

**LINEARIZATION INSTABILITY IN GRAVITY THEORIES**

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## ABSTRACT

### LINEARIZATION INSTABILITY IN GRAVITY THEORIES

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In a nonlinear theory, such as gravity, physically relevant solutions are usually hard to find. Therefore, starting from a background exact solution with symmetries, one uses the perturbation theory, which albeit approximately, provides a lot of information regarding a physical solution. But even this approximate information comes with a price: the basic premise of a perturbative solution is that it should be improvable. Namely, by going to higher order perturbation theory, one should be able to improve and better approximate the physical problem or the solution. While this is often the case in many theories and many background solutions, there are important cases where the linear perturbation theory simply fails for various reasons. This issue is well known in the context of general relativity through the works that started in the early 1970s, but it has only been recently studied in modified gravity theories. This

thesis is devoted to the study of linearization instability in generic gravity theories where there are spurious solutions to the linearized equations which do not come from the linearization of possible exact solutions. For this purpose we discuss the Taub charges, the ADT charges and the quadratic constraints on the linearized solutions. We give the three dimensional chiral gravity and the  $D$  dimensional critical gravity as explicit examples and give a detailed ADM analysis of the topologically massive gravity with a cosmological constant.

Keywords: Linearization instability, Second order perturbation theory, Constraint equation analysis, Taub charges, ADT charges, Chiral gravity, Critical gravity, Topologically massive gravity.

## ÖZ

### KÜTLE ÇEKİM KURAMLARINDA DOĞRUSALLAŞTIRMA KARARSIZLIĞI

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Kütle çekim kuramı gibi doğrusal olmayan kuramlarda fiziksel olarak uygun olan çözümleri bulmak zordur. Bu yüzden, simetrisi olan bir arkaplan gerçek çözümünden başlayarak, pertürbasyon kuramının kullanılması, yaklaşık da olsa fiziksel çözüme ait bir çok bilgi sağlar. Fakat bu yaklaşık bilgi de bir bedel ile elde edilir: yaklaşık çözümün temel özelliği geliştirilebilir olmasıdır. Şöyle ki, bir üst mertebeye pertürbasyon kuramına gidildiğinde, fiziksel problem ya da çözüm geliştirilebilmeli ve daha yaklaşık olmalıdır. Bu birçok kuramda ve birçok arkaplan çözümünde böyle olsa da, doğrusal pertürbasyon kuramının bazı sebeplerden geçersiz olduğu önemli durumlar vardır. Bu sorun genel görelilik bağlamında 1970'lerin başlarında başlayan çalışmalar aracılığıyla sağlam bir zemine oturdu, fakat modifiye edilmiş kütle çe-

kim teorilerinde sadece son zamanlarda çalışılmaktadır. Bu tez, genel kütle çekim kuramlarında doğrusallaştırılmış denklemlerin bazı çözümlerinin mümkün bir gerçek çözümün doğrusallaştırılmasından elde edilemediği genel kütle çekim kuramları ile ilgilidir. Bu amaçla Taub yüklerini, ADT yüklerini ve doğrusallaştırılmış denklemler üzerindeki ikinci derece kısıtlayıcı denklemleri tartışacağız. Üç boyutlu kiral kütle çekim kuramını ve  $D$  boyutlu kritik kütle çekim kuramını açık örnekler olarak vereceğiz ve kozmolojik sabitli topolojik kütle kuramının detaylı ADM analizini vereceğiz.

Anahtar Kelimeler: Doğrusallaştırma kararsızlığı, İkinci derece pertürbasyon kuramı, Kısıtlayıcı denklem analizi, Taub yükleri, ADT yükleri, Kiral kütle çekim kuramı, Kritik kütle çekim kuramı, Topolojik kütle çekim kuramı.



*To my beloved parents*

*Selime Altaş and Özcan Altaş*

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## LIST OF ABBREVIATIONS

AdS	anti-de Sitter
ADM	Arnowitt-Deser-Misner
ADT	Abbott-Deser-Tekin
dS	de Sitter
GR	General Relativity
MMG	Minimal Massive Gravity
TMG	Topologically Massive Gravity



# CHAPTER 1

## INTRODUCTION

In nonlinear theories such as Einstein's general relativity or its modifications, extensions with higher powers of curvature, physically interesting solutions with few or no symmetries are usually analytically unavailable. Therefore one relies on the perturbation theory of some sort which often involves an exact background solution (with symmetries) and perturbations or deviations from this background solution. Generally, perturbation theory works fine in the sense that it can be improved order by order in some small parameter. But there are some important cases where perturbation theory fails as a method. This thesis is devoted to a detailed study of a phenomenon called "*linearization instability*" which refers to the failure of the first order perturbation theory in the following sense: not every solution of the linearized equations can be improved to get exact solutions, even in principle. In the following section we give a brief review of linearization instability. Throughout the thesis, we work both with the index free and local coordinate forms of the relevant tensors.

### 1.1 Linearization instability in brief

A nonlinear equation  $F(x) = 0$  is said to be linearization stable at a solution  $x_0$  if every solution  $\delta x$  to the linearized equation  $F'(x_0) \cdot \delta x = 0$  is tangent to a curve of solutions to the original nonlinear equation. In some nonlinear theories, not all solutions to the linearized field equations represent linearized versions of exact (non-linear) solutions. As a common algebraic example, let us consider the following:

*Example* : Let us consider a function  $F$  from  $\mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  such that  $F(x, y) = x(x^2 + y^2)$ . In the domain of definition, the solution set to  $F(x, y) = 0$  is given as

$(0, y)$  which is the  $y$  axis.

Linearization of this exact solution set is then tangent to the  $y$  axis and can be shown as  $(0, \frac{\partial}{\partial y})$  or  $\text{span}\{\frac{\partial}{\partial y}\}$ , which is one dimensional.

Now let us consider the particular solution  $(0, 0)$  and linearize the equation around it. The linearized equation is simply  $(3x^2 + y^2) \delta x + 2xy\delta y = 0$ . For  $(0, 0)$  there is no constraint on the linearized solutions, then the solution set is  $(\delta x, \delta y)$  where  $\delta x$  and  $\delta y$  are arbitrary. Although  $\delta x$  can be arbitrary with this approach, we know that it cannot be from the linearization of the exact solution. Only  $(0, \frac{\partial}{\partial y})$  is allowed to be integrable to an exact solution. Consequently, the exact solution  $(0, 0)$  is linearization unstable and the perturbation theory fails about it.  $F(x, y) = x(x^2 + y^2) = 0$ , where  $x, y$  are real, exact solution space is one dimensional given as  $(0, y)$ , and the linearized solution space is also one dimensional  $(0, \delta y)$  as long as  $y \neq 0$ . But at exactly the solution  $(0, 0)$ , the linearized solution space is two dimensional  $(\delta x, \delta y)$  and so there are clearly linearized solutions with  $\delta x \neq 0$ , which do not come from the linearization of any exact solution. The existence of such spurious solutions depends on the particular theory at hand and the background solution (with its symmetries and topology) about which linearization is carried out. If such so called "non-integrable" solutions exist, perturbation theory in some directions of solution space fails and we say that the theory is not linearization stable at a nonlinear exact solution. See Figure 1.1 for a depiction of the function and the solution set.

What we have just described is not an exotic phenomenon: *a priori* no nonlinear theory is immune to linearization instability: one must study the problem case by case. For example, pure general relativity is linearization stable in Minkowski spacetime (with a non-compact Cauchy surface) [1], hence perturbation theory makes sense, but it is not linearization stable on a background with compact Cauchy surfaces that possesses at least one Killing symmetry [2] which is the case when the Cauchy surface is a flat 3-torus [3]: on  $T^3 \times \mathbb{R}$ , at second order of the perturbation theory, one must go back and readjust the first order perturbative solution.

As gravity is our main interest here, let us consider some nonlinear gravity field equations in a coordinate chart as  $\mathcal{E}_{\mu\nu} = 0$ , which admits  $\bar{g}_{\mu\nu}$  as an exact solution, if every solution  $h_{\mu\nu}$  of the linearized field equations  $\mathcal{E}^{(1)}(\bar{g}) \cdot h = 0$  is tangent to an

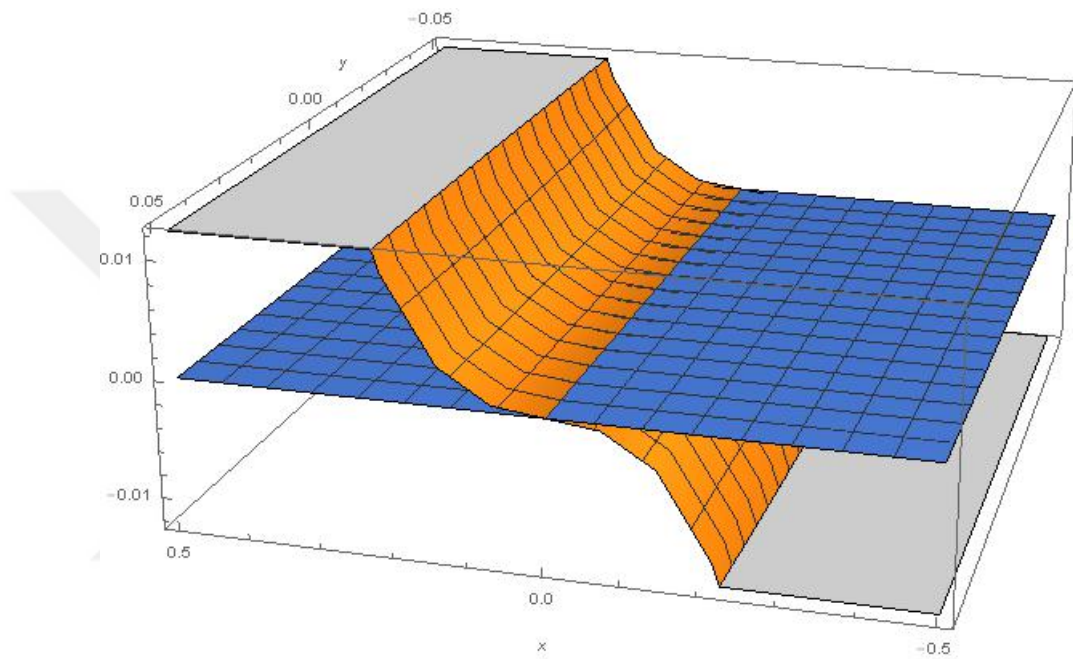


Figure 1.1: The blue surface represents the  $(x, y)$  plane and the orange surface shows the  $F$  function, also the intersection of the surfaces is the  $y$  axis which is the exact solution. As we mentioned above, linearization of the exact solution is tangent to the  $y$  axis. Conversely, solution of the linearized equation around the particular solution  $(0, 0)$  is  $(\delta x, \delta y)$ . It is obvious that this solution is out of the linearized solution space.

exact solution  $g_{\mu\nu}(\lambda)$  such that  $g_{\mu\nu}(0) =: \bar{g}_{\mu\nu}$  and  $\frac{dg_{\mu\nu}}{d\lambda}|_{\lambda=0} =: h_{\mu\nu}$  then, according to our definition above, the theory is linearization stable. Otherwise it is linearization unstable. In general, we do not have a theorem stating the *necessary and sufficient* conditions for the linearization stability of a generic gravity theory about a given exact solution. We shall give a detailed discussion on generic gravity models in the next section based on our recent work [4]. For a brief note, let us consider the following: defining the second order perturbation as  $\frac{d^2 g_{\mu\nu}}{d\lambda^2}|_{\lambda=0} =: k_{\mu\nu}$ , if the following second order equation

$$(\mathcal{E})^{(2)}(\bar{g}) \cdot [h, h] + (\mathcal{E})^{(1)}(\bar{g}) \cdot k = 0, \quad (1.1)$$

has a solution for  $k_{\mu\nu}$  without a constraint on the linear solution  $h_{\mu\nu}$ , then the theory is linearization stable. Of course, at this stage it is not clear that there will arise no further constraints on the linear theory beyond the second order perturbation theory. In fact, besides Einstein's theory, this problem has not been worked out, to the best of our knowledge. But in Einstein's gravity, as the constraint equations are related to the zeros of the moment map, one knows that there will be no further constraint for the linear theory coming from higher order perturbation theory beyond the second order [5]. In Einstein's gravity for compact Cauchy surfaces without a boundary, the necessary and sufficient conditions are known for linearization stability [2, 6–8].

In practice, it is very hard to show that (1.1) is satisfied for *all* linearized solutions, therefore, one resorts to a weaker condition by contracting that equation with a Killing vector field and integrates over a hypersurface to obtain  $Q_{Taub}[\bar{\xi}] + Q_{ADT}[\bar{\xi}] = 0$  where the Taub charge [9] is defined as

$$Q_{Taub}[\bar{\xi}] := \int_{\Sigma} d^3\Sigma \sqrt{\gamma} \hat{n}^\nu \bar{\xi}^\mu (\mathcal{E}_{\mu\nu})^{(2)} \cdot [h, h], \quad (1.2)$$

and the Abbott-Deser-Tekin (ADT) charge [10, 11] is defined as

$$Q_{ADT}[\bar{\xi}] := \int_{\Sigma} d^3\Sigma \sqrt{\gamma} \hat{n}^\nu \bar{\xi}^\mu (\mathcal{E}_{\mu\nu})^{(1)} \cdot k. \quad (1.3)$$

As it appears in the second order perturbation theory, the Taub charge is not a widely known quantity in physics, therefore a more detailed account of it will be given in the next chapter following the relevant discussion of [4]. The ADT charge can be expressed as a boundary integral. For the case of compact Cauchy surfaces without a boundary,  $Q_{ADT} = 0$ , and hence one must have  $Q_{Taub} = 0$  which leads to the

aforementioned quadratic integral constraint on the linearized perturbation  $h_{\mu\nu}$  as the integral in (2.13) should be zero. This is the case for Einstein's gravity, for example, on a flat 3-torus:  $Q_{Taub}$  does not vanish automatically and so the first order perturbative result  $h$  is constrained. On the other hand, for extended gravity theories (such as the ones we are interested in this thesis),  $Q_{ADT}$  vanishes for a different reason, even for non-compact surfaces, as in the case of AdS. The reason is that for some tuned values of the parameters in the theory, the contribution to the conserved charges from various tensors cancel each other exact, yielding non-vacuum solutions that carry the (vanishing) charges of the vacuum. This is the source of the linearization instability.

Chapter II of this thesis is devoted to a detailed study of linearization instability in generic gravity theories. We give  $D$ -dimensional critical and three dimensional chiral gravity theories (which both received interest in the recent literature) as two interesting examples of theories that exhibit the kind of linearization instability we mentioned above.

In Chapter III, we give a discussion of the initial value formulation and the ADM decomposition of topologically massive gravity, study its constraints on a spacelike surface and give a second proof of linearization instability at the chiral limit of the theory. This second proof is not based on the charge construction but is based directly on the field equations (especially the constraints of the theory).

We relegate some of the computations to the appendices: In Appendix A, we give details of second order perturbation theory in the context of Riemannian geometry and compute the relevant expanded tensors up to second order and discuss the gauge invariance issues of the second order Einstein's tensor. Note that, in contrast to the first order Einstein's tensor, the second order one is not gauge invariant, therefore one must be very careful about any result (such as the Taub charge construction) based on the second order perturbation theory. So we discuss the gauge transformation properties of the relevant tensors under small gauge transformations (infinitesimal diffeomorphisms).

In Appendix B, we compute explicitly the form of the  $K_{\mu\nu}$  tensor for the case of Einstein's theory in AdS and Minkowski spaces. This is relevant for the proof of the linearization stability of the Minkowski space.

Appendix C is devoted to a detailed construction of the ADM formulation of the topologically massive gravity directly from the field equations and the action. In most of that appendix we work with nonzero lapse and shift functions, but at the end we restrict to the Gaussian normal coordinates for the particular goal of studying the linearization instability issue on the spacelike initial value surface for AdS.



## CHAPTER 2

### LINEARIZATION INSTABILITY FOR GENERIC GRAVITY IN AdS

There is an <sup>1</sup> interesting conundrum in nonlinear theories, such as Einstein's gravity or its modifications with higher curvature terms: exact solutions without symmetries (which are physically interesting) are hard to find, hence one resorts to symmetric "background" solutions and develops a perturbative expansion about them. But it turns out that exactly at the symmetric solutions, namely about solutions having Killing vector fields, naive first order perturbation theory fails under certain conditions. The set of solutions to Einstein's equations forms a smooth manifold except at the solutions with infinitesimal symmetries and spacetimes with compact Cauchy surfaces where there arise conical singularities in the solution space. Namely, perturbation theory in non-linear theories can yield results which are simply wrong in the sense that *some* perturbative solutions cannot be obtained from the linearization of exact solutions. Roughly speaking, the process of first linearizing the field equations and then finding the solutions to those linearized equations; and the process of linearization of exact solutions to the non-linear equations can yield different results if certain necessary criteria are not met with regard to the background solution about which perturbation theory is carried out. Actually, the situation is more serious: linearized field equations can have spurious solutions which do not come from exact solutions. This could happen for various reasons and the failure of the first order perturbation theory can be precisely defined, as we shall do below. Figure 2.1 summarizes the results.

Let us give a couple of early observations in this issue in the context of general relativity (GR) before we start the discussion in generic gravity. One clear way to see the failure of the perturbation theory is through the *initial value* formulation of the

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<sup>1</sup> This chapter was published as Phys. Rev. D 97, 024028 on 24 January 2018.

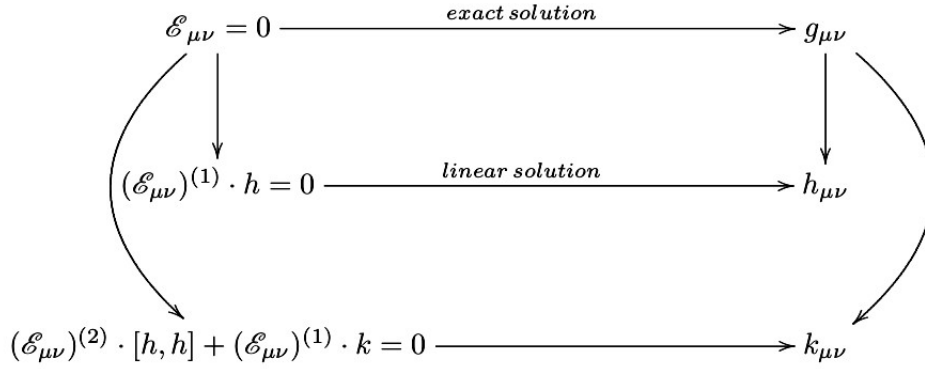


Figure 2.1: The vertical straight arrows show first order linearization while the curved ones show second order linearization. For a linearization stable theory, the diagram makes sense and the solution to the linearized equation  $h$  is not further restricted at the second order which means that there is a symmetric tensor  $k$  that satisfies the second order equation in the bottom left. The details of the symbols are explained in the next section.

theory for globally hyperbolic, oriented, time-orientable spacetimes with the topology  $\mathcal{M} \approx \Sigma \times \mathbb{R}$ , where  $\Sigma$  is a spacelike Cauchy surface on which the induced Riemannian metric  $\gamma$  and the extrinsic curvature  $K$  (as well as matter content of the theory) are defined. [It is also common to formulate the constraint equations in terms of  $\gamma$  and a tensor density of weight 1 defined as  $\pi := \sqrt{\det\gamma}(K - \gamma\text{tr}_\gamma K)$  which is the conjugate momentum of the induced metric  $\gamma$ .] We shall consider the matter-free case through-out this chapter. Since GR is nonlinear, the initial data cannot be arbitrarily prescribed: they must satisfy the so called Hamiltonian and momentum constraints  $\Phi_i(\gamma, K) = 0$  with  $i \in \{1, 2, 3, 4\}$  in four dimensions. If a given initial data  $(\bar{\gamma}, \bar{K})$  solving the constraints is not isolated, meaning the linearized constraint equations  $\delta\Phi_i(\bar{\gamma}, \bar{K}) \cdot [\delta\gamma, \delta K] = 0$  allow *viable* linearized solutions  $(\delta\gamma, \delta K)$ , then the theory is said to be *linearization stable* about the initial Cauchy data. Deser and Brill [3] showed that in GR with a compact Cauchy surface having the topology of a 3-torus, there are strong constraints on the perturbations of the initial data. Any such perturbation leads to contradictions in the sense that bulk integrals of conserved mass and angular momenta do not vanish, while since there is no boundary, they must vanish in this compact space: hence the background is an isolated solution. Put in another

way, the linearized field equations about the background have solutions which do not come from the linearization of exact solutions. This happens because, as we shall see below, the linearized equations of the theory are not *sufficient* to constrain the linearized solutions: quadratic constraints on the linearized solutions, in the form of an integral (so called *Taub conserved quantity* first introduced in [9] for each Killing vector field), arise.

Most of the work regarding the linearization stability or instability in gravity has been in the context of GR with or without matter and with compact or with non-compact Cauchy surfaces. A nice detailed account of all these in the context of GR is given in the book [12]. See also [13] where a chapter is devoted to this issue and the Taub conserved quantity construction which is not widely known in the physics community. Our goal here is to extend the discussion to generic gravity theories: we show that if the field equations of the theory are defined by the Einstein tensor plus a covariantly conserved two tensor, then a new source of linearization instability that does not exist in GR arises, especially in de Sitter or Anti-de Sitter backgrounds, with non-compact Cauchy surfaces. This happens because in these backgrounds there are special *critical* points in the space of parameters of the theory which conspire to cancel the conserved charge (mass, angular momentum, *etc.*) of non-perturbative objects (black holes) or the energies of the perturbative excitations. One needs to understand the origin of this rather interesting phenomenon that non-vacuum objects have the same charges as the vacuum. To give an example of this phenomenon let us note that this is exactly what happens in chiral gravity [14–17] in 2+1 dimensions where the Einstein tensor is augmented with the Cotton tensor and the cosmological constant times the metric (namely a special limit of the cosmological topologically massive gravity [18]). In AdS, at the chiral point, the contribution of the Cotton tensor and the Einstein tensor in AdS cancel each other at the level of the conserved charges. Exactly at that point, new ghost-like solutions, the so called log modes arise [19] and if the boundary conditions are not those of Brown-Henneaux type [20], then these modes are present in the theory with negative energies. This would mean that the theory has no vacuum. But it was argued in [15, 17] that chiral gravity in AdS has a linearization instability which would remedy this problem. A similar phenomenon occurs in critical gravity in all dimensions [21,22]. Here we give a systematic discussion of the linearization stability

and instability in generic gravity theories and study these two theories as examples. We will not follow the route of defining the theory in the 3+1 setting and considering the instability problem on the Cauchy data. The reason for this is the following: in GR for asymptotically flat spacetimes, splitting the problem into the constraints on the Cauchy data and the evolution of the 3-metric and the extrinsic curvature turns the stability problem to a problem in elliptic operator theory which is well-developed and sufficient to rigorously prove the desired results. In the initial value formulation setting, the problem becomes a problem of determining the surjectivity of a linear operator, namely the linearized constraint operator. But this method is not convenient for our purposes since the source of the linearization instability in the extended gravity models that we shall discuss is quite different and so the full spacetime formulation is much better-suited for our problem. In GR as noted in the abstract, what saves the Minkowski space from the linearization instability is its non-compact Cauchy surfaces as was shown by Choquet-Bruhat and Deser [1]. This result is certainly consistent with the non-zero conserved charges (ADM mass or angular momentum) that can be assigned to an asymptotically-flat 3 dimensional Cauchy surface.

The layout of the chapter is as follows: In section II, we discuss the linearization stability in generic gravity theory and derive the second order constraints on the solutions of the linearized field equations. Of course these constraints are all related to the diffeomorphism invariance and the Bianchi identities of the theory. Hence we give a careful discussion of the linearized forms of the field equations and their gauge invariance properties. As the second order perturbation theory about a generic background is quite cumbersome in the local coordinates, we carry out the index-free computations in the bulk of the chapter and relegate some parts of the component-wise computations to the appendices. In section II, we establish the relation between the Taub conserved quantities coming from the second order perturbation theory and the Abbott-Deser-Tekin (ADT) charges coming from the first order perturbation theory. We study the linearization stability and instability of the Minkowski space, chiral gravity and critical gravity as examples. In the forth-coming chapter, we shall give a more detailed analysis of the chiral gravity discussion in the initial value formulation context.

## 2.1 Linearization Stability in Generic Gravity

Let us consider the matter-free equation of a generic gravity theory in a  $D$ -dimensional spacetime, whose dynamical field is the metric tensor  $g$  only. In the index-free notation the covariant two-tensor equation reads

$$\mathcal{E}(g) = 0, \quad (2.1)$$

together with the covariant divergence condition which comes from the diffeomorphism invariance of the theory

$$\delta_g \mathcal{E}(g) = 0, \quad (2.2)$$

where  $\delta_g$  denotes the divergence operator with respect to the metric  $g$ . (As usual, one uses the musical isomorphism to extend the divergence from the contravariant tensors to the covariant ones.) Here we generalize the discussion in [6, 7] given for Einstein's theory to generic gravity. Let us assume that there is a one-parameter family of solutions to (2.1) denoted as  $g(\lambda)$  which is at least twice differentiable with respect to  $\lambda$  parameterizing the solution set. Then we can explore the consequences of this assumption with the help of the following identifications :

$$\bar{g} := g(\lambda) \Big|_{\lambda=0}, \quad h := \frac{d}{d\lambda} g(\lambda) \Big|_{\lambda=0}, \quad k := \frac{d^2}{d\lambda^2} g(\lambda) \Big|_{\lambda=0}. \quad (2.3)$$

At this stage there is of course no immediate relation between the two covariant tensor fields  $h$  (the first derivative of the metric) and  $k$  (the second derivative of the metric) but, as we shall see later, consistency of the theory, *i.e.* the first order linearized and second order linearized forms of the field equations will relate them. We would first like to find that relation.

We assume that  $\bar{g}$  exactly solves the vacuum equations  $\mathcal{E}(\bar{g}) = 0$  and we compute the first derivative of the field equations with respect to  $\lambda$  and evaluate it at  $\lambda = 0$  as

$$\frac{d}{d\lambda} \mathcal{E}(g(\lambda)) \Big|_{\lambda=0} = D\mathcal{E}(g(\lambda)) \cdot \frac{dg(\lambda)}{d\lambda} \Big|_{\lambda=0} = 0, \quad (2.4)$$

where  $D$  denotes the Fréchet derivative and the center-dot denotes "along the direction of the tensor that comes next" and we have used the chain rule. In local coordinates, this equation is just the first order "linearization" of the field equations (2.1) which we shall denote as  $(\mathcal{E}_{\mu\nu})^{(1)} \cdot h = 0$ . It is important to understand that solutions

of (2.4) yield all possible  $h$  tensors (up to diffeomorphisms), which are tangent to the exact solution  $g(\lambda)$  at  $\lambda = 0$  in the space of solutions. To understand if there are any *further* constraints on the linearized solutions  $h$ , let us consider the second derivative of the field equation with respect to  $\lambda$  and evaluate it at  $\lambda = 0$  to arrive at

$$\begin{aligned} & \left. \frac{d^2}{d\lambda^2} \mathcal{E}(g(\lambda)) \right|_{\lambda=0} \\ &= \left( D^2 \mathcal{E}(g(\lambda)) \cdot \left[ \frac{dg(\lambda)}{d\lambda}, \frac{dg(\lambda)}{d\lambda} \right] + D \mathcal{E}(g(\lambda)) \cdot \frac{d^2 g(\lambda)}{d\lambda^2} \right) \Big|_{\lambda=0} = 0, \end{aligned} \quad (2.5)$$

where we have used the common notation for the second Fréchet derivative in the first term and employed the chain rule when needed. We can write (2.5) in local coordinates as

$$(\mathcal{E}_{\mu\nu})^{(2)} \cdot [h, h] + (\mathcal{E}_{\mu\nu})^{(1)} \cdot k = 0, \quad (2.6)$$

where again  $(\mathcal{E}_{\mu\nu})^{(2)} \cdot [h, h]$  denotes the second order linearization of the field equation about the background  $\bar{g}$ . Even though this equation is rather simple, it is important to understand its meaning to appreciate the rest of the discussion. This is the equation given in the bottom-left corner of Figure 2.1. Given a solution  $h$  of  $(\mathcal{E}_{\mu\nu})^{(1)} \cdot h = 0$ , equation (2.6) *determines* the tensor field  $k$ , which is the second order derivative of the metric  $g(\lambda)$  at  $\lambda = 0$ . If such a  $k$  can be found then there is no further constraint on the linearized solution  $h$ . In that case, the field equations are said to be linearization stable at the exact solution  $\bar{g}$ . This says that the infinitesimal deformation  $h$  is tangent to a full (exact) solution and hence it is integrable to a full solution. Of course, what is tacitly assumed here is that in solving for  $k$  in (2.6), one cannot change the first order solution  $h$ , it must be kept intact for the perturbation theory to make any sense.

We can understand these results from a more geometric vantage point as follows. For the spacetime manifold  $\mathcal{M}$ , let  $\mathcal{S}$  denote the set of solutions of the field equations  $\mathcal{E}(g) = 0$ . The obvious question is (in a suitable Sobolev topology), when does this set of solutions form a smooth manifold whose tangent space at some "point"  $\bar{g}$  is the space of solutions ( $h$ ) to the linearized equations? The folklore in the physics literature is not to worry about this question and just assume that the perturbation theory makes sense and the linearized solution can be improved to get better solutions, or the linearized solution is assumed to be integrable to a full solution. But as we have given examples above, there are cases when the perturbation theory fails and the

set  $\mathcal{S}$  has a conical singularity instead of being a smooth manifold. One should not confuse this situation with the case of dynamical instability as the latter really allows a "motion" or perturbation about a given solution. Here linearization instability refers to a literal break-down of the first order perturbation theory. It is somewhat a non trivial matter to show that there are no further constraints beyond the second order perturbation theory: In Einstein's gravity, this is related to the fact that constraint equations are related to zeros of the moment maps [5]. For generic gravity, this issue deserves to be further studied.

### 2.1.1 Taub conserved quantities and ADT charges

So far, in our discussion we have not assumed anything about whether the spacetime has a compact Cauchy surface or not. First, let us now assume that the spacetime has a compact spacelike Cauchy surface and has at least one Killing vector field. Then we can get an *integral constraint* on  $h$ , without referring to the  $k$  tensor as follows. Let  $\bar{\xi}$  be a Killing vector field of the metric  $\bar{g}$ , then the following vector field <sup>2</sup>

$$T := \bar{\xi} \cdot D^2 \mathcal{E}(\bar{g}) \cdot [h, h], \quad (2.7)$$

is divergence free, since  $\delta_{\bar{g}} D^2 \mathcal{E}(\bar{g}) \cdot [h, h] = 0$  due to the linearized Bianchi identity. Then we can integrate  $T$  over a compact hypersurface  $\Sigma$  and observe that the integral (for the sake of definiteness, here we consider the 3+1 dimensional case)

$$\int_{\Sigma} d^3 \Sigma \sqrt{\gamma} T \cdot \hat{n}_{\Sigma} \quad (2.8)$$

is independent of hypersurface  $\Sigma$  where  $\gamma$  is the pull-back metric on the hypersurface and  $\hat{n}_{\Sigma}$  is the unit future pointing normal vector. Let us restate the result in a form that we shall use below: given *two* compact disjoint hypersurfaces  $\Sigma_1$  and  $\Sigma_2$  (as shown in Figure 2.2) in the spacetime  $\mathcal{M}$ , we have the statement of the "charge conservation" as the equality of the integration over the two hypersurfaces

$$\int_{\Sigma_1} d^3 \Sigma_1 \sqrt{\gamma_{\Sigma_1}} T \cdot \hat{n}_{\Sigma_1} = \int_{\Sigma_2} d^3 \Sigma_2 \sqrt{\gamma_{\Sigma_2}} T \cdot \hat{n}_{\Sigma_2}. \quad (2.9)$$

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<sup>2</sup> For the lack of a better notation, note that  $\bar{\xi}$  is contracted with the covariant background tensor with a center dot which we shall employ in what follows and it should not be confused with the center dot in the Fréchet derivative.

We can now go to (2.6) and after contracting it with the Killing tensor  $\bar{\xi}$ , and integrating over  $\Sigma$ , we obtain the identity

$$\int_{\Sigma} d^3 \Sigma \sqrt{\gamma} \bar{\xi}^{\mu} \hat{n}^{\nu} (\mathcal{E}_{\mu\nu})^{(2)} \cdot [h, h] = - \int_{\Sigma} d^3 \Sigma \sqrt{\gamma} \bar{\xi}^{\mu} \hat{n}^{\nu} (\mathcal{E}_{\mu\nu})^{(1)} \cdot k. \quad (2.10)$$

Let us study the right-hand side more carefully. In a generic theory, this conserved Killing charge is called the Abbott-Deser-Tekin (ADT) charge ( for further details on the ADT charges, please see the recent review [23] and the relevant references therein )when the symmetric two-tensor  $k$  is the just the linearized two tensor  $h$  [10, 11]. Once the field equations of the theory are given, it is possible, albeit after some lengthy computation, to show that one can write the integral on the right-hand side as a total derivative.

$$\bar{\xi}^{\mu} (\mathcal{E}_{\mu\nu})^{(1)} \cdot h = \bar{\nabla}_{\alpha} (\mathcal{F}^{\alpha}{}_{\nu\mu} \bar{\xi}^{\mu}), \quad (2.11)$$

with an anti-symmetric tensor  $\mathcal{F}$  in  $\alpha$  and  $\nu$ . Hence if the Cauchy surface is compact without a boundary, the ADT charge vanishes identically, namely

$$Q_{ADT} [\bar{\xi}] := \int_{\Sigma} d^3 \Sigma \sqrt{\gamma} \hat{n}^{\nu} \bar{\xi}^{\mu} (\mathcal{E}_{\mu\nu})^{(1)} \cdot h = 0, \quad (2.12)$$

which via (2.10) says that one has the vanishing of the integral on the left hand-side which is called the Taub conserved quantity:

$$Q_{Taub} [\bar{\xi}] := \int_{\Sigma} d^3 \Sigma \sqrt{\gamma} \hat{n}^{\nu} \bar{\xi}^{\mu} (\mathcal{E}_{\mu\nu})^{(2)} \cdot [h, h] = 0, \quad (2.13)$$

which must be *automatically* satisfied for the case when  $h$  is an integrable deformation. Otherwise this equation is a second order constraint on the linearized solutions. Even though the ADT potential  $\mathcal{F}$  was explicitly found for a large family of gravity theories, such as Einstein's gravity [10], quadratic gravity [11],  $f(Riem)$  theories [24], and some examples will be given below, we can still refine the above argument of the vanishing of both the ADT and Taub conserved quantities without referring to the ADT potential (or more explicitly without referring to (2.11)). The following argument was given for Einstein's gravity in [7] which immediately generalizes to the most general gravity as follows: consider the ADT charge (2.12) and assume that in the spacetime one has two disjoint compact hypersurfaces  $\Sigma_1$  and  $\Sigma_2$  as above. Then the statement of conservation of the charge is simply

$$Q_{ADT} (\bar{\xi}, \Sigma_1) = Q_{ADT} (\bar{\xi}, \Sigma_2). \quad (2.14)$$

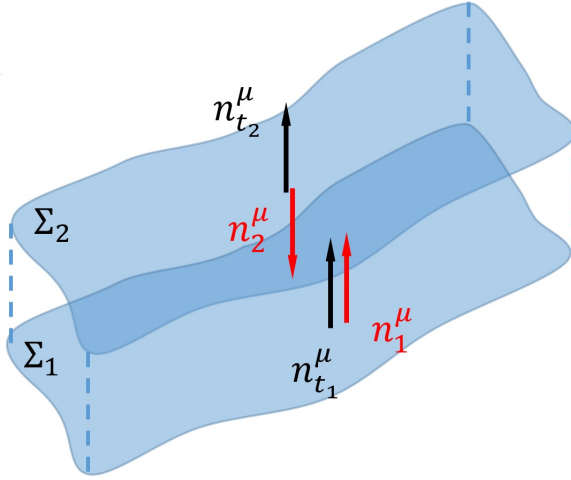


Figure 2.2: Disjoint hypersurfaces  $\Sigma_1$  and  $\Sigma_2$  are shown along with their timelike unit normals. The figure is taken from [23].

