# HEURISTICS FOR THE PROBABILISTIC CAPACITATED MULTI-FACILITY WEBER PROBLEM

by

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# ABSTRACT


## HEURISTICS FOR THE PROBABILISTIC CAPACITATED MULTI-FACILITY WEBER PROBLEM

In this study, we consider the probabilistic capacitated multi-facility Weber problem with general distance functions and probability density functions where customer locations are assumed to be random variables. The aim of the problem is to find supply points and allocations by minimizing the expected sum of demand weighted distances. Since customer locations are random, it is not always possible to obtain analytical expressions for expected distances between facilities and customers. First, we used general distance functions and probability distributions for our problem. Afterwards, we specialized the problem using two distance functions to examine different instances: Euclidean and rectilinear distances. For the Euclidean distance, we assume that customer locations follow symmetric bivariate normal and symmetric bivariate exponential probability distributions. In addition to these two probability distributions, symmetric bivariate uniform probability distribution is also assumed for the rectilinear distance.

Solving this problem to optimality with known optimization techniques is very hard since the objective function of the problem is neither convex nor concave. In order to solve the problem, we have developed four heuristic methods. The first one is the probabilistic alternating location-allocation heuristic and the other three of them depend on discrete approximations. Furthermore, we have also developed three approximation methods for calculating the expected distances for different situations. These new approaches are implemented and computational results based on extensive experiments are also provided.

# ÖZET

## RASSAL BÜTÜNLEŞİK YER SEÇİMİ -TAŞIMA PROBLEMLERİNİ ÇÖZMEK İÇİN BULUŞSAL YÖNTEMLER

Bu çalışmada, rassal bütünleşik yer seçimi-taşıma problemleri müşteri koordinatlarının rassal değişken olduğu varsayılarak genel uzaklık fonksiyonları ve olasılık yoğunluk fonksiyonları için ele alınmaktadır. Problem kısaca istem ağırlıklı uzaklıklar toplamının beklenen değerini en küçükleyen tesis yerlerinin ve dağıtım planının bulunmasıdır. Müşteri koordinatlarının rassallığı nedeniyle tesisler ve müşteriler arasındaki uzaklıkların ifade edilmesi çok zordur. İlkin, problem genel uzaklık fonksiyonları ve olasılık dağılımları için ele alınmıştır. Daha sonra bazı özel durumların incelenebilmesi için iki değişik uzaklık fonksiyonu incelenmiştir: Öklid uzaklık fonksiyonu ve yatay-dikey uzaklık fonksiyonu. Öklid uzaklık fonksiyonunun kullanıldığı durumlarda müşteri koordinatlarının dengeli iki değişkenli normal dağılıma ve dengeli iki değişkenli üstel dağılıma uydukları varsayımı yapılmıştır. Yatay-dikey uzaklık fonksiyonunun kullanıldığı durumlarda ise müşteri koordinatlarının normal ve üstel dağılıma ek olarak dengeli iki değişkenli düzgün dağılıma uyduğu varsayımı yapılmıştır.

Bu problemin eniyi çözümünü bilinen eniyileme yöntemleriyle bulmak çok zordur çünkü problemin amaç fonksiyonu ne içbükey ne de dışbükeydir. Bu nedenle, problemi çözmek için bu çalışmada dört sezgisel yöntem geliştirilmiştir. Birinci yöntem olasılıksal yer seçimi-taşıma algoritmasıdır. Diğerleriyse ayrık yaklaşıklamayı taban alırlar. Bunun dışında, genel durumlarda beklenen uzaklık fonksiyonunun hesaplanabilmesi için üç ayrı yaklaşıklama yöntemi geliştirilmiştir. Bütün bu yeni yaklaşımlarla çok sayıda bilgisayısal deney yapılmış ve sonuçları verilmiştir.

# TABLE OF CONTENTS

# LIST OF FIGURES

# LIST OF TABLES

# LIST OF SYMBOLS/ABBREVIATIONS

| | |
|---|---|
| $c_{ij}$ | Unit transportation cost per unit distance between facility $i$ and customer $j$ |
| $d_j$ | Demand of customer $j$ |
| $d(\mathbf{x_i}, \mathbf{a_j})$ | Distance between points $x_i$ and $a_j$ |
| $e_j$ | Lower bound for customer $j$'s abscissa |
| $f_j(a_j)$ | Probability density function of customer $j$'s location |
| $f_j$ | Upper bound for customer $j$'s abscissa |
| $g_j$ | Lower bound for customer $j$'s ordinate |
| $h_j$ | Upper bound for customer $j$'s ordinate |
| $H(.,.,.)$ | Confluent hypergeometric function |
| $h$ | Number of horizontal cells in Cellular heuristic |
| $i$ | Facility index |
| $j$ | Customer index |
| $k$ | Candidate point index |
| $m$ | Total number of facilities to be located |
| $n$ | Total number of customers |
| $o$ | Total number of candidate points |
| $s_i$ | Capacity of facility $i$ |
| $T$ | Number of PALA runs in discrete approximations |
| $\mathbf{x_i}^{(r)}$ | Coordinates of facility $i$ obtained in the $r^{th}$ iteration of the Weiszfeld's type iterative equations. |
| $u_{ikj}$ | Amount of goods to be sent from facility $i$ to customer $j$ when located at candidate point $k$ |
| $v$ | Number of vertical cells in the Cellular heuristic |
| $w_{ij}$ | The amount shipped from facility $i$ to customer $j$ |
| $y_{ik}$ | Binary variable to determine whether the facility $i$ is located at candidate point $k$ or not |
| $\Gamma(.)$ | Gamma function |

| | |
|---|---|
| $\epsilon$ | A very small quantity |
| $\boldsymbol{\mu_j}$ | Mean location vector of customer $j$ |
| $\rho_{ij}$ | Covariance between vector coordinates $i$ and $j$ |
| $\sigma_j$ | Standard deviation of customer $j$'s location |

| | |
|---|---|
| ACMFWP | Approximated capacitated multi-facility Weber problem |
| ALA | Alternating location-allocation |
| APCMFWP | Approximated probabilistic capacitated multi-facility Weber problem |
| CMFWP | Capacitated multi-facility Weber problem |
| HAP | Hyperboloid approximation procedure |
| LAP | Location-allocation problem |
| LP | Linear programming |
| MFWP | Multi-facility Weber problem |
| MILP | Mixed integer linear program |
| MPALA | Multi-start probabilistic alternating location-allocation |
| PALA | Probabilistic alternating location-allocation |
| PCMFWP | Probabilistic capacitated multi-facility Weber problem |
| PWP | Probabilistic Weber problem |

# 1. INTRODUCTION

In a generic location problem the aim is to determine where to locate facilities such as plants, distribution centers, warehouses and factories. Major parameters of location problems are demands, distances, costs and timing namely planning periods. These parameters can be estimated during the modelling process. If the parameters are given and assumed fixed then the models which contributes the decision about the facility locations are said to be deterministic models. However, estimations may include errors because of the poor measurement or methods (Snyder 2005). On the contrary, if there is no information about the parameters then the decision techniques that helps under uncertainty can be used. However, most of the time, some partial informations can be collected about the parameters. In this case, stochastic programming techniques are applicable for solving the problem.

Weber problem is one of the type of location problems which has already studied by many researchers (Wesolowsky 1993). In Weber problem the aim is to find a supply point in a plane such that the sum of demand weighted distances between suppy point and demand points is minimized.

The extensions of the Weber problem can be constructed by altering its parameters. In the first extension, one can change the number of supply points that are to be placed. If the aim is to locate a single supply point then the problem is called single facility Weber problem. On the other hand, if the number of supply points is more than one, then the problem is named as multi-facility Weber problem (MFWP).

The second extension can be obtained when the supply points are assumed to have limited capacities. When the capacity is issued the problem can be classified as capacitated or uncapacitated. For the naming convention, if the facilities of the multi-facility Weber problem have limited capacities then the problem is said to be capacitated multi-facility Weber problem (CMFWP).

In the third extension, the coordinates of the demand points may not be known exactly. As it is mentioned before, there may exist some partial information about the coordinates of the demand points. Specifically, coordinates of the demand points may fit in a probabilistic distribution. In addition, travel times, price of the products or travel cost can be the random elements (Louveaux 1986). In these cases, the problem can be solved by using stochastic programming techniques. According to Aly and White (1978), there are four approaches for solving this type of stochastic location problem: expected value criterion, portfolio criterion, aspiration-level criterion and fractile criterion. In expected value criterion, locations of the supply points are determined by minimizing the adapted expected-cost function. On the other hand, portfolio criterion minimizes the variance of the cost function. In aspiration-level criterion, there is a specific value where the aim is to locate facilities such that the probability of cost being less than the specific value is maximized. Finally, the fractile criterion method locates the supply points where the value on cumulative distribution of total cost is minimized which exhibits the acceptable probability of cost exceeding that value.

In Weber problem, distances between supply point and the demand points can be measured in a various way. Table 1.1 shows the most frequently used distance functions in location theory (Francis *et al.* 1992). In Table 1.1, $\mathbf{x_1} = (x_{11}, x_{12})$ and $\mathbf{x_2} = (x_{21}, x_{22})$

Table 1.1. Most frequently used distance functions in location theory.

| **Distance** | **Formula** |
|---|---|
| Euclidean | $d(\mathbf{x_1}, \mathbf{x_2}) = \left(\lvert x_{11} - x_{21}\rvert^2 + \lvert x_{12} - x_{22}\rvert^2\right)^{\frac{1}{2}}$ |
| Squared Euclidean | $d(\mathbf{x_1}, \mathbf{x_2}) = \left(\lvert x_{11} - x_{21}\rvert^2 + \lvert x_{12} - x_{22}\rvert^2\right)$ |
| Rectilinear | $d(\mathbf{x_1}, \mathbf{x_2}) = \lvert x_{11} - x_{21}\rvert + \lvert x_{12} - x_{22}\rvert$ |
| $\ell_p$ distance | $d(\mathbf{x_1}, \mathbf{x_2}) = (\lvert x_{11} - x_{21}\rvert^p + \lvert x_{12} - x_{22}\rvert^p)^{1/p}$ $p \geq 1$ |

illustrates the coordinates of two points. In $\ell_p$ norm, when $p = 1$ the distance is said rectilinear or Mannhattan distance and when $p = 2$ the distance is called Euclidean distance.

In this study, we are trying to solve probabilistic capacitated multi-facility Weber problem (PCMFWP) where distances between facilities and demand points are

Euclidean, rectilinear and $\ell_p$ distance with $1 < p < 2$. The deterministic version of the problem has been very attractive for researchers and there are many exact and approximate algorithms exist in literature. However, deterministic version of the problem is not usefull in most of the real world situations. For instance, locating emergency stations emergency calls are randomly distributed coordinate vectors. Thus, it is appropriate to assume that coordinates of the demand points are random variables.

In capacitated multi-facility Weber problem, locations of the facilities and allocations should be determined simultaneously for minimizing the cost function. Therefore, these type of problems are also called location-allocation problems (LAP). CMFWP is very hard to solve optimally because it is a non-convex optimization problem. However, many efficient heuristic methods can be found in the literature. One of the most usefull heuristic method for solving CMFWP is the alternating location-allocation (ALA) algorithm (Cooper 1978). The exact solution method also developed by Cooper (1972). However, this method is based on explicit enumeration of all the extreme points of the transportation polytope. Thus, the exact solution method (Cooper 1978) is not useful when the size of the problem increases. On the other hand, ALA heuristic converges to a local optimum solution very fast. Therefore, the final solution of the algorithm depends on the initial facility locations. In order to increase the quality of the solution, Aras *et al.* (2006) offered some heuristic methods; $p$-median heuristic, Cellular heuristic and Lagrangean heuristic.

Several attempts also have been made to solve probabilistic single-facility Weber problem with the Euclidean distance function. Cooper (1974) and Katz and Cooper (1974) assume that demand points are independently distributed with bivariate normal probability function. Katz and Cooper (1976) also tried to solve the same problem where the demand points fit in bivariate exponential and bivariate symetric exponential distributions. After these studies, Wesolowsky (1977) studied the same problem with rectangular distances where the customer coordinates fit in bivariate uniform, bivariate exponential and bivariate normal distributions. There are also studies (Aly and White 1978, Rao and Varma 1985) which are related to multifacility location problem where demand points are assumed to be a random variables.

In this study, we first develop some heuristics for solving the PCMFWP both with Euclidean and rectilinear distance functions wgich are then generated for more general distance functions. In Euclidean distance case, we assume that customer locations are distributed with symmetric bivariate exponential and symmetric bivariate normal probability distributions. In rectilinear distance case, we assume that customer locations fit in symmetric bivariate uniform, symmetric bivariate exponential and symmetric bivariate normal probability distributions. Since this problem is a stochastic optimization problem we use the expectation criterion method (Aly and White 1978). First of all, in order to solve all these special cases of the PCMFWP, we develop probabilistic alternating location-allocation heuristics (PALA). Afterwards, we also develop probabilistic discrete approximation heuristic and two cellular heuristic depending on Aras *et al.* (2006). Second, we try to develop Taylor Approximation based heuristics. The reason why we try Taylor Approximation is to develop generic solution procedures for any probability distribution assumptions. Finally, we assume that we do not know the probability distribution of the demand points and the only information is some sample points for each demand points. After that, we again develop modified version of the probabilistic ALA for this situation.

In the remainder of the thesis, we first review the literature which are related to PCMFWP. Afterwards, the formulation of the problem is given. In the fourth part, we explain how to approximate the expected distance function. As a sequence, we explain our solution methods and we illustrate the computational results of the methods. Eventually, we conclude the thesis by making our comments.

# 2. LITERATURE SURVEY

In this work, we concentrate on the PCMFWP where facilities have finite capacities and customer locations are random vectors. Therefore, a detailed critical survey of the PCMFWP and related problems are examined in this section. Some of the works which we review are also presented in the survey by Snyder (2005). This is more general survey since it covers many aspects of uncertainty in location theory.

## 2.1. Deterministic Weber Problem

Brimberg and Love (1993) studied on the single facility Weber problem where distances are modeled with the $\ell_p$ norm. Previously, an iterative solution tecnique for the Euclidean distance is introduced by Weiszfeld (1937) whose convergence is studied in Katz (1974). Brimberg and Love (1993) studied Weiszfeld procedure for $1 \leq p \leq 2$ and proved a descent property. They also showed that although the descent property does not hold for values $p \geq 2$, Weiszfeld's procedure converges globally. In addition, the authors also showed the nonconvergence of the procedure for $p > 2$. When $p > 2$, the descent property does not hold and there may exist two or more limit points for Weiszfeld's procedure.

## 2.2. Deterministic Multi-Facility Weber Problem

Cooper (1963) studied the location-allocation problems and derived the exact extremal equations for this class of location problems. In the problem, locations of the $n$ demand points are assumed to be known. The aim is to locate $m$ facilities such that the sum of distances between demand points and the allocated facilities are minimized. Therefore, the problem is not only about finding the locations of the facilities, but also about finding the allocations of the facilities. Obviously, the allocation problem is easy to solve when the locations of the facilities are known. Therefore, Cooper derived a formula for finding the best locations of the facilities. However, it is also admitted by the author, when the size of the problem is getting larger, solving these equations

requires a great amount of time. Thus, Cooper (1963) proposed a heuristic method in order to solve the problem efficiently. According to this heuristic method, the set of $m$ facility locations are selected from the set of $n$ demand points, then the allocations are determined. After allocations are determined, the optimum facility locations are found by solving the extremal equations which are derived at the beginning of this study. In addition to this heuristic method, Cooper (1963) also announced several other heuristics in the same work.

One year later, Cooper (1964) developed new heuristic methods for location-allocation problems. At the beginning of the study, formulations for obtaining lower and upper bounds on the objective function of the location-allocation problems are derived. After that, four new heuristic methods are introduced. The first heuristic method selects a set of $m$ facility locations among $n$ customer locations, alternatively demand points. Here, $m$ represents the number of facilities, which are tried to be located in order to minimize the sum of distances between $n$ customers and the allocated facilities. As a sequence, the allocations are determined and the exact extremal equations are applied. In this method, all possible subsets of $m$ demand points, from the total set of $n$ demand points are tried in order to find the optimum solution. The second approach depends on the first heuristic. In the first heuristic, which he called destination subset algorithm, all possible subsets are tried for solving the problem, however, in the second algorithm the set of $m$ facility locations among $n$ customer locations is selected randomly. The third approach is also inspired from the destination subset algorithm. At the beginning of this algorithm, all possible sets of two points from the set of $n$ customer locations are selected as a location of two facilities and the two points which have the minimum value of the sum of distances between demand points and the allocated two facilities are selected as a solution. After that, the third location for the third facility is seeked from the remaining set of $(n-2)$ demand points, and then the fourth facility is located and this iterative procedure is applied until all facilities are located. When all facilities are located, the allocations are determined by assigning each customer to its closest facility. Conversely, the last algorithm, alternate location-allocation heuristic, does not resemble the previous algorithm. The idea is to divide the demand points into $m$ subsets. After that, the single facility location prob-

lem is solved by using the exact extremal equations for each subset. As a sequence, each demand point is examined whether the closest facility of that demand point is changed or not. If the closest facility is different from the previous allocated facility then the new allocations are determined. These location and allocation processes are applied sequentially until there is no changes in the allocations. Eventually, all of these proposed heuristic methods are compared by using some randomly generated problems. The result of these experiments shows that random destination algorithm and the subsequent improvement by a single calculation of exact extremal equations is the best for large location-allocation problems.

Hansen *et al.* (1998) developed a $p$-median problem based heuristic in order to solve the uncapacitated multi-facility Weber problem. In $p$-median problem, facilities must be located among a given set of candidate points. Hansen *et al.* (1998) claimed that most of the time some of the locations chosen for a facility in the multi-facility Weber problem coincide with demand points. Besides, they asserted that solving the $p$-median problem is easier than solving the multi-facility Weber problem. As a result, the authors developed a heuristic method where demand points are assumed to be the candidate points for the facility locations. After the $p$-median problem is solved, the points selected by the $p$-median solution are used as an initial points for Cooper's alternate location-allocation (ALA) heuristic (Cooper 1964). In the study, the $p$-median heuristic is compared with the projection method (Bongartz *et al.* 1994) and with the multistart alternate location-allocation heuristic (MALT)(Cooper 1963). In the computational part of the study, it is shown that $p$-median heuristic finds better results than the MALT. However, when the the size of the problem increases, the run time of the $p$-median based heuristic increases very much. In order to shorten the run time of the heuristic, the authors offered some heuristic methods for solving the $p$-median problem such as variable neighborhood heuristics.

