A CONSTRUCTION FOR THE REAL NUMBERS

by

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This is to certify that we have read this thesis and that in our opinion it is fully adequate, in scope and quality, as a thesis for the degree of Master of Science.

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ABSTRACT

A CONSTRUCTION FOR THE REAL NUMBERS

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In this thesis, a construction of the field of real numbers is given in terms of the equivalence classes of slopes.

Keywords: Slopes; Real Numbers; Cauchy Sequences; Dedekind Cuts; Axioms of Real Numbers.

ÖZET

REEL SAYILARIN BİR İNŞASI

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Bu tezde gerçel sayılar cismi eğimlerin denklik sınıfları kullanılarak verilmistir.

Anahtar Kelimeler: Eğim ; Reel Sayılar; Cauchy Dizileri; Dedekind Kesitleri; Reel Sayıların Aksiyomları.

To my family

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CHAPTER 1

AXIOMS OF THE REAL NUMBER SYSTEMS

In this chapter, we give definition of the notions group, ring, field, partial order and the axiomatic definition of real numbers.

1.1 Groups

Definition 1.1. Let G be a nonempty set and

$$\cdot : G \times G \to G$$

be a function.The pair $(G, \cdot)$ is called a *group* if the followings are satisfied.
(we write ab for $\cdot(a, b)$).

i) (Associativity) For all a, b and c in G,

$$(ab)c = a(bc).$$

ii) (Identity Element) There exists an element e in G, such that for every element a in G,

$$ea = ae = a$$

holds. With respect to this property e is unique, and it is called the identity element.

iii) (Inverse Element) For each a in G, there exists an element b in G such that

$$ab = ba = e.$$

1.2 Rings And Fields

We give the definition of ring as follows:

Definition 1.2. (Ring) Let $(\mathbb{R}, +, \cdot)$ be a triple such that $(\mathbb{R}, +)$ is commutative group and

$$\cdot : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$$

be a function satisfying the following properties:
(we write $\cdot(a, b) = ab$)

i) Multiplication is associative: $(ab)c = a(bc)$

ii) Multiplication distributes over addition: $a(b + c) = (ab) + (ac)$

iii) $(a + b)c = (ac) + (bc)$.

Then the triple $(\mathbb{R}, +, \cdot)$ is called a ring. If there exists $1 \in \mathbb{R}$ such that $1x = x1$ for all $x \in \mathbb{R}$ then 1 is called the identity element of $\mathbb{R}$, and in this case $\mathbb{R}$ is called a *ring* with identity.

Example 1.1. Let $\mathbb{Z}$ be the set of integers. $(\mathbb{Z}, +, \cdot)$ is a ring with identity.

Definition 1.3. (Field) Let $(\mathbb{R}, +, \cdot)$ be a ring with identity $1 \in \mathbb{R}$. If for each $0 \neq x$, there exists $x^{-1} \in \mathbb{R}$ such that

$$xx^{-1} = x^{-1}x = 1,$$

then the triple $(\mathbb{R}, +, \cdot)$ is called a *field*.

Example 1.2. The ring $(\mathbb{Z}, +, \cdot)$ is not a field. But, $(\mathbb{Q}, +, \cdot)$ is a field, where $\mathbb{Q}$ is the set of rational number.

1.3 Partial Orders

Let X be a nonempty set. A nonempty subset $\leq$ of $X \times X$ called a *relation* on X.

Definition 1.4. (Partial Orders) Let $\leq$ be a relation on a set X. If the followings are satisfied the pair $(X, \leq)$ is called a *partially ordered set*.

i) $a \leq a$ for all a in X (reflexivity)

ii) $a \leq b$ and $b \leq a$ then $a = b$ (antisymmetry)

iii) $a \leq b$ and $b \leq c$ then $a \leq c$ (transitivity).

1.4 Axioms of Real Number Systems

Consider $(\mathbb{R}, +, \cdot, 0, 1, \leq)$ where

i) $(\mathbb{R}, +)$ is a group with zero element 0.

ii) $(\mathbb{R}, +, \cdot)$ is a field with identity element 1, and $0 \neq 1$.

iii) $(\mathbb{R}, \leq)$ is a partial ordered set such that each two elements of $\mathbb{R}$ are comparable.

iv) $supA$ exists in $\mathbb{R}$ whenever A is bounded above.

Then sextuple $(\mathbb{R}, +, \cdot, 0, 1, \leq)$ is called a system of real numbers.

CHAPTER 2

CONSTRUCTION OF REAL NUMBERS

The real numbers can also be constructed from the rational numbers in various ways. Two of the most prominent methods are due to Richard Dedekind and Augustin-Louis Cauchy. This chapter discusses the construction of the real numbers from the rational numbers.

2.1 Construction by Caucy Sequences

Consider the set of the rational numbers $\mathbb{Q}$ as field, partially ordered set (with respect to natural order) and a metric space with metric $d(x, y) = |x - y|$.

A sequence (x_n) in $\mathbb{Q}$ is called a *Cauchy Sequence* if for each $0 < \epsilon \in \mathbb{Q}$, there exists $n_0 \in \mathbb{N}$ such that $|x_n - x_m| < \epsilon$ for all $n, m \geq n_0$.

Let $\mathbb{R}$ be the set of Cauchy Sequences in $\mathbb{Q}$. For each $(x_n), (y_n) \in \mathbb{R}$ define

$$(x_n) = (y_n) \Leftrightarrow lim(x_n - y_n) = 0.$$

One can show that $\equiv$ defines an equivalence relation. The equivalence class (x_n) is denoted by $[(x_n)]$. Let

$$\mathbb{R} = \{[(x_n)]|(x_n) \in \mathbb{R}\}$$

Define

$$[(x_n)] + [(y_n)] = [(x_n + y_n)]$$

$$[(x_n)] \cdot [(y_n)] = [(x_n \cdot y_n)]$$

$$0 = [(0)]$$

$$1 = [(1)]$$

$[(x_n)] \leq [(y_n)] \Leftrightarrow \exists\, n_0$ such that $x_n \leq y_n;\ \forall\, n \geq n_0$.

Then $(\mathbb{R}, +, \cdot, 0, 1, \leq)$ is a system of real numbers.

2.2 Construction by Dedekind Cuts

There is another construction of the real number system, which is given as follows; A nonempty subset α of the rational numbers is called a *Dedekind Cut* if,

i) $\mathbb{Q} \backslash \alpha \neq \emptyset$ $(x \in \mathbb{Q} : x \notin \alpha)$

ii) If $x, y \in \mathbb{Q}$, $x < y$ and $y \in \alpha$ then $x \in \alpha$.

iii) There is no $x \in \alpha$ such that $y \leq x$ for all $y \in \mathbb{Q}$ (that is maximum of α does not exists)

Let $\mathbb{R}$ be the set of cuts of $\mathbb{Q}$. For each $\alpha, \beta \in \mathbb{R}$ define

$$\alpha + \beta = \{x + y | x \in \alpha, y \in \beta\}$$

$$\alpha \cdot \beta = \{x \cdot y | x \in \alpha, y \in \beta\}$$

$$0 = \{x \in \mathbb{Q} | x < 0\}$$

$$1 = \{x \in \mathbb{Q} | x < 1\}$$

$$\alpha \leq \beta \Leftrightarrow \alpha \subseteq \beta$$

Then $(\mathbb{R}, +, \cdot, 0, 1, \leq)$ is a system of real numbers.

CHAPTER 3

ANOTHER CONSTRUCTION FOR THE REAL NUMBERS

In this chapter we will study another construction of the real numbers from the group of integers.

3.1 Slopes

Let $(\mathbb{Z}, +)$ be the set of *integers* together with the arithmetic operation of addition. The basic objects in our construction are slopes.

Definition 3.1. (Slope) A *slope* is by definition a map $\lambda : \mathbb{Z} \to \mathbb{Z}$, with the property that the set

$$\{\lambda(m+n) - \lambda(m) - \lambda(n) | m, n \in Z\}$$

is finite.

Example 3.1. Let $\lambda : \mathbb{Z} \to \mathbb{Z}$ be defined by $\lambda(x) = c$.

$$\{\lambda(m+n) - \lambda(m) - \lambda(n) | m, n \in \mathbb{Z}\} = \{c - c - c | m, n \in \mathbb{Z}\} = \{-c | m, n \in Z\}$$

is finite. Then we can say constant function is a slope.

Example 3.2. Let $\lambda : \mathbb{Z} \to \mathbb{Z}$ be defined by $\lambda(x) = x$.

$$\{\lambda(m+n) - \lambda(m) - \lambda(n) | m, n \in \mathbb{Z}\} = \{m + n - m - n | m, n \in \mathbb{Z}\} = \{0 | m, n \in Z\}.$$

is finite. Then we can say *unit function* is slope.

Example 3.3. Let $\lambda : \mathbb{Z} \to \mathbb{Z}$ be defined by $\lambda(x) = x^2$.

$$\{\lambda(m+n) - \lambda(m) - \lambda(n) | m, n \in \mathbb{Z}\} = \{(m+n)^2 - m^2 - n^2 | m, n \in \mathbb{Z}\} = \{2mn | m, n \in Z\}.$$

is infinite. Then $\lambda(x) = x^2$ is not a slope.

Example 3.4. Let $\lambda : \mathbb{Z} \to \mathbb{Z}$ be defined by $\lambda(x) = cx + d$.

$$\{\lambda(m+n) - \lambda(m) - \lambda(n) | m, n \in \mathbb{Z}\}$$

$$= \{c(m+n) + d - cm - d - cn - d | m, n \in \mathbb{Z}\}$$

$$= \{cm + cn + d - cm - d - cn - d | m, n \in Z\}$$

$$= \{-d | m, n \in Z\}$$

is finite. Then we can say $\lambda(x) = cx + d$ is a slope.

Example 3.5. If λ_1 and λ_2 are slopes. Is $\lambda = \lambda_1 + \lambda_2$ a slope?

$$\{\lambda(m+n) - \lambda(m) - \lambda(n) | m, n \in \mathbb{Z}\}$$

$$= \{\lambda_1(m+n) - \lambda_1(m) - \lambda_1(n) + \lambda_2(m+n) - \lambda_2(m) - \lambda_2(n) | m, n \in \mathbb{Z}\}$$

$$\subseteq \{\lambda_1(m+n) - \lambda_1(m) - \lambda_1(n) | m, n \in \mathbb{Z}\} + \{\lambda_2(m+n) - \lambda_2(m) - \lambda_2(n) | m, n \in \mathbb{Z}\}$$

is finite. Then λ is a slope.

Example 3.6. If λ_1 and λ_2 are slopes. Is $\lambda = \lambda_1 \cdot \lambda_2$ a slope?

Let $\lambda_1(x) = x$ and $\lambda_2(x) = x$. $\lambda = x \cdot x = x^2$ is not slope.

Example 3.7. If λ_1 and λ_2 are slopes. Is $\lambda = \lambda_1^{\lambda_2}$ a slope?

Let $\lambda_1(x) = x$ and $\lambda_2(x) = 2$. Then $\lambda = x^2$ is not slope.

Definition 3.2. (Equivalent slope): If the set $\{\lambda_1(n) - \lambda_2(n)\}$ is finite, then two slopes λ_1 and λ_2 are equivalent.