Now let  $k$  be a two tensor which is  $k_1$  and non-zero on  $\Sigma_1$  and  $k_2$  and zero near  $\Sigma_2$ , then  $Q_{ADT}(\bar{\xi}, \Sigma_2) = 0$  so  $Q_{ADT}(\bar{\xi}, \Sigma_1) = 0$  which in turn yields the vanishing of the Taub conserved quantities via (2.10).

To summarize the results obtained so far, let us note that assuming an integrable infinitesimal deformation  $h$ , which is by definition a solution to the linearized field equations about a background  $\bar{g}$  solution, we arrived at (2.6). And the discussion after that equation showed that Taub conserved quantities constructed with a Killing vector field, from the second order linearization,  $(\mathcal{E}_{\mu\nu})^{(2)} \cdot [h, h]$ , and the ADT charges constructed from the first order linearization,  $(\mathcal{E}_{\mu\nu})^{(1)} \cdot h$ , vanish identically for the case of compact Cauchy hypersurfaces without a boundary. If these integrals do not vanish, then there is a contradiction and the the linearized solution  $h$  is further constraint. Hence it is not an integrable deformation, namely,  $h$  is not in the tangent space about the point  $\bar{g}$  in the space of solutions. For Einstein's theory with compact Cauchy surfaces, it was shown that the necessary condition for linearization stability is the absence of Killing vector fields [2, 8]. As noted above, the interesting issue is that further study reveals that besides the quadratic constraint, there are no other constraints on the solutions to the linearized equations [5].

### 2.1.2 Gauge invariance of the charges

Of course there is one major issue that we still must address that is the gauge-invariance (or coordinate independence) of the above construction which we show now. Following [7], first let us consider a (not necessarily small) diffeomorphism  $\varphi$  of the spacetime as  $\varphi : \mathcal{M} \rightarrow \mathcal{M}$ . Then we demand that having obtained our rank two tensor  $\mathcal{E}(g)$  from a diffeomorphism invariant action (or from a diffeomorphism invariant action up to a boundary term as in the case of topologically massive gravity) we have a global statement of diffeomorphism invariance as

$$\mathcal{E}(\varphi^*g) = \varphi^*\mathcal{E}(g), \quad (2.15)$$

which states that  $\mathcal{E}$  evaluated for the pull-back metric is equivalent to the pull-back of  $\mathcal{E}$  evaluated for  $g$ . Let us now consider a one-parameter family of diffeomorphisms as  $\varphi_\lambda$ , generated by a vector field  $X$  well-defined on some region of spacetime. Let  $\varphi_0$  be the identity diffeomorphism denoted as  $\varphi_0 = I_{\mathcal{M}}$ . Then we can differentiate (2.15) with respect to  $\lambda$  once to get

$$\frac{d}{d\lambda}\mathcal{E}(\varphi_\lambda^*g) = \frac{d}{d\lambda}\varphi_\lambda^*\mathcal{E}(g), \quad (2.16)$$

which, after making use of the chain rule, yields

$$D\mathcal{E}(\varphi_\lambda^*g) \cdot \frac{d}{d\lambda}\varphi_\lambda^*g = \varphi_\lambda^*\left(\mathcal{L}_X\mathcal{E}(g)\right), \quad (2.17)$$

where  $\mathcal{L}_X$  is the Lie derivative with respect to the vector field  $X$ . Taking the derivative of the last equation with respect to  $g$  yields

$$D^2\mathcal{E}(g) \cdot \left(h, \mathcal{L}_Xg\right) + D\mathcal{E}(g) \cdot \mathcal{L}_Xh = \mathcal{L}_X\left(D\mathcal{E}(g) \cdot h\right). \quad (2.18)$$

In components, and after setting  $\lambda = 0$ , equation (2.17) reads, respectively

$$\delta_X(\mathcal{E}_{\mu\nu})^{(1)} \cdot h = \mathcal{L}_X\mathcal{E}_{\mu\nu}(\bar{g}), \quad (2.19)$$

and equation (2.18) reads

$$\delta_X(\mathcal{E}_{\mu\nu})^{(2)} \cdot [h, h] + (\mathcal{E}_{\mu\nu})^{(1)} \cdot \mathcal{L}_Xh = \mathcal{L}_X(\mathcal{E}_{\mu\nu})^{(1)} \cdot h, \quad (2.20)$$

where  $\delta_X(\mathcal{E}_{\mu\nu})^{(1)} \cdot h$  denotes the variation of the background tensor  $(\mathcal{E}_{\mu\nu})^{(1)} \cdot h$  under the flow of  $X$  or under the infinitesimal diffeomorphisms. Since  $\mathcal{E}_{\mu\nu}(\bar{g}) = 0$ , (2.19) says that  $(\mathcal{E}_{\mu\nu})^{(1)} \cdot h$  is gauge invariant:  $\delta_X(\mathcal{E}_{\mu\nu})^{(1)} \cdot h = 0$ . Similarly (2.20) yields

$$\delta_X(\mathcal{E}_{\mu\nu})^{(2)} \cdot [h, h] + (\mathcal{E}_{\mu\nu})^{(1)} \cdot \mathcal{L}_Xh = 0, \quad (2.21)$$

since  $(\mathcal{E}_{\mu\nu})^{(1)} \cdot h = 0$  by assumption, the right hand side of (2.20) vanishes. It is worth stressing that since generically  $(\mathcal{E}_{\mu\nu})^{(1)} \cdot \mathcal{L}_X h$  is not zero, the second order expansion  $(\mathcal{E}_{\mu\nu})^{(2)} \cdot [h, h]$  is not gauge invariant but transforms according to (2.21). Gauge invariance of the Taub conserved quantity and the ADT charge follows immediately from (2.21). Contracting that equation with the Killing vector field  $\bar{\xi}$  and integrating over the Cauchy surface, one finds

$$\int_{\Sigma} d^3 \Sigma \sqrt{\gamma} n^\nu \left[ \bar{\xi}^\mu \delta_X (\mathcal{E}_{\mu\nu})^{(2)} \cdot [h, h] + \bar{\xi}^\mu (\mathcal{E}_{\mu\nu})^{(1)} \cdot \mathcal{L}_X h \right] = 0. \quad (2.22)$$

Since we have already shown that the second term can be written as a divergence we can drop it out, the remaining part is the Taub conserved quantity which is shown to be gauge invariant, by this construction. The above discussion has been for a generic gravity theory based on the metric tensor as the only dynamical field, let us consider Einstein's gravity as an explicit example.

### 2.1.3 Linearization stability in Einstein's gravity

Let  $Ein$  denote the  $(0, 2)$  Einstein tensor, and  $h$  denote a symmetric two tensor field as described above and  $X$  be a vector field, then the effect of infinitesimal one-parameter diffeomorphisms generated by  $X$  follows as

$$DEin(g) \cdot \mathcal{L}_X g = \mathcal{L}_X Ein(g), \quad (2.23)$$

which in local coordinates reads

$$\delta_X (G_{\mu\nu})^{(1)} \cdot h = \mathcal{L}_X \bar{G}_{\mu\nu}, \quad (2.24)$$

where  $G_{\mu\nu} := Ein(e_\mu, e_\nu)$  and  $Ein := Ric - \frac{1}{2} Rg$ . We have already given the proof of the above equation for a generic theory in the previous part, but it pays to do it more explicitly in Einstein's theory: so it follows as

$$\delta_X (G_{\mu\nu})^{(1)} \cdot h = \delta_X (R_{\mu\nu})^{(1)} \cdot h - \frac{1}{2} \bar{g}_{\mu\nu} \delta_X (R)^{(1)} \cdot h - \frac{1}{2} \bar{R} \delta_X h_{\mu\nu}, \quad (2.25)$$

which just comes from the definition of the linearized Einstein tensor. Then one can rewrite the above expression as desired:

$$\delta_X (G_{\mu\nu})^{(1)} \cdot h = \mathcal{L}_X \left( \bar{R}_{\mu\nu} - \frac{1}{2} \bar{g}_{\mu\nu} \bar{R} \right) = \mathcal{L}_X \bar{G}_{\mu\nu}. \quad (2.26)$$

At the second order of linearization, one has

$$D^2 Ein(g) \cdot (h, \mathcal{L}_X g) + DEin(g) \cdot \mathcal{L}_X h = \mathcal{L}_X \left( DEin(g) \cdot h \right), \quad (2.27)$$

whose local version reads

$$\delta_X (G_{\mu\nu})^{(2)} \cdot [h, h] + (G_{\mu\nu})^{(1)} \cdot \mathcal{L}_X h = \mathcal{L}_X (G_{\mu\nu})^{(1)} \cdot h. \quad (2.28)$$

The explicit proof of this expression is rather long, hence we relegate it to Appendix A.

Now let us study the linearization stability of a particular solution to Einstein's gravity with a cosmological constant. Let  $\bar{g}$  solve the cosmological Einstein's field equations then the equation relevant to the study of linearization stability of this solution is (2.6) which now reads

$$(\mathcal{G}_{\mu\nu})^{(2)} \cdot [h, h] + (\mathcal{G}_{\mu\nu})^{(1)} \cdot k = 0, \quad (2.29)$$

where  $(\mathcal{G}_{\mu\nu})^{(1)} \cdot k$  is a simple object but the the second order object  $(\mathcal{G}_{\mu\nu})^{(2)} \cdot [h, h]$  is quite cumbersome. It is very hard to use this equation to show that for a generic background  $\bar{g}_{\mu\nu}$ , a  $k_{\mu\nu}$  can be found or cannot be found that satisfy (2.29). Therefore one actually resorts to a weaker (sufficiency) condition that the Taub charges vanish which, as we have seen, results from integrating this equation after contracting with a Killing vector field  $\bar{\xi}^\mu$ . To set the stage for generic gravity theories about their AdS backgrounds, let us study (2.29) in AdS and flat spaces. In that case one can plug an explicit ansatz as follows: assume that such a  $k$  exists in the form

$$k_{\mu\nu} = a h_{\mu\beta} h_\nu^\beta + b h h_{\mu\nu} + \bar{g}_{\mu\nu} (c h_{\alpha\beta}^2 + d h^2), \quad (2.30)$$

where  $k := k_{\mu\nu} \bar{g}^{\mu\nu}$  and  $a, b, c, d$  are constants to be determined and all the raising and lowering is done with the background AdS metric  $\bar{g}$ . Here we shall work in  $D$  spacetime dimensions. Inserting  $k_{\mu\nu}$  as given in (2.30) in  $(\mathcal{G}_{\mu\nu})^{(1)} \cdot k$ , and choosing  $a = 1$  and  $b = -\frac{1}{2}$ , one arrives at

$$(\mathcal{G}_{\mu\nu})^{(2)} \cdot [h, h] + (\mathcal{G}_{\mu\nu})^{(1)} \cdot k =: K_{\mu\nu}, \quad (2.31)$$

where  $K_{\mu\nu}$  is a tensor which must vanish if the background is linearization stable. Its explicit form is worked out in Appendix B. Let us consider the transverse traceless gauge, and make use of the field equations and the linearized field equations: Namely

let us use  $\bar{R}_{\mu\nu} = \frac{2\Lambda}{D-2}\bar{g}_{\mu\nu}$  and  $(\mathcal{G}_{\mu\nu})^{(1)} \cdot h = 0$ , which in this gauge reads  $\bar{\square}h_{\mu\nu} = \frac{4\Lambda}{(D-1)(D-2)}h_{\mu\nu}$  to arrive at

$$K_{\mu\nu} = \bar{\nabla}_\alpha H^\alpha{}_{\mu\nu} + \frac{\Lambda}{D-2} \left( c(D-2) + \frac{1}{2} \right) \bar{g}_{\mu\nu} h_{\alpha\beta}^2 - \frac{1}{4} \bar{\nabla}_\nu h^{\alpha\beta} \bar{\nabla}_\mu h_{\alpha\beta} - \frac{\Lambda D}{(D-1)(D-2)} h_{\mu\beta} h_\nu^\beta, \quad (2.32)$$

where the divergence piece is given as

$$\begin{aligned} H^\alpha{}_{\mu\nu} := & -\frac{1}{4} \bar{g}_{\mu\nu} h_{\sigma\beta} \bar{\nabla}^\beta h^{\sigma\alpha} + \left( c(2-D) - \frac{1}{2} \right) \delta_\nu^\alpha h^{\sigma\beta} \bar{\nabla}_\mu h_{\sigma\beta} \\ & + \left( c(D-2) + \frac{5}{8} \right) \bar{g}_{\mu\nu} h_{\sigma\beta} \bar{\nabla}^\alpha h^{\sigma\beta} \\ & + \frac{1}{2} (h^{\alpha\beta} \bar{\nabla}_\beta h_{\nu\mu} + h_{\beta\nu} \bar{\nabla}_\mu h^{\alpha\beta} + h_{\beta\mu} \bar{\nabla}_\nu h^{\alpha\beta} - h_{\mu\beta} \bar{\nabla}^\alpha h_\nu^\beta - h_{\mu\beta} \bar{\nabla}^\beta h_\nu^\alpha). \end{aligned} \quad (2.33)$$

In the transverse-traceless gauge, the coefficient  $d$  is not fixed and can be set to zero.  $K_{\mu\nu}$  has a single parameter  $c$ , that one can choose to fix the stability of the flat space-time (which was proven by [1] using the linearization of the constraints on a non-compact Cauchy surface in Minkowski space). Before looking at the flat space case, let us note that one has  $\bar{\nabla}_\mu K^{\mu\nu} = 0$  as expected. Let us consider the flat space with  $\Lambda = 0$  and use the Cartesian coordinates so that  $\bar{\nabla}_\alpha \rightarrow \partial_\alpha$ . The corresponding linearized field equations become

$$\partial^2 h_{\mu\nu} = 0, \quad (2.34)$$

together with the gauge choices  $\partial_\mu h^{\mu\nu} = 0 = h$ . The general solution of (2.34) can be exactly constructed as a superposition of plane-wave solutions, hence it suffices to study the linearized stability of flat space against the plane-wave modes which we take to be the real part of

$$h_{\mu\nu} = \varepsilon_{\mu\nu} e^{ik \cdot x}, \quad (2.35)$$

together with  $k^\mu \varepsilon_{\mu\nu} = 0$ ,  $\varepsilon_\mu^\mu = 0$  and  $k^2 = 0$ , which follow from the gauge condition and (2.34). In a compact space without a boundary,  $k = 0$  mode should also be considered, in that case one has the solution  $h_{\mu\nu} = \varepsilon_{\mu\nu} (c_1 t + c_2)$  which gives rise to linearization instability [25] for the case of the torus. Evaluating  $K_{\mu\nu}$  for the solution 2.35, one arrives at

$$K_{\mu\nu} = k_\nu k_\mu \varepsilon_{\alpha\beta} \varepsilon^{\alpha\beta} e^{ik \cdot x} \left( 2c(D-2) + \frac{5}{4} \right), \quad (2.36)$$

which vanishes for the choice

$$c = -\frac{5}{8(D-2)}. \quad (2.37)$$

So (2.29) is satisfied for

$$k_{\mu\nu} = h_{\mu\beta}h_{\nu}^{\beta} - \frac{5}{8(D-2)}\bar{g}_{\mu\nu}h_{\alpha\beta}^2 \quad (2.38)$$

and therefore there is no further constraint on the linearized solutions (2.35) and the Minkowski space is linearization stable. Next we move on to quadratic gravity theory.

#### 2.1.4 Linearization instability beyond Einstein's theory

One of the reasons that lead us to study the linearization instability in generic gravity theories is an observation made in [11] where conserved charges of generic gravity theories for asymptotically AdS backgrounds were constructed.<sup>3</sup>The observation was that in AdS backgrounds, the conserved energy and angular momenta vanish in generic gravity theories for all asymptotically AdS solutions at some particular values of the parameters defining the theory (in fact a whole section in that chapter was devoted for the zero energy issue). This apparent infinite degeneracy of the vacuum for AdS spaces, is in sharp contrast to the flat space case where the unique zero energy is attained only by the Minkowski space, namely the classical ground state. Let us expound upon this a little more: for *all* purely metric based theories, the energy (mass) of the space-time that asymptotically approaches the flat space at spatial infinity is given by the ADM formula [27]

$$M_{ADM} = \frac{1}{\kappa} \oint_{\partial\Sigma} dS_i (\partial_j h^{ij} - \partial^i h^j{}_j). \quad (2.39)$$

It is well-known that  $M_{ADM} \geq 0$ , which is known as the positive energy theorem [28, 29]. An important part of this theorem is that the vacuum, namely the flat space-time with  $M_{ADM} = 0$ , is unique (up to diffeomorphisms of course) [30, 31]. It should be also noted that, the ADM mass is defined in flat Cartesian coordinates but it was shown to be coordinate invariant. Here one must be very careful, if proper decaying conditions are not realized for  $h_{ij}$ , *any* (positive, negative, finite or divergent) value

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<sup>3</sup> For an earlier zero energy result in the context of asymptotically flat backgrounds for purely quadratic gravity in four dimensions, see [26].

of mass can be assigned to the flat space. It is exactly these properties of the ADM formula that made it a useful tool in geometry: without even referring to Einstein's equations, one can take (2.39) to be a geometric invariant of an asymptotically flat manifold, modulo some decaying conditions on the first and the second fundamental forms of the spacelike surface.

Once one deviates from asymptotic flatness, then as we have noted, for higher derivative theories there are critical points which seem to make the vacuum infinitely degenerate, namely, the corresponding mass formula assigns any solution of the theory the same zero charge. Naively, one can try to understand the meaning of vanishing charges for non-vacuum solutions (namely, non-maximally symmetric solutions) as follows:

- There is a confinement of the relevant perturbations (in the weak coupling), just-like in QCD in the strong coupling of color charge; and so a non-vacuum solution such as the proton has zero total color charge, same as the vacuum. In the case of QCD, perturbation theory might yield spurious states that cannot freely exist, such as quarks, as also noted in [15]. In gravity confinement would mean, confinement of mass-energy or some other properties under consideration such as chirality. But this would be highly unphysical because if there are no other conserved charges to suppress the creation of confined mass, then the vacuum state of gravity would be infinitely degenerate and creating confined mass would cost nothing.
- The second possibility is that perturbation theory about a given background solution, be it the maximally symmetric vacuum or not, may simply fail to exist just because the background solution is an isolated solution in the solution space. Namely, the solution space may fail to be a smooth manifold.

In fact, as discussed above, linearization of non-linear equations such as Einstein's gravity and Yang-Mill's theory showed that naive first order perturbation theory fails generically when the background has a Killing symmetry. To be more specific we consider two recent examples: the chiral gravity in 2+1 dimensions which is a special case of topologically massive gravity with a cosmological constant and the critical gravity which is a specific example of quadratic gravity in AdS. These examples can

be easily extended, as the phenomenon we discuss is quite generic and take place whenever Einstein's theory with a cosmological constant is modified with some curvature terms.

To see how perturbation theory can fail let us go back to the necessary condition (2.6) and contract it with the Killing vector  $\bar{\xi}^\mu$  to obtain

$$\bar{\xi}^\mu (\mathcal{E}_{\mu\nu})^{(2)} \cdot [h, h] + \bar{\xi}^\mu (\mathcal{E}_{\mu\nu})^{(1)} \cdot k = 0. \quad (2.40)$$

In some modified gravity theories one finds that the second term can be written as

$$\bar{\xi}^\mu (\mathcal{E}_{\mu\nu})^{(1)} \cdot k = c(\alpha_i, \bar{R}) \bar{\nabla}_\alpha \mathcal{F}_1^{\alpha\nu} + \bar{\nabla}_\alpha \mathcal{F}_2^{\alpha\nu}, \quad (2.41)$$

where  $c(\alpha_i, \bar{R})$  is a constant determined by the parameters  $\alpha_i$  of the theory as well as the curvature invariants (symbolically denoted above as  $\bar{R}$ ) of the background metric.  $\mathcal{F}_i^{\alpha\nu}$  are antisymmetric background tensors. It turns out that for asymptotically AdS spacetimes  $\mathcal{F}_2^{\alpha\nu}$  vanishes identically at the boundary as it involves higher derivative terms of the perturbation, while  $\mathcal{F}_1^{\alpha\nu}$  need not if there are not so fast decaying fields such as for example the Kerr-AdS black holes. On the other hand for the particular choice of the parameters  $c(\alpha_i, \bar{R}) = 0$ , one arrives at the constraint that again the Taub charges must vanish identically

$$Q_{Taub}[\bar{\xi}] = \oint_{\Sigma} d^{D-1} \Sigma \sqrt{\gamma} \bar{\xi}^\mu (\mathcal{E}_{\mu\nu})^{(2)} \cdot [h, h] = 0. \quad (2.42)$$

But this time we have the additional non-trivial equation

$$\oint_{\Sigma} d^{D-1} \Sigma \sqrt{\gamma} \bar{\xi}^\mu (\mathcal{E}_{\mu\nu})^{(1)} \cdot h \neq 0. \quad (2.43)$$

In general it is very hard to satisfy these two conditions simultaneously for all solutions. Therefore some solutions to the linearized equations  $h$  turn out to be not integrable to a full solution, hence the linearization instability of the AdS background in these critical theories. Let us stress that we have not assumed that the Cauchy surfaces are compact: this type of linearization instability arises even in the non-compact case.

### 2.1.5 Linearization instability in quadratic gravity

The message we would like to convey is a rather universal one in all generic higher derivative gravity theories, but for the sake of being concrete and yet sufficiently general, we shall consider the quadratic gravity theory with the action (in  $D$  dimensions)

$$I = \int d^D x \sqrt{-g} \left( \frac{1}{\kappa} (R - 2\Lambda_0) + \alpha R^2 + \beta R_{\mu\nu}^2 + \gamma (R_{\mu\nu\rho\sigma}^2 - 4R_{\mu\nu}^2 + R^2) \right), \quad (2.44)$$

where the last term is organized into the Gauss-Bonnet form, which vanishes identically for  $D = 3$  and becomes a surface term for  $D = 4$ . But for  $D \geq 5$ , it contributes to the field equations with at most second order derivatives in the metric, just like the Einstein-Hilbert part. Conserved gravitational charges of this theory in its asymptotically AdS backgrounds were constructed in [11] following the background space-time techniques developed in [10] which is an extension of the ADM approach [27]. For any theory with a Lagrangian density  $\mathcal{L} = \frac{1}{\kappa} (R - 2\Lambda_0) + f(R_{\sigma\rho}^{\mu\nu})$ , for a generic differentiable function  $f$  of the Riemann tensor and its contractions, the conserved charges follow from those of (2.44), as shown in [24] since any such theory can be written as a quadratic theory with effective coupling constants as far as its energy properties and particle content are concerned [32]. In what follows, we quote some of the computations done in [11] here to make the ensuing discussion complete. The field equations that follow from (2.44) are

$$\begin{aligned} \mathcal{E}_{\mu\nu}[g] &= \frac{1}{\kappa} (R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R) + 2\alpha R (R_{\mu\nu} - \frac{1}{4} g_{\mu\nu} R) \\ &+ (2\alpha + \beta) (g_{\mu\nu} \square - \nabla_\mu \nabla_\nu) R + 2\gamma \left\{ R R_{\mu\nu} - 2R_{\mu\sigma\nu\rho} R^{\sigma\rho} \right. \\ &\left. + R_{\mu\sigma\rho\tau} R_\nu^{\sigma\rho\tau} - 2R_{\mu\sigma} R_\nu^\sigma - \frac{1}{4} g_{\mu\nu} (R_{\tau\lambda\rho\sigma}^2 - 4R_{\sigma\rho}^2 + R^2) \right\} \\ &+ \beta \square (R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R) + 2\beta (R_{\mu\sigma\nu\rho} - \frac{1}{4} g_{\mu\nu} R_{\sigma\rho}) R^{\sigma\rho} = 0. \end{aligned} \quad (2.45)$$

As we shall study the stability/instability of the non-flat maximally symmetric solution (or solutions), let  $\bar{g}$  represent such a solution with the curvature tensors normalized as

$$\bar{R}_{\mu\rho\nu\sigma} = \frac{2\Lambda}{(D-1)(D-2)} (\bar{g}_{\mu\nu} \bar{g}_{\rho\sigma} - \bar{g}_{\mu\sigma} \bar{g}_{\rho\nu}), \quad (2.46)$$

$$\bar{R}_{\mu\nu} = \frac{2\Lambda}{D-2} \bar{g}_{\mu\nu}, \quad (2.47)$$

$$\bar{R} = \frac{2D\Lambda}{D-2}. \quad (2.48)$$

The field equations reduce to a single quadratic equation :

$$\frac{\Lambda - \Lambda_0}{2\kappa} + k\Lambda^2 = 0, \quad k \equiv (D\alpha + \beta) \frac{(D-4)}{(D-2)^2} + \gamma \frac{(D-3)(D-4)}{(D-1)(D-2)}. \quad (2.49)$$

For generic values of the parameters of the theory, of course, there may not be real solution and so the theory may not posses a maximally symmetric vacuum, but here we assume that there is a real solution to this algebraic equation (so  $8\Lambda_0 k \kappa + 1 \geq 0$ ) and study the linearization stability of this solution, which we call the (classical) vacuum or the background. One can then linearize the field equations (2.45) about the vacuum and get at the linear order

$$c_1 (\mathcal{G}_{\mu\nu})^{(1)} + (2\alpha + \beta) \left( \bar{g}_{\mu\nu} \bar{\square} - \bar{\nabla}_\mu \bar{\nabla}_\nu + \frac{2\Lambda}{D-2} \bar{g}_{\mu\nu} \right) (R)^{(1)} + \beta \left( \bar{\square} (\mathcal{G}_{\mu\nu})^{(1)} - \frac{2\Lambda}{D-1} \bar{g}_{\mu\nu} (R)^{(1)} \right) = 0, \quad (2.50)$$

where the constant in front of the first term is

$$c_1 \equiv \frac{1}{\kappa} + \frac{4\Lambda D}{D-2} \alpha + \frac{4\Lambda}{D-1} \beta + \frac{4\Lambda(D-3)(D-4)}{(D-1)(D-2)} \gamma, \quad (2.51)$$

and the linearized (background) tensors read

$$(\mathcal{G}_{\mu\nu})^{(1)} = (R_{\mu\nu})^{(1)} - \frac{1}{2} \bar{g}_{\mu\nu} (R)^{(1)} - \frac{2\Lambda}{D-2} h_{\mu\nu}, \quad (2.52)$$

which is just the linearized cosmological Einstein's tensor given in terms of the linearized Ricci tensor and the linearized scalar curvature :

$$\begin{aligned} (R_{\mu\nu})^{(1)} &= \frac{1}{2} \left( \bar{\nabla}^\sigma \bar{\nabla}_\mu h_{\nu\sigma} + \bar{\nabla}^\sigma \bar{\nabla}_\nu h_{\mu\sigma} - \bar{\square} h_{\mu\nu} - \bar{\nabla}_\mu \bar{\nabla}_\nu h \right), \\ (R)^{(1)} &= -\bar{\square} h + \bar{\nabla}^\sigma \bar{\nabla}^\mu h_{\sigma\mu} - \frac{2\Lambda}{D-2} h. \end{aligned} \quad (2.53)$$

Given a background Killing vector  $\bar{\xi}$ , (there are  $D(D+1)/2$  number of Killing vectors for this space and the arguments work for any one of these) if we had not truncated the expansion of the field equations at  $\mathcal{O}(h)$  but collected all the non-linear terms on the right-hand side, we would have gotten

$$\bar{\xi}^\mu (\mathcal{E}_{\mu\nu})^{(1)} \cdot h := \bar{\xi}^\mu T_{\mu\nu} [h^2, h^3, \dots h^n \dots]. \quad (2.54)$$

where  $T_{\mu\nu}[h^2, h^3, \dots h^n \dots]$  represents all the higher order terms (and if there is a matter source with compact support of energy-momentum tensor, it also includes that). The next step is the crucial step: as was shown in [11], one can write (2.54) as a divergence of two pieces as described by (2.41)

$$\bar{\xi}^\mu (\mathcal{E}_{\mu\nu})^{(1)} \cdot h = c \bar{\nabla}_\alpha \mathcal{F}_1^\alpha{}_\nu + \bar{\nabla}_\alpha \mathcal{F}_2^\alpha{}_\nu, \quad (2.55)$$

where the constant  $c_1$  given in (2.51) is shifted due to the  $\beta$  term as

$$c \equiv c_1 + \frac{4\Lambda}{(D-1)(D-2)}\beta. \quad (2.56)$$

The explicit forms of the  $\mathcal{F}_i^{\mu\rho}$  tensors are found to be

$$\mathcal{F}_1^{\mu\rho} = 2\bar{\xi}_\nu \bar{\nabla}^{[\mu} h^{\rho]\nu} + 2\bar{\xi}^{[\mu} \bar{\nabla}^{\rho]} h + 2h^{\nu[\mu} \bar{\nabla}^{\rho]} \bar{\xi}_\nu + 2\bar{\xi}^{[\rho} \bar{\nabla}_\nu h^{\mu]\nu} + h \bar{\nabla}^\mu \bar{\xi}^\rho, \quad (2.57)$$

and

$$\begin{aligned} \mathcal{F}_2^{\mu\rho} = & (2\alpha + \beta) \left( 2\bar{\xi}^{[\mu} \bar{\nabla}^{\rho]} (R)^{(1)} + (R)^{(1)} \bar{\nabla}^\mu \bar{\xi}^\rho \right) \\ & + 2\beta \left( \bar{\xi}^\sigma \bar{\nabla}^{[\rho} (\mathcal{G}^{\mu]}{}_\sigma)^{(1)} + (\mathcal{G}^{[\rho\sigma])^{(1)}} \bar{\nabla}^{\mu]} \bar{\xi}_\sigma \right). \end{aligned} \quad (2.58)$$

For asymptotically AdS spacetimes,  $\mathcal{F}_2^{\mu\rho}$  vanishes at spatial infinity due to the vanishing of both of  $(R)^{(1)}$  and  $(\mathcal{G}_{\mu\sigma})^{(1)}$ . As discussed in the previous section, vanishing of the constant  $c$  leads to two strong constraints (2.42) and (2.43) on the linearized solution  $h$  which is a statement of the instability of the background solution. Note that, for this higher order theory, we have not assumed that the spatial hypersurface is compact. (In fact, to be more accurate, AdS is not globally hyperbolic and does not have a Cauchy surface but one can work in the double cover which does).

The point at which  $c = 0$  is the point when the mass of the spin-2 massive mode also vanishes and further, assuming  $4\alpha(D-1) + D\beta = 0$ , one can also decouple the massive spin-0 mode in this theory and arrive at the so called *critical gravity* defined in  $D = 4$  [21] for generic  $D$  in [22]. All these conditions are compatible with the existence of a maximally symmetric vacuum. For critical gravity, the apparent mass and angular momenta of all black holes and perturbative excitations with asymptotically AdS conditions vanish.<sup>4</sup> But as we have seen here, perturbation theory used for both

<sup>4</sup> The energy of the perturbative bulk excitations can be constructed using the Ostrogradsky Hamiltonian [22].

the excitations and construction of conserved quantities does not work exactly at the critical point: namely, the theory for the AdS background is not linearization stable. At the chiral point, there arise exact log-modes in chiral gravity [33, 34] which are of the wave type but they do not correspond to the linearized log-modes of [19].

Just for the sake of completeness, let us note that if  $c \neq 0$ , then the perturbation theory makes sense and the conserved charges of the theory for any asymptotically AdS solutions (such as the Kerr-AdS black holes) are simply given in terms of the conserved charges of the same solution in Einstein's gravity as

$$\frac{Q_{\text{quad}}(\bar{\xi})}{Q_{\text{Einstein}}(\bar{\xi})} = -\beta m_g^2, \quad (2.59)$$

where  $m_g$  is the mass of the spin-2 graviton given as

$$-\beta m_g^2 = \frac{1}{\kappa} + \frac{4\Lambda(D\alpha + \beta)}{D-2} + \frac{4\Lambda(D-3)(D-4)}{(D-1)(D-2)}\gamma. \quad (2.60)$$

In (2.59),  $Q_{\text{Einstein}}(\bar{\xi})$  refers to (with  $\kappa_{\text{Newton}} = 1$ ) the conserved charge (mass, angular momenta) in the cosmological Einstein's theory.

### 2.1.6 Linearization instability in chiral gravity

A model of quantum gravity even in the simpler  $2 + 1$  dimensional setting has been rather elusive. One of the latest promising proposals was the so called chiral gravity [14] which is a specific limit of topologically massive gravity (TMG) [18] with the asymptotically AdS boundary conditions. TMG, as opposed to Einstein's gravity has non-trivial local dynamics hence in this respect, it might be more relevant to the four dimensional gravity both at the classical and quantum level. The crux of the arguments of the quantum version chiral gravity is that the bulk theory is dual to a unitary and chiral conformal field theory (CFT) on the two dimensional boundary, whose symmetry is known to be one of the two copies of the Virasoro algebra [20]. Finding the correct conformal field theory would amount to defining the quantum gravity via the AdS/CFT duality [35]. But immediately after the proposal of chiral gravity, it was realized that the theory has arbitrarily negative energy log modes that appear exactly at the chiral point and not only the dual CFT is not unitary (but a logarithmic one), but apparently chiral gravity does not have even a classical vacuum [19]. If true, this of

course would be disastrous for chiral gravity. But later it was argued in [15, 17] that chiral gravity has linearization instability against these log modes in AdS: namely, these perturbative negative energy solutions do not actually come from the linearization of any exact solution. If that is the case, then linearization instability saves chiral gravity certainly at the classical level and perhaps at the quantum level. Here we give further arguments of the existence of linearization instability in chiral gravity.

The field equations of topologically massive gravity [18] with a negative cosmological constant ( $\Lambda := -\frac{1}{\ell^2}$ ) is

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - \frac{1}{\ell^2}g_{\mu\nu} + \frac{1}{\mu}C_{\mu\nu} = 0, \quad (2.61)$$

where the Cotton tensor in terms of the anti-symmetric tensor and the covariant derivative of the Schouten tensor reads

$$C_{\mu\nu} = \eta_{\mu}{}^{\alpha\beta}\nabla_{\alpha}S_{\beta\nu}, \quad S_{\mu\nu} = R_{\mu\nu} - \frac{1}{4}g_{\mu\nu}R. \quad (2.62)$$

The boundary theory has two copies of the Virasoro algebra [20] for asymptotically AdS boundary conditions given as

$$c_{R/L} = \frac{3l}{2G_3} \left( 1 \pm \frac{1}{\mu l} \right), \quad (2.63)$$

and the bulk theory has a single helicity 2 mode with a mass-square

$$m_g^2 = \mu^2 - \frac{1}{l^2}. \quad (2.64)$$

It was shown in [36] that the contraction of the Killing vector ( $\bar{\xi}$ ) with the linearized equations coming from (2.61) yields

$$\bar{\xi}^{\mu} \left( (\mathcal{G}_{\mu\nu})^{(1)} + \frac{1}{\mu}(C_{\mu\nu})^{(1)} \right) = \bar{\nabla}_{\alpha}\mathcal{F}_1^{\alpha}{}_{\nu}[\bar{\Xi}] + \bar{\nabla}_{\alpha}\mathcal{F}_3^{\alpha}{}_{\nu}[\bar{\xi}], \quad (2.65)$$

where  $\mathcal{F}_1^{\mu\rho}$  was given in (2.57) whereas one finds  $\mathcal{F}_3^{\mu\rho}$  to be

$$\mathcal{F}_3^{\mu\rho}[\bar{\xi}] = \eta^{\mu\rho\beta} (\mathcal{G}_{\nu\beta})^{(1)} \bar{\xi}^{\nu} + \eta^{\nu\rho\beta} (\mathcal{G}^{\mu}{}_{\beta})^{(1)} \bar{\xi}_{\nu} + \eta^{\mu\nu\beta} (\mathcal{G}^{\rho}{}_{\beta})^{(1)} \bar{\xi}_{\nu}, \quad (2.66)$$

where a new (twisted) Killing vector ( $\bar{\Xi}$ ) appears:

$$\bar{\Xi}^{\alpha} := \bar{\xi}^{\alpha} + \frac{1}{2\mu}\eta^{\alpha\beta\nu}\bar{\nabla}_{\beta}\bar{\xi}_{\nu}. \quad (2.67)$$

The conserved charges of TMG for asymptotically AdS backgrounds read as an integral over the circle at infinity as

$$Q[\bar{\xi}] = \frac{1}{8\pi G_3} \oint_{\partial\mathcal{M}} dS_i \left( \mathcal{F}_1^{0i}[\bar{\Xi}] + \frac{1}{2\mu} \mathcal{F}_3^{0i}[\bar{\xi}] \right). \quad (2.68)$$

Once again for the asymptotically AdS cases  $\mathcal{F}_3^\alpha{}_\nu[\bar{\xi}]$  vanishes identically on the boundary as it involves the linearized Einstein tensor at infinity. For generic values of  $\mu$  and  $\ell$ , the first term, that is  $\mathcal{F}_1^\alpha{}_\nu[\bar{\Xi}]$  gives the conserved charges for the corresponding Killing vector. But, for  $\mu^2\ell^2 = 1$ , as was shown in [37] the angular momentum and the energy of the rotating black hole solutions with the rotation parameter ( $j$ ) and the mass ( $m$ ) related as ( $j = m\ell$ ) (the extremal BTZ black hole) vanishes identically. This particular point was further studied in [14] where it was argued and conjectured that the theory, so called *chiral* gravity, as one of the central charges noted above (2.63) becomes zero, makes sense both classically and quantum mechanically.