Gamal and Salhi (2003) also developed a heuristic method for solving uncapacitated multi facility Weber problem which is named as cellular heuristic. Gamal and Salhi (2003) use Cooper's iterative location-allocation algorithm (Cooper 1964) as a base for their cellular heuristic. As the authors claimed that they take into account pre-

vious information which is obtained from the ALA heuristic. In the algorithm, ALA heuristic is solved for a pre-determined times (K). Afterwards, all facility locations gathered from this multistart algorithm are used as a base for the cellular heuristic. In the cellular heuristic, the biggest rectangle which covers all of the candidate locations is drawn. After that, this rectangle is divided into cells and all candidate points are assigned to related cells. The way how cells are constructed is presented in the study. Consequently, centroids of the each non-empty cells are assumed to be candidate points. Furthermore, $p$ facilities are selected from these centroids by using frequency and compatibility criteria. At the end, ALA heuristic is applied once more by starting of these $p$ points. In the study, some test problems are solved by the cellular heuristic and the results are compared with the optimum solutions. According to authors, the cell-based approach can improve the solutions which are found by the multi-start ALA heuristic especially for the problem with equal weight. Besides, the efficiency and the effectiveness of the algorithm is emphasized. However, when the number $p$ is large, the deviation from the optimal solution increases.

### 2.3. Deterministic Capacitated Multi-Facility Weber Problem

In order to solve the deterministic CMFWP, Cooper (1972) developed an exact algorithm. In his previous works (Cooper 1963, Cooper 1964), Cooper studied on the uncapacitated version of the problem. In this study, Cooper proved that the objective function of the multifacility Weber problem is neither a convex function nor a concave function. After that, Cooper (1972) proved that the optimal solution of the capacitated multi-facility Weber problem occurs at an extreme point of the tranportation polytope. Consequently, the exact algorithm for the CMFWP based on the enumeration of the extreme points of transportation polytope is developed. To clarify, in the first step of the algorithm, all extreme points of the transportation polytope are found and then the alternating location-allocation heuristic is applied for all single facility Weber problem which is again developed by Cooper (1964). However, as also admitted by the author, this method can be used for solving relatively small problems. In addition to exact algorithm Cooper (1972) also construct an efficient alternating transportation-location heuristic method. In this algorithm, facilities are located on the demand points

randomly and the allocations are determined by using these locations. As a sequence, facilities are relocated by using the ALA algorithm. After that, new allocations are determined by solving the transportation problem. These two steps, location and allocation, are applied until no further changes on the allocation or facility locations is obtained. According to Cooper (1972), this algorithm yields a convergent monotone nonincreasing sequence of the objective function. However, as Cooper stated, this algorithm does not quarentee to find the global minimum. On the other hand, the computational results show that this algorithm mostly finds a solution within the ten percent of the optimal solution.

The CMFWP where the distances are assumed to be rectilinear is also studied by Aras *et al.* (2006). At the beginning of the study, authors proposed a new integer linear programming formulation for the problem. As it is admitted, the idea behind this formulation depends on the study of Wendell and Hurter (1973). Wendell and Hurter (1973) showed that CMFWP with rectilinear distance has always an optimal solution with facilities located at the intersection points of vertical and horizantal lines drawn through the customer locations. In addition to this result, it is stated that only the intersection points which lie in the convex hull of customer locations should be considered as candidate points for the optimal facility locations.Indeed, this model can be solved by standard optimization packages. However, as the size of the problem increases the computation becomes prohibitive. Therefore, three new heuristics are developed for solving this mathematical model. The first one is the Lagrangean Relaxation method where the demand constraint is relaxed. The second method is the $p$-median heuristic where the customer locations are considered as candidate locations for facilities and the third heuristic is the modified version of the cellular heuristic which was originally studied by Gamal and Salhi (2003). In the cellular heuristic, centroids of the non-empty cells are used as candidate points. Finally, all of these heuristics are tested on some test problems which are available in the literature. The deviations from the optimal results and the computational times are given in order to make comparisons.

Aras *et al.* (2006) also studied the CMFWP with Euclidean, squared Euclidean and $\ell_p$ distances. In the work, three new heuristic methods are proposed which are

based on an approximated mixed integer linear program. The first heuristic is the Lagrange Relaxation and subgradient optimization method of the approximated mixed integer linear program. The second one is the $p$-median heuristic where the customer locations are assumed as potential facility sites. The third heuristic, cellular heuristic, tries also to reduce the number of good candidate points, but performs this in an intelligent way by using the results generated by ALA type two-phase heuristic. At the end of these heuristics, two-phase method is applied in order to improve the results. Furthermore, these three heuristics are applied to some problem sets which are collected from the literature. It is observed that the results are accurate and efficient.

## 2.4. Probabilistic Weber Problem

The first method to solve the probabilistic Weber problem is due to Cooper (1974). In this work, Cooper assumed that demand points are bivariate normal random variables. The aim of the problem is to minimize the sum of expected weighted distance between destinations and a center. Cooper formulated the objective function and proved that the objective function is strictly convex. After all, Cooper constructed a globally convergent iterative method. However, Cooper does not consider any other probabilisty distributions. He concluded that he gained extremely inaccurate results when he tried to solve 200 randomly generated problems.

Katz and Cooper (1974) also studied probabilistic Weber problem. The authors developed linearly convergent descent method for this class of problems. For the numerical computations, the authors tried different initial solutions for the source coordinates and their method always converges to the same solution rapidly. Thus the iterative solution method is independent from the initial solution. They also mathematically proved that their iteration is globally convergent. In their examples, 30 destinations are chosen and their bivariate normal density function parameters are generated randomly as the destination weights. The results and some formulas which are used to solve this problem are given in the study. Furthermore, the authors suggested to use Steffensen's iteration for quicker computation. However, they did not try to solve the problem by Steffensen's iteration.

Two years later, Katz and Cooper (1976) improved further their previous results on the probabilistic Wber problem. This time they proved that optimal location for the facility, $P$, is not necessarily in the convex hull of customers, namely $P_j$'s. However, the optimal location for the facility in the original deterministic case is in the convex hull of customer locations. On the other hand, the authors showed the sufficient conditions for the optimal facility location to be in the convex hull for the probabilistic case. In this study, symmetric bivariate normal, bivariate exponential and symmetric bivariate exponential distributions are assumed for customer locations. After the problem is modeled, the authors developed upper bounds for the objective function and some mathematically equivalent expressions for eliminating the double integral computations which require long quadratures. In their equivalent formulas, the authors suggested to use Bessel functions in order to evaluate double integrals. Besides, they use Steffensen's iteration method in numerical computations for reducing the computation times. The authors also made comparisons between the exact results of the objective functions and its upper bounds. All numerical results and their example problems are given in the work.

Wesolowsky (1977) developed an exact solution method for the probabilistic Weber problem with rectilinear distance. According to Wesolowsky, rectilinear norm is more appropriate to measure urban distances. In the study, the situations where the demand points are distributed with symmetric bivariate normal, symmetric bivariate exponential and symmetric bivariate uniform are studied. Wesolowsky, showed that the expected cost function is seperable. The computational results showed that the type of the distribution assumptions for the demand points has little effect on the optimum location of the facility. According to author, this result is obvious when the distances between destination means are very large when compared with the standard deviations of the distributions.

### 2.5. Probabilistic Multi-Facility Weber Problem

Aly and White (1978) considered the probabilistic multifacility Weber problem. They partitioned the facilities into two subsets, the old and the new ones, and assumed

that there can be transportations between new facilities, although this is not possible for the old ones. However, Aly and White (1978) do not make any probabilistic distribution assumptions about customer coordinates. In order to solve the problem, they take the derivatives of the expected objective function with respect to facility coordinates. Afterwards, they applied the hyperboloid approximation procedure in order to avoid the undefined partial derivatives.

Rao and Varma (1985) studied also the probabilistic multi-facility minisum location problem with Euclidean distances. The authors assumed that there are $m$ existing facilities whose coordinates are distibuted with symmetric bivariate normal distribution and the objective is to locate $n$ new facilities by minimizing sum of the weighted expected distances among the facilities. While formulating the model, weighted distances between new facilities are included into the model. After modeling the problem, the authors proved that the objective function which is to be minimized is strictly convex. After showing this property, the derivatives of the objective function are taken and a formula for finding a global minimum is obtained. After that, an iteration scheme is introduced which is also proved to be a descent method. Furthermore, the authors solved an example problem by Weiszfield type iteration method and the Steffensen's iteration method. The authors made a conclusion that by using Steffensen's iteration method, the objective function converges to vicinity of the optimal solution with less iteration. It is also claimed that initial solutions do not have any influence on the required number of iterations.

One year later, Louveaux (1986) studied the discrete probabilistic location models. In the study, demands, production costs, transportation costs and the selling prices are assumed to be random variables. At the beginning of the study, the author gives the formulation of simple plant location problem. As a sequence, the random variables are introduced and the models are reconstructed with these random variables. Louveaux (1986) also emphasized the budget constraint where the $p$ value could be obtained. In the final part of the study, the price system is included in the stochastic $p$-median problem so that the relation between stochastic versions of the simple plant location problem and the stochastic version of the $p$-median problem is discussed. However,

solution methods to solve these types of problems are not given in the study.

After six years from the work by Louveaux (1986), the probabilistic discrete facility location problem where demands, prices, production and transportation costs are random variables is studied by Louveaux and Peeters (1992). In order to solve the problem, a heuristic dual based solution procedure is offered. This procedure may thought to be an extension of the solution which was proposed by Erlenkotter for the deterministic case. After constructing mathematical model of the probabilistic version of the problem, integrality restriction is relaxed and the dual of the model is obtained. Afterwards, condensed dual of the model is obtained by making some substitutions. Furthermore, the solution procedure for the condensed dual problem is given. It is known that there may be duality gap between the solutions of the dual and primal problem. In order to decrease this gap, some primal-dual adjustment procedures are given in the study.

Many variants of the probabilistic location problems again studied by Louveaux, (1993) in a similar fashion as author's previous study (Louveaux 1986). In this study, typical elements, which can be a random variable in the discrete location problems are introduced. After that, Louveaux classified the problems such as stochastic location problems on networks, stochastic queuing location problems, discrete stochastic location and set covering problems. In each class, sub problems which are obtained by changing the parameters are modeled and each of the models is shown in detail. However, the author do not attempt to give the solution procedure for each problem since the aim of the study is generalization and the classification of the probabilistic discrete location problems.

In a recent study, Altınel *et al.* (2005) developed two heuristics for solving the probabilistic multi-facility Weber problem. The first heuristic is based on the Cooper's ALA heuristic (Cooper 1964). In multi-facility Weber problem, if the locations of the facilities are known then the allocation can be determined very easily by assigning each customers to the closest facility. On the other hand, if the allocations are known then the locations of the facilities can be found by using Weiszfeld's algorithm (Weiszfeld

1937). However, in the probabilistic version, customer locations are not known exactly. Furthermore, customers locations are assumed to fit in symmetric bivariate normal distribution. Thus, in this study the authors minimize the sum of expected distances. Therefore, an ALA type local search procedure is developed by the authors. In addition, the second heuristic is based on the first procedure. However, in the second one, variable neigborhood search (VNS) is combined with this procedure. Therefore variable neighborhood search improves the local search procedure as expected which is also shown by computational results.

### 2.6. Probabilistic Capacitated Multi-Facility Weber Problem

Williams (1963) studied probabilistic transportation problem in which the stock-out penalty costs and the inventory carrying costs namely salvage values are considered. In the problem, there are $n$ destinations which have probabilistic demands and there are $m$ prospective facilities which are assumed to be have finite capacities. Moreover, the unit transportation costs between sources and destinations are also considered. In the study, nothing is stated about the distances between sources and destinations. The aim is to minimize fatal expected transportation cost. First of all, a model is build for an uncapacitated single source and a single destination problem. As the author stated, the problem with uncapacitated sources would not be sophisticated then a single uncapacitated source. Then, the problem is generalized for $n$ destinations and $m$ capacitated sources. Furthermore, Williams showed the convexity of the objective function when the stock-out costs are larger than the salvage values. After all, the author showed that the problem could be solvable by applying Dantzig Decomposition (Dantzig 1955).

Cooper (1978) also studied the probabilistic location-allocation problem where the demand points are random variables. In this study, Cooper developed both an exact algorithm and a heuristic method for solving the PCMFWP where these two algorithm is inherited from his early studies (Cooper 1972, Cooper 1974). At the beginning of the study, Cooper proved that the objective function of the probabilistic version of the CMFWP is neither a convex function nor a concave function. Besides,

the author stated that the exact solution of the problem is found on the extreme points of the transportation polhedron. After that Cooper derived one iterative Weiszfeld's type procedure. In the exact solution method, all of the basic feasible solutions of the transportation problem is determined. In consequence, these basic feasible solutions are used as the allocation and the location problem is solved by using this procedure. As a result, all local minimum points are determined by this enumeration method. Therefore, the global minimum point can easily be found among these local minimums. However, the exact algorithm can be used to solve the small instances of the problem. It is obvious because the number of the basic feasible solutions of the transportation problem is very large to enumerate in a reasonable time. In order to handle this diffuculty, Cooper also developed a transportation-location heuristic which is similar to the one of his previous study (Cooper 1972). In this heuristic, first a feasible allocation is obtained. Then, by using this allocations, a location problem for each facility is solved by using the new Weiszfeld like procedure. Moreover, this solution equations are applied to the adjacent extreme points of the starting feasible solution. Briefly, this procedure is applied like the neighborhood search. In addition, both the exact algorithm and the heuristic method is applied to the example problem.

# 3. PROBLEM FORMULATION

CMFWP is formulated as the following mathematical programming problem:

$$\min \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij} w_{ij} d(\mathbf{x_i}, \mathbf{a_j}) \tag{3.1}$$

s.t.

$$\sum_{i=1}^{m} w_{ij} = d_j \qquad j = 1, \ldots, n; \tag{3.2}$$

$$\sum_{j=1}^{n} w_{ij} = s_i \qquad i = 1, \ldots, m; \tag{3.3}$$

$$w_{ij} \geq 0 \qquad i = 1, \ldots, m;\ j = 1, \ldots, n. \tag{3.4}$$

Here, $n$ represents the number of customers and $m$ represents the number of facilities to be located. In Equation (3.1), $c_{ij}$ is the unit transportation cost between facility $i$ and customer $j$ and $w_{ij}$ is the amount shipped from facility $i$ to customer $j$. Moreover, $d(\mathbf{x_i}, \mathbf{a_j})$ is the distance function between facility $i$ and customer $j$. As familiarized in the introduction section, distance can be measured in a various ways. In distance function, $\mathbf{x_i}$ represents the unknown coordinates of the facility $i$ and $\mathbf{a_j}$ represents the coordinates of the customer $j$. In the formulation, Equation (3.2) assures that demands of the each customer must be satisfied where $d_j$ represents the demand of the customer $j$. Similarly, Equation (3.3) quarantee that each facility can not serve more than its capacity and in this constraint $s_i$ represents the capacity of the facility $i$. In addition, this formulation assumes that the problem is balanced namely total supply is equal to the total demand. If total supply is greater than the total demand then the problem can be balanced with a dummy customer with zero unit transportation cost. However, if the total demand is greater than the total supply then there is no feasible solution to the problem. It is obvious that, each customer can be served from different facilities. On the other hand, if one wants every customer to be served by a single facility then additional binary variables and constraints can be added to the formulation.

As stated in the previous sections, when the locations of the facilities are known, determination of the allocations is just a transportation problem. On the contrary, when the allocations are known, the problem reduces to $m$ single facility Weber problem and they can be solved by the Weiszfeld's iterative algorithm (Weiszfeld 1937). As a consequence, Cooper (1972) showed that an optimal solution of this problem always occurs at an extreme point of the transportation polyhedron. Although pure location and pure allocation problems are easy to solve, the CMFWP is hard to solve. As Sherali *et al.* (1988) have shown that the CMFWP with Euclidean distance is NP-hard.

In PMFWP, it is assumed that customer locations are random vectors and not known exactly. We therefore minimize the expected total cost and the objective function of the PCMFWP becomes

$$\min \; E\left[\sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}w_{ij}d(\mathbf{x_i},\mathbf{a_j})\right]. \tag{3.5}$$

In Equation (3.5), expectation operator can be taken into the summation. Moreover, we can take $c_{ij}$ and $w_{ij}$ outside the expectation operator since they are not random variables. As a consequence, the objective function becomes

$$\min \; \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}w_{ij}E\left[d(\mathbf{x_i},\mathbf{a_j})\right]. \tag{3.6}$$

Now, let us assume that joint probability density function of the customer $j$'s coordinates is $f_j(a_j)$. Then, the general form of the expected distance function between facility $i$ and customer $j$ in two dimensional space becomes

$$E\left[d(\mathbf{x_i},\mathbf{a_j})\right] = \iint d(\mathbf{x_i},\mathbf{a_j})f_j(a_j)\,da_{j1}\,da_{j2}. \tag{3.7}$$

If Equation (3.7) is substituted in Equation (3.6) then the objective function becomes

$$\min \; \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}w_{ij}\iint d(\mathbf{x_i},\mathbf{a_j})f_j(a_j)\,da_{j1}\,da_{j2}. \tag{3.8}$$

Clearly, Expression (3.8) is the generic objective function formula for different probability distributions and distance functions. Moreover, the constraints for PCMFWP is the same with the deterministic version; and they are given with (3.2)-(3.4). As a result, the mathematical formulation of the PCMFWP can be stated as

$$\min \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}w_{ij} \iint_{S_j} d(\mathbf{x_i}, \mathbf{a_j}) f_j(a_j)\, da_{j1}\, da_{j2} \tag{3.9}$$

s.t.

$$\sum_{i=1}^{m} w_{ij} = d_j \qquad j = 1, \ldots, n; \tag{3.10}$$

$$\sum_{j=1}^{n} w_{ij} = s_i \qquad i = 1, \ldots, m; \tag{3.11}$$

$$w_{ij} \geq 0 \qquad i = 1, \ldots, m;\ j = 1, \ldots, n. \tag{3.12}$$

Notice that, under any probability distribution function and any distance norm, the constraints remain the same. On the other hand, some adjustments are needed on the objective function. In the thesis, the locations of the customers are assumed to fit in various probability distributions and distances are assumed to be Euclidean or rectilinear. In Euclidean distance case, we assume that customer locations are distributed with symmetric bivariate normal and symmetric bivariate exponential probability density functions. Similarly, in the rectilinear distance case, we assume that customer locations are distributed with symmetric bivariate uniform, symmetric bivariate normal and symmetric bivariate exponential probability density functions. Now, let us derive the objective function for each case.