Example 3.8. Let $\lambda_1, \lambda_2 : \mathbb{Z} \to \mathbb{Z}$ be two slopes with $\lambda_1(\mathbb{Z})$ and $\lambda_2(\mathbb{Z})$ are finite. Are $\lambda_1(\mathbb{Z})$ and $\lambda_2(\mathbb{Z})$ equivalent?
$\{\lambda_1(n) - \lambda_2(n) | n \in \mathbb{Z}\} \subseteq \{\lambda_1(n) - \lambda_2(m) | m, n \in \mathbb{Z}\} = \lambda_1(\mathbb{Z}) - \lambda_2(\mathbb{Z})$ is finite. Then $\lambda_1(\mathbb{Z})$ and $\lambda_2(\mathbb{Z})$ are equivalent.

Example 3.9. Let $\lambda_1, \lambda_2 : \mathbb{Z} \to \mathbb{Z}$ be two slopes with $\lambda_1(\mathbb{Z})$ is finite and $\lambda_2(\mathbb{Z})$ is infinite. Are $\lambda_1(\mathbb{Z})$ and $\lambda_2(\mathbb{Z})$ equivalent?
$\{\lambda_1(n) - \lambda_2(n) | n \in \mathbb{Z}\}$ be finite. $\{\lambda_1(n) - \lambda_2(n) | n \in \mathbb{Z}\} = \{\lambda_1(n_i) - \lambda_2(n_i) | i = 1, 2, 3, .., k\}$. $n \in \mathbb{Z}$ and $n \notin \{n_1, n_2, n_3, .., n_k\}$. So $\{\lambda_1(n) - \lambda_2(n) | n \in \mathbb{Z}\}$ is not finite. And $\lambda_1(\mathbb{Z})$ and $\lambda_2(\mathbb{Z})$ are not equivalent.

3.2 Real Numbers In Terms Of Slopes

Definition 3.3. (Relation) Let A and B two sets. A *relation* $\equiv$ from A into B is a subset of the cartesian product $A \times B$. a $\equiv$ b can be read as "a is equivalent to b".

Definition 3.4. (Equivalence Relation) Let A be a nonempty set. A relation $\equiv$ on A is called an *equivalence relation* on A if the followings hold:

i) $a \equiv a$ for all $a \in A$ (reflexive)

ii) if $a \equiv b$, then $b \equiv a$ for all $a, b \in A$ (symmetric)

iii) if $a \equiv b$ and $b \equiv c$ then $a \equiv c$ for all $a, b, c \in A$ (transitive)

An equivalence relation is therefore a relation which is reflexive, symmetric and transitive.

Definition 3.5. (Equivalence Class) Let $\equiv$ an equivalence relation on a nonempty set A, and let a be an element of A. The *equivalence class* of a is defined to be the set of all elements of A that are equivalent to a.

The equivalence class of a will be denoted by $[a]$.

$$[a] = \{x \in A : x \equiv a\}$$

The equivalence classes $[a]$ are subsets of A.

Theorem 3.1. *Let λ_1 and λ_2 be two slopes. Define*

$$\lambda_1 \equiv \lambda_2 \Leftrightarrow \{\lambda_1(n) - \lambda_2(n) | n \in \mathbb{Z}\}$$

is finite. Then $\equiv$ defines an equivalence relation on slope $(\mathbb{Z})$.

Proof.

i) $\{\lambda(n) - \lambda(n) | n \in \mathbb{Z}\} = \{0\}$ *is finite. So* $\lambda \equiv \lambda$.

ii) $\lambda_1 \equiv \lambda_2 \Rightarrow \{\lambda_1(n) - \lambda_2(n) | n \in \mathbb{Z}\} = -\{\lambda_2(n) - \lambda_1(n) | n \in \mathbb{Z}\}$. *So* $\{\lambda_2(n) - \lambda_1(n) | n \in \mathbb{Z}\}$ *is finite. Then* $\lambda_2 \equiv \lambda_1$

iii) $\lambda_1 \equiv \lambda_3 \Leftrightarrow \lambda_1 \equiv \lambda_2$ *and* $\lambda_2 \equiv \lambda_3$. $\{\lambda_1(n) - \lambda_2(n) | n \in \mathbb{Z}\}$ *and* $\{\lambda_2(n) - \lambda_3(n) | n \in \mathbb{Z}\}$ *are finite. Then* $\{\lambda_1(n) - \lambda_3(n) | n \in \mathbb{Z}\}$ *is finite. So* $\lambda_1 \equiv \lambda_3$.

□

Definition 3.6. A *real number* is an equivalence class of slopes.

Definition 3.7. The equivalence class of $\lambda \in slope(\mathbb{Z})$ is denoted by

$$[\lambda] = \{\mu \in slope(\mathbb{Z}) | \lambda \equiv \mu\}$$

Definition 3.8. Define $\mathbb{R} = \{[\lambda] | \lambda \in slope(\mathbb{Z})\}$

Definition 3.9. Each element of $\mathbb{R}$ is called a real number.

3.3 Integers

Let $\mathbb{R}$ denote the set of real numbers.

Example 3.10. For $j \in \mathbb{Z}$ let $\bar{j} : \mathbb{Z} \to \mathbb{Z}$ be the map $\bar{j}(n) = nj$. The map $\bar{j}$ is additive, i.e. satisfies $\bar{j}(n+m) = \bar{j}(n) + \bar{j}(m)$, and hence is a slope for which the expression $\bar{j}(n+m) - \bar{j}(n) - \bar{j}(m)$ takes only the value 0.
Because $\bar{j} : \mathbb{Z} \to \mathbb{Z}$ and $\bar{j}(n) = nj$. Then $\{\bar{j}(n+m) - \bar{j}(n) - \bar{j}(m) | m, n \in \mathbb{Z}\} = \{0\}$ is finite, then $\bar{j}$ is a slope.

We identify an integer $j \in \mathbb{Z}$ with the real number represented by the slope $\bar{j}$. After this identification the set of integers $\mathbb{Z}$ becomes a subset of the set of real numbers $\mathbb{R}$. We see that among the real numbers the integers appear as those real numbers, which are representable by an additive slope.

Example 3.11. Let $\psi : \mathbb{Z} \to \mathbb{R}$ be defined by $j \to [j]$. Is ψ one to one?
$\psi(j_1) = \psi(j_2) \Rightarrow [j_1] = [j_2] \Rightarrow \bar{j_1} = \bar{j_2} \Rightarrow \{\bar{j_1}(n) - \bar{j_2}(n) | n \in \mathbb{Z}\} = \{nj_1 - nj_2 | n \in \mathbb{Z}\}$
$= \{n(j_1 - j_2) | n \in \mathbb{Z}\}$ is finite. So $j_1 = j_2$.
If $j_1 \neq j_2$ then $\{\bar{j_1}(n) - \bar{j_2}(n) | n \in \mathbb{Z}\}$ is not finite and $\psi(j_1) \neq \psi(j_2)$. So ψ is one to one.

Example 3.12. For $p, q \in \mathbb{Z}$, $q > 0$, let the map $\phi : \mathbb{Z} \to \mathbb{Z}$ be defined by

$$\phi(n) = min\{k \in \mathbb{N} | qk \geq pn\}$$

and by

$$\phi(-n) = -\phi(n) for n \in \mathbb{Z}, n < 0.$$

The map ϕ is a slope representing the rational number p/q, i.e. the slope ϕ represents a real number which is a solution of the equation $qx = p$.

The definition of $\phi(n)$ for positive n is just the smallest integer k with $qk \geq pn$. So for all integers n, $n > 0$, we have the inequalities

$$q\phi(n) \geq pn,$$

$$q(\phi(n) - 1) < pn.$$

Remember $\phi(n) = -\phi(-n)$ for integers n with $n < 0$. We conclude for all integers n,

$$-q \leq q\phi(n) - pn \leq q,$$

$$-q \leq q\phi(m) - pm \leq q,$$

$$-q \leq q\phi(n + m) - p(n + m) \leq q.$$

Hence by adding the above inequalities for $n + m$, n, m we get

$$-3q \leq q(\phi(n + m) - \phi(n) - \phi(m)) \leq 3q.$$

So

$$-3 \leq \phi(n + m) - \phi(n) - \phi(m) \leq 3,$$

all integers n,m. So $\phi(n + m) - \phi(n) - \phi(m)$ takes at most 7 values.

This will become clear, when we have defined multiplication of real numbers in our setting. As for the integers we identify the set of rational numbers $\mathbb{Q}$ with a subset of $\mathbb{R}$. One can characterize the rational numbers as those real numbers, which are representable by a slope λ, such that for some integer $q > 0$ the map $n \in \mathbb{Z} \to \lambda(qn) \in \mathbb{Z}$ is additive.

Example 3.13. Let $r \geq 1$ be a rational number. For $n \in \mathbb{N}$ let $L_r(n)$ the greatest $k \in \mathbb{N}$ with $10^k \leq r^n$.

$$L_r(n) = max\{k : 10^k \leq r^n\}$$

$r = 1$

$L_1(1) = max\{k : 10^k \leq 1\} = 0$

$L_1(2) = max\{k : 10^k \leq 1\} = 0$

So $L_1(n) = 0$

$r = 2$

$L_2(1) = max\{k : 10^k \leq 2\} = 0$

$L_2(2) = max\{k : 10^k \leq 4\} = 0$

$L_2(4) = max\{k : 10^k \leq 16\} = 1$

$L_2(7) = max\{k : 10^k \leq 128\} = 2$

$r = 10$

$L_{10}(1) = max\{k : 10^k \leq 10\} = 1$

$L_{10}(2) = max\{k : 10^k \leq 10^2\} = 2$

$L_{10}(3) = max\{k : 10^k \leq 10^3\} = 3$

$L_{10}(n) = n$

The number $L_r(n) + 1$ is the number of digits before the period of the decimal expansion of r^n. We set $L_r(-n) = -L_r(n)$ for negative $n \in \mathbb{Z}$.

$r = 1$

$L_1(1) + 1 = 1 \Rightarrow 1^1 = 1$

$L_1(2) + 1 = 1 \Rightarrow 1^2 = 1$

$r = 2$

$L_2(4) + 1 = 2 \Rightarrow 2^4 = 16$

$L_2(7) + 1 = 3 \Rightarrow 2^7 = 128$

a) Now we will show that the map $n \in \mathbb{Z} \to L_r(n) \in \mathbb{Z}$ is a slope.

$L_r(m + n) = k_1 \Rightarrow 10^{k_1} \leq r^{m+n}$

$L_r(m) = k_2 \Rightarrow 10^{k_2} \leq r^m$

$L_r(n) = k_3 \Rightarrow 10^{k_3} \leq r^n$

Then $10^{k_2+k_3} \leq r^{n+m}$.

