)-Lebesgue point of f

Theorem 4.8. .*Let $K_n(t)$ be nonnegative on $[0,\infty)$ monotone decreasing even kernel and $\int_{-\infty}^{+\infty} K_n(t)dt = 1$. Also, let μ be absolutely continuous function and satisfying*

$$\Delta_n(\mu) = \int_0^{\infty} \mu(h) K_n(t)dt$$

and

$$K_n(\delta) = o(\Delta_n(\mu))$$

for arbitraray $\delta > 0$. As $n \to \infty$,right side of inequality converges to 0

$$\int_{\delta}^{\infty} K_n(t)dt = o(\Delta_n(\mu)) \ \ (n \to \infty)\,.$$

Then at every (μ, p, m)-Lebesgue point of f function, we have

$$\lim_{n\to\infty} \frac{|(L_n^m f)(x) - f(x)|}{\Delta_n(\mu)} = 0.$$

Proof. Let

$$F(t) = \int_0^t \left|\left[\Delta_t^m + \Delta_{-t}^m\right] f(x)\right|^p d\zeta$$

be defined as a function

$$dF(t) = \left|\left[\Delta_t^m + \Delta_{-t}^m\right] f(x)\right|^p dt$$

x is (μ, p, m)-Lebesgue point of f .

$$\lim_{h\to 0} \frac{1}{\mu(h)} \left(\int_0^h \left|\left[\Delta_t^m + \Delta_{-t}^m\right] f(x)\right|^p dt \right)^{\frac{1}{p}} = 0$$

by using the function $F(t)$, one can write

$$\lim_{h\to 0}\frac{1}{\mu(h)}F(h)=0$$

If $\forall\varepsilon>0$, $\exists\delta>0$ $\ni h\leq\delta$,then $F(h)<\varepsilon\mu(h)$

$$2^{\frac{p}{p}^{'}}\left|(L_n^m f)(x)-f(x)\right|\leq\left(\int_0^{\delta}+\int_{\delta}^{\infty}\right)\left|\left[\Delta_t^m+\Delta_{-t}^m\right]f(x)\right|^p K_n(t)\,dt=I_1+I_2$$

using by integration by parts for integral I_1

$$\begin{aligned}I_1 &= K_n(\delta)F(\delta)-\int_0^{\delta}F(t)dK_n(t)\\ &= K_n(\delta)F(\delta)+\int_0^{\delta}F(t)d\left(-K_n(t)\right)\end{aligned}$$

And then

$$I_1\leq\varepsilon\mu(h)K_n(\delta)+\varepsilon\int_0^{\delta}\mu(t)d\left(-K_n(t)\right)$$

using by integration by parts again

$$I_1<\varepsilon\int_0^{\delta}\mu(t)K_n(t)dt$$

For I_2

$$I_2\leq M_1K_n(\delta)\left\|f\right\|_p+M_2\left|f(x)\right|^p\int_{\delta}^{\infty}K_n(t)\,dt$$

We can write $M_1=(2^m-1)^{p+1}$, $M_2=2^p$

$$2^{\frac{p}{p}^{'}}\left|(L_n^m f)(x)-f(x)\right|^p\leq\varepsilon\int_{\delta}^{\infty}K_n(t)\,\mu'(t)dt+M_1K_n(\delta)\left\|f\right\|_p+M_2\left|f(x)\right|^p\int_{\delta}^{\infty}K_n(t)\,dt$$

$$\lim_{n\to\infty} \frac{|(L_n^m f)(x) - f(x)|}{\Delta_n(\mu)} = 0.$$

The proof is completed.

CHAPTER 5

MELLIN TYPE NONLINEAR M-SINGULAR INTEGRAL OPERATOR

Bardaro, Vinti and Karsli studied on pointwise convergence of Mellin type nonlinear m-singular integral operators.

Mellin type nonlinear m-singular integral operator of the form :

$$\left(T_{\lambda}^{m} f\right)(x) = \int_{0}^{\infty} K_{\lambda}\left(z, \sum_{k=1}^{m} (-1)^{k-1} \binom{m}{k} f\left(xz^{k}\right)\right) \frac{dz}{z}$$

where $x > 0$, $m \in \mathbb{N}$ with kernel satisfying some suitable singularity assumptions.