**1**. In the Euclidean distance case, we first assumed that customer coordinates are independent, symmetric, bivariate normal random variables with mean $\mu_{j1}$ and $\mu_{j2}$ and standard deviations $\sigma_{j1} = \sigma_{j2} = \sigma_j$ $j = 1, 2, \ldots, n$. The probability density function of

the bivariate normal distribution is

$$f_j(a_j) = \left(2\pi\sigma_{j1}\sigma_{j2}\sqrt{1-\rho_j^2}\right)^{-1} \exp\Bigg\{ -\frac{1}{2\sqrt{1-\rho_j^2}}\Bigg[ \left(\frac{a_{j1}-\mu_{j1}}{\sigma_{j1}}\right)^2 - 2\rho_j \left(\frac{a_{j1}-\mu_{j1}}{\sigma_{j1}}\right)\left(\frac{a_{j2}-\mu_{j2}}{\sigma_{j2}}\right) + \left(\frac{a_{j2}-\mu_{j2}}{\sigma_{j2}}\right)^2 \Bigg]\Bigg\} \quad (3.13)$$

where $\rho_j$ is the correlation coefficient. Since, we use independent random variables $\rho_j$ is equal to zero. Therefore, Equation (3.13) becomes

$$f_j(a_j) \quad = \quad (2\pi\sigma_{j1}\sigma_{j2})^{-1} \exp\Bigg\{ \; - \; \frac{1}{2}\Bigg[ \left(\frac{a_{j1}-\mu_{j1}}{\sigma_{j1}}\right)^2 \; + \; \left(\frac{a_{j2}-\mu_{j2}}{\sigma_{j2}}\right)^2 \Bigg]\Bigg\}. \quad (3.14)$$

When we substitute Equation (3.14) and the Euclidean distance function into the Equation (3.7), the expected distance between facility $i$ and customer $j$ for this case becomes

$$E\left[d(\mathbf{x_i}, \mathbf{a_j})\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sqrt{(x_{i1}-a_{j1})^2 + (x_{i2}-a_{j2})^2}\,(2\pi\sigma_{j1}\sigma_{j2})^{-1} \exp\Bigg\{ -\frac{1}{2}\Bigg[ \left(\frac{a_{j1}-\mu_{j1}}{\sigma_{j1}}\right)^2 + \left(\frac{a_{j2}-\mu_{j2}}{\sigma_{j2}}\right)^2 \Bigg]\Bigg\} da_{j1}\, da_{j2} \quad (3.15)$$

and then the objective function for this case is directly obtained by putting the Equation (3.15) into the Equation (3.8) which is

$$\min \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij} w_{ij} \Bigg[ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sqrt{(a_{j1}-x_{i1})^2 + (a_{j2}-x_{i2})^2}\,(2\pi\sigma_{j1}\sigma_{j2})^{-1} \exp\Bigg\{ -\frac{1}{2}\Bigg[ \left(\frac{a_{j1}-\mu_{j1}}{\sigma_{j1}}\right)^2 + \left(\frac{a_{j2}-\mu_{j2}}{\sigma_{j2}}\right)^2 \Bigg]\Bigg\} da_{j1}\; da_{j2}\Bigg]. \quad (3.16)$$

In order to eliminate double integral computation, Aly and White (1978) derived the following equivalent closed form of the expected distance:

$$E\left[d(\mathbf{x_i}, \mathbf{a_j})\right] = \sqrt{\frac{\pi}{2}}\sigma_j H\left[-\frac{1}{2}, 1, -\frac{\lambda_{ij}^2}{2\sigma_j^2}\right]. \tag{3.17}$$

In Equation (3.17), $\lambda_{ij}$ is the Euclidean distance between facility $i$ and customer $j$'s mean coordinate ($\boldsymbol{\mu_j}$), namely:

$$\lambda_{ij} = \sqrt{(x_{i1} - \mu_{j1})^2 + (x_{i2} - \mu_{j2})^2}. \tag{3.18}$$

Moreover, in Equation (3.17) $H\left[.,.,.\right]$ is the confluent hypergeometric function of the form

$$H(a, b, t) = \sum_{k=0}^{\infty} \frac{\Gamma(a+k)}{\Gamma(a)} \frac{\Gamma(b)}{\Gamma(b+k)} \frac{t^k}{k!} \tag{3.19}$$

where $\Gamma(.)$ is the gamma function. Eventually, when we substitute all piece of equations into the Equation (3.8) the final objective function for this case is given below:

$$\min \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij} w_{ij} \sqrt{\frac{\pi}{2}}\sigma_j H\left[-\frac{1}{2}, 1, -\frac{\lambda_{ij}^2}{2\sigma_j^2}\right]. \tag{3.20}$$

**2**. We also assume that customer locations are associated with symmetric bivariate exponential distribution with mean $(\mu_{j1}, \mu_{j2})$ and standard deviation $\left(\sqrt{2}/\lambda_{j1}, \sqrt{2}/\lambda_{j2}\right)$, $j = 1, \ldots, n$, which is given by

$$f_j\left(a_j\right) = \frac{1}{4}\lambda_{j1}\lambda_{j2}\exp\left\{-\lambda_{j1}\left|a_{j1} - \mu_{j1}\right| - \lambda_{j2}\left|a_{j2} - \mu_{j2}\right|\right\}, \tag{3.21}$$

where $|.|$ is the absolute value. In this case, when we substitute Equation (3.21) and the Euclidean distance function into the Equation (3.7), we obtain the expected distance

function as:

$$E\left[d(\mathbf{x_i}, \mathbf{a_j})\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sqrt{(x_{i1}-a_{j1})^2+(x_{i2}-a_{j2})^2}\frac{1}{4}\lambda_{j1}\lambda_{j2} \exp\left\{-\lambda_{j1}\left|a_{j1}-\mu_{j1}\right| - \lambda_{j2}\left|a_{j2}-\mu_{j2}\right|\right\} da_{j1}\, da_{j2} \quad (3.22)$$

and the objective function for this case is easily obtained by substituting Equation (3.22) into the Equation (3.8), which is

$$\min \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}w_{ij} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sqrt{(a_{j1}-x_{i1})^2+(a_{j2}-x_{i2})^2}\ \frac{1}{4}\lambda_{j1}\lambda_{j2} \exp\left\{-\lambda_{j1}\left|a_{j1}-\mu_{j1}\right| - \lambda_{j2}\left|a_{j2}-\mu_{j2}\right|\right\} da_{j1}\, da_{j2}\Bigg]. \quad (3.23)$$

**3**. In the rectilinear distance case, we first assume that each customer location is specified by a symmetric bivariate uniform probability density function

$$f_j\left(a_j\right) = \frac{1}{(f_j-e_j)(h_j-g_j)} \quad (3.24)$$

where $f_j \geq e_j$, $e_j \leq a_{j1} \leq f_j$ and $h_j \geq g_j$, $g_j \leq a_{j2} \leq h_j$. When we substitute Equation (3.24) and the rectilinear distance function into the Equation (3.7), expected distance function becomes:

$$E\left[d(\mathbf{x_i}, \mathbf{a_j})\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(\left|a_{j1}-x_{i1}\right| + \left|a_{j2}-x_{i2}\right|\right) \frac{1}{(f_j-e_j)(h_j-g_j)}\, da_{j1}\, da_{j2} \quad (3.25)$$

and by substituting the Equation (3.25) into the Equation (3.8) the objective function becomes

$$\min \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}w_{ij}\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(|a_{j1}-x_{i1}|+|a_{j2}-x_{i2}|\right)\right. \\ \left.\frac{1}{(f_j-e_j)(h_j-g_j)}\,da_{j1}\,da_{j2}\right]. \quad (3.26)$$

**4**. We assume that customer locations follows a symmetric bivariate normal probability distribution which has already given in Equation (3.16). However, this time we use the rectilinear distance function. Thus, the expected distance function for this case is

$$E\left[d(\mathbf{x_i},\mathbf{a_j})\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(|a_{j1}-x_{i1}|+|a_{j2}-x_{i2}|\right)\left(2\pi\sigma_{j1}\sigma_{j2}\right)^{-1} \\ \exp\left\{-\frac{1}{2}\left[\left(\frac{a_{j1}-\mu_{j1}}{\sigma_{j1}}\right)^2+\left(\frac{a_{j2}-\mu_{j2}}{\sigma_{j2}}\right)^2\right]\right\}da_{j1}\,da_{j2} \quad (3.27)$$

and as a consequence, the objective function for the rectilinear distance - symmetric bivariate normal probability distribution case is

$$\min \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}w_{ij}\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(|a_{j1}-x_{i1}|+|a_{j2}-x_{i2}|\right)\left(2\pi\sigma_{j1}\sigma_{j2}\right)^{-1}\right. \\ \left.\exp\left\{-\frac{1}{2}\left[\left(\frac{a_{j1}-\mu_{j1}}{\sigma_{j1}}\right)^2+\left(\frac{a_{j2}-\mu_{j2}}{\sigma_{j2}}\right)^2\right]\right\}da_{j1}\,da_{j2}\right]. \quad (3.28)$$

**5**. Finally, we assume that customer locations are distributed with symmetric bivariate exponential probability density function which has already given in Equation (3.16). When we put this probability density function and the rectilinear distance

function into the related fields, we obtain the expected distance function as

$$E\left[d(\mathbf{x_i}, \mathbf{a_j})\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(|a_{j1} - x_{i1}| + |a_{j2} - x_{i2}|\right) \frac{1}{4}\lambda_{j1}\lambda_{j2} \exp\left\{-\lambda_{j1}\left|a_{j1} - \mu_{j1}\right| - \lambda_{j2}\left|a_{j2} - \mu_{j2}\right|\right\} da_{j1}\, da_{j2}. \quad (3.29)$$

and finally, the objective function for this case becomes

$$\min \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij} w_{ij} \left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(|a_{j1} - x_{i1}| + |a_{j2} - x_{i2}|\right) \frac{1}{4}\lambda_{j1}\lambda_{j2} \exp\left\{-\lambda_{j1}\left|a_{j1} - \mu_{j1}\right| - \lambda_{j2}\left|a_{j2} - \mu_{j2}\right|\right\} da_{j1}\, da_{j2}\right]. \quad (3.30)$$

# 4. APPROXIMATING THE EXPECTED DISTANCE FUNCTION

## 4.1. Approximating $E[d(\mathbf{x_i}, \mathbf{a_j})]$ by using mean, variance and covariance

In the thesis, we make an assumption that customer coordinates are random variables, because it is more suitable for modeling some real world situations. For instance, Altınel *et al.* (2005) give an example of emergency station location problem and Cooper (1974) gives an example of water quality management problem. Consequently, there may exist situations with known joint probability distributions for customer coordinates. To illustrate, we assume that customer coordinates are symmetric bivariate normal random variables where $a_{j1}$ and $a_{j2}$ are independent normal random variables with mean $\mu_{j1}$ and $\mu_{j2}$ and standard deviations for $\sigma_{j1} = \sigma_{j2} = \sigma_j$ $j = 1, 2, \ldots, n$. Afterwards, we use the expectation method in order to minimize the total demand weighted distances between facilities and customers. As a sequence, we obtain an equation as (3.6). In this equation we are faced with expected distance function, $E\left[d(\mathbf{x_i}, \mathbf{a_j})\right]$. In some cases, it may be impossible to derive a closed form expression for the expectation $E\left[d(\mathbf{x_i}, \mathbf{a_j})\right]$. In symmetric bivariate normal probability distribution assumption, it is possible to have a closed form of the expectation as Equation (3.17) when the Euclidean distance is used. On the other hand, when we use the rectilinear distance function, we are unable to derive the closed form of the expectation and we are faced with a double integral as in the Equation (3.28). There may also some situations where we can not estimate the distribution of the random variables but we may know the distribution parameters such as means and standard deviations. Therefore, we need some approximation methods in order to handle this problem. The following approximation provide a solution for these situations. Here we provide a formula for continuously twice differentiable functions of two dimensional random vectors.

Let $d(\mathbf{x}, \mathbf{a})$ be a two-vector function where $\mathbf{x}$ is deterministic in $\mathbb{R}^n$ and $\mathbf{a}$ is random

in $\mathbb{R}^n$ with known mean vector $\boldsymbol{\mu} = (\mu_1\ \mu_2 \dots \mu_n)^T$, variance vector $\boldsymbol{\sigma^2} = (\sigma_1^2\ \sigma_2^2 \dots \sigma_n^2)^T$ and covariance matrix $[\rho_{ij}\ i, j = 1, \dots, n]$. The second order Taylor approximation of $d(\mathbf{x}, \mathbf{a})$ with respect to $\mathbf{a}$ around $\boldsymbol{\mu}$ is

$$d(\mathbf{x}, \mathbf{a}) \approx d(\mathbf{x}, \boldsymbol{\mu}) + \nabla_a d(\mathbf{x}, \boldsymbol{a})(\mathbf{a} - \boldsymbol{\mu}) + \frac{1}{2}(\mathbf{a} - \boldsymbol{\mu})^T \nabla_a^2 d(\mathbf{x}, \boldsymbol{\mu})(\mathbf{a} - \boldsymbol{\mu}) \tag{4.1}$$

Here $\nabla_a d(\mathbf{x}, \boldsymbol{\mu})$ and $\nabla_a^2 d(\mathbf{x}, \boldsymbol{\mu})$ denote respectively the gradient and Hessian matrix of $d(\mathbf{x}, \boldsymbol{\mu})$ both with respect to $\mathbf{a}$ and evaluated at the mean vector $\boldsymbol{\mu}$. The Equation (4.1) can also be expressed as

$$d(\mathbf{x}, \mathbf{a}) \approx d(\mathbf{x}, \boldsymbol{\mu}) + \sum_{j=1}^{n} \frac{\partial d(\mathbf{x}, \boldsymbol{\mu})}{\partial a_j}(a_j - \mu_j) + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2 d(\mathbf{x}, \boldsymbol{\mu})}{\partial a_i \partial a_j}(a_i - \mu_i)(a_j - \mu_j), \tag{4.2}$$

and when we take the expectation of this equation it becomes

$$\begin{aligned} E\left[d(\mathbf{x}, \mathbf{a})\right] \approx d(\mathbf{x}, \boldsymbol{\mu}) + \sum_{j=1}^{n} \frac{\partial d(\mathbf{x}, \boldsymbol{\mu})}{\partial a_j}\left(E\left[a_j\right] - \mu_j\right) \\ + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2 d(\mathbf{x}, \boldsymbol{\mu})}{\partial a_i \partial a_j} E\left[(a_i - \mu_i)(a_j - \mu_j)\right]. \end{aligned} \tag{4.3}$$

As can be observed, $E\left[d(\mathbf{x}, \boldsymbol{\mu})\right] = d(\mathbf{x}, \boldsymbol{\mu})$ since $d(\mathbf{x}, \boldsymbol{\mu})$ is a constant statistically. Besides, the second term is equal to zero since $E\left[a_j\right] = \mu_j$, and $E\left[(a_i - \mu_i)(a_j - \mu_j)\right] = \sigma_i^2$ when $i = j$ and $E\left[(a_i - \mu_i)(a_j - \mu_j)\right] = \rho_{ij}$ in other cases. As a result, the last equation becomes

$$E\left[d(\mathbf{x}, \mathbf{a})\right] \approx d(\mathbf{x}, \boldsymbol{\mu}) + \frac{1}{2}\sum_{i=1}^{n} \sigma_i^2 \frac{\partial^2 d(\mathbf{x}, \boldsymbol{\mu})}{\partial a_i^2} + \frac{1}{2}\sum_{i=1}^{n}\sum_{\substack{j=1 \\ i \neq j}}^{n} \rho_{ij} \frac{\partial^2 d(\mathbf{x}, \boldsymbol{\mu})}{\partial a_i \partial a_j}. \tag{4.4}$$

The last expression clearly reduces to

$$E\left[d(\mathbf{x}, \mathbf{a})\right] \approx d(\mathbf{x}, \boldsymbol{\mu}) + \frac{1}{2}\sum_{i=1}^{n} \sigma_i^2 \frac{\partial^2 d(\mathbf{x}, \boldsymbol{\mu})}{\partial a_i^2} \tag{4.5}$$

when random variables $a_j\ j = 1, \dots, n$ are independent ($\rho_{ij} = 0, \forall\ i, j$ and $i \neq j$).

As a result we can use Equation (4.5) to approximate $E\left[d(\mathbf{x_i}, \mathbf{a_j})\right]$ by replacing $d(\mathbf{x}, \mathbf{a})$ with $d(\mathbf{x_i}, \mathbf{a_j})$ provided that the mean vector $\boldsymbol{\mu_j} = (\mu_{j1}, \mu_{j2})$ and the variance vector $\boldsymbol{\sigma_j^2} = \left(\sigma_{j1}^2, \sigma_{j2}^2\right)^T$. Then, the approximation becomes

$$E\left[d(\mathbf{x_i}, \mathbf{a_j})\right] \approx d(\mathbf{x_i}, \boldsymbol{\mu_j}) + \frac{1}{2}\sum_{p=1}^{2} \sigma_{jp}^2 \frac{\partial^2 d(\mathbf{x_i}, \boldsymbol{\mu}_j)}{\partial a_{jp}^2}. \tag{4.6}$$

Since we use Euclidean and rectilinear distance functions in the thesis, we need to derive the approximation formula for these distance functions. Let us derive the formula for the Euclidean distance case where we assume that $\sigma_{j1} = \sigma_{j2} = \sigma_j \; j = 1, 2, \ldots, n$. When we look at the Equation (4.6), we need to obtain the second order derivatives of the Euclidean distance function with respect to random variables $a_{j1}$ and $a_{j2}$. The distance function is given as

$$d(\mathbf{x_i}, \mathbf{a_j}) = \sqrt{(x_{i1} - a_{j1})^2 + (x_{i2} - a_{j2})^2} \tag{4.7}$$

where the first and second order derivatives are

$$\frac{\partial d(\mathbf{x_i}, \mathbf{a_j})}{\partial a_{j1}} = \frac{(a_{j1} - x_{i1})}{d(\mathbf{x_i}, \mathbf{a_j})} \tag{4.8}$$

$$\frac{\partial d(\mathbf{x_i}, \mathbf{a_j})}{\partial a_{j2}} = \frac{(a_{j2} - x_{i2})}{d(\mathbf{x_i}, \mathbf{a_j})} \tag{4.9}$$

$$\frac{\partial^2 d(\mathbf{x_i}, \mathbf{a_j})}{\partial a_{j1}^2} = \frac{d(\mathbf{x_i}, \mathbf{a_j}) - \frac{(a_{j1} - x_{i1})^2}{d(\mathbf{x_i}, \mathbf{a_j})}}{\left[d(\mathbf{x_i}, \mathbf{a_j})\right]^2} \tag{4.10}$$

$$\frac{\partial^2 d(\mathbf{x_i}, \mathbf{a_j})}{\partial a_{j2}^2} = \frac{d(\mathbf{x_i}, \mathbf{a_j}) - \frac{(a_{j2} - x_{i2})^2}{d(\mathbf{x_i}, \mathbf{a_j})}}{\left[d(\mathbf{x_i}, \mathbf{a_j})\right]^2}. \tag{4.11}$$