$k_2 + k_3 \leq L_r(n+m)$

$L_r(n) + L_r(m) \leq L_r(n+m)$

$0 \leq L_r(n+m) - L_r(n) - L_r(m).$

$10^{k_1} \leq r^{n+m}$

$10^{k_2} \leq r^n$

$10^{k_3} \leq r^m$

Then $10^{k_2+k_3} \leq r^{n+m}$

$k_2 + k_3 \leq L_r(n+m)$

$k_2 + k_3 + 1 > L_r(n+m)$

$L_r(n+m) + k_2 + k_3 < L_r(n) + L_r(m) + k_2 + k_3 + 1$

$L_r(n+m) - L_r(n) - L_r(m) < 1$

So $0 \leq L_r(n+m) - L_r(n) - L_r(m) < 1$

Hence $L_r(n+m) - L_r(n) - L_r(m)$ takes at most 1 values.

Then $L_r(n)$ is a slope.

b) Show $[L_{ab}] = [L_a] + [L_b]$ for rational numbers $a, b \geq 1$.

The basic thing is that for two integers u,v the number of digits ND in decimal notation of u,v and uv behaves almost additive.

$ND(u, v) = ND(u) + ND(v) + error$

$|error| \leq 2$

It follows that $L_a : \mathbb{N} \to \mathbb{N}$ is a slope.

$L_a(n) = ND(a^n) - 1$

It follows that L_{ab} and $L_a + L_b$ are equivalent slope a, b integers > 0.

So the corresponding real numbers satisfie

$[L_{ab}] = [L_a + L_b] = [L_a] + [L_b].$

c) Compute $[L_1]$, $[L_{10}]$. Guess what is $[L_r]$.

$r = 1$

$L_1(1) = 0$

$L_1(2) = 0$

$L_1(3) = 0$

$L_1(n) = 0$

$[L_1] = 0$

$r = 10$

$L_{10}(1) = 1$

$L_{10}(2) = 2$

$L_{10}(3) = 3$

$L_{10}(n) = n$

$[L_{10}] = n$

Indeed $[L_r]$ is a logarithm but not the natural one. ln with base e, but $log10$ with base 10.

d) Define L_r for rational numbers $r > 0$.

$$L_r = -L_{\frac{1}{r}}$$

for $0 < r < 1$.

Definition 3.10. We now define the basic arithmetic operations such as addition and multiplication of real numbers. Let a, $b \in \mathbb{R}$ be real numbers. Let α , β be slopes representing the real numbers a and b.

The map $\alpha + \beta : \mathbb{Z} \to \mathbb{Z}$, which is defined by

$$(\alpha + \beta)(n) = \alpha(n) + \beta(n),$$

is again a slope and its equivalence class is independent of the choice of representatives α, β for a, b. We define the sum $a + b \in \mathbb{R}$ of $a, b \in \mathbb{R}$ as the equivalence class of the slope $\alpha + \beta : \mathbb{Z} \to \mathbb{Z}$.

The composition $\alpha o \beta : \mathbb{Z} \to \mathbb{Z}$ is again a slope, and we define the product $ab \in \mathbb{R}$ as the equivalence class of the composition $\alpha o \beta : \mathbb{Z} \to \mathbb{Z}$.

Example 3.14. Let α and β be two slopes. $\alpha' \in [\alpha]$ and $\beta' \in [\beta]$. Show that $\alpha' + \beta' \in [\alpha + \beta]$.

Proof. $\{(\alpha + \beta)(n) - (\alpha' + \beta')(n) | n \in \mathbb{N}\} = \{\alpha(n) - \alpha'(n) + \beta(n) - \beta'(n) | n \in \mathbb{N}\}$
$\subseteq \{\alpha(n) - \alpha'(n) | n \in \mathbb{N}\} \cup \{\beta(m) - \beta\beta'(m) | m \in \mathbb{N}\}$.

So $\{(\alpha + \beta)(n) - (\alpha' + \beta')(n) | n \in \mathbb{N}\}$ is finite. Then $\alpha' + \beta' \in [\alpha + \beta]$. □

Example 3.15. Let $\alpha = \alpha'$ and $\beta = \beta'$. Show that $\alpha + \beta = \alpha' + \beta'$.

Proof. $\{(\alpha + \beta)(n) - (\alpha' + \beta')(n) | n \in \mathbb{Z}\} = \{\alpha(n) - \alpha'(n) + \beta(n) - \beta'(n) | n \in \mathbb{Z}\}$
$\subseteq \{\alpha(n) - \alpha'(n) | n \in \mathbb{Z}\} \cup \{\beta(m) - \beta'(m) | m \in \mathbb{Z}\}$ is finite. Then $\alpha + \beta = \alpha' + \beta'$. □

Definition 3.11. Let α and β be two slopes. Let $\alpha = [\alpha]$ and $\beta = [\beta]$. We define

$$a + b = [\alpha + \beta]$$

Lemma 3.1. *Let the slopes α, α' represent $a \in \mathbb{R}$ and the slopes β, β' represent $b \in R$. Then the compositions $\alpha \ o \ \beta$ and $\alpha' \ o \ \beta'$ are equivalent slopes.*

Proof. We first show that the map $\alpha o \beta$ is a slope. Let E_α and E_β be finite subsets in $\mathbb{Z}$, such that

$$\alpha(n + m) - \alpha(n) - \alpha(m) \in E_\alpha$$

and

$$\beta(n + m) - \beta(n) - \beta(m) \in E_\beta$$

for $n, m \in \mathbb{Z}$. Hence, for $n, m \in \mathbb{Z}$ there exist u , $u' \in E_\alpha$, $v \in E_\beta$. Then

$$\alpha o \beta(n) + \alpha o \beta(m) - \alpha(\beta(n) + \beta(m)) = u \in E_\alpha$$

$\Rightarrow \alpha o \beta(n) + \alpha o \beta(m) = \alpha(\beta(n) + \beta(m)) + u$

$$\beta(n) + \beta(m) - \beta(n + m) = v \in E_\beta$$

$\Rightarrow \beta(n+m) = \beta(n) + \beta(m) - v$

$\Rightarrow \alpha[\beta(n+m)] = \alpha[\beta(n) + \beta(m) - v]$

$\Rightarrow \alpha o\beta(n+m) = \alpha[\beta(n) + \beta(m) - v]$

$$\alpha[\beta(n) + \beta(m)] + \alpha(-v) - \alpha[\beta(n) + \beta(m) - v] = u' \in E_\alpha$$

$\Rightarrow \alpha[\beta(n) + \beta(m) - v] = \alpha(\beta(n) + \beta(m)) + \alpha(-v) - u'$.

Hence $\alpha o\beta(n) + \alpha o\beta(m) - \alpha o\beta(n+m) =$

$\alpha[\beta(n) + \beta(m)] + u - \alpha[\beta(n) + \beta(m) - v]=$

$\alpha[\beta(n) + \beta(m)] + u - [\alpha(\beta(n) + \beta(m) + \alpha(-v) - u']$

$u - \alpha(-v) - u'$.

We conclude that the expression $\alpha o\beta(n) + \alpha o\beta(m) - \alpha o\beta(n+m)$, $n, m \in \mathbb{Z}$, takes its values in the finite set $\{u - \alpha(-v) - u' | u, u' \in E_\alpha, v \in E_\beta\}$. Hence, the map $\alpha\ o\ \beta$ and, with the same justification, also the map $\alpha'\ o\ \beta'$ are slopes.

And now we will show that the compositions $\alpha o\beta$ and $\alpha' o\beta'$ are equivalent slopes.

Let $E_{\alpha,\alpha'}$ and $E_{\beta,\beta'}$ be finite sets such that we have $\alpha(n) - \alpha'(n) \in E_{\alpha,\alpha'}$ and $\beta(n) - \beta'(n) \in E_{\beta,\beta'}$ for $n \in \mathbb{Z}$. Hence, for $n \in \mathbb{Z}$ there exist $r \in E_{\alpha,\alpha'}$, $s \in E_{\beta,\beta'}$ and $u \in E_\alpha$ with. Then

$$\beta'(n) - \beta(n) = s \in E_{\beta,\beta'}$$

$\Rightarrow \beta(n) = \beta'(n) - s$

$\Rightarrow \alpha[\beta(n)] = \alpha[\beta'(n) - s]$

$\Rightarrow \alpha o\beta(n) = \alpha[\beta'(n) - s]$ Then

$$\alpha'(\beta'(n)) - \alpha(\beta'(n)) = r \in E_{\alpha,\alpha'}$$

$\Rightarrow \alpha'(\beta'(n)) = \alpha(\beta'(n)) + r$

$\Rightarrow \alpha' o\beta'(n) = \alpha(\beta'(n)) + r$.

Then

$$\alpha(\beta'(n)) + \alpha(-s) - \alpha(\beta'(n) - s) = u \in E_\alpha$$

$\Rightarrow \alpha(\beta'(n) - s) = \alpha(\beta'(n)) + \alpha(-s) - u$

So $\alpha o\beta(n) - \alpha' o\beta'(n)$

$= \alpha(\beta'(n) - s) - [\alpha(\beta'(n) + r]$

$=\alpha(\beta'(n)) + \alpha(-s) - u - (\alpha(\beta'(n)) + r)$

$=\alpha(-s) - r - u.$

We conclude that the expression $\alpha o\beta(n) - \alpha' o\beta'(n)$, $n \in \mathbb{Z}$, takes its values in a finite set. Hence, the slopes $\alpha o\beta$ and $\alpha' o\beta'$ are equivalent. □

Definition 3.12. (Positive Slope) : A slope $\lambda : \mathbb{Z} \to \mathbb{Z}$ is positive if

$$\{\lambda(n) | n \in \mathbb{N}, \lambda(n) \leq 0\}$$

is finite and $\lambda(\mathbb{Z})$ is infinite. ($\mathbb{N} = \{n \in \mathbb{Z}, n \geq 0\}$)

Remark 3.1. Let λ_1 and λ_2 be two slopes. If $\lambda_1 = \lambda_2$ and λ_1 is positive then λ_2 is positive. So we can define the following.

Definition 3.13. A real number a is positive (we write $a > 0$ and $0 < a$) if λ is positive where $a = [\lambda]$.

Example 3.16. Let λ_1 and λ_2 are positive. Is $\lambda_1 + \lambda_2$ positive?

λ_1 is positive $\to \{\lambda_1(n) | n \in \mathbb{N}, \lambda_1(n) \leq 0\}$ is finite.

λ_2 is positive $\to \{\lambda_2(n) | n \in \mathbb{N}, \lambda_2(n) \leq 0\}$ is finite.

$\lambda = \lambda_1 + \lambda_2 = \{\lambda_1(n) | n \in \mathbb{N}, \lambda_1(n) \leq 0\} + \{\lambda_2(n) | n \in \mathbb{N}, \lambda_2(n) \leq 0\}$ is finite.

λ_1 is positive $\to \{\lambda_1(n), n \in \mathbb{Z}\}$ is infinite.

λ_2 is positive $\to \{\lambda_2(n), n \in \mathbb{Z}\}$ is infinite.

$\lambda = \lambda_1 + \lambda_2 = \{\lambda_1(n), n \in \mathbb{Z}\} + \{\lambda_2(n), n \in \mathbb{Z}\}$ is infinite.

So $\{\lambda(n) | n \in \mathbb{Z}\}$ is infinite.

Hence $\lambda = \lambda_1 + \lambda_2$ is positive.

From this definition of positivity we obtain the ordering of the real numbers as usual in the following way. If a is positive, we say that $a > 0$ and $0 < a$ hold. The real number a is defined to be less then the real number b if there exists a positive real number t with $b = a + t$. If a is less then b, we say that $a < b$ holds.