Classically the theory should have a stable vacuum and quantum mechanically, it should have a dual healthy boundary conformal field theory. In [14] it was shown that all the bulk excitations have vanishing energy exactly at the chiral point. Later new log modes that were not accounted for were found in [19] which violated the existence of a ground state (namely, these modes have arbitrarily large negative energy compared to the zero energy of the vacuum). For further work on chiral gravity, see [38,39]. In [15] and [17] it was argued that the AdS has linearization instability in chiral gravity against these log modes. Here, our construction lends support to these arguments.

For the sake of concreteness, let us consider the background metric as

$$\bar{g} = - \left( 1 + \frac{r^2}{\ell^2} \right) dt^2 + \frac{dr^2}{1 + \frac{r^2}{\ell^2}} + r^2 d\phi^2, \quad (2.69)$$

then for  $\bar{\xi} = (-1, 0, 0)$  referring to the time-like energy Killing vector, one finds the twisted Killing vector to be

$$\bar{\Xi} = \left( -1, 0, -\frac{1}{\ell^2\mu} \right). \quad (2.70)$$

For this  $\bar{\Xi}$  to be a time-like Killing vector for *all*  $r$  including the boundary at  $r \rightarrow \infty$ , one can see that (excluding the trivial  $\mu \rightarrow \infty$  case) one must set  $\mu^2\ell^2 = 1$ , which is

the chiral gravity limit. To further see this chiral gravity limit, let us recast  $\mathcal{F}_1^{\mu\rho}[\Xi]$  using the the superpotential  $\mathcal{K}^{\mu\alpha\nu\beta}$  is defined by [10]

$$\mathcal{K}^{\mu\nu\alpha\beta} := \frac{1}{2}(\bar{g}^{\mu\beta}\tilde{h}^{\nu\alpha} + \bar{g}^{\nu\alpha}\tilde{h}^{\mu\beta} - \bar{g}^{\mu\nu}\tilde{h}^{\alpha\beta} - \bar{g}^{\alpha\beta}\tilde{h}^{\mu\nu}), \quad \tilde{h}^{\mu\nu} := h^{\mu\nu} - \frac{1}{2}\bar{g}^{\mu\nu}h, \quad (2.71)$$

which yields

$$\mathcal{F}_1^{\mu\rho}[\Xi] = \Xi_\nu \bar{\nabla}_\beta \mathcal{K}^{\mu\rho\nu\beta} - \mathcal{K}^{\mu\sigma\nu\rho} \bar{\nabla}_\sigma \Xi_\nu. \quad (2.72)$$

For all asymptotically AdS solutions with the Brown-Henneaux boundary conditions, one can show that

$$\mathcal{F}_1^{\mu\rho}[\Xi] = \left(1 - \frac{1}{\ell^2\mu^2}\right) \mathcal{F}_1^{\mu\rho}[\bar{\xi}], \quad (2.73)$$

which vanishes at the chiral point. So exactly at this point, there exists second order integral constraints on the linearized solutions as discussed in the previous section. The log-modes of [19] do not satisfy these integral constraints and so fail to be integrable to full solutions.<sup>5</sup>

Let us compute the value of the Taub conserved quantity for the log solution which was given in the background with the global coordinates for which the metric reads

$$ds^2 = \ell^2(-\cosh^2 \rho d\tau^2 + \sinh^2 \rho d\phi^2 + d\rho^2). \quad (2.74)$$

For the coordinates  $u = \tau + \phi$ ,  $v = \tau - \phi$ , at exactly in the chiral point, one has the following additional solution

$$\begin{aligned} h_{\mu\nu} = & -\tanh^2 \rho (\sin(2u)\tau + \cos(2u) \ln \cosh \rho) \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & -4 \sinh^{-2} 2\rho \end{pmatrix}_{\mu\nu} \\ & + \frac{\sinh \rho}{\cosh^3 \rho} (\cos(2u)\tau - \sin(2u) \ln \cosh \rho) \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}_{\mu\nu} \end{aligned} \quad (2.75)$$

Considering the Killing vector  $\bar{\xi} = (-1, 0, 0)$  one finds the result of the integral in (2.13) to be non-vanishing

$$Q_{Taub}[\bar{\xi}] = \frac{\pi}{2\ell} \left(3\tau^2 - \frac{161}{72}\right), \quad (2.76)$$

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<sup>5</sup> See [40] a nice compilation of possible applications of logarithmic field theories in the context of holography and gravity.

which shows that this log mode is not in the tangent space of the solution space of chiral gravity around the  $AdS_3$  metric.

In this chapter, we have shown that at certain *critical parameter* values of extended gravity theories in constant curvature backgrounds, perturbation theory fails. Our arguments provide support to the discussion given by [15, 17] regarding the linearization instability in three dimensional chiral gravity and extend the discussion to generic gravity theories in a somewhat former form. The crucial point is that even in spacetimes with non-compact Cauchy surfaces, linearization instability can exist for background metrics with at least one Killing vector field. Our computation also sheds light on the earlier observations [11] that at certain critical values of the parameters defining the theory, conserved charges of all solutions, such as black holes, excitations vanish identically.<sup>6</sup> For example, Kerr-AdS black hole metrics have the same mass and angular momentum as the AdS background. This leads to a rather non-physical infinite degeneracy of the vacuum: for example, creating back holes costs nothing which is unacceptable. With our discussion above, it is now clear that, perturbation theory which is used to define boundary integrals of the conserved Killing charges does not make sense exactly at the critical values of the parameters. Therefore one really needs a new method to find/define conserved charges in these theories at their critical points. One such method was proposed in for quadratic theories [42] and in [43] for TMG.

We must note that, for asymptotically flat spacetimes, the ADM mass is the correct definition of mass-energy for any metric-based theory of gravity. Therefore, the stability of the Minkowski space as was shown for Einstein's theory by Choquet-Bruhat and Deser [1] is valid for all higher derivative models as long as one considers the non-compact Cauchy surfaces and asymptotically flat boundary conditions. But once a cosmological constant is introduced, the problem changes dramatically as we have shown: the ADM mass-energy (or angular momentum) expressions are modified and conserved charges get contributions from each covariant tensors added to the field equations. Once such a construction is understood, it is clear that some theories will have identically vanishing charges for all solutions with some fixed boundary conditions, which is a signal of linearization instability.

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<sup>6</sup> For a recent review of conserved charges in generic gravity theories see the book [41].

It is also important to realize that, linearization instability of certain background solutions in some theories is not bad as it sounds: for example chiral gravity is a candidate both as a non-trivial classical and quantum gravity theory in  $AdS_3$  with a two dimensional chiral conformal field theory induced on the boundary. But it has log-mode solutions which appear as ghosts in the classical theory and negative norm states in the quantum theory. It just turns out that chiral gravity in  $AdS_3$  has linearization instability along these log-modes: namely, they do not have vanishing Taub conserved quantities which is a constraint for all integrable solutions. Therefore, they cannot come from linearization of exact solutions. A similar phenomenon takes place for the minimal massive gravity [44] which was proposed as a possible solution to the bulk-boundary unitarity clash in three dimensional gravity theories and as a viable model that has a healthy dual conformal field theory on the boundary of  $AdS_3$ . It was shown recently in [45] that this theory only makes sense at the chiral point [46, 47] and hence linearization instability arises at that point which can save the theory from its log-modes. Let us note that we have also computed the second order constraint in the minimal massive gravity, namely the Taub conserved quantity and found that it is non-vanishing.

In the discussion of linearization stability and instability of a given exact solution in the context of general relativity, we noted that to make use of the powerful techniques of elliptic operator theory, one rewrites the four dimensional Einstein's theory as a dynamical system with constraints on a spacelike Cauchy surface and the evolution equations. As the constraints are intact, initial Cauchy data uniquely defines a spacetime (modulo some technical assumptions). Therefore, to study the linearization stability one can simply study the linearization stability of the constraints on the surface where the metric tensor field is positive-definite. All these arguments boil down to showing that the initial background metric is not a singular point and that the space of solutions around the initial metric is an open subset (in fact a submanifold) of all solutions. This can be shown by proving the surjectivity of the operators that appear in the linearized constraints. A similar construction, dynamical formulation of the higher derivative models studied here in AdS and the surjectivity of the relevant linear maps would be highly valuable. For the case of the cosmological Einstein's theory, such a construction was carried out in [48] where it was observed that certain

strong decays lead to linearization instability even for non-compact Cauchy surfaces with hyperbolic asymptotics.



## CHAPTER 3

### LINEARIZATION INSTABILITY IN CHIRAL GRAVITY

Quantum gravity <sup>1</sup> is elusive not mainly because we lack computational tools, but because we do not know *what* to compute and so how to define the theory for a generic spacetime. One possible exception and a promising path is the case of asymptotically anti-de Sitter (AdS) spacetimes for which a dual quantum conformal field theory that lives on the boundary of a bulk spacetime with gravity would amount to a definition of quantum gravity. But, even for this setting, we do not have a realistic four dimensional example. In three dimensions, the situation is slightly better: the cosmological Einstein's theory (with  $\Lambda < 0$ ) has a black hole solution [49] and possesses the right boundary symmetries (a double copy of the centrally extended Virasoro algebra [20]) for a unitary two dimensional conformal field theory. But as the theory has no local dynamics (namely gravitons), it is not clear exactly how much one can learn from this model as far as quantum gravity is concerned. Having said that, even for this ostensibly simple model, we still do not yet have a quantum gravity theory. Recasting Einstein's gravity in terms of a solvable Chern-Simons gauge theory is a possible avenue [50], but this only works for non-invertible dreibein which cannot be coupled to generic matter.

A more realistic gravity in three dimensions is the topologically massive gravity (TMG) [18] which has black hole solutions as well as a dynamical massive graviton. But the apparent problem with TMG is that the bulk graviton and the black hole cannot be made to have positive energy generally. This obstruction to a viable classical and perhaps quantum theory was observed to disappear in an important work [14], where it was realized that at a "chiral point" defined by a tuned topological mass in terms of the AdS radius, one of the Virasoro algebras has a vanishing central charge

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<sup>1</sup> This chapter appeared as Phys. Rev. D 97, 124068 on 27 June 2018.

(and so admits a trivial unitary representation) and the other has a positive nonzero central charge with unitary nontrivial representations, the theory has a positive energy black hole and zero energy bulk gravitons. This tuned version of TMG, called "chiral gravity", seems to be a viable candidate for a well-behaved classical and quantum gravity.

One of the main objections raised against the chiral gravity is that it possesses a negative energy perturbative log-mode about the AdS vacuum which ruins the unitarity of the putative boundary CFT [19]. Of course if this is the case, chiral gravity is not even viable at the classical level, since it does not have a vacuum. It was argued in [15, 17] that chiral gravity could survive if the theory is linearization unstable about its AdS solution. This means that there would be perturbative modes which cannot be obtained from any exact solution of the theory. In fact, these arguments were supported with the computations given in [4] where it was shown that the Taub charges which are functionals quadratic in the perturbative modes that must vanish identically due to background diffeomorphism invariance, do not vanish for the log-mode that ruins the chiral gravity. This means that the log-mode found from the linearized field equations is an artifact of the linearized equations and does not satisfy the global constraints coming from the Bianchi identities.

In this part, we give a direct proof of the linearization instability of chiral gravity in AdS using the constraint analysis of the full TMG equations defined on a spacelike hypersurface. The crux of the argument that we shall lay out below is the following: the linearized constraint equations of TMG show that there are inconsistencies exactly at the chiral point. Namely perturbed matter fields do not determine the perturbations of the metric components on the spacelike hypersurface and there are unphysical constraints on matter perturbations besides the usual covariant conservation.

To support our local analysis on the hypersurface, we compute the symplectic structure (that carries all the information about the phase space of the theory) for all perturbative solutions of the linearized field equations and find that the symplectic 2-form is degenerate and so non-invertible hence these modes do not approximate (*i.e.* they are not tangent to) actual nonlinear solutions. The symplectic 2-form evaluated for the log-mode is time-dependent (hence not coordinate-invariant) and vanishes at the

initial value surface and grows unbounded in the future.

To carry out the constraint analysis and their linearizations (which will yield possible nearby solutions to exact solution), we shall use the field equations instead of the TMG action as the latter is not diffeomorphism invariant which complicates the discussion via the introduction of tensor densities (momenta) instead of tensors. We shall also work in the metric formulation instead of the first order one as there can be significant differences between the two formulations. Before we indulge into the analysis, let us note that the linearization instability that arises in the perturbative treatment of nonlinear theories and can be confused with dynamical or structural instability, as both are determined with the same linearization techniques. The difference is important: the latter refers to a real instability of a system such as the instability of the vacuum in a theory with ghosts such as the  $R + \beta R_{\mu\nu}^2$  theory with  $\beta \neq 0$ , this is simply not physically acceptable. On the other hand linearization instability refers to the failure of perturbation theory for a given background solution and one should resort to another method to proceed. From the point of view of the full solution space of the theory, this means that this (possibly infinite dimensional) space is not a smooth manifold but it has conical singularities around certain solutions. Let us expound on this a little bit.

### 3.1 ADM decomposition of TMG

Before restricting to the chiral gravity limit, we first study the full TMG field equations coupled with matter fields as an initial value problem, hence we take

$$\mathcal{E}_{\mu\nu} = G_{\mu\nu} + \Lambda g_{\mu\nu} + \frac{1}{\mu} C_{\mu\nu} = \kappa \tau_{\mu\nu}. \quad (3.1)$$

The ADM [27] decomposition of the metric reads

$$ds^2 = -(n^2 - n_i n^i) dt^2 + 2n_i dt dx^i + \gamma_{ij} dx^i dx^j, \quad (3.2)$$

where  $(n, n_i)$  are lapse and shift functions and  $\gamma_{ij}$  is the  $2D$  spatial metric. From now on, the Greek indices will run over the full spacetime, while the Latin indices will run over the hypersurface  $\Sigma$ , as  $i, j, \dots = 1, 2$ . The spatial indices will be raised and lowered by the  $2D$  metric. The extrinsic curvature  $(k_{ij})$  of the surface is given as

$$2nk_{ij} = \dot{\gamma}_{ij} - 2D_{(i} n_{j)}, \quad (3.3)$$

where  $D$  is the covariant derivative compatible with  $\gamma_{ij}$  and  $\dot{\gamma}_{ij} := \partial_0 \gamma_{ij}$  and the round brackets denote symmetrization with a factor of  $1/2$ . With the convention  $R_{\rho\sigma} = \partial_\mu \Gamma_{\rho\sigma}^\mu - \partial_\rho \Gamma_{\mu\sigma}^\mu + \Gamma_{\mu\nu}^\mu \Gamma_{\rho\sigma}^\nu - \Gamma_{\sigma\nu}^\mu \Gamma_{\mu\rho}^\nu$ , one finds the hypersurface components of the three dimensional Ricci tensor as

$$R_{ij} = {}^{(2)}R_{ij} + k k_{ij} - 2k_{ik} k_j^k + \frac{1}{n}(\dot{k}_{ij} - n^k D_k k_{ij} - D_i \partial_j n - 2k_{k(i} D_{j)} n^k), \quad (3.4)$$

where  ${}^{(2)}R_{ij}$  is the Ricci tensor of the hypersurface and  $k \equiv \gamma^{ij} k_{ij}$ . Similarly one find the twice projection to the normal of the surface as

$$R_{00} = \frac{n^i n^j}{n}(\dot{k}_{ij} - n^k D_k k_{ij} - D_i \partial_j n - 2k_{kj} D_i n^k) - n^2 k_{ij}^2 + n^i n^j ({}^{(2)}R_{ij} + k k_{ij} - 2k_{ik} k_j^k) + n(D_k \partial^k n - \dot{k} - n^k D_k k + 2n^k D_m k_k^m). \quad (3.5)$$

On the other hand, projecting once to the surface and once normal to the surface yields

$$R_{0i} = \frac{n^j}{n}(\dot{k}_{ij} - n^k D_k k_{ij} - D_i \partial_j n - 2k_{k(i} D_{j)} n^k) + n^j ({}^{(2)}R_{ij} + k k_{ij} - 2k_{ik} k_j^k) + n(D_i k + D_m k_i^m). \quad (3.6)$$

We also need the 3D scalar curvature in terms of the hypersurface quantities which can be found as

$$R = {}^{(2)}R + k^2 - k_{ij}^2 + \frac{2}{n}(\dot{k} + n k_{ij}^2 - D_i D^i n - n^i D_i k). \quad (3.7)$$

Given the Schouten tensor  $S_{\mu\nu} := R_{\mu\nu} - \frac{1}{4} R g_{\mu\nu}$ , the Cotton tensor is defined as

$$C_{\mu\nu} := \frac{1}{2} \epsilon^{\rho\alpha\beta} (g_{\mu\rho} \nabla_\alpha S_{\beta\nu} + g_{\nu\rho} \nabla_\alpha S_{\beta\mu}), \quad (3.8)$$

where  $\epsilon^{\rho\alpha\beta}$  is the totally antisymmetric tensor which splits as  $\epsilon^{0mn} = \frac{1}{n} \epsilon^{mn} = \frac{1}{n} \gamma^{-\frac{1}{2}} \epsilon^{mn}$  where  $\epsilon^{mn}$  is the antisymmetric symbol. Just as we have done the ADM decomposition of the Ricci tensor, a rather lengthy computation yields the following expressions,

for the projections of the Cotton tensor

$$\begin{aligned}
2nC_{ij} = & \epsilon^{mn}n_i(D_m S_{nj} - k_{mj}(D_r k_n^r - \partial_n k)) \\
& + \epsilon^m{}_i \left\{ \dot{S}_{mj} - nk_j^k S_{mk} - S_{mk} D_j n^k \right. \\
& - (\partial_j n + n^r k_{rj})(D_s k_m^s - \partial_m k) \\
& - D_m(n^r S_{rj} + n(D_r k_j^r - D_j k)) \\
& \left. + k_{mj}(D_k \partial^k n - \dot{k} + n^k D_s k_k^s + n(\frac{R}{4} - k_{rs}^2)) \right\} \\
& + i \leftrightarrow j,
\end{aligned} \tag{3.9}$$

and

$$C_{i0} = n^j C_{ij} - \frac{\epsilon^{mn}}{2}(nA_{mni} - n_i B_{mn} - \gamma_{in}(C_m + nE_m)) \tag{3.10}$$

and

$$\begin{aligned}
C_{00} = & n^i n^j C_{ij} \\
& - \epsilon^{mn}(nn^i A_{mni} - (n_i n^i - n^2)B_{mn} - n_n(C_m + nE_m)),
\end{aligned} \tag{3.11}$$

where we have defined the following tensors

$$A_{mni} \equiv D_m S_{ni} - k_{mi}(D_r k_n^r - \partial_n k), \tag{3.12}$$

$$B_{mn} \equiv D_m D_r k_n^r - k_m^k S_{kn}, \tag{3.13}$$

$$E_m \equiv 2k_{rs} D_m k^{rs} - \frac{1}{4} \partial_m R + k_m^k (D_r k_k^r - \partial_k k), \tag{3.14}$$

$$\begin{aligned}
C_m \equiv & \partial_0 D_r k_m^r - S_m^k (\partial_k n + n^r k_{rk}) - D_m D_k \partial^k n \\
& - D_m (n^k D_s k_k^s) + k_m^k S_{kr} n^r + \partial_m n (k_{rs}^2 - \frac{R}{4}).
\end{aligned} \tag{3.15}$$

Using the above decomposition, we can recast the ADM form of the full TMG equations as

$$\mathcal{E}_{ij} = \kappa \tau_{ij} = S_{ij} - \frac{1}{4} \gamma_{ij} R + \Lambda \gamma_{ij} + \frac{1}{\mu} C_{ij} \tag{3.16}$$

and

$$\begin{aligned}
\mathcal{E}_{0i} = & \kappa \tau_{0i} = n^j \mathcal{E}_{ij} + n(D_r k_i^r - \partial_i k) \\
& - \frac{1}{2\mu} \epsilon^{mn}(nA_{mni} - n_i B_{mn} - \gamma_{in}(C_m + nE_m))
\end{aligned} \tag{3.17}$$

and

$$\begin{aligned}\mathcal{E}_{00} = \kappa\tau_{00} &= 2n^i \mathcal{E}_{0i} - n^i n^j \mathcal{E}_{ij} - \Lambda n^2 - \frac{1}{\mu} \epsilon^{mn} n^2 B_{mn} \\ &+ n(D_k \partial^k n - \dot{k} + n^k D_k k + n(\frac{R}{2} - k_{rs}^2)).\end{aligned}\quad (3.18)$$

From  $\mathcal{E}_{0i}$ , we get the momentum constraint as

$$\begin{aligned}\Phi_i &= \kappa(\tau_{0i} - n^j \tau_{ij}) = n(D_r k_i^r - \partial_i k) \\ &+ \frac{1}{2\mu} \epsilon^{mn} (n_i B_{mn} - n A_{mni} + \gamma_{in} C_m + n \gamma_{in} E_m)\end{aligned}\quad (3.19)$$

and from  $\mathcal{E}_{00}$  we get the Hamiltonian constraint as

$$\begin{aligned}\Phi &= \frac{\kappa}{n^2} (\tau_{00} - 2n^i \tau_{0i} + n^i n^j \tau_{ij}) \\ &+ \frac{1}{2} ({}^{(2)}R + k^2 - k_{ij}^2 - 2\Lambda) \\ &- \frac{1}{\mu} \epsilon^{mn} (D_m D_r k_n^r - k_m^k S_{kn}),\end{aligned}\quad (3.20)$$

where in the last equation we made use of the explicit form of  $R$  given in (3.7) which for TMG is  $R = 6\Lambda - 2\kappa\tau$ . From now on, for our purposes, it will suffice to work in the Gaussian normal coordinates with  $n = 1$  and  $n_i = 0$  for which  $k_{ij} = \frac{1}{2} \dot{\gamma}_{ij}$  and the constraints reduce to

$$\begin{aligned}&\frac{\epsilon^{mn}}{4\mu} (\dot{\gamma}_{im} \dot{\gamma}^{ik} ({}^{(2)}R_{kn} - \dot{\gamma}_{kp} \dot{\gamma}_{sn} \dot{\gamma}^{ps} - \ddot{\gamma}_{kn}) - 2D_m D^k \dot{\gamma}_{kn}) \\ &- \frac{1}{8} \dot{\gamma}_{ij} (\dot{\gamma}_{ab} \dot{\gamma}^{ab} \dot{\gamma}^{ij} + \dot{\gamma}^{ij}) = \kappa\tau_{00} + \Lambda - \frac{{}^{(2)}R}{2}\end{aligned}\quad (3.21)$$

and

$$\begin{aligned}&\frac{\epsilon^m{}_i}{8\mu} (\dot{\gamma}^{kp} (2D_k \dot{\gamma}_{pm} - D_m \dot{\gamma}_{kp}) + 2D^k \ddot{\gamma}_{km} - \dot{\gamma}_{mk} \dot{\gamma}^{kl} D^p \dot{\gamma}_{pl}) \\ &- \frac{\epsilon^{mn}}{8\mu} \left( \dot{\gamma}_{ab} \dot{\gamma}^{ab} D_m \dot{\gamma}_{in} - 2\dot{\gamma}^{ks} D_m (\dot{\gamma}_{kn} \dot{\gamma}_{si}) \right. \\ &\left. + 2D_m \ddot{\gamma}_{in} - \dot{\gamma}_{mi} D^k \dot{\gamma}_{kn} \right) \\ &+ \frac{1}{2} (D^k \dot{\gamma}_{ki} - \dot{\gamma}^{ab} D_i \dot{\gamma}_{ab}) = \kappa\tau_{0i} + \frac{1}{2\mu} \epsilon^{mn} D_m ({}^{(2)}R_{ni}).\end{aligned}\quad (3.22)$$

Furthermore, taking a conformally flat  $2D$  metric on  $\Sigma$ , we have  $\gamma_{ij} = e^\varphi \delta_{ij}$ , where  $\varphi = \varphi(t, x_i)$ ,  $k_{ij} = \frac{1}{2} \dot{\varphi} \gamma_{ij}$  and the  $2D$  Ricci tensor becomes

$${}^{(2)}R_{ij} = -\frac{1}{4} \gamma_{ij} e^{-\varphi} (2D_k \partial_k \varphi + \partial_k \varphi \partial_k \varphi),\quad (3.23)$$

whereas the 3D Ricci tensor reads

$$R_{ij} = \frac{1}{2}\gamma_{ij}(-D^k\partial_k\varphi + \dot{\varphi}^2 + \ddot{\varphi} - \frac{1}{2}\partial^k\varphi\partial_k\varphi) \quad (3.24)$$

and the 3D scalar curvature is

$$R = -D^k\partial_k\varphi + \frac{3}{2}\dot{\varphi}^2 + 2\ddot{\varphi} - \frac{1}{2}\partial^k\varphi\partial_k\varphi. \quad (3.25)$$

With all these results in hand, one can obtain from the constraint equations the following relation

$$\partial_i\dot{\varphi} = -J_i + \frac{1}{2\mu}\epsilon^m{}_i\dot{\varphi}\partial_m\dot{\varphi}, \quad (3.26)$$

where we have introduced the "source current" which, on the hypersurface, reads

$$J_i := 2\kappa\tau_{0i} + \frac{\kappa}{\mu}\epsilon^m{}_i\partial_m\tau_{00}. \quad (3.27)$$

Contracting (3.26) with the epsilon-tensor, one arrives at

$$\frac{2\mu}{\dot{\varphi}}\epsilon^{mi}\partial_m\dot{\varphi}\left(1 + \frac{\dot{\varphi}^2}{4\mu^2}\right) = -\frac{2\mu}{\dot{\varphi}}\epsilon^{mi}J_m + J^i. \quad (3.28)$$

In the case of vacuum,  $\tau_{\mu\nu} = 0$ , and so  $J_i = 0$ , the unique solution to (3.28) is of the form  $\varphi_0 = ct$ , where  $c$  is a constant which can be found from the trace equation that reads  $R = 6\Lambda$ . So  $c = 2\sqrt{\Lambda} \equiv \frac{2}{\ell}$ , which is the de Sitter (dS) solution and  $\ell > 0$  is its radius. Turning on a compactly supported matter perturbation with  $\delta\tau_{\mu\nu} \neq 0$ , one has  $\delta J_i \neq 0$  and perturbing the constraint equations about  $\varphi_0$  as  $\varphi = \varphi_0 + \delta\varphi$ , we find a linearized constraint equation

$$\begin{aligned} & \mu\left(1 + \frac{1}{\mu^2\ell^2}\right)\epsilon^m{}_i\partial_m\delta\dot{\varphi} \\ & = (\partial_i + \frac{1}{\mu\ell}\epsilon^m{}_i\partial_m)\kappa\delta\tau_{00} + 2\mu(\epsilon_i{}^m + \frac{1}{\mu\ell}\delta^m{}_i)\kappa\delta\tau_{0m}, \end{aligned} \quad (3.29)$$

from which, for the dS case, one can solve the perturbation ( $\delta\varphi$ ) and hence the perturbed metric by integration in terms of the perturbed matter fields on the hypersurface. Hence dS is linearization stable in TMG for any finite value of  $\mu\ell$ . The other linearized constraints are compatible with this solution. Our computation has been analytic in  $\ell$ , hence, we can do the following "Wick" rotation to study the AdS case:  $x^i \rightarrow ix^i$ ,  $t \rightarrow it$ ,  $\ell \rightarrow i\ell$  yielding  $\Lambda = -\frac{1}{\ell^2}$  with the Gaussian normal form of the (signature changed) metric  $ds^2 = dt^2 - e^{-2t/\ell}(dx^2 + dx^2)$ . Then for AdS, (3.29)

becomes

$$\begin{aligned} & \mu \left(1 - \frac{1}{\mu^2 \ell^2}\right) \epsilon^m{}_i \partial_m \delta \dot{\varphi} \\ &= - \left(\partial_i - \frac{1}{\mu \ell} \epsilon^m{}_i \partial_m\right) \kappa \delta \tau_{00} - 2\mu \left(\epsilon_i{}^m + \frac{1}{\mu \ell} \delta^m{}_i\right) \kappa \delta \tau_{0m} \end{aligned} \quad (3.30)$$

and once again the perturbation theory is valid for *generic* values of  $\mu\ell$  in AdS as in the case of dS. But at the chiral point,  $\mu\ell = 1$ , the left-hand side vanishes identically and there is an unphysical constraint on the matter perturbations  $\delta\tau_{0m}$  and  $\delta\tau_{00}$  in addition to their background covariant conservation. Moreover, the metric perturbation is not determined by the matter perturbation. What this says is that in the chiral gravity limit of TMG, for AdS, the exact AdS solution is linearization unstable. The above computation has been a local one, and does not depend on the fact that AdS does not have a Cauchy surface on which one can define the initial value problem. AdS requires initial and boundary values together, but what we have computed is a necessary condition for such a formulation (not a sufficient one) and AdS in chiral gravity does not satisfy the necessary conditions for the initial-boundary value problem.

### 3.2 Symplectic structure of TMG

Let us give another argument for the linearization instability of AdS making use of the symplectic structure of TMG which was found in [51] following [52] as  $\omega := \int_{\Sigma} d\Sigma_{\alpha} \sqrt{|g|} \mathcal{J}^{\alpha}$ , where  $\Sigma$  is the hypersurface.  $\omega$  is a closed ( $\delta\omega = 0$ ) non-degenerate (except for gauge directions) 2-form for full TMG including chiral gravity. Here the on-shell covariantly conserved symplectic current reads

$$\begin{aligned} \mathcal{J}^{\alpha} &= \delta\Gamma^{\alpha}_{\mu\nu} \wedge \left(\delta g^{\mu\nu} + \frac{1}{2} g^{\mu\nu} \delta \ln g\right) \\ &\quad - \delta\Gamma^{\nu}_{\mu\nu} \wedge \left(\delta g^{\alpha\mu} + \frac{1}{2} g^{\alpha\mu} \delta \ln g\right) \\ &\quad + \frac{1}{\mu} \epsilon^{\alpha\nu\sigma} \left(\delta S^{\rho}_{\sigma} \wedge \delta g_{\nu\rho} + \frac{1}{2} \delta\Gamma^{\rho}_{\nu\beta} \wedge \delta\Gamma^{\beta}_{\sigma\rho}\right). \end{aligned} \quad (3.31)$$

What is important to understand is that  $\omega$  is a gauge invariant object on the solution space, say  $\mathcal{Z}$ , and also on the (more relevant) quotient  $\mathcal{Z}/Diff$  which is the phase space and  $Diff$  is the group of diffeomorphisms. Therefore, even without knowing

the full space of solutions, by studying the symplectic structure, one gains a lot of information for both classical and quantum versions of the theory. Perturbative solutions live in the tangent space of the phase space and hence they are crucial in the discussion. We refer the reader to [51] for a full discussion of this.

Let us show that for the linearized solutions of chiral gravity given in [14] the symplectic 2-form is degenerate and hence not invertible. In the global coordinates, the background metric reads

$$ds^2 = \ell^2 \left( -\cosh^2 \rho d\tau^2 + \sinh^2 \rho d\phi^2 + d\rho^2 \right), \quad (3.32)$$

defining  $u = \tau + \phi$ ,  $v = \tau - \phi$ , making use of the  $SL(2, R) \times SL(2, R)$ , [14] found all the primary states (but one) and their descendants. The primary solutions are

$$h_{\mu\nu} = \Re \left\{ e^{-i\Delta\tau - iS\phi} F_{\mu\nu}(\rho) \right\}, \quad (3.33)$$

where the real part is taken and the background tensor reads

$$F_{\mu\nu}(\rho) = f(\rho) \begin{pmatrix} 1 & \frac{S}{2} & \frac{2i}{\sinh 2\rho} \\ \frac{S}{2} & 1 & \frac{iS}{\sinh 2\rho} \\ \frac{2i}{\sinh 2\rho} & \frac{iS}{\sinh 2\rho} & -\frac{4}{\sinh^2 2\rho} \end{pmatrix} \quad (3.34)$$

and  $f(\rho) = (\cosh \rho)^{-\Delta} \sinh^2 \rho$ , where  $\Delta \equiv h + \bar{h}$  and  $S \equiv h - \bar{h}$ . Components of the symplectic current for these modes (for generic  $\mu\ell$ ) can be found as

$$\begin{aligned} \mathcal{J}^\tau &= \frac{(4 - S^2)(S + 2\mu\ell)\Delta}{8\mu\ell^7 (\cosh \rho)^{2(1+\Delta)}} \sin(2\Delta\tau + 2S\phi), \\ \mathcal{J}^\phi &= -\frac{2 \coth^2 \rho}{S + 2\mu\ell} \mathcal{J}^\tau, \\ \mathcal{J}^\rho &= -\frac{(S\Delta + 4\mu\ell) \coth \rho + (\Delta - 2)\mu\ell \sinh 2\rho}{\Delta(S + 2\mu\ell)} \mathcal{J}^\tau, \end{aligned} \quad (3.35)$$

which yield a vanishing  $\omega$  at the chiral limit since for left, right and massive modes we have  $S^2 = 4$  and the relevant symplectic current  $\mathcal{J}^\tau$  vanishes identically, hence the solution is not viable. Moreover, one can show that its Taub charge diverges, while its ADT charge is for the background Killing vector  $(-1, 0, 0)$  is

$$Q_{ADT} = -\lim_{r \rightarrow \infty} \frac{\sin(\pi S) \cos(2\pi S + \Delta t)}{4\pi S 2^{2-\Delta} \ell} \Delta(2\Delta + S - 2) e^{r(2-\Delta)}, \quad (3.36)$$

which vanishes for the massive mode  $\Delta = S = 2$ . In addition to the above solutions,

there is an additional the log-mode given in [19] which reads

$$h_{\mu\nu} = f_1(\tau, \rho) \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}_{\mu\nu} + f_2(\tau, \rho) \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & -\frac{4}{\sinh^2 2\rho} \end{pmatrix}_{\mu\nu}, \quad (3.37)$$

where the two functions are given as

$$f_1(\tau, \rho) = \frac{\sinh \rho}{\cosh^3 \rho} (\tau \cos 2u - \sin 2u \ln \cosh \rho), \quad (3.38)$$

$$f_2(\tau, \rho) = -\tanh^2 \rho (\tau \sin 2u + \cos 2u \ln \cosh \rho). \quad (3.39)$$

The components of the symplectic current for this mode read

$$\begin{aligned} \mathcal{J}^\tau &= \frac{1}{\mu\ell^7} \tau ((1 - \mu\ell) \cosh 2\rho + 1) \operatorname{sech}^{10} \rho, \\ \mathcal{J}^\phi &= -\frac{2}{\mu\ell^7} \tau (1 - \mu\ell) \operatorname{sech}^8 \rho, \\ \mathcal{J}^\rho &= \frac{1}{\rho^6} \tanh \rho \operatorname{sech}^8 \rho (4(\log^2 \cosh \rho + \tau^2) + \log \operatorname{sech} \rho), \end{aligned} \quad (3.40)$$

which yield a linearly growing  $\omega$  in  $\tau$  and vanishes on the initial value surface. What all these say is that first order perturbation theory simply fails in chiral gravity limit of TMG. If the theory makes any sense at the classical and/or quantum level one must resort to a new method to carry out computations. This significantly affects its interpretation in the context of AdS/CFT as the perturbed metric couples to the energy-momentum tensor of the boundary CFT. This of course does not say anything about the solutions of the theory which are not globally AdS and one might simply have to define the theory in a different background.