Now, we can substitute the second order derivatives and the distance function into Equation (4.6). Since we assume that $\sigma_{j1} = \sigma_{j2} = \sigma_j \; j = 1, 2, \ldots, n$, we can make some simplification on the equation. Therefore, the approximating function of the

expected distance function where Euclidean norm is used is

$$E\left[d(\mathbf{x_i}, \mathbf{a_j})\right] \approx \sqrt{(x_{i1} - \mu_{j1})^2 + (x_{i2} - \mu_{j2})^2} + \frac{1}{2}\sigma_j^2 \left[\frac{1}{\sqrt{(x_{i1} - \mu_{j1})^2 + (x_{i2} - \mu_{j2})^2}}\right]. \quad (4.12)$$

Notice that, when the inside of the square root becomes zero, the division in the second term of Equation (4.12) is undefined. Figure 4.1 illustrates this problem clearly. In Figure 4.1, we assume that there is a customer with $\mu_{j1} = 0$, $\mu_{j2} = 0$ and $\sigma_{j1} = \sigma_{j2} = 20$ and we plot the expected distances where $x$ and $y$ coordinates are between 100 and -100. In Figure 4.1, when we are getting closer to the customer means simultaneously, the

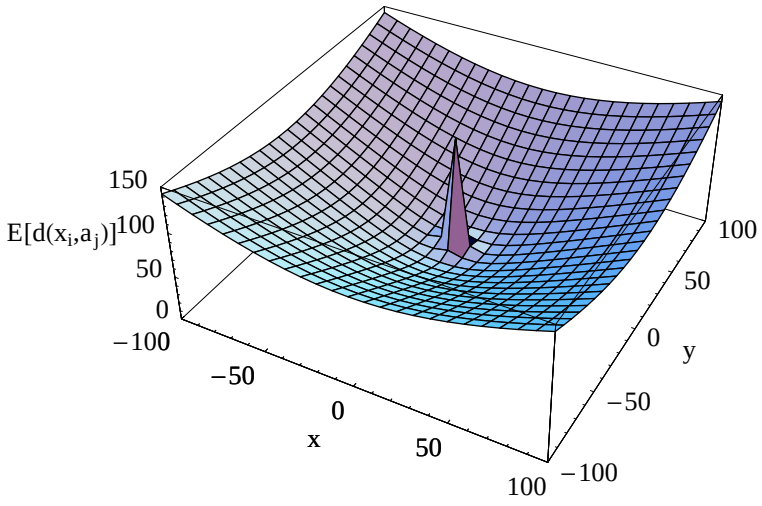



Figure 4.1. Taylor Approximation for Euclidean distance

expected distance is getting smaller. However, beginning with some points, expected distance starts to get higher. Intuitively, expected distance should decrease when we are getting closer to $(0, 0)$. Moreover, Figure 4.2 illustrates the expected distance graph in which the expected distances are calculated exactly with Equation (3.17) by using the same parameters of Figure 4.1. In order to see the difference between the approximation and the exact calculation, we overlap Figure 4.1 and Figure 4.2 in Figure 4.3. Obviously, Taylor approximation diverts from the exact method around customer means, $(0, 0)$. Hence, we can improve the approximation method by cutting

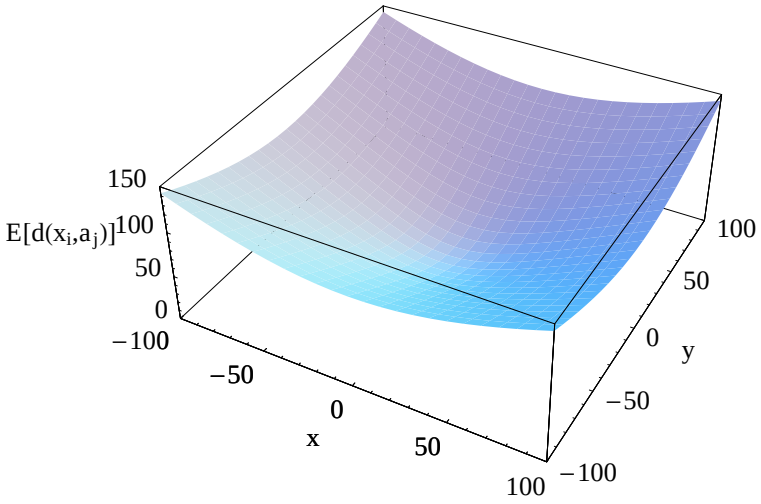



Figure 4.2. Exact expected distances for the Euclidean distance

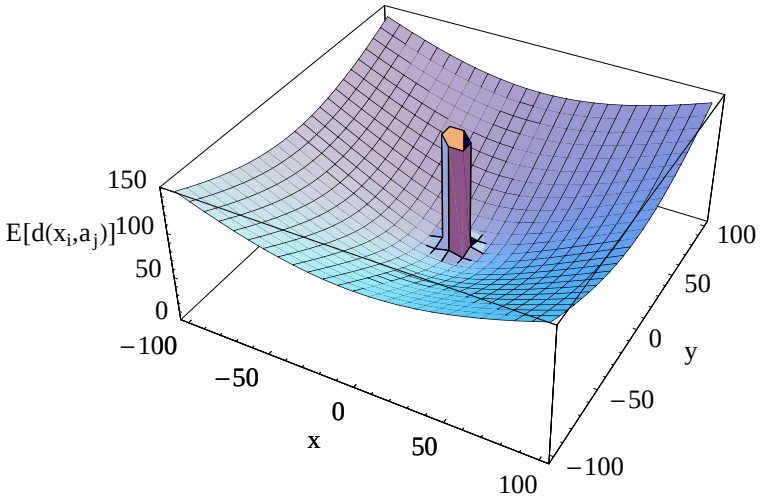



Figure 4.3. Overlapping graph of the exact expected distances and the Taylor Approximation for the Euclidean distance

the pipe shape where the approximation starts to get worse and by fixing the value of the approximation at that point. Let us name Equation (4.12) as $h(x)$. In order to find the points where approximation starts to get worse, we can set the first derivative of $h(x)$ with respect to $x_{i1}$ and $x_{i2}$ zero and solving this equation in terms of $x_{i1}$ and

$x_{i2}$ will give us the stationary points:

$$\frac{\partial h(x)}{\partial x_{ik}} = \frac{x_{ik} - \mu_{jk}}{\sqrt{(x_{i1} - \mu_{j1})^2 + (x_{i2} - \mu_{j2})^2}} \left[ 1 - \frac{\sigma_j^2}{2\sqrt{(x_{i1} - \mu_{j1})^2 + (x_{i2} - \mu_{j2})^2}} \right] = 0$$

$$\implies \begin{cases} x_{ik} = \mu_{jk} \\ \sqrt{(x_{i1} - \mu_{j1})^2 + (x_{i2} - \mu_{j2})^2} = \pm \frac{\sigma_j}{\sqrt{2}} \end{cases} \tag{4.13}$$

where $i = 1, \ldots, m;\ j = 1, \ldots, n;\ k = 1, 2$. In the solution set (4.13), first solution is obvious since it makes the denominator zero in Equation (4.12). However, this is not the solution which we are seeking for. On the other hand, we have two alternatives in the second solution where one of them gives the complex number solution. Whereas, the remaining one defines the circle,

$$\sqrt{(x_{i1} - \mu_{j1})^2 + (x_{i2} - \mu_{j2})^2} = \frac{\sigma_j}{\sqrt{2}}, \tag{4.14}$$

whose radius is $\frac{\sigma_j}{\sqrt{2}}$. Properly, this circle is the solution which we are seeking for. Now, we need to find the value of the approximation function at this point in order to obtain the approximation. It is very simple to find the value of the function such that

$$\begin{aligned} E\left[d(\mathbf{x_i}, \mathbf{a_j})\right] &= \overbrace{\sqrt{(x_{i1} - \mu_{j1})^2 + (x_{i2} - \mu_{j2})^2}}^{\sigma_j/\sqrt{2}} + \frac{1}{2}\sigma_j^2 \overbrace{\left((x_{i1} - \mu_{aj1})^2 + (x_{i2} - \mu_{j2})^2\right)^{\frac{-1}{2}}}^{\sqrt{2}/\sigma_j} \\ &= \frac{\sigma_j}{\sqrt{2}} + \frac{1}{2}\sigma_j^2 \frac{\sqrt{2}}{\sigma_j} = \frac{\sigma_j}{\sqrt{2}} + \frac{\sigma_j}{\sqrt{2}} = \sqrt{2}\sigma_j. \end{aligned}$$

Finally, the improved approximation function for $E\left[d(\mathbf{x_i}, \mathbf{a_j})\right]$ is

$$E[d(\mathbf{x_i}, \mathbf{a_j})] \approx \begin{cases} h(x) & \text{if } \sqrt{(x_{i1} - \mu_{j1})^2 + (x_{i2} - \mu_{j2})^2} \geq \frac{\sigma_j}{\sqrt{2}} \\ \sqrt{2}\sigma_j & \text{if } \sqrt{(x_{i1} - \mu_{j1})^2 + (x_{i2} - \mu_{j2})^2} < \frac{\sigma_j}{\sqrt{2}} \end{cases} \tag{4.15}$$

where $i = 1, \ldots, m;\ j = 1, \ldots, n$. Moreover, if we plot the improved approximation function with the same parameters of Figure 4.1, then it will enable us to observe the improvement. Therefore, Figure 4.4 illustrates the improved case and also, for

more consideration, Figure 4.5 is obtained by superposing Figure 4.4 and Figure 4.2.

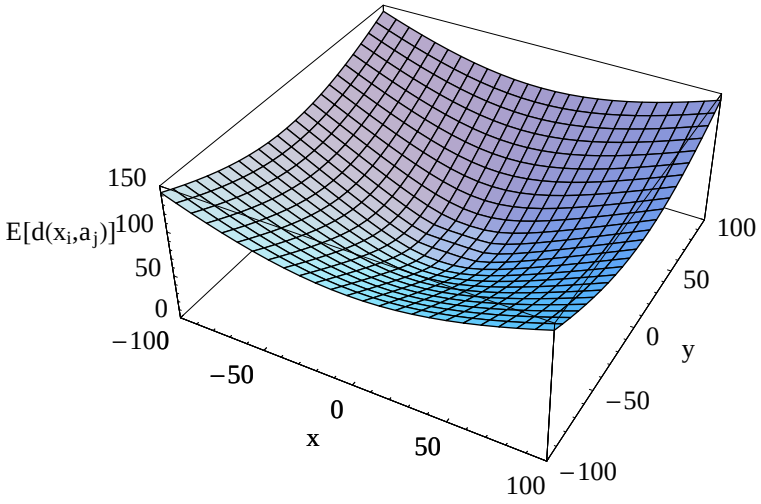



Figure 4.4. Improved Taylor Approximation for the Euclidean distance

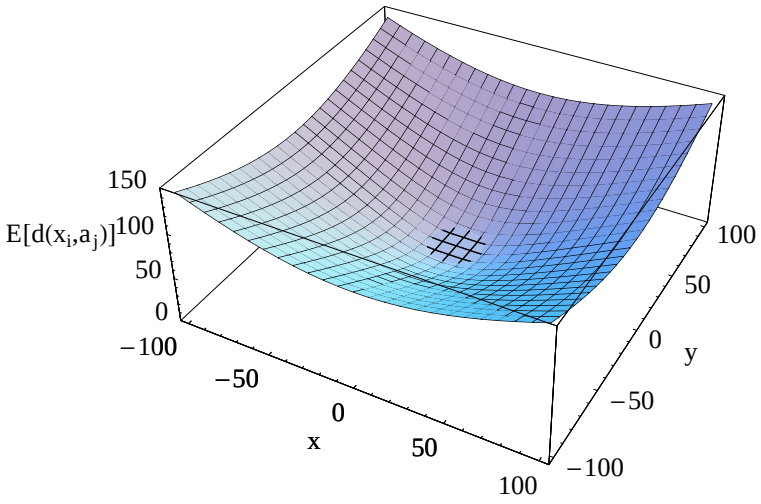



Figure 4.5. Superposition of the improved Taylor Approximation and exact expected distances for the Euclidean distance

Clearly, the approximation function and the exact function are very close to each other outside the patched area. On the other hand, patched area lies above the exact function since the meshes are bold. Here, we can calculate the error between the approximation and the exact functions. At the origin $(0, 0)$, where the exact function, Equation (3.17),

reaches its minimum. At the point $(0,0)$, Confluent Hypergeometric function is equal to one and the Equation (3.17) is equal to $\sqrt{\frac{\pi}{2}}\sigma_j$. On the other hand, the value of the approximation function is $\sqrt{2}\sigma_j$. Therefore, the approximation error is

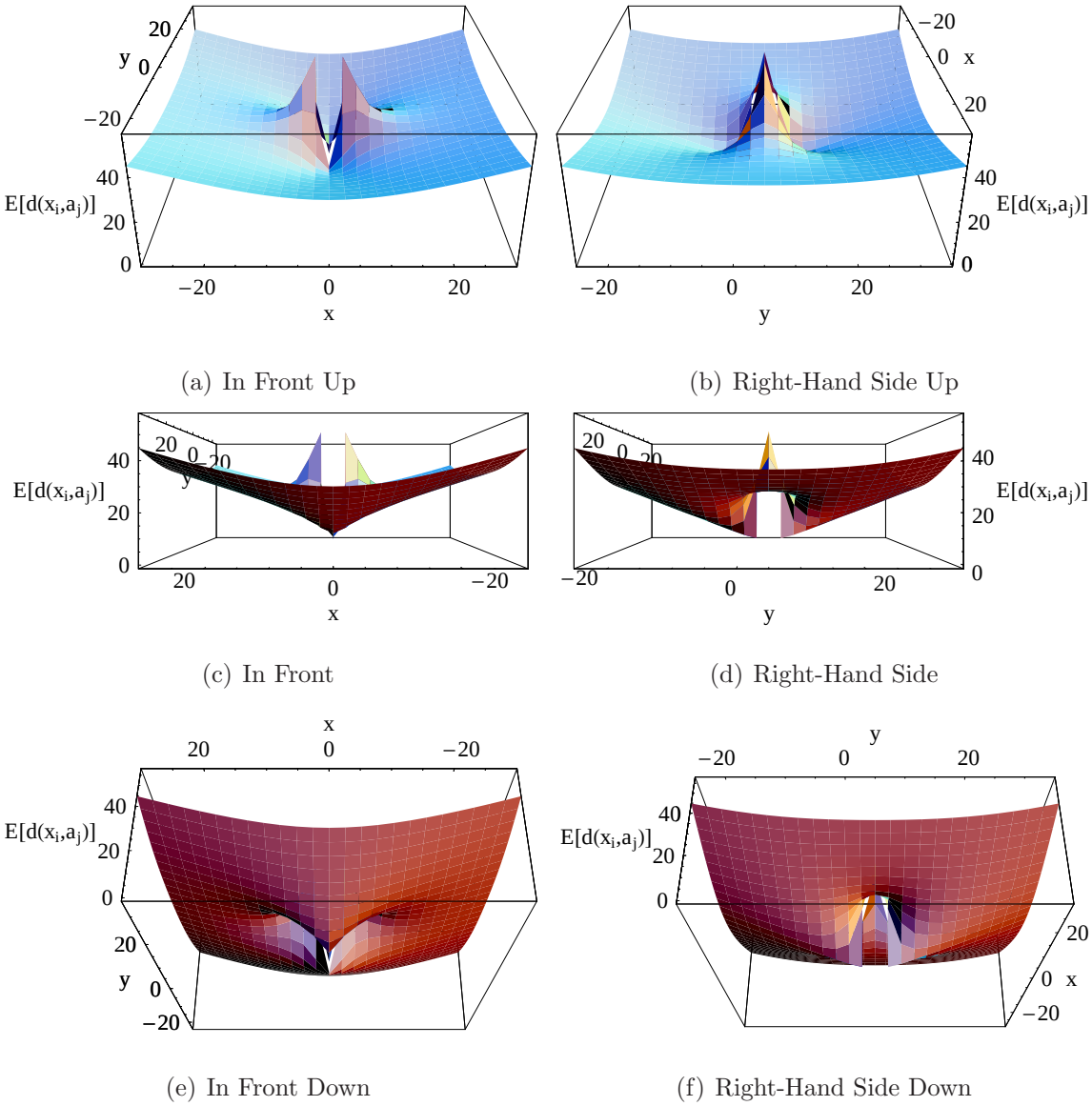
$$\sqrt{2}\sigma_j - \sqrt{\frac{\pi}{2}}\sigma_j = \left(\frac{2-\sqrt{\pi}}{\sqrt{2}}\right)\sigma_j = 0.160899\sigma_j. \tag{4.16}$$

We can also obtain the approximation function for the Euclidean distance case where $\sigma_{j1} \neq \sigma_{j2}$ for $j = 1, 2, \ldots, n$. Since we have already obtained the partial derivatives of the distance function, Equations (4.8)-(4.11), we can directly substitute the Euclidean distance and the partial derivatives functions into the Equation (4.6). Therefore, by making some simplifications we obtain the approximation function as

$$E[d(\mathbf{x_i}, \mathbf{a_j})] \approx \sqrt{(x_{i1}-\mu_{j1})^2 + (x_{i2}-\mu_{j2})^2} + \frac{\sigma_{j1}^2(x_{i2}-\mu_{j2})^2 + \sigma_{j2}^2(x_{i1}-\mu_{j1})^2}{2((x_{i1}-\mu_{j1})^2 + (x_{i2}-\mu_{j2})^2)^{3/2}}. \tag{4.17}$$

Let us visualize the approximation function (4.17) as we did previously. Assume that we have a customer with the parameters $\mu_{j1} = 0$, $\mu_{j2} = 5$ and $\sigma_{j1} = 3$, $\sigma_{j2} = 20$. In Figure 4.6, we graph the approximation distances to this customer by using Equation (4.17) where $x$ ranges from -30 to 30 and $y$ ranges from -25 to 35. In Figure 4.6, there are six graphs of the function from different perspectives. It is obvious that the function is undetermined at the point $(0,5)$ since this point makes the denominator of the second term zero. In order to handle this problem, Hyperboloid Approximation Procedure (HAP) can be used which is developed by Eyster *et al.* (1973). In HAP, $\epsilon$ (a small positive quantity) is added to the Euclidean distance formula in order to avoid knife-edged derivatives. Inspiring from this procedure, we can add a small positive number, $\epsilon$, to the denominator of the second term. However, we should not add this quantity everytime because there are two huge cliffs in both side of the customer means. Therefore, it is appropriate to add this small quantity whenever the denominator of the second term in Equation (4.17) is equals to zero. Besides, adding this small quantity does not harm the orijinal function because whenever a facility is located at the customer means the expected distance is saved and it becomes zero which

is more logical. Some readers might have a question in their mind if it is possible to




(a) In Front Up

(b) Right-Hand Side Up

(c) In Front

(d) Right-Hand Side

(e) In Front Down

(f) Right-Hand Side Down

Figure 4.6. Taylor Approximation for the Euclidean distance

improve the equation (4.17) as we improve the approximation function (4.15). We have already tried to solve the first order conditions. However, this process does not lead us to reasonable results. One important observation is that there exist three main local minima at this function; first one is the customer mean, the second and the third one are located symmetrically in both side of the customer mean which are clearly seen in figure 4.6.