We illustrate the definitions by examples before stating and verifying that the set $\mathbb{R}$ with the addition + , multiplication · and order relation < satisfies all the axioms of the real numbers, i.e. of a complete totally ordered archimedean field.

Definition 3.14. (Odd map) A map $f : \mathbb{Z} \to \mathbb{Z}$ is called odd if for all $n \in \mathbb{Z}$ the property $f(-n) = -f(n)$ holds. An odd map $f : \mathbb{Z} \to \mathbb{Z}$ is determined by its restriction to $\mathbb{N}_+ = \{n \in \mathbb{Z} | n > 0\}$.

Definition 3.15. (Odd Slope) Let λ be an arbitrary slope. Then the map $\kappa : \mathbb{Z} \to \mathbb{Z}$ with $\kappa(0) = 0$ defined by

$$\kappa(n) = \lambda(n), n > 0,$$

and by

$$\kappa(n) = -\lambda(-n), n < 0,$$

is an *odd slope*, which is equivalent to the slope λ.

Example 3.17. If $f = \{n | n > 0\} \to \mathbb{Z}$ a map then $\bar{f}(n) = \{f(n), n > 0, -f(-n), n < 0\}$ is an odd map.

Proof. $n > 0 \Rightarrow \bar{f}(-n) = -\bar{f}(-(-n)) = -\bar{f}(n)$.
$n < 0 \Rightarrow \bar{f}(-n) = f(-n) = -\bar{f}(n)$.
$\bar{f}(n) = -\bar{f}(-n)$. □

Example 3.18. Let λ be a slope. Define $\kappa : \mathbb{Z} \to \mathbb{Z}$ by

$$\kappa(n) = \{\lambda(n), n \geq 0, -\lambda(-n), n < 0\}.$$

Then κ is an odd slope.

Example 3.19. Let $\lambda : \mathbb{Z} \to \mathbb{Z}$ be a slope and $\kappa : \mathbb{Z} \to \mathbb{Z}$ be a function.

$\{\lambda(n) - \kappa(n) | n \in \mathbb{Z}\}$ is finite. Then κ is a slope.

Proof. $\{\kappa(n) - \lambda(n) | n \in \mathbb{Z}\}$

$\{\kappa(n) - \lambda(n) | n > 0\} \cup \{\kappa(n) - \lambda(n) | n < 0\} \cup \{\kappa(n) - \lambda(n) | n = 0\}$

$\{\lambda(n) - \lambda(n) | n > 0\} \cup \{-\lambda(-n) - \lambda(n) | n < 0\} \cup \{0 - 0 | n = 0\}$ □

Theorem 3.2. *Let $\lambda : \mathbb{Z} \to \mathbb{Z}$ be an odd function. Then the followings are equivalent.*

i) *λ is a slope.*

ii) *$\{\lambda(n+m) - \lambda(n) - \lambda(m) | n, m \in \mathbb{N}_+\}$ is finite.*

Proof. $i \Rightarrow ii$: λ is a slope $\Rightarrow \{\lambda(n+m) - \lambda(n) - \lambda(m) | n, m \in \mathbb{Z}\}$ is finite.

$\{\lambda(n+m) - \lambda(n) - \lambda(m) | n, m \in \mathbb{N}_+\} \cup \{\lambda(n+m) - \lambda(n) - \lambda(m) | n, m \in \mathbb{Z}, m, n \leq 0\}$

So $\{\lambda(n+m) - \lambda(n) - \lambda(m) | n, m \in \mathbb{N}_+\}$ is finite.

$ii \Rightarrow i$: $\{\lambda(n+m) - \lambda(n) - \lambda(m) | n, m \in \mathbb{N}_+\}$ is finite.

$n = 0, m = 0 \Rightarrow \{\lambda(0) - \lambda(0) - \lambda(0) | n, m = 0\} = \{-\lambda(0) | n, m = 0\}$ is finite.

$n < 0, m < 0 \Rightarrow \{\lambda(n+m) - \lambda(n) - \lambda(m) | m, n < 0\} = \{\lambda(-n-m) - \lambda(-n) - \lambda(-m) | (-n), (-m) > 0\}$ is finite.

Then $\{\lambda(n+m) - \lambda(n) - \lambda(m) | n, m \in \mathbb{N}_+\}$ is finite. So λ is a slope. □

Theorem 3.3. *Let $\lambda : \mathbb{Z} \to \mathbb{Z}$ be a slope. Define $\kappa : \mathbb{Z} \to \mathbb{Z}$ by*

$$\kappa(\mathbb{Z}) = \{\lambda(\mathbb{Z}) : \mathbb{Z} \geq 0, -\lambda(-\mathbb{Z}), \mathbb{Z} < 0\}.$$

i) *κ is odd.*

ii) *κ is a slope.*

iii) *λ and κ are equivalent.*

Corollary : For each real number $a \in \mathbb{R}$ there exists an odd slope λ such that $a = [\lambda]$.

Proof. $i)$ $\mathbb{Z} > 0$, $\kappa(-Z) = -\lambda(-(\mathbb{Z})) = -\lambda(\mathbb{Z}) = -\kappa(\mathbb{Z})$

$\mathbb{Z} < 0$, $\kappa(-\mathbb{Z}) = \lambda(-\mathbb{Z}) = -\kappa(\mathbb{Z})$

Then $\kappa(\mathbb{Z}) = -\kappa(-\mathbb{Z})$. So $\kappa(\mathbb{Z})$ is an odd map.

$ii)$ κ is a slope $\Rightarrow \{\kappa(n+m) - \kappa(n) - \kappa(m) | m, n \in \mathbb{Z}\}$ is finite.

$\{\kappa(n+m) - \kappa(n) - \kappa(m) | m, n \in \mathbb{Z}\}$

$=\{\kappa(n+m) - \kappa(n) - \kappa(m) | m > 0, n > 0\}$.........(1)

$\cup \{\kappa(n+m) - \kappa(n) - \kappa(m) | m < 0, n < 0\}$.....(2)

$\cup \{\kappa(n+m) - \kappa(n) - \kappa(m) | m < 0, n > 0, m+n \geq 0\}$...(3)

$\cup \{\kappa(n+m) - \kappa(n) - \kappa(m) | m < 0, n > 0, m+n \leq 0\}$...(4)

$\cup \{\kappa(n+m) - \kappa(n) - \kappa(m) | m > 0, n < 0, m+n \geq 0\}$...(5)

$\cup \{\kappa(n+m) - \kappa(n) - \kappa(m) | m < 0, n < 0, m+n \leq 0\}$...(6)

(1) $\{\lambda(n+m) - \lambda(n) - \lambda(m) | m > 0, n > 0\}$ is finite.

(2) $\{-\lambda(-n-m) + \lambda(-n) + \lambda(-m) | m < 0, n < 0\}$

$=-\{\lambda(-n-m) - \lambda(-n) - \lambda(-m) | m < 0, n < 0\}$ is finite.

(3) $\{\lambda(n+m) - \lambda(n) - \lambda(m) | m < 0, n > 0, m+n \geq 0\}$

$n > -m$, $n = -m + k$ $k \in \mathbb{N}_+$

$=\{\lambda(n+m) - \lambda(-m+k) + \lambda(-m) | m < 0, n > 0, m+n > 0\}$

$=\{\lambda(n+m) - \lambda(-m+k) + \lambda(k) + \lambda(-m) | m < 0, n > 0, m+n > 0\}$

$=\{\lambda(n+m) - \lambda(k) - [\lambda(-m+k) - \lambda(k) - \lambda(-m)] | m < 0, n > 0, m+n > 0\}$ is finite.

(4) $\{-\lambda(-n-m) - \lambda(n) + \lambda(-m) | m < 0, n > 0, m+n < 0\}$

$m < -n$, $m = -n - k$, $n = -m - k$

$=\{-\lambda(-n-m) - \lambda(-m-k) + \lambda(-m) | m < 0, n > 0, m+n < 0\}$

$=\{-\lambda(-n-m) - \lambda(-m-k) + \lambda(-m) - \lambda(-k) + \lambda(-k) | m < 0, n > 0, m+n < 0\}$

$=\{-\lambda(-n-m) - \lambda(-k) - [\lambda(-m-k) - \lambda(-m) - \lambda(-k)] | m < 0, n > 0, m+n < 0\}$

is finite.

(5) $\{\lambda(n+m) - \lambda(n) - \lambda(m) | m > 0, n > 0\}$

$m > -n$, $m = -n + k$, $k \in \mathbb{N}_+$

$=\{\lambda(n+m)+\lambda(-n)-\lambda(m)|m>0, n<0, m+n>0\}$

$=\{\lambda(n+m)+\lambda(-n)-\lambda(-n+k)+\lambda(k)-\lambda(k)|m>0, n<0, m+n>0\}$

$=\{\lambda(n+m)-\lambda(k)-[\lambda(-n+k)-\lambda(-n)-\lambda(k)]|m>0, n<0, m+n>0\}$ is finite.

(6) $\{-\lambda(-n-m)+\lambda(-n)-\lambda(m)|m>0, n<0, m+n<0\}$

$n<-m$, $n=-m-k$, $m=-n-k$

$=\{-\lambda(-n-m)+\lambda(-n)-\lambda(-n-k)|m>0, n<0, m+n<0\}$

$=\{-\lambda(-n-m)+\lambda(-n)-\lambda(-n-k)-\lambda(-k)+\lambda(-k)|m>0, n<0, m+n<0\}$

$=\{-\lambda(-n-m)-\lambda(-k)-[\lambda(-n-k)-\lambda(-n)-\lambda(-k)|m>0, n<0, m+n<0\}$ is finite.

Hence $\kappa:\mathbb{Z}\to\mathbb{Z}$ is finite. So κ is a slope.

iii) λ and κ are equivalent.

$\{\lambda(n)-\kappa(n)|n\in\mathbb{Z}\}$

$\{\lambda(n)-\kappa(n)|n>0\}\cup\{\lambda(n)-\kappa(n)|n=0\}\cup\{\lambda(n)-\kappa(n)|n<0\}$

$\{\lambda(n)-\lambda(n)|n>0\}\cup\{\lambda(n)-0|n=0\}\cup\{\lambda(n)--\lambda(-n)|n<0\}$

$\{0\}\cup\{\lambda(n)\}\cup\{\lambda(n)+\lambda(-n)|n<0\}$

$\{0\}\cup\{\lambda(n)\}\cup\{\lambda(n+(-n))+\lambda(n)+\lambda(-n)-\lambda(n+(-n))|n<0\}$

$\{0\}\cup\{\lambda(n)\}\cup\{\lambda(n+(-n))-[-\lambda(n)-\lambda(-n)+\lambda(n+(-n))|n<0\}$

$\{0\}\cup\{\lambda(n)\}\cup\{\lambda(0)-[\lambda(n+(-n))-\lambda(n)-\lambda(-n)|n<0\}$ is finite.

So λ and κ are equivalent.

$[\lambda]=[\kappa]$.