In this chapter we have studied a frequently recurring problem [53], for example it also appears in critical gravity [54, 55]. Linearized solutions by definition satisfy the linearized equations but this is not sufficient; they should also satisfy a quadratic constraint to actually represent linearized versions of exact solutions. This deep result comes from the Bianchi identities and their linearizations and it is connected to the conserved quantities. With the observation of gravity waves, research in general relativity and its modifications, extensions has entered an exciting era in which many theories might be possibly tested. One major tool of computation in nonlinear theories, such as gravity, is perturbation theory from which one obtains a lot of information and the gravitational wave physics is no exception as one uses the tools

of perturbation theory to obtain the wave profile far away from the sources. Therefore, the issue of linearization instability arises in any use of perturbation theory as the examples provided here and before [4] show even for the ostensibly safe case of spacetimes with noncompact Cauchy surfaces.





## CHAPTER 4

### CONCLUSIONS

In Einstein's general relativity, perturbation theory about a background exact solution fails if the background spacetime has a Killing symmetry and a compact (without a boundary) spacelike Cauchy surface. This failure, dubbed as *linearization instability*, shows itself as a nonintegrability of the perturbative infinitesimal deformation to a finite deformation of the background. Namely, the linearized field equations have spurious solutions which cannot be obtained from the linearization of some exact solutions. In absence of the knowledge of exact solutions, in practice, one can show the failure of the linear perturbation theory by showing that a certain quadratic (integral) constraint, that is the vanishing of the so-called Taub charge, on the linearized solutions is not satisfied. This is the case for compact Cauchy surfaces ( or in the absence of Cauchy surfaces as in the case of AdS, for spacelike surfaces which constitute part of the initial-boundary value problem).

For noncompact Cauchy surfaces, the situation is different and for example, Minkowski space, having a noncompact Cauchy surface, is linearization stable. If this were not the case, one could not trust perturbation theory in a Minkowski background, including all the computations related to the gravitational waves. In this thesis, we have studied the linearization instability in generic metric theories of gravity where Einstein's theory is modified with additional curvature terms. Of course the problem of validity of perturbation theory becomes much more complicated as one usually lacks the tools of the elliptic operator theory on a spacelike hypersurface. Our main finding is that, unlike the case of general relativity, for modified theories even in the noncompact Cauchy surface cases (or spacelike surfaces), there are some theories which show linearization instability about their anti-de Sitter backgrounds. Recent  $D$  dimensional critical and three dimensional chiral gravity theories are two such exam-

ples. We have discussed them, especially the chiral gravity case, in great detail. This observation sheds light on the paradoxical behavior of vanishing conserved charges (mass, angular momenta) for nonvacuum solutions, such as black holes, in these extended theories. This vanishing of conserved charges for nonvacuum solutions for certain theories in asymptotically AdS spacetimes was discussed at length in [11]. At the time this zero-energy problem was not properly understood. Let us explain why in some extended theories the conserved charges vanish identically. This vanishing happens as the charge is a global quantity constructed with the help of the Stokes' theorem and asymptotic Killing symmetries where the perturbation theory is sufficient. In perturbation theory, about a constant curvature background (such as the AdS spacetime) all higher order curvature terms in the field equations contribute to the conserved charges in an additive manner which always vanishes for some particular combination of parameters of the theory. Here we have shown that, exactly at that point, the perturbation theory fails and therefore one needs other methods to define conserved quantities.

As a second, perhaps more direct proof of the linearization instability in chiral gravity, we have carried out a detailed analysis of the constraints and their linearizations on a spacelike hypersurface, we have shown that the topologically massive gravity (which is a dynamical theory of gravity in three dimensions) has a linearization instability at the chiral gravity limit about its  $AdS_3$  vacuum. We have also calculated the symplectic structure (that is built from the perturbative tensors and that carries all the information about the classical theory as well as its linearizations) for all the known perturbative modes, including the log-mode, for the linearized field equations and find it to be degenerate (non-invertible). Hence these modes do not approximate any possible exact solutions and so do not belong to the linearized phase space of the theory. Naive perturbation theory fails: the linearized field equations are necessary but not sufficient in finding viable linearized solutions. This proof supports the construction given in [15] where it was shown that the linearization of all exact solutions of chiral gravity around  $AdS_3$  has positive energy. This has important consequences for both classical and possible quantum versions of the theory which need further scrutiny.

In the linearization stability problem of general relativity, one can show that the first order perturbation theory can receive constraints at most from the second order per-

turbation theory. Namely, there will be no more constraints from the higher order perturbation theory. This issue is an outstanding problem in generic modified gravity theories which needs to be studied further. For example, we do not know what are the necessary and sufficient conditions for linearization stability of a third order theory such as topologically massive gravity.





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## APPENDIX A

### SECOND ORDER PERTURBATION THEORY AND GAUGE INVARIANCE ISSUES

#### A.1 Second order perturbation theory

Let us summarize some results about the second order perturbation theory (see also [56]). By definition one has

$$g_{\mu\nu} := \bar{g}_{\mu\nu} + \tau h_{\mu\nu}, \quad (\text{A.1})$$

where  $\tau$  is introduced to keep the order of the expansions. The inverse of the metric tensor up to and including the second order is

$$g^{\mu\nu} = \bar{g}^{\mu\nu} + \tau h^{\mu\nu} + \tau^2 h_{\alpha}^{\mu} h^{\alpha\nu} + O(\tau^3). \quad (\text{A.2})$$

Let  $T$  be a generic tensor or a geometrical object, then it can be expanded as

$$T = \bar{T} + \tau T^{(1)} + \tau^2 T^{(2)} + O(\tau^3). \quad (\text{A.3})$$

For the Christoffel connection we have

$$\Gamma_{\mu\nu}{}^{\gamma} = \bar{\Gamma}_{\mu\nu}{}^{\gamma} + \tau (\Gamma_{\mu\nu}{}^{\gamma})^{(1)} + \tau^2 (\Gamma_{\mu\nu}{}^{\gamma})^{(2)}, \quad (\text{A.4})$$

where the first order term is

$$(\Gamma_{\mu\nu}{}^{\gamma})^{(1)} = \frac{1}{2} (\bar{\nabla}_{\mu} h_{\nu}^{\gamma} + \bar{\nabla}_{\nu} h_{\mu}^{\gamma} - \bar{\nabla}^{\gamma} h_{\mu\nu}), \quad (\text{A.5})$$

and the second order one is

$$(\Gamma_{\mu\nu}{}^{\gamma})^{(2)} = -h^{\gamma\delta} (\Gamma_{\mu\nu\delta})^{(1)}. \quad (\text{A.6})$$

As is clear from (A.5) the linearized Christoffel connection is a background tensor (so is the second order one), we can raise and lower the indices with the background metric and its inverse

$$(\Gamma_{\mu\nu\delta})^{(1)} = \bar{g}_{\gamma\delta} (\Gamma_{\mu\nu}{}^{\gamma})^{(1)}. \quad (\text{A.7})$$

Please note that the up index is lowered to the third position, there is a symmetry only in the first two indices. The first order linearized Riemann tensor is

$$(R^\rho{}_{\mu\sigma\nu})^{(1)} = \bar{\nabla}_\sigma(\Gamma_{\nu\mu}{}^\rho)^{(1)} - \bar{\nabla}_\nu(\Gamma_{\sigma\mu}{}^\rho)^{(1)}, \quad (\text{A.8})$$

and the second order linearized Riemann tensor is

$$(R^\rho{}_{\mu\sigma\nu})^{(2)} = \bar{\nabla}_\sigma(\Gamma_{\nu\mu}{}^\rho)^{(2)} - \bar{\nabla}_\nu(\Gamma_{\sigma\mu}{}^\rho)^{(2)} + (\Gamma_{\mu\nu}{}^\alpha)^{(1)}(\Gamma_{\sigma\alpha}{}^\rho)^{(1)} - (\Gamma_{\mu\sigma}{}^\alpha)^{(1)}(\Gamma_{\nu\alpha}{}^\rho)^{(1)}. \quad (\text{A.9})$$

The first order linearized Ricci tensor is

$$(R_{\mu\nu})^{(1)} = \bar{\nabla}_\sigma(\Gamma_{\mu\nu}{}^\sigma)^{(1)} - \bar{\nabla}_\nu(\Gamma_{\sigma\mu}{}^\sigma)^{(1)}, \quad (\text{A.10})$$

and the second order linearized Ricci tensor is

$$(R_{\mu\nu})^{(2)} = \bar{\nabla}_\sigma(\Gamma_{\nu\mu}{}^\sigma)^{(2)} - \bar{\nabla}_\nu(\Gamma_{\sigma\mu}{}^\sigma)^{(2)} + (\Gamma_{\mu\nu}{}^\alpha)^{(1)}(\Gamma_{\sigma\alpha}{}^\sigma)^{(1)} - (\Gamma_{\mu\sigma}{}^\alpha)^{(1)}(\Gamma_{\nu\alpha}{}^\sigma)^{(1)}. \quad (\text{A.11})$$

We shall need the explicit form of it in terms of the  $h_{\mu\nu}$  field, which reads

$$(R_{\mu\nu})^{(2)} = -\frac{1}{2}\bar{\nabla}_\rho(h^{\rho\beta}(\bar{\nabla}_\mu h_{\nu\beta} + \bar{\nabla}_\nu h_{\mu\beta} - \bar{\nabla}_\beta h_{\nu\mu})) + \frac{1}{2}\bar{\nabla}_\nu(h^{\rho\beta}\bar{\nabla}_\mu h_{\rho\beta}) - \frac{1}{4}(\bar{\nabla}_\mu h_{\rho\beta})\bar{\nabla}_\nu h^{\rho\beta} + \frac{1}{4}(\bar{\nabla}^\beta h)(\bar{\nabla}_\mu h_{\nu\beta} + \bar{\nabla}_\nu h_{\mu\beta} - \bar{\nabla}_\beta h_{\nu\mu}) + \frac{1}{2}(\bar{\nabla}_\beta h_{\nu\alpha})\bar{\nabla}^\beta h_\mu^\alpha - \frac{1}{2}(\bar{\nabla}_\beta h_{\nu\alpha})\bar{\nabla}^\alpha h_\mu^\beta. \quad (\text{A.12})$$

The first order linearized scalar curvature is

$$(R)^{(1)} = \bar{\nabla}_\alpha \bar{\nabla}_\beta h^{\alpha\beta} - \bar{\square}h - \bar{R}_{\mu\nu}h^{\mu\nu}, \quad (\text{A.13})$$

and the second order linearized scalar curvature is

$$(R)^{(2)} = \bar{R}_{\mu\nu}h_\alpha^\mu h^{\alpha\nu} - (R_{\mu\nu})^{(1)}h^{\mu\nu} + \bar{g}^{\mu\nu}(R_{\mu\nu})^{(2)}. \quad (\text{A.14})$$

Explicitly we have a rather cumbersome result for the last expression which reads

$$(R)^{(2)} = \frac{1}{4}(\bar{\nabla}_\sigma h_{\rho\beta})\bar{\nabla}^\sigma h^{\rho\beta} - \frac{1}{2}(\bar{\nabla}_\sigma h_{\rho\beta})\bar{\nabla}^\rho h^{\sigma\beta} + \frac{1}{2}\bar{\nabla}^\sigma(h^{\rho\beta}\bar{\nabla}_\sigma h_{\rho\beta}) + \frac{1}{4}(\bar{\nabla}^\beta h)(2\bar{\nabla}_\sigma h_\beta^\sigma - \bar{\nabla}_\beta h) - \frac{1}{2}\bar{\nabla}_\rho(h^{\rho\beta}(2\bar{\nabla}_\sigma h_\beta^\sigma - \bar{\nabla}_\beta h)) - \frac{1}{2}h^{\rho\beta}(2\bar{\nabla}_\sigma \bar{\nabla}_\rho h_\beta^\sigma - \bar{\square}h_{\rho\beta} - \bar{\nabla}_\rho \bar{\nabla}_\beta h) + \bar{R}_{\rho\beta}h^{\rho\alpha}h_\alpha^\beta. \quad (\text{A.15})$$

Here without going into too much detail, let us summarize some of the relevant formulas that we use in the bulk of the second chapter to show various expressions, such as the gauge transformation of the linearized tensors, second order forms of the tensors *etc.*

## A.2 Some useful identities that involve Lie and covariant derivatives

Lie and covariant derivatives do not commute so we shall need the following expressions. Let  $X$  be a vector field on our spacetime manifold with a metric  $\bar{g}$  and let  $T$  be a  $(0, 2)$  background tensor field. Then in components one has the Lie derivative of  $T$  with respect to the vector field  $X$  as

$$\mathcal{L}_X T_{\rho\sigma} = X^f \bar{\nabla}_f T_{\rho\sigma} + (\bar{\nabla}_\rho X^f) T_{f\sigma} + (\bar{\nabla}_\sigma X^f) T_{\rho f}. \quad (\text{A.16})$$

Taking the covariant derivative of the last expression gives

$$\begin{aligned} \bar{\nabla}_\mu \mathcal{L}_X T_{\rho\sigma} &= (\bar{\nabla}_\mu X^f) \bar{\nabla}_f T_{\rho\sigma} + X^f \bar{\nabla}_\mu \bar{\nabla}_f T_{\rho\sigma} + (\bar{\nabla}_\mu \bar{\nabla}_\rho X^f) T_{f\sigma} \\ &+ (\bar{\nabla}_\rho X^f) \bar{\nabla}_\mu T_{f\sigma} + (\bar{\nabla}_\mu \bar{\nabla}_\sigma X^f) T_{\rho f} + (\bar{\nabla}_\sigma X^f) \bar{\nabla}_\mu T_{\rho f}. \end{aligned} \quad (\text{A.17})$$

Now let us change the order of the differentiations, we have

$$\begin{aligned} \mathcal{L}_X \bar{\nabla}_\mu T_{\rho\sigma} &= X^f \bar{\nabla}_f \bar{\nabla}_\mu T_{\rho\sigma} + (\bar{\nabla}_\mu X^f) \bar{\nabla}_f T_{\rho\sigma} + (\bar{\nabla}_\rho X^f) \bar{\nabla}_\mu T_{f\sigma} \\ &+ (\bar{\nabla}_\sigma X^f) \bar{\nabla}_\mu T_{\rho f}. \end{aligned} \quad (\text{A.18})$$

Subtracting the last two expressions gives

$$\begin{aligned} \bar{\nabla}_\mu \mathcal{L}_X T_{\rho\sigma} - \mathcal{L}_X \bar{\nabla}_\mu T_{\rho\sigma} &= X^f \bar{\nabla}_\mu \bar{\nabla}_f T_{\rho\sigma} - X^f \bar{\nabla}_f \bar{\nabla}_\mu T_{\rho\sigma} + (\bar{\nabla}_\mu \bar{\nabla}_\rho X^f) T_{f\sigma} \\ &+ (\bar{\nabla}_\mu \bar{\nabla}_\sigma X^f) T_{\rho f}, \end{aligned} \quad (\text{A.19})$$

where

$$[\bar{\nabla}_\mu, \bar{\nabla}_f] T_{\rho\sigma} = \bar{R}_{\mu f \rho}{}^\lambda T_{\lambda\sigma} + \bar{R}_{\mu f \sigma}{}^\lambda T_{\lambda\rho}. \quad (\text{A.20})$$

Then we obtain

$$\begin{aligned} \bar{\nabla}_\mu \mathcal{L}_X T_{\rho\sigma} - \mathcal{L}_X \bar{\nabla}_\mu T_{\rho\sigma} &= (\bar{\nabla}_\mu \bar{\nabla}_\rho X^f + X^\lambda \bar{R}_{\mu\lambda\rho}{}^f) T_{f\sigma} \\ &+ (\bar{\nabla}_\mu \bar{\nabla}_\sigma X^f + X^\lambda \bar{R}_{\mu\lambda\sigma}{}^f) T_{\rho f}. \end{aligned} \quad (\text{A.21})$$

Let  $\delta_X$  denote the gauge transformation generated by the vector field  $X$ , such that

$$\delta_X h_{\mu\nu} = \bar{\nabla}_\mu X_\nu + \bar{\nabla}_\nu X_\mu. \quad (\text{A.22})$$

From the definition of the first order linearized Christoffel connection, we have

$$\begin{aligned} \delta_X (\Gamma_{\mu\nu}{}^\gamma)^{(1)} &= \frac{1}{2} (\bar{\nabla}_\mu \bar{\nabla}_\nu X^\gamma + \bar{\nabla}_\mu \bar{\nabla}^\gamma X_\nu + \bar{\nabla}_\nu \bar{\nabla}_\mu X^\gamma + \bar{\nabla}_\nu \bar{\nabla}^\gamma X_\mu \\ &\quad - \bar{\nabla}^\gamma \bar{\nabla}_\mu X_\nu - \bar{\nabla}^\gamma \bar{\nabla}_\nu X_\mu), \end{aligned} \quad (\text{A.23})$$

which can be expressed as

$$\begin{aligned} \delta_X(\Gamma_{\mu\nu}{}^\gamma)^{(1)} &= \bar{\nabla}_\mu \bar{\nabla}_\nu X^\gamma \\ &+ \frac{1}{2} \left( [\bar{\nabla}_\mu, \bar{\nabla}^\gamma] X_\nu + [\bar{\nabla}_\nu, \bar{\nabla}_\mu] X^\gamma + [\bar{\nabla}_\nu, \bar{\nabla}^\gamma] X_\mu \right). \end{aligned} \quad (\text{A.24})$$

By using the following identity

$$[\bar{\nabla}_\nu, \bar{\nabla}_\mu] X^\gamma = \bar{R}_{\nu\mu}{}^{\gamma\sigma} X_\sigma, \quad (\text{A.25})$$

and the first Bianchi identity

$$\bar{R}_{\alpha\beta\gamma\delta} + \bar{R}_{\beta\gamma\alpha\delta} + \bar{R}_{\gamma\alpha\beta\delta} = 0, \quad (\text{A.26})$$

the gauge transformation of the first order linearized Christoffel connection reads,

$$\delta_X(\Gamma_{\mu\nu}{}^\gamma)^{(1)} = \bar{\nabla}_\mu \bar{\nabla}_\nu X^\gamma + \bar{R}^\gamma{}_{\nu\sigma\mu} X^\sigma. \quad (\text{A.27})$$

The symmetry in  $\mu$  and  $\nu$  is not explicit on the right hand side, but it is in fact symmetric. Making use of this in (A.21) one finds

$$\bar{\nabla}_\mu \mathcal{L}_X T_{\rho\sigma} = \mathcal{L}_X \bar{\nabla}_\mu T_{\rho\sigma} + T_{\alpha\sigma} \delta_X(\Gamma_{\mu\rho}{}^\alpha)^{(1)} + T_{\rho\alpha} \delta_X(\Gamma_{\mu\sigma}{}^\alpha)^{(1)}. \quad (\text{A.28})$$

Similarly, the Lie derivative of a tensor with one upper and one lower indices is

$$\mathcal{L}_X T_\rho{}^\sigma = X^f \bar{\nabla}_f T_\rho{}^\sigma + (\bar{\nabla}_\rho X^f) T_f{}^\sigma - (\bar{\nabla}_f X^\sigma) T_\rho{}^f, \quad (\text{A.29})$$

and the covariant differentiation of the result gives

$$\begin{aligned} \bar{\nabla}_\mu \mathcal{L}_X T_\rho{}^\sigma &= (\bar{\nabla}_\mu X^f) \bar{\nabla}_f T_\rho{}^\sigma + X^f \bar{\nabla}_\mu \bar{\nabla}_f T_\rho{}^\sigma + (\bar{\nabla}_\mu \bar{\nabla}_\rho X^f) T_f{}^\sigma \\ &+ (\bar{\nabla}_\rho X^f) \bar{\nabla}_\mu T_f{}^\sigma - (\bar{\nabla}_\mu \bar{\nabla}_f X^\sigma) T_\rho{}^f - (\bar{\nabla}_f X^\sigma) \bar{\nabla}_\mu T_\rho{}^f. \end{aligned} \quad (\text{A.30})$$

Reversing the order of the derivatives, we have

$$\begin{aligned} \mathcal{L}_X \bar{\nabla}_\mu T_\rho{}^\sigma &= X^f \bar{\nabla}_f \bar{\nabla}_\mu T_\rho{}^\sigma + (\bar{\nabla}_\mu X^f) \bar{\nabla}_f T_\rho{}^\sigma \\ &+ (\bar{\nabla}_\rho X^f) \bar{\nabla}_\mu T_f{}^\sigma - (\bar{\nabla}_f X^\sigma) \bar{\nabla}_\mu T_\rho{}^f, \end{aligned} \quad (\text{A.31})$$

and subtracting the last two expressions we obtain

$$\begin{aligned} \bar{\nabla}_\mu \mathcal{L}_X T_\rho{}^\sigma - \mathcal{L}_X \bar{\nabla}_\mu T_\rho{}^\sigma &= X^f [\bar{\nabla}_\mu, \bar{\nabla}_f] T_\rho{}^\sigma \\ &+ (\bar{\nabla}_\mu \bar{\nabla}_\rho X^f) T_f{}^\sigma - (\bar{\nabla}_\mu \bar{\nabla}_f X^\sigma) T_\rho{}^f. \end{aligned} \quad (\text{A.32})$$

By using

$$[\bar{\nabla}_\mu, \bar{\nabla}_f] T_\rho{}^\sigma = \bar{R}_{\mu f \rho}{}^\lambda T_\lambda{}^\sigma + \bar{R}_{\mu f}{}^\sigma{}_\lambda T_\rho{}^\lambda, \quad (\text{A.33})$$

and the expression for the gauge transformation of the first order linearized Christoffel connection (A.27), we can rewrite the result as

$$\bar{\nabla}_\mu \mathcal{L}_X T_\rho{}^\sigma = \mathcal{L}_X \bar{\nabla}_\mu T_\rho{}^\sigma + T_\alpha{}^\sigma \delta_X (\Gamma_{\mu\rho}{}^\alpha)^{(1)} - T_\rho{}^\alpha \delta_X (\Gamma_{\mu\alpha}{}^\rho)^{(1)}. \quad (\text{A.34})$$

Applying the same procedure for the case of any three lower index tensor, we arrive at the relations

$$\begin{aligned} \bar{\nabla}_\mu \mathcal{L}_X T_{\rho\sigma\gamma} = \mathcal{L}_X \bar{\nabla}_\mu T_{\rho\sigma\gamma} + T_{\alpha\sigma\gamma} \delta_X (\Gamma_{\mu\rho}{}^\alpha)^{(1)} + T_{\rho\alpha\gamma} \delta_X (\Gamma_{\mu\sigma}{}^\alpha)^{(1)} \\ + T_{\rho\sigma\alpha} \delta_X (\Gamma_{\mu\gamma}{}^\alpha)^{(1)}, \end{aligned} \quad (\text{A.35})$$

and

$$\begin{aligned} \bar{\nabla}_\mu \mathcal{L}_X T_{\rho\gamma}{}^\sigma = \mathcal{L}_X \bar{\nabla}_\mu T_{\rho\gamma}{}^\sigma + T_{\alpha\gamma}{}^\sigma \delta_X (\Gamma_{\mu\rho}{}^\alpha)^{(1)} + T_{\rho\alpha}{}^\sigma \delta_X (\Gamma_{\mu\gamma}{}^\alpha)^{(1)} \\ - T_{\rho\gamma}{}^\alpha \delta_X (\Gamma_{\mu\alpha}{}^\rho)^{(1)}. \end{aligned} \quad (\text{A.36})$$

In fact in the most general case of a  $(m, n)$  tensor  $T$  we have

$$\begin{aligned} \bar{\nabla}_\mu \mathcal{L}_X T_{\rho_1 \rho_2 \dots \rho_n}{}^{\sigma_1 \sigma_2 \dots \sigma_m} = \mathcal{L}_X \bar{\nabla}_\mu T_{\rho_1 \rho_2 \dots \rho_n}{}^{\sigma_1 \sigma_2 \dots \sigma_m} \\ + T_{\alpha \rho_2 \dots \rho_n}{}^{\sigma_1 \sigma_2 \dots \sigma_m} \delta_X (\Gamma_{\mu \rho_1}{}^\alpha)^{(1)} + T_{\rho_1 \alpha \dots \rho_n}{}^{\sigma_1 \sigma_2 \dots \sigma_m} \delta_X (\Gamma_{\mu \rho_2}{}^\alpha)^{(1)} \\ + \dots + T_{\rho_1 \rho_2 \dots \alpha}{}^{\sigma_1 \sigma_2 \dots \sigma_m} \delta_X (\Gamma_{\mu \rho_n}{}^\alpha)^{(1)} - T_{\rho_1 \rho_2 \dots \rho_n}{}^{\alpha \sigma_2 \dots \sigma_m} \delta_X (\Gamma_{\mu \alpha}{}^{\sigma_1})^{(1)} \\ - T_{\rho_1 \rho_2 \dots \rho_n}{}^{\sigma_1 \alpha \dots \sigma_m} \delta_X (\Gamma_{\mu \alpha}{}^{\sigma_2})^{(1)} - \dots - T_{\rho_1 \rho_2 \dots \rho_n}{}^{\sigma_1 \sigma_2 \dots \alpha} \delta_X (\Gamma_{\mu \sigma_m}{}^\rho)^{(1)}. \end{aligned} \quad (\text{A.37})$$

### A.3 Gauge transformation issues

Using the formulas we discussed above, let us find how the first order linearized Einstein tensor changes under the gauge transformations generated by the flow of the vector field  $X$ . In the index free notation one has

$$DEin(g) \cdot (\mathcal{L}_X g) = \mathcal{L}_X Ein(g), \quad (\text{A.38})$$

and in a local coordinate chart it can be expressed as

$$\delta_X (G_{\mu\nu})^{(1)} \cdot h = \mathcal{L}_X \bar{G}_{\mu\nu}. \quad (\text{A.39})$$

The center dot notation means that the operator on the left is evaluated at  $h$ . This notation seems a little bit redundant at the first order, but it becomes useful in the second order. Let us compute the left hand side of the last equation explicitly, we can write

$$\delta_X(G_{\mu\nu})^{(1)} \cdot h = \delta_X(R_{\mu\nu})^{(1)} \cdot h - \frac{1}{2}\bar{g}_{\mu\nu}\delta_X(R)^{(1)} \cdot h - \frac{1}{2}\bar{R}\delta_X h_{\mu\nu}. \quad (\text{A.40})$$

In order to find the change of the first order linearized Ricci tensor under the gauge transformation, let us compute the transformation of the linear order perturbation of the Riemann tensor under the gauge transformation. Using the previous equations (A.8, A.27), we have

$$\delta_X(R^\rho{}_{\mu\sigma\nu})^{(1)} \cdot h = \bar{\nabla}_\sigma(\bar{\nabla}_\nu\bar{\nabla}_\mu X^\rho + \bar{R}^\rho{}_{\mu\alpha\nu}X^\alpha) - \bar{\nabla}_\nu(\bar{\nabla}_\sigma\bar{\nabla}_\mu X^\rho + \bar{R}^\rho{}_{\mu\alpha\sigma}X^\alpha), \quad (\text{A.41})$$

which can be reexpressed as

$$\begin{aligned} \delta_X(R^\rho{}_{\mu\sigma\nu})^{(1)} \cdot h &= [\bar{\nabla}_\sigma, \bar{\nabla}_\nu] \bar{\nabla}_\mu X^\rho + \bar{R}^\rho{}_{\mu\alpha\nu} \bar{\nabla}_\sigma X^\alpha - \bar{R}^\rho{}_{\mu\alpha\sigma} \bar{\nabla}_\nu X^\alpha \\ &+ X^\alpha (\bar{\nabla}_\sigma \bar{R}^\rho{}_{\mu\alpha\nu} - \bar{\nabla}_\nu \bar{R}^\rho{}_{\mu\alpha\sigma}). \end{aligned} \quad (\text{A.42})$$

By using the identity

$$[\bar{\nabla}_\sigma, \bar{\nabla}_\nu] \bar{\nabla}_\mu X^\rho = \bar{R}_{\sigma\nu\mu}{}^\alpha \bar{\nabla}_\alpha X^\rho + \bar{R}_{\sigma\nu}{}^\rho{}_\alpha \bar{\nabla}_\mu X^\alpha, \quad (\text{A.43})$$

and the second Bianchi identity

$$\bar{\nabla}_\mu \bar{R}_{\nu\rho\alpha\beta} + \bar{\nabla}_\alpha \bar{R}_{\nu\rho\beta\mu} + \bar{\nabla}_\beta \bar{R}_{\nu\rho\mu\alpha} = 0, \quad (\text{A.44})$$

we can write

$$\begin{aligned} \delta_X(R^\rho{}_{\mu\sigma\nu})^{(1)} \cdot h &= X^\alpha \bar{\nabla}_\alpha \bar{R}^\rho{}_{\mu\sigma\nu} + (\bar{\nabla}_\mu X^\alpha) \bar{R}^\rho{}_{\alpha\sigma\nu} + (\bar{\nabla}_\sigma X^\alpha) \bar{R}^\rho{}_{\mu\alpha\nu} \\ &+ (\bar{\nabla}_\nu X^\alpha) \bar{R}^\rho{}_{\mu\sigma\alpha} - (\bar{\nabla}_\alpha X^\rho) \bar{R}^\alpha{}_{\mu\sigma\nu}, \end{aligned} \quad (\text{A.45})$$

where the right hand side of the equation is the Lie derivative of the Riemann tensor with respect to vector field  $X$ . So we can express the gauge transformation of the first order linearized Riemann tensor as

$$\delta_X(R^\rho{}_{\mu\sigma\nu})^{(1)} \cdot h = \mathcal{L}_X \bar{R}^\rho{}_{\mu\sigma\nu}. \quad (\text{A.46})$$

Contracting and renaming the indices we obtain the change of the Ricci tensor under the gauge transformation as

$$(R_{\mu\nu})^{(1)} \cdot h = \mathcal{L}_X \bar{R}_{\mu\nu}. \quad (\text{A.47})$$

Using the previous results, for the first order linearized scalar curvature we have

$$\delta_X(R)^{(1)} \cdot h = \bar{g}^{\mu\nu} \mathcal{L}_X \bar{R}_{\mu\nu} + \bar{R}_{\mu\nu} \mathcal{L}_X \bar{g}^{\mu\nu}, \quad (\text{A.48})$$

which becomes

$$\delta_X(R)^{(1)} \cdot h = \mathcal{L}_X \bar{R}. \quad (\text{A.49})$$

Inserting these results in equation (A.40), the gauge transformation of the first order linearized Einstein tensor can be expressed as

$$\delta_X(G_{\mu\nu})^{(1)} \cdot h = \mathcal{L}_X \bar{R}_{\mu\nu} - \frac{1}{2} \bar{g}_{\mu\nu} \mathcal{L}_X \bar{R} - \frac{1}{2} \bar{R} \mathcal{L}_X \bar{g}_{\mu\nu}. \quad (\text{A.50})$$

By combining the Lie derivative terms, it also can be written as

$$\delta_X(G_{\mu\nu})^{(1)} \cdot h = \mathcal{L}_X \left( \bar{R}_{\mu\nu} - \frac{1}{2} \bar{g}_{\mu\nu} \bar{R} \right), \quad (\text{A.51})$$

where the right hand side of the last expression shows the Lie derivative of the Einstein tensor which is evaluated at the background metric. Then this construction proves equation (A.39). Let us note that for  $\bar{G}_{\mu\nu} = 0$ , that is for solutions of Einstein's theory,  $\delta_X(G_{\mu\nu})^{(1)} \cdot h$ , that is the first order linearized Einstein's tensor is gauge invariant.