In the rectilinear distance case, deriving the approximation function for the expected distance function is easier than the Euclidean distance case because the second

order derivatives of the rectilinear distance function with respect to $x_{i1}$ and $x_{i2}$ is equal to zero. Therefore, the second order Taylor approximation for the rectilinear distance is

$$E\left[d(\mathbf{x_i}, \mathbf{a_j})\right] \approx d(\mathbf{x_i}, \boldsymbol{\mu}_j) = |x_{j1} - \mu_{j1}| + |x_{i2} - \mu_{j2}| \tag{4.18}$$

which is rectilinear distance to mean customer location. Remark that, Equation (4.18) does not depend on the standard deviations of the customer location distributions. Therefore, this equation can be used both for $\sigma_{j1} = \sigma_{j2}$ and $\sigma_{j1} \neq \sigma_{j2}$ cases.

Now, let us visualize the approximation function for the rectilinear distance case. Assume that there is a customer whose location fits in a bivariate uniform distribution with parameters $e_j = -5$, $f_j = 5$, $g_j = -5$ and $h_j = 5$. Therefore, when the mean and the standard deviation of the customer is calculated, we obtain $\mu_{j1} = 0$, $\mu_{j2} = 0$ and $\sigma_{j1} = 2.8867$, $\sigma_{j2} = 2.8867$. In figure 4.7, we graph the expected distances for this customer by using the approximation function. Moreover, in order to make comparison

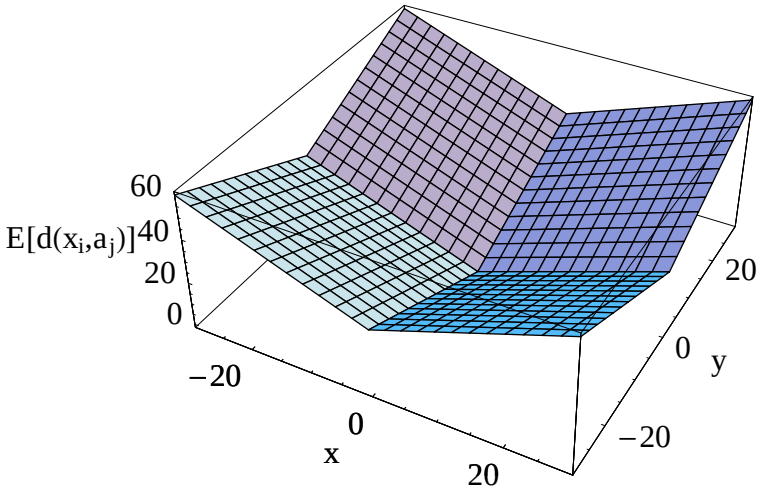



Figure 4.7. Taylor Approximation for the rectilinear distance

we also graph an exact calculation of the expected distance function where we use the same location assumptions about the customer. Figure 4.8 is the graph of the exact expectation and Figure 4.9 is the superposition graph of Figure 4.7 and Figure 4.8. As

it is seen in Figure 4.9, the approximation function is very close to the exact function of the expected distance. However, when $x_{i1} = 0$ or $x_{i2} = 0$, the exact function slightly lies above the approximation.

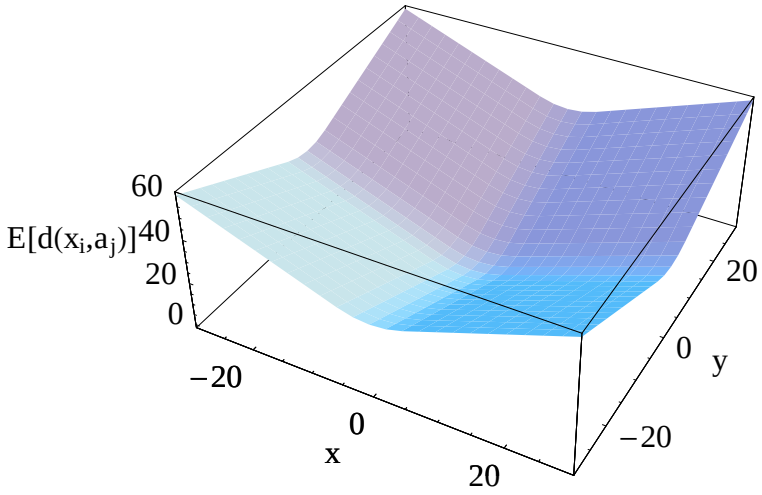



Figure 4.8. Exact expected distances for the rectilinear distance

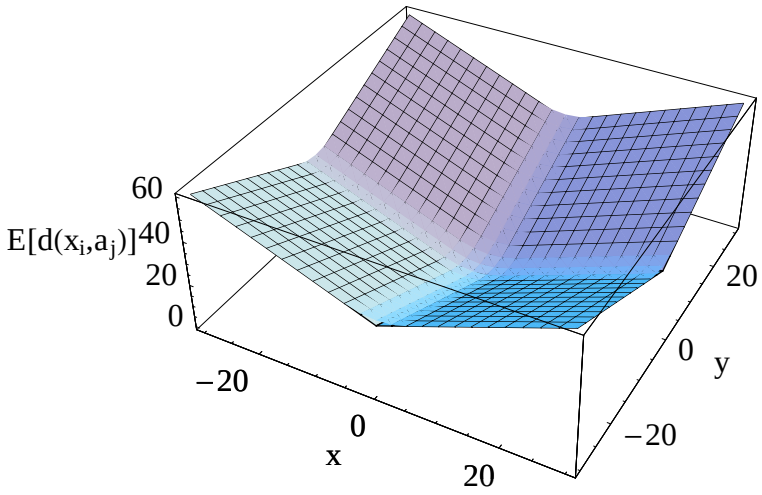



Figure 4.9. Superposition of the exact expected distances and Taylor Approximation for the rectilinear distance

## 4.2. Approximating $E[d(\mathbf{x_i}, \mathbf{a_j})]$ by estimated parameters

In section (4.1), we assume that we know the exact parameters of the joint probability distribution of the customer locations. However, sometimes we may not know the probability distribution. Assume that we have a large enough random sample of location vectors for customers. In this situation we are able to estimate sample parameters such as average, variance and covariance. Therefore, we can use the second order Taylor approximation method which is developed in section (4.1). Namely for the Euclidean distance case Equations (4.15) and (4.17) and for the rectilinear distance case Equation (4.18) can be used. Now, let us introduce the common unbiased estimators for the parameters $\mu_j$ and $\sigma_j$ which can be used in the Taylor approximation functions. The well known unbiased estimator for $\mu_{jk}$ is

$$\hat{a}_{jk} = \frac{1}{|S_j|} \sum_{a_{jk} \in S_j} a_{jk} \tag{4.19}$$

and the unbiased estimator for the standard deviation is

$$S_{jk} = \sqrt{\frac{1}{|S_j| - 1} \sum_{a_j \in S_j} (a_{jk} - \hat{a}_{jk})^2} \tag{4.20}$$

for $k = 1, 2$ where $S_j$ denotes the sample vector of customer $j$.

## 4.3. Approximating $E[d(\mathbf{x_i}, \mathbf{a_j})]$ by the average distance

In the previous two sections, we approximate the expected distance by using mean, variance and covariance and their estimators. We can also approximate the expected distance function by calculating the average distance to the customer location sample. Let us assume that we have a large enough random sample of location vectors for customers and let $S_j$ denote the set of vectors for customer $j$. Then we can use the simple average

$$\frac{1}{|S_j|} \sum_{a_j \in S_j} d(\mathbf{x_i}, \mathbf{a_j}) \tag{4.21}$$

as an estimator of the expected distance. Such sample can be either directly given, or a distribution function from which we can generate location vectors can be provided. Now, let us consider the Euclidean and rectilinear distances in particular. The average distance for the Euclidean distance function is

$$E[d(\mathbf{x_i}, \mathbf{a_j})] \approx \frac{1}{|S_j|} \sum_{a_j \in S_j} \sqrt{(x_{i1} - a_{j1})^2 + (x_{i2} - a_{j2})^2}, \tag{4.22}$$

and the objective function of the PCMFWP becomes

$$\min \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} w_{ij} \left[ \frac{1}{|S_j|} \sum_{a_j \in S_j} \sqrt{(x_{i1} - a_{j1})^2 + (x_{i2} - a_{j2})^2} \right]. \tag{4.23}$$

when (4.22) is used to estimate $E[d(\mathbf{x_i}, \mathbf{a_j})]$. Similarly we have

$$\frac{1}{|S_j|} \sum_{a_j \in S_j} (|x_{i1} - a_{j1}| + |x_{i2} - a_{j2}|) \tag{4.24}$$

as the estimator of the expected distance in case of the rectilinear distance and the objective function of the PCMFWP can be replaced with its estimator

$$\min \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} w_{ij} \left[ \frac{1}{|S_j|} \sum_{a_j \in S_j} |x_{i1} - a_{j1}| + |x_{i2} - a_{j2}| \right]. \tag{4.25}$$

# 5. SOLUTION METHODS

Solving the PCMFWP to optimality is a difficult task even for small-sized instances. As a result, efficient and accurate heuristics are necessary. For solving the PCMFWP, we first develop a probabilistic variant of the Cooper's alternating location-allocation heuristic (Cooper 1978) which we call probabilistic alternating location allocation (PALA) heuristic. In PALA, we use both exact and approximating methods to evaluate the expected distances. Afterwards, we make some adaptations on the deterministic version of the discrete approximation in order to find a good solution method for the PCMFWP. Furthermore, we develop two grid based discrete approximation methods in which the number of candidate locations are decreased by using problem structure implicitly. In addition to these grid based approximations, we also developed $p$-median type discrete approximation method. In the following subsections, these heuristic methods are described in detail.

## 5.1. Probabilistic Alternating Location-Allocation (PALA) Heuristic

In the alternating location-allocation heuristic (ALA), the problem is decomposed into two parts. In the first part, when the allocations are known, the aim is to locate $m$ facilities by minimizing the total demand weighted distances. Therefore, the problem becomes $m$ single facility Weber problems and they can be solved by using Weiszfeld type iterative formulations (Weiszfeld 1937). On the other hand, when the locations of the facilities are known, the problem is to determine the allocations with the same objective function, namely it becomes a regular transportation problem. To sum up, ALA heuristics starts from a given set of initial locations and alternately solves location and allocation subproblems until no improvement is possible. However, when the customer locations are random variables it is hard to calculate the distances between customers and facilities. As it is stated in the previous chapters, we use the expectation method. Therefore, we need to calculate the expected distances between customers and facilities. In the previous two chapters, the exact expected distance and the approximating expected distance formulas has been introduced. Consequently, Algorithm 1 is the

generic probabilistic location-allocation (PALA) heuristic for different probability distribution and distance function assumptions. In the first step of the algorithm, convex

**Algorithm 1** Probabilistic Alternating Location-Allocation (PALA) Heuristic

1. Find the convex hull of the customer location means $\mu_{j1}, \mu_{j2}$ $j = 1, \ldots, n$.
2. Locate the facilities randomly inside the convex hull which is determined in step 1.
3. Calculate the expected distance $E\left[d(\mathbf{x_i}, \mathbf{a_j})\right]$ between each customer-facility pair and define cost coefficients $c_{ij} = c'_{ij} E\left[d(\mathbf{x_i}, \mathbf{a_j})\right]$ where $c'_{ij}$ is the unit transportation cost per unit expected distance between facility $i$ and customer $j$.
4. Solve the transportation problem
$$\left\{ \min \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} w_{ij} : \sum_{i=1}^{m} w_{ij} = d_j, \sum_{j=1}^{n} w_{ij} = s_i, w_{ij} \geq 0 \; i = 1, \ldots, m; \; j = 1, \ldots, n \right\}$$
to find a set of allocations $w_{ij}$.
5. Using $w_{ij}$ solve $m$ probabilistic single-facility Weber problems by an algorithm appropriate for the probability distribution and distance function assumptions.
6. Repeat steps 3, 4 and 5 until either the facility locations $\mathbf{x_i} = (x_{i1}, x_{i2})^T$ or the allocations $w_{ij}$ remain unchanged.
7. Report final facility locations and allocations.

hull of the customer location means can be determined with some efficient methods; such as Quick Hull (Barber *et al.* 1996) and Graham Scan (Graham 1972) methods. It is shown by Katz and Cooper (1976) that an optimal facility location for a probabilistic single facility Weber problem exists in the convex hull of the customer location means where the customer coordinates are jointly and independently distributed according to symmetric bivariate probability distribution with mean $\mu_{j1}$ and $\mu_{j2}$ and standard deviations for $\sigma_{j1} = \sigma_{j2} = \sigma_j$ $j = 1, 2, \ldots, n$. Therefore, in our cases, the optimal facility locations for PCMFWP exist in the convex hull of the mean customer locations. Thus, in the second step of the algorithm the facilities are randomly located in the convex hull of the customer means.

In the third step, expected distance between each customer-facility pair is calculated with one of the appropriate Equations (3.17), (3.22), (3.25), (3.27), (3.29), (4.15),

(4.17), (4.18). After calculating the expected distances, the transportation problem in step 4 is solved in order to determine the allocations. Furthermore, when the allocations are determined the problem is decomposed into $m$ probabilistic single facility Weber problems which minimizes the expected cost

$$\min E[z_i] = \sum_{j=1}^{n_i} c_{ij} w_{ij} E[d(\mathbf{x_i}, \mathbf{a_j})] \tag{5.1}$$

with respect to $\mathbf{x_i} = (x_{i1}, x_{i2})^T$ for $i = 1, \ldots, m$. Notice that the expected cost $E[z_i]$ is a function of the coordinates $\mathbf{x_i} = (x_{i1}, x_{i2})^T$ of facility $i$ and $n_i$ is the number customers which are partially or fully served from facility $i$. The solution of the probabilistic single facility Weber problem is going to be explained in the following subsections where we explain how to solve Equation (5.1) for various probability distributions and distance functions assumptions.

The sixth step is the stopping conditions for the PALA heuristic. The algorithm stops when the allocations or the facility locations remain the same. Furthermore, there may exist negligible changes in the location of the facilities because of the iterative solution technique of the probabilistic single-facility Weber problem. In order to handle this problem, the changes of the facility location can be compared with a small quantity such that if the change in the location is smaller than the small quantity then the algorithm should stop. Finally, the local optimal facility locations is obtained after stopping the location and the allocation phases.

### 5.1.1. Solution of the Probabilistic Weber Problem (PWP) With Exact Expected Distances

The probabilistic single facility Weber problem can be solved by using Weiszfeld type iterative equations (Weiszfeld 1937). An iterative algorithm for finding an optimal facility location for PWP is obtained formally by setting $\nabla_{xi} E[z_i] = 0$, by solving this equation in the form $x_i = H(x_i)$, and then by passing to the iteration scheme $x_i^{n+1} = H(x_i^n)$ (Cooper 1974, Katz and Cooper 1976, Cooper 1978). However, this procedure

assumes that objective function is strictly convex. Therefore, at the beginning of the algorithm, necessary and sufficient conditions should be checked. Now, let us apply this procedure to the Equation (5.1). First of all, we need to solve the equations $\nabla_{x_i} E[z_i] = 0$ which are also equal to

$$\frac{\partial E[z_i]}{\partial x_{i1}} = \sum_{j=1}^{n_i} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{c_{ij} w_{ij} (x_{i1} - a_{j1}) f_j(a_j)}{d(\mathbf{x_i}, \mathbf{a_j})} \, da_{j1} \, da_{j2} = 0; \tag{5.2}$$

$$\frac{\partial E[z_i]}{\partial x_{i2}} = \sum_{j=1}^{n_i} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{c_{ij} w_{ij} (x_{i2} - a_{j2}) f_j(a_j)}{d(\mathbf{x_i}, \mathbf{a_j})} \, da_{j1} \, da_{j2} = 0. \tag{5.3}$$

We know that there is a unique minimum point $\mathbf{x_i}^* = (x_{i1}^*, x_{i2}^*)$ which will satisfy Equations (5.2) and (5.3). When we solve Equations (5.2) and (5.3) we obtain

$$x_{i1} = \frac{\sum_{j=1}^{n_i} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{c_{ij} w_{ij} x_{i1} f_j(a_j)}{d(\mathbf{x_i}, \mathbf{a_j})} \, da_{j1} \, da_{j2}}{\sum_{j=1}^{n_i} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{c_{ij} w_{ij} f_j(a_j)}{d(\mathbf{x_i}, \mathbf{a_j})} \, da_{j1} \, da_{j2}}; \tag{5.4}$$

$$x_{i2} = \frac{\sum_{j=1}^{n_i} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{c_{ij} w_{ij} x_{i2} f_j(a_j)}{d(\mathbf{x_i}, \mathbf{a_j})} \, da_{j1} \, da_{j2}}{\sum_{j=1}^{n_i} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{c_{ij} w_{ij} f_j(a_j)}{d(\mathbf{x_i}, \mathbf{a_j})} \, da_{j1} \, da_{j2}}. \tag{5.5}$$

Now, let the superscript on $x_{i1}$ and $x_{i2}$ indicate the iteration parameter. The notation $d(\mathbf{x_i}^{(r)}, \mathbf{a_j})$ means that $d(\mathbf{x_i}, \mathbf{a_j})$ is evaluated for the values $\mathbf{x_i}^{(r)} = (x_{i1}^{(r)}, x_{i2}^{(r)})$, i.e. , the values of $x_{i1}$ and $x_{i2}$ obtained at the $r^{th}$ iteration. Therefore, the iteration formulas are

$$x_{i1}^{(r+1)} = \frac{\sum_{j=1}^{n_i} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{c_{ij} w_{ij} x_{i1}^{(r)} f_j(a_j)}{d(\mathbf{x_i}^{(r)}, \mathbf{a_j})} \, da_{j1} \, da_{j2}}{\sum_{j=1}^{n_i} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{c_{ij} w_{ij} f_j(a_j)}{d(\mathbf{x_i}^{(r)}, \mathbf{a_j})} \, da_{j1} \, da_{j2}} \tag{5.6}$$

and

$$x_{i2}^{(r+1)} = \frac{\sum_{j=1}^{n_i} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{c_{ij} w_{ij} x_{i2}^{(r)} f_j(a_j)}{d(\mathbf{x_i}^{(r)}, \mathbf{a_j})} \, da_{j1} \, da_{j2}}{\sum_{j=1}^{n_i} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{c_{ij} w_{ij} f_j(a_j)}{d(\mathbf{x_i}^{(r)}, \mathbf{a_j})} \, da_{j1} \, da_{j2}}. \tag{5.7}$$

Moreover, a suitable set of starting values $\mathbf{x_i}^{(0)} = (x_{i1}^{(0)}, x_{i2}^{(0)})$ to begin the iteration is given by

$$x_{i1}^{(0)} = \sum_{j=1}^{n_i} c_{ij} w_{ij} \mu_{j1} \Big/ \sum_{j=1}^{n_i} c_{ij} w_{ij}, \tag{5.8}$$

$$x_{i2}^{(0)} = \sum_{j=1}^{n_i} c_{ij} w_{ij} \mu_{j2} \Big/ \sum_{j=1}^{n_i} c_{ij} w_{ij} \tag{5.9}$$

where $x_{i1}^{(0)}$ and $x_{i2}^{(0)}$ are simply the weighted mean values of the bivariate probability density function.

