ANALYSIS ON SELF-SIMILAR SETS

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By
Hayriye Sıla Kesimal
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We certify that we have read this thesis and that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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ABSTRACT

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Self-similar sets are one class of fractals that are invariant under geometric similarities. In this thesis, we study on self-similar sets. We give the definition of a self-similar set K and present the proof the existence theorem of such a set. We define the shift space. We define a relation between the shift space and K. We show the self-similarity of the shift space. We define overlapping set, critical and post-critical set for a self-similar set. We give the characterization of K by the periodic sequences in the shift space.

We give the notion of a self-similar structure and define a self-similar set purely topologically. We give its local topology. We define isomorphism between self-similar structures so that we can have a classification of self-similar structures. We point out that the critical set for a self-similar structure provides us with a characterization for determining the topological structure of a self-similar structure. We define the notion of minimality for a self-similar structure and give a characterization theorem for investigating the minimality of a self-similar structure. We define a post-critically finite self-similar structure.

Keywords: Self-similar set, Shift space, Shift Map, Critical Set, Post-critical Set, Self-similar Structure, Post-critically finite set.

ÖZET

KENDİNE BENZER KÜMELER ÜZERİNDE ANALİZ

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Kendine benzer kümeler fraktalların geometrik benzerlik dönüşümü altında değişmez olan bir sınıfıdır. Bu tezde kendine benzer kümeler uzerinde çalıştık. Kendine benzer küme K'nin tanımını verdik ve bu kümelerin varlık teoreminin kanıtını sunduk. Kaydırma uzayını tanımladık. Kaydırma uzayı ve kendine benzer küme K arasında bir bağıntı tanımladık. Çakışma kümesini, kritik ve post-kritik kümelerini tanıttık. Kaydırma uzayında periyodik dizileri tanımladık. K'yi tanımlamada periyodik dizileri kullanarak bir karakterizasyon verdik.

Kendine benzer kümeleri topolojik olarak tanımlamak amacıyla kendine benzer yapıyı tanımladık. Kendine benzer yapının lokal topolojisini açıkladık. Kendine benzer yapılar arasında izomorfizmi tanımladık. Kendine benzer yapı için çakışma kümesi, kritik ve post-kritik kümeleri tanıttık ve bu kümelerin kendine benzer yapı uzerinde belirlediği topolojik yapıyı karakterize ettik. Minimal olan kendine benzer yapıyı açıkladık ve bu yapının araştırılması için bir karakterizasyon teoremi verdik. Kaydırma uzayının parçalanması tanımını yaptık. Post-kritik sonlu kümeyi tanımladık.

Anahtar sözcükler: Kendine benzer küme, Kendine Benzer Yapı, Kaydırma Uzayı, Sağa Kaydırma Operatörü, Post-kritik sonlu küme.

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I dedicate this thesis to my dear mother Dilek Paksoy Pekin.

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Chapter 1

Introduction

The notion of a *fractal* was firstly introduced by Mandelbrot in [5]. As he stated in [5], he coined the word fractal from the Latin word *fractus* which means irregular. Also, the verb form of fractus corresponds "to break" in meaning. He claimed that many patterns in nature are highly irregular and fragmented when it is compared to Euclidean geometry. So, he conceived and developed a new geometry of nature so that it is used in several fields.

In [3], without a general definition fractals are considered to have some characteristics. Some of these characteristics are :

(i) It is highly irregular that it cannot be described with the standard geometry.

(ii) Its irregularity can be seen for arbitrary scalings.

(iii) Often it has some sort of self-similarity, maybe in a statistical way.

As Mandelbrot stated in [5] fractals tend to be *scaling* that the irregularity or fragmentation is identical at all scales. Also, for his purpose he considered both regularities and irregularities to be statistical.

Mandelbrot also stated in [5] that most classical fractals are invariant under certain transformations of scale. There is another notion as *self-similarity* which is an older idea than that the idea of fractals. A fractal which is invariant under ordinary *geometric similarity* (see Definition 2.0.3) is called *self-similar*. *Cantor's middle third set* is a typical example for a self-similar fractal. It is invariant under the set of similarities.

Besides many fractals, one class of them attracted attention for studies, namely self-similar sets. For example, Koch curve is a well-known example of it. Though, mathematicians defined self-similar sets differently. In [2], it is considered as a set in a complete metric space which is invariant under a contraction, in other words:

A non-empty subset K in a complete metric space (X, d) is self-similar if

$$K = f_1(K) \cup f_2(K) \cup ... \cup f_n(K)$$

where $\{f_1, f_2, ..., f_n\}$ is a finite set of contractions.(See Definition 2.0.1)

In [4], a self-similar set is considered in a more restricted sense. He called a nonempty subset K in a complete metric space (X, d) self-similar if

$$K = f_1(K) \cup f_2(K) \cup ... \cup f_n(K)$$

where $\{f_1, f_2, ..., f_n\}$ is a set of contractions which are *similitudes* (See Definition 2.0.2), in addition to K having a separation condition.

In [8], it is proved that

$$K = \overline{\left(\bigcup_{1 \leq i_1, ..., i_n \leq m;\ n \geq 1} \mathrm{Fix}(f_{i_1} \circ f_{i_2} \circ ... \circ f_{i_n}) \right)},$$

where $\mathrm{Fix}(f)$ denotes the set of fixed points of f.

On the other hand, [1] generalized the definition of self-similarity to *weak contractions* where (X, d) is a complete and separable metric space. Then a compact set K is self-similar if

$$K = \Sigma_{\lambda \in \Lambda} f_\lambda(K)$$

where $\{f_\lambda\}_{\lambda \in \Lambda}$ is a set of weak contractions on the set $\mathcal{C}(X)$ of compact subsets of X and the index set Λ is either finite or $\mathbb{N}$.

In this thesis, the first chapter is the Introduction part.

In the second chapter we will give some preliminary concepts and results that are needed for the presentation in the other chapters. The sources are [7], [9], [12] and [14].

The third chapter, based on [2], is devoted to a short presentation of what is now called symbolic dynamics associated to an iterated function system. We will first see the construction of self-similar sets associated to an iterated function system of contractions on a complete metric space by means of the contraction principle applied on the complete space of nonempty compact subsets and endowed with the Hausdorff metric. Then, taking the alphabet of the indices of the contraction functions, there is defined the *shift space* which is the abstract space of self-similar sets. To do so, there is defined a metric in which the shift space becomes a compact metric space. On this space, there is a backward shift σ as well as the forward shifts σ_k, that play the role of the iterated function system. By pointing out the relation between the shift space and a self-similar set K there is constructed a structure that contains all the necessary information of the dynamical system.

The fourth chapter is also based on [2]. For this chapter, we present the concept of a *self-similar structure* that provides an abstract topological description of a self-similar set. On the one hand, self-similar sets yield self-similar structures provided that all contractions are injective, in particular, if they are similarities. On the other hand, in the definition of a self-similar structure the generating functions are not supposed to be contractions. The next step is the classification of self-similar structures through isomorphism classes. Next, we introduce the *minimality* condition for a self-similar structure. Lastly, we introduce the definition of a *post-critically finite*, self-similar structure, which is very important for analysis on self-similar sets, and present a theorem that gives many equivalent characterizations of minimality.

Chapter 2

Preliminaries

Definition 2.0.1 (*Contraction*)**.** Let (X, d_X) and (Y, d_Y) be metric spaces. A map $f\colon X \to Y$ is called a *contraction* if

$$L = \sup_{x,y\in X, x\neq y} \frac{d_Y(f(x), f(y))}{d_X(x,y)} < 1,$$

where L is called the *Lipschitz constant* of f.

Definition 2.0.2 (*Similitude*)**.** Let (X, d) be a metric space. A map $f\colon X \to X$ is called a *similitude*, equivalently, a *similarity*, if $d(f(x), f(y)){=}rd(x, y)$ for all x,$y \in X$ and some fixed r.

Remark 2.0.3. A *similarity* of a Euclidean space $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is a bijection such that for any two points x and y we have

$$d(f(x), f(y)) = rd(x, y),$$

where d is the Euclidean distance. In other words, it is a map from the space onto itself that multiplies all distances by the same positive real number r. A similarity $f\colon \mathbb{R}^n \to \mathbb{R}^n$ with ratio r takes the form

$$f(x) = rAx + t$$

where A is an $n\, x\, n$ orthogonal matrix and $t \in \mathbb{R}^n$ is a translation vector. A similarity preserves ratios of distances. It therefore transforms figures into similar figures. It preserves both collinearity and angles. If $r < 1$ then the similarity becomes a contraction.

Theorem 2.0.4 (*Contraction Mapping Theorem*)**.** Let (X, d) be a complete metric space and $f: X \to X$ be a contraction with respect to d. Then f has a unique fixed point x such that $f(x)$=x.

Theorem 2.0.5 (*Baire's Category Theorem*)**.** Let (X, d) be a complete metric space and let $\{E_i\}_i$ be a countable collection of nowhere dense subsets of X. Then X cannot be written as the union of E_i's:

$$X \neq \bigcup_i E_i.$$

Definition 2.0.6 (*Local Base*)**.** Let X be a topological space and let, for an arbitrary point $x \in$ X, $\mathcal{N}_x$ be the collection of all neighborhoods of $x \in X$. A *local base* at x is any set $\mathcal{B} \subset \mathcal{N}_x$ for which each element $U \in \mathcal{N}_x$ includes some member of $\mathcal{B}$.

Remark 2.0.7. Let (X, d) be a metric space. At $x \in X$, $\{B_{1/n}(x)\}$ forms a *local base.* Here, $B_r(x)$=$\{\ y \in X \mid d(x, y) \leq r\}$ is the closed ball of center x with radius r>0.

Definition 2.0.8. For A,$B \in \mathcal{C}(X)$, where $\mathcal{C}(x)$ is the collection of all non-empty compact subsets of X, let us define

$$d_H(A, B) = \inf\{\epsilon > 0 \mid A \subseteq N_\epsilon(B) \text{ and } B \subseteq N_\epsilon(A)\}$$

where

$$\begin{aligned} N_\epsilon(A) &= \{x \in X \mid d(x, y) \leq \epsilon \ \text{ for some } y \in A\} \\ &= \bigcup_{y \in A} B_\epsilon(y). \end{aligned}$$

Remark 2.0.9. Observe that $N_\epsilon(A)$ is closed. To show this, let $x \in \overline{N_\epsilon(A)}$, where $\overline{N_\epsilon(A)}$ denotes the closure of $N_\epsilon(A)$. Then, there exist a sequence $(x_n)_{n\geq 1}$ $\in N_\epsilon(A)$ such that $d(x_n, x) \to 0$ as $n \to \infty$. Since for all n, $x_n \in N_\epsilon(A)$, we have $d(x_n, a_n) \leq \epsilon$ for some $a_n \in A$. Taking into account that A is compact, if follows that there exists $a \in A$ and a subsequence $(a_{k_n})_n$ such that $a_{k_n} \to a$ as $n \to +\infty$.

Therefore, by passing to subsequences, without loss of generality we can assume that $a_n \to a$ as $n \to +\infty$. By using the *Triangle Inequality*, we have

$$\begin{aligned} d(x,a) &\le d(x,x_n) + d(x_n,a_n) + d(a_n,a) \\ &\le d(x,x_n) + \epsilon + d(a_n,a), \quad n \ge 1, \end{aligned}$$

and hence, letting $n \to +\infty$ in the previous inequality

$$d(x,a) \le \epsilon.$$

Therefore, $x \in N_\epsilon(A)$ and we have proven that $N_\epsilon(A)$ is closed.