Using the above results, let us find how the second order linearized form of the Einstein tensor transforms under the gauge transformations generated by the flow of the vector field  $X$ . In the index-free notation one has

$$\begin{aligned} D^2 \text{Ein}(g) \cdot (h, \mathcal{L}_X g) + D \text{Ein}(g) \cdot \mathcal{L}_X h \\ = \mathcal{L}_X (D \text{Ein}(g) \cdot h), \end{aligned} \quad (\text{A.52})$$

which reads in local coordinates as

$$\delta_X(G_{\mu\nu})^{(2)} \cdot [h, h] + (G_{\mu\nu})^{(1)} \cdot \mathcal{L}_X h = \mathcal{L}_X (G_{\mu\nu})^{(1)} \cdot h. \quad (\text{A.53})$$

Let us prove this. By definition we have

$$\begin{aligned} \delta_X(G_{\mu\nu})^{(2)} \cdot [h, h] &= \delta_X(R_{\mu\nu})^{(2)} \cdot [h, h] - \frac{1}{2} \bar{g}_{\mu\nu} \delta_X(R)^{(2)} \cdot [h, h] \\ &\quad - \frac{1}{2} (R)^{(1)} \cdot h \delta_X h_{\mu\nu} - \frac{1}{2} h_{\mu\nu} \delta_X (R)^{(1)} \cdot h. \end{aligned} \quad (\text{A.54})$$

Let us calculate the right hand side of the equation term by term. The first term reads

$$\begin{aligned} \delta_X(R_{\mu\nu})^{(2)} \cdot [h, h] &= -(\delta_X h^{\rho\beta}) (\bar{\nabla}_\rho (\Gamma_{\nu\mu\beta})^{(1)} - \bar{\nabla}_\nu (\Gamma_{\rho\mu\beta})^{(1)}) \\ &\quad - h^{\rho\beta} \delta_X (\bar{\nabla}_\rho (\Gamma_{\nu\mu\beta})^{(1)} - \bar{\nabla}_\nu (\Gamma_{\rho\mu\beta})^{(1)}) \\ &\quad - \delta_X ((\Gamma_{\mu\nu}^\alpha)^{(1)} (\Gamma_\sigma^\sigma{}_\alpha)^{(1)} - (\Gamma_{\mu\sigma}^\alpha)^{(1)} (\Gamma_\nu^\sigma{}_\alpha)^{(1)}). \end{aligned} \quad (\text{A.55})$$

Since one has

$$\delta_X h^{\rho\beta} = -\mathcal{L}_X \bar{g}^{\rho\beta}, \quad (\text{A.56})$$

the first two terms on the right hand side of the (A.55) can be written as

$$\begin{aligned} & -(\delta_X h^{\rho\beta}) (\bar{\nabla}_\rho (\Gamma_{\nu\mu\beta})^{(1)} - \bar{\nabla}_\nu (\Gamma_{\rho\mu\beta})^{(1)}) \\ & = \mathcal{L}_X (R_{\mu\nu})^{(1)} \cdot h - \bar{g}^{\rho\beta} \mathcal{L}_X (\bar{\nabla}_\rho (\Gamma_{\nu\mu\beta})^{(1)} - \bar{\nabla}_\nu (\Gamma_{\rho\mu\beta})^{(1)}). \end{aligned} \quad (\text{A.57})$$

By using the definition of the linearized Riemann tensor, the terms on the second line of the (A.55) yield

$$\begin{aligned} & h^\rho{}_\beta \delta_X (\bar{\nabla}_\rho (\Gamma_{\nu\mu}{}^\beta)^{(1)} - \bar{\nabla}_\nu (\Gamma_{\rho\mu}{}^\beta)^{(1)}) = h^\rho{}_\beta \delta_X (R^\beta{}_{\mu\rho\nu})^{(1)} \cdot h \\ & = h^\rho{}_\beta \mathcal{L}_X \bar{R}^\beta{}_{\mu\rho\nu}. \end{aligned} \quad (\text{A.58})$$

Collecting these expressions we have

$$\begin{aligned} & \delta_X (R_{\mu\nu})^{(2)} \cdot [h, h] = \mathcal{L}_X (R_{\mu\nu})^{(1)} \cdot h - \bar{g}^{\rho\beta} \mathcal{L}_X (\bar{\nabla}_\rho (\Gamma_{\nu\mu\beta})^{(1)} - \bar{\nabla}_\nu (\Gamma_{\rho\mu\beta})^{(1)}) \\ & - \delta_X ((\Gamma_{\mu\nu}{}^\alpha)^{(1)} (\Gamma_\sigma{}^\sigma{}_\alpha)^{(1)} - (\Gamma_{\mu\sigma}{}^\alpha)^{(1)} (\Gamma_\nu{}^\sigma{}_\alpha)^{(1)}) \\ & - h^\rho{}_\beta \delta_X (R^\beta{}_{\mu\rho\nu})^{(1)} \cdot h. \end{aligned} \quad (\text{A.59})$$

Using the identity given in the equation (A.35) we can write the following equalities

$$\begin{aligned} & \mathcal{L}_X \bar{\nabla}_\rho (\Gamma_{\nu\mu\beta})^{(1)} = \bar{\nabla}_\rho \mathcal{L}_X (\Gamma_{\nu\mu\beta})^{(1)} - (\Gamma_{\sigma\mu\beta})^{(1)} \delta_X (\Gamma_{\rho\nu}{}^\sigma)^{(1)} \\ & - (\Gamma_{\nu\sigma\beta})^{(1)} \delta_X (\Gamma_{\rho\mu}{}^\sigma)^{(1)} - (\Gamma_{\nu\mu\sigma})^{(1)} \delta_X (\Gamma_{\rho\beta}{}^\sigma)^{(1)}, \end{aligned} \quad (\text{A.60})$$

$$\begin{aligned} & \mathcal{L}_X \bar{\nabla}_\nu (\Gamma_{\rho\mu\beta})^{(1)} = \bar{\nabla}_\nu \mathcal{L}_X (\Gamma_{\rho\mu\beta})^{(1)} - (\Gamma_{\sigma\mu\beta})^{(1)} \delta_X (\Gamma_{\nu\rho}{}^\sigma)^{(1)} \\ & - (\Gamma_{\rho\sigma\beta})^{(1)} \delta_X (\Gamma_{\nu\mu}{}^\sigma)^{(1)} - (\Gamma_{\rho\mu\sigma})^{(1)} \delta_X (\Gamma_{\nu\beta}{}^\sigma)^{(1)}. \end{aligned} \quad (\text{A.61})$$

Inserting the results we have

$$\begin{aligned} & \delta_X (R_{\mu\nu})^{(2)} \cdot [h, h] = \mathcal{L}_X (R_{\mu\nu})^{(1)} \cdot h \\ & - \bar{g}^{\rho\beta} (\bar{\nabla}_\rho \mathcal{L}_X (\Gamma_{\nu\mu\beta})^{(1)} - \bar{\nabla}_\nu \mathcal{L}_X (\Gamma_{\rho\mu\beta})^{(1)}) + (\Gamma_{\nu\sigma}{}^\rho)^{(1)} \delta_X (\Gamma_{\rho\mu}{}^\sigma)^{(1)} \\ & + (\Gamma_{\nu\mu\sigma})^{(1)} \delta_X (\Gamma_\rho{}^{\rho\sigma})^{(1)} - (\Gamma_{\rho\sigma}{}^\rho)^{(1)} \delta_X (\Gamma_{\nu\mu}{}^\sigma)^{(1)} - (\Gamma_{\rho\mu\sigma})^{(1)} \delta_X (\Gamma_\nu{}^{\rho\sigma})^{(1)} \\ & - h^\rho{}_\beta \delta_X (R^\beta{}_{\mu\rho\nu})^{(1)} \cdot h - \delta_X ((\Gamma_{\mu\nu}{}^\alpha)^{(1)} (\Gamma_\sigma{}^\sigma{}_\alpha)^{(1)} - (\Gamma_{\mu\sigma}{}^\alpha)^{(1)} (\Gamma_\nu{}^\sigma{}_\alpha)^{(1)}). \end{aligned} \quad (\text{A.62})$$

Decomposing the first order linearized Christoffel connection as

$$(\Gamma_{\nu\sigma}{}^\rho)^{(1)} = \bar{\nabla}_\nu h_\sigma{}^\rho - (\Gamma_\nu{}^\rho{}_\sigma)^{(1)}, \quad (\text{A.63})$$

and

$$(\Gamma_{\rho\sigma}{}^\rho)^{(1)} = \bar{\nabla}_\rho h_{\sigma}{}^\rho - (\Gamma_{\rho}{}^\rho{}_\sigma)^{(1)}, \quad (\text{A.64})$$

we obtain

$$\begin{aligned} \delta_X(R_{\mu\nu})^{(2)} \cdot [h, h] &= \mathcal{L}_X(R_{\mu\nu})^{(1)} \cdot h \\ &- \bar{g}^{\rho\beta} (\bar{\nabla}_\rho \mathcal{L}_X(\Gamma_{\nu\mu\beta})^{(1)} - \bar{\nabla}_\nu \mathcal{L}_X(\Gamma_{\rho\mu\beta})^{(1)}) \\ &- (\Gamma_{\nu}{}^\rho{}_\sigma)^{(1)} \delta_X(\Gamma_{\rho\mu}{}^\sigma)^{(1)} + (\bar{\nabla}_\nu h_{\sigma}{}^\rho) \delta_X(\Gamma_{\rho\mu}{}^\sigma)^{(1)} + (\Gamma_{\nu\mu\sigma})^{(1)} \delta_X(\Gamma_{\rho}{}^{\rho\sigma})^{(1)} \\ &- (\bar{\nabla}_\rho h_{\sigma}{}^\rho) \delta_X(\Gamma_{\nu\mu}{}^\sigma)^{(1)} + (\Gamma_{\rho}{}^\rho{}_\sigma)^{(1)} \delta_X(\Gamma_{\nu\mu}{}^\sigma)^{(1)} - (\Gamma_{\rho\mu\sigma})^{(1)} \delta_X(\Gamma_{\nu}{}^{\rho\sigma})^{(1)} \\ &- h^\rho{}_\beta \delta_X(R^\beta{}_{\mu\rho\nu})^{(1)} \cdot h - \delta_X((\Gamma_{\mu\nu}{}^\alpha)^{(1)}(\Gamma_{\sigma}{}^\sigma{}_\alpha)^{(1)} - (\Gamma_{\mu\sigma}{}^\alpha)^{(1)}(\Gamma_{\nu}{}^\sigma{}_\alpha)^{(1)}). \end{aligned} \quad (\text{A.65})$$

Since we have

$$\begin{aligned} &- (\Gamma_{\nu}{}^\rho{}_\sigma)^{(1)} \delta_X(\Gamma_{\rho\mu}{}^\sigma)^{(1)} + (\Gamma_{\nu\mu\sigma})^{(1)} \delta_X(\Gamma_{\rho}{}^{\rho\sigma})^{(1)} + (\Gamma_{\rho}{}^\rho{}_\sigma)^{(1)} \delta_X(\Gamma_{\nu\mu}{}^\sigma)^{(1)} \\ &- (\Gamma_{\rho\mu\sigma})^{(1)} \delta_X(\Gamma_{\nu}{}^{\rho\sigma})^{(1)} = \delta_X((\Gamma_{\rho}{}^\rho{}_\sigma)^{(1)}(\Gamma_{\nu\mu}{}^\sigma)^{(1)} - (\Gamma_{\rho\mu\sigma})^{(1)}(\Gamma_{\nu}{}^{\rho\sigma})^{(1)}), \end{aligned} \quad (\text{A.66})$$

we can write

$$\begin{aligned} \delta_X(R_{\mu\nu})^{(2)} \cdot [h, h] &= \mathcal{L}_X(R_{\mu\nu})^{(1)} \cdot h \\ &- \bar{g}^{\rho\beta} (\bar{\nabla}_\rho \mathcal{L}_X(\Gamma_{\nu\mu\beta})^{(1)} - \bar{\nabla}_\nu \mathcal{L}_X(\Gamma_{\rho\mu\beta})^{(1)}) \\ &+ (\bar{\nabla}_\nu h_{\sigma}{}^\rho) \delta_X(\Gamma_{\rho\mu}{}^\sigma)^{(1)} - (\bar{\nabla}_\rho h_{\sigma}{}^\rho) \delta_X(\Gamma_{\nu\mu}{}^\sigma)^{(1)} - h^\rho{}_\beta \delta_X(R^\beta{}_{\mu\rho\nu})^{(1)} \cdot h. \end{aligned} \quad (\text{A.67})$$

Expressing the terms which involve the Lie derivative of the Christoffel connection in terms of the linear order perturbation of the metric tensor, we can write

$$\begin{aligned} \delta_X(R_{\mu\nu})^{(2)} \cdot [h, h] &= \mathcal{L}_X(R_{\mu\nu})^{(1)} \cdot h + (\bar{\nabla}_\nu h_{\sigma}{}^\rho) \delta_X(\Gamma_{\rho\mu}{}^\sigma)^{(1)} \\ &- (\bar{\nabla}_\rho h_{\sigma}{}^\rho) \delta_X(\Gamma_{\nu\mu}{}^\sigma)^{(1)} - h^\rho{}_\beta \delta_X(R^\beta{}_{\mu\rho\nu})^{(1)} \cdot h \\ &- \frac{1}{2} \bar{g}^{\rho\beta} (\bar{\nabla}_\rho \mathcal{L}_X(\bar{\nabla}_\nu h_{\mu\beta} + \bar{\nabla}_\mu h_{\nu\beta} - \bar{\nabla}_\beta h_{\mu\nu}) - \bar{\nabla}_\nu \mathcal{L}_X \bar{\nabla}_\mu h_{\rho\beta}), \end{aligned} \quad (\text{A.68})$$

and using the identity given in the equation (A.28) we have

$$\begin{aligned} \delta_X(R_{\mu\nu})^{(2)} \cdot [h, h] &= \mathcal{L}_X(R_{\mu\nu})^{(1)} \cdot h + h_{\sigma}{}^\rho (\bar{\nabla}_\rho \delta_X(\Gamma_{\nu\mu}{}^\sigma)^{(1)} - \bar{\nabla}_\nu \delta_X(\Gamma_{\rho\mu}{}^\sigma)^{(1)}) \\ &- \frac{1}{2} \bar{g}^{\rho\beta} (\bar{\nabla}_\rho \bar{\nabla}_\nu \mathcal{L}_X h_{\mu\beta} + \bar{\nabla}_\rho \bar{\nabla}_\mu \mathcal{L}_X h_{\nu\beta} - \bar{\nabla}_\rho \bar{\nabla}_\beta \mathcal{L}_X h_{\mu\nu} - \bar{\nabla}_\nu \bar{\nabla}_\mu \mathcal{L}_X h_{\rho\beta}) \\ &- h^\rho{}_\beta \delta_X(R^\beta{}_{\mu\rho\nu})^{(1)} \cdot h, \end{aligned} \quad (\text{A.69})$$

where the second line is the first order linearization of the Ricci tensor, with a minus sign, which is evaluated at  $\mathcal{L}_X h$ . Also, from the definition of the first order linearized Riemann tensor we get

$$h_{\sigma}{}^\rho (\bar{\nabla}_\rho \delta_X(\Gamma_{\nu\mu}{}^\sigma)^{(1)} - \bar{\nabla}_\nu \delta_X(\Gamma_{\rho\mu}{}^\sigma)^{(1)}) = h_{\sigma}{}^\rho \delta_X(R^\sigma{}_{\mu\rho\nu})^{(1)} \cdot h, \quad (\text{A.70})$$

and finally one can express the following equation

$$\delta_X(R_{\mu\nu})^{(2)} \cdot [h, h] = \mathcal{L}_X(R_{\mu\nu})^{(1)} \cdot h - (R_{\mu\nu})^{(1)} \cdot \mathcal{L}_X h. \quad (\text{A.71})$$

By using the expression for the second order linearized scalar curvature, we can write

$$\begin{aligned} \delta_X(R)^{(2)} \cdot [h, h] &= \bar{R}_{\rho\sigma} h^{\sigma\lambda} \delta_X h_{\lambda}{}^{\rho} + \bar{R}_{\rho\sigma} h_{\lambda}{}^{\rho} \delta_X h^{\sigma\lambda} - \delta_X h^{\sigma\rho} (R_{\rho\sigma})^{(1)} \cdot h \\ &\quad - h^{\sigma\rho} \delta_X (R_{\rho\sigma})^{(1)} \cdot h + \bar{g}^{\sigma\lambda} \delta_X (R_{\rho\sigma})^{(2)} \cdot [h, h]. \end{aligned} \quad (\text{A.72})$$

Using the result for the gauge transformation of the second order linearized Ricci tensor we have

$$\begin{aligned} \delta_X(R)^{(2)} \cdot [h, h] &= \bar{R}_{\rho\sigma} h^{\sigma\lambda} \delta_X h_{\lambda}{}^{\rho} + \bar{R}_{\rho\sigma} h_{\lambda}{}^{\rho} \delta_X h^{\sigma\lambda} + \mathcal{L}_X \bar{g}^{\sigma\rho} (R_{\rho\sigma})^{(1)} \cdot h \\ &\quad - h^{\sigma\rho} \mathcal{L}_X \bar{R}_{\rho\sigma} + \bar{g}^{\sigma\lambda} \left( \mathcal{L}_X (R_{\rho\sigma})^{(1)} - (R_{\rho\sigma})^{(1)} \cdot \mathcal{L}_X h \right). \end{aligned} \quad (\text{A.73})$$

This expression can be recast as

$$\begin{aligned} \delta_X(R)^{(2)} \cdot [h, h] &= \bar{R}_{\rho\sigma} h^{\sigma\lambda} \delta_X h_{\lambda}{}^{\rho} + \bar{R}_{\rho\sigma} h_{\lambda}{}^{\rho} \delta_X h^{\sigma\lambda} + \mathcal{L}_X (\bar{g}^{\sigma\rho} (R_{\rho\sigma})^{(1)} \cdot h) \\ &\quad - \bar{g}^{\sigma\rho} \mathcal{L}_X (R_{\rho\sigma})^{(1)} \cdot h - \mathcal{L}_X (h^{\sigma\rho} \bar{R}_{\rho\sigma}) + (\mathcal{L}_X h^{\sigma\rho}) \bar{R}_{\rho\sigma} + \bar{g}^{\sigma\lambda} \mathcal{L}_X (R_{\rho\sigma})^{(1)} \cdot h \\ &\quad - \bar{g}^{\sigma\rho} (R_{\rho\sigma})^{(1)} \cdot \mathcal{L}_X h. \end{aligned} \quad (\text{A.74})$$

Combining the Lie derivative terms we obtain

$$\begin{aligned} \delta_X(R)^{(2)} \cdot [h, h] &= \bar{R}_{\rho\sigma} h^{\sigma\lambda} \delta_X h_{\lambda}{}^{\rho} + \bar{R}_{\rho\sigma} h_{\lambda}{}^{\rho} \delta_X h^{\sigma\lambda} \\ &\quad + \mathcal{L}_X (\bar{g}^{\sigma\rho} (R_{\rho\sigma})^{(1)} \cdot h - h^{\sigma\rho} \bar{R}_{\rho\sigma}) + (\mathcal{L}_X h^{\sigma\rho}) \bar{R}_{\rho\sigma} - \bar{g}^{\sigma\rho} (R_{\rho\sigma})^{(1)} \cdot \mathcal{L}_X h, \end{aligned} \quad (\text{A.75})$$

where

$$\bar{g}^{\sigma\rho} (R_{\rho\sigma})^{(1)} \cdot h - h^{\sigma\rho} \bar{R}_{\rho\sigma} = (R)^{(1)} \cdot h. \quad (\text{A.76})$$

After a straightforward calculation, for the remaining terms we can write

$$\bar{R}_{\rho\sigma} h^{\sigma\lambda} \delta_X h_{\lambda}{}^{\rho} + \bar{R}_{\rho\sigma} h_{\lambda}{}^{\rho} \delta_X h^{\sigma\lambda} + (\mathcal{L}_X h^{\sigma\rho}) \bar{R}_{\rho\sigma} = \bar{R}^{\rho\sigma} \mathcal{L}_X h_{\sigma\rho}, \quad (\text{A.77})$$

and then we obtain

$$\delta_X(R)^{(2)} \cdot [h, h] = \mathcal{L}_X (R)^{(1)} \cdot h - \left( \bar{g}^{\sigma\rho} (R_{\rho\sigma})^{(1)} \cdot \mathcal{L}_X h - \bar{R}^{\rho\sigma} \mathcal{L}_X h_{\sigma\rho} \right), \quad (\text{A.78})$$

where the terms in the parenthesis forms the linearized scalar curvature evaluated at  $\mathcal{L}_X h$ . As a result the gauge transformation of the second order linearized scalar curvature becomes

$$\delta_X(R)^{(2)} \cdot [h, h] = \mathcal{L}_X (R)^{(1)} \cdot h - (R)^{(1)} \cdot \mathcal{L}_X h. \quad (\text{A.79})$$

Using our results in the equation (A.54) we have

$$\begin{aligned} \delta_X(G_{\mu\nu})^{(2)} \cdot [h, h] &= \mathcal{L}_X(R_{\mu\nu})^{(1)} \cdot h - (R_{\mu\nu})^{(1)} \cdot \mathcal{L}_X h - \frac{1}{2}h_{\mu\nu}\delta_X(R)^{(1)} \cdot h \\ &- \frac{1}{2}\bar{g}_{\mu\nu} \left( \mathcal{L}_X(R)^{(1)} \cdot h - (R)^{(1)} \cdot \mathcal{L}_X h \right) - \frac{1}{2}\delta_X h_{\mu\nu}(R)^{(1)} \cdot h, \end{aligned} \quad (\text{A.80})$$

and it can be expressed as

$$\begin{aligned} \delta_X(G_{\mu\nu})^{(2)} \cdot [h, h] &= \mathcal{L}_X \left( (R_{\mu\nu})^{(1)} \cdot h - \frac{1}{2}\bar{g}_{\mu\nu}(R)^{(1)} \cdot h - \frac{1}{2}h_{\mu\nu}\bar{R} \right) \\ &- \left( (R_{\mu\nu})^{(1)} \cdot \mathcal{L}_X h - \frac{1}{2}\bar{g}_{\mu\nu}(R)^{(1)} \cdot \mathcal{L}_X h - \frac{1}{2}\bar{R}\mathcal{L}_X h_{\mu\nu} \right). \end{aligned} \quad (\text{A.81})$$

Here the total Lie derivative terms express the Lie derivative of the first order linearized Einstein tensor and the second line is the first order linearized Einstein tensor evaluated at  $\mathcal{L}_X h$ . We can express the final result as

$$\delta_X(G_{\mu\nu})^{(2)} \cdot [h, h] = \mathcal{L}_X(G_{\mu\nu})^{(1)} \cdot h - (G_{\mu\nu})^{(1)} \cdot \mathcal{L}_X h, \quad (\text{A.82})$$

which is the desired formula that was used in the text to study the gauge-invariance properties of the Taub charges. Observe that if  $h_{\mu\nu}$  solve the linearized equations, the first term on the right hand side vanishes, but the second one does not vanish. Therefore, unlike the first order linearized Einstein tensor, the second order linearized Einstein tensor is not gauge invariant.



## APPENDIX B

### EXPLICIT FORM OF THE K-TENSOR IN ADS

Here let us depict some of the intermediate steps leading to (2.32). Assuming a general form for the second order perturbation of the metric tensor,  $k_{\mu\nu}$ , as

$$k_{\mu\nu} = a h_{\mu\beta} h_{\nu}^{\beta} + b h h_{\mu\nu} + \bar{g}_{\mu\nu} (c h_{\alpha\beta}^2 + d h^2). \quad (\text{B.1})$$

The trace of the  $k_{\mu\nu}$  tensor is

$$k = \bar{g}^{\mu\nu} k_{\mu\nu} = (a + cD) h_{\alpha\beta}^2 + (b + dD) h^2, \quad (\text{B.2})$$

where the constants  $a, b, c, d$  are to be determined. The first order Ricci operator evaluated at the  $k_{\mu\nu}$  tensor is

$$(R_{\mu\nu})^{(1)} \cdot k = \frac{1}{2} (\bar{\nabla}_{\alpha} \bar{\nabla}_{\mu} k_{\nu}^{\alpha} + \bar{\nabla}_{\alpha} \bar{\nabla}_{\nu} k_{\mu}^{\alpha} - \bar{\square} k_{\mu\nu} - \bar{\nabla}_{\mu} \bar{\nabla}_{\nu} k), \quad (\text{B.3})$$

whose explicit form follows as

$$\begin{aligned} (R_{\mu\nu})^{(1)} \cdot k &= \frac{a}{2} (\bar{\nabla}_{\alpha} \bar{\nabla}_{\mu} h^{\alpha\beta} h_{\beta\nu} + \bar{\nabla}_{\alpha} \bar{\nabla}_{\nu} h^{\alpha\beta} h_{\beta\mu} - \bar{\square} h_{\nu}^{\beta} h_{\beta\mu} - \bar{\nabla}_{\mu} \bar{\nabla}_{\nu} h_{\alpha\beta}^2) \\ &+ \frac{b}{2} (\bar{\nabla}_{\alpha} \bar{\nabla}_{\mu} h h_{\nu}^{\alpha} + \bar{\nabla}_{\alpha} \bar{\nabla}_{\nu} h h_{\mu}^{\alpha} - \bar{\square} h h_{\mu\nu} - \bar{\nabla}_{\mu} \bar{\nabla}_{\nu} h^2) \\ &+ \frac{c}{2} (\bar{\nabla}_{\nu} \bar{\nabla}_{\mu} h_{\alpha\beta}^2 + \bar{\nabla}_{\mu} \bar{\nabla}_{\nu} h_{\alpha\beta}^2 - \bar{g}_{\mu\nu} \bar{\square} h_{\alpha\beta}^2 - D \bar{\nabla}_{\mu} \bar{\nabla}_{\nu} h_{\alpha\beta}^2) \\ &+ \frac{d}{2} (\bar{\nabla}_{\nu} \bar{\nabla}_{\mu} h^2 + \bar{\nabla}_{\mu} \bar{\nabla}_{\nu} h^2 - \bar{g}_{\mu\nu} \bar{\square} h^2 - D \bar{\nabla}_{\mu} \bar{\nabla}_{\nu} h^2). \end{aligned} \quad (\text{B.4})$$

We should set  $a = 1$  and  $b = -1/2$  to get the second order linearized Ricci tensor given in (A.12)

$$\begin{aligned} (R_{\mu\nu})^{(1)} \cdot k &= \frac{1}{2} \bar{\nabla}_{\alpha} (h^{\alpha\beta} (\bar{\nabla}_{\mu} h_{\nu\beta} + \bar{\nabla}_{\nu} h_{\mu\beta})) \\ &+ \frac{1}{2} \bar{\nabla}_{\alpha} (h_{\beta\nu} \bar{\nabla}_{\mu} h^{\alpha\beta} + h_{\beta\mu} \bar{\nabla}_{\nu} h^{\alpha\beta} + h^{\alpha\beta} \bar{\nabla}_{\beta} h_{\nu\mu} - \bar{\nabla}^{\alpha} (h_{\nu}^{\beta} h_{\beta\mu})) \\ &- \bar{\nabla}_{\nu} (h^{\alpha\beta} \bar{\nabla}_{\mu} h_{\alpha\beta}) - \frac{1}{4} h \bar{\nabla}_{\alpha} (\bar{\nabla}_{\mu} h_{\nu}^{\alpha} + \bar{\nabla}_{\nu} h_{\mu}^{\alpha} - \bar{\nabla}^{\alpha} h_{\mu\nu}) + \frac{1}{2} \bar{\nabla}_{\nu} (h \bar{\nabla}_{\mu} h) \\ &- \frac{1}{4} \bar{\nabla}_{\alpha} (h_{\nu}^{\alpha} \bar{\nabla}_{\mu} h + h_{\mu}^{\alpha} \bar{\nabla}_{\nu} h - h_{\mu\nu} \bar{\nabla}^{\alpha} h) - \frac{1}{4} \bar{\nabla}^{\beta} h (\bar{\nabla}_{\mu} h_{\nu\beta} + \bar{\nabla}_{\nu} h_{\mu\beta} - \bar{\nabla}_{\beta} h_{\mu\nu}) \\ &+ \frac{c}{2} ((2 - D) \bar{\nabla}_{\nu} \bar{\nabla}_{\mu} h_{\alpha\beta}^2 - \bar{g}_{\mu\nu} \bar{\square} h_{\alpha\beta}^2) + \frac{d}{2} ((2 - D) \bar{\nabla}_{\nu} \bar{\nabla}_{\mu} h^2 - \bar{g}_{\mu\nu} \bar{\square} h^2). \end{aligned} \quad (\text{B.5})$$

Using the expression for  $(R_{\mu\nu})^{(2)} \cdot [h, h]$ , finally the Ricci tensor evaluated at the  $k_{\mu\nu}$  tensor becomes

$$\begin{aligned}
(R_{\mu\nu})^{(1)} \cdot k &= -(R_{\mu\nu})^{(2)} \cdot [h, h] - \frac{3}{4} \bar{\nabla}_\nu h^{\alpha\beta} \bar{\nabla}_\mu h_{\alpha\beta} + \frac{1}{2} \bar{\nabla}_\alpha h_{\mu\beta} \bar{\nabla}^\alpha h_\nu^\beta \\
&+ \frac{1}{2} \bar{\nabla}_\alpha (h_{\beta\nu} \bar{\nabla}_\mu h^{\alpha\beta} + h_{\beta\mu} \bar{\nabla}_\nu h^{\alpha\beta} + h^{\alpha\beta} \bar{\nabla}_\beta h_{\nu\mu} - \bar{\nabla}^\alpha (h_\nu^\beta h_{\beta\mu})) \\
&- \frac{h}{2} (R_{\mu\nu})^{(1)} \cdot h + \frac{1}{2} \bar{\nabla}_\nu (h \bar{\nabla}_\mu h) - \frac{1}{4} \bar{\nabla}_\alpha (h_\nu^\alpha \bar{\nabla}_\mu h + h_\mu^\alpha \bar{\nabla}_\nu h - h_{\mu\nu} \bar{\nabla}^\alpha h) \\
&- \frac{1}{2} h^{\alpha\beta} \bar{\nabla}_\nu \bar{\nabla}_\mu h_{\alpha\beta} + \frac{c}{2} ((2-D) \bar{\nabla}_\nu \bar{\nabla}_\mu h_{\alpha\beta}^2 - \bar{g}_{\mu\nu} \bar{\square} h_{\alpha\beta}^2) \\
&- \frac{1}{4} h \bar{\nabla}_\nu \bar{\nabla}_\mu h - \frac{1}{2} \bar{\nabla}_\alpha h_{\mu\beta} \bar{\nabla}^\beta h_\nu^\alpha + \frac{d}{2} ((2-D) \bar{\nabla}_\nu \bar{\nabla}_\mu h^2 - \bar{g}_{\mu\nu} \bar{\square} h^2). \quad (\text{B.6})
\end{aligned}$$

For the first order linearized scalar curvature which is evaluated at the  $k_{\mu\nu}$  tensor we have

$$(R)^{(1)} \cdot k = \bar{g}^{\mu\nu} (R_{\mu\nu})^{(1)} \cdot k - \bar{R}_{\mu\nu} k^{\mu\nu}, \quad (\text{B.7})$$

which is explicitly

$$\begin{aligned}
(R)^{(1)} \cdot k &= -\bar{g}^{\mu\nu} (R_{\mu\nu})^{(2)} \cdot [h, h] - \frac{3}{4} \bar{\nabla}^\mu h^{\alpha\beta} \bar{\nabla}_\mu h_{\alpha\beta} + \frac{1}{2} \bar{\nabla}_\alpha h_{\mu\beta} \bar{\nabla}^\alpha h^{\beta\mu} \\
&- \frac{1}{2} \bar{\nabla}_\alpha h_{\mu\beta} \bar{\nabla}^\beta h^{\mu\alpha} + \frac{1}{2} \bar{\nabla}_\alpha (2h_\beta^\mu \bar{\nabla}_\mu h^{\alpha\beta} + h^{\alpha\beta} \bar{\nabla}_\beta h - \bar{\nabla}^\alpha (h_{\beta\mu}^2)) \\
&- \frac{1}{2} h^{\alpha\beta} \bar{\square} h_{\alpha\beta} - \frac{h}{2} \bar{g}^{\mu\nu} (R_{\mu\nu})^{(1)} \cdot h - \frac{1}{4} h \bar{\square} h + \frac{1}{2} \bar{\nabla}^\mu (h \bar{\nabla}_\mu h) \\
&+ c(1-D) \bar{\square} h_{\alpha\beta}^2 + d(1-D) \bar{\square} h^2 - \bar{R}_{\mu\nu} \left( h_\beta^\mu h^{\beta\nu} - \frac{1}{2} h h^{\mu\nu} + \bar{g}^{\mu\nu} (c h_{\alpha\beta}^2 + d h^2) \right) \\
&- \frac{1}{4} \bar{\nabla}_\alpha (2h^{\alpha\mu} \bar{\nabla}_\mu h - h \bar{\nabla}^\alpha h). \quad (\text{B.8})
\end{aligned}$$

By using

$$(R)^{(2)} \cdot [h, h] = \bar{R}_{\mu\nu} h_\alpha^\mu h^{\alpha\nu} - h^{\mu\nu} (R_{\mu\nu})^{(1)} + \bar{g}^{\mu\nu} (R_{\mu\nu})^{(2)} \cdot [h, h], \quad (\text{B.9})$$

we obtain

$$\begin{aligned}
(R)^{(1)} \cdot k &= -(R)^{(2)} \cdot [h, h] - h^{\mu\nu} (R_{\mu\nu})^{(1)} \cdot h - \frac{5}{4} \bar{\nabla}^\mu h^{\alpha\beta} \bar{\nabla}_\mu h_{\alpha\beta} \\
&+ \frac{1}{2} \bar{\nabla}_\alpha h_{\mu\beta} \bar{\nabla}^\beta h^{\mu\alpha} + \frac{1}{2} h^{\mu\beta} (2\bar{\nabla}_\alpha \bar{\nabla}_\mu h_\beta^\alpha - \bar{\nabla}_\alpha \bar{\nabla}^\alpha h_{\mu\beta} - \bar{\nabla}_\mu \bar{\nabla}_\beta h) \\
&+ \frac{1}{2} h^{\alpha\beta} \bar{\nabla}_\alpha \bar{\nabla}_\beta h - h^{\alpha\beta} \bar{\square} h_{\alpha\beta} - \frac{h}{2} (R)^{(1)} \cdot h + \frac{1}{2} h \bar{\square} h + \frac{3}{4} \bar{\nabla}^\mu h \bar{\nabla}_\mu h \\
&+ c(1-D) \bar{\square} h_{\alpha\beta}^2 + d(1-D) \bar{\square} h^2 - \bar{R} (c h_{\alpha\beta}^2 + d h^2), \quad (\text{B.10})
\end{aligned}$$

where

$$\frac{1}{2} h^{\mu\beta} (2\bar{\nabla}_\alpha \bar{\nabla}_\mu h_\beta^\alpha - \bar{\square} h_{\mu\beta} - \bar{\nabla}_\mu \bar{\nabla}_\beta h) = h^{\mu\beta} (R_{\mu\beta})^{(1)} \cdot h. \quad (\text{B.11})$$

Finally the Ricci tensor evaluated at the  $k_{\mu\nu}$  tensor becomes:

$$\begin{aligned}
(R)^{(1)} \cdot k &= -(R)^{(2)} \cdot [h, h] - \frac{5}{4} \bar{\nabla}^\mu h^{\alpha\beta} \bar{\nabla}_\mu h_{\alpha\beta} + \frac{1}{2} \bar{\nabla}_\alpha h_{\mu\beta} \bar{\nabla}^\beta h^{\mu\alpha} \\
&+ \frac{1}{2} h^{\alpha\beta} \bar{\nabla}_\alpha \bar{\nabla}_\beta h - h^{\alpha\beta} \bar{\square} h_{\alpha\beta} - \frac{h}{2} (R)^{(1)} \cdot h + \frac{1}{2} h \bar{\square} h + \frac{3}{4} \bar{\nabla}^\mu h \bar{\nabla}_\mu h \\
&+ c(1-D) \bar{\square} h_{\alpha\beta}^2 + d(1-D) \bar{\square} h^2 - \bar{R}(ch_{\alpha\beta}^2 + dh^2). \tag{B.12}
\end{aligned}$$

We can express the first order linearized cosmological Einstein tensor evaluated at the  $k_{\mu\nu}$  tensor as

$$(\mathcal{G}_{\mu\nu})^{(1)} \cdot k = (R_{\mu\nu})^{(1)} \cdot k - \frac{1}{2} \bar{g}_{\mu\nu} (R)^{(1)} \cdot k - \frac{2\Lambda}{(D-2)} k_{\mu\nu}, \tag{B.13}$$

where we made use of the fact that the background is an AdS spacetime with the normalization  $\bar{R}_{\mu\nu} = \frac{2\Lambda}{D-2} \bar{g}_{\mu\nu}$ . Inserting our results in the last expression we have

$$\begin{aligned}
(\mathcal{G}_{\mu\nu})^{(1)} \cdot k &= -(R_{\mu\nu})^{(2)} \cdot [h, h] + \frac{1}{2} \bar{g}_{\mu\nu} (R)^{(2)} \cdot [h, h] \\
&- \frac{h}{2} \left( (R_{\mu\nu})^{(1)} \cdot h - \frac{1}{2} \bar{g}_{\mu\nu} (R)^{(1)} \cdot h - \frac{2\Lambda}{(D-2)} h_{\mu\nu} \right) + \frac{1}{2} \bar{\nabla}_\alpha h_{\mu\beta} \bar{\nabla}^\alpha h_\nu^\beta \\
&+ \frac{1}{2} \bar{\nabla}_\alpha (h_{\beta\nu} \bar{\nabla}_\mu h^{\alpha\beta} + h_{\beta\mu} \bar{\nabla}_\nu h^{\alpha\beta} + h^{\alpha\beta} \bar{\nabla}_\beta h_{\nu\mu} - \bar{\nabla}^\alpha (h_\nu^\beta h_{\beta\mu})) \\
&- \frac{3}{4} \bar{\nabla}_\nu h^{\alpha\beta} \bar{\nabla}_\mu h_{\alpha\beta} - \frac{1}{2} h^{\alpha\beta} \bar{\nabla}_\nu \bar{\nabla}_\mu h_{\alpha\beta} - \frac{1}{4} h \bar{\nabla}_\nu \bar{\nabla}_\mu h + \frac{1}{2} \bar{\nabla}_\nu (h \bar{\nabla}_\mu h) \\
&- \frac{1}{4} \bar{\nabla}_\alpha (h_\nu^\alpha \bar{\nabla}_\mu h + h_\mu^\alpha \bar{\nabla}_\nu h - h_{\mu\nu} \bar{\nabla}^\alpha h) + \frac{c}{2} (2-D) \bar{\nabla}_\nu \bar{\nabla}_\mu h_{\alpha\beta}^2 \\
&+ \frac{d}{2} (2-D) \bar{\nabla}_\nu \bar{\nabla}_\mu h^2 - \frac{1}{2} \bar{g}_{\mu\nu} \left( -\frac{5}{4} \bar{\nabla}^\sigma h^{\alpha\beta} \bar{\nabla}_\sigma h_{\alpha\beta} + \frac{1}{2} \bar{\nabla}_\alpha h_{\sigma\beta} \bar{\nabla}^\beta h^{\sigma\alpha} \right) \\
&- \frac{1}{2} \bar{g}_{\mu\nu} \left( \frac{1}{2} h^{\alpha\beta} \bar{\nabla}_\alpha \bar{\nabla}_\beta h - h^{\alpha\beta} \bar{\square} h_{\alpha\beta} + \frac{1}{2} h \bar{\square} h + \frac{3}{4} \bar{\nabla}^\sigma h \bar{\nabla}_\sigma h \right) \\
&- \frac{1}{2} \bar{g}_{\mu\nu} (c(2-D) \bar{\square} h_{\alpha\beta}^2 + d(2-D) \bar{\square} h^2) - \bar{R}(ch_{\alpha\beta}^2 + dh^2) \\
&- \frac{1}{2} \bar{\nabla}_\alpha h_{\mu\beta} \bar{\nabla}^\beta h_\nu^\alpha - \frac{2\Lambda}{(D-2)} (h_{\mu\beta} h_\nu^\beta + \bar{g}_{\mu\nu} (ch_{\alpha\beta}^2 + dh^2)), \tag{B.14}
\end{aligned}$$

where

$$-(R_{\mu\nu})^{(2)} \cdot [h, h] + \frac{1}{2} \bar{g}_{\mu\nu} (R)^{(2)} \cdot [h, h] = -(\mathcal{G}_{\mu\nu})^{(2)} \cdot [h, h] - \frac{1}{2} h_{\mu\nu} (R)^{(1)} \cdot h. \tag{B.15}$$