The iteration process described by Equations (5.6) and (5.7) has some properties such as:

- The iteration is monotone.
- The iteration is globally convergent.
- The order of the convergence of the iteration is linear.

The proofs of these results are given by Katz and Cooper (1974). We can now derive the iteration schemes for our cases which has already introduced in Chapter (3). In Euclidean distance-symmetric bivariate normal probability distribution case, by using the Equations (3.17) and (5.1) the PWP becomes

$$\min\ E\left[z_i\right] = \sum_{j=1}^{n_i} c_{ij} w_{ij} \sqrt{\frac{\pi}{2}} \sigma_j H\left[-\frac{1}{2}, 1, -\frac{\lambda_{ij}^2}{2\sigma_j^2}\right]. \tag{5.10}$$

In this case, the iterative scheme for Equation (5.10) follows from Aly and White (1978). Therefore, the iteration scheme for Euclidean distance–symmetric bivariate normal probability distribution case is:

$$x_{ik}^{(r+1)} = \frac{\sum\limits_{j=1}^{n_i} \frac{c_{ij} w_{ij}}{\sigma_j} \mu_{jk} H\left(\frac{1}{2}, 2, -\frac{(\lambda_{ij}^{(r)})^2}{\sigma_j^2}\right)}{\sum\limits_{j=1}^{n_i} \frac{c_{ij} w_{ij}}{\sigma_j} H\left(\frac{1}{2}, 2, -\frac{(\lambda_{ij}^{(r)})^2}{\sigma_j^2}\right)} \text{ for } k = 1, 2. \tag{5.11}$$

In the Euclidean distance–bivariate exponential probability distribution case, the PWP can be obtained by substituting the Equation (3.22) into the Equation (5.1) such that

$$\min E[z_i] = \sum_{j=1}^{n_i} c_{ij} w_{ij} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sqrt{(x_{i1} - a_{j1})^2 + (x_{i2} - a_{j2})^2} \frac{1}{4} \lambda_{j1} \lambda_{j2} \exp\{-\lambda_{j1} |a_{j1} - \mu_{j1}| - \lambda_{j2} |a_{j2} - \mu_{j2}|\} \, da_{j1} \, da_{j2}. \quad (5.12)$$

Moreover, we obtain the iteration scheme for this case by substituting the Equation (3.22) into the Equations (5.6) and (5.7) such that

$$x_{ik}^{(r+1)} = \left[ \sum_{j=1}^{n_i} \frac{c_{ij} w_{ij}}{\sigma_j} \mu_{jk} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sqrt{(x_{i1}^{(r)} - a_{j1})^2 + (x_{i2}^{(r)} - a_{j2})^2} \frac{1}{4} \lambda_{j1} \lambda_{j2} \exp\{-\lambda_{j1} |a_{j1} - \mu_{j1}| - \lambda_{j2} |a_{j2} - \mu_{j2}|\} \, da_{j1} \, da_{j2} \right] \Bigg/ \left[ \sum_{j=1}^{n_i} \frac{c_{ij} w_{ij}}{\sigma_j} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sqrt{(x_{i1}^{(r)} - a_{j1})^2 + (x_{i2}^{(r)} - a_{j2})^2} \frac{1}{4} \lambda_{j1} \lambda_{j2} \exp\{-\lambda_{j1} |a_{j1} - \mu_{j1}| - \lambda_{j2} |a_{j2} - \mu_{j2}|\} \, da_{j1} \, da_{j2} \right] \text{ for } k = 1, 2. \quad (5.13)$$

In the rectilinear distance case, as Wesolowsky (1977) stated, PWP is decomposed into two parts with respect to coordinates. To illustrate, below is the PWP for the rectilinear distance:

$$\min E[z_i] = \sum_{j=1}^{n_i} c_{ij} w_{ij} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (|x_{i1} - a_{j1}| + |x_{i2} - a_{j2}|) f_j(a_j) \, da_{j1} \, da_{j2}. \quad (5.14)$$

It becomes equivalent to

$$\min E[z_i] = \sum_{j=1}^{n_i} c_{ij} w_{ij} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \{|x_{i1} - a_{j1}| f_j(a_j) + |x_{i2} - a_{j2}| f_j(a_j)\} \, da_{j1} \, da_{j2}. \quad (5.15)$$

Let the marginal probability densities be $f_{j1}(a_j)$ and $f_{j2}(a_j)$. Then the Equation (5.15)

can be rewritten as

$$\min E\left[z_i\right] = \overbrace{\sum_{j=1}^{n_i} c_{ij} w_{ij} \int_{-\infty}^{\infty} |x_{i1} - a_{j1}| f_{j1}(a_j)\, da_{j1}}^{E_1[z_i]} + \overbrace{\sum_{j=1}^{n_i} c_{ij} w_{ij} \int_{-\infty}^{\infty} |x_{i2} - a_{j2}| f_{j2}(a_j)\, da_{j2}}^{E_2[z_i]}. \quad (5.16)$$

It is obvious from Equation (5.16) that minimization of the first sum, $E_1[z_i]$, gives the optimum $x_{i1}^*$ and minimization of the second sum, $E_2[z_i]$, gives the optimum $x_{i2}^*$ (Wesolowsky 1977). Moreover, Wesolowsky (1977) has already derived the independent equations like $E_1[z_i]$ and $E_2[z_i]$ for the bivariate uniform, bivariate normal and bivariate exponential probability distribution cases. Since these equations are convex, Wesolowsky (1977) takes the derivatives of these functions with respect to $x_{i1}$ and $x_{i2}$ and then tries to find the root of the equations. For bivariate normal and bivariate exponential probability distribution cases Wesolowsky (1977) finds the roots by using bisection. On the other hand, for the bivariate uniform probability distribution case, Wesolowsky (1977) developed a method for finding the root of the equations. However, we catch a typing error in the proposed method. In the study, there is a term $(1 - I_i)$ in the denominator of (9) which should be written as $(1 + I_i)$. Here, it is appropriate to state that the bivariate uniform probability distribution case can also be solved by the bisection method. Now, let us restate these equations. To clarify, we will give the equations only for one dimension, namely just for $E_1[z_i]$. Moreover, the equations for $E_1[z_i]$ can easily be adjusted for $E_2[z_i]$. For the bivariate uniform distribution case, simplified version of $E_1[z_i]$ can be given as:

$$E\left[z_i\right]_x = -\sum_{j=1}^{n_i} \frac{c_{ij} w_{ij} R_j}{2(f_j - e_j)} [I_j (x_{i1} - e_j)^2 + (x_{i1} - f_j)^2]. \quad (5.17)$$

Here $R_j = 1$ if $x_{i1} \geq f_j$ otherwise $R_j = -1$ and $I_j = 1$ if $e_j \leq x_{i1} < f_j$ otherwise

$I_j = -1$. The partial derivative of Equation (5.17) with respect to $x_{ij}$ is

$$\frac{\partial E\left[z_i\right]_x}{\partial x_{i1}} = -\sum_{j=1}^{n_i} \frac{c_{ij} w_{ij} R_j}{2(f_j - e_j)} [I_j (x_{i1} - e_j)^2 + (x_{i1} - f_j)^2]. \tag{5.18}$$

For the bivariate exponential probability distribution case, simplified $E_1[z_i]$ is given as

$$E_1\left[z_i\right] = \sum_{j=1}^{n_i} \frac{c_{ij} w_{ij}}{\lambda_{j1}} \lambda_{j1} (x_{i1} - \mu_{j1}) I_j + \exp[-\lambda_{j1} (x_{i1} - \mu_{j1}) I_j] \tag{5.19}$$

where $I_j = 1$ if $x_{i1} \geq \mu_{j1}$ otherwise $I_j = -1$ and the partial derivative of the Equation (5.19) with respect to $x_{i1}$ is given by

$$\frac{\partial E_1\left[z_i\right]}{\partial x_{i1}} = \sum_{j=1}^{n_i} c_{ij} w_{ij} I_j \{1 - \exp[-\lambda_{j1} (x_{i1} - \mu_{j1}) I_j]\} \tag{5.20}$$

Finally, for the bivariate normal distribution case, $E_1[z_i]$ is

$$E_1\left[z_i\right] = \sum_{j=1}^{n_i} \int_{-\infty}^{\infty} \frac{c_{ij} w_{ij} |x_{i1} - a_{aj1}|}{\sqrt{2\pi}\sigma_{j1}} \exp\left\{ -\frac{1}{2} \left[ \frac{(a_{j1} - \mu_{j1})}{\sigma_{j1}} \right]^2 \right\} da_{j1} \tag{5.21}$$

and the simplified partial derivative of this equation with respect to $x_{i1}$ is given as

$$\frac{\partial E_1\left[z_i\right]}{\partial x_{i1}} = \sum_{j=1}^{n_i} c_{ij} w_{ij} \left[ 1 - 2\Pr\left( z \geq \frac{x_{i1} - \mu_{j1}}{\sigma_{j1}} \right) \right] \tag{5.22}$$

where $z$ has a standardized normal distribution.

To sum up, in order to solve Euclidean distance PWP with bivariate normal probability distribution, the iteration scheme (5.11) can be used. Similarly, for solving the symmetric bivariate exponential probability distribution case the iterative scheme (5.13) can be used. On the other hand, rectilinear distance PWP is decomposed respect to coordinate variables, $x_{i1}$ and $x_{i2}$. Therefore, these two parts can be minimized separately. As a sequence, Wesolowsky (1977) has already derived the equations for solving the rectilinear distance PWP with symmetric bivariate uniform, exponential

and normal probability distribution cases. Respectively, Equations (5.18), (5.20) and (5.22) can be used to solve rectilinear distance PWP with symmetric bivariate uniform, bivariate exponential and bivariate normal probability distribution cases.

### 5.1.2. Solution of the Probabilistic Weber Problem (PWP) Using Approximations

In Chapter 4, we have introduced how to approximate expected distance functions and we obtain some approximation functions for different situations. To illustrate, in the Euclidean distance case Equation (4.15) is used to approximate the expected distance function when customer locations are associated with a bivariate probability distribution with mean $\mu_{j1}$ and $\mu_{j2}$ and with equal standard deviations, $\sigma_{j1} = \sigma_{j2} = \sigma_j$. Besides, Equation (4.17) can be used when the distribution is not symmetric, namely $\sigma_{j1} \neq \sigma_{j2}$. In the rectilinear distance case, Equation (4.18) can be used to approximate the expected distances. Moreover, when the sample data is used to approximate the expected distances, Equation (4.21) can be used. However, when one of these approximating functions is used in the PALA heuristic, we need to solve PWP with the appropriate methods. In the previous section, we explain the solution methods for the PWP where expected distances are calculated exactly. Here, we are going to explain the solution methods of PWP with approximating expected distances.

Let us begin with the Euclidean distance for symmetric distributions (i.e. $\sigma_{j1} = \sigma_{j2} = \sigma_j$). When the approximation function for this case, Equation (4.15), is inspected the convexity of the function is obvious. The Equation (4.15) is convex but it is not strictly convex. Furthermore, before improving this function, we obtain the aproximation function as Equation (4.12) which is absolutely non-convex function. However, in order to improve the function, we chop the function where it starts to get worse and we patch the function at this point. Therefore, the function is convex which is also illustrated in Figure (4.4). In addition, when we use Equation (4.15) the PWP

becomes

$$\min \sum_{j=1}^{n_i} c_{ij} w_{ij} \begin{cases} d(\mathbf{x_i}, \boldsymbol{\mu_j}) + \sigma_j^2 \left(2d(\mathbf{x_i}, \boldsymbol{\mu_j})\right)^{-1} & \text{if } d(\mathbf{x_i}, \boldsymbol{\mu_j}) \geq \sigma_j/\sqrt{2} \\ \sqrt{2}\sigma_j & \text{if } d(\mathbf{x_i}, \boldsymbol{\mu_j}) < \sigma_j/\sqrt{2} \end{cases} \tag{5.23}$$

where $d(\mathbf{x_i}, \boldsymbol{\mu_j})$ is the Euclidean distance between facility $i$ and the coordinate distribution means of customer $j$. In Equation (5.23), we multiply the convex functions with some positive constants and we sum the resulting functions. As it is known, when we multiply a convex function with a positive constant we obtain a convex function and when we add two convex functions we again obtain a convex function. Therefore, Equation (5.23) is a convex function and it can be solved by an unconstrained minimization techniques (Bazaraa *et al.* 2006).

For an asymmetric probability distribution of the coordinates, we have derived the approximation function as Equation (4.17). When this function is used as an approximation of the expected distance function, the PWP for this case becomes

$$\min \sum_{j=1}^{n_i} c_{ij} w_{ij} \sqrt{(x_{i1} - \mu_{j1})^2 + (x_{i2} - \mu_{j2})^2} + \frac{\sigma_{j1}^2 (x_{i2} - \mu_{j2})^2 + \sigma_{j2}^2 (x_{i1} - \mu_{j1})^2}{2((x_{i1} - \mu_{j1})^2 + (x_{i2} - \mu_{j2})^2)^{3/2}}. \tag{5.24}$$

Since Equation (4.17) is a non-convex function, the objective function is the sum of $n$ nonconvex functions, which makes it probably nonconvex; we have not be able to know it. For this reason, we can only suggest to use an unconstrained minimization techniques to compute a stationary point, which can be optimal. Remark that, the objective function becomes undefined when a facility is located on one of the customer means, namely when $x_{i1} = \mu_{j1}$ and $x_{i2} = \mu_{j2}$. In this situation, as expressed earlier in Chapter 4, we add a small quantity, $\epsilon$.

For the rectilinear distance case, solving PWP is more easier than the Euclidean distance case because the approximation function does not include any standard deviation terms or any fractional terms which is clearly seen in Equation (4.18). If Equation

(4.18) is used as an approximation to expected distances then PWP can be stated as

$$\min \sum_{j=1}^{n_i} c_{ij} w_{ij} (|x_{i1} - \mu_{j1}| + |x_{i2} - \mu_{j2}|). \tag{5.25}$$

It is easy to show that Equation (4.18) is a piecewise linear and convex function. Therefore, (5.25) is a convex minimization problem. However, it is not differentiable at every point. Therefore, (5.25) can be solved by using a descent method which does not require derivative evaluations.

In Sections (4.2) and (4.3), we explain how to approximate the expected distances when a sample is available. One way to have an approximation is to estimate the parameters using the sample. In Section (4.2), we have introduced unbiased estimators for the parameters mean and standard deviation. After estimating the parameters, one of the appropriate Equations (4.15), (4.17) or (4.18) can be used and the PWP can be solved.

The second approximation method is based on the estimation of the expected distance by mean of its average. Equation (4.22) is used to approximate the expected distance function in the Euclidean distance case and the PWP becomes

$$\min z = \sum_{j=1}^{n_i} c_{ij} w_{ij} \frac{1}{|S_j|} \sum_{a_j \in S_j} \sqrt{(x_{i1} - a_{j1})^2 + (x_{i2} - a_{j2})^2}. \tag{5.26}$$

It is trivial to observe that Equation (5.26) is strictly convex and differentiable. Therefore, we can derive Weiszfeld type iterative equations for solving this PWP. In order to derive the equations, we need to solve $\nabla_{x_i} z = 0$. When we take the partial derivatives of Equation (5.26) with respect to $x_{i1}$ and $x_{i2}$, the equations becomes

$$\frac{\partial z}{\partial x_{i1}} = \sum_{j=1}^{n_i} c_{ij} w_{ij} \frac{1}{|S_j|} \sum_{a_j \in S_j} \frac{(x_{i1} - a_{j1})}{\sqrt{(x_{i1} - a_{j1})^2 + (x_{i2} - a_{j2})^2}} = 0, \tag{5.27}$$

$$\frac{\partial z}{\partial x_{i2}} = \sum_{j=1}^{n_i} c_{ij} w_{ij} \frac{1}{|S_j|} \sum_{a_j \in S_j} \frac{(x_{i2} - a_{j2})}{\sqrt{(x_{i1} - a_{j1})^2 + (x_{i2} - a_{j2})^2}} = 0. \tag{5.28}$$

When we solve these equations in terms of $x_{i1}$ and $x_{i2}$ and when we pass these solutions to the iteration scheme, we obtain the iterative equations as

$$x_{i1}^{(r+1)} = \frac{\sum_{j=1}^{n_i} c_{ij} w_{ij} \frac{1}{|S_j|} \sum_{a_j \in S_j} \frac{a_{j1}}{\sqrt{(x_{i1}^{(r)}-a_{j1})^2+(x_{i2}^{(r)}-a_{j2})^2}}}{\sum_{j=1}^{n_i} c_{ij} w_{ij} \frac{1}{|S_j|} \sum_{a_j \in S_j} \frac{1}{\sqrt{(x_{i1}^{(r)}-a_{j1})^2+(x_{i2}^{(r)}-a_{j2})^2}}}, \tag{5.29}$$

$$x_{i2}^{(r+1)} = \frac{\sum_{j=1}^{n_i} c_{ij} w_{ij} \frac{1}{|S_j|} \sum_{a_j \in S_j} \frac{a_{j2}}{\sqrt{(x_{i1}^{(r)}-a_{j1})^2+(x_{i2}^{(r)}-a_{j2})^2}}}{\sum_{j=1}^{n_i} c_{ij} w_{ij} \frac{1}{|S_j|} \sum_{a_j \in S_j} \frac{1}{\sqrt{(x_{i1}^{(r)}-a_{j1})^2+(x_{i2}^{(r)}-a_{j2})^2}}}. \tag{5.30}$$

When rectilinear distance is used in the average distance approximation, the PWP can be formulated specifically by using the Equations (5.1) and (4.24), which is

$$\min z = \sum_{j=1}^{n_i} c_{ij} w_{ij} \frac{1}{|S_j|} \sum_{a_j \in S_j} (|x_{i1} - a_{j1}| + |x_{i2} - a_{j2}|). \tag{5.31}$$

Since rectilinear distance function is convex but not differantible, (5.31) is a convex minimization problem; but it is not differantible. Therefore, we can not derive iterative solution equations. However, we can solve (5.31) with a derivative free unconstrained minimization techniques.