Theorem 2.0.10. d_H is a metric on $\mathcal{C}(X)$, called the *Hausdorff* metric.

Proof. Let A,B,$C \in \mathcal{C}(X)$. Then,

(i) $d_H(A) \ge 0$ by definition.

(ii) $d_H(A,B)$=$d_H(B,A)$, obviously.

(iii) Assume that A=B. Let $a \in A$. Then, $a \in B$, and $a \in N_{1/n}(B)$, for n>0. Hence, $A \subseteq N_{1/n}(B)$. In the same way, $B \subseteq N_{1/n}(A)$. Therefore, $d_H(A,B)$=0.

For the other implication, let $d_H(A,B)$=0. Then, for all $n \in \mathbb{N}$, we have $A \subseteq N_{1/n}(B)$ and $B \subseteq N_{1/n}(A)$. Let $a \in A$. Then, there exists $(b_n)_{n\in\mathbb{N}} \in B$ such that $d(a,b_n) \le \frac{1}{n}$ for all $n \in \mathbb{N}$. Hence a is an adherent point for B. Since B is a compact subset of a complete metric space X, it is closed. Therefore, $a \in B$.

In the same way, $B \subseteq A$. Hence, A=B.

(iv) Let $d_H(A,B)$$\le r$ and $d_H(B,C)$$\le s$. Then, $N_r(A) \supseteq B$ and $N_s(B) \supseteq C$. Hence, $N_{r+s}(A) \supseteq C$. In the same way, $N_{r+s}(C) \supseteq A$. Therefore, we have $d_H(A,C) \le r+s$. On the other hand, there exists $(r_n)_{n\in\mathbb{N}}$ converging to $\mathrm{d}_H(A,B)$ and there exists $(s_n)_{n\in\mathbb{N}}$ converging to $d_H(B,C)$ as $n \to \infty$. Hence, we have $N_{r_n+s_n}(A) \supseteq C$ and $N_{r_n+s_n}(C) \supseteq A$. Since $r_n \to d_H(A,B)$ and $s_n \to d_H(B,C)$ as $n \to \infty$, we have r_n+$s_n \to d_H(A,B)$+ $d_H(B,C)$. Therefore,

$$d_H(A,C) \le d_H(A,B) + d_H(B,C).$$

□

Chapter 3

Self-similar Sets

This chapter is based on [2].

3.1 Existence and Uniqueness of a Self-similar Set

Definition 3.1.1. Let (X, d) be a complete metric space. A non-empty compact subset $K \subseteq X$ is *self-similar* if there exists a finite set of maps $\{f_1, f_2, ..., f_n\}$ such that

$$K = f_1(K) \cup f_2(K) \cup ... \cup f_n(K).$$

Theorem 3.1.2. Let (X, d) be a complete metric space and $f_i \colon X \to X$ be a contraction for $i \in \{1, 2, ..., N\}$. Then, there exists a unique non-empty compact subset $K \subseteq X$ such that

$$K = f_1(K) \cup f_2(K) \cup ... \cup f_N(K).$$

Remark 3.1.3. For a continuous function $f \colon X \to X$, there is an induced map

$$f^* \colon \mathcal{C}(X) \to \mathcal{C}(X),\ \ f^*(A) = f(A)$$

where $\mathcal{C}(X)$=$\{ A \subseteq X \mid A$ is a non-empty compact set$\}$. Since compactness is preserved under continuous transformations, f^* is well-defined.

Now, for $Y \in \mathcal{C}$(X) let us define

$$F(Y) = \bigcup_{1\leq i\leq N} f_i^*(Y). \tag{3.1}$$

In order to prove the theorem, we need to show the existence and uniqueness of a fixed point of F. For this, let us define a metric on $\mathcal{C}$(X) which makes it into a complete metric space and show that F is a contraction with respect to that metric.

Theorem 3.1.4. $(\mathcal{C}(X), d_H)$ is a complete metric space, where d_H is the Hausdorff metric.(see Definition 2.0.8, Theorem 2.0.10)

Proof. Let $(A_n)_{n\geq 1}$ be a Cauchy sequence in $(\mathcal{C}(X), d_H)$. Let us also define

$$A = \bigcap_{n\geq 1} B_n \text{ where } B_n = \overline{\bigcup_{k\geq n} A_k},$$

where $\bar{A}$ denotes the closure of A.

We will show that $A \in \mathcal{C}(X)$ and the sequence $(A_n)_{n\geq 1}$ converges to A with respect to the Hausdorff metric.

Let us first show that $A \in \mathcal{C}(X)$. For any ϵ>0, there exists $m \in \mathbb{N}$ such that $N_{\epsilon/2}(A_m) \supseteq A_k$ for all k≥m. We also know that A_m is compact, hence it is totally bounded. Then, there exists an $\epsilon/2$-net S of A_m. Hence,

$$\bigcup_{s\in S} B_{\epsilon/2}(s) \supseteq A_m \text{ and } \bigcup_{s\in S} B_{\epsilon}(s) \supseteq N_{\epsilon/2}(A_m) \supseteq \bigcup_{k\geq m} A_k.$$

Since $\bigcup_{s\in S} B_\epsilon(s)$ is closed, $\bigcup_{s\in S} B_\epsilon(s) \supseteq \overline{\bigcup_{k\geq m} A_k} = B_m$. So, S is an ϵ-net of B_m. If we add ϵ-nets of $A_1, ..., A_{m-1}$, we obtain an ϵ-net of B_1. Therefore, B_1 is totally bounded. Also, B_1 is complete since it is a closed subset of a complete metric space (X, d). Therefore, B_1 is compact.

As B_n is a decreasing sequence of nonempty compact sets, using *Cantor's Intersection Theorem*,

$$A = \bigcap_{n\geq 1} B_n$$

is compact and non-empty.

Now, we will show that $A_n \to A$ in the Hausdorff metric. Since $(A_n)_{n\geq 1}$ is Cauchy, we know that for all $\epsilon>0$, there exists $m\in\mathbb{N}$ such that $N_\epsilon(A_m) \supseteq A_k$ for all $k \geq m$. Then, $N_\epsilon(A_m) \supseteq \bigcup_{k\geq m} A_k$ and $N_\epsilon(A_m) \supseteq \overline{\bigcup_{k\geq m} A_k}$=B$_m$, as $N_\epsilon(A_m)$ is closed. Therefore,

$$N_\epsilon(A_m) \supseteq B_m \supseteq A.$$

Also, since $(B_m)_{m\geq 1}$ is Cauchy as well, for all $r>0$ and for every $m\in\mathbb{N}$ we can choose b_m such that

$$d(b_m, b_{m+1}) \leq r/2^m,\ m\in\mathbb{N}.$$

Hence,

$$d(b_m, a) \leq \epsilon$$

for sufficiently large m, where $a\in A$. Therefore, $b_m \in N_\epsilon(A)$ and $B_m \subseteq N_\epsilon(A)$. So,

$$N_\epsilon(A) \supseteq B_m \supseteq A_m$$

for sufficiently large m.
Therefore,

$$d_H(A_m, A) \leq \epsilon$$

and $A_m \underset{m\to\infty}{\longrightarrow} A$ in d_H. We have proven that $(\mathcal{C}(X), d_H)$ is complete. □

Now, we have a complete metric space and want to show that F has a unique fixed point. The last step is to show that F is a *contraction* with respect to d_H.

In order to prove that F is a *contraction*, we need two lemmas.

Lemma 3.1.5. For any A,B $\in \mathcal{C}$(X), and $f\colon X \to X$ a contraction with the contraction ratio r, we have

$$d_H(f^*(A), f^*(B)) \leq r d_H(A, B).$$

Proof. Let $N_s(A) \supseteq B$ and $N_s(B) \supseteq A$. Then, for any $x \in N_s(A)$, we have $d(x, y) \leq s$ for some $y \in A$. Since f is a contraction with the contraction ratio r, we have

$$d(f(x), f(y)) \leq rd(x, y) \leq rs.$$

Therefore, $N_{sr}(f^*(A)) \supseteq f^*(N_s(A)) \supseteq f^*(B)$.

In the same way, $N_{sr}(f^*(B)) \supseteq f^*(A)$. Therefore, $d_H(f^*(A), f^*(B)) \leq rs$. □

Lemma 3.1.6. For any A_1,A_2,B_1,$B_2 \in \mathcal{C}(X)$, we have

$$d_H(A_1 \cup A_2, B_1 \cup B_2) \leq \max\{d_H(A_1, B_1), d_H(A_2, B_2)\}.$$

Proof. If $r > \max\{d_H(A_1, B_1), d_H(A_2, B_2)\}$, then $N_r(A_1) \supseteq B_1$ and $N_r(A_2) \supseteq B_2$. Hence $N_r(A_1 \cup A_2) \supseteq B_1 \cup B_2$. Similarly, $N_r(B_1 \cup B_2) \supseteq A_1 \cup A_2$. Hence $r \geq d_H(A_1 \cup A_2, B_1 \cup B_2)$. □

By using these two lemmas, now we show that F is a contraction with respect to d_H.

Proposition 3.1.7. Let $f_i \colon X \to X$ be contractions for $i = 1, 2, \ldots, N$ and let $F \colon \mathcal{C}(X) \to \mathcal{C}(X)$ be defined as in (3.1). Then F is a contraction with respect to the Hausdorff metric $d_{H.}$

Proof. We have,

$$\begin{aligned} d_H(F(A), F(B)) &= d_H\Big(\bigcup_{1\leq j\leq N} f_j^*(A), \bigcup_{1\leq j\leq N} f_j^*(B)\Big) \\ &\leq \max_{1\leq j\ \leq N} d_H(f_j^*(A), f_j^*(B)) \end{aligned} \tag{3.2}$$

by using Lemma 3.1.6 repeatedly. Also, by Lemma 3.1.5, for all $j = 1, \ldots, N$,

$$d_H(f_j^*(A), f_j^*(B)) \leq r_j d_H(A, B), \tag{3.3}$$

where r_j is the Lipschitz constant of f_j.

Let $R = \max_{1 \leq i \leq N} r_j < 1$, then considering 3.2 and 3.3, we have

$$d_H(F(A), F(B)) \leq R d_H(A, B).$$

□

Now, let us restate the Theorem 3.1.2.

Theorem 3.1.2.* Let (X, d) be a complete metric space, and $f_i : X \to X$ be a contraction for $i \in \{1, 2, .., N\}$. Let also $f^* : \mathcal{C}(X) \to \mathcal{C}(X)$ be the induced mapping of f, see (3.1.3). If we define $F: \mathcal{C}(X) \to \mathcal{C}(X)$, $F(A) = \bigcup_{1 \leq i \leq N} f_i^*(A)$, then F has a unique fixed point K. In fact, for all $A \in \mathcal{C}(X)$,

$$F^n(A) \underset{n \to \infty}{\longrightarrow} K$$

where F^n is the nth iteration of F.

Proof. This is a direct consequence of Theorem 2.0.4 and Proposition 3.1.7. □

3.2 The Shift Space

Definition 3.2.1. The collection of all finite words on the symbol set $\{1, 2, \ldots, N\}$ with length m, $m \geq 1$ is defined as

$$W_m = \{1, 2, ..., N\}^m = \{w_1 w_2 ... w_m : w_i \in \{1, 2, ..., N\}\},$$

where $w = w_1 w_2 ... w_m$, $w_i \in \{1, 2, ..., N\}$, is called a finite *word* with length m on the symbol set $\{1, 2, ..., N\}$, for any $N \in \mathbb{N}$.