Then the linearized Einstein tensor evaluated at the  $k_{\mu\nu}$  tensor can be found as

$$\begin{aligned}
(\mathcal{G}_{\mu\nu})^{(1)} \cdot k &= -(\mathcal{G}_{\mu\nu})^{(2)} \cdot [h, h] - \frac{1}{2} h_{\mu\nu} (R)^{(1)} \cdot h - \frac{h}{2} (\mathcal{G}_{\mu\nu})^{(1)} \cdot h \\
&- \frac{3}{4} \bar{\nabla}_\nu h^{\alpha\beta} \bar{\nabla}_\mu h_{\alpha\beta} + \frac{1}{2} \bar{\nabla}_\alpha h_{\mu\beta} \bar{\nabla}^\alpha h_\nu^\beta - \frac{1}{2} \bar{\nabla}_\alpha h_{\mu\beta} \bar{\nabla}^\beta h_\nu^\alpha \\
&+ \frac{1}{2} \bar{\nabla}_\alpha (h_{\beta\nu} \bar{\nabla}_\mu h^{\alpha\beta} + h_{\beta\mu} \bar{\nabla}_\nu h^{\alpha\beta} + h^{\alpha\beta} \bar{\nabla}_\beta h_{\nu\mu} - \bar{\nabla}^\alpha (h_\nu^\beta h_{\beta\mu}))
\end{aligned}$$

$$\begin{aligned}
& -\frac{1}{2}h^{\alpha\beta}\bar{\nabla}_\nu\bar{\nabla}_\mu h_{\alpha\beta} - \frac{1}{4}h\bar{\nabla}_\nu\bar{\nabla}_\mu h + \frac{1}{2}\bar{\nabla}_\nu(h\bar{\nabla}_\mu h) + \frac{d}{2}(2-D)\bar{\nabla}_\nu\bar{\nabla}_\mu h^2 \\
& -\frac{1}{4}\bar{\nabla}_\alpha(h_\nu^\alpha\bar{\nabla}_\mu h + h_\mu^\alpha\bar{\nabla}_\nu h - h_{\mu\nu}\bar{\nabla}^\alpha h) + \frac{c}{2}(2-D)\bar{\nabla}_\nu\bar{\nabla}_\mu h_{\alpha\beta}^2 \\
& -\frac{1}{2}\bar{g}_{\mu\nu}\left(-\frac{5}{4}\bar{\nabla}^\sigma h^{\alpha\beta}\bar{\nabla}_\sigma h_{\alpha\beta} + \frac{1}{2}\bar{\nabla}_\alpha h_{\sigma\beta}\bar{\nabla}^\beta h^{\sigma\alpha}\right) \\
& -\frac{1}{2}\bar{g}_{\mu\nu}\left(\frac{1}{2}h^{\alpha\beta}\bar{\nabla}_\alpha\bar{\nabla}_\beta h - h^{\alpha\beta}\bar{\square}h_{\alpha\beta} + \frac{1}{2}h\bar{\square}h + \frac{3}{4}\bar{\nabla}^\sigma h\bar{\nabla}_\sigma h\right) \\
& -\frac{1}{2}\bar{g}_{\mu\nu}(c(2-D)\bar{\square}h_{\alpha\beta}^2 + d(2-D)\bar{\square}h^2) - \bar{R}(ch_{\alpha\beta}^2 + dh^2) \\
& -\frac{2\Lambda}{D-2}(h_{\mu\beta}h_\nu^\beta + \bar{g}_{\mu\nu}(ch_{\alpha\beta}^2 + dh^2)). \tag{B.16}
\end{aligned}$$

We can express the last equation in a more compact form as

$$(\mathcal{G}_{\mu\nu})^{(1)} \cdot k = -(\mathcal{G}_{\mu\nu})^{(2)} \cdot [h, h] + K_{\mu\nu}, \tag{B.17}$$

where the final form of the  $K_{\mu\nu}$  tensor is

$$\begin{aligned}
K_{\mu\nu} &= -\frac{1}{2}h_{\mu\nu}(R)^{(1)} \cdot h - \frac{h}{2}(\mathcal{G}_{\mu\nu})^{(1)} \cdot h - \frac{3}{4}\bar{\nabla}_\nu h^{\alpha\beta}\bar{\nabla}_\mu h_{\alpha\beta} + \frac{1}{2}\bar{\nabla}_\alpha h_{\mu\beta}\bar{\nabla}^\alpha h_\nu^\beta \\
& -\frac{1}{2}\bar{\nabla}_\alpha h_{\mu\beta}\bar{\nabla}^\beta h_\nu^\alpha + \frac{1}{2}\bar{\nabla}_\alpha(h_{\beta\nu}\bar{\nabla}_\mu h^{\alpha\beta} + h_{\beta\mu}\bar{\nabla}_\nu h^{\alpha\beta} + h^{\alpha\beta}\bar{\nabla}_\beta h_{\nu\mu} - \bar{\nabla}^\alpha(h_\nu^\beta h_{\beta\mu})) \\
& -\frac{1}{4}h\bar{\nabla}_\nu\bar{\nabla}_\mu h + \frac{1}{2}\bar{\nabla}_\nu(h\bar{\nabla}_\mu h) - \frac{1}{4}\bar{\nabla}_\alpha(h_\nu^\alpha\bar{\nabla}_\mu h + h_\mu^\alpha\bar{\nabla}_\nu h - h_{\mu\nu}\bar{\nabla}^\alpha h) \\
& + \frac{c}{2}(2-D)\bar{\nabla}_\nu\bar{\nabla}_\mu h_{\alpha\beta}^2 + \frac{d}{2}(2-D)\bar{\nabla}_\nu\bar{\nabla}_\mu h^2 - \frac{2\Lambda}{D-2}(h_{\mu\beta}h_\nu^\beta + \bar{g}_{\mu\nu}(ch_{\alpha\beta}^2 + dh^2)) \\
& -\frac{1}{2}h^{\alpha\beta}\bar{\nabla}_\nu\bar{\nabla}_\mu h_{\alpha\beta} - \frac{1}{2}\bar{g}_{\mu\nu}\left(-\frac{5}{4}\bar{\nabla}^\sigma h^{\alpha\beta}\bar{\nabla}_\sigma h_{\alpha\beta} + \frac{1}{2}\bar{\nabla}_\alpha h_{\sigma\beta}\bar{\nabla}^\beta h^{\sigma\alpha}\right) \\
& -\frac{1}{2}\bar{g}_{\mu\nu}\left(+\frac{1}{2}h^{\alpha\beta}\bar{\nabla}_\alpha\bar{\nabla}_\beta h - h^{\alpha\beta}\bar{\square}h_{\alpha\beta} + \frac{1}{2}h\bar{\square}h + \frac{3}{4}\bar{\nabla}^\sigma h\bar{\nabla}_\sigma h\right) \\
& -\frac{1}{2}\bar{g}_{\mu\nu}(c(2-D)\bar{\square}h_{\alpha\beta}^2 + d(2-D)\bar{\square}h^2) - \bar{R}(ch_{\alpha\beta}^2 + dh^2). \tag{B.18}
\end{aligned}$$

Using  $(\mathcal{G}_{\mu\nu})^{(1)} \cdot h = 0$  and choosing the transverse traceless gauge for the sake of simplicity, which yields  $\bar{\square}h_{\mu\nu} = \frac{4\Lambda}{(D-1)(D-2)}h_{\mu\nu}$ , the  $K$  tensor becomes

$$\begin{aligned}
K_{\mu\nu} &= \frac{1}{2}h^{\alpha\beta}\bar{\nabla}_\alpha\bar{\nabla}_\beta h_{\nu\mu} - \frac{1}{2}\bar{\nabla}_\alpha h_{\mu\beta}\bar{\nabla}^\alpha h_\nu^\beta - \frac{1}{2}\bar{\nabla}_\alpha h_{\mu\beta}\bar{\nabla}^\beta h_\nu^\alpha \\
& + \frac{1}{2}\bar{\nabla}_\alpha h_{\beta\nu}\bar{\nabla}_\mu h^{\alpha\beta} + \frac{1}{2}\bar{\nabla}_\alpha h_{\beta\mu}\bar{\nabla}_\nu h^{\alpha\beta} + \left(c(2-D) - \frac{1}{2}\right)h^{\alpha\beta}\bar{\nabla}_\nu\bar{\nabla}_\mu h_{\alpha\beta} \\
& + \left(\frac{\Lambda}{(D-2)(D-2)}(cD^2 + cD - 6c + 2)\right)\bar{g}_{\mu\nu}h_{\alpha\beta}^2 \\
& + \left(c(D-2) + \frac{5}{8}\right)\bar{g}_{\mu\nu}\bar{\nabla}^\sigma h^{\alpha\beta}\bar{\nabla}_\sigma h_{\alpha\beta} - \frac{1}{4}\bar{g}_{\mu\nu}\bar{\nabla}_\alpha h_{\sigma\beta}\bar{\nabla}^\beta h^{\sigma\alpha} \\
& + \left(c(2-D) - \frac{3}{4}\right)\bar{\nabla}_\nu h^{\alpha\beta}\bar{\nabla}_\mu h_{\alpha\beta} - \frac{2\Lambda}{(D-1)(D-2)}h_{\mu\beta}h_\nu^\beta. \tag{B.19}
\end{aligned}$$

Using the following identities

$$h^{\alpha\beta}\bar{\nabla}_\alpha\bar{\nabla}_\beta h_{\nu\mu} = \bar{\nabla}_\alpha(h^{\alpha\beta}\bar{\nabla}_\beta h_{\nu\mu}), \quad (\text{B.20})$$

$$\bar{\nabla}_\alpha h_{\mu\beta}\bar{\nabla}^\alpha h_\nu^\beta = \bar{\nabla}_\alpha(h_{\mu\beta}\bar{\nabla}^\alpha h_\nu^\beta) - \frac{4\Lambda}{(D-1)(D-2)}h_{\mu\beta}h_\nu^\beta, \quad (\text{B.21})$$

$$\bar{\nabla}_\alpha h_{\mu\beta}\bar{\nabla}^\beta h_\nu^\alpha = \bar{\nabla}_\alpha(h_{\mu\beta}\bar{\nabla}^\beta h_\nu^\alpha) - \frac{2\Lambda D}{(D-1)(D-2)}h_{\mu\beta}h_\nu^\beta, \quad (\text{B.22})$$

$$\bar{\nabla}_\alpha h_{\beta\nu}\bar{\nabla}_\mu h^{\alpha\beta} = \bar{\nabla}_\alpha(h_{\beta\nu}\bar{\nabla}_\mu h^{\alpha\beta}) - \frac{2\Lambda D}{(D-1)(D-2)}h_{\mu\beta}h_\nu^\beta \quad (\text{B.23})$$

and

$$\bar{\nabla}_\alpha h_{\beta\mu}\bar{\nabla}_\nu h^{\alpha\beta} = \bar{\nabla}_\alpha(h_{\beta\mu}\bar{\nabla}_\nu h^{\alpha\beta}) - \frac{2\Lambda D}{(D-1)(D-2)}h_{\mu\beta}h_\nu^\beta, \quad (\text{B.24})$$

$$\begin{aligned} & \left(c(2-D) - \frac{1}{2}\right)h^{\alpha\beta}\bar{\nabla}_\nu\bar{\nabla}_\mu h_{\alpha\beta} + \left(c(2-D) - \frac{3}{4}\right)\bar{\nabla}_\nu h^{\alpha\beta}\bar{\nabla}_\mu h_{\alpha\beta} \\ &= \left(c(2-D) - \frac{1}{2}\right)\bar{\nabla}_\nu(h^{\alpha\beta}\bar{\nabla}_\mu h_{\alpha\beta}) - \frac{1}{4}\bar{\nabla}_\nu h^{\alpha\beta}\bar{\nabla}_\mu h_{\alpha\beta}, \end{aligned} \quad (\text{B.25})$$

$$\bar{g}_{\mu\nu}\bar{\nabla}^\sigma h^{\alpha\beta}\bar{\nabla}_\sigma h_{\alpha\beta} = \bar{\nabla}_\sigma(\bar{g}_{\mu\nu}h_{\alpha\beta}\bar{\nabla}^\sigma h^{\alpha\beta}) - \frac{4\Lambda}{(D-1)(D-2)}\bar{g}_{\mu\nu}h_{\alpha\beta}^2, \quad (\text{B.26})$$

$$\bar{g}_{\mu\nu}\bar{\nabla}_\alpha h_{\sigma\beta}\bar{\nabla}^\beta h^{\sigma\alpha} = \bar{\nabla}_\alpha(\bar{g}_{\mu\nu}h_{\sigma\beta}\bar{\nabla}^\beta h^{\sigma\alpha}) - \frac{2\Lambda D}{(D-1)(D-2)}\bar{g}_{\mu\nu}h_{\alpha\beta}^2, \quad (\text{B.27})$$

we can express the  $K$  tensor in a more compact form as

$$\begin{aligned} K_{\mu\nu} &= \bar{\nabla}_\alpha H^\alpha{}_{\mu\nu} + \frac{\Lambda}{(D-2)}\left(c(D-2) + \frac{1}{2}\right)\bar{g}_{\mu\nu}h_{\alpha\beta}^2 - \frac{1}{4}\bar{\nabla}_\nu h^{\alpha\beta}\bar{\nabla}_\mu h_{\alpha\beta} \\ &\quad - \frac{\Lambda D}{(D-1)(D-2)}h_{\mu\beta}h_\nu^\beta, \end{aligned} \quad (\text{B.28})$$

where

$$\begin{aligned} H^\alpha{}_{\mu\nu} &= \frac{1}{2}(h^{\alpha\beta}\bar{\nabla}_\beta h_{\nu\mu} + h_{\beta\nu}\bar{\nabla}_\mu h^{\alpha\beta} + h_{\beta\mu}\bar{\nabla}_\nu h^{\alpha\beta} - h_{\mu\beta}\bar{\nabla}^\alpha h_\nu^\beta - h_{\mu\beta}\bar{\nabla}^\beta h_\nu^\alpha) \\ &\quad - \frac{1}{4}\bar{g}_{\mu\nu}h_{\sigma\beta}\bar{\nabla}^\beta h^{\sigma\alpha} + \left(c(2-D) - \frac{1}{2}\right)\delta_\nu^\alpha h^{\sigma\beta}\bar{\nabla}_\mu h_{\sigma\beta} \\ &\quad + \left(c(D-2) + \frac{5}{8}\right)\bar{g}_{\mu\nu}h_{\sigma\beta}\bar{\nabla}^\alpha h^{\sigma\beta}. \end{aligned} \quad (\text{B.29})$$

In this gauge the coefficient  $d$  is not fixed but can be set to zero.  $K_{\mu\nu}$  has a single parameter which one can choose to fix the stability of the flat spacetime. In the Chapter 2, we use this expression to study the linearization stability of the Minkowski space by choosing  $\Lambda = 0$ .



## APPENDIX C

### ADM FORMALISM OF TOPOLOGICALLY MASSIVE GRAVITY

The topologically massive gravity is a higher order gravity theory that involves the third order derivative Cotton tensor, hence the ADM construction is somewhat cumbersome. In this appendix, we give a full account of this. What is also important is that as the action of the theory is only diffeomorphism invariant up to a boundary term, canonical ADM analysis should better be carried out at the field equation level. For the purpose of completeness we give the computation at the level of the action also. For this purpose we need to find the ADM decomposition of Christoffel symbol, the Ricci tensor, the scalar curvature and Cotton tensor. Let us compute the corresponding quantities step by step.

#### C.1 ADM split of the Christoffel symbol

We denote the full  $2 + 1$  dimensional metric with  $g_{\mu\nu}$  and the ADM decomposition of the metric is

$$ds^2 = (n_i n^i - n^2) dt^2 + 2n_i dt dx^i + \gamma_{ij} dx^i dx^j, \quad (\text{C.1})$$

where  $n$  is lapse function and  $n_i$  is shift vector both of which are functions of all coordinates. The spatial indices can be raised and lowered with the 2 dimensional spatial metric  $\gamma_{ij}$ . We will denote the spacetime coordinates with the Greek indices and the space coordinates with the Latin indices as  $\mu, \nu, \rho, \dots = 0, 1, 2$  and  $i, j, k, \dots = 1, 2$  respectively. The components of the three dimensional metric tensor are then  $g_{00} = -(n^2 - n_i n^i)$ ,  $g_{0i} = n_i$  and  $g_{ij} = \gamma_{ij}$ . Similarly, components of the inverse metric are  $g^{00} = -\frac{1}{n^2}$ ,  $g^{0i} = \frac{1}{n^2} n^i$  and  $g^{ij} = \gamma^{ij} - \frac{1}{n^2} n^i n^j$ . We define the extrinsic curvature tensor,  $k_{ij}$ , of the surface

$$k_{ij} := \frac{1}{2n} (\dot{\gamma}_{ij} - D_i n_j - D_j n_i), \quad (\text{C.2})$$

where  $D_i$  is the covariant derivative which is compatible with  $\gamma$ , namely  $D\gamma = 0$  and an over dot denotes the time derivative.  $\Gamma$  denotes Christoffel symbol of the three dimensional space with the well known definition

$$\Gamma_{\nu\rho}^{\mu} = \frac{1}{2}g^{\mu\sigma} (\partial_{\nu}g_{\rho\sigma} + \partial_{\rho}g_{\nu\sigma} - \partial_{\sigma}g_{\nu\rho}) \quad (\text{C.3})$$

and  $\gamma$  denotes the Christoffel symbol of the two dimensional surface, which is compatible with the spatial metric  $\gamma$  as

$$\gamma_{ij}^k = \frac{1}{2}\gamma^{kp} (\partial_i\gamma_{jp} + \partial_j\gamma_{ip} - \partial_p\gamma_{ij}). \quad (\text{C.4})$$

Now we can find the relations between the Christoffel symbol components of the three and two dimensional spaces. We can express the three dimensional Christoffel connection which consist of only time components as

$$\Gamma_{00}^0 = \frac{1}{2}g^{0\sigma} (2\partial_0g_{\sigma 0} - \partial_{\sigma}g_{00}), \quad (\text{C.5})$$

where  $\sigma$  can be a space or time component as we expressed above. Considering the possible cases we obtain

$$\Gamma_{00}^0 = \frac{1}{2}g^{00}\partial_0g_{00} + \frac{1}{2}g^{0k} (2\partial_0g_{k0} - \partial_kg_{00}), \quad (\text{C.6})$$

and inserting the components of the metric and the inverse metric tensor we have

$$\Gamma_{00}^0 = \frac{1}{2n^2}(2n\dot{n} - n_in_j\dot{\gamma}^{ij}) + \frac{n^k}{2n^2} (2n\partial_k n - 2\partial_k n_in^i - n_in_j\partial_k\gamma^{ij}). \quad (\text{C.7})$$

Expressing the partial derivatives in terms of covariant derivatives and by using  $\dot{\gamma}^{ij} = -\gamma^{ik}\gamma^{lj}\dot{\gamma}_{kl} = -2nk^{ij} - D^i n^j - D^j n^i$  we obtain the final result as

$$\Gamma_{00}^0 = \frac{1}{n} (\dot{n} + n^k (\partial_k n + n^i k_{ik})). \quad (\text{C.8})$$

Now let us compute the Christoffel symbol with only one lower space component. It can be expressed as

$$\Gamma_{0i}^0 = \frac{1}{2}g^{00}\partial_i g_{00} + \frac{1}{2}g^{0k} (\partial_0 g_{ki} + \partial_i g_{k0} - \partial_k g_{i0}), \quad (\text{C.9})$$

and substituting the corresponding metric tensor components in the last equation we have

$$\Gamma_{0i}^0 = \frac{1}{n} (\partial_i n + n^k k_{ik}). \quad (\text{C.10})$$

The Christoffel symbol which has only one lower time component can be expressed as

$$\Gamma_{0j}^i = \frac{1}{2}g^{0i}\partial_j g_{00} + \frac{1}{2}g^{ik}(\partial_0 g_{kj} + \partial_j g_{k0} - \partial_k g_{j0}), \quad (\text{C.11})$$

and in terms of covariant derivative it becomes

$$\Gamma_{0j}^i = -\frac{1}{n}n^i(\partial_j n + k_{kj}n^k) + nk_j^i + D_j n^i. \quad (\text{C.12})$$

With upper time and lower space components the Christoffel connection is

$$\Gamma_{ij}^0 = \frac{1}{2}g^{0k}(\partial_i g_{j0} + \partial_j g_{i0} - \partial_0 g_{ij}) + \frac{1}{2}g^{0k}(\partial_i g_{kj} + \partial_j g_{ki} - \partial_k g_{ij}) \quad (\text{C.13})$$

and it can be compactly expressed as

$$\Gamma_{ij}^0 = \frac{1}{n}k_{ij}. \quad (\text{C.14})$$

We can express the Christoffel symbol which consists of only space components as follows

$$\Gamma_{ij}^k = \gamma_{ij}^k + \frac{1}{2}g^{k0}(\partial_i g_{0j} + \partial_j g_{0i} - \partial_0 g_{ij}), \quad (\text{C.15})$$

or equivalently, this can be expressed as

$$\Gamma_{ij}^k = \gamma_{ij}^k - \frac{n^k}{n}k_{ij}. \quad (\text{C.16})$$

The last component that we need to compute is the Christoffel symbol with upper space and lower time components, explicitly it is

$$\Gamma_{00}^i = \frac{1}{2}g^{0i}\partial_0 g_{00} + \frac{1}{2}g^{ik}(2\partial_0 g_{0k} - \partial_k g_{00}), \quad (\text{C.17})$$

and substituting the metric tensor components it can also be expressed as

$$\begin{aligned} \Gamma_{00}^i &= \frac{n^i}{n^2}(-n\dot{n} + \dot{n}_k n^k - nn_k n_r k^{kr} - n_k n_r D^k n^r) \\ &\quad + \left( \gamma^{ik} - \frac{1}{n^2}n^i n^k \right) (\dot{n}_k + n\partial_k n - D_k n_r n^r). \end{aligned} \quad (\text{C.18})$$

After a lengthy calculation we obtain

$$\Gamma_{00}^i = -\frac{1}{n}(n^i \dot{n} + n^i n_k n_r k^{kr} + n^i n^k \partial_k n) + \gamma^{ik}(\dot{n}_k + n\partial_k n - n^r D_k n_r), \quad (\text{C.19})$$

and by using  $\gamma^{ik}\dot{n}_k = \dot{n}^i - n_k \dot{\gamma}^{ik}$  we can express the last equation as

$$\Gamma_{00}^i = -\frac{n^i}{n}(\dot{n} + n^k(\partial_k n + n^l k_{kl})) + n(\partial^i n + 2n^k k_k^i) + \dot{n}^i + n^k D_k n^i. \quad (\text{C.20})$$

To compute the decomposition of the field equations, we need to compute additional tensor quantities such that Ricci tensor components, the scalar curvature and the Cotton tensor.

## C.2 ADM split of the Ricci tensor and the scalar curvature

The three dimensional Ricci tensor which is compatible with the metric  $g$  can be expressed as

$$R_{\rho\sigma} = \partial_\mu \Gamma_{\rho\sigma}^\mu - \partial_\rho \Gamma_{\mu\sigma}^\mu + \Gamma_{\mu\nu}^\mu \Gamma_{\rho\sigma}^\nu - \Gamma_{\sigma\nu}^\mu \Gamma_{\mu\rho}^\nu. \quad (\text{C.21})$$

We can express the hypersurface projection of the three dimensional Ricci tensor as

$$\begin{aligned} R_{ij} = & \partial_0 \Gamma_{ij}^0 + \partial_k \Gamma_{ij}^k - \partial_i \Gamma_{0j}^0 - \partial_i \Gamma_{kj}^k + (\Gamma_{00}^0 + \Gamma_{k0}^k) \Gamma_{ij}^0 + (\Gamma_{0k}^0 + \Gamma_{lk}^l) \Gamma_{ij}^k \\ & - \Gamma_{0i}^0 \Gamma_{0j}^0 - \Gamma_{jk}^0 \Gamma_{0i}^k - \Gamma_{0j}^k \Gamma_{ik}^0 - \Gamma_{jm}^k \Gamma_{ik}^m, \end{aligned} \quad (\text{C.22})$$

and by inserting the expressions for the three dimensional Christoffel connection we obtain

$$\begin{aligned} R_{ij} = & \quad {}^{(2)}R_{ij} + k k_{ij} - 2k_{ik} k_j^k \\ & + \frac{1}{n} \left( \dot{k}_{ij} - n^k D_k k_{ij} - D_i \partial_j n - k_{kj} D_i n^k - k_{ki} D_j n^k \right), \end{aligned} \quad (\text{C.23})$$

where  ${}^{(2)}R_{ij}$  denotes the Ricci tensor of the hypersurface, which is explicitly

$${}^{(2)}R_{ij} = \partial_k \gamma_{ij}^k - \partial_i \gamma_{kj}^k + \gamma_{kl}^k \gamma_{ij}^l - \gamma_{jl}^k \gamma_{ki}^l. \quad (\text{C.24})$$

The three dimensional Ricci tensor component which is orthogonal to the hypersurface is

$$R_{00} = \partial_k \Gamma_{00}^k - \partial_0 \Gamma_{0k}^k + \Gamma_{k0}^k \Gamma_{00}^0 - \Gamma_{k0}^m \Gamma_{0m}^k + (\Gamma_{mk}^m - \Gamma_{k0}^0) \Gamma_{00}^k \quad (\text{C.25})$$

and it can be written as

$$\begin{aligned} R_{00} = & \frac{1}{n} n^i n^j \left( \dot{k}_{ij} - n^k D_k k_{ij} - D_i \partial_j n - 2k_{kj} D_i n^k \right) - n^2 k_{ij}^2 \\ & + n^i n^j \left( {}^{(2)}R_{ij} + k k_{ij} - 2k_{ik} k_j^k \right) + n \left( D_k \partial^k n - \dot{k} - n^k D_k k + 2n^k D_m k_k^m \right). \end{aligned} \quad (\text{C.26})$$

By using equation (C.23) we can express the result in a compact form as

$$\begin{aligned} R_{00} = & n^i n^j R_{ij} - n^2 k_{ij}^2 + n n^k (D_m k_k^m - D_k k) \\ & + n \left( D_k \partial^k n - \dot{k} + n^k D_m k_k^m \right). \end{aligned} \quad (\text{C.27})$$

The Ricci tensor with once time and once space projection is

$$R_{0i} = \partial_k \Gamma_{0i}^k - \partial_0 \Gamma_{ki}^k + \Gamma_{k0}^k \Gamma_{0i}^0 + \Gamma_{km}^m \Gamma_{0i}^k - \Gamma_{ik}^0 \Gamma_{00}^k - \Gamma_{ik}^m \Gamma_{m0}^k \quad (\text{C.28})$$

and in terms of 2 + 1 dimensional decomposition it can be written as

$$R_{0i} = \frac{1}{n} n^j \left( \dot{k}_{ij} - n^k D_k k_{ij} - D_i \partial_j n - k_{kj} D_i n^k - k_{ki} D_j n^k \right) + n^j \left( {}^{(2)}R_{ij} + k k_{ij} - 2k_{ik} k_j^k \right) + n (D_i k + D_m k_i^m), \quad (\text{C.29})$$

which can be similarly expressed in terms of  $R_{ij}$  by using the equation (C.23) as

$$R_{0i} = n^j R_{ij} + n (D_m k_i^m - D_i k). \quad (\text{C.30})$$

Since we know the decomposition of the three dimensional Ricci tensor components, we can compute the decomposition of the three dimensional scalar curvature  $R$ . Contracting the Ricci tensor with the inverse metric tensor yields the scalar curvature as

$$R = R_{\mu\nu} g^{\mu\nu} = R_{00} g^{00} + 2R_{0i} g^{0i} + R_{ij} g^{ij}, \quad (\text{C.31})$$

and by using our results we have

$$R = {}^{(2)}R + k^2 + k_{ij}^2 + \frac{2}{n} \left( \dot{k} - D_i D^i n - n^i D_i k \right), \quad (\text{C.32})$$

where  ${}^{(2)}R$  is scalar curvature of the hypersurface which is compatible with spatial metric  $\gamma$ .

### C.3 ADM split of the TMG action

The ADM decomposition of the TMG action is known in the literature [57]. Here, as a complementary exercise to our decomposition of the field equations, we will construct the decomposition of the action in the metric formulation. From the actual physical point of view, one should not have a physical difference between the two formulations. But because of the possible boundary terms, and due to the fact that the TMG action is only diffeomorphism invariant up to a boundary term, the canonical variables and the form of the actions look quite different. We have chosen to work with the field equations, since the action formulation involves tensor densities. But the following computation is still valuable and can be used for the Hamiltonian formulation of the theory.

For simplicity let us analyse the TMG action in two parts.

### C.3.1 The Einstein-Hilbert action

The Einstein-Hilbert Lagrangian is

$$\mathcal{L}_{EH} = \sqrt{-g} (R - 2\Lambda), \quad (\text{C.33})$$

where  $\sqrt{-g} = n\sqrt{\gamma}$  and substituting the three dimensional curvature scalar, which was given in (C.32) in the action, we arrive at

$$\mathcal{L}_{EH} = n\sqrt{\gamma} \left( {}^{(2)}R + k^2 + k_{ij}^2 + \frac{2}{n} \left( \dot{k} - D_i D^i n - n^i D_i k \right) - 2\Lambda \right). \quad (\text{C.34})$$

We can express the term which has explicit time dependence as

$$\dot{k}\sqrt{\gamma} = \partial_0 (k\sqrt{\gamma}) - k\partial_0\sqrt{\gamma}, \quad (\text{C.35})$$

where

$$\partial_0\sqrt{\gamma} = \frac{1}{2}\sqrt{\gamma}\gamma^{ij}\partial_0\gamma_{ij} = \sqrt{\gamma} (nk + D_i n^i) \quad (\text{C.36})$$

and then we obtain

$$\dot{k}\sqrt{\gamma} = \partial_0 (k\sqrt{\gamma}) - \sqrt{\gamma} (nk^2 + kD_i n^i). \quad (\text{C.37})$$

Inserting the last expression in the action we have

$$\begin{aligned} \mathcal{L}_{EH} = n\sqrt{\gamma} \left( {}^{(2)}R - k^2 + k_{ij}^2 + \Lambda \right) + \partial_0 (2\sqrt{\gamma}k) \\ - D_i (2\sqrt{\gamma} (D^i n + kn^i)), \end{aligned} \quad (\text{C.38})$$

where the last three terms are boundary terms. So up to boundary terms ADM formalism of the Einstein-Hilbert action becomes

$$\mathcal{L}_{EH} = \sqrt{-g} (R + \Lambda) = n\sqrt{\gamma} \left( {}^{(2)}R - k^2 + k_{ij}^2 - 2\Lambda \right). \quad (\text{C.39})$$

### C.3.2 The Chern-Simons action

The Chern-Simons Lagrangian is

$$\mathcal{L}_{CS} = \frac{1}{2\mu} \sqrt{-g} \epsilon^{\rho\nu\mu} \left( \Gamma_{\rho\gamma}^\sigma \partial_\nu \Gamma_{\mu\sigma}^\gamma + \frac{2}{3} \Gamma_{\rho\gamma}^\sigma \Gamma_{\nu\delta}^\gamma \Gamma_{\mu\sigma}^\delta \right). \quad (\text{C.40})$$

2 + 1 dimensional decomposition of the three index  $\epsilon$  tensor gives

$$\begin{aligned} \mathcal{L}_{CS} = \frac{1}{2\mu} \sqrt{-g} & \left( \epsilon^{n0m} \left( \Gamma_{n\gamma}^\sigma \partial_0 \Gamma_{m\sigma}^\gamma + \frac{2}{3} \Gamma_{n\gamma}^\sigma \Gamma_{0\delta}^\gamma \Gamma_{m\sigma}^\delta \right) \right. \\ & \left. + \epsilon^{0mn} \left( \Gamma_{0\gamma}^\sigma \partial_m \Gamma_{n\sigma}^\gamma + \frac{2}{3} \Gamma_{0\gamma}^\sigma \Gamma_{m\delta}^\gamma \Gamma_{n\sigma}^\delta \right) + \epsilon^{mn0} \left( \Gamma_{m\gamma}^\sigma \partial_n \Gamma_{0\sigma}^\gamma + \frac{2}{3} \Gamma_{m\gamma}^\sigma \Gamma_{n\delta}^\gamma \Gamma_{0\sigma}^\delta \right) \right). \end{aligned} \quad (\text{C.41})$$

Here the terms with the coefficient  $\frac{2}{3}$  are identical, by expressing these terms together we have

$$\mathcal{L}_{CS} = \frac{1}{2\mu} \epsilon^{mn} \left( \Gamma_{n\gamma}^\sigma \partial_0 \Gamma_{m\sigma}^\gamma + \Gamma_{0\gamma}^\sigma \partial_m \Gamma_{n\sigma}^\gamma + \Gamma_{m\gamma}^\sigma \partial_n \Gamma_{0\sigma}^\gamma + 2 \Gamma_{n\gamma}^\sigma \Gamma_{0\delta}^\gamma \Gamma_{m\sigma}^\delta \right). \quad (\text{C.42})$$

Let us compute the ADM decomposition of the Chern-Simons action term by term. The first term can be decomposed as

$$\epsilon^{mn} \Gamma_{n\gamma}^\sigma \partial_0 \Gamma_{m\sigma}^\gamma = \epsilon^{mn} \left( \Gamma_{n0}^0 \partial_0 \Gamma_{m0}^0 + \Gamma_{nk}^0 \partial_0 \Gamma_{m0}^k + \Gamma_{n0}^k \partial_0 \Gamma_{mk}^0 + \Gamma_{nl}^k \partial_0 \Gamma_{mk}^l \right). \quad (\text{C.43})$$

In terms of the ADM decomposition the terms in the right hand side of the last equation can be recast as follows

$$\begin{aligned} \epsilon^{mn} \Gamma_{n0}^0 \partial_0 \Gamma_{m0}^0 & \quad (\text{C.44}) \\ & = \frac{\epsilon^{mn}}{n^2} \left( (\partial_n n + n^k k_{kn}) \partial_m \dot{n} + \partial_n n \partial_0 (n^l k_{ml}) + n^k k_{kn} \partial_0 (n^l k_{ml}) \right) \end{aligned}$$

and

$$\begin{aligned} \epsilon^{mn} \Gamma_{nk}^0 \partial_0 \Gamma_{m0}^k & = \epsilon^{mn} \left( \frac{1}{n^3} \left( \dot{n} n^k k_{kn} \partial_m n \right) \right. \\ & \left. - \frac{1}{n^2} \left( k_{kn} \partial_0 (n^k \partial_m n) + n^k k_{kn} n^l \dot{k}_{lm} \right) + \frac{1}{n} k_{nk} \partial_0 D_m n^k + k_{nk} \dot{k}_m^k \right) \end{aligned} \quad (\text{C.45})$$

and

$$\begin{aligned} \epsilon^{mn} \Gamma_{n0}^k \partial_0 \Gamma_{mk}^0 & = \epsilon^{mn} \left( -\frac{1}{n^2} \left( \dot{k}_{km} n^k (\partial_m n + n^l k_{ln}) + \dot{n} k_{km} D_n n^k \right) \right. \\ & \left. + \frac{1}{n^3} \left( \dot{n} n^k k_{km} \partial_n n \right) + \frac{1}{n} \dot{k}_{mk} D_n n^k + \dot{k}_{mk} k_n^k \right). \end{aligned} \quad (\text{C.46})$$