□

3.4 A Construction of Square Root of 2

We will construct a slope that represents the number $\sqrt{2}$. Let $\rho:\mathbb{Z}\to\mathbb{Z}$ be the odd map defined by

$$\rho(n)=min\{k\in\mathbb{N}|2n^2\leq k^2\}, n\in\mathbb{N}_+.$$

From the definition of $\rho(n)$ we have for $n \in \mathbb{N}_+$:

$\rho(n) = min\{k \in \mathbb{N} | 2n^2 \leq k^2\} \Rightarrow 2n^2 \leq \rho(n)^2$

$n^2 \leq 2n^2 \leq \rho(n)^2 \Rightarrow n^2 \leq \rho(n)^2 \Rightarrow n \leq \rho(n)$

$2n \in \{k \in \mathbb{N} | 2n^2 \leq k^2\} \Rightarrow \rho(n) \leq 2n.$

So $n \leq \rho(n) \leq 2n$

$(\rho(n) - 1)^2 \notin \{k \in \mathbb{N} | 2n^2 \leq k^2\} \Rightarrow 2n^2 \nleq (\rho(n) - 1)^2 \Rightarrow (\rho(n) - 1)^2 < 2n^2$

So we have for the inequalities :

$n \leq \rho(n) \leq 2n, 2n^2 \leq k^2\}, (\rho(n) - 1)^2 \leq 2n^2$

Then $\rho(n) - 1)^2 \leq 2n^2 \Rightarrow \rho(n)^2 + 1 - 2\rho(n) \leq 2n^2 \Rightarrow \rho(n)^2 \leq 2n^2 + 2\rho(n) * 1$

And $2n^2 + 2\rho(n) - 1 \leq 2n^2 + 2 + 4n \Rightarrow 2\rho(n) - 1 \leq 2 + 4n \Rightarrow \rho(n) \leq 2n.$

Hence $2n^2 \leq \rho(n)^2 \leq 2n^2 + 2\rho(n) - 1 \leq 2(n+1)^2$

For $n, m \in \mathbb{N}_+$, $2n^2 \leq \rho(n)^2 \leq 2(n+1)^2$ and $2m^2 \leq \rho(m)^2 \leq 2(m+1)^2 \Rightarrow 4n^2m^2 \leq \rho(n)^2\rho(m)^2 \leq 4(n+1)^2(m+1)^2 \Rightarrow 2mn \leq \rho(n)\rho(m) \leq 2(n+1)(m+1).$

Hence, for $n, m \in \mathbb{N}_+$ we deduce $2mn \leq \rho(n)\rho(m) \leq 2(n+1)(m+1).$

The map ρ is a slope, since for $n, m \in \mathbb{N}_+$ we estimate

$x = [-\rho(n+m) + \rho(n) + \rho(m)] \cdot [\rho(n+m) + \rho(n) + \rho(m)]$

$= -\rho(n+m)^2 + \rho(n)^2 + \rho(m)^2 + 2\rho(n)\rho(m)$ by

$-4n - 4m - 2 = -2(n+m+1)^2 + 2n^2 + 2m^2 + 4nm \leq x \leq$

$-2(n+m)^2 + 2(n+1)^2 + 2(m+1)^2 + 4(n+1)(m+1) = 8m + 8n + 8$

and with $\rho(n+m) + \rho(n) + \rho(m) \geq n + m + 1,$

we conclude $|\rho(n+m) - \rho(n) - \rho(m)| \leq 8.$

First we will show that $-2 - 4n - 4m = -2(n+m+1)^2 + 2n^2 + 2m^2 + 4nm.$

★ $-2(n+m+1)^2 + 2n^2 + 2m^2 + 4nm$

$= -2(n^2 + m^2 + 2mn + 1 + 2n + 2m) + 2n^2 + 2m^2 + 4mn$

$= -2n^2 - 2m^2 - 4mn - 2 - 4n - 4m + 2n^2 + 2m^2 + 4mn$

$= -2 - 4n - 4m$

★ We will show that $-2(m+n+1)^2 + 2n^2 + 2m^2 + 4mn \leq -\rho(n+m)^2 + \rho(n)^2 + \rho(m)^2 + 2\rho(n)\rho(m).$

$2n^2 \leq \rho(n)^2$ and $2m^2 \leq \rho(m)^2$ then $2n^2 + 2m^2 \leq \rho(n)^2 + \rho(m)^2$

$2n^2 \leq \rho(n)^2$ and $2m^2 \leq \rho(m)^2$ then $4m^2n^2 \leq \rho(n)^2\rho(m)^2 \Rightarrow 2mn \leq \rho(n)\rho(m) \Rightarrow 4mn \leq 2\rho(n)\rho(m)$.

$k = m + n \Rightarrow \rho(k)^2 \leq 2(k+1)^2 \Rightarrow \rho(n+m)^2 \leq 2(m+n+1)^2 \leq -\rho(n+m)^2$.

So $-2(m+n+1)^2 + 2n^2 + 2m^2 + 4mn \leq -\rho(n+m)^2 + \rho(n)^2 + \rho(m)^2 + 2\rho(n)\rho(m)$.

★ $\rho(n)^2 \leq 2(n+1)^2$ and $\rho(m)^2 \leq 2(m+1)^2$ then $\rho(n)^2 + \rho(m)^2 \leq 2(n+1)^2 + 2(m+1)^2$.

$\rho(n)^2 \leq 2(n+1)^2$ and $\rho(m)^2 \leq 2(m+1)^2$ then $\rho(n)^2 + \rho(m)^2 \leq 4(n+1)^2(m+1)^2 \Rightarrow \rho(n)\rho(m) \leq 2(n+1)(m+1) \Rightarrow 2\rho(n)\rho(m) \leq 4(n+1)(m+1)$.

$n + m = k \Rightarrow 2k^2 \leq \rho(k)^2 \Rightarrow 2(n+m)^2 \leq \rho(n+m)^2 \Rightarrow -\rho(n+m)^2 \leq -2(n+m)^2$.

So $-\rho(n+m)^2 + \rho(n)^2 + \rho(m)^2 + 2\rho(n)\rho(m) \leq -2(n+m)^2 + 2(n+1)^2 + 2(m+1)^2 + 4(n+1)(m+1)$

★ $8m + 8n + 8 = -2(n+m)^2 + 2(n+1)^2 + 2(m+1)^2 + 4(n+1)(m+1)$

$= -2(n^2 + m^2 + 2mn) + 2(n^2 + 1 + 2n) + 2(m^2 + 1 + m) + 4(mn + m + n + 1)$

$= -2n^2 - 2m^2 - 4mn + 2n^2 + 2 + 4n + 2m^2 + 2 + 4m + 4mn + 4m + 4n + 4$

$= 2 + 4n + 2 + 4m + 4m + 4n + 4$

$= 8m + 8n + 8$

★ $n \leq \rho(n)$, $m \leq \rho(m)$ and $1 \leq \rho(n+m)$

So $n + m + 1 \leq \rho(n) + \rho(m) + \rho(n+m)$

★ $x \leq 8(m+n+1)$

$-\rho(n+m)^2 + \rho(n)^2 + \rho(m)^2 + 2\rho(n)\rho(m) \leq 8(m+n+1)$

$[-\rho(n+m)+\rho(n)+\rho(m)].[\rho(n+m)+\rho(n)+\rho(m)] \leq 8(m+n+1) \leq 8(\rho(n+m)+\rho(n)+\rho(m))$

$-\rho(n+m) + \rho(n) + \rho(m) \leq 8$

$-(\rho(n+m) + \rho(n) + \rho(m)) \leq -(n+m+1)$

$-4(\rho(n+m) + \rho(n) + \rho(m)) \leq -4(n+m+1)$

$-4(\rho(n+m) + \rho(n) + \rho(m)) \leq -4(n+m+1) \leq [-\rho(n+m) + \rho(n) + \rho(m)].[\rho(n+m) + \rho(n) + \rho(m))]$

Hence $-4 \leq -\rho(n+m) + \rho(n) + \rho(m) \leq 8$.

The equivalence class of ρ is a positive real number a satisfying $a^2 = 2$. Indeed, the number a^2 is represented by the composition $\rho o \rho$.

We have for $n \in \mathbb{N}_+$ the inequalities :

$2n^2 \leq \rho(n)^2 \Rightarrow 4n^2 \leq 2\rho(n)^2$

$\rho(n) = k \Rightarrow 2k^2 \leq \rho(k)^2 \Rightarrow 2\rho(n)^2 \leq \rho(\rho(n))^2$.

$\rho(n) = k \Rightarrow$

$4n^2 \leq 2\rho(n)^2 \leq \rho(\rho(n))^2 \leq 2\rho(n+1)^2 \leq 4n^2 + 8n + 2 \leq 4(n+1)^2$

showing $2n \leq \rho(\rho(n)) \leq 2n + 2$ which means that the slopes $\rho o \rho$ and $\bar{2}$ are equivalent and represent the integer 2. Hence ρ represents the square root $\sqrt{2}$ of 2 , which is the length of the diagonal of a unit square and can not be represented as a fraction $\frac{p}{q}$.

3.5 Roots Of Polynomial

We will now construct a real number that is a root of the polynomial

$$p(x) = x^5 + x - 3.$$

Let the odd map α: $\mathbb{Z} \to \mathbb{Z}$ be defined by

$$\alpha(n) = min\{k \in \mathbb{N} | 3n^5 \leq k^5 + n^4 k\}$$

$$= min\{k \in \mathbb{N} | p(\frac{k}{n}) \geq 0, n \in \mathbb{N}_+\}$$

From the definition of $\alpha(n)$ we conclude

$$p(\frac{\alpha(n) - 1}{n}) < 0 \leq p(\frac{\alpha(n)}{n})$$

From $p(2) > 0$, $p(1) = -1$ and the monotonicity of the map $r \in \mathbb{Q} \to p(r) \in \mathbb{Q}$ we deduce,

$$\frac{\alpha(n)-1}{n} \leq 2, \frac{\alpha(n)}{n} \geq 1.$$

We will show that the map α is a slope and represents the real root a of the equation $x^5 + x - 3$. This root a can not be represented by a compound radical expression after the work of Paolo Ruffuni (1762-1822) and of Niels Henrik Abel (18021829), see [1,16,17].

First we show that α is a slope. For $n, m \in \mathbb{N}_+$ fixed let a_-, a_+ be the rational numbers

$$a_- = max\{\frac{\alpha(n)-1}{n}, \frac{\alpha(m)-1}{m}, \frac{\alpha(m+n)-1}{m+n}\}$$

and

$$a_+ = min\{\frac{\alpha(n)}{n}, \frac{\alpha(m)}{m}), \frac{\alpha(m+n)}{m+n}\}$$

We have $a_- \leq a_+$. From the monotonicity of p and the definition of α we deduce $p(a_-) \leq 0, p(a_+) \geq 0$. Let A be any rational number with $a_- \leq A \leq a_+$. From

$$\frac{\alpha(n)-1}{n} \leq a_- \leq A \leq a_+ \leq \frac{\alpha(n)}{n}$$

we deduce the inequality $|\alpha(n) - nA| \leq 1$. We also obtain in the same way the inequalities $|\alpha(m) - mA| \leq 1$ and $|\alpha(m+n) - (m+n)A| \leq 1$. From last three inequalities it follows $|\alpha(m+n) - \alpha(m) - \alpha(n)| \leq 3$, hence α is a slope.