- We call $\emptyset$ the *empty word* and $W_0 = \{\emptyset\}$.

We also define,

$$W_* = \bigcup_{m \geq 1} W_m.$$

Definition 3.2.2. The collection of all infinite sequences on the symbol set $\{1, 2, ..., N\}$,

$$\Sigma = \{1, 2, ..., N\}^{\mathbb{N}} = \{\omega_1\omega_2... : \omega_i \in \{1, 2, ..., N\}\}$$

is called the *shift space* with N symbols.

Definition 3.2.3. We define the *branches* of Σ as

$$\Sigma_k = \{k\omega_2\omega_3...|\ \omega_2\omega_3... \in \Sigma\},\ \text{where } k \in \{1, 2, ..., N\}.$$

Definition 3.2.4. For any $\omega = \omega_1\omega_2 \cdots \in \Sigma$, let us define backward shift

$$\sigma \colon \Sigma \to \Sigma,\ \sigma(\omega) = \omega_2\omega_3 \dots$$

Then $\sigma \colon \Sigma \to \Sigma$ is called the *shift map*.

• We also consider the forward shifts $\sigma_k \colon \Sigma \to \Sigma$, for every $k \in \{1, 2, ..., N\}$ and $\omega = \omega_1\omega_2 \ldots \in \Sigma$ by

$$\sigma_k \colon \Sigma \to \Sigma,\ \sigma_k(\omega) = k\omega_1\omega_2 \dots$$

Remark 3.2.5. For all k, σ_k is a *right inverse* of σ since

$$\sigma(\sigma_k(\omega)) = \sigma(k\omega), \omega = \omega_1\omega_2\ \dots$$

But it is not a *left inverse* since, for $\omega_1 \neq k$,

$$\sigma_k(\sigma(\omega)) = \sigma_k(\omega_2\omega_3...) = k\omega_2\omega_3 \dots,$$

• σ_k is a left inverse of $\sigma|\Sigma_k$ for $k \in\{1, 2, ..., N\}$.

3.3 Topology of the Shift Space

Now, we will define a metric on the shift space Σ which makes it into a complete metric space.

Definition 3.3.1. For $\omega = \omega_1\omega_2...$, $\tau = \tau_1\tau_2... \in \Sigma$ and $0 < r < 1$, let us define

$$\rho_r(\omega, \tau) = r^m,$$

where $m+1 \in \mathbb{N}$ is the minimum such that $\omega_1\omega_2...\omega_m = \tau_1\tau_2...\tau_m$ and $\omega_{m+1} \neq \tau_{m+1}$. Also, let us define $\rho_r(\omega, \tau) = 0$ when $\omega = \tau$.

Theorem 3.3.2. ρ_r is a metric on Σ.

Proof. Let ω, τ, $\kappa \in \Sigma$. Then,

(i) $\rho_r \geq 0$ by definition.

(ii) $\omega = \tau \Rightarrow \rho_r(\omega, \tau) = 0$ by definition.

For the other direction, let $\rho_r(\omega, \tau) = 0$. Then, $\omega = \tau$ by definition.

(iii) $\rho_r(\omega, \tau) = \rho_r(\tau, \omega)$, clearly.

(iv) Let ω, τ, $\kappa \in \Sigma$ such that $\omega_1...\omega_k = \tau_1...\tau_k$ with $\omega_{k+1} \neq \tau_{k+1}$ and $\tau_1...\tau_m = \kappa_1...\kappa_m$ with $\tau_{m+1} \neq \kappa_{m+1}$.

- **Case1:**

Let $k > m$. Then, $\omega_1\omega_2...\omega_m = \kappa_1\kappa_2...\kappa_m$. Hence

$$\rho_r(\omega, \kappa) = r^m \leq r^k + r^m = \rho_r(\omega, \tau) + \rho_r(\tau, \kappa).$$

- **Case 2:**

Let $m > k$. Then, $\omega_1....\omega_k = \kappa_1...\kappa_k$. Hence,

$$\rho_r(\omega, \kappa) = r^k \leq r^k + r^m = \rho_r(\omega, \tau) + \rho_r(\tau, \kappa).$$

- **Case 3:**

Let $k = m$. Then,

$$\rho_r(\omega, \kappa) = r^m \leq r^m + r^m = \rho_r(\omega, \tau) + \rho_r(\tau, \kappa).$$

Therefore, anyways

$$\rho_r(\omega, \kappa) \leq \rho_r(\omega, \tau) + \rho_r(\tau, \kappa).$$

□

Proposition 3.3.3. Σ is a compact metric space with the metric ρ_r.

Proof. Since sequentially compactness characterizes compactness in a metric space, we will show that Σ is sequentially compact with the metric ρ_r.

Let $(\omega^{(n)})_{n\geq 1}$ be a sequence in Σ. Then, we choose $\tau = \tau_1\tau_2 \ldots \in \Sigma$ inductively in the following way. Firstly, since the set of symbols $\{1, \ldots, N\}$ is finite, there exists $\tau_1 \in \{1, \ldots, N\}$ and an infinite subset $I_1 \subseteq \mathbb{N}$ such that $\omega_1^{(n)} = \tau_1$ for all $n \in I_1$. Let n_1 the least of this infinite set. Then, there exists $\tau_2 \in \{1, 2, \ldots, N\}$ and an infinite subset $I_2 \subseteq I_1$ such that $\omega_2^{(n)} = \tau_2$ for all $n \in I_2$. Let n_2 be the least of I_2. We continue by induction and find $\tau = \tau_1\tau_2 \ldots \in \Sigma$ and a subsequence $(\omega^{(n_i)})_{n_i\geq 1}$ such that

$$\begin{aligned}
\omega^{(n_1)} =& \tau_1\omega_2^{(n_1)}\omega_3^{(n_1)} \cdots \\
\omega^{(n_2)} =& \tau_1\tau_2\omega_3^{(n_2)} \cdots \\
&\vdots \\
\omega^{(n_i)} =& \tau_1\tau_2 \cdots \tau_i \cdots \\
&\vdots
\end{aligned}$$

Since $\lim\limits_{i\to\infty} \rho_r(\omega^{(n_i)}, \tau) = 0,$ ω^{n_i} convergences to $\tau \in \Sigma$.

Therefore for every sequence in $\Sigma,$ there exists a subsequence that is convergent to an element in Σ. So, Σ is compact. $\square$

Corollary 3.3.4. (Σ,ρ_r) is a complete metric space.

Proof. Since (Σ,ρ_r) is a compact metric space, it is complete and totally bounded. $\square$

3.4 Self-similarity of the Shift Space Σ

Theorem 3.4.1. Consider the complete metric space (Σ,ρ_r). Then σ_i (see definition 3.2.4) is a contraction for all $i \in \{1, 2, ..., N\}$. Moreover, Σ is a self-similar set with respect to the set of contractions $\{\sigma_1, \sigma_2, ..., \sigma_N\}$.

Proof. Let us first show that σ_i is a contraction with respect to ρ_r. If $\omega,\tau \in \Sigma$ and $\omega \neq \tau$, then

$$\frac{\rho_r(\sigma_i(\omega), \sigma_i(\tau))}{\rho_r(\omega, \tau)} = \frac{\rho_r(i\omega, i\tau)}{\rho_r(\omega, \tau)} = r$$

where $\omega_1...\omega_m=\tau_1...\tau_m$, and $\omega_{m+1} \neq \tau_{m+1}$. So, $\sup\limits_{\omega,\tau\in\Sigma} \frac{\rho_r(\sigma_i(\omega), \sigma_i(\tau))}{\rho_r(\omega, \tau)} = r$, where $0<r<1$ is fixed. Therefore, σ_i is a contraction for all $i \in \{1, 2, ..., N\}$.

Now, since $\{\sigma_1, \sigma_2, ..., \sigma_N\}$ is a finite set of contractions in (Σ,ρ_r), there exists a unique non-empty compact subset $K \subseteq \Sigma$ by the Theorem 3.1.2. Since

$$\Sigma = \sigma_1(\Sigma) \cup \sigma_2(\Sigma) \cup ... \cup \sigma_N(\Sigma),$$

K=Σ is the self-similar set. □

3.5 Relation between Σ and K

From now on K will always be the self-similar set corresponding to the contractions $f_1, \ldots, f_N$ on the complete metric space X throughout this chapter.

Notation 3.5.1. For any $w \in W_*$, we define

$$K_w = f_{w_1w_2...w_m}(K),$$

where f is a function from K to itself and f_w=$f_{w_1} \circ f_{w_2} \circ \ldots \circ f_{w_m}$ for w=$w_1w_2 \ldots w_m$.

Definition 3.5.2. For any ω=$\omega_1\omega_2 \ldots \in \Sigma$ let us define $\pi : \Sigma \to K$, $\omega\mapsto \{\pi(\omega)\}$=$\bigcap\limits_{k\geq 1} K_{\omega_1\omega_2...\omega_k}$.

Proposition 3.5.3. π is a function.

Proof. Let us first show that $\bigcap\limits_{k\geq 1} K_{\omega_1\omega_2\ldots\omega_k}$ is non-empty. We know that

$$K_{\omega_1\ldots\omega_k} = f_{\omega_1\ldots\omega_k}(K).$$

Since $f_{\omega_1\ldots\omega_k}(f_{\omega_{k+1}}(K)) \subseteq f_{\omega_1\ldots\omega_k}(K)$, $K_{\omega_1\ldots\omega_k}$ is a nested decreasing sequence of sets for all $k \in \mathbb{N}$. Also K_w=$f_w(K)$, w=$\omega_1 \ldots \omega_k$ is compact since K is compact and f_w is continuous. By *Finite Intersection Property*,

$$\bigcap_{k\geq 1} K_{\omega_1\omega_2\ldots\omega_k}$$

is non-empty.

Secondly, we will show that $\bigcap\limits_{k\geq 1} K_{\omega_1\omega_2\ldots\omega_k}$ contains only one point. For that, we need to show that diam($\bigcap\limits_{k\geq 1}$ $\mathrm{K}_{\omega_1\ldots\omega_k}$)=0.

Let r_i be the contraction ratio for the contraction f_i, $i \in \{1, 2, ..., N\}$ and R=$\max\limits_{i=1,2,\ldots,N} r_i$. Since diam($f_i$(K)) $\leq r_i$ diam(K), $i \in \{1, 2, ..., N\}$,

$$\mathrm{diam}(f_i(K)) \leq R\mathrm{diam}(K).$$

Therefore, diam($f_{\omega_1\omega_2\ldots\omega_{k-1}}(f_{\omega_k}(K)) \leq \mathrm{R}^k\mathrm{diam}(K)$. If we take the limit of both sides as $k \to \infty$, by using the Squeeze theorem we get

$$\lim_{k\to\infty} \mathrm{diam}(K_{\omega_1\ldots\omega_k}) = 0.$$

Also, since

$$\lim_{k\to\infty} \mathrm{diam}(K_{\omega_1\ldots\omega_k}) = \mathrm{diam}(\lim_{k\to\infty} K_{\omega_1\ldots\omega_k}) \text{ and } \lim_{k\to\infty} K_{\omega_1\ldots\omega_k} = \bigcap_{k\geq 1} K_{\omega_1\omega_2\ldots\omega_k}$$

, we have

$$\mathrm{diam}(\bigcap_{k\geq 1} K_{\omega_1\ldots\omega_k}) = 0.$$

□

Theorem 3.5.4. π is a surjective, continuous function such that the following diagram commutes,

$$\begin{array}{ccc} \Sigma & \xrightarrow{\sigma_i} & \Sigma \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ K & \xrightarrow{f_i} & K \end{array},$$

that is, for any i=1,2,...,N, we have $\pi \circ \sigma_i = f_i \circ \pi$.