Here $\wedge$ is a non-empty set of indices with topology and λ_0 is accumulation point of $\wedge$ in this topology.

Bardaro and Mantellini investigated the pointwise convergence of families of nonlinear Mellin type convulation operators, defined as

$$\left(T_{w} f\right)(s) = \int_{0}^{\infty} K_{w}\left(ts^{-1}, f\left(t\right)\right) \frac{dt}{t}$$

where $w > 0$, $s > 0$ at Lebesgue points. These operators are examples of operators with homogenous kernel with degree 0 with respect to the multiplicative topological group $\mathbb{R}^{+}$.

As a continuation of this , recently Bardaro, Karsli, and Vinti obtained point-

wise convergence for the family of operators of the form

$$(T_\lambda f)(x) = \int_0^\infty f(z)\, L_\lambda(x,z)\, \frac{dz}{z}$$

where $L_\lambda : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}_0^+$ and $x > 0$,$\lambda > 0$

L_λ is a family of kernels satisfying a general property of homogeneity on the kernels L_λ. Very recently the same authors also obtained similar results for its nonlinear counterpart.

The aim of this paper is to give pointwise convergence results for a family of Mellin type nonlinear m-singular integral operators of the form

$$(T_\lambda^m f)(x) = \int_0^\infty K_\lambda\left(z, \sum_{k=1}^m (-1)^{k-1} \binom{m}{k} f\left(xz^k\right)\right) \frac{dz}{z} \tag{5.1}$$

where $K_\lambda : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ is a family of kernels satisfying a $(L,\varphi)-$Lipschitz condition of type

$$|K_\lambda(s,u) - K_\lambda(s,v)| \leq L_\lambda(s)\varphi(|u-v|)$$

where $\varphi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is continuous and concave with $\varphi(0) = 0$ and $\varphi(u) > 0$ for $u > 0$

$(L_\lambda)_{\lambda\in\wedge}$ is a family of functions satisfying suitable assumptions.

The function L_λ defined by the Lipschitz condition, we will use the following notations:

$$A_{L_\lambda} := \int_0^\infty L_\lambda(z)\frac{dz}{z}$$

and

$$A_{L_\lambda}(U_\delta) := \int_{\mathbb{R}^+\setminus U_\delta}^\infty L_\lambda(z)\frac{dz}{z},$$

for any $\delta > 1$ and $U_\delta = \left[\frac{1}{\delta}, \delta\right] \in U(1)$.

We will put

$$A_{L_\lambda}(U_\delta) = A_\lambda(U_\delta),$$

for any $U_\delta \in U(1)$.

Now we will give folowing definitions.

Definition 5.1. We will say that $(K_\lambda)_{\lambda \in \wedge} \in K$ is singular if

i) $\sup\limits_{\lambda \in \wedge} A_\lambda := A < \infty$

ii) For every $z \in \mathbb{R}^+$ and $u \in \mathbb{R}$, we have

$$\lim_{\lambda \to \lambda_0} \left| \int_0^\infty K_\lambda(z,u) \frac{dz}{z} - u \right| = 0$$

iii) For every$U_\delta \in U(1)$, we have

$$A_\lambda(U_\delta) \to 0,$$

as $\lambda \to \lambda_0$.

Moreover we assume that

iv) For any $\delta > 1$,

$$\lim_{\lambda \to \lambda_0} \sup_{z \in \mathbb{R}^+ \setminus U_\delta} [L_\lambda(z)] = 0$$

v)There exists $\delta_0 > 1$ such that $L_\lambda(z)$ is non-decreasing on $\left(\frac{1}{\delta_0}, 1\right]$ and non-increasing on $[1, \delta_0)$ as a function of z, for each $\lambda \in \wedge$.

Definition 5.2. We will say that a point $x \in \mathbb{R}^+$ is a m-Lebesgue point of the integrable function g, if

$$\lim_{z\to 1}\left|\frac{1}{\log z}\int_1^z\left|\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}g\left(xz^k\right)\right|\frac{dz}{z}\right|=0 \tag{5.2}$$

if $m=1$ then m-Lebesgue point reduce to the Lebesgue point.as considered for the multiplicative group.