## 5.2. Probabilistic Discrete Approximations

In order to increase the readibility, we first describe briefly the deterministic approximation for the CMFWP which is due to Aras *et al.* (2006). It is possible to state that a set of optimal locations for the facilities lies in the convex hull of the customer locations for the CMFWP with rectilinear, Euclidean, squared Euclidean and $\ell_p$ norm distance functions with $1 \leq p \leq 2$ as a consequence of the results by Hansen *et al.* (1980). In addition, Wendell and Hurter (1973) showed that, rectilinear CMFWP has always an optimal solution with facilities located at the intersection points of vertical and horizontal lines drawn through the customer locations. However, this property is not valid for the CMFWP with other distance functions. On the other hand, for the rectilinear problem an equivalent mixed integer program can be used to obtain an

approximate solution which uses the points of an arbitrary grid lying in the convex hull of customer locations.

In the following an approximating mixed integer linear program (MILP) formulation is given which can be referred as ACMFWP:

$$\min \sum_{i=1}^{m}\sum_{k=1}^{o}\sum_{j=1}^{n} c_{ikj}u_{ikj} \tag{5.32}$$

s.t.

$$\sum_{i=1}^{m}\sum_{k=1}^{o} u_{ikj} = d_j \qquad j = 1,\ldots,n; \tag{5.33}$$

$$\sum_{k=1}^{o} y_{ik} = 1 \qquad i = 1,\ldots,m; \tag{5.34}$$

$$\sum_{j=1}^{n} u_{ikj} = s_i y_{ik} \qquad i = 1,\ldots,m; k = 1,\ldots,o; \tag{5.35}$$

$$u_{ikj} \geq 0,\ y_{ik} \in \{0,1\} \qquad i = 1,\ldots,m; k = 1,\ldots,o; j = 1,\ldots,n. \tag{5.36}$$

In the formulation $k = 1, 2, \ldots, o$ denotes the grid points within the convex hull of the customers and $u_{ikj}$ denotes the amount shipped from facility $i$ located at point $k$ to customer $j$. Variables $y_{ik}$ are binary variables denoting whether facility $i$ is located at point $k$. $c_{ikj}$ is the unit cost of shipment from facility $i$ located at point $k$ to customer $j$. Here, $c_{ikj}$ is obtained by multiplying the unit shipment cost per unit distance from facility $i$ located at point $k$, $c'_{ik}$, with the distance between point $k$ and customer $j$, namely $c_{ikj} = c'_{ik}d(\mathbf{x_k}, \mathbf{a_j})$. Here, $d(\mathbf{x_k}, \mathbf{a_j})$ is the distance function.

In this mathematical model, constraints (5.33) assure that the demand of each customer is satisfied. Constraints (5.34) quarantee that each facility $i$ is located at exactly one of the candidate points $k = 1, \ldots, o$. Constraints (5.35) ensure that the total amount shipped from facility $i$ located at point $k$ is equal to its capacity. Here, it is assumed that the problem is balanced, namely total supply of the facilities is equal to the total demand of customers. Moreover, this model allows locating more than one

facility at the same point.

The ACMFWP can be solved to optimality using standard methods of integer programming. Although, some commercial solvers are available to solve these types of problem, the computation time may increase exponentially with its size. On the contrary, in order to have better solutions, the ACMFWP has to be formulated by using large number of grid points which increase the size of the MILP.

It is obvious that the solution of the ACMFWP does not ensure an optimal solution for the CMFWP if not we are extremely lucky in identifying a grid in the plane which includes optimal facility sites. Basicaly, when the number of candidate points selected from the convex hull of customer locations goes to infinity, we would obtain an optimal solution to the original CMFWP. Thus, there is a trade-off between the quality of the solution by the MILP and the computation time required to solve it. Although ACMFWP is used to solve CMFWP, we can also use this discrete formulation for solving PCMFWP. As it is emphasized in the previous sections, we use the expectation criteria in order to handle this problem. Therefore, we can minimize the expectation of the Equation (5.32). When we take the expectation of the Equation (5.32), the objective function becomes

$$\min \sum_{i=1}^{m} \sum_{k=1}^{o} \sum_{j=1}^{n} E[c_{ikj}] u_{ikj} \tag{5.37}$$

where $c_{ikj} = c'_{ik} d(\mathbf{x_k}, \mathbf{a_j})$. It is obvious that $c'_{ik}$ is constant and it can be taken out of the expectation operator. Remark that, $c'_{ik}$ is equal to $c_{ij}$ in our representation, so we can use $c_{ij}$ instead of $c'_{ik}$. Thus, the Equation (5.37) can be rewritten as

$$\min \sum_{i=1}^{m} \sum_{k=1}^{o} \sum_{j=1}^{n} c_{ij} E[d(\mathbf{x_k}, \mathbf{a_j})] u_{ikj}. \tag{5.38}$$

In Section (3) and (4) calculation methods of the expected distance functions for different situations has been introduced. To be more specific, $E[d(\mathbf{x_i}, \mathbf{a_j})]$ can be calculated with one of the appropriate expressions (3.17), (3.22), (3.25), (3.27), (3.29), (4.15),

(4.17), (4.18). Moreover, the constraints of the ACMFWP does not include any random variables. Thus, the constraints (5.33)-(5.36) remain the same for the probabilistic case. For the naming convention, we can call the probabilistic version of the ACMFWP as APCMFWP which has a formulation of

$$\min \sum_{i=1}^{m}\sum_{k=1}^{o}\sum_{j=1}^{n} c_{ij}E[d(\mathbf{x_k}, \mathbf{a_j})]u_{ikj}$$

$$\text{s.t. Equations } (5.33) - (5.36) \text{ hold.}$$

As stated earlier, when the number of candidate points ($o$) increases, the amount of time required to solve ACMFWP is also increased drastically since it is a mixed integer programming problem. The same properties also hold for the APCMFWP. On the other hand, even if we assume that the solution space of the PCMFWP is limited with the convex hull of the customer means we still have infinitely many candidate points. Therefore, we need to decrease the number candidate points.. Hence, we have developed two grid based solution methods, *cellular heuristic (CH)* and *grids on means heuristic (GOMH)*, and a $p$-median type solution method which is named as *mean discrete location-allocation heuristic (MDLAPH)*. In the following subsections, we will explain these methods in detail.

### 5.2.1. Cellular Heuristic

The selection of the candidate points is crucial since the solution quality of the APCMFWP depends heavily on the candidate points. Therefore, good selection methods for obtaining better candidate facility locations are necessary. Gamal and Salhi (2003) proposed a cellular heuristic to solve Euclidean distance uncapacitated MFWP. In this approach, a rectangle is drawn which covers all customer locations, and divided into the cells for a fixed resolution. Furthermore, these cells are considered as a potential for containing the optimal facility locations. Besides, Aras *et al.* (2006) adopt this idea for the solution of the rectilinear distance CMFWP. At the beginning, Aras *et al.* (2006) apply ALA type two-phase heuristic for a large number of times starting with initial facility locations chosen arbitrarily from the set of the intersection points of the

lines drawn through the customer locations. Then they draw a rectangle which covers all the points obtained from the two-phase runs. After that, the rectangle is divided into cells and the centroids of the cells are calculated by using the points residing in each cell. Finally, these centroid points are used to formulate ACMFWP. Inspiring from these cellular heuristic methods, we can developed a cellular heuristic for solving PCMFWP.

Algorithm 2 is the generic cellular heuristic method for solving PCMFWP with various distance function and probability distribution assumptions. In the first step, PALA method is solved for pre-determined $T$ times. Remember that, PALA method has been introduced in Section 5.1 for all assumptions. Here, if $T$ is selected very large, the number of the candidate facility locations will be very high. In this situation, we may obtain better results but the solution of the APCMFWP may last very long. On the other hand, if $T$ is selected very small then the quality of the solution may not be satisfiable. As a result, there is a tradeoff between the solution quality and the required computational time.

**Algorithm 2** Cellular Heuristic

1. Apply PALA method for the pre-determined times, $T$, with random initial facility locations and keep the solution points in each trial.

2. Draw the smallest rectangle which covers all facility locations obtained in step 1 and then divide the rectangle $h$ times in horizantal axis and $v$ times in vertical axis with equal lenghts. Here $h$ and $v$ determine the resolutions of the grid.

3. Find the centroid of the points contained within the cells.

4. Solve the APCMFWP
$$\left\{ \min \sum_{i=1}^{m} \sum_{k=1}^{o} \sum_{j=1}^{n} c_{ij} E[d(\mathbf{x_k}, \mathbf{a_j})] u_{ikj} : \text{s.t. Equations } (5.33) - (5.36) \right\}$$
by using the centroid points and the candidate customer locations.

5. Apply PALA heuristic starting at the facility locations obtained from the optimal solution of the APCMFWP.

6. Report final facility locations obtained from the final run of the PALA heuristic.

In the second step, cells are constructed. Let us explain the construction of the cells in detail. Let $\mathbf{x_i}^{(t)} = (x_{i1}^{(t)}, x_{i2}^{(t)})^T : \ i = 1, \ldots, m; \ t = 1, \ldots, T$ be the coordinates of facility $i$ obtained in the $t^{th}$ PALA run. We define the minimum and maximum of the $x$-coordinates obtained in $T$ runs as $x_{1min} = \min_{i,t} x_{i1}^{(t)}$ and $x_{1max} = \max_{i,t} x_{i1}^{(t)}$, respectively. $x_{2min} = \min_{i,t} x_{i2}^{(t)}$ and $x_{2max} = \max_{i,t} x_{i2}^{(t)}$ are defined similarly. Let $L_x$ and $L_y$ represent the width and height of the cells, which are determined as $L_x = (x_{1max} - x_{1min})/h$ and $L_y = (x_{2max} - x_{2min})/v$ where $h$ and $v$ denote the number of cells along the $x$-axis and $y$-axis,respectively. Creating a large number of cells by choosing high values for $h$ and $v$ would require a large number of PALA runs in order to reach a meaningful distribution of locally optimal locations over the cells. On the other hand, having only a few cells would result in the aggregation of different location in one cell. In this case, the information conveyed by the facility locations of different PALA runs would be lost.

The third step is to assign each facility location $\mathbf{x_i}^{(t)} : \ i = 1, \ldots, m; \ t = 1, \ldots, T$ found by the PALA heuristic into the appropriate cell. To do so, each cell is defined by its bottom left corner with coordinates $(x_{c_x}, y_{c_y})^T : \ 1 \leq c_x \leq v; \ 1 \leq c_y \leq h$ where $x_{c_x} = x_{1min} + (c_x - 1)L_x$ and $y_{c_y} = x_{2min} + (c_y - 1)L_y$. As a result, facility $i$ with coordinates $(x_{i1}^t, x_{i2}^t)^T$ found in the $t^{th}$ PALA run belongs to the cell with bottom left corner $(x_{c_x}, y_{c_y})$ if and only if $x_{c_x} \leq x_{i1}^t < x_{c_{x+1}}$ and $y_{c_y} \leq x_{i2}^t < y_{c_{y+1}}$. After assignning facility locations to the related cells, the centroids of the non-empty cells are computed.

In the fourth step APCMFWP is solved by using the centroid points and the customer location means as candidate points. Customer location means are also considered as candidate points in order to improve the quality of the APCMFWP. While solving the APCMFWP, $E[d(\mathbf{x_k}, \mathbf{a_j})]$ is calculated with the same method used in the first step for PALA runs.

In the fifth step, PALA heuristic is applied for the last time by using the facility locations obtained from APCMFWP. Here, the PALA heuristic may improve the solution of the algorithm or not. If the improvement is not possible, the PALA heuristic would directly give the solution of the APCMFWP. Finally, the facility locations

obtained from the PALA heuristic is reported in the last step.

For more consideration and clarification of the construction of the cells, we visualize an example in Figure 5.1 where the number of customers is 25 and the number of facilities to be located is six. In Figure 5.1, big blue squares illustrate the customer location means, small green squares illustrate the optimal facilitity locations obtained from $T = 100$ PALA runs and the moderate-sized red squares illustrate the centroids of the cells. Moreover, light blue lines illustrate the convex hull of the customer location means. In this example, we choose the parameters $h = 5$ and $v = 5$. Therefore, we obtain 25 cells which can be examined from the figure.

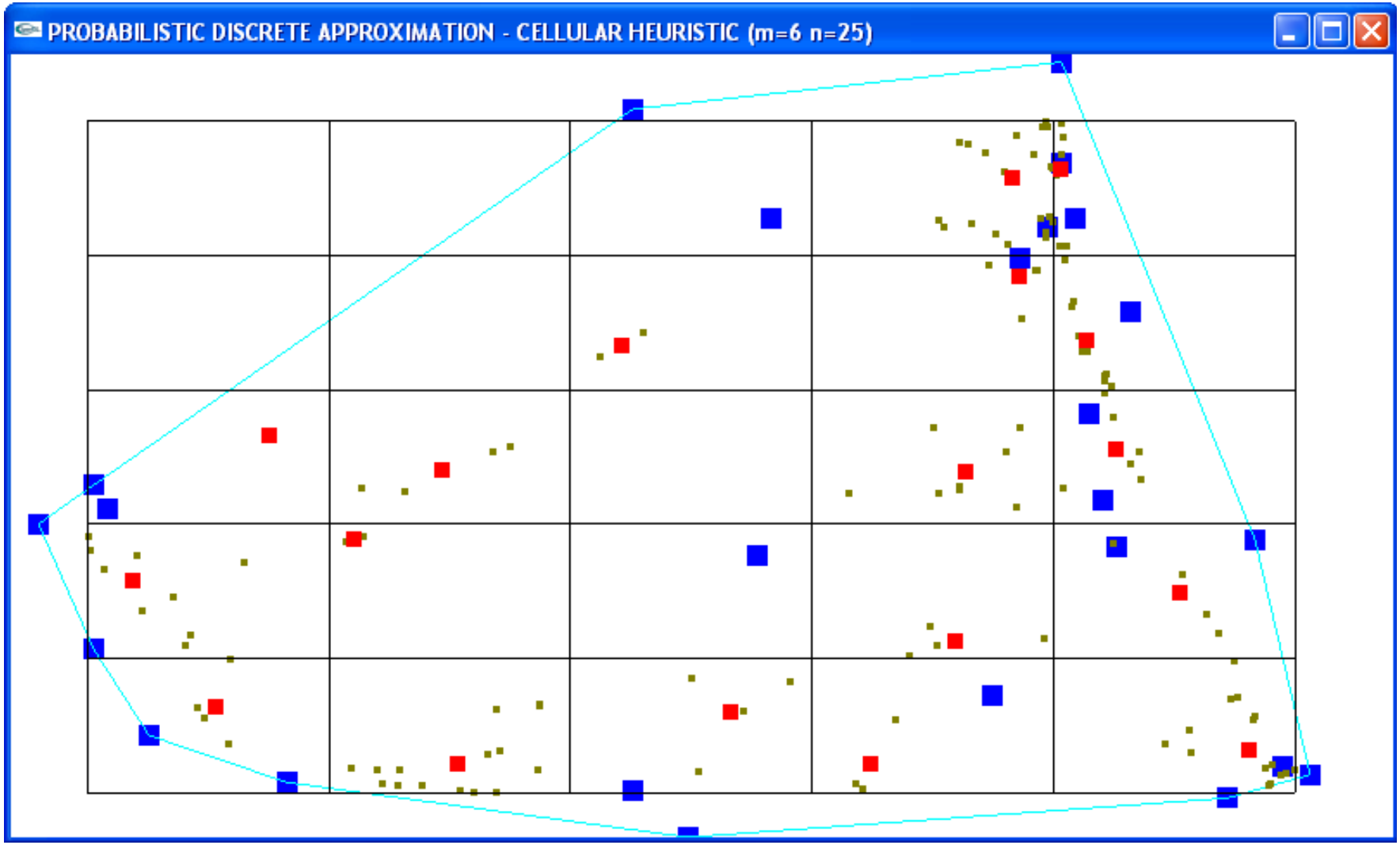



Figure 5.1. An example screen shot for the construction of the cells in Cellular Heuristic

### 5.2.2. Grids On Means Heuristic

Previously, we have emphasized that the rectilinear distance location problem has always an optimal solution with facilities located at the intersection points of vertical and horizontal lines drawn through the customer locations (Wendell and Hurter 1973). However, this property does not hold for the Euclidean distance location problem. On the other hand, this property can be useful while constructing the cells of the grid based approximation. Therefore, we can make some adoptation on the cellular heuristic which has been introduced in the previous section. The cells can be constructed by drawing horizontal and vertical lines through the customer means. That is why this method is named as *Grids On Means Heuristic*. Algorithm 3 is the generic grids on means heuristic which can be used to solve PCMFWP. The only difference between

**Algorithm 3** Grids On Means Heuristic

1. Apply PALA method for the pre-determined times, $T$, with random initial facility locations and keep the solution points in each trial.
2. Draw horizantal and vertical lines through each customer means and obtain cells.
3. Find the centroid of the points contained within the cells.
4. Solve the APCMFWP
$$\left\{ \min \sum_{i=1}^{m} \sum_{k=1}^{o} \sum_{j=1}^{n} c'_{ik} E[d(\mathbf{x_k}, \mathbf{a_j})] u_{ikj} : \text{s.t. Equations } (5.33) - (5.36) \right\}$$
by using the centroid points and the candidate customer locations.
5. Apply PALA heuristic starting at the facility locations obtained from the optimal solution of the APCMFWP.
6. Report final facility locations obtained from the final run of the PALA heuristic.

cellular heuristic and the grids on means heuristic is the construction of the cell. In the second step of the grids on means method, cells are constructed by drawing horizontal and vertical lines through the customer means. To clarify the construction of the cells, we visualize the second and the third steps of the algorithm in Figure 5.2 by using the same example used in Figure 5.1. Furthermore, remaining steps of the Algorithm 3 are the same with the steps of Algorithm 2. The readers are referred to Section 5.2.1 in

order to obtain more details about the same steps of the grids on means methods.