Proof. Let us first show that for all $i \in \{1, 2, ..., N\}$ and for all $\omega \in \Sigma$, we have

$$\pi \circ \sigma_i(\omega) = f_i \circ \pi(\omega).$$

Indeed, by Definition 3.5.2

$$\pi(\sigma_i(\omega)) = \bigcap_{k\geq 1} K_{i\omega_1\omega_2...\omega_k} = \bigcap_{k\geq 1} f_i(f_{\omega_1\omega_2...\omega_k}(K)).$$

Since, $f_i(\bigcap_{k\geq 1} K_{\omega_1\omega_2...\omega_k}) \subseteq \bigcap_{k\geq 1} f_i(K_{\omega_1\omega_2...\omega_k}) = \{x\}$ for some $x \in K$, it follows that

$$f_i(\bigcap_{k\geq 1} K_{\omega_1\omega_2...\omega_k}) = \bigcap_{k\geq 1} f_i(K_{\omega_1\omega_2...\omega_k}),$$

hence, taking into account that $\bigcap_{k\geq 1} K_{\omega_1\omega_2...\omega_k} = \{\pi(\omega)\}$ (see Definition 3.5.2), we have

$$\pi(\sigma_i(\omega)) = f_i(\bigcap_{k\geq 1} K_{\omega_1\omega_2...\omega_k}) = f_i(\pi(\omega)).$$

Now, let us show that π is continuous. Let, for $\omega, \tau \in \Sigma$, $\rho_r(\omega, \tau) \leq r^k$. Then, $\omega_1\omega_2...\omega_k = \tau_1\tau_2...\tau_k$, and $\omega_{k+1} \neq \tau_{k+1}$ by the definition of ρ_r. For $w = \omega_1...\omega_k = \tau_1...\tau_k$, we have

$$d(\pi(\omega),\pi(\tau)) = d(\pi(w\omega_{k+1}...),\pi(w\tau_{k+1}...)) = d(f_w(\pi(\omega')), f_w(\pi(\tau'))),$$

where $\omega' = \omega_{k+1}\omega_{k+2}\ldots$, and $\tau' = \tau_{k+1}\tau_{k+2}\ldots$

Also, since f_i's are contractions for all $i \in \{1, 2, ..., N\}$, $f_{\omega_1...\omega_k}$ is also a contraction. For the contraction ratio r_i of f_i, let $R = \max_i r_i$. Then, for any k_1, $k_2 \in K$,

$$d(f_i(k_1), f_i(k_2)) \leq Rd(k_1, k_2),$$

and

$$d(f_w(\pi(\omega^{'})), f_w(\pi(\tau^{'}))) \leq R^k d(\pi(\omega^{'}), \pi(\tau^{'})).$$

Therefore,

$$d(f_w(\pi(\omega^{'})), f_w(\pi(\tau^{'}))) \leq R^k \text{diam}(K).$$

Since $\text{diam}(K)$ is finite from the fact that K is compact, π is continuous.

Lastly we show that π is surjective. It is clear that

$$\Sigma = \Sigma_1 \cup \Sigma_2 \cup ... \cup \Sigma_N.$$

Then,

$$\pi(\Sigma) = \pi(\sigma_1(\Sigma) \cup \sigma_2(\Sigma) ... \cup \sigma_N(\Sigma)).$$

Since,

$$\pi(\sigma_1(\Sigma) \cup \sigma_2(\Sigma) ... \cup \sigma_N(\Sigma)) = \pi(\sigma_1(\Sigma)) \cup \pi(\sigma_2(\Sigma)) \cup ... \cup \pi(\sigma_N(\Sigma)),$$

we have

$$\begin{aligned}\pi(\Sigma) =& \pi(\sigma_1(\Sigma)) \cup \pi(\sigma_2(\Sigma)) \cup ... \cup \pi(\sigma_N(\Sigma)) \\ =& f_1(\pi(\Sigma)) \cup f_2(\pi(\Sigma)) \cup ... \cup f_N(\pi(\Sigma)).\end{aligned}$$

From the fact that K is the unique self-similar set with respect to the set of contractions $\{f_1, f_2, ..., f_N\}$, we get that $\pi(\Sigma)=K$. It follows that π is surjective. □

Corollary 3.5.5. If π is injective, then it is a homeomorphism between Σ and K.

Proof. Let π be injective. Since π is also surjective, π^{-1} exists. We also know that π is a continuous function. Since Σ is compact and π is continuous, π^{-1} is continuous as well. Therefore, π is a homeomorphism between Σ and K. □

Remark 3.5.6. The self-similar set K is the quotient space of Σ with the equivalence relation $\sim$ defined by π. If we define the relation $\sim$ such that $\omega \sim \tau \Leftrightarrow \pi(\omega)=\pi(\tau)$, then the quotient space of Σ is homeomorphic to K.

3.6 The Overlapping Set

While defining the self-similar set K, there may be the case that $f_i(K)\cap f_j(K)\neq\emptyset$ for $i\neq j \in S=\{1,2,...,N\}$. In other words, there may be overlaps. Now, we will define the overlapping set and some other sets related to it.

Definition 3.6.1. Let K be the self-similar set with respect to f_i, where $i\in S=\{1,2,...,N\}$. Then we define,

$$\begin{aligned} C_K &= \bigcup_{i,j\in S, i\neq j} (f_{\mathrm{i}}(\mathrm{K}) \cap f_{\mathrm{j}}(\mathrm{K})), \\ \mathcal{C} &= \pi^{-1}(C_K), \\ \mathcal{P} &= \bigcup_{n\geq 1} \sigma^n(\mathcal{C}), \text{ and} \\ V_0 &= \pi(\mathcal{P}). \end{aligned}$$

C_K is called the *overlapping set for* K, $\mathcal{C}$ is called the *critical set* and $\mathcal{P}$ is called the *post critical set.*

Now we will characterize the elements of the overlap set C_K associated to the self-similar set K.

Proposition 3.6.2. Let K be the self-similar set with respect to *injective* functions f_i, for $i \in \{1,2,...,N\}$ and let ω, $\tau \in \Sigma$ such that $\omega_1\omega_2...\omega_k=\tau_1\tau_2...\tau_k$ and $\omega_{k+1} \neq \tau_{k+1}$. Then, $\pi(\omega) = \pi(\tau)$ if and only if $\pi(\sigma^k\omega) = \pi(\sigma^k\tau)$.

Proof. Let $\pi(\omega) = \pi(\tau)$, where $\omega, \tau \in \Sigma$ and $\omega_1\omega_2...\omega_k = \tau_1\tau_2...\tau_k$. Then, by using the equality $f_i \circ \pi=\pi \circ \sigma_i$, $i \in \{1,2,...,N\}$ for k times, see Theorem 3.5.4, we have

$$f_{\omega_1...\omega_k}(\pi(\omega_{k+1}...)) = f_{\omega_1...\omega_k}(\pi(\tau_{k+1}...)).$$

By the injectivity of f_i's,

$$\pi(\omega_{k+1}...) = \pi(\tau_{k+1}...),$$

hence,

$$\pi(\sigma^k \omega) = \pi(\sigma^k \tau).$$

For the other direction, let $\pi(\sigma^k \omega) = \pi(\sigma^k \tau)$. If we take the image of both sides under $f_{\omega_1 \dots \omega_k}$, we have $\pi(\omega) = \pi(\tau)$. □

Remark 3.6.3. Let K be the self-similar set with respect to the injective contractions $f_1 \dots f_n$. If $\pi(\omega)=\pi(\tau)$ for ω, $\tau \in \Sigma$, $\omega \neq \tau$, then $\pi(\sigma^n(\omega))=\pi(\sigma^n(\tau)) \in \mathrm{C}_K$, where $\omega_1 \dots \omega_n=\tau_1 \dots \tau_n$.

3.7 Characterization of the Self-similar Set K by Iterated Functions

Notation 3.7.1. We will define $\dot{w}$ as

$$\dot{w} = www...$$

where $w \in W_* \setminus W_0$.

Theorem 3.7.2. $\pi(\dot{w})$ is the unique fixed point of f_w. Moreover, the set of $\pi(\dot{w})$, where $w \in W_*$, is *dense* in K.

Proof. Since f_w is a contraction in a complete metric space X, it has a unique fixed point by the Contraction Mapping Theorem.

By using the equality we proved in Theorem 3.5.4,

$$\pi(w\dot{w}) = f_w(\pi(\dot{w})).$$

Therefore, $\pi(\dot{w})$ is the unique fixed point of f_w.

Now, we will show that the set of fixed points $\pi(\dot{w})$, $w \in W_*$ is dense in K. Let $\omega = \omega_1\omega_2... \in \Sigma$, and $w = \omega_1\omega_2...\omega_k \in W_*$. Since, $w = w_1w_2...w_k = \omega_1\omega_2...\omega_k$, $\rho_r(\omega, \dot{w}) \leq r^k$. As $k \to \infty$, we have

$$\rho_r(\omega, \dot{w}) = 0.$$

Since π is continuous $d(\pi(\omega), \pi(\dot{w})) = 0$ as $k \to \infty$. It says that for any $\omega \in \Sigma$ there exists $\pi(\dot{w})$, $w = \omega_1...\omega_k \in W_*$ such that $\pi(\dot{w}) \to \pi(\omega)$, $k \to \infty$. Since $\pi(\Sigma)$=K,

$$K = \overline{\{\pi(\dot{w}) : w \in W_*,\ w \neq \emptyset\}}.$$

□

3.8 Examples of Self-similar Sets

Example 3.8.1. ***Cantor(Middle-third) Set*** Let us consider $X = [0, 1]$ with the Euclidean distance, and let $f_1(x) = \frac{1}{3}x$ and $f_2(x) = \frac{1}{3}(x - 1) + 1$. Then, $f_1([0, 1]) = [0, 1/3]$ and $f_2([0, 1]) = [2/3, 1]$.

Let

$$A_1 = [0, 1/3] \cup [2/3, 1].$$

Now, let

$$A_2 = f_1(A_1) \cup f_2(A_1) = [0, 1/9] \cup [2/9, 1/3] \cup [2/3, 7/9] \cup [8/9, 1].$$

If we generalize this, A_{k+1} is the set which we have by removing the middle third of every closed interval in A_k. The Cantor set which we denote by K is the limiting set of this process:

$$K = \bigcap_{n=1}^{\infty} A_n.$$

The self-similar set K with respect to the set of contractions $\{f_1, f_2\}$ is the *Cantor set*.(See Figure 3.1) K is compact by π becoming a homeomorphism between the topological Cantor set Σ and K. (See 3.8.2.) Also, $K{=}f_1(K) \cup f_2(K)$. To show this, let $k \in f_1(K) \cup f_2(K){=}\frac{1}{3}K \cup \left(\frac{1}{3} K{+}\frac{2}{3} \right)$. Let $k \in \frac{1}{3}K$. Then, $3k \in K$. To prove that $k \in K$, we need to show that $k \in A_n$ for all $n{\geq}1$. What we have is $3k \in A_n$ for every $n{\geq}1$. In particular, $3k \in A_{n-1}$. Since $A_n{=}\frac{1}{3}A_{n-1} \cup \left(\frac{1}{3}A_{n-1}{+}\frac{2}{3} \right)$, $k \in A_n$. Therefore, $k \in K$. The other inclusion is obvious.