Let we will define m_0

$$m_0:=\max_{k=1,\dots,m}\left(\begin{array}{c} m \\ k \end{array}\right).$$

As it is well known, we have

$$m_0=\left(\begin{array}{c} m \\ \frac{m}{2} \end{array}\right),$$

if m is even and

$$m_0=\left(\begin{array}{c} m \\ \frac{m+1}{2} \end{array}\right),$$

if m is odd.

Lemma 5.1. *Let $\psi\in\Psi$. Then if $x\in\mathbb{R}^+$ is a m-Lebesgue point of the integrable function g we have*

$$\left|\int_1^z\psi\left(\left|\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}g\left(xz^k\right)\right|\right)\frac{dz}{z}\right|=o\left(|\log z|\right)\qquad as\quad z\to 1. \tag{5.3}$$

Proof. In order to prove the lemma we will show the following two statements:

$$\left| \int_1^z \psi \left(\left| \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} g\left(xz^k\right) \right| \right) \frac{dz}{z} \right| = o\left(|\log z|\right) \qquad as \quad z \to 1^+$$

$$\left| \int_z^1 \psi \left(\left| \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} g\left(xz^k\right) \right| \right) \frac{dz}{z} \right| = o\left(|\log z|\right) \qquad as \quad z \to 1^-$$

Since ψ is concave, by a Jensen type inequality, one has $z < 1$ and $z > 1$, respectively,

$$\begin{aligned} & \frac{1}{-\log z} \int_z^1 \psi \left(\left| \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} g\left(xz^k\right) \right| \right) \frac{dz}{z} \\ \leq \quad & \psi \left(\frac{1}{-\log z} \int_z^1 \left| \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} g\left(xz^k\right) \right| \frac{dz}{z} \right) \end{aligned}$$

and

$$\begin{aligned} & \frac{1}{\log z} \int_1^z \psi \left(\left| \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} g\left(xz^k\right) \right| \right) \frac{dz}{z} dt \\ \leq \quad & \psi \left(\frac{1}{\log z} \int_1^z \left| \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} g\left(xz^k\right) \right| \frac{dz}{z} \right). \end{aligned}$$

Hence by continuity of ψ and $\psi(0) = 0$, we reach the desired result.

5.1 Pointwise Convergence

The following theorem gives a pointwise approximation of the integral operators (5.1) to the function$f$ at m-Lebesgue points of f.

Theorem 5.1. *Let $\psi \in \Psi$. Suppose that $(K_\lambda)_{\lambda \in \wedge} \in K$ is singular kernel satisfying a $(L, \psi)-$Lipschitz condition (iv) and (v). Let $Dom(T_\lambda) \cap L\mu^1(\mathbb{R}^+)$ and such*

that

$\psi \circ m_0 |f| \in L\mu^1(\mathbb{R}^+)$.Then at every point for which (5.2) holds, we have

$$\lim_{\lambda \to \lambda_0} |(T_\lambda^m f)(x) - f(x)| = 0$$

Proof. From the definition of the integral (5.1) and singularity condition (ii), we write

$$\begin{aligned} |(T_\lambda^m f)(x) - f(x)| &= \int_0^\infty K_\lambda \left(z, \sum_{k=1}^m (-1)^{k-1} \binom{m}{k} f\left(xz^k\right) \right) \frac{dz}{z} - \int_0^\infty K_\lambda (z, f(x)) \frac{dz}{z} \\ &\quad + \int_0^\infty K_\lambda (z, f(x)) \frac{dz}{z} - f(x) \end{aligned}$$

$$\begin{aligned} &\leq \int_0^\infty \left| K_\lambda \left(z, \sum_{k=1}^m (-1)^{k-1} \binom{m}{k} f\left(xz^k\right) \right) - K_\lambda (z, f(x)) \right| \frac{dz}{z} \\ &\quad + \left| \int_0^\infty K_\lambda (z, f(x)) \frac{dz}{z} - f(x) \right| \end{aligned}$$

Applying the $(L, \psi)-$Lipschitz condition on the kernel function, one has

$$\begin{aligned} |(T_\lambda^m f)(x) - f(x)| &\leq \int_0^\infty L_\lambda(z) \psi \left(\left| \sum_{k=1}^m (-1)^{k-1} \binom{m}{k} f\left(xz^k\right) \right| \right) \\ &\quad + \left| \int_0^\infty K_\lambda (z, f(x)) \frac{dz}{z} - f(x) \right| \end{aligned}$$