Let a be the real number that is represented by the slope α. We show $p(a) = 0$ by showing that the slope $\alpha^{o5} + \alpha - \bar{3}$ is bounded. Here we have used the notation α^{o5} for the $e - th$ iterate, $e \in \mathbb{N}$, of α. It is not at all easy to handle directly iterates of slopes.

The following estimate helps out and is proved by induction upon the exponent $e \in \mathbb{N}_+$

$$|n^{e-1}\alpha^{oe}(n) - \alpha(n)^e| \leq n^{e-1}(1 + |\alpha(1)| + S_\alpha)^{e-1}, n \in \mathbb{N}_+,$$

where the quantity

$$S_\alpha = max|\alpha(u + v) - \alpha(u) - \alpha(v)|, u, v \in \mathbb{Z}$$

measures the non-additivity of the slope α. Note also $|\alpha(n)| \leq |n|(|\alpha(1)| + S_\alpha)$. It follows that the slope α^{oe} is equivalent to the odd slope defined by

$$n \in \mathbb{N}_+ \to [\frac{\alpha(n)^e}{n^{e-1}}]_{Gauss}.$$

So the slope $\alpha^{o5} + \alpha - \bar{3}$ is equivalent to the odd slope ε defined by

$$n \in \mathbb{N}_+ \to [\frac{\alpha(n)^5}{n^4}]_{Gauss} + \alpha(n) - 3n.$$

The slope ε is bounded, since for $n \in \mathbb{N}_+$ we have with the monotonicity of p' the inequalities

$$0 \leq \varepsilon(n) = np(\frac{\alpha(n)}{n}) \leq n(p(\frac{\alpha(n) - 1}{n} + \frac{1}{n})) \leq$$
$$n(p(\frac{\alpha(n) - 1}{n} + \frac{p'(2)}{n}) \leq p'(2) = 81$$

If one does not want the use of the derivative p', one can with the binomial formula prove for $0 \leq x \leq x + h \leq 2$, $0 < h < 1$, the inequality $p(x + h) - p(x) = (x + h)5 + (x + h) - x^5 - x \leq 212h$ and obtain the bound

$$0 \leq \varepsilon(n) \leq 212.$$

It follows the claim $p(a) = 0$. Here $[?]_{Gauss}$ is the Gaussian integral part bracket

$$[x]_{Gauss} = max\{k \in \mathbb{Z} | k \leq x\}, x \in \mathbb{R}.$$

3.6 A Construction of π

Let $\beta : \mathbb{Z} \to \mathbb{Z}$ be the odd map with

$$\beta(n) = \sharp\{(p, q) \in \mathbb{Z} \times \mathbb{Z} | p^2 + q^2 \leq n\}; n > 0.$$

Unit squares in the plane with centers at the lattice points $(p, q) \in \mathbb{Z}\times\mathbb{Z}$, $p^2+q^2 \leq n$, cover the disk of radius $\sqrt{n} - \frac{1}{2}\sqrt{2}$ and are contained in the disk of radius $\sqrt{n} + \frac{1}{2}\sqrt{2}$. Hence

$$|\beta(n) - n\pi| \leq 2\sqrt{2}\sqrt{n}, n \in \mathbb{N}.$$

It follows that the odd map $\bar{\beta}$ defined by $n \in \mathbb{N}_+ \to [\frac{\beta(n^2)}{n}]$ is a slope. The map β is not a slope since the quantity $s(n) = \beta(n) - \beta(n-1) - \beta(1)$ is not bounded. One has for instance $s(5^u) = 4u - 1, u \in \mathbb{N}_+$, see [3] about Jacobi's Theorem.

This part is less self contained as explained in the beginning of the chapter. The aim is not to define real numbers but merely to show that famous real numbers can be represented by slopes.

$\beta(n)$ counts the points in the euclidian plane with integral coordinates (p, q) and with $p^2 + q^2 \leq n$.

Consider the squares of size 1, sides parallel to the coordinate axis, and center such a point (p, q). These squares of area 1 cover the disk with center $(0, 0)$ and radius $\sqrt{n} - \frac{1}{2}\sqrt{2}$ and moreover lie in the disk of radius $\sqrt{n} + \frac{1}{2}\sqrt{2}$. So we get an estimate for $\beta(n)$ by considering area's.

The estimate is $|\beta(n) - n\pi| \leq 2\sqrt{2}\sqrt{n}$.

Substituting for n its square n^2 we get $|\beta(n^2) - n^2\pi| \leq 2\sqrt{2}n$, also we have

$$|\frac{\beta(n^2)}{n} - n\pi| \leq 2\sqrt{2}.$$

This shows that $\bar{\beta}(n) = \frac{\beta(n^2)}{n}$ is a slope.

Now $\beta(n) - \beta(n-1)$ counts the number of times the number n can be written as $p^2 + q^2$, p,q integers.

5 can be written 8 times as $p^2 + q^2$, 5^u can be written $4(u+1)$ times as $p^2 + q^2$, $\beta(1) = 5$, so putting $n = 5^u$ we have $\beta(n) - \beta(n-1) - \beta(1) = 4(u+1) - 5 = 4u - 1$, hence the map $n \to \beta(n)$ is not a slope.

The slope $\bar{\beta}$ represents the area π of the unit disk in the plane, see Chap. 1 in Anschauliche Geometrie of David Hilbert and S. Cohn-Vossen [7]. Johann Heinrich Lambert (1728-1777) has proved in his communication of 1761 to the Academy in Berlin [10] that the number π is not a rational number and has conjectured that π is not root of a polynomial equation with integral coefficients, see [12], [5]. Carl Louis Ferdinand von Lindemann [13] has proved this conjecture in 1882, see [2].

3.7 A Construction of e

The number e appeared in the sixteenth century, when it was noticed that the expression $(1+\frac{1}{n})^n$ for compound interest increases with n to a certain value $2,7182818...$, see the book "*e* The Story of a Number" by Eli Maor [M]. The number e became of central importance in Mathematics since its interpretations in Geometry and Analysis by Gregoire de Saint-Vincent (1584-1667). It is not obvious to define the number e with a slope. We use the solution to a problem, see [4], of Jakob Steiner (1796- 1863) and define for $n \in \mathbb{N}$, $n > 0$, the integer $\epsilon(n)$ to be the natural number k, $k > 0$, such that the expression $(\frac{k}{n})^{\frac{n}{k}}$ takes its maximal value. The corresponding odd function $\epsilon : \mathbb{Z} \to \mathbb{Z}$ is a slope representing the number e.

The classical construction in textbooks of the system of real numbers is based on Dedekind cuts or on Cauchy sequences $(r_n)_{n\in\mathbb{N}}$ of rational numbers. The present construction by slopes is related to the classical ones as follows: To a slope λ corresponds a Dedekind cut (A, B) by setting

$$A = \{\frac{p}{q} \in \mathbb{Q} | \bar{p} \leq \lambda o \bar{q}\}$$

and

$$B = \{\frac{p}{q} \in \mathbb{Q} | \lambda o \bar{q} \leq \bar{p}\}$$

and also a Cauchy sequence $(r_n)_{n\in\mathbb{N}}$ by setting $r_n = \frac{\lambda(n+1)}{n+1}$.

3.8 Well Adjusted Slopes

Definition 3.16. (Well Adjusted Slope) We call a slope λ well adjusted if it is odd and satisfies the inequalities;

$$-1 \leq \lambda(m+n) - \lambda(m) - \lambda(n) \leq 1$$

One can say that a well adjusted slope need not be an additive map from $\mathbb{Z}$ to $\mathbb{Z}$, but deviates as little as possible from being additive. Each slope is equivalent to a well adjusted slope, as shows the Concentration Lemma below. So in particular, a real number can be represented by a well adjusted slope.

Definition 3.17. (Optimal Euclidean Division) For integers p, q, $q \neq 0$, the result of optimal euclidean division of p by q will be denoted by $p : q$. The optimal euclidean division is the integer $r = p : q \in \mathbb{Z}$ that satisfies the inequalities

$$2p - |q| \leq 2qr < 2p + |q|.$$

where $|q| = max\{q, -q\}$ is the absolute value of q. For instance $4 : 7 = 1$ but $3 : 7 = 0$. If $\frac{p}{q}$, $p, q \in \mathbb{Z}, q \neq 0$, denotes the fraction, then we have $|\frac{p}{q} - p : q| \leq \frac{1}{2}$.

$$2p - |q| \leq 2qr < 2p + |q|$$

$$\frac{2p - |q|}{2q} \leq r < \frac{2p + |q|}{2q}, q > 0$$

$$\frac{p}{q} - \frac{1}{2} \leq r < \frac{p}{q} + \frac{1}{2}$$

$$\frac{p}{q} - \frac{1}{2} \leq p : q < \frac{p}{q} + \frac{1}{2}$$

$$\frac{-1}{2} \leq p : q - \frac{p}{q} < \frac{1}{2}$$

$$|p : q - \frac{p}{q}| < \frac{1}{2}$$

$$|p : q - \frac{p}{q}| < \frac{1}{2}$$

Lemma 3.2. *Let $q \in \mathbb{N}_+$ and $a, b, c \in \mathbb{Z}$ be such that $-q \leq a - b - c \leq q$. Then we have*

$$-1 \leq a : 3q - b : 3q - c : 3q \leq 1.$$

Proof.

$$0 \leq |a : 3q - b : 3q - c : 3q|$$

$$= |a : 3q - b : 3q - c : 3q - \frac{a}{3q} + \frac{a}{3q} - \frac{b}{3q} + \frac{b}{3q} - \frac{c}{3q} + \frac{c}{3q}|$$

$$|(a : 3q - \frac{a}{3q}) + (-b : 3q + \frac{b}{3q}) + (-c : 3q + \frac{c}{3q}) + (\frac{a}{3q} - \frac{b}{3q} - \frac{c}{3q})|$$

$$\leq |\frac{a}{3q} - a : 3q| + |\frac{b}{3q} - b : 3q| + |\frac{c}{3q} - c : 3q| + |\frac{a}{3q} - \frac{b}{3q} - \frac{c}{3q}|$$

$$\leq \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{3} = \frac{11}{6} < 2.$$

Hence $-1 \leq a : 3q - b : 3q - c : 3q \leq 1.$ □

Lemma 3.3. *Let $n, m \in \mathbb{N}_+$ and $c \in \mathbb{Z}$. Then we have*

$$-1 \le c : m(n+m) - c : n(n+m) - c : nm \le 1.$$

Proof. *The integer*

$$c : m(n+m) - c : n(n+m) - c : nm$$

differs from the integer

$$\frac{c}{m(n+m)} - \frac{c}{n(n+m)} - \frac{c}{nm} = 0.$$

$$|[c : m(n+m) - c : n(n+m) - c : nm] - [\frac{c}{m(n+m)} - \frac{c}{n(n+m)} - \frac{c}{nm}]|$$

$$= [c : m(n+m) - \frac{c}{m(n+m)}] + [-c : n(n+m) + \frac{c}{n(n+m)}] + [-c : nm + \frac{c}{nm}]$$

$$\le |c : m(n+m) - \frac{c}{m(n+m)}| + | - c : n(n+m) + \frac{c}{n(n+m)}| + | - c : nm + \frac{c}{nm}|$$

$$\le \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = \frac{3}{2} < 2.$$

Then

$$-1 \le c : m(n+m) - c : n(n+m) - c : nm \le 1.$$

□

Lemma 3.4. *(Concentration Lemma) Let λ be a slope. Let $s \in \mathbb{N}_+$ be such that for all $n, m \in \mathbb{Z}$ we have*

$$-s \le \lambda(m+n) - \lambda(m) - \lambda(n) \le s.$$

Let $\lambda' : \mathbb{Z} \to \mathbb{Z}$ be defined by

$$\lambda'(n) = \lambda(3sn) : 3s, n \in \mathbb{Z}.$$

Then the map λ' is a well adjusted slope, which is equivalent to the slope λ.