Remark 3.8.2. In this example, for the self-similar set K,

$$K_1 = f_1(K) \subseteq f_1([0,1]) = [0,1/3], \text{ and}$$

$$K_2 = f_2(K) \subseteq f_1([0,1]) = [2/3,1].$$

Hence,

$$C_K = \bigcup_{1 \leq i < j \leq 2} f_i(K) \cap f_j(K) = \emptyset.$$

It says that π is a homeomorphism between Σ (Topological Cantor set) and K (Cantor set). Hence, a self-similar set that is homeomorphic to Σ shares some characteristics with the Cantor set. In general, the overlapping set C_K is non-empty.

Figure 3.1: Cantor set, [17]

Example 3.8.3. ***Sierpinski Gasket*** Let us consider $X{=}\mathbb{C}$ with the Euclidean distance. Let us start with a solid equilateral triangle, T, and let $\{p_1, p_2, p_3\}$ be the vertices of it. Let also define $f_i(z) = \frac{z{-}p_i}{2} + p_i$ for i=1,2,3.

Firstly, observe that $f_i(p_i) = p_i$ and $f_i(p_j) = \frac{p_i{+}p_j}{2}$. We can see that this transformation sends two vertices to middle points while fixing one of the vertices. Note that f_1 and f_2 are similarity transformations. Then, we can see that

$\bigcup_{i\in\{1,2,3\}} f_i(T) \subset T$. Hence $K \subset T$. With this construction, at first step we remove the interior of the middle triangle and have three remaining equilateral triangles. If we continue this process for every triangle in itself, the limiting set will be the Sierpinski Gasket.(See Figure 3.2) Now, we will explain what we have done by using the notation we have defined.

(a) First Iteration (b) Sierpinski Gasket

Figure 3.2: Sierpinski Gasket [15]

As p_i is the fixed point of f_i, where $i \in \{1,2\}$, p_i=$\pi(\dot{i})$. Hence, p_1=$\pi(\dot{1})$, and p_2=$\pi(\dot{2})$. If we call the middle point of the $\overline{p_1p_2}$ side of T as q_3, then $f_1(\pi(\dot{2}))$=$f_2(\pi(\dot{1}))$=q_3. So, $\{1\dot{2}, 2\dot{1}\} \subseteq \pi^{-1}(q_3)$. We can easily see that $\pi^{-1}(q_3) \subseteq \{1\dot{2}, 2\dot{1}\}$. (See 4.6.3 for the proof.) Therefore,

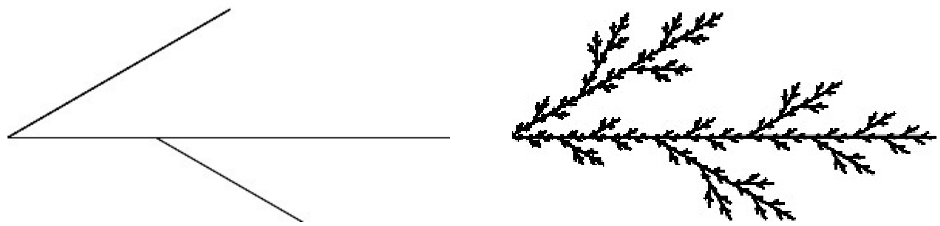
$$\pi^{-1}(q_3) = \{1\dot{2}, 2\dot{1}\}.$$

In the same way,

$$\pi^{-1}(q_2) = \{1\dot{3}, 3\dot{2}\} \text{ and } \pi^{-1}(q_1) = \{2\dot{3}, 3\dot{2}\}.$$

Now, from all of these we can conclude that for $\omega, \tau \in \Sigma$, $\omega \neq \tau$ such that $\pi(\omega)$=$\pi(\tau)$, there exists $w \in W_*$ such that

$$\{\omega, \tau\} = \{w1\dot{2}, w2\dot{1}\} \text{ or } \{w1\dot{3}, w3\dot{2}\} \text{ or } \{w2\dot{3}, w3\dot{2}\}.$$



(a) First Iteration (b) Hata's Tree-like Set

Figure 3.3: Hata's Tree-like Set [13]

Example 3.8.4. ***Hata's Tree-like Set*** Let us consider $X = \mathbb{C}$ with the Euclidean distance. Let us also define $f_1(z) = c\bar{z}$, $f_2(z) = (1 - |c|^2)\bar{z} + |c|^2$ for all $z \in \mathbb{C}$ and $|c|$,$|1 - c| \in (0,1)$. The self-similar structure K with respect to $\{f_1, f_2\}$ is called Hata's tree-like set". Since f_1 and f_2 are similarities, we can represent them in a matrix form. For $z= \begin{bmatrix} x \\ y \end{bmatrix}$, $z{=}x{+}iy$, $x,y \in \mathbb{R}$ *and* $c{=}c_1{+}ic_2$, $c_1,c_2 \in \mathbb{R}$,

$$f_1 \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c_1 & c_2 \\ c_2 & -c_1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, \; f_2 \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 - |c|^2 & 0 \\ 0 & |c|^2 - 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} |c|^2 \\ 0 \end{bmatrix}.$$

Note that since f_1, f_2 are similarities (See 2.0.3.), they map straight lines to straight lines. For the approximation of Hata's tree-like set, let us define

$$A = \{t \mid 0 \le t \le 1\} \cup \{ct \mid 0 \le t \le 1\}, \text{ where } |c|, |1 - c| \in (0, 1).$$

Then,

$$f_1(0) = 0 = \pi(\dot{1}), \text{and } f_2(1) = 1 = \pi(\dot{2})$$

$$f_1(1) = c = \pi(1\dot{2}), f_2(0) = |c|^2 = \pi(2\dot{1}) = \pi(11\dot{2}) = f_1(c),$$

$$f_2(c) = (1 - |c|^2)\bar{c} + |c|^2 = \pi(21\dot{2}).$$

Hence $f_1(A) \cup f_2(A) \supset A$. (See Figure 3.3*)*

Now, if we define

$$A_m = \bigcup_{w \in W_m} f_w(A),$$

then, A_m is a monotone increasing sequence and the limiting set

$$K = \overline{\bigcup_{m \geq 0} A_m}$$

is Hata's tree-like set. Now, let $k_1, k_2 \in K$. Then, there exists $(x_m)_{m\geq 0}$, $(y_m)_{m\geq 0}$ $\in \bigcup_{m\geq 0} A_m$ such that $x_m \to k_1$ and $y_m \to k_2$ as $m \to \infty$. Since A is bounded and f_w is a contraction, A_m is bounded. Hence, $d(x_m, y_m) \leq \operatorname{diam}(A_m)$ for all $m \geq 0$. Now, by *Triangle Inequality*

$$d(k_1, k_2) \leq d(k_1, x_m) + d(x_m, y_m) + d(y_m, k_2),$$

$d(k_1, k_2) \leq \epsilon$ for sufficiently large m. Therefore, K is bounded. Since K is also closed, it is compact.

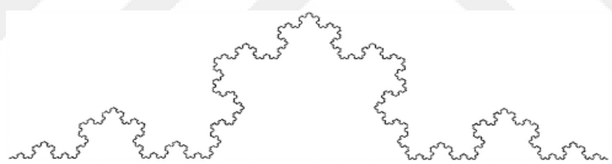


Figure 3.4: Koch curve, [16]

Example 3.8.5. ***Koch Curve*** Let $X = \mathbb{C}$ and T be a triangle domain with its boundary. Let also the set of the vertices of T be $\{0, a, 1\}$, where $a \in \{z \mid z \in \mathbb{C}, |z|^2 + |1-z|^2 < 1\}$. Let us now define $f_1(z) = a\bar{z}$ and $f_2(z) = (1-a)(\bar{z}-1)+1$.

The construction of the Koch curve is starting by dividing every line segment into three equal segments and replacing the middle segment by two sides of an equilateral triangle in which the sides have the same length as the segment being removed. The Koch curve is the limiting set which is an actual curve since there exists a homeomorphism between [0,1] (for $a = \frac{1}{2}$) and itself. (See 4.2.2 and [1] for the theory.) Therefore, it is compact.

Now,

$$f_1(0) = 0, f_1(a) = |a|^2, \ f_1(1) = a, \text{and}$$

$$f_2(0) = a, \ f_2(1) = 1, \ f_2(a) = |1-a|^2.$$

From these, we can see that $f_1(T) \cup f_2(T) \subseteq T$. Hence, the self-similar set $K_a \subseteq T$.

Also,

$$\pi(\dot{1}) = 0, \pi(\dot{2}) = 1, \text{and}$$

$$\pi(1\dot{2}) = \pi(2\dot{1}) = a.$$

For $a = \frac{1}{2} + i\frac{1}{2\sqrt{3}}$, K_a is called the Koch curve.

Chapter 4

Self-similar Structure

For this chapter, we follow again [2].

4.1 Self-similar Structure

Now, we will first define what a self-similar structure is. Our aim at giving this notion is assigning a topological structure to a self-similar set.

Definition 4.1.1. Let K be a *compact* and *metrizable* topological space and $f_i \colon K \to K$ be a continuous injection for all $i \in S$, where S is a finite symbol set. Then $(K, S, \{f_i\}_{i \in S})$ is called a *self-similar structure* if there exists a *surjective, continuous* map $\pi : \Sigma \to K$ such that the following diagram commutes. (Compare with the Definition 3.2.2.)

$$\begin{array}{ccc} \Sigma & \xrightarrow{\sigma_i} & \Sigma \\ {\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle\pi} \\ K & \xrightarrow{f_i} & K \end{array}$$

i.e. $\pi \circ \sigma_i = f_i \circ \pi$, where σ_i, $i \in S$ are the shift maps defined in the Definition 3.2.4.

We will denote the self-similar structure $(K, S, \{f_i\}_{i\in S})$ by $\mathcal{L}$.

Example 4.1.2. Let X=[0,1] with Euclidean distance and let $f_1(x) = \frac{1}{3}x$ and $f_2(x) = \frac{1}{3}(x-1)+1$. We know from Chapter 3 (see example 3.8.1) that the self-similar set K with respect to $\{f_1, f_2\}$ is the Cantor set. Since f_1 and f_2 are injective, $(K, \{1,2\}, \{f_1, f_2\})$ is a self-similar structure.

Remark 4.1.3. When the contractions are *injective*, a self-similar set defines a self-similar structure. In Chapter 3 (see examples 3.8.1, 3.8.4, 3.8.3, 3.8.5), all self-similar sets are self-similar structures.

Proposition 4.1.4. Let $\mathcal{L}$ be a self-similar structure. Then, the continuous, surjective mapping π between Σ and K is unique. Moreover, for any ω=$\omega_1\omega_2$... $\in \Sigma$, we have

$$\{\pi(\omega)\} = \bigcap_{m\geq 1} K_{\omega_1\ldots\omega_m},$$

(See Notation 3.5.1)

Proof. It is obvious that $\pi(\omega) \in \bigcap_{m\geq 1} K_{\omega_1\ldots\omega_m}$ from the above diagram.