Let $1 < \delta < \delta_0$ be fixed and put $U_\delta = \left[\frac{1}{\delta}, \delta\right]$

Firstly we consider $I_1(x,\lambda)$.

$$I_1(x,\lambda) = \left\{ \int_{\frac{1}{\delta}}^{\delta} + \int_{\mathbb{R}^+\setminus U_\delta} \right\} L_\lambda(z)\psi\left(\left| \sum_{k=1}^{m} (-1)^{k-1} \binom{m}{k} f\left(xz^k\right) \right| \right) \frac{dz}{z}$$

$$\begin{aligned} &= \int_{\frac{1}{\delta}}^{\delta} L_\lambda(z)\psi\left(\left| \sum_{k=1}^{m} (-1)^{k-1} \binom{m}{k} f\left(xz^k\right) \right| \right) \frac{dz}{z} \\ &\quad + \int_{\mathbb{R}^+\setminus U_\delta} L_\lambda(z)\psi\left(\left| \sum_{k=1}^{m} (-1)^{k-1} \binom{m}{k} f\left(xz^k\right) \right| \right) \frac{dz}{z} \\ &= I_{1,1}(x,\lambda) + I_{1,2}(x,\lambda) \end{aligned}$$

Since $\delta > 1$, according to condition (v), we split $I_{1,1}(x,\lambda)$ as follows:

$$\begin{aligned} I_{1,1}(x,\lambda) &= \left\{ \int_{\frac{1}{\delta}}^{1} + \int_{1}^{\delta} \right\} L_\lambda(z)\psi\left(\left| \sum_{k=1}^{m} (-1)^{k-1} \binom{m}{k} f\left(xz^k\right) \right| \right) \frac{dz}{z} \\ &= \left\{ \int_{\frac{1}{\delta}}^{1} + \int_{1}^{\delta} \right\} L_\lambda(z)\psi\left(\left| (-1)^{m+1} \sum_{k=1}^{m} (-1)^{m-k} \binom{m}{k} f\left(xz^k\right) \right| \right) \frac{dz}{z} \end{aligned}$$

$$\begin{aligned} &= \left\{ \int_{\frac{1}{\delta}}^{1} + \int_{1}^{\delta} \right\} L_\lambda(z)\psi\left(\left| \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} f\left(xz^k\right) \right| \right) \frac{dz}{z} \\ &= I_{1,1,1}(x,\lambda) + I_{1,1,2}(x,\lambda) \end{aligned}$$

Let

$$F(z) = \int_{1}^{z} \psi\left(\left| \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} f\left(xy^k\right) \right| \right) \frac{dy}{y}. \tag{5.4}$$

According to (5.3) we can assume that $\delta > 1$ is such that

$$|F(z)| \leq \varepsilon \log z \tag{5.5}$$

for every $z \in (1, \delta]$.

we now estimate $I_{1,1,1}(x, \lambda)$ and $I_{1,1,2}(x, \lambda)$

$$I_{1,1,2}(x, \lambda) = \int_1^{\delta} L_\lambda(z) \psi \left(\left| \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} f\left(xz^k\right) \right| \right) \frac{dz}{z}$$

from (5.4), we can rewrite $I_{1,1,2}(x, \lambda)$ as

$$|I_{1,1,2}(x, \lambda)| = \left| \int_1^{\delta} L_\lambda(z) dF(z) \right|.$$

Integration by parts gives

$$\begin{aligned} |I_{1,1,2}(x, \lambda)| &= \left| F(\delta) L_\lambda(\delta) - \int_1^{\delta} F(z) dL_\lambda(z) \right| \\ &\leq |F(\delta)| \, L_\lambda(\delta) - \int_1^{\delta} |F(z)| \, d\left[-L_\lambda(z)\right]. \end{aligned} \tag{5.6}$$

According to condition (iv) when $z \in [1, \delta)$, $d\left[-L_\lambda(z)\right]$ is positive. Thus we can apply (5.5) to the right hand side of the inequality (5.6) Hence we get

$$|I_{1,1,2}(x, \lambda)| \leq \varepsilon \log \delta L_\lambda(\delta) + \varepsilon \int_1^{\delta} \log z \, d\left[-L_\lambda(z)\right].$$