Proof. *By induction on $t \in \mathbb{N}_+$ we prove $-s(t-1) \leq \lambda(tn) - t\lambda(n) \leq s(t-1)$. Let $t = 1$. Then $0 \leq \lambda(n) - \lambda(n) \leq 0 \Rightarrow 0 \leq 0 \leq 0$. So it's true.*

Let $t = t+1$. Then $-s((t+1)-1) \leq \lambda((t+1)n) - (t+1)\lambda(n) \leq s((t+1)-1)$

$-s((t+1)-1) \leq \lambda(tn+n) - t\lambda(n) - \lambda(n) \leq s((t+1)-1)$

$-s((t+1)-1) \leq \lambda(tn+n) - t\lambda(n) - \lambda(n) + \lambda(tn) - \lambda(tn) \leq s((t+1)-1)$

$-s((t+1)-1) \leq \lambda(tn+n) - \lambda(tn) - \lambda(n) + \lambda(tn) - t\lambda(n) \leq s((t+1)-1)$

We use $-s \leq \lambda(m+n) - \lambda(m) - \lambda(n) \leq s$ and $-s(t-1) \leq \lambda(tn) - t\lambda(n) \leq s(t-1)$ we get

$-s - s(t-1) \leq \lambda(tn+n) - \lambda(tn) - \lambda(n) + \lambda(tn) - t\lambda(n) \leq s + s(t-1)$

$-s((t-1)+1) \leq \lambda(tn+n) - \lambda(tn) - \lambda(n) + \lambda(tn) - t\lambda(n) \leq s((t-1)+1)$

$-s((t+1)-1) \leq \lambda(tn+n) - \lambda(tn) - \lambda(n) + \lambda(tn) - t\lambda(n) \leq s((t+1)-1)$

Setting $t = 3s$ we get

$-s(t-1) \leq \lambda(tn) - t\lambda(n) \leq s(t-1)$

$-s(3s-1) \leq \lambda(3sn) - 3s\lambda(n) \leq s(3s-1)$

and after dividing by $3s$

$$\frac{-s(3s-1)}{3s} \leq \frac{\lambda(3sn)}{3s} - \frac{3s\lambda(n)}{3s} \leq \frac{s(3s-1)}{3s}$$

$$\frac{1-3s}{3} \leq \frac{\lambda(3sn)}{3s} - \lambda(n) \leq \frac{3s-1}{3}$$

$$\frac{1}{3} \leq \frac{\lambda(3sn)}{3s} - \lambda(n) \leq \ s - \frac{1}{3}$$

and using $|\frac{\lambda(3sn)}{3s} - \lambda'(n)| \leq \frac{1}{2}$ we get

$$\frac{-1}{2} \leq \frac{\lambda(3sn)}{3s} - \lambda'(n) \leq \frac{1}{2}$$

$$\frac{-1}{2} + \lambda'(n) \leq \frac{\lambda(3sn)}{3s} \leq \frac{1}{2} + \lambda'(n)$$

$$\frac{-1}{2} - \lambda'(n) \leq \frac{-\lambda(3sn)}{3s} \leq \frac{1}{2} - \lambda'(n)$$

$$\frac{1}{3} - s - \frac{1}{2} - \lambda'(n) \leq -\lambda(n) \leq s - \frac{1}{3} + \frac{1}{2} - \lambda'(n)$$

$$-s + (\frac{1}{3} - \frac{1}{2}) \leq \lambda'(n) - \lambda(n) \leq s - (\frac{1}{3} - \frac{1}{2})$$

which shows for the integers $s, \lambda'(n), \lambda(n)$ *the inequality*

$$-s \leq \lambda'(n) - \lambda(n) \leq s$$

and hence the equivalence of the slopes λ *and* λ' *. With lemma 2 we deduce from*

$$-s \leq \lambda(3sn + 3sm) - \lambda(3sn) - \lambda(3sm) \leq s$$

the inequality

$$-1 \leq \lambda'(n + m) - \lambda'(n) - \lambda'(m) \leq 1$$

Hence the slope λ' *is well adjusted.* □

A well adjusted slope λ has the following properties:

i) $|\lambda(n+1) - \lambda(n)| \leq |\lambda(1)| + 1$

ii) if for some $k \in \mathbb{N}_+$ we have $\lambda(k) > 1$ (or $\lambda(k) < -1$), then we have for any $n \in \mathbb{N}_+$ the inequality $\lambda(n) \geq -1 + n : k$ (or $\lambda(n) \leq +1 - n : k$),

iii) if for some $k \in \mathbb{Z}$ we have $\lambda(k) > 1$, then for $v \in \mathbb{Z}$ the set $\{n \in \mathbb{Z} | \lambda(n) = v\}$ is finite and has fewer then $k + 1$ elements,

iv) if for some $k \in \mathbb{Z}$ we have $\lambda(k) > 1$, then for any $v \in \mathbb{Z}$, there exists $n \in \mathbb{Z}$ with $|v\lambda(n)| \leq |\lambda(1)| + 1$

v) the real number x represented by λ satisfies $x > 0$ if and only if there exists $a \in \mathbb{N}$ with $\lambda(a) > 1$,

vi) let y be a real number represented by a well adjusted slope κ. We have $x > y$ if and only if there exists $n \in \mathbb{N}_+$ with $\lambda(n) > 2 + \kappa(n)$.

We claim that the map σ is a slope. Indeed, for $u \in \mathbb{N}_+$ let $\delta_u \in \Delta$ be a slope, which attains at u the value $max\{\delta(u) | \delta \in \Delta\}$. So, we have $\delta_u(u) = \sigma(u)$. For $p, N \in \mathbb{N}_+$ put $q = pN$. We compare δ_p, δ_q at p and q as follows.

Lemma 3.5. *Let λ be a slope. If λ takes infinitely many values, then there exist b, $B \in \mathbb{N}_+$ such that the following inequalities hold:*

$$|\lambda(n+k) - \lambda(n)| \leq kb, n \in \mathbb{Z}, k \in \mathbb{N}$$

$$|\lambda(n+kB) - \lambda(n)| \geq k, n \in \mathbb{Z}, k \in \mathbb{N}$$

In particular, the slope λ takes each value at most $2B - 1$ times.

Proof. *Let λ be a slope with $|\lambda(m+n) - \lambda(n) - \lambda(m)| \leq s$.*

We have $|\lambda(n+1) - \lambda(n) - \lambda(1)| \leq s$.

$|\lambda(n+1)-\lambda(n)-\lambda(1)|+|\lambda(1)| \leq s+|\lambda(1)|.$

$|\lambda(n+1)-\lambda(n)-\lambda(1)+\lambda(1)| \leq s+|\lambda(1)|$

$|\lambda(n+1)-\lambda(n)| \leq s+|\lambda(1)|$

$|\lambda(n+2)-\lambda(n)| \leq |\lambda(n+2)-\lambda(n+1)|+|\lambda(n+1)-\lambda(n)|.$

$\leq |\lambda(1)|+s+|\lambda(1)|+s = 2(|\lambda(1)|+s)$

$|\lambda(n+k)-\lambda(n)| \leq k(|\lambda(1)+s|)$

$b = |\lambda(1)|+s \rightarrow |\lambda(n+k)-\lambda(n)| \leq kb, n \in \mathbb{Z}, k \in \mathbb{N}$

Let λ takes infinetely many values.

Let B such that $|\lambda(B)| \geq s+1$

Assume $\lambda(B) > 0$

We have $\lambda(n+B)-\lambda(n) \geq \lambda(B)-s \geq 1$

$\lambda(n+2B)-\lambda(n+B) \geq 1$

$\lambda(n+kB)-\lambda(n+(k-1)B) \geq 1.$

Hence $\lambda(n+kB)-\lambda(n) \geq k$ with the assumption $\lambda(B) < 0$, a slight modification gives also the inequality.

So we have $|\lambda(n+kB)-\lambda(n)| \geq k$. □

Example 3.20. There are many well known examples of well adjusted slopes. For a natural number $a > 0$, let its length $L(a)$ be 1 less then the number of digits that appear in the decimal notation of a. The odd function $\lambda : \mathbb{Z} \rightarrow \mathbb{Z}$ defined by $\lambda(n) = L(a^n)$, $n > 0$, is a well adjusted slope that represents the real number $Log_{10}(a)$. The importance of the Log_{10} function lies in the property:

$$Log_{10}(ab) = Log_{10}(ab) + Log_{10}(ab)$$

The length function L satisfies this property with error at most 1. We have:

$$-1 \leq L(ab) - L(a) - L(b) \leq 1$$

hence L appears as a well adjusted map that transform multiplication of natural numbers in addition of natural numbers. We get finally the following definition for $Log_{10}(x)$ of a real number x; $x \geq 1$:

$$Log_{10}(x) = [n \in \mathbb{N} \rightarrow L([x_n]_{Gauss})]_{ec}$$

where $[?]_{ec}$ means equivalence class of ?. For $0 < x \leq 1$ we define

$$Log_{10}(x) = -Log_{10}(\frac{1}{x}).$$

3.9 The Axioms

We now state, partially in abbreviated form, the axioms for a complete totally ordered field, that are satisfied by the quadruple $(\mathbb{R}, +, \cdot, <)$. The presentation of the axioms is slightly redundant.

1. The pair $(\mathbb{R}, +)$ is an abelian group.
2. The triple $(\mathbb{R}, +, \cdot)$ is a field.
3. The quadruple $(\mathbb{R}, +, \cdot, <)$ is an archimedean, complete, totally ordered field.

Complete ordered field,for any non-empty subset T bounded from above in $\mathbb{R}$ there exists a least upper bound in $\mathbb{R}$ called the supremum of T. It will be denoted by $SupT$. Archimedian ordered field, for $a \in \mathbb{R}$, $a > 0$ and $A \in \mathbb{R}$ there exists $N \in \mathbb{N}$ such that $Na > A$.

We now begin the verification of the axioms for the system of real numbers, that we have introduced above. We leave out those verifications, that are straightforward and can be done without using well adjusted slopes as representatives.

The addition + of integers makes $\mathbb{Z}$ into an abelian group $(\mathbb{Z}, +)$. It follows easily that $(\mathbb{R}, +)$ is also an abelian group.

Proof. $\mathbb{R} = \{[\lambda] : \lambda \text{ is a slope}\}$

Now we will show that the pair $(\mathbb{R}, +)$ is an abelian group.