Now, let $x \in \bigcap_{m\geq 1} K_{\omega_1\ldots\omega_m}$. Then, there exists $\tau^m \in \Sigma_{\omega_1\omega_2\ldots\omega_m}$, $m\geq 1$ such that $\pi(\tau^m) = x$. Observe that $\rho_r(\tau^m, \omega) \to 0$, as $m\to\infty$. By the continuity of π, d($\pi(\tau^m)$,$\pi(\omega)$) $\to 0$ as $m\to\infty$. Therefore, x=$\pi(\omega)$. □

4.2 Isomorphism between Self-similar Structures

Definition 4.2.1. Let $\mathcal{L}_1 = (K_1, S_1, \{f_i\}_{i\in S_1})$ and $\mathcal{L}_2 = (K_2, S_2, \{f_i\}_{i\in S_2})$ be two self-similar structures. Let also $\Sigma(S_1)$ and $\Sigma(S_2)$ be the shift spaces on the symbol sets S_1 and S_2, respectively. We say $\mathcal{L}_1$ and $\mathcal{L}_2$ are *isomorphic* if there exists a bijection ρ between S_1 and S_2 such that the following diagram induces a well-defined homeomorphism between K_1 and K_2.

$$\begin{array}{ccc} \Sigma(S_1) & \xrightarrow{\pi_1} & K_1 \\ {\scriptstyle I_\rho}\downarrow & & \downarrow{\scriptstyle g} \\ \Sigma(S_2) & \xrightarrow{\pi_2} & K_2 \end{array}$$

Here, the map I_ρ is the bijective map induced by ρ, naturally. In other words, ρ induces a bijective map between $\Sigma(S_1)$ and $\Sigma(S_2)$ by

$$I_\rho\colon \Sigma(S_1) \to \Sigma(S_2)\ ,\, I_\rho(\omega_1\omega_2...) = \rho(\omega_1)\rho(\omega_2)....$$

Example 4.2.2. Let $X=\mathbb{C}$ and T be the triangle domain with its boundary such that the set of the vertices of T be $\{0,\, a,\, 1\}$, where $a \in \{z \mid z \in \mathbb{C},\, |z|^2 + |1-z|^2 <1\}$. Let us now define $f_1(z) = a\bar{z}$ and $f_2(z) = (1-a)(\bar{z}-1)+1$.
We know that there exists a self-similar set K with respect to the set contractions $\{f_1, f_2\}$ from the Chapter 3.

Observe that for $a=\dfrac{1}{2}$, $K=[0,1]$. Also, for $a=\frac{1}{2}+i\,\dfrac{1}{2\sqrt{3}}$, the self-similar structure K is the Koch curve. By *de Rham*'s theorem in [1] , for contractions f_1, f_2 of $\mathbb{R}^n$ satisfying the condition

$$f_1(\mathrm{Fix}(f_2)) = f_2(\mathrm{Fix}(f_1)),$$

the functional equation $G(t) = \begin{cases} f_1(G(2t)) & 0 \le t \le \frac{1}{2} \\ f_2(G(2t-1)) & \frac{1}{2} \le t \le 1 \end{cases}$ has a *unique continuous* solution.

We can easily see that $f_1(\pi(\dot{2})) = f_1(1) = a = f_2(0) = f_2(\pi(\dot{1}))$. Observe that

$$G([0,1]) = G\big([0,\frac{1}{2}] \cup [\frac{1}{2},1]\big) = f_1(G[0,1]) \cup f_2(G[0,1]).$$

Hence, $G([0,1])=K$. In [1], it is also investigated the case where G is a homeomorphism and is generalized to *weak contractions.*

Also, by the theory in [1], K is a *simple arc* since $\{f_1, f_2\}$ is a set of injective contractions of X such that $\mathrm{Fix}(f_1)=0\neq 1=\mathrm{Fix}(f_2)$ and $K_1 \cap K_2$ contains exactly one point, namely a. Therefore, there exists a homeomorphism between [0,1] and K. Also, since there exists a bijective map between S_1 and S_2, we can easily see that I_ρ is continuous. Therefore, these two self-similar structures are isomorphic. In the same way, for every $a \in \{z \mid z \in \mathbb{C},\, |z|^2+|1-z|^2<1\}$, all self-similar structures are isomorphic.

4.3 Local Topology of a Self-similar Structure

In this section, d will be the metric producing the topology of K.

Theorem 4.3.1. Let $\mathcal{L}$=$(K, S, \{f_i\}_{i\in S})$ be a self-similar structure. For all $k \in K$, and for all m>0, let us define

$$K_{m,k} = \bigcup_{w\in W_m, k\in K_w} K_w.$$

Then it defines a local base for a fixed $k \in K$, and arbitrary m>0.

Proof. Let us first show that $K_{m,k}$ is indeed a neighborhood of k.

Let $(k_m)_{m\geq 1}$ be a sequence in K converging to k as $m\to\infty$. Let also ω^m be a sequence in Σ such that $\pi(\omega^m)$=k_m, for all $m\geq 1$. By the compactness of Σ, we have the existence of a subsequence ω^{m_i}, $i \geq 1$ that converges to $\omega \in \Sigma$. By the continuity of π, $d(\pi(\omega^{m_i}), \pi(\omega)) \to 0$ as $i \to \infty$. Hence, $\pi(\omega)$=k. It says that $k_{m_i} \in K_{m,k}$ when i is sufficiently large.

Now, let us show that $\sup_{w\in W_m} \text{diam}(K_w) \to 0$ as $m\to\infty$. Assume not. Then, there exists a sequence $(w_m)_{m\geq 1}$ with $w_m \in W_m$ for $m\geq 1$, such that $\inf_{w_m} \text{diam}(K_w) >$ 0 as $m\to\infty$. Now, let $\tau^m \in \Sigma_{w_m}$ for $m\geq 1$. By compactness, we know that there exists a subsequence τ^{m_i} converging to $\omega \in \Sigma$. Note that $K_{w_1 w_2 \ldots w_m w_{m+1}} \subseteq K_{w_1 w_2 \ldots w_m}$ and $\text{diam}(K_{w_1\ldots w_{m+1}}) \leq \text{diam}(K_{w_1\ldots w_m})$, for all m. Hence, $\liminf_{m\geq 1}$ $\text{diam}(K_{\omega_1\ldots\omega_m}) > 0$ as $m \to \infty$. It contradicts with $\text{diam}(\bigcap_{m\geq 1} K_{\omega_1\omega_2\ldots\omega_m}) \to 0$ as $m\to\infty$. (Compare with the Proposition 3.5.3.) □

Definition 4.3.2. For a self-similar structure $\mathcal{L}$=$(K, S, \{f_i\}_{i\in S})$, we define *the overlapping set, the critical set, the post-critical set* as follows:

$$C_{\mathcal{L},K} = \bigcup_{i,j\in S; i\neq j} (f_i(K) \cap f_j(K)),$$

$$\mathcal{C}_{\mathcal{L}} = \pi^{-1}(C_{\mathcal{L},K}),$$

$$\mathcal{P}_{\mathcal{L}} = \bigcup_{m \geq 1} \sigma^m(\mathcal{C}_{\mathcal{L}}).$$

Also, we define

$$V_0 = \pi(\mathcal{P}_{\mathcal{L}}).$$

Proposition 4.3.3. Let $\mathcal{L}$ be a self-similar structure. Then, $\mathcal{C}_{\mathcal{L}}$=$\emptyset$ if and only if π is injective.

Proof. (:$\Rightarrow$) Assume that π is not injective. Then, there exists ω,τ; $\omega \neq \tau \in \Sigma$ such that $\pi(\omega) = \pi(\tau)$. Let $\omega_1 \ldots \omega_k = \tau_1 \ldots \tau_k$ with $\omega_{k+1} \neq \tau_{k+1}$. Then, for $w = \omega_1 \ldots \omega_k$, we have

$$f_w(\pi(\omega_{k+1}\omega_{k+2}\ldots))=f_w(\pi(\tau_{k+1}\tau_{k+2}\ldots)).$$

By the injectivity of $f_i$âs, $i \in S$, we have

$$\pi(\omega_{k+1}\omega_{k+2}\ldots) = \pi(\tau_{k+1}\tau_{k+2}\ldots).$$

Hence, $\sigma^k(\omega)$, $\sigma^k(\tau) \in \mathcal{C}_{\mathcal{L}}$. Therefore, $\mathcal{C}_{\mathcal{L}}$ is not empty.

($\Leftarrow$:) Assume that $\mathcal{C}_{\mathcal{L}}$ is not empty. Then, there exists ω, $\tau \in \Sigma$; $\omega \neq \tau$ such that $\pi(\omega)$=$\pi(\tau)$. Therefore, π is not injective. □

Remark 4.3.4. Observe that the *critical set* (*hence the post critical set*) provides a way to determine the topological structure of a self-similar structure. If π is injective, then π is a homeomorphism between K and the Topological Cantor set Σ.

Proposition 4.3.5. $\pi^{-1}(V_0) = \mathcal{P}_{\mathcal{L}}$.

Proof. $\mathcal{P}_{\mathcal{L}} \subseteq \pi^{-1}(V_0)$ by definition.

For the other inclusion, let $\omega \in \pi^{-1}(V_0)$. Then, there exists τ'=$w\tau \in \mathcal{C}_{\mathcal{L}}$, where $w \in W_*$ such that $\pi(\omega)$=$\pi(\tau)$. If we set ω'=$w\omega$, then

$$\pi(\omega') = f_w(\pi(\omega)) = f_w(\pi(\tau)) = \pi(\tau').$$

Therefore, $\omega' \in \mathcal{C}_{\mathcal{L}}$ and $\sigma^k(\omega')$=$\omega \in \mathcal{P}_{\mathcal{L}}$, where $k = |w|$. □

4.4 Minimality of a Self-similar Structure

Now, we will define a new notion starting from the overlap set definition. A self-similar structure may have overlaps. In other words, there may be unnecessary symbols or words while defining a self-similar structure.

Let us firstly give an example to this.

Example 4.4.1. Let K=[0, 1]. Let also S={1, 2} and $f_1(x)=\frac{3}{4}x$, $f_2(x)=\frac{3}{4}x+\frac{1}{4}$. We can easily see that the triple $(K, \{1, 2\}, \{f_1, f_2\})$ defines a self-similar structure. For w_1=11 and w_2=22,

$$[0, 1] = f_{11}[0, 1] \cup f_{22}[0, 1].$$

Hence, we do not need the words $12, 21 \in W_*$ to define K. Therefore, there are unnecessary words.

Remark 4.4.2. Let $S=\{1, 2, ..., N\}$, and let $W \subseteq W_* \setminus W_0$ be a finite subset. Then for the subset W, we can define a new self-similar structure in a natural way by,

$$\pi_W : \Sigma(W) \to K(W), \ \pi_W(\omega) = \pi(\omega)$$

since for $\omega \in \Sigma(W)$, ω=$\omega_1\omega_2 \ldots \in \Sigma(S)$. Hence, $\pi_W := \pi_{|\Sigma(W)}$. It is now easy to see that $\mathcal{L}(W) = (K(W), W, \{f_w\}_{w \in W})$ is a self-similar structure itself from the fact that $\mathcal{L}$ is a self-similar structure as follows:

- As f_i's are injections, f_w, $w \in W$ is also an injection,
- For $w \in$ W, $f_w \circ \pi = \pi \circ \sigma_w$ is obvious from Definition 4.1.1, and
- π_W is surjective from the fact that π is surjective.

Definition 4.4.3. A self-similar structure that is defined on a collection of non-empty words which does not have any unnecessary word is called a *minimal* self-similar structure.

Now, we will state and prove a theorem that provides us with some characterizations of a minimal self-similar structure.