We also have the last piece as

$$\begin{aligned} \epsilon^{mn} \Gamma_{nl}^k \partial_0 \Gamma_{mk}^l & = \epsilon^{mn} \left( \left( \frac{1}{n^2} \left( k_{nl} n^k \partial_0 (n^l k_{mk}) + \dot{n} k_{km} n^l \dot{\gamma}_{nl}^k \right) \right. \right. \\ & \left. \left. - \frac{1}{n} \left( \dot{\gamma}_{nl}^k \partial_0 (n^l k_{mk}) + \dot{\gamma}_{mk}^l k_{nl} n^k \right) + \dot{\gamma}_{mk}^l \dot{\gamma}_{nl}^k \right). \end{aligned} \quad (\text{C.47})$$

Combining these we obtain the following identity

$$\begin{aligned} \varepsilon^{mn} \Gamma_{n\gamma}^\sigma \partial_0 \Gamma_{m\sigma}^\gamma &= \varepsilon^{mn} \left( \frac{1}{n^2} \left( \partial_n n \partial_m \dot{n} + 2 \partial_n n \dot{n}^k k_{km} - \dot{n} k_{km} \partial_n n^k \right) \right. \\ &\left. + \frac{1}{n} \left( k_{nk} D_m \dot{n}^k + \dot{k}_{mk} \partial_n n^k - \gamma_{nl}^k \dot{n}^l k_{mk} \right) + \dot{k}_{mk} k_n^k + k_{nk} \dot{k}_m^k + \gamma_{nl}^k \dot{\gamma}_{mk}^l \right). \end{aligned} \quad (\text{C.48})$$

Similarly the second term can be decomposed as

$$\varepsilon^{mn} \Gamma_{0\gamma}^0 \partial_m \Gamma_{n\sigma}^\gamma = \varepsilon^{mn} \left( \Gamma_{00}^0 \partial_m \Gamma_{n0}^0 + \Gamma_{0k}^0 \partial_m \Gamma_{n0}^k + \Gamma_{00}^k \partial_m \Gamma_{nk}^0 + \Gamma_{0l}^k \partial_m \Gamma_{nk}^l \right), \quad (\text{C.49})$$

where the right hand side of the equation can be expressed term by term as

$$\begin{aligned} \varepsilon^{mn} \Gamma_{00}^0 \partial_m \Gamma_{n0}^0 &= \varepsilon^{mn} \left( -\frac{1}{n^3} \left( \dot{n} n^r k_{rn} \partial_m n + n^r k_{rn} \partial_m n n^k (\partial_k n + n^l k_{lk}) \right) \right. \\ &\left. + \frac{1}{n^2} \left( \dot{n} \partial_m (n^r k_{rn}) + n^k (\partial_k n + n^l k_{lk}) \partial_m (n^r k_{rn}) \right) \right) \end{aligned} \quad (\text{C.50})$$

and

$$\begin{aligned} \varepsilon^{mn} \Gamma_{0k}^0 \partial_m \Gamma_{n0}^k &= \varepsilon^{mn} \left( \frac{1}{n^3} \left( n^r k_{rn} \partial_m n n^k (\partial_k n + n^l k_{lk}) \right) \right. \\ &\left. + \frac{1}{n} \left( (\partial_k n + n^l k_{lk}) (\partial_m D_n n^k + \partial_m n k_n^k) + \partial_m k_n^k (\partial_k n + n^l k_{lk}) \right) \right. \\ &\left. - \frac{1}{n^2} \left( n^k (\partial_k n + n^l k_{lk}) \partial_m (n^r k_{rn}) + \partial_m n^k (\partial_k n + n^l k_{lk}) (\partial_n n + k_{ln} n^l) \right) \right) \end{aligned} \quad (\text{C.51})$$

and

$$\begin{aligned} \varepsilon^{mn} \Gamma_{00}^k \partial_m \Gamma_{nk}^0 &= \varepsilon^{mn} \left( \frac{1}{n^3} \left( n^r k_{kn} \partial_m n n^k (\partial_r n + n^l k_{lr}) + \dot{n} \partial_m n n^k k_{kn} \right) \right. \\ &\left. - \frac{1}{n^2} \left( n^k n^r \partial_m k_{kn} (\partial_r n + n^l k_{lr}) + \partial_m n k_{kn} (\dot{n}^k + n^l D_l n^k) + \dot{n} n^k \partial_m k_{kn} \right) \right. \\ &\left. + \frac{1}{n} \left( \partial_m k_{kn} (\dot{n}^k + n^l D_l n^k) - \partial_m n k_n^k (\partial_k n + 2n^l k_{lk}) \right) \right. \\ &\left. + \partial_m k_n^k (\partial_k n + 2n^l k_{lk}) \right) \end{aligned} \quad (\text{C.52})$$

also and

$$\begin{aligned}
\varepsilon^{mn} \Gamma_{0l}^k \partial_m \Gamma_{nk}^l &= \varepsilon^{mn} \left( \frac{1}{n^3} \left( -n^l k_{kn} \partial_m n n^k (\partial_l n + n^r k_{lr}) \right) \right. \\
&+ \frac{1}{n^2} \left( n^k \partial_m (k_{kn} n^l) (\partial_l n + n^r k_{lr}) + \partial_m n n^l k_{kn} D_l n^k \right) \\
&- \frac{1}{n} \left( \partial_m (k_{kn} n^l) D_l n^k - \partial_m n k_n^k n^l k_{lk} + n^k \partial_m \gamma_{nk}^l (\partial_l n + k_{rl} n^r) \right) \\
&\left. + \partial_m \gamma_{nk}^l (n k_l^k + D_l n^k) - k_l^k \partial_m (k_{kn} n^l) \right). \tag{C.53}
\end{aligned}$$

Collecting the pieces one has the identity,

$$\begin{aligned}
\varepsilon^{mn} \Gamma_{0\gamma}^\sigma \partial_m \Gamma_{n\sigma}^\gamma &= \varepsilon^{mn} \left( \partial_m k_n^k \partial_k n + \partial_m k_{nk} \partial^k n + \partial_m k_n^k n^l k_{kl} \right. \\
&+ \partial_m k_{nk} n^l k_l^k + D_l n^k \partial_m \gamma_{nk}^l - k_l^k k_{nk} \partial_m n^l + n k_l^k \partial_m \gamma_{n\sigma k}^l \\
&+ \frac{1}{n} \left( \gamma_{nl}^k \partial_m n^l \partial_k n + \gamma_{nr}^k \partial_m n^r n^l k_{lk} + \dot{n}^k \partial_m k_{nk} - D_l n^k k_{nk} \partial_m n^l \right) \\
&\left. + \frac{1}{n^2} \left( \dot{n} k_{kn} \partial_m n^k - \partial_m n^k \partial_k n \partial_n n - \partial_m n^k \partial_n n n^l k_{kl} - \dot{n}^l \partial_m n k_{ln} \right) \right). \tag{C.54}
\end{aligned}$$

The third term can be decomposed as

$$\varepsilon^{mn} \Gamma_{m\gamma}^\sigma \partial_n \Gamma_{0\sigma}^\gamma = \varepsilon^{mn} (\Gamma_{m0}^0 \partial_n \Gamma_{00}^0 + \Gamma_{mk}^0 \partial_n \Gamma_{00}^k + \Gamma_{m0}^k \partial_n \Gamma_{0k}^0 + \Gamma_{ml}^k \partial_n \Gamma_{0k}^l), \tag{C.55}$$

where

$$\begin{aligned}
\varepsilon^{mn} \Gamma_{m0}^0 \partial_n \Gamma_{00}^0 &= \varepsilon^{mn} \left( -\frac{1}{n^3} \left( n^l k_{ml} n^k \partial_n n (\partial_k n + n^l k_{lk}) + n^l k_{ml} \partial_n n \dot{n} \right) \right. \\
&\left. + \frac{1}{n^2} \left( n^l k_{ml} \partial_n \dot{n} + \partial_m n \partial_n \dot{n} + (\partial_m n + n^l k_{ml}) \partial_n (n^k (\partial_k n + n^l k_{lk})) \right) \right) \tag{C.56}
\end{aligned}$$

and

$$\begin{aligned}
\varepsilon^{mn} \Gamma_{mk}^0 \partial_n \Gamma_{00}^k &= \varepsilon^{mn} \left( \frac{1}{n^3} \left( n^k k_{km} \partial_n n (\dot{n} + n^l (\partial_l n + n^r k_{rl})) \right) \right. \\
&- \frac{1}{n^2} \left( k_{km} \partial_n (\dot{n} n^k) + k_{km} \partial_n (n^k n^l (\partial_l n + n^r k_{rl})) \right) \\
&+ \frac{1}{n} \left( k_{km} \partial_n \dot{n}^k + k_{km} \partial_n n^l D_l n^k + k_{km} \partial_n n (\partial^k n + 2n^l k_l^k) \right) \\
&\left. + k_{km} n^r \partial_n D_r n^k + k_{km} \partial_n (\partial^k n + 2n^l k_l^k) \right) \tag{C.57}
\end{aligned}$$

and

$$\begin{aligned}
\varepsilon^{mn} \Gamma_{m0}^k \partial_n \Gamma_{0k}^0 &= \varepsilon^{mn} \left( \frac{1}{n^3} \left( n^k \partial_n n k_{rm} n^r (\partial_k n + n^l k_{kl}) \right) \right. \\
&- \frac{1}{n^2} \left( (n^k \partial_m n + n^k k_{rm} n^r) \partial_n (\partial_k n + n^l k_{kl}) + \partial_n n D_m n^k (\partial_k n + n^l k_{kl}) \right) \\
&+ \frac{1}{n} \left( D_m n^k (\partial_k n + n^l k_{kl}) - k_m^k \partial_n n (\partial_k n + n^l k_{kl}) \right) \\
&\left. + k_m^k \partial_n (\partial_k n + n^l k_{kl}) \right) \quad (C.58)
\end{aligned}$$

and

$$\begin{aligned}
\varepsilon^{mn} \Gamma_{ml}^k \partial_n \Gamma_{0k}^l &= \varepsilon^{mn} \left( -\frac{1}{n^3} \left( n^k \partial_n n k_{lm} n^l (\partial_k n + n^l k_{kl}) \right) \right. \\
&+ \frac{1}{n^2} \left( n^l \partial_n n \gamma_{ml}^k (\partial_k n + n^l k_{kl}) + n^k k_{lm} \partial_n (n^l (\partial_k n + n^l k_{kl})) \right) \\
&- \frac{1}{n} \left( \gamma_{ml}^k \partial_n (n^l (\partial_k n + n^l k_{kl})) + n^k k_{lm} \partial_n D_k n^l + n^k k_{lm} \partial_n n k_k^l \right) \\
&\left. + \gamma_{ml}^k \partial_n (D_k n^l + n k_k^l) - n^k \partial_n k_k^l k_{lm} \right). \quad (C.59)
\end{aligned}$$

Then the third term becomes,

$$\begin{aligned}
\varepsilon^{mn} \Gamma_{m\gamma}^\sigma \partial_n \Gamma_{0\sigma}^\gamma &= \varepsilon^{mn} \left( k_m^k n^l \partial_n k_{kl} + \gamma_{mk}^l \partial_n D_l n^k + \gamma_{mk}^l \partial_n (n k_l^k) \right. \\
&- k_{mk} n^l \partial_n k_l^k + k_m^k \partial_n \partial_k n + k_{mk} \partial_n \partial^k n + 2k_{mk} n^l \partial_n k_l^k + 3k_{mk} \partial_n n^l k_l^k \\
&+ \frac{1}{n} \left( \partial_m n^l \partial_n (\partial_l n + k_{lk} n^k) - \gamma_{mr}^k \partial_n n^r (\partial_k n + n^l k_{lk}) + D_l n^k k_{mk} \partial_n n^l + \partial_n \dot{n}^k k_{mk} \right) \\
&\left. \frac{1}{n^2} \left( -\dot{n} k_{km} \partial_n n^k + 2\partial_n n^k \partial_k n \partial_m n + 2\partial_n n^k \partial_m n n^l k_{kl} + \partial_m n \partial_n \dot{n} \right) \right). \quad (C.60)
\end{aligned}$$

The last term also can be decomposed as

$$\begin{aligned}
\varepsilon^{mn} \Gamma_{n\gamma}^\sigma \Gamma_{0\delta}^\gamma \Gamma_{m\sigma}^\delta &= \varepsilon^{mn} \left( \Gamma_{n0}^0 (\Gamma_{m0}^0 \Gamma_{00}^0 + \Gamma_{0k}^0 \Gamma_{m0}^k) + \Gamma_{nk}^0 (\Gamma_{m0}^0 \Gamma_{00}^k + \Gamma_{0l}^k \Gamma_{m0}^l) \right. \\
&\left. + \Gamma_{n0}^l (\Gamma_{ml}^0 \Gamma_{00}^0 + \Gamma_{0k}^0 \Gamma_{ml}^k) + \Gamma_{nk}^l (\Gamma_{ml}^0 \Gamma_{00}^k + \Gamma_{0r}^k \Gamma_{ml}^r) \right), \quad (C.61)
\end{aligned}$$

where the decomposition of the right hand side of the equation can be expressed term

by term as

$$\begin{aligned} \varepsilon^{mn} \Gamma_{n0}^0 (\Gamma_{m0}^0 \Gamma_{00}^0 + \Gamma_{0k}^0 \Gamma_{m0}^k) &= \varepsilon^{mn} \left( \frac{1}{n^2} \left( D_m n^k (\partial_n n + k_{ln} n^l) (\partial_k n + n^r k_{rk}) \right) \right. \\ &\left. + \frac{1}{n} \left( k_m^k (\partial_n n + k_{ln} n^l) (\partial_k n + n^r k_{rk}) \right) \right) \end{aligned} \quad (C.62)$$

and

$$\begin{aligned} \varepsilon^{mn} \Gamma_{nk}^0 (\Gamma_{m0}^0 \Gamma_{00}^k + \Gamma_{0l}^k \Gamma_{m0}^l) &= \varepsilon^{mn} \left( -\frac{1}{n^3} \left( \partial_m n k_{ln} n^l \dot{n} \right) + k_{nr} (k_m^k D_k n^r + k_k^r D_m n^k) \right. \\ &+ \frac{1}{n^2} \left( \dot{n}^k k_{nk} (\partial_m n + k_{lm} n^l) - D_m n^k n^l k_{nl} (\partial_k n + n^r k_{rk}) \right) \\ &\left. + \frac{1}{n} \left( k_n^k (\partial_m n + 2k_{lm} n^l) (\partial_k n + n^r k_{rk}) + D_m n^k D_k n^r k_{nr} \right) \right) \end{aligned} \quad (C.63)$$

and

$$\begin{aligned} \varepsilon^{mn} \Gamma_{n0}^l (\Gamma_{ml}^0 \Gamma_{00}^0 + \Gamma_{0k}^0 \Gamma_{ml}^k) &= \varepsilon^{mn} \left( -\frac{1}{n^3} \left( \partial_n n k_{lm} n^l \dot{n} \right) \right. \\ &+ \frac{1}{n^2} \left( \dot{n} k_{mk} D_n n^k - \gamma_{mp}^k n^p (\partial_n n + k_{ln} n^l) (\partial_k n + n^r k_{rk}) \right) \\ &\left. + \frac{1}{n} \left( \gamma_{mp}^k D_n n^p (\partial_k n + n^r k_{rk}) + \gamma_{mp}^k k_k^p (\partial_k n + n^r k_{rk}) \right) \right) \end{aligned} \quad (C.64)$$

and

$$\begin{aligned} \varepsilon^{mn} \Gamma_{nk}^l (\Gamma_{ml}^0 \Gamma_{00}^k + \Gamma_{0r}^k \Gamma_{ml}^r) &= \varepsilon^{mn} \left( \gamma_{nl}^p k_{pm} (\partial^l n + n^r k_r^l) - \gamma_{mp}^k k_{nr} n^p k_k^r + \gamma_{mp}^k \gamma_{nl}^p (\partial_k n^l + n k_k^l) \right. \\ &+ \frac{1}{n^2} \left( -\dot{n} \gamma_{np}^k n^p k_{km} - \dot{n}^r k_{rn} n^k k_{km} + \gamma_{mp}^k n^p k_{ln} n^l (\partial_k n + n^r k_{rk}) \right) \\ &- \frac{1}{n} \left( k_n^k n^l k_{lm} (\partial_k n + n^r k_{rk}) + \gamma_{mp}^k \gamma_{nl}^p n^l (\partial_k n + n^r k_{rk}) \right. \\ &\left. - \gamma_{np}^k \dot{n}^p k_{mk} + \gamma_{mp}^k n^p k_{nl} D_k n^l \right) \end{aligned} \quad (C.65)$$

Finally collecting all the pieces together 2 + 1 dimensional decomposition of the last

term is

$$\begin{aligned}
\varepsilon^{mn} \Gamma_{n\gamma}^\sigma \Gamma_{0\delta}^\gamma \Gamma_{m\sigma}^\delta &= \varepsilon^{mn} \left( \frac{1}{n} \left( \gamma_{ml}^k \partial_n n^l (\partial_k n + n^r k_{rk}) + D_l n^k k_{nk} \partial_m n^l + \gamma_{nk}^r k_{mr} \dot{n}^k \right) \right. \\
&+ \frac{1}{n^2} \left( \dot{n} k_{km} \partial_n n^k + \partial_m n^k \partial_k n \partial_n n + \partial_m n^k \partial_n n n^l k_{kl} + \partial_m n \dot{n}^k k_{nk} \right) \\
&+ k_m^k k_{nl} D_k n^l + k_k^l k_{nl} \partial_m n^k + \gamma_{mr}^k \gamma_{nl}^r (D_k n^l + n k_k^l) + \gamma_{mr}^k k_n^r (k_{kl} n^l + \partial_k n) \\
&\left. + \gamma_{nl}^r k_{mr} (\partial^l n + n^k k_k^l) \right). \tag{C.66}
\end{aligned}$$

Using these terms we obtain the ADM decomposition of the Chern-Simons action as follows

$$\begin{aligned}
\mathcal{L}_{CS} &= \frac{1}{2\mu} \varepsilon^{mn} \left( \frac{1}{n^2} \left( \partial_n n \dot{n}^k k_{mk} - \partial_m n^k \partial_k n \partial_n n - \partial_m n^k \partial_n n n^l k_{kl} + \dot{n} k_{ln} \partial_m n^l \right) \right. \\
&+ \frac{1}{n} \left( \partial_m n^k \partial_n (\partial_k n + k_{kl} n^l) + \dot{n}^k \partial_m k_{nk} + \dot{k}_{km} \partial_n n^k \right) \\
&+ \dot{k}_{mk} k^k + k_{nk} \dot{k}_m^k + \gamma_{nl}^k \dot{\gamma}_{mk}^l + 2D_m k_{nk} (\partial^k n + n^l k_l^k) \\
&+ 2k_{mk} D_n (\partial^k n + n^l k_l^k) + \partial_m \gamma_{nk}^l (D_l n^k + n k_k^l) \\
&\left. + 2\gamma_{mr}^k \gamma_{nl}^r (D_k n^l + n k_k^l) + \gamma_{mr}^k \partial_n (D_k n^r + n k_k^r) + 2k_m^k k_{nl} D_k n^l \right). \tag{C.67}
\end{aligned}$$

Here the terms in the first two lines can be written as surface terms and therefore we will not take them into account in the canonical analysis but they are needed for the purposes of conserved charge computation. The first two terms in the third line are identical which is obvious from

$$\varepsilon^{mn} k_{nk} \dot{k}_m^k = \partial_0 (\varepsilon^{mn} k_{nk} k_m^k) - \varepsilon^{mn} \dot{k}_{nk} k_m^k = \varepsilon^{mn} \dot{k}_{mk} k_n^k, \tag{C.68}$$

where the total time derivative term vanishes due to the antisymmetry. Now let us focus on the non tensorial terms and try to express them in a tensorial form. The terms with the two dimensional Christoffel symbol can be written as

$$\begin{aligned}
&\varepsilon^{mn} \left( \gamma_{nl}^k \dot{\gamma}_{mk}^l + \partial_m \gamma_{nk}^l (D_l n^k + n k_k^l) + 2\gamma_{mr}^k \gamma_{nl}^r (D_k n^l + n k_l^k) \right. \\
&\left. + \gamma_{mr}^k \partial_n (D_k n^r + n k_k^r) \right) \\
&= \varepsilon^{mn} \left( \gamma_{nl}^k \dot{\gamma}_{mk}^l + 2 (D_l n^k + n k_l^k) (\partial_m \gamma_{nk}^l + \gamma_{mr}^l \gamma_{nk}^r) \right). \tag{C.69}
\end{aligned}$$

By using the definition of the two dimensional Riemann tensor, which is explicitly

$${}^{(2)}R^l{}_{kmn} = \frac{1}{2} (\delta_m^l \gamma_{kn} - \delta_n^l \gamma_{km}) {}^{(2)}R, \quad (\text{C.70})$$

we can write

$$\varepsilon^{mn} (D_l n^k + n k_l^k) (\partial_m \gamma_{nk}^l + \gamma_{mr}^l \gamma_{nk}^r) = \frac{1}{2} \varepsilon^{mn} (D_l n^k + n k_l^k) {}^{(2)}R^l{}_{kmn} \quad (\text{C.71})$$

and it reduces to

$$\varepsilon^{mn} (D_l n^k + n k_l^k) (\partial_m \gamma_{nk}^l + \gamma_{mr}^l \gamma_{nk}^r) = \frac{1}{2} \varepsilon^{mn} D_m n_n {}^{(2)}R. \quad (\text{C.72})$$

The remaining term that we need to compute is the term which involves a time derivative of the two dimensional Christoffel symbol and it can be expressed as

$$\begin{aligned} \varepsilon^{mn} \gamma_{nl}^k \dot{\gamma}_{mk}^l &= \frac{1}{2} \varepsilon^{mn} \left( \gamma_{nl}^k 2 \dot{\gamma}^{lr} \gamma_{nl}^k \gamma_{mk}^p \gamma_{pr} - \dot{\gamma}_{kr} \partial_m (\gamma_{nl}^k \gamma^{lr}) \right. \\ &\quad \left. - \dot{\gamma}_{mr} \partial_k (\gamma_{nl}^k \gamma^{lr}) + \dot{\gamma}_{km} \partial_r (\gamma_{nl}^k \gamma^{lr}) \right). \end{aligned} \quad (\text{C.73})$$

Ignoring the surface terms and using the metric compatibility condition we have

$$\begin{aligned} \varepsilon^{mn} \gamma_{nl}^k \dot{\gamma}_{mk}^l &= \frac{1}{2} \varepsilon^{mn} \left( 2 \dot{\gamma}^{lr} \gamma_{nl}^k \gamma_{mk}^p \gamma_{pr} - \dot{\gamma}_{kr} \gamma^{lr} \partial_m \gamma_{nl}^k + \dot{\gamma}_{kr} \gamma^{pr} \gamma_{mp}^l \gamma_{nl}^k \right. \\ &\quad \left. - \dot{\gamma}_{mr} \gamma^{lr} \partial_k \gamma_{nl}^k + \dot{\gamma}_{mr} \gamma^{pr} \gamma_{pk}^l \gamma_{nl}^k + \dot{\gamma}_{mr} \gamma^{pl} \gamma_{pk}^r \gamma_{nl}^k \right. \\ &\quad \left. + \dot{\gamma}_{mk} \gamma^{pr} \partial_p \gamma_{nr}^k - \dot{\gamma}_{mk} \gamma^{pr} \gamma_{pr}^l \gamma_{nl}^k - \dot{\gamma}_{mk} \gamma^{pr} \gamma_{pl}^l \gamma_{nr}^k \right). \end{aligned} \quad (\text{C.74})$$

Again by using the expression of the two dimensional Riemann tensor which was given in equation (C.70) we have

$$\varepsilon^{mn} (-\dot{\gamma}_{kr} \gamma^{lr} \partial_m \gamma_{nl}^k + \dot{\gamma}_{kr} \gamma^{pr} \gamma_{mp}^l \gamma_{nl}^k) = -\frac{1}{2} \varepsilon^{mn} \dot{\gamma}_{mn} {}^{(2)}R = 0 \quad (\text{C.75})$$

and we can express some of the remaining terms as

$$\varepsilon^{mn} (\dot{\gamma}_{mk} \gamma^{pr} \partial_p \gamma_{nr}^k + \dot{\gamma}_{mr} \gamma^{pl} \gamma_{pk}^r \gamma_{nl}^k - \dot{\gamma}_{mk} \gamma^{pr} \gamma_{pr}^l \gamma_{nl}^k) = \varepsilon^{mn} \dot{\gamma}_{mk} \gamma^{pr} \partial_n \gamma_{pr}^k. \quad (\text{C.76})$$

Finally we obtain

$$\begin{aligned} 2\varepsilon^{mn} \gamma_{nl}^k \dot{\gamma}_{mk}^l &= \varepsilon^{mn} \left( 2 \dot{\gamma}^{lr} \gamma_{nl}^k \gamma_{mk}^p \gamma_{pr} \right. \\ &\quad \left. + \dot{\gamma}_{mk} \gamma^{pr} (\partial_n \gamma_{pr}^k - \gamma_{pl}^l \gamma_{nr}^k) - \dot{\gamma}_{mr} \gamma^{lr} (\partial_k \gamma_{nl}^k - \gamma_{np}^k \gamma_{kl}^p) \right) \end{aligned} \quad (\text{C.77})$$

and we can write the term that involves a time derivative of the connection in a symmetric form as,

$$\varepsilon^{mn}\gamma_{nl}^k\dot{\gamma}_{mk}^l =: A^{ab}\dot{\gamma}_{ab}, \quad (\text{C.78})$$

where we have defined a new two index non-tensor quantity which can be expressed as

$$A^{ab} = \frac{1}{4} \left( -2\varepsilon^{mn}\gamma_{nl}^k\gamma_{mk}^b\gamma^{la} + \varepsilon^{an} \left( \gamma^{pr} (\partial_n\gamma_{pr}^b - \gamma_{pl}^l\gamma_{nr}^b) - \gamma^{rb} (\partial_k\gamma_{nr}^k - \gamma_{np}^k\gamma_{kr}^p) \right) \right) + a \leftrightarrow b. \quad (\text{C.79})$$

Consequently all two dimensional Christoffel symbol terms together can be written as

$$\frac{1}{2\mu}\varepsilon^{mn} \left( \gamma_{nl}^k\dot{\gamma}_{mk}^l + \partial_m\gamma_{nk}^l (D_l n^k + n k_k^l) + 2\gamma_{mr}^k\gamma_{nl}^r (D_k n^l + n k_l^k) + \gamma_{mr}^k\partial_n (D_k n^r + n k_k^r) \right) = \frac{1}{2\mu} (A^{ab}\dot{\gamma}_{ab} + \varepsilon^{mn} D_m n_n^{(2)} R). \quad (\text{C.80})$$

Substituting the result in the ADM decomposition of the Chern-Simons action, which is given in (C.68) we obtain

$$\mathcal{L}_{CS} = \frac{1}{2\mu}\varepsilon^{mn} \left( 2\dot{k}_{mk}k_n^k + 2D_mk_{nk} (\partial^k n + n^l k_l^k) + 2k_{mk}D_n (\partial^k n + n^l k_l^k) + D_m n_n^{(2)} R + 2k_m^k k_{nl} D_k n^l \right) + \frac{1}{2\mu} A^{ab}\dot{\gamma}_{ab} \quad (\text{C.81})$$

and by using  $\dot{\gamma}_{ab} = 2nk_{ab} + D_a n_b + D_b n_a$  we have

$$\begin{aligned} \mathcal{L}_{CS} = & \frac{1}{\mu}\varepsilon^{mn} \left( D^k (D_mk_{nk}n) + D_n (k_{mk}\partial^k n) - D^k (nD_n k_{mk}) \right. \\ & + D_n (k_{mk} (n^l k_l^k)) + D_k (k_m^k k_{nl} n^l) + D_m \left( \frac{1}{2} n_n^{(2)} R \right) - \frac{1}{2} n_n \partial_m^{(2)} R \\ & - D_k (k_m^k k_{nl}) n^l + \dot{k}_{mk} k_n^k + D_mk_{nk} n^l k_l^k \\ & \left. - D_n k_{mk} n^l k_l^k - nD^k D_mk_{nk} + nD^k D_n k_{mk} \right) \\ & + \frac{1}{\mu} \left( A^{ab}\gamma_{ab}^r n_r - \partial_a A^{ab} n_b + A^{ab} n k_{ab} + \partial_a (A^{ab} n_b) \right). \quad (\text{C.82}) \end{aligned}$$

Finally up to boundary terms the ADM decomposition of the Chern-Simons La-

grangian is

$$\begin{aligned} \mathcal{L}_{CS} = & \frac{1}{\mu} \left( \epsilon^{mn} \dot{k}_{mk} k_n^k + n (A^{ab} k_{ab} + 2\epsilon^{mn} D^k D_n k_{mk}) \right) \\ & + \frac{1}{\mu} n_a \left( \frac{1}{2} \epsilon^{an} \partial_n^{(2)} R - 2\epsilon^{mn} D_n k_{mk} k^{ak} - \epsilon^{mn} D_k (k_m^k k_n^a) - (\partial_b A^{ba} - A^{cb} \gamma_{cb}^a) \right). \end{aligned} \quad (\text{C.83})$$

### C.3.3 The TMG action

Combining all the above results the Lagrangian of the topologically massive gravity up to a boundary term in the ADM formulation reads

$$\begin{aligned} \mathcal{L}_{TMG} = & n\sqrt{\gamma} (-2\Lambda + {}^{(2)}R - k^2 + k_{ab}^2) + v^{ab} (\dot{\gamma}_{ab} - 2n k_{ab} - 2D_a n_b) \\ & + \frac{1}{\mu} \sqrt{\gamma} \left( \epsilon^{mn} \dot{k}_{mk} k_n^k + n \left( \gamma^{-\frac{1}{2}} A^{ab} k_{ab} + 2\epsilon^{mn} D^k D_n k_{mk} \right) \right) \\ & + \frac{1}{\mu} \sqrt{\gamma} n_a \left( \frac{1}{2} \epsilon^{an} \partial_n^{(2)} R - 2\epsilon^{mn} D_n k_{mk} k^{ak} - \epsilon^{mn} D_k (k_m^k k_n^a) \right. \\ & \left. - \gamma^{-\frac{1}{2}} (\partial_b A^{ba} - A^{cb} \gamma_{cb}^a) \right), \end{aligned} \quad (\text{C.84})$$

where we have introduced the Lagrange multiplier  $v^{ab}$ . As usual we can define the conjugate momenta of the two dimensional metric tensor as

$$\pi^{ab} = \frac{\delta \mathcal{L}_{TMG}}{\delta \dot{\gamma}_{ab}} = v^{ab} \quad (\text{C.85})$$

and the conjugate momenta of the extrinsic curvature tensor as

$$\Pi^{ab} = \frac{\delta \mathcal{L}_{TMG}}{\delta \dot{k}_{ab}} = \frac{1}{\mu} \sqrt{\gamma} \epsilon^{an} k_n^b. \quad (\text{C.86})$$

### C.3.4 The Hamiltonian of TMG

It is obvious from equation (C.86) that  $k$  and  $\Pi$  are not independent, therefore the system is constrained. Or to stress this: we cannot solve the velocities  $\dot{k}_{ab}$  in terms of their momenta  $\Pi^{ab}$ , so we have to include this dependency as a constraint equation by introducing Lagrange multipliers in the Hamiltonian. Therefore the Hamiltonian of TMG is

$$\mathcal{H}_{TMG} = \pi^{ab} \dot{\gamma}_{ab} + \Pi^{ab} \dot{k}_{ab} - \mathcal{L}_{TMG} + f_{ab} \left( \Pi^{ab} - \frac{1}{\mu} \sqrt{\gamma} \epsilon^{an} k_n^b \right), \quad (\text{C.87})$$

here just like  $A^{ab}$ ,  $\pi^{ab}$  is not a tensor. Inserting the TMG action, which was expressed in equation (C.84), Hamiltonian of the TMG becomes

$$\begin{aligned}
\mathcal{H}_{TMG} = & \partial_a (\pi^{ab} n_b) + f_{ab} \left( \Pi^{ab} - \frac{1}{\mu} \sqrt{\gamma} \epsilon^{an} k_n{}^b \right) \\
& + n \sqrt{\gamma} \left( 2\Lambda - {}^{(2)}R + k^2 - k_{ab}^2 - \frac{2}{\mu} \epsilon^{mn} D^k D_n k_{mk} + \gamma^{-\frac{1}{2}} \left( 2\pi^{ab} - \frac{1}{\mu} A^{ab} \right) k_{ab} \right) \\
& + \sqrt{\gamma} n_a \left( -\frac{1}{2\mu} \epsilon^{an} \partial_n {}^{(2)}R + \frac{1}{\mu} \epsilon^{mn} (2D_n k_{mk} k^{ak} + D_k (k_m{}^k k_n{}^a)) \right) \\
& + \gamma^{-\frac{1}{2}} \partial_b \left( \frac{1}{\mu} A^{ba} - 2\pi^{ab} \right) + \gamma^{-\frac{1}{2}} \left( \frac{1}{\mu} A^{bc} - 2\pi^{cb} \right) \gamma_{cb}^a. \tag{C.88}
\end{aligned}$$

We can introduce the following tensor

$$P^{ab} = \gamma^{-\frac{1}{2}} \left( 2\pi^{ab} - \frac{1}{\mu} A^{ab} \right) \tag{C.89}$$

to recast the Hamiltonian as

$$\begin{aligned}
\mathcal{H}_{TMG} = & n \sqrt{\gamma} \left( 2\Lambda - {}^{(2)}R + k^2 - k_{ab}^2 - \frac{2}{\mu} \epsilon^{mn} D^k D_n k_{mk} + P^{ab} k_{ab} \right) \\
& + \sqrt{\gamma} n_a \left( -\frac{1}{2\mu} \epsilon^{an} \partial_n {}^{(2)}R + \frac{2}{\mu} \epsilon^{mn} D_n k_{mk} k^{ak} + \frac{1}{\mu} \epsilon^{mn} D_k (k_m{}^k k_n{}^a) + D_b P^{ab} \right) \\
& + f_{ab} \left( \Pi^{ab} - \frac{1}{\mu} \sqrt{\gamma} \epsilon^{an} k_n{}^b \right), \tag{C.90}
\end{aligned}$$

where  $f_{ab}$  is a Lagrange multiplier.

### C.3.5 The Constraint equations of TMG

We can now obtain the constraint equations of TMG from the above Hamiltonian. Variations of  $n_a$ ,  $n$  and  $f_{ab}$  yields three constraint equations.

The Hamiltonian constraint equation can be obtained from the variation with respect to  $n$  as

$$(\Phi_1) = \frac{\delta \mathcal{H}_{TMG}}{\delta n} = -\frac{2}{l^2} - {}^{(2)}R + k^2 - k_{ab}^2 - \frac{2}{\mu} \epsilon^{mn} D^k D_n k_{mk} + P^{ab} k_{ab}, \tag{C.91}$$

and the momentum constraint equations can be obtained from the variation with respect to  $n_a$  as

$$\begin{aligned}
(\Phi_2)^a = & \frac{\delta \mathcal{H}_{TMG}}{\delta n_a} = \frac{2}{\mu} \epsilon^{mn} D_n k_{mk} k^{ak} + \frac{1}{\mu} \epsilon^{mn} D_k (k_m{}^k k_n{}^a) \\
& - \frac{1}{2\mu} \epsilon^{an} \partial_n {}^{(2)}R - D_b P^{ba} \tag{C.92}
\end{aligned}$$

and also we have an additional constraint equation which comes from the variation of the Hamiltonian with respect to Lagrange multiplier  $f_{ab}$  as

$$(\Phi_3)^{ab} = \frac{\delta \mathcal{H}_{TMG}}{\delta f_{ab}} = \Pi^{ab} - \frac{1}{2\mu} \sqrt{\gamma} (\epsilon^{an} k_n{}^b + \epsilon^{bn} k_n{}^a). \quad (\text{C.93})$$

In contrast to the constraint equations directly obtained from the field equations, it seems there arise three constraint equations from the Hamiltonian formalism of TMG. But actually the last constraint equation does not say anything new about the theory, it only repeats the relation between the extrinsic curvature tensor and its conjugate momenta, where the relation is obvious in equation (C.86).