$\forall([\alpha], [\beta]) \in \mathbb{R} \times \mathbb{R}$

$$+ : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$$

$$([\alpha], [\beta]) \to [\alpha] + [\beta] = [\alpha + \beta]$$

$\forall[\alpha], [\beta] \in \mathbb{R}$,$([\alpha] + [\beta]) = [\alpha + \beta] = ([\beta] + [\alpha]) = [\beta] + [\alpha]$ (the commutative law)

$\forall[\alpha], [\beta], [\gamma] \in \mathbb{R}$,$[\alpha] + ([\beta] + [\gamma]) = [\alpha] + [\beta + \gamma] = [\alpha + \beta + \gamma] = [\alpha + \beta] + [\gamma] = ([\alpha] + [\beta]) + [\gamma]$ (the associtive law)

There exists ana element $[0] \in \mathbb{R}$ such that $[\alpha] + [0] = [\alpha + 0] = [\alpha]$ for all $[\alpha] \in \mathbb{R}$.

For each $[\alpha] \in \mathbb{R}$ there exists an element $[-\alpha] \in \mathbb{R}$ such that $[\alpha]+[-\alpha] = [\alpha+(-\alpha)] = [\alpha - \alpha] = [0]$.

Now we will show that the triple $(\mathbb{R}, +, \cdot)$ is a field, i.e. verifies all the field axioms.

$$\cdot : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$$

$$([\alpha], [\beta]) \to [\alpha] \cdot [\beta] = [\alpha o \beta]$$

We must show that $[\alpha] \cdot [\beta] = [\beta] \cdot [\alpha]$

$\Rightarrow [\alpha o \beta] = [\beta o \alpha]$

$\Rightarrow |\alpha o \beta(n) - \beta o \alpha(n)|$ is finite.

$$n\alpha(\beta(n)) = \alpha(n\beta(n)) + E_1 = \alpha(\beta(n)n) + E_1 = \alpha(n)\beta(n) + E_2 + E_1$$

with

$$|E_1| \leq |n|S_\alpha$$

and

$$|E_2| \leq |\beta(n)|S_\alpha \leq |n|(|\beta(1) + S_\beta|)S_\alpha.$$

It follows

$$|\alpha o\beta(n) - \beta o\alpha(n)| \leq S_\alpha(1 + |\beta(1)| + S_\beta) + S_\beta(1 + |\alpha(1)| + S_\alpha)$$

showing that the slopes $\alpha o\beta$ and $\beta o\alpha$ are equivalent, hence the multiplication is commutative.

Multiplication is associative since the composition of maps is.
$\forall[\alpha], [\beta], [\gamma] \in \mathbb{R}$,$[\alpha] \cdot ([\beta] \cdot [\gamma]) = [\alpha] \cdot [\beta o\gamma] = [\alpha o\beta o\gamma] = [\alpha o\beta] \cdot [\gamma] = ([\alpha] \cdot [\beta]) \cdot [\gamma]$

Let 1 be the real number represented by the identity map $IdZ : \mathbb{Z} \to \mathbb{Z}$. Clearly we have for a real number x the properties $1x = x1 = x$, which makes 1 into the unit element of the multiplication $\cdot$ of $\mathbb{R}$.

$$\exists[1] \in \mathbb{R} : \forall[\alpha] \in \mathbb{R}, [\alpha] \cdot [1] = [\alpha o1] = [\alpha]$$

We now construct a right inverse for $x \in \mathbb{R}$, $x \neq 0$, i.e. an element $y \in \mathbb{R}$ satisfying $xy = 1$.

Let α be a well adjusted representing slope for x. It follows, that for each $v \in \mathbb{Z}$ we may choose $n_v \in \mathbb{Z}$ with $|v - \alpha(n_v)| \leq |\alpha(1)| + 1$. We define a map $\beta : \mathbb{Z} \to \mathbb{Z}$ by $\beta(v) = n_v$.

We claim that the map β is a slope. Indeed, for $v, w \in \mathbb{Z}$ we have

$$|\alpha(\beta(v + w) - \beta(v) - \beta(w))| = |\alpha(n_{v+w} - n_v - n_w)| \leq$$

$$|(v+w)-v-w|+2+3(|\alpha(1)|+1)=3|\alpha(1)|+5.$$

Since α takes each value only finitely many times, we conclude that the set $\{\beta(v+w)-\beta(v)-\beta(w)|v,w\in\mathbb{Z}\}$ is finite.

For $v\in\mathbb{Z}$ we have $\alpha o\beta(v)=\alpha(n_v)$, so the slopes $\alpha o\beta$ and Id_Z are equivalent, since $|v-\alpha(n_v)|\leq|\alpha(1)|+1$ holds. It follows that $xy=1$. □

The pair $(\mathbb{R},<)$ is an order relation. First we prove that the relation $<$ is total. Let x, y be real numbers represented by the slopes α and β. We consider the slope $\delta=\alpha-\beta$, which represents the number $x-y$. Let δ' be the well adjusted slope equivalent to δ given by the concentration lemma. If $\delta'(n)\in\{-1,0,1\}$ for all $n\in\mathbb{Z}$ then we have $x=y$. If $x\neq y$ we have for some $n\in\mathbb{N}$ either $\delta'(n)>1$, or $\delta'(n)<-1$. In the first case we have $x>y$ and in the second case we have $x<y$. The case $x=y$ excludes $x<y$ and $x>y$. The cases $x<y$ and $x>y$ exclude each other. It remains only to prove transitivity. Let x,y,z be real numbers with $x>y$ and $y>z$, which are represented by the slopes α,β and γ. Let δ_1, δ_2 be well adjusted slopes equivalent to the slopes $\alpha-\beta$ and $\beta-\gamma$. So for some $n\in\mathbb{N}_+$ and $m\in\mathbb{N}_+$ we have $\delta_1(n)>1$ and $\delta_2(m)>1$. So $\delta_1(n+n)\geq\delta_1(n)+\delta_1(n)-1\geq 2+2-1=3$. Hence $\delta_1(2n)>2$. By induction on k, $\delta_1(kn)>k$. So $\delta_1(nm)>m$. Also $\delta_2(nm)>n$. It follows that $(\delta_1+\delta_2)(nm)\geq n+1+m+1$. Then $(\delta_1+\delta_2)(nm)>2$. The well adjusted slope δ_{12} equivalent to $\delta_1+\delta_2$ will satisfy $\delta_{12}(nm)>1$ and hence we have $x>z$.

The quadruple $(\mathbb{R},+,\cdot,<)$ is an ordered field. Let x , y be reals satisfying $x<y$ and let t be real. We represent x,y,t by the well adjusted slopes α,β and τ. Since $x<y$ there exists $b\in\mathbb{N}$ with $\alpha(bn)<\alpha(bn)-n$, $n\in\mathbb{N}_+$. Hence, $\alpha(bn)+\tau(bn)<\alpha(bn)+\tau(bn)$ - n , $n\in\mathbb{N}_+$, showing the monotonicity property for translations $x+t<y+t$. If $t>0$, for some $d\in\mathbb{N}$ we have $\tau(dn)>n$, $n\in\mathbb{N}_+$, hence $\tau(\alpha(bdn))<\tau(\alpha(dbn))-n$, $n\in\mathbb{N}_+$, showing the monotonicity property for the stretching $tx<ty$.

We now prove the archimedean property. Let $a\in\mathbb{R}$ with $a>0$ and $A\in\mathbb{R}$ be given. We construct $N \in \mathbb{N}$ such that $Na > A$, as follows. We represent a and A by well adjusted slopes λ and Λ. Since $a > 0$ holds, we may choose $n \in \mathbb{N}_+$ with $\lambda(n) > 1$. Then $\lambda(2n) > 2$. Define

$$N = 1 + max\{\Lambda(2n), 0\}.$$

Let κ be the well adjusted slope equivalent to the slope $N\lambda$. We have

$$\kappa(2n) > N\lambda(2n) - N > 2 + \Lambda(2n).$$

Hence $Na > A$.

Finally, we will establish the completeness. Let D be a non-empty subset in $\mathbb{R}$ bounded from above by $m \in \mathbb{R}$. So for $x \in D$ we have the inequality $x \leq m$. Let Δ be a set of well adjusted slopes representing the real numbers in the set D. Let μ be a well adjusted slope representing m. For every $n \in \mathbb{N}$ and every $\delta \in \Delta$ we have

$$\delta(n) < \mu(n) + 2.$$

It follows that for $n \in \mathbb{N}_+$ the non-empty set $\{\delta(n)|\delta \in \Delta\}$ is bounded from above by $\mu(n) + 2$. Let $\sigma : \mathbb{Z} \to \mathbb{Z}$ be the odd map defined by

$$\sigma(n) = max\{\delta(n)|\delta \in \Delta\}.$$

We claim that the map σ is a slope. Indeed, for $u \in \mathbb{N}_+$ let $\delta_u \in \Delta$ be a slope, which attains at u the value $max\{\delta(u)|\delta \in \Delta\}$. So, we have $\delta_u(u) = \sigma(u)$. For $p, N \in \mathbb{N}_+$ put $q = pN$. We compare δ_p, δ_q at p and q as follows. We have

$$\delta_q(q) : N \leq \delta_p(p) + 1$$

since $|\delta_q(q) : N - \delta(p)| \leq 1$ and $\delta_q(p) \leq \delta_p(p)$. We also have

$$N\delta_p(p) \leq \delta_p(q) + N \leq \delta_q(q) + N.$$

We conclude that for all p, $N \in \mathbb{N}_+$

$$|\delta_p(p) - \delta_{pN}(pN) : N| \leq 1.$$

Hence for $n, m \in \mathbb{N}_+$, where we put $c = \delta_{nm(n+m)}(nm(n+m))$, the following inequalities hold:

$$|\sigma(n) - c : m(n + m)| \leq 1,$$

$$|\sigma(m) - c : n(n + m)| \leq 1,$$

$$|\sigma(n + m) - c : nm| \leq 1.$$

For instance, the first inequality is obtained with $p = n$, $N = m(n + m)$, $q = Np$ and by comparing δ_p and δ_q at the point p. From

$$|c : nm - c : m(n + m) - c : n(n + m)| \leq 1$$

it follows that for all $n, m \in \mathbb{N}_+$ we have

$$|\sigma(n + m) - \sigma(n) - \sigma(m)| \leq 1 + 3 = 4$$

which proves our claim.

Let s be the real number represented by the slope σ. For all $x \in D$ we have the inequality $x \leq s$, since for a slope $\delta \in \Delta$ representing x the inequalities

$$\delta(n) \leq \delta_n(n) = \sigma(n), n \in \mathbb{N}_+,$$

hold. So $s \in \mathbb{R}$ is an upper bound for D.

In order to prove that s is the least upper bound of D, we show that no $t \in \mathbb{R}$ with $t < s$ is an upper bound of D. Indeed, let τ be a well adjusted slope for $t \in \mathbb{R}$ with $t < s$. There exists $n \in \mathbb{N}_+$ with $\tau(n) < \sigma(n) - 2$. Let x in D be represented by δ_n. We have $\delta_n(n) > \tau(n) + 2$, hence $x > t$ and t is not an upper bound for D.

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