Theorem 4.4.4. Let $\mathcal{L}$=$(K, S, \{f_i\}_{i\in S})$ be a self-similar structure. If $\mathcal{L}$ satisfies any one of the following equivalent conditions, then it is *minimal.*

(i) If for a closed subset $\mathcal{A} \subseteq \Sigma$, $\pi(\mathcal{A}) = K$ then $\mathcal{A} = \Sigma$.
(ii) If for a subset $W \subseteq W_m$, $K(W) = K$ then $W = W_m$.
(iii) For any $v \in W_m$, $K_v \nsubseteq \bigcup_{w\in W_m\setminus\{v\}} K_w$.
(iv) $C_{\mathcal{L}} \nsupseteq K_w$, *for any* $w \in W_*$.
(v) int$(\mathcal{C}_{\mathcal{L}})$=$\emptyset$
(vi) int$(\mathcal{P}_{\mathcal{L}})$=$\emptyset$ (vi') $\mathcal{P}_{\mathcal{L}} \neq \Sigma$.
(vii) int(V_0)=$\emptyset$ (vii') $V_0 \neq K$.

Proof. $(ii) \Rightarrow (iii)$: Let $K_v \subseteq \bigcup_{w\in W_m\setminus\{v\}} K_w$ for some $v \in W_m$. Observe that Σ= $\bigcup_{w\in W_m} \Sigma_w$. Hence, K= $\bigcup_{w\in W_m} K_w$. Since $K_v \subseteq \bigcup_{w\in W_m\setminus\{v\}} K_w$, K= $\bigcup_{w\in W_m\setminus\{v\}} K_w$. Recall that for a subset W $\subseteq W_* \setminus W_0$, we can define a new self-similar structure $\mathcal{L}(W)$ with a map $\pi_W : \Sigma(W) \to K(W)$.(See Remark 4.4.2) Therefore, for W=$W_m\setminus\{v\}\subsetneq W_m$, we have $K(W)$=K.

$(iii) \Rightarrow (i)$: Let $\mathcal{A} \subsetneq \Sigma$ be a closed subset such that $\pi(\mathcal{A})$=K. Since $\mathcal{A}$ is closed, $\mathcal{A}^c$ is open and it contains some $B_\epsilon(\omega)$ for ϵ >0, for all $\omega \in \mathcal{A}^c$. Hence, $\mathcal{A}^c \supseteq \Sigma_v$ for some $v \in W_*$, where v=$\omega_1 \ldots \omega_m$. Now, let $w \in W_* \setminus \{v\}$ such that $|w|$=$|v|$. For $\tau \in \bigcup_{w\in W_m\setminus\{v\}} \Sigma_w$, $\tau \notin \mathcal{A}^c$ and $\tau \in \mathcal{A}$. Hence, $\bigcup_{w\in W_m\setminus\{v\}} \Sigma_w \subseteq \mathcal{A}$. Therefore, K_v $\in \bigcup_{w\in W_m\setminus\{v\}} K_w$.

$(iv) \Rightarrow (v)$: Let int$(\mathcal{C}_{\mathcal{L}}) \neq \emptyset$. Hence for all $\omega \in \mathcal{C}_{\mathcal{L}}$, $\mathcal{C}_{\mathcal{L}} \supseteq \Sigma_v$ for some $v \in W_*$, where v=$\omega_1 \ldots \omega_n$. Therefore, $C_{\mathcal{L}} = \pi(\mathcal{C}_{\mathcal{L}}) \supseteq \pi(\Sigma_v) = K_v$ for some $v \in W_*$.

$(vi') \Rightarrow (vi)$: Assume that int$(\mathcal{P}_{\mathcal{L}}) \neq \emptyset$. Then, $\mathcal{P}_{\mathcal{L}} \supseteq \Sigma_v$ for some $v \in W_*$. Let $|v|$=n. Then, $\sigma^n(\mathcal{P}_{\mathcal{L}}) \supseteq \Sigma$. Since $\mathcal{P}_{\mathcal{L}} \supset \sigma^n(\mathcal{P}_{\mathcal{L}})$,(See Definition 4.3.2) we have $\mathcal{P}_{\mathcal{L}} \supset \Sigma$. Therefore, $\mathcal{P}_{\mathcal{L}}$=$\Sigma$.

$(v) \Rightarrow (vi')$: Let $\mathcal{P}_{\mathcal{L}} = \Sigma$. Since $\mathcal{P}_{\mathcal{L}} = \bigcup_{n\geq 1} \sigma^n(\mathcal{C}_{\mathcal{L}})$ is complete, by *Baire's Category Theorem*, for some n, $\sigma^n(\mathcal{C}_{\mathcal{L}})$ is not rare. Hence, int$(\sigma^n(\mathcal{C}_{\mathcal{L}})) \neq \emptyset$ for some $n \geq 1$ and $\sigma^n(\mathcal{C}_{\mathcal{L}}) \supseteq \Sigma_w$ for some $w \in W_*$. If $|w| = m$, then $\sigma^{m+n}(\mathcal{C}_{\mathcal{L}}) \supseteq \Sigma$. Therefore,

$\sigma^k(\mathcal{C}_{\mathcal{L}}) \supseteq \Sigma$, where $k = m + n$. We can write $\sigma^k(\mathcal{C}_{\mathcal{L}}) = \bigcup_{v \in W_k} \sigma^k(\mathcal{C}_{\mathcal{L}} \cap \Sigma_v)$. If we use *Baire's Category Theorem* again, we can see that $\sigma^k(\mathcal{C}_{\mathcal{L}} \cap \Sigma_v) \supseteq \Sigma_{v'}$ for some $v' \in W_*$, $v \in W_k$. Hence, $\mathcal{C}_{\mathcal{L}} \supseteq \Sigma_{vv'}$.

$(vi) \Rightarrow (vii)$: We know that $\pi^{-1}(V_0)$=$\mathcal{P}_{\mathcal{L}}$. Hence, $\mathrm{int}(\pi^{-1}(V_0)) \subseteq \mathrm{int}(\mathcal{P}_{\mathcal{L}})$. Also, by the continuity of π, $\pi^{-1}(\mathrm{int}(V_0)) \subseteq \mathrm{int}(\pi^{-1}(V_0))$. Therefore, $\mathrm{int}(\mathcal{P}_{\mathcal{L}})=\emptyset$ implies that $\mathrm{int}(V_0)=\emptyset$.

$(i) \Rightarrow (iv)$: Assume that $\mathrm{C}_{\mathcal{L}} \supseteq K_w$ for some $w \in W_*$. Let $k \in K_w$, where $w=iw'$; $i \in$ S, $w' \in W_*$. Since $k \in C_{\mathcal{L}}$, $k \in K_v$ for some $v=jv' \in W_*$; $j \neq i$. Then, for $\mathcal{A}= \bigcup_{v \in W_m \setminus \{w\}} \Sigma_v$, $K=\pi(\mathcal{A})$ and $\mathcal{A}$ is closed since it is a finite union of closed sets, namely $\Sigma_w=f_w(\Sigma)$.

(vi') $\Rightarrow$ (iii): Assume that $K_v \subseteq \bigcup_{w \in W_m \setminus \{v\}} K_w$ for some $v \in \mathrm{W}_m$. Then, $C_{\mathcal{L},K}$ is not empty.(See Definition 4.3.2) Now, let $\omega \in \Sigma$. Then, there exists $\tau=v\tau' \in \Sigma$ such that $k=\pi(w\omega)=\pi(\tau)$. Since $k \in C_{\mathcal{L},K}$, $w\omega \in \mathcal{C}_{\mathcal{L}}$. Therefore, $\omega \in \mathcal{P}_{\mathcal{L}}$.

$(vii) \Rightarrow (vii')$: We know that $\pi^{-1}(V_0)$=$\mathcal{P}_{\mathcal{L}}$(see Proposition 4.3.5). If V_0=K, then $\pi^{-1}(K)$=$\mathcal{P}_{\mathcal{L}}$=$\Sigma$. Hence $\mathrm{int}(\mathcal{P}_{\mathcal{L}}) \neq \emptyset$ and $\mathrm{int}(\mathrm{V}_0) \neq \emptyset$.

$(vii') \Rightarrow (vi')$: Assume that $\mathcal{P}_{\mathcal{L}}$=$\Sigma$. Again by using $\pi^{-1}(V_0)$=$\mathcal{P}_{\mathcal{L}}$, $\pi^{-1}(V_0)$=Σ. Hence V_0=$\pi(\Sigma)$=K.

$(i) \Rightarrow (ii)$: Let $K(W)$=K for $W \subseteq W_m$. Then, $\pi_W : \Sigma(W) \to K(W)=K$. Since $\Sigma(W) \subseteq \Sigma$ is closed, there exists $\mathcal{A}$=$\Sigma(W)$ such that $\pi(\mathcal{A})$=K. □

4.5 Partition of $\Sigma(S)$

Definition 4.5.1. A finite subset $\Lambda \subseteq W_*(S)$, where S is a finite symbol set, is a *partition* of $\Sigma(S)$ if

(P1) $\Sigma(S) = \bigcup_{w \in \Lambda} \Sigma_w$,
(P2) For all $w \neq v \in \Lambda$, $\Sigma_w \cap \Sigma_v=\emptyset$,

where Σ_w=$\{w\omega \mid \omega \in \Sigma\}$.

Let us characterize the axiom (P2)

Case1: Let $|w| = |v| = n$; $w \neq v$. Then, $\Sigma_w \cap \Sigma_v = \emptyset$.

Case2: Let $|w|{=}m$, $|v|{=}n$ and let $m{>}n$.

- If $w = v_1 v_2 ... v_n w_{n+1} ... w_m$, then $\Sigma_w \subseteq \Sigma_v$, hence $\Sigma_w \cap \Sigma_v \neq \emptyset$.
- If $w_1 \neq v_1$, then

$$\Sigma_w \cap \Sigma_v = \emptyset.$$

Example 4.5.2. $\Lambda = W_m(S)$, $m{>}0$ is a partition of $\Sigma = \Sigma(S)$. It is easy to see that

$$\Sigma = \bigcup_{w \in W_m} \Sigma_w$$

as for all $\omega \in \Sigma$, $\omega = \omega_1 \omega_2 ... \omega_m ... \in \Sigma_w$ for some $w = \omega_1 \omega_2 ... \omega_m \in W_m$.

(P2) is clear.

Definition 4.5.3. Let Λ be a partition of Σ. Λ' is a *refinement* of Λ if, for any $w \in \Lambda$, $v \in \Lambda'$, either

$$\Sigma_w \cap \Sigma_v = \emptyset$$

or

$$\Sigma_v \subseteq \Sigma_w.$$

Remark 4.5.4. For a self-similar structure $\mathcal{L}$ and a partition Λ of Σ, we can define the self-similar structure $\mathcal{L}(\Lambda){=}(K(\Lambda), \Lambda, \{f_w\}_{w \in \Lambda})$ that is similar to defining a new self-similar structure for $W {\subseteq} W_* \setminus W_0$.(See 4.4.2) Since Λ is a partition of Σ, $\Sigma(\Lambda) = \Sigma$. Hence,

$$\pi_\Lambda : \Sigma(\Lambda) = \Sigma \to K(\Lambda) = K, \ \pi_\Lambda(\omega) = \pi(\omega).$$

Notation 4.5.5. For a self-similar structure $\mathcal{L}$, and a partition Λ we will define

$$V_{\mathcal{L},\Lambda} = \bigcup_{v \in \Lambda} f_v(V_0).$$

(See Definition 4.3.2 for the definition of V_0.)

Remark 4.5.6. Notice that for a refinement Λ' of Λ,

$$V_{\mathcal{L},\Lambda} \subseteq V_{\mathcal{L},\Lambda'}$$

since, for $k{=}\pi(v\omega) \in V_{\mathcal{L},\Lambda}$, where $v \in \Lambda$, $\omega \in \mathcal{P}$, there exists $w = v\omega_1\omega_2...\omega_n \in \Lambda'$ by the definition of a refinement. Since $\omega_{n+1}\omega_{n+1}... \in \mathcal{P}$, $k = \pi(w\omega_{n+1}\omega_{n+2}...) \in V_{\mathcal{L},\Lambda'}$.