Using by integration by parts again, we obtain

$$|I_{1,1,2}(x, \lambda)| \leq \varepsilon \int_1^{\delta} L_\lambda(z) \frac{dz}{z} \tag{5.7}$$

Using the same arguments and taking into consideration the function

$$G(z) = \int_z^1 \psi\left(\left|\sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} f\left(xy^k\right)\right|\right) \frac{dy}{y}$$

for $z \in (0,1)$, we obtain

$$|I_{1,1,1}(x,\lambda)| \leq \varepsilon \int_{\frac{1}{\delta}}^{1} L_\lambda(z) \frac{dz}{z} \tag{5.8}$$

As to the $I_{1,2}(x,\lambda)$, since ψ is non-decreasing and subadditive, we obtain the following inequality

$$\begin{aligned} I_{1,2}(x,\lambda) &= \int_{\mathbb{R}^+\setminus U_\delta} \psi\left(\left|\sum_{k=0}^{m} (-1)^{k-1} \binom{m}{k} f\left(xz^k\right)\right|\right) L_\lambda(z) \frac{dz}{z} \\ &\leq \int_{\mathbb{R}^+\setminus U_\delta} \psi\left(\left|\sum_{k=0}^{m} (-1)^{k-1} \binom{m}{k} f\left(xz^k\right)\right|\right) L_\lambda(z) \frac{dz}{z} \\ &\quad + \psi(|f(x)|) \int_{\mathbb{R}^+\setminus U_\delta} L_\lambda(z) \frac{dz}{z} \\ &\leq m \sup_{z\in\mathbb{R}^+\setminus U_\delta} [L_\lambda(z)] \left\|\psi \circ m_0 |f|\right\|_{L^1_\mu(\mathbb{R}^+)} + \psi(|f(x)|) A_\lambda(U_\delta) \end{aligned} \tag{5.9}$$

Collecting the estimates (5.7), (5.8) and (5.9), we have

$$\begin{aligned} \left|\left(T_\lambda^m f\right)(x) - f(x)\right| \;\leq\; & \varepsilon \int_{\frac{1}{\delta}}^{1} L_\lambda(z) \frac{dz}{z} + m \sup_{z\in\mathbb{R}^+\setminus U_\delta} [L_\lambda(z)] \left\|\psi \circ m_0 |f|\right\|_{L^1_\mu(\mathbb{R}^+)} \\ & + \psi(|f(x)|) A_\lambda(U_\delta) + \left| \int_0^\infty K_\lambda\left(z, f\left(x\right)\right) \frac{dz}{z} - f(x) \right| \end{aligned}$$

in view of the condition (i-iv) tends to zero as $\lambda \to \lambda_0$.

Thus the proof of the theorem is completed.

REFERENCES

[1] Altomare, F. and Campiti, M. (1994), Korovkin-type approximation theory and its applications, Walter de Gruyter, 627, Berlin and New York.

[2] Bardaro, C. Karsli, H. and Vinti, G. (2013), On pointwise convergence of Mellin type nonlinear m-singular integral operators, Communications on Applied Nonlinear Analysis, 2, 25-39.

[3] Butzer, P. L. and Nessel, R. J. (1971), Fourier analysis and approximation, Academic Pres, 553, New York and London.

[4] Korovkin, P. (1960), Linear Operators and Aproximation theory, Hindustan Publishing Corp., 22, Delhi.

[5] Kreyszig, E. (1978), Introductory functional analysis with applications, John Wiley and Sons. Inc., New York, Santa Barbara, London, Sydney, Toronto.

[6] Mamedov, R. (1962), Direct and converse theorems in the theory of approximation of functions by m-singular integrals, Soviet Math. Dokl., 3;711-714.

[7] Mamedov, R. (1967), Fonksiyonların lineer operatörlerle yakınlaşması, Azerbaycan Devlet Neşriyatı, 215, Bakü.

[8] Mamedov, R. (1963), On the order of convergence of m-singular integrals at generalized Lebesgue points and in the space $L_p(-\infty, \infty)$, Izv Akad. Nauk SSSR Ser. Mat., 27, 287-304.

[9] Natanson, I. P. (1955/1964), Theory of functions of a real variable, vol. 2, Frederick Ungar Publishing Co., New York.

[10] Stein, E. M. (1970), Singular integrals and differentiability properties of functions, Princeton University Press, 287, New Jersey.