#### C.4 ADM split of the Cotton tensor

To carry out the full ADM analysis, we need to split the Cotton tensor on the hypersurface and normal to the hypersurface. In local coordinates the Cotton tensor is defined as

$$C_{\mu\nu} = \frac{1}{2} \epsilon^{\rho\alpha\beta} (g_{\mu\rho} \nabla_\alpha S_{\beta\nu} + g_{\nu\rho} \nabla_\alpha S_{\beta\mu}), \quad (\text{C.94})$$

here  $\epsilon^{\rho\alpha\beta}$  is the antisymmetric tensor in interchanging of any two indices and it can be expressed in terms of the two dimensional antisymmetric tensor,  $\epsilon^{mn}$ , where  $\epsilon^{0mn} = \epsilon^{n0m} = \epsilon^{mn0} = \frac{1}{n} \epsilon^{mn}$  and  $\epsilon^{mn} = \gamma^{-\frac{1}{2}} \epsilon^{mn}$ . Also  $S_{\mu\nu}$  is the Schouten tensor which is defined in three dimensions as

$$S_{\mu\nu} = R_{\mu\nu} - \frac{1}{4} g_{\mu\nu} R. \quad (\text{C.95})$$

Let us find the 2 + 1 dimensional decomposition of the Cotton tensor step by step. Since there are summations over Greek indices we can decompose them as summations over time and space components. Decomposing  $\rho$  gives

$$C_{\mu\nu} = \frac{1}{2} \epsilon^{0\alpha\beta} (g_{\mu 0} \nabla_\alpha S_{\beta\nu} + g_{\nu 0} \nabla_\alpha S_{\beta\mu}) + \frac{1}{2} \epsilon^{k\alpha\beta} (g_{\mu k} \nabla_\alpha S_{\beta\nu} + g_{\nu k} \nabla_\alpha S_{\beta\mu}) \quad (\text{C.96})$$

and decomposing  $\alpha$  and  $\beta$  indices gives

$$C_{\mu\nu} = \frac{1}{2} \epsilon^{0mn} (g_{\mu 0} \nabla_m S_{n\nu} + g_{\nu 0} \nabla_m S_{n\mu}) + \frac{1}{2} \epsilon^{k0m} (g_{\mu k} \nabla_0 S_{m\nu} + g_{\nu k} \nabla_0 S_{m\mu}) + \frac{1}{2} \epsilon^{km0} (g_{\mu k} \nabla_m S_{0\nu} + g_{\nu k} \nabla_m S_{0\mu}). \quad (\text{C.97})$$

Using the expression for three index  $\epsilon$  tensor and renaming the indices we can write

$$C_{\mu\nu} = \frac{1}{2n} \epsilon^{mn} (g_{\mu 0} \nabla_m S_{n\nu} + g_{\nu 0} \nabla_m S_{n\mu} + g_{\mu n} \nabla_0 S_{m\nu} + g_{\nu n} \nabla_0 S_{m\mu} - g_{\mu n} \nabla_m S_{0\nu} - g_{\nu n} \nabla_m S_{0\mu}) \quad (\text{C.98})$$

and in terms of the partial derivatives it is

$$C_{\mu\nu} = \frac{1}{2n} \epsilon^{mn} \left( g_{\mu 0} (\partial_m S_{n\nu} - \Gamma_{m\nu}^0 S_{n0} - \Gamma_{m\nu}^k S_{nk}) + g_{\nu 0} (\partial_m S_{n\mu} - \Gamma_{m\mu}^0 S_{n0} - \Gamma_{m\mu}^k S_{nk}) + g_{\mu n} (\partial_0 S_{m\nu} - \Gamma_{0\nu}^0 S_{m0} - \Gamma_{0\nu}^k S_{mk} - \partial_m S_{0\nu} + \Gamma_{m\nu}^0 S_{00} + \Gamma_{m\nu}^k S_{0k}) + g_{\nu n} (\partial_0 S_{m\mu} - \Gamma_{0\mu}^0 S_{m0} - \Gamma_{0\mu}^k S_{mk} - \partial_m S_{0\mu} + \Gamma_{m\mu}^0 S_{00} + \Gamma_{m\mu}^k S_{0k}) \right). \quad (\text{C.99})$$

By using the general expression above we can obtain the Cotton tensor components.

Hypersurface projection of the Cotton tensor is

$$C_{ij} = \frac{1}{2n} \epsilon^{mn} \left( g_{i0} (\partial_m S_{nj} - \Gamma_{mj}^0 S_{n0} - \Gamma_{mj}^k S_{nk}) + g_{in} (\partial_0 S_{mj} - \Gamma_{0j}^0 S_{m0} - \Gamma_{0j}^k S_{mk} - \partial_m S_{0j} + \Gamma_{mj}^0 S_{00} + \Gamma_{mj}^k S_{0k}) \right) + i \leftrightarrow j \quad (\text{C.100})$$

and projection once to the normal and once to the surface yields

$$C_{0i} = \frac{1}{2n} \epsilon^{mn} \left( g_{00} (\partial_m S_{ni} - \Gamma_{mi}^0 S_{n0} - \Gamma_{mi}^k S_{nk}) + g_{i0} (\partial_m S_{n0} - \Gamma_{m0}^0 S_{n0} - \Gamma_{m0}^k S_{nk}) + g_{0n} (\partial_0 S_{mi} - \Gamma_{0i}^0 S_{m0} - \Gamma_{0i}^k S_{mk} - \partial_m S_{0i} + \Gamma_{mi}^0 S_{00} + \Gamma_{mi}^k S_{0k}) + g_{in} (\partial_0 S_{m0} - \Gamma_{00}^0 S_{m0} - \Gamma_{00}^k S_{mk} - \partial_m S_{00} + \Gamma_{m0}^0 S_{00} + \Gamma_{m0}^k S_{0k}) \right). \quad (\text{C.101})$$

Projecting twice to the surface yields

$$C_{00} = \frac{1}{n} \epsilon^{mn} \left( g_{00} (\partial_m S_{n0} - \Gamma_{m0}^0 S_{n0} - \Gamma_{m0}^k S_{nk}) + g_{0n} (\partial_0 S_{m0} - \Gamma_{00}^0 S_{m0} - \Gamma_{00}^k S_{mk} - \partial_m S_{00} + \Gamma_{m0}^0 S_{00} + \Gamma_{m0}^k S_{0k}) \right). \quad (\text{C.102})$$

Before finding the explicit expressions for the components of the Cotton tensor we need to express the components of the Schouten tensor. From the definition of the Schouten tensor, one has

$$S_{ij} = R_{ij} - \frac{1}{4} \gamma_{ij} R \quad (\text{C.103})$$

and by using the equations (C.30, C.32) projection once to the surface and once to normal yields

$$S_{i0} = n^j S_{ij} + n (D_m k_i^m - D_i k) \quad (\text{C.104})$$

and from equations (C.27, C.32) projection twice normal to the surface gives

$$\begin{aligned} S_{00} &= n^i n^j S_{ij} - n^2 k_{ij}^2 + n n^k (D_m k_k^m - D_k k) \\ &+ n \left( D_k \partial^k n - \dot{k} + n^k D_m k_k^m \right) + \frac{1}{4} n^2 R. \end{aligned} \quad (\text{C.105})$$

Before finding the Cotton tensor components, let us compute some frequently appearing expressions. These are

$$\partial_m S_{nj} - \Gamma_{mj}^0 S_{n0} - \Gamma_{mj}^k S_{nk} = D_m S_{nj} - k_{mj} (D_r k_n^r - D_n k), \quad (\text{C.106})$$

$$\begin{aligned} \partial_m S_{n0} - \Gamma_{m0}^0 S_{n0} - \Gamma_{m0}^k S_{nk} &= n^r (D_m S_{nr} - k_{mr} (D_s k_n^s - D_n k)) \\ &+ n (D_m D_r k_n^r - k_m^k S_{kn}), \end{aligned} \quad (\text{C.107})$$

$$\begin{aligned} \partial_0 S_{mj} - \Gamma_{0j}^0 S_{m0} - \Gamma_{0j}^k S_{mk} &= \dot{S}_{mj} - n k_j^k S_{mk} - S_{mk} D_j n^k \\ &- (\partial_j n + n^r k_{rj}) (D_s k_m^s - D_m k) \end{aligned} \quad (\text{C.108})$$

and

$$\begin{aligned} -\partial_m S_{0j} + \Gamma_{mj}^0 S_{00} + \Gamma_{mj}^k S_{0k} &= D_m (-n^r S_{rj} - n (D_r k_j^r - D_j k)) \\ &+ k_{mj} \left( D_k \partial^k n - \dot{k} + n^k D_s k_k^s + n \left( \frac{1}{4} R - k_{rs}^2 \right) \right), \end{aligned} \quad (\text{C.109})$$

$$\begin{aligned} \partial_0 S_{m0} - \Gamma_{00}^0 S_{m0} - \Gamma_{00}^k S_{mk} &= n \left( \partial_0 D_r k_m^r - \partial_m \dot{k} - S_m^k (\partial_k n + n^r k_{rk}) \right) \\ &n^r \left( \dot{S}_{mr} - n k_j^k S_{mk} - S_{mk} D_r n^k - (\partial_r n + n^s k_{sr}) (D_k k_m^k - D_m k) \right) \end{aligned} \quad (\text{C.110})$$

and

$$\begin{aligned} -\partial_m S_{00} + \Gamma_{m0}^0 S_{00} + \Gamma_{m0}^k S_{0k} &= n^r D_m (-n^s S_{rs} - n (D_s k_r^s - \partial_r k)) \\ + n^r k_{mr} \left( D_k \partial^k n - \dot{k} + n^k D_s k_k^s + n \left( \frac{1}{4} R - k_{rs}^2 \right) \right) &+ n \partial_m n \left( k_{rs}^2 - \frac{1}{4} R \right) \\ - n^k \partial_m n (D_s k_k^s - D_k k) - n \partial_m \left( D_k \partial^k n - \dot{k} + n^k D_s k_k^s \right) &- \frac{1}{4} n^2 \partial_m R \\ + 2n^2 k_{rs} D_m k^{rs} + n k_m^k S_{kr} n^r + n^2 k_m^k (D_s k_k^s - D_k k). \end{aligned} \quad (\text{C.111})$$

Making use of these identities, we can finally obtain the components of the Cotton tensor as follows: full hypersurface projection of the Cotton tensor becomes

$$\begin{aligned}
C_{ij} = & \frac{1}{2n} \epsilon^{mn} \left( n_i (D_m S_{nj} - k_{mj} (D_r k_n^r - \partial_n k)) \right. \\
& + \gamma_{in} \dot{S}_{mj} - n k_j^k S_{mk} - S_{mk} D_j n^k - (\partial_j n + n^r k_{rj}) (D_s k_m^s - \partial_m k) \\
& + D_m (-n^r S_{rj} - n (D_r k_j^r - D_j k)) \\
& \left. + k_{mj} \left( D_k \partial^k n - \dot{k} + n^k D_s k_k^s + \frac{n}{4} R - n k_{rs}^2 \right) \right) + i \leftrightarrow j \quad (C.112)
\end{aligned}$$

and projection once to the surface once normal to the surface becomes

$$\begin{aligned}
C_{i0} = & n^j C_{ij} + \frac{1}{2} \epsilon^{mn} \left( -n (D_m S_{ni} - k_{mi} (D_r k_n^r - \partial_n k)) \right. \quad (C.113) \\
& + n_i (D_m D_r k_n^r - k_m^k S_{kn}) + \gamma_{in} \partial_0 D_r k_m^r - S_m^k (\partial_k n + n^r k_{rk}) \\
& - D_m D_k \partial^k n - D_m (n^k D_s k_k^s) + k_m^k S_{kr} n^r + \partial_m n \left( k_{rs}^2 - \frac{1}{4} R \right) \\
& \left. + n \left( 2k_{rs} D_m k^{rs} - \frac{1}{4} \partial_m R + k_m^k (D_r k_k^r - \partial_k k) \right) \right).
\end{aligned}$$

Projecting twice to the normal of the surface yields

$$\begin{aligned}
C_{00} = & n^i n^j C_{ij} + \epsilon^{mn} \left( -n n^r (D_m S_{nr} - k_{mr} (D_s k_n^s - D_n k)) \right. \quad (C.114) \\
& + (n_i n^i - n^2) (D_m D_r k_n^r - k_m^k S_{kn}) + n_n \partial_0 D_r k_m^r - S_m^k (\partial_k n + n^r k_{rk}) \\
& - D_m D_k \partial^k n - D_m (n^k D_s k_k^s) + k_m^k S_{kr} n^r \\
& \left. + \partial_m n \left( k_{rs}^2 - \frac{1}{4} R \right) + n \left( 2k_{rs} D_m k^{rs} - \frac{1}{4} \partial_m R + k_m^k (D_r k_k^r - \partial_k k) \right) \right).
\end{aligned}$$

For notational simplicity, let us define the following tensors

$$A_{mni} := D_m S_{ni} - k_{mi} (D_r k_n^r - \partial_n k), \quad (C.115)$$

$$B_{mn} := D_m D_r k_n^r - k_m^k S_{kn}, \quad (C.116)$$

$$E_m := 2k_{rs} D_m k^{rs} - \frac{1}{4} \partial_m R + k_m^k (D_r k_k^r - \partial_k k), \quad (C.117)$$

$$\begin{aligned}
C_m := & \partial_0 D_r k m - S_m^k (\partial_k n + n^r k_{rk}) - D_m D_k \partial^k n - D_m (n^k D_s k_k^s) \\
& + k_m^k S_{kr} n^r + \partial_m n \left( k_{rs}^2 - \frac{1}{4} R \right) \quad (C.118)
\end{aligned}$$

and reexpress  $C_{i0}$  and  $C_{00}$  in terms of  $C_{ij}$  and the above tensors as follows

$$C_{i0} = n^j C_{ij} + \frac{1}{2} \epsilon^{mn} (-n A_{mni} + n_i B_{mn} + \gamma_{in} (C_m + n E_m)) \quad (\text{C.119})$$

and

$$C_{00} = n^i n^j C_{ij} + \epsilon^{mn} (-n n^r A_{mnr} + (n_r n^r - n^2) B_{mn} + n_n (C_m + n E_m)). \quad (\text{C.120})$$

Obtaining the last two equations has been our main goal in this part, let us now move on to the field equations.

## C.5 ADM split of the TMG field equations

Matter-coupled cosmological TMG field equations are

$$\mathcal{E}_{\mu\nu} = G_{\mu\nu} + \Lambda g_{\mu\nu} + \frac{1}{\mu} C_{\mu\nu} = \kappa \tau_{\mu\nu}, \quad (\text{C.121})$$

where  $G_{\mu\nu}$  is the three dimensional Einstein tensor with the definition

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R. \quad (\text{C.122})$$

By using the Cotton and Ricci tensor components and the scalar curvature, we obtain the ADM decomposition of the TMG field equation components. Full hypersurface projection of the field equation is

$$\mathcal{E}_{ij} = S_{ij} - \frac{1}{4} \gamma_{ij} R + \Lambda \gamma_{ij} + \frac{1}{\mu} C_{ij} = \kappa \tau_{ij}, \quad (\text{C.123})$$

while projection once to the surface and once normal to the surface gives

$$\begin{aligned} \mathcal{E}_{0i} &= n^j \mathcal{E}_{ij} + n (D_r k_i^r - \partial_i k) \\ &+ \frac{1}{2\mu} \epsilon^{mn} (n_i B_{mn} - n A_{mni} + \gamma_{in} (C_m + n E_m)) = \kappa \tau_{0i} \end{aligned} \quad (\text{C.124})$$

and projection twice to the normal yields

$$\begin{aligned} \mathcal{E}_{00} &= 2n^i \mathcal{E}_{0i} - n^i n^j \mathcal{E}_{ij} + n \left( D_k \partial^k n - \dot{k} + n^k D_k k + n \left( \frac{1}{2} R - k_{rs}^2 \right) \right) \\ &- \Lambda n^2 - \frac{1}{\mu} \epsilon^{mn} n^2 B_{mn} = \kappa \tau_{00}. \end{aligned} \quad (\text{C.125})$$

The field equation components can be used to find the constraint equations by setting  $\mathcal{E}_{ij}$  to  $\kappa\tau_{ij}$  and  $\mathcal{E}_{0i}$  to  $\kappa\tau_{0i}$ . Then we obtain the momentum constraint equation of TMG as

$$\begin{aligned}\Phi_i &:= n(D_r k_i^r - \partial_i k) + \frac{1}{2\mu}\epsilon^{mn}(-nA_{mni} + n_i B_{mn} + \gamma_{in}(C_m + nE_m)) \\ &= \kappa(\tau_{0i} - n^j \tau_{ij})\end{aligned}\quad (\text{C.126})$$

and the Hamiltonian constraint equation as

$$\begin{aligned}\Phi &:= D_k \partial^k n - \dot{k} + n^k D_k k + n\left(\frac{1}{2}R - k_{rs}^2\right) - \Lambda n - \frac{1}{\mu}\epsilon^{mn}nB_{mn} \\ &= \kappa(\tau_{00} - 2n^i \tau_{0i} + n^i n^j \tau_{ij}).\end{aligned}\quad (\text{C.127})$$

Substituting the corresponding tensors, the explicit form of the momentum constraint equation is

$$\begin{aligned}\Phi_i &= n(D_r k_i^r - \partial_i k) + \frac{1}{2\mu}\epsilon^{mn}\left(-n(D_m S_{ni} - k_{mi}(D_r k_n^r - \partial_n k))\right. \\ &+ n_i(D_m D_r k_n^r - k_m^k S_{kn}) + \gamma_{in}\left\{n\left(2k_{rs}D_m k^{rs} - \frac{1}{4}\partial_m R + k_m^k(D_r k_k^r - \partial_k k)\right)\right. \\ &+ \partial_0 D_r k_m^r - S_m^k(\partial_k n + n^r k_{rk}) - D_m D_k \partial^k n - D_m(n^k D_s k_k^s) + k_m^k S_{kr} n^r \\ &\left.\left. + \partial_m n\left(k_{rs}^2 - \frac{1}{4}R\right)\right\}\right) = \kappa(\tau_{0i} - n^j \tau_{ij})\end{aligned}\quad (\text{C.128})$$

and the explicit form of the Hamiltonian constraint equation is

$$\begin{aligned}\Phi &= D_k \partial^k n - \dot{k} + n^k D_k k + n\left(\frac{1}{2}R - k_{rs}^2\right) - \frac{1}{\mu}\epsilon^{mn}n(D_m D_r k_n^r - k_m^k S_{kn}) \\ &\quad - \Lambda n = \kappa(\tau_{00} - 2n^i \tau_{0i} + n^i n^j \tau_{ij}).\end{aligned}\quad (\text{C.129})$$

By using the ADM formulation of the scalar curvature,  $R = 2(3\Lambda - \kappa\tau)$  for the TMG case, the Hamiltonian constraint equation reduces to

$$\begin{aligned}\Phi &= \frac{1}{2}\left({}^{(2)}R + k^2 - k_{ij}^2\right) - \Lambda - \frac{1}{\mu}\epsilon^{mn}(D_m D_r k_n^r - k_m^k S_{kn}) \\ &= \frac{\kappa}{n^2}(\tau_{00} - 2n^i \tau_{0i} + n^i n^j \tau_{ij}).\end{aligned}\quad (\text{C.130})$$

## C.6 Constraint equations and field equations in Gaussian normal coordinates

Up to this point we have worked with generic lapse and shift functions. Dues to the complicated expressions in this generic form, let us restrict the discussion to the Gaussian normal coordinates. Of course this is always possible in a small neighborhood of the spacetime manifold. Coordinate singularities might develop which would make these coordinates unsuitable, but this does not change the relevant discussion here. In the Gaussian normal coordinates, one has

$$ds^2 = -dt^2 + \gamma_{ij} dx^i dx^j, \quad (\text{C.131})$$

where we took  $n = 1$ ,  $n_i = 0$  and with this setting the Hamiltonian constraint equation which is given with equation (C.130) reduces to

$$\Phi = \frac{1}{2} \left( {}^{(2)}R + k^2 - k_{ij}^2 \right) - \Lambda - \frac{1}{\mu} \epsilon^{mn} \left( D_m D_r k_n^r - k_m^k S_{kn} \right) = \kappa \tau_{00} \quad (\text{C.132})$$

and the momentum constraint equations given in equation (C.128) reduce to

$$\begin{aligned} \Phi_i &= D_r k_i^r - \partial_i k + \frac{1}{2\mu} \epsilon^{mn} \left( -D_m S_{ni} + k_{mi} (D_r k_n^r - \partial_n k) \right. \\ &\quad \left. + \gamma_{in} \left( 2k_{rs} D_m k^{rs} - \frac{1}{4} \partial_m R + k_m^k (D_r k_k^r - \partial_k k) + \partial_0 D_r k_m^r \right) \right) = \kappa \tau_{0i}. \end{aligned} \quad (\text{C.133})$$

The extrinsic curvature tensor of the surface in these coordinates simply becomes  $k_{ij} = \frac{1}{2} \dot{\gamma}_{ij}$  and from equation (C.23), three dimensional Ricci tensor can be expressed as

$$R_{ij} = {}^{(2)}R_{ij} + k k_{ij} - 2k_{ik} k_j^k + \dot{k}_{ij}. \quad (\text{C.134})$$

From equation (C.32) scalar curvature reduces to

$$R = {}^{(2)}R + k^2 + k_{ij}^2 + 2\dot{k}. \quad (\text{C.135})$$

We can compute the Schouten tensor by using these expressions and find the expressions of the constraints in the Gaussian normal coordinates. As a final result for the Hamiltonian constraint equation we obtain the following expression

$$\begin{aligned} -\frac{1}{2\mu} \epsilon^{mn} \left( D_m D^k \dot{\gamma}_{kn} - \frac{1}{2} \dot{\gamma}_{sm} \gamma^{ks} {}^{(2)}R_{kn} - \dot{\gamma}_{kp} \dot{\gamma}_{sn} \gamma^{ps} - \ddot{\gamma}_{kn} \right) \\ + \frac{1}{8} \dot{\gamma}_{ij} \left( \dot{\gamma}_{ab} \gamma^{ab} \gamma^{ij} + \dot{\gamma}^{ij} \right) = \tau_{00} + \Lambda - \frac{1}{2} {}^{(2)}R \end{aligned} \quad (\text{C.136})$$

and for the momentum constraint equations we have

$$\begin{aligned}
& -\frac{1}{8\mu}\epsilon^{mn}(\dot{\gamma}_{ab}\gamma^{ab}D_m\dot{\gamma}_{in} - 2\dot{\gamma}^{ks}D_m(\dot{\gamma}_{kn}\dot{\gamma}_{si}) + 2D_m\ddot{\gamma}_{in} - \dot{\gamma}_{mi}D^k\dot{\gamma}_{kn}) \\
& + \frac{1}{8\mu}\epsilon^m{}_i(2\dot{\gamma}^{kp}D_k\dot{\gamma}_{pm} - \dot{\gamma}^{ks}D_m\dot{\gamma}_{ks} + 2D^k\ddot{\gamma}_{km} - \dot{\gamma}_{mk}\gamma^{kl}D^p\dot{\gamma}_{pl}) \\
& + \frac{1}{2}(D^k\dot{\gamma}_{ki} - \gamma^{ab}D_i\dot{\gamma}_{ab}) = \tau_{0i} + \frac{1}{2\mu}\epsilon^{mn}D_m^{(2)}R_{ni}. \quad (\text{C.137})
\end{aligned}$$

Furthermore if we take a conformally flat 2D metric on  $\Sigma$ , which is always possible, we have  $\gamma_{ij} = e^\varphi\delta_{ij}$ , where  $\varphi = \varphi(t, x_i)$  and the metric of the Gaussian normal coordinates becomes

$$ds^2 = -dt^2 + e^\varphi\delta_{ij}dx^i dx^j. \quad (\text{C.138})$$

The Ricci tensor of the two dimensional surface is then

$${}^{(2)}R_{ij} = -\frac{1}{2}\gamma_{ij}e^{-\varphi}\nabla^2\varphi, \quad (\text{C.139})$$

where  $\nabla^2 = \partial_x^2 + \partial_y^2 = \partial_k\partial_k$  here. The scalar curvature of the hypersurface  $\Sigma$  can be expressed as

$${}^{(2)}R = -e^{-\varphi}\partial_k\partial_k\varphi = -\frac{1}{2}e^{-\varphi}(2D_k\partial_k\varphi + \partial_k\varphi\partial_k\varphi). \quad (\text{C.140})$$

The Hamiltonian and the momentum constraints can be expressed as the following equations respectively

$$\frac{1}{4}\dot{\varphi}^2 = \kappa\tau_{00} + \Lambda - \frac{1}{2}{}^{(2)}R, \quad (\text{C.141})$$

$$-\frac{1}{2}\partial_i\dot{\varphi} = \kappa\tau_{0i} + \frac{1}{2\mu}\epsilon^{mn}D_m^{(2)}R_{ni}. \quad (\text{C.142})$$

The relation between the Ricci tensor and the scalar curvature of the hypersurface is

$${}^{(2)}R_{ij} = \frac{1}{2}\gamma_{ij}{}^{(2)}R \quad (\text{C.143})$$

and by using this relation we can express the momentum constraint equations as follows

$$\partial_i\dot{\varphi} = -2\kappa\tau_{0i} - \frac{1}{2\mu}\epsilon^m{}_i D_m^{(2)}R. \quad (\text{C.144})$$

By using the equation (C.141) we can write

$${}^{(2)}R = 2(\kappa\tau_{00} + \Lambda) - \frac{1}{2}\dot{\varphi}^2 \quad (\text{C.145})$$

and substituting the last expression in equation (C.144) we obtain

$$\partial_i \dot{\varphi} = -2\kappa\tau_{0i} - \frac{1}{\mu}\epsilon^m{}_i \kappa\tau_{00} + \frac{1}{2\mu}\epsilon^m{}_i \dot{\varphi} \partial_m \dot{\varphi}. \quad (\text{C.146})$$

For simplicity let us define the current tensor  $J$  as

$$J_i := 2\kappa\tau_{0i} + \frac{1}{\mu}\epsilon^m{}_i \kappa\tau_{00} \quad (\text{C.147})$$

and reexpress the constraint equation as

$$\partial_i \dot{\varphi} = -J_i + \frac{1}{2\mu}\epsilon^m{}_i \dot{\varphi} \partial_m \dot{\varphi}. \quad (\text{C.148})$$

Contracting the last equation with  $\epsilon^{ki}$  gives

$$\partial_i \dot{\varphi} = \frac{2\mu}{\dot{\varphi}} \epsilon_i{}^m J_m + \frac{2\mu}{\dot{\varphi}} \epsilon_i{}^m \dot{\varphi} \partial_m \dot{\varphi}, \quad (\text{C.149})$$

which has the same left hand side with the equation (C.146). This yields the equality

$$-J_i + \frac{1}{2\mu}\epsilon^m{}_i \dot{\varphi} \partial_m \dot{\varphi} = \frac{2\mu}{\dot{\varphi}} \epsilon_i{}^m J_m + \frac{2\mu}{\dot{\varphi}} \epsilon_i{}^m \dot{\varphi} \partial_m \dot{\varphi}. \quad (\text{C.150})$$

Let us reexpress our result by using the definition of the  $J$  tensor which was given in equation (C.147) as

$$\begin{aligned} \frac{2\mu}{\dot{\varphi}} \epsilon^m{}_i \partial_m \dot{\varphi} \left(1 + \frac{\dot{\varphi}^2}{4\mu^2}\right) &= \frac{2}{\dot{\varphi}} \left( \partial_i + \frac{\dot{\varphi}}{2\mu} \epsilon^m{}_i \partial_m \right) \kappa\tau_{00} \\ &+ \frac{4\mu}{\dot{\varphi}} \left( \epsilon_i{}^m + \frac{\dot{\varphi}}{2\mu} \delta^m{}_i \right) \kappa\tau_{0m}. \end{aligned} \quad (\text{C.151})$$

Note that in the vacuum case, the right hand side of the last equation becomes zero and the only solution is  $\varphi_0 = ct$  with a constant  $c$  which is the de Sitter solution and the constant can be obtained from the trace equation as  $c = \frac{2}{l}$ .

### C.6.1 Perturbations around the de Sitter space

We can compute the perturbed constraint equations around the background solution given by  $\varphi_0$ . The perturbation is defined as  $\varphi = \varphi_0 + \delta\varphi$ . Since  $\varphi_0 = ct$  any arbitrary spatial derivative of the  $\varphi_0$  will be zero ( $\partial_i \dot{\varphi}_0 = 0$ ). Nonvanishing terms of the perturbation of the final result which is given in equation (C.151) is then

$$\begin{aligned} \frac{2\mu}{\dot{\varphi}_0} \epsilon^m{}_i \partial_m \delta\dot{\varphi} \left(1 + \frac{\dot{\varphi}_0^2}{4\mu^2}\right) &= \frac{2}{\dot{\varphi}_0} \left( \partial_i + \frac{\dot{\varphi}_0}{2\mu} \epsilon^m{}_i \partial_m \right) \kappa\delta\tau_{00} \\ &+ \frac{4\mu}{\dot{\varphi}_0} \left( \epsilon_i{}^m + \frac{\dot{\varphi}_0}{2\mu} \delta^m{}_i \right) \kappa\delta\tau_{0m}, \end{aligned} \quad (\text{C.152})$$

where we can express the result in terms of the solution constant  $c$  as

$$\begin{aligned} \frac{2\mu}{c} \epsilon^m{}_i \partial_m \delta \dot{\varphi} \left( 1 + \frac{c^2}{4\mu^2} \right) &= \frac{2}{c} \left( \partial_i + \frac{c}{2\mu} \epsilon^m{}_i \partial_m \right) \kappa \delta \tau_{00} \\ &+ \frac{4\mu}{c} \left( \epsilon_i{}^m + \frac{c}{2\mu} \delta^m{}_i \right) \kappa \delta \tau_{0m}. \end{aligned} \quad (\text{C.153})$$

### C.6.2 From de Sitter to anti de Sitter

Wick rotating the parameters as  $x_i \rightarrow ix_i$ ,  $t \rightarrow it$ ,  $c \rightarrow ic$ , one obtains the AdS space and the Gaussian normal coordinates metric (C.138) becomes

$$ds^2 = dt^2 - e^{-ct} (dx^2 + dx^2). \quad (\text{C.154})$$

At the chiral point  $\mu^2 l^2 = 1$ , we have  $c^2 = -\frac{4}{l^2}$  and left hand side of the perturbed equation (C.153) is zero but since the right hand side of the equation is not zero, there exists a linearization instability at this point.

### C.6.3 TMG field equations in Gaussian normal coordinates

We can also analyse the ADM decomposition of the TMG field equations in Gaussian normal coordinates. The ADM decomposition of the Cotton tensor, which is given in equation (C.112), reduces to

$$\begin{aligned} C_{ij} = & \frac{1}{2} \epsilon^{mn} \left( \gamma_{in} \left( \dot{S}_{mj} - k_j^k S_{mk} - D_m (D_r k_j^r - D_j k) \right. \right. \\ & \left. \left. - k_{mj} \left( \dot{k} - \frac{1}{4} R + k_{rs}^2 \right) \right) \right) + i \leftrightarrow j \end{aligned} \quad (\text{C.155})$$

and in terms of the metric components it can be expressed as

$$C_{ij} = \frac{1}{4} \epsilon^{mn} (\gamma_{in} D_m \partial_j \dot{\varphi} + \gamma_{jn} D_m \partial_i \dot{\varphi}). \quad (\text{C.156})$$

Now we can express the TMG field equations as follows

$$\mathcal{E}_{ij} = -\frac{1}{4} \gamma_{ij} (\dot{\varphi}^2 + 2\ddot{\varphi}) + \Lambda \gamma_{ij} + \frac{1}{4\mu} (\epsilon^m{}_i D_m \partial_j \dot{\varphi} + \epsilon^m{}_j D_m \partial_i \dot{\varphi}) \quad (\text{C.157})$$

and writing the covariant derivatives as partial derivatives we have

$$\begin{aligned} \mathcal{E}_{ij} = & -\frac{1}{4}\gamma_{ij} (\dot{\varphi}^2 + 2\ddot{\varphi}) + \Lambda\gamma_{ij} + \frac{1}{8\mu} \left( 2\epsilon^m{}_i \partial_m \partial_j \dot{\varphi} + 2\epsilon^m{}_j \partial_m \partial_i \dot{\varphi} \right. \\ & \left. - \epsilon^m{}_i (\partial_m \varphi \partial_j \dot{\varphi} + \partial_j \varphi \partial_m \dot{\varphi}) - \epsilon^m{}_j (\partial_m \varphi \partial_i \dot{\varphi} + \partial_i \varphi \partial_m \dot{\varphi}) \right). \end{aligned} \quad (\text{C.158})$$





## CURRICULUM VITAE

### PERSONAL INFORMATION

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### EDUCATION

<b>Degree</b>	<b>Institution</b>	<b>Year of Graduation</b>
B.S.	Ondokuz Mayıs University	2008
High School	Ordu Anadolu Lisesi	2003

### PROFESSIONAL EXPERIENCE

<b>Year</b>	<b>Place</b>	<b>Enrollment</b>
2011-	Middle East Technical University 1	Research Assistant
2010-2011	Karamanoğlu Mehmet Bey University 2	Research Assistant

### TEACHING SKILLS

Freshman Physics Lab. (Kinematics, Electricity), laboratory assistant.

Phys 507, Quantum Mechanics I, teaching assistant.

Phys 508, Quantum Mechanics II, teaching assistant.

Phys 513, Gravitation and Cosmology I, teaching assistant.

Phys 514, Gravitation and Cosmology II, teaching assistant.

Phys 548, Supersymmetry and Supergravity, teaching assistant.

## **PUBLICATIONS**

### **International Journal Publications**

1) *Linearization instability of chiral gravity*, Emel Altas and Bayram Tekin, Phys. Rev. D 97, 124068 Published 27 June 2018.

2) *Linearization instability for generic gravity in AdS spacetime*, Emel Altas and Bayram Tekin, Phys. Rev. D 97, 024028 (2018) - Published 24 January 2018.

3) *Holographically viable extensions of topologically massive and minimal massive gravity?*, Emel Altas and Bayram Tekin, Phys. Rev. D 93, 025033 (2016) - Published 28 January 2016.

4) *Exact solutions and the consistency of 3D minimal massive gravity*, Emel Altas and Bayram Tekin, Phys. Rev. D 92, 025033 (2015) - Published 20 July 2015.

### **Conferences, Meetings and Courses Attended**

#### **Oral presentations:**

1) *Recent Trends in String Theory and Related Topics*, Tehran- Iran, May 2016.

Title of talk: On Exact Solutions and the Consistency of 3D Minimal Massive Gravity.

2) *Workshop on topics in Three Dimensional Gravity*, ICTP, Trieste- Italy, March 2016.

Title of talk: Holographically Viable Extensions of Topologically Massive and Minimal Massive Gravity?

### **Poster presentations:**

1) *3rd Karl Schwarzschild Meeting on Gravitational Physics*, KSM 2017, Frankfurt-Germany, July 2017.

Title of the poster: Linearization Instability For Generic Gravity In AdS.

2) *The 25th Workshop on General Relativity and Gravitation in Japan*, JGRG25, Kyoto-Japan, December 2015.

Title of the poster: On Exact Solutions and the Consistency of 3D Minimal Massive Gravity.

### **Meetings:**

1) *GGI Lectures on the Theory of Fundamental Interactions*, Winter school, Firenze-Italy, January 2018.

2) *13th Workshop on Quantization, Dualities, and Integrable Systems*, Istanbul, April 2014.

3) *16th Workshop on Quantization, Dualities, and Integrable Systems*, Istanbul, April 2018.