In case of $\Lambda{=}W_m,\ \ m \geq 1$,

$$V_{\mathcal{L},W_{m+1}} = \bigcup_{i \in S} f_i(V_{\mathcal{L},W_m})$$

since

$$V_{\mathcal{L},W_m} = \bigcup_{w \in W_m} f_w(V_0),$$

and for all $v \in W_{m+1}$, it follows that $v = iw$ for some $i \in$ S.

Proposition 4.5.7. Let $\mathcal{L}$ be a self-similar structure and let us define

$$V_{\mathcal{L},W_*} = \bigcup_{m \geq 0} V_{\mathcal{L},W_m}.$$

Then, $V_{\mathcal{L},W_*}$ is dense in K if V_0 is not empty.

Proof. Let $k{=}\pi(\omega) {\in}\ K$ for $\omega \in \Sigma$. Let us set $x_n = \pi(\omega_1\omega_2...\omega_n\omega')$, where $\omega' \in \mathcal{P}$. Then $\pi(\omega_1\omega_2...\omega_n\omega') \in V_{\mathcal{L},W_n}$ for $n \geq 0$, therefore $x_n \in V_{\mathcal{L},W_*}$.

Now, $\rho_r(\omega, \tau_n) = r^n \to 0$ as $n \to \infty$, where $\tau_n = \omega_1\omega_2...\omega_n\omega'$. Since π is continuous, $d(k, x_n){=}d(\pi(\omega), \pi(\tau_n)) \to 0$ as $n{\to}\infty$. □

Now, we will give one proposition about the relation between $\mathcal{P}$ and $\mathcal{P}(\Lambda)$.

Proposition 4.5.8. Let $\mathcal{L}$ be a self-similar structure and let Λ be a partition of Σ. Then, $\mathcal{P}_{\mathcal{L}(\Lambda)} \subseteq \mathcal{P}_{\mathcal{L}}$.

Proof. Let $\omega = \omega_1\omega_2... \in \mathcal{P}(\Lambda)$, where $\omega_i \in \Lambda$. Then, there exists $w \in W_*(\Lambda)\backslash W_0$, and $\tau \in \Sigma(\Lambda)$; $\omega'{=}w\omega \neq \tau$ such that $\pi(w\omega){=}\pi(\tau)$. Since $\omega' \neq \tau$, it follows that $\omega'_1 \neq \tau_1$ where $\omega'_i{,}\tau_i \in W_*(\Lambda)$.

Now, let $\omega'_1{=}\alpha_1...\alpha_m \in W_m$, and $\tau_1{=}\beta_1...\beta_n \in W_n$. We can find k such that $\alpha_1...\alpha_k{=}\beta_1...\beta_k$, and $\alpha_{k+1} \neq \beta_{k+1}$. Hence, $\pi(\sigma^k(\alpha_1\alpha_2...)){=}\pi(\sigma^k(\beta_1\beta_2...))$. Since we swap two elements in $\Sigma(S)$, $\sigma^k(\omega') \in \mathcal{C}_{\mathcal{L}}$, where $\omega'{=}w\omega$. Therefore, $\omega{\in}\mathcal{P}_{\mathcal{L}}$. □

Proposition 4.5.9. Let $\mathcal{L}$ be a self-similar structure and let Λ=W_m, m>0. Then, $\mathcal{P}_{\mathcal{L}(\Lambda)}$=$\mathcal{P}_{\mathcal{L}}$.

Proof. We know that $\mathcal{P}_{\mathcal{L}(W_m)} \subseteq \mathcal{P}_{\mathcal{L}}$ by Proposition 4.5.8.

For the other implication, let ω=$\omega_1\omega_2...\in \mathcal{P}_{\mathcal{L}}$, where $\omega_i \in S$. Then, there exists $w \in W_*\setminus W_0$ and $\tau \in \Sigma$ such that $w_1 \neq \tau_1$, and $\pi(w\omega)$=$\pi(\tau)$. For $v \in W_* \setminus W_0$, we can set vw=$\alpha_1\alpha_2...\alpha_k$, and $v\tau$=$\beta_1\beta_2...$ by the definition of a partition, where α_i,$\beta_i \in W_m$, and $\alpha_1 \neq \beta_1$. We can also set ω=$\gamma_1\gamma_2...$, where $\gamma_i \in W_m$. Then, since $\pi(\alpha_1\alpha_2...\alpha_k\gamma_1\gamma_2...)$=$\pi(\beta_1\beta_2...)$, $\alpha_1...\alpha_k\gamma_1\gamma_2... \in \mathcal{C}_{\mathcal{L}(W_m)}$. Hence, ω=$\gamma_1\gamma_2... \in \mathcal{P}_{\mathcal{L}(W_m)}$. □

4.6 Post-critically Finite Self-similar Structures

Now, we will give one definition on the post critical set $\mathcal{P}_{\mathcal{L}}$ which is important for further studies.

Definition 4.6.1. Let $\mathcal{L}$ be a self-similar structure. $\mathcal{L}$ is said *post critically finite* if the post critical set $\mathcal{P}_{\mathcal{L}}$ is a finite set.

Proposition 4.6.2. Let $\mathcal{L}$ be a post-critically finite self-similar structure. If k is the fixed point of f_v where $v \in W_*$, then $\pi^{-1}(k)$=$\{\dot{v}\}$.

Proof. It is obvious that $\dot{v} \in \pi^{-1}(k)$.

For the other inclusion, let $\omega \in \Sigma$ such that $\omega \neq \dot{v}$, and $\pi(\omega)$=k. We can assume without loss of generality that $\omega_1 \neq v$. Let us define a sequence $\omega^{(n)}$=$(\sigma_v)^n$ ω, where $(\sigma_v)^n$ is the operator obtained by applying the operator σ_v n times. Then, for all n≥1, $f_v(f_v...(f_v(\pi(\omega))))$=$k$ from the fact that k is the fixed point of f_v. Hence, $\omega^n \in \pi^{-1}(k)$ for any $n \geq 1$ and $\pi^{-1}(k)$ has infinitely many elements. It says that $\mathcal{C}_{\mathcal{L}}$ is not finite. Therefore, $\mathcal{P}_{\mathcal{L}}$ is not finite. Since we start with a post-critically finite self-similar structure, $\pi^{-1}(k)$=$\{\dot{v}\}$. □

Example 4.6.3. Let K be the Hata's tree-like set as a self-similar structure with respect to the contractions $f_1(z)$=c$\bar{z}$, $f_2(z)$=$(1-|c|^2)\bar{z}$+$|c|^2$. Recall from the previous chapter that we defined

$$A=\{t:t\in[0,1]\}\cup\{ct:t\in[0,1]\}$$

for the explicit construction of Hata's tree-like set. In fact,

$$K=\overline{\bigcup_{m>0}A_m},\quad \text{where } A_m=\bigcup_{w\in W_m}f_w(A).$$

Now, observe that $f_1(A)\cap f_2(A)$=$\{|c|^2\}$. Hence, $f_1(K)\cap f_2(K)$=$\{|c|^2\}$. Also observe that $\pi(11\dot{2})$=$\pi(2\dot{1})$=$|c|^2$. Hence, $11\dot{2}$, $2\dot{1}\in\pi^{-1}(|c|^2)$.

Let us now show that $\pi^{-1}(|c|^2)\subseteq\{11\dot{2},2\dot{1}\}$. Let for $\omega\notin\{11\dot{2},2\dot{1}\}$, $\pi(\omega)$=$|c|^2$. For $\omega=\omega_1\omega_2...$ assume without loss of generality that ω_1=1. Then, $\pi(\omega)$=$f_1(\pi(\omega_2\omega_3...))$=$|c|^2$ if and only if $\pi(\omega_2\omega_3...)$=c. Now, assume that ω_2=2. Then, $f_2(\pi(\omega_3\omega_4...))$=c. But it is not possible since c $\notin f_2(A)$. Therefore, ω_2=1 and $\pi(\omega_3\omega_4...)$=1. Now, assume that ω_3=1. In the same way, $1\notin f_1(A)$ and ω_3=2. Since $f_2(\pi(\omega_4\omega_5...))$=1 if and only if $\pi(\omega_4\omega_5...\omega_m...)$=1 and 1 is the fixed point of f_2, by doing induction on m, ω'=$\omega_3\omega_4...$=$\dot{2}$. In the same way, starting with ω_1=2, ω=$2\dot{1}$. Therefore, $\pi^{-1}(|c|^2)$=$\{11\dot{2},2\dot{1}\}$.

Now, we know that $\mathcal{C}_{\mathcal{L},K}$=$\{11\dot{2},2\dot{1}\}$. Hence, $\mathcal{P}_{\mathcal{L},K}$=$\{1\dot{2},\dot{2},\dot{1}\}$ and V_0=$\{c,1,0\}$.

Example 4.6.4. Let K be the Sierpinski Gasket. We know that the Sierpinski Gasket is the self-similar structure with respect to the set of injective contractions $\{f_j(z)=\frac{z+p_j}{2}:j=1,2,3\}$. We also know from the Chapter 3 (see Example 3.8.3) that the only intersection point of $f_1(T)$ and $f_2(T)$, hence the only intersection point of $f_1(K)$ and $f_2(K)$ is the middle point of $|p_1p_2|$, namely q_3. Therefore, $\pi^{-1}(q_3)=\{1\dot{2},2\dot{1}\}$. In the same way, the other overlaps come from the other middle points, namely q_1=$f_2(K)\cap f_3(K)$ and q_2=$f_1(K)\cap f_3(K)$. So, $\mathcal{C}_{\mathcal{L},K}$=$\{1\dot{2},2\dot{1},1\dot{3},3\dot{1},2\dot{3},3\dot{2}\}$, and $\mathcal{P}_{\mathcal{L}}$=$\{\dot{1},\dot{2},\dot{3}\}$. Also, $V_0=\{p_1,p_2,p_3\}$.

There are also examples of self-similar structures that are not post-critically finite.

Example 4.6.5. Let $X{=}\mathbb{C}$ and let $p_1{=}0$, $p_2{=}1/2$, $p_3{=}1$, $p_4{=}\frac{1}{2}i$, $p_5{=}i$, $p_6{=}\frac{1}{2}{+}i$, $p_7{=}1{+}i$, $p_8{=}1{+}\frac{1}{2}i$. If we set, $f_j(z){=}\frac{z+2p_j}{3}$ for j=1,2,...,8. Then the self-similar structure with respect to them is called the *Sierpinski Carpet*. There are infinitely many overlaps between f_j's. To show this, let $k \in K$. Since π is surjective there exists $\omega \in \Sigma$ such that $\pi(\omega){=}k$. Then, $f_{\omega_1}(\sigma(\omega)){=}k$. Assume without loss generality that $\omega_1{=}p_3$.Then $k{=}f_{p_3}(z)$ for some $z \in \mathbb{C}$. Now, $f_{p_3}(z){=}\dfrac{z+2p_3}{3}{=}\dfrac{z+2}{3}$. For $z{=}x{+}iy$, if we choose $z'{=}x{+}1{+}iy$, $f_{p_3}(z){=}\dfrac{x+1+iy+2.1/2}{3}{=}f_{p_2}(z')$. Since k is arbitrary, there are infinitely many overlaps. Hence $C_{\mathcal{L},K}$ is an infinite set. Therefore, $\mathcal{C}_{\mathcal{L}}$ and $\mathcal{P}_{\mathcal{L}}$ are infinite sets and K is not post critically finite.

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