### 5.2.3. Mean DLAP Heuristic

Cooper (1963) developed a method, *destination subset algorithm*, for solving Euclidean distance uncapacitated MFWP which selects a subset of $m$ facilities from the total set of $n$ customers as locations to locate facilities. Furthermore, Cooper (1963) stated that if all possible subsets are considered then one of the these should provide a close approximation to the optimum location of facities and would yield the correct allocations. Furthermore, $p$-median heuristic developed by Hansen *et al.* (1998) for the Euclidean distance uncapacitated MFWP is also based on the observation that in an optimal solution to this problem the facility locations are usually close to the customer locations. Besides, this $p$-median approximation also applied to the rectilinear distance capacitated MFWP by Aras *et al.* (2006) and the results are found quite promising. Therefore, it is reasonable to use customer location means as candidate points. Algorithm 4 is the generic algorithm which can be used for solving APCMFWP. In the formulation, APCMFWP is solved by using the customer coordinate means. After that, the result is improved by applying the PALA method which starts from the facility locations found by the APCMFWP. Notice that, various calculation methods of the $E[d(\mathbf{x_k}, \mathbf{a_j})]$ has already been introduced in Sections (3) and (4).

**Algorithm 4** Mean DLAP Heuristic

1. Solve the APCMFWP
$$\left\{ \min \sum_{i=1}^{m} \sum_{k=1}^{o} \sum_{j=1}^{n} c_{ij} E[d(\mathbf{x_k}, \mathbf{a_j})] u_{ikj} : \text{s.t. Equations } (5.33) - (5.36) \right\}$$
by using the customer location means as candidate points.

2. Apply PALA heuristic by using the optimal facility locations obtained from APCMFWP.

3. Report final facility locations obtained from PALA heuristic.

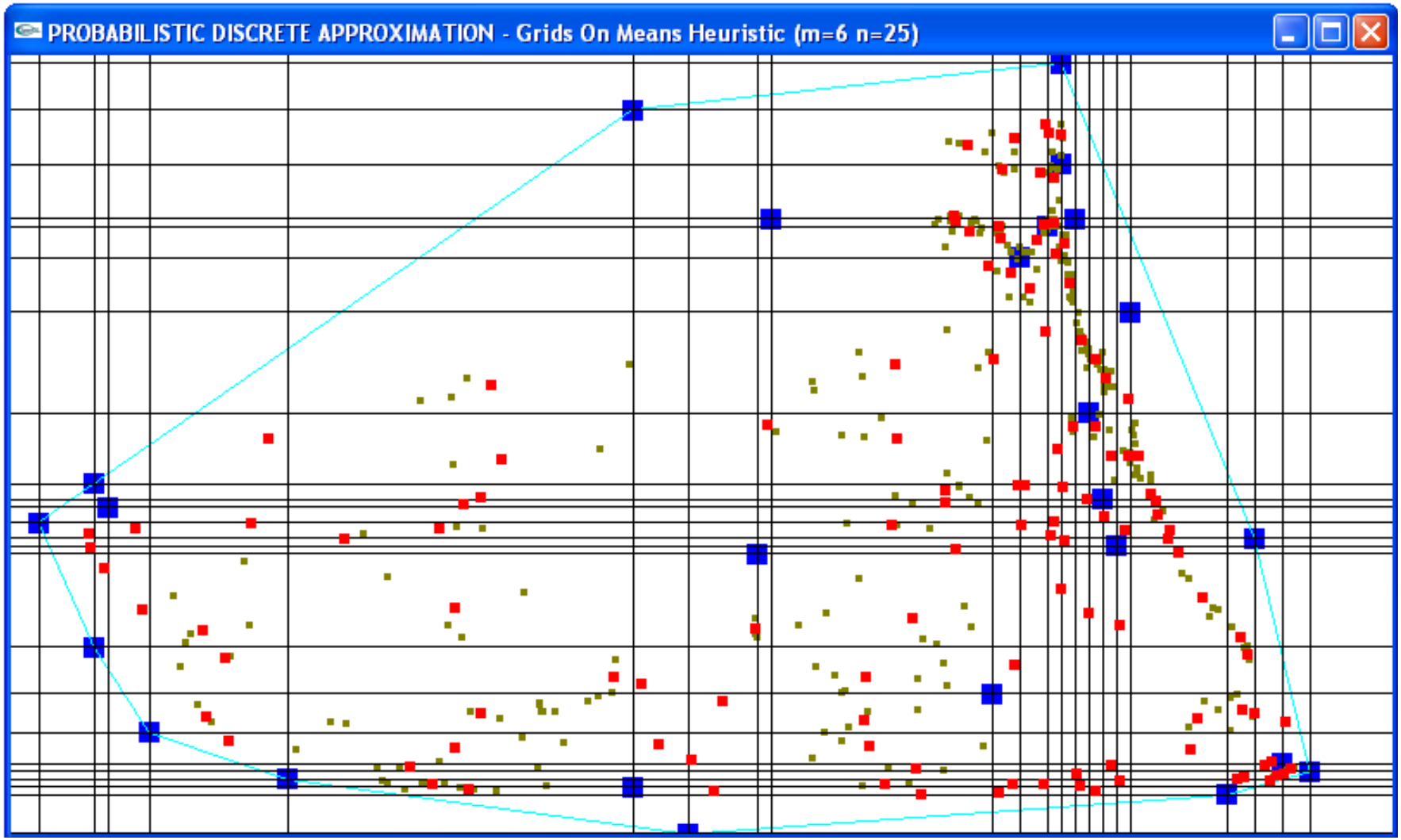



Figure 5.2. An example screen shot for the construction of the cells in Grids on Means Heuristic

# 6. EXPERIMENTAL RESULTS

## 6.1. Quality of the Approximations for Evaluating $E[d(\mathbf{x_i}, \mathbf{a_j})]$

Before using the approximation methods for the expected distance functions, it is appropriate to investigate the quality of them. In order to investigate the quality of the approximation methods, we plot two dimensional graphs for different situations. First of all, we investigate the Euclidean distance case. In this case, we select symmetric bivariate normal probability distribution for satifying the probabilistic customer location assumption. In Figure 6.1 we assume that we have a customer with means $\mu_{j1} = 0$ and $\mu_{j2} = 0$ and with the standard deviations $\sigma_{j1} = \sigma_{j2} = \sigma_j = 20$. Furthermore, we assume that we locate a facility where its $y$ coordinate is zero and its $x$ coordinate differs in the range $[-60, +60]$. In this graph, we plot the expected distances by using the exact method (Equation (3.17)), Taylor's approximation (Equation 4.12) and the average distance (Equation 4.22) with 25, 100 and 1000 samples generated from the distribution. As it is seen in Figure 6.1, when the sample size increases average

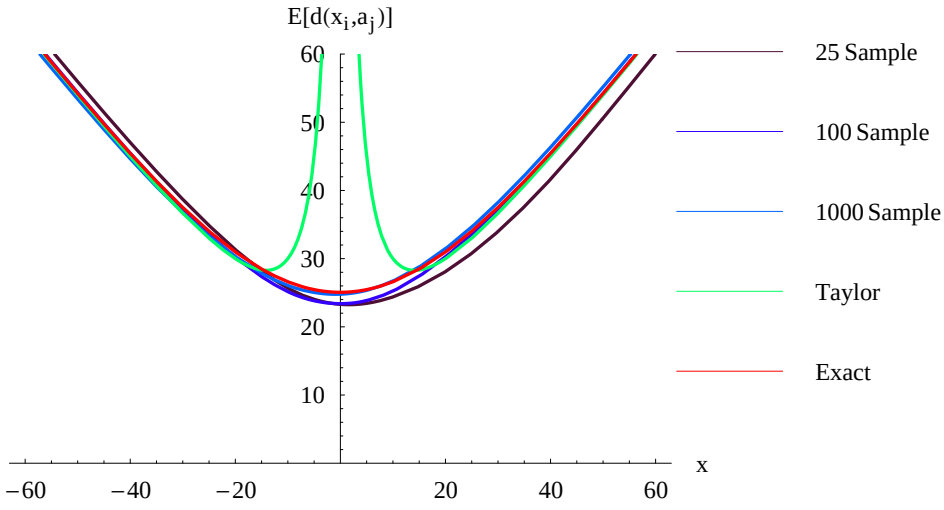



Figure 6.1. Sensitivity for sample size for symmetric distribution: Average distance vs Taylor's approximation for the Euclidean norm

distance to sample is getting closer to the exact expected distance. Furthermore, as stated earlier, Taylor's approximation is very close to the exact expected distance when

distance to the customer mean is larger than $\sqrt{2}\sigma_j$. On the other hand, when distance to customer mean is smaller than $\sqrt{2}\sigma_j$, Taylor's approximation starts to give large values for the expectation (asymptotically goes to infinity as $x$ approaches to $\mu$). In order to handle this problem, we have improved the function which is given by the Equation (4.15). For supporting our comments, we also plot the exact expected distance, improved Taylor approximation and average distance to sample method with 1000 samples in Figure 6.2.

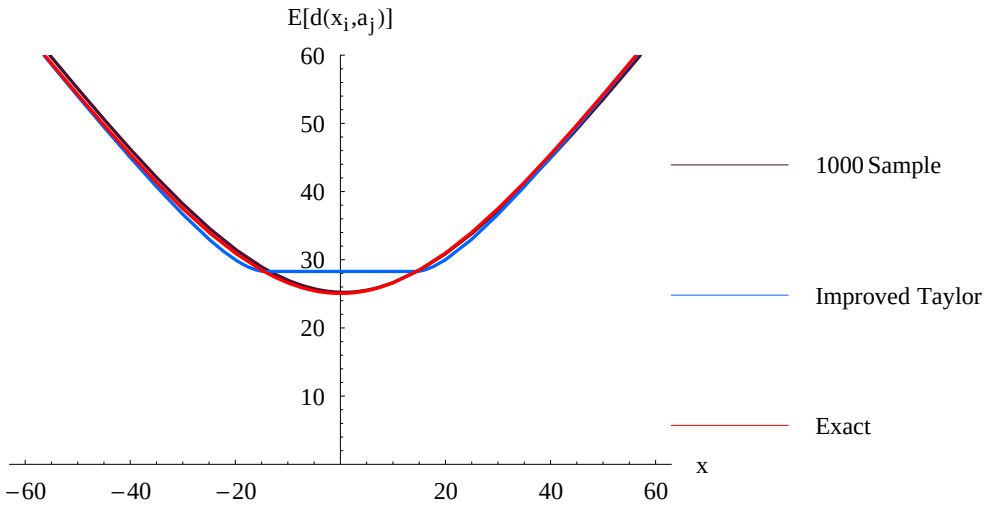



Figure 6.2. Improving Taylor approximation

In the previous illustrations, we have investigated the equal standard deviations situation. However, in some situations, standard deviations may not be equal to each other (i.e. an asymmetric distribution). For these circumstances, we have developed a Taylor approximation formula for approximating the expected distance function which has given by Equation (4.17). Moreover, we can use Equation (4.17) when we have samples of the customer locations. In Section 4.2, we have given the unbiased point estimators of the parameters mean and standard deviation. When means and standard deviations are estimated from samples, it is almost impossible to have equal standard deviations. Therefore, it is appropriate to use Equation (4.17) as an approximation of the expected distance function. Now, let us use the same samples and parameters of Figure 6.1. After evaluating the point estimators with the help of the Equations (4.19) and (4.20), we plot the approximating expected distances in Figure 6.3 by using

Equation (4.17). As it is seen in the figure, when the sample size increases the standard deviations are getting closer to each other. Therefore, sample approximations are also getting closer to the Taylor approximation. Now, let us investigate a different example.

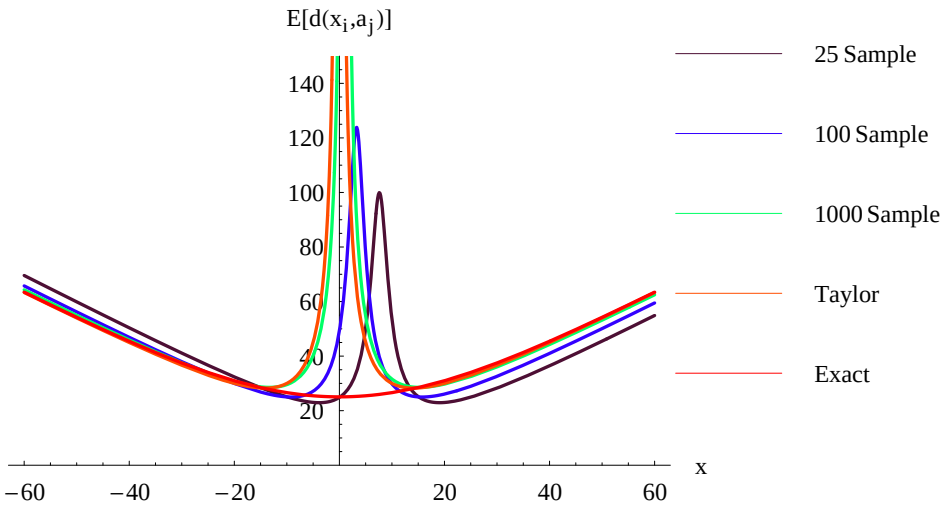



Figure 6.3. Sensitivity to sample size for symmetric distributions: Average distance vs Taylor approximation with estimated distribution parameters for the Euclidean norm

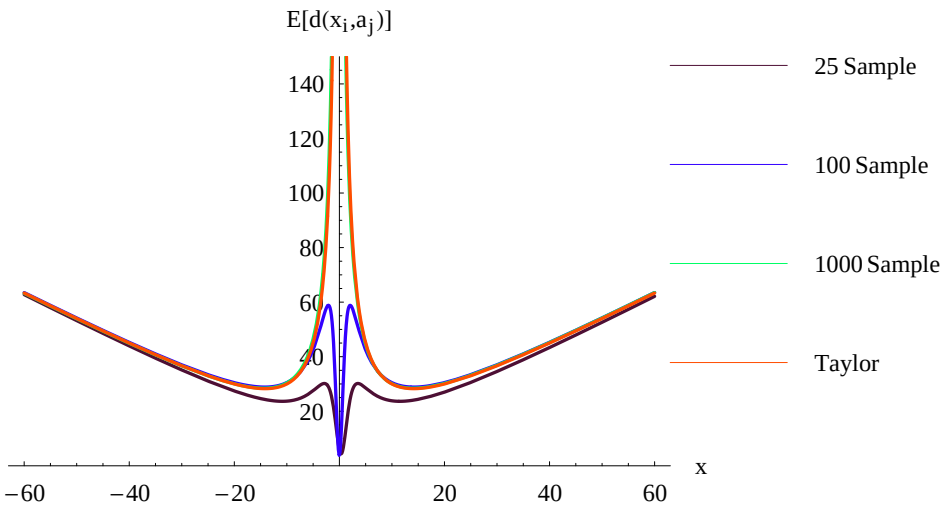



Figure 6.4. Sensitivity to sample size for asymmetric distributions: Taylor approximation with exact and estimated parameters for the Euclidean norm

In Figure 6.4, we again assume that we have a customer with means $\mu_{j1} = 0$ and $\mu_{j2} = 0$ and with standard deviations $\sigma_{j1} = 3$ and $\sigma_{j2} = 20$. Furthermore, we have generated 25, 100 and 1000 samples from the distribution. After evaluating the parameters with the help of the Equations (4.19) and (4.20), we plot the approximating expected distances

by using Equation (4.17). Besides, we also plot the Taylor approximation (Equation (4.17)) by using the distribution parameters directly. On the other hand, we omit the plot of the exact expected distances since the double integral requires considerable computational effort. Thus, it is hard to compare the approximations with the exact expected distances. The only result from this graph is that as the sample size increases, sample approximation is changing its shape and it resembles the shape of the Taylor approximation. In addition, sample approximations have the minimum value near the customer location mean, $(0, 0)$.

In the rectilinear distance case, we use symmetric bivariate uniform probability distribution for satisfying the probabilistic customer location assumption. The second order Taylor approximation for the rectilinear distance has been given in the Equation 4.18. As it is seen from the equation, the approximation does not include any standard deviation terms. Moreover, it is just the regular rectilinear distance to the customer mean. For investigating the quality of the function, we assume that we have a customer with the location parameters $e_j = -5$, $f_j = 5$, $g = -5$, and $h = 5$. When we calculate the means and the standard deviations of this bivariate uniform distribution, we obtain $\mu_{j1} = 0$, $\mu_{j2} = 0$, $\sigma_{j1} = 2.8867$ and $\sigma_{j2} = 2.8867$. In Figure 6.5, we plot the exact expected distance function (Equation (3.25)), Taylor approximation (Equation (4.18)) and average distance to (Equation (4.24)) with 25, 100 and 1000 samples. In this figure, we have to remark that $x$ coordinate is changing in the range $[-10, +10]$ and the $y$ coordinate is set to zero which is also the mean of the customer. When we look at Figure 6.5, it is obvious that average distance to sample method is very close to the exact expected distance function. Moreover, the approximating distance function with 1000 sample is almost the same with the exact expected distance function. On the other hand, there is a remarkable gap between Taylor's approximation and the exact expected distance function. However, this graph can be deceptive for the Taylor approximation. In Figure 4.9, we have already plot graphs of the Taylor approximation and the exact expected distance function. In Figure 4.9, we see that Taylor approximation follows closely the exact expected distance function except through the lines $y = 0$ and $x = 0$. To prove these comments, let us replot Figure 6.5 where $y = x$. Figure 6.6 is the different view of the Figure 6.5. As it can be seen, Taylor approximation is very close

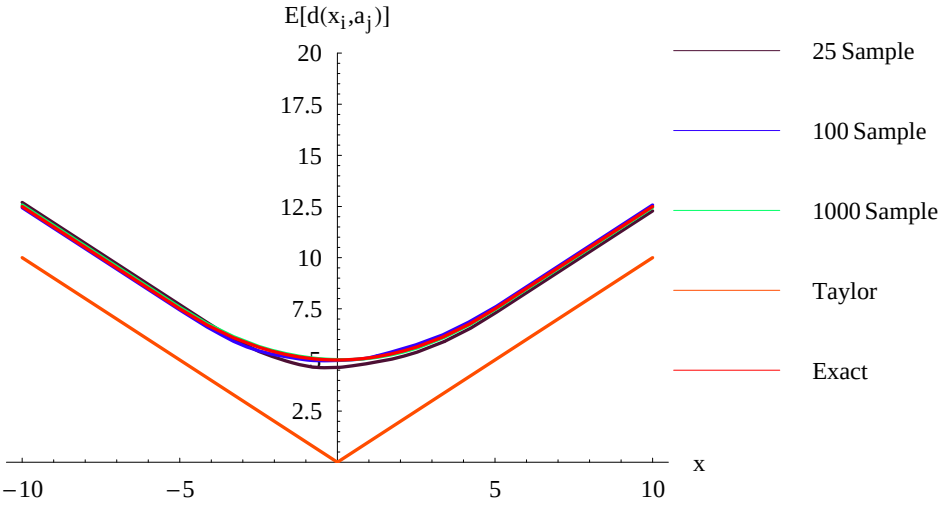



Figure 6.5. Sensitivity to sample size for symmetric distributions: Average distance vs Taylor approximation for the rectilinear norm

to the exact expected distance except around the customer means. Notice that, average distance to sample method is very close to the exact expected distances in both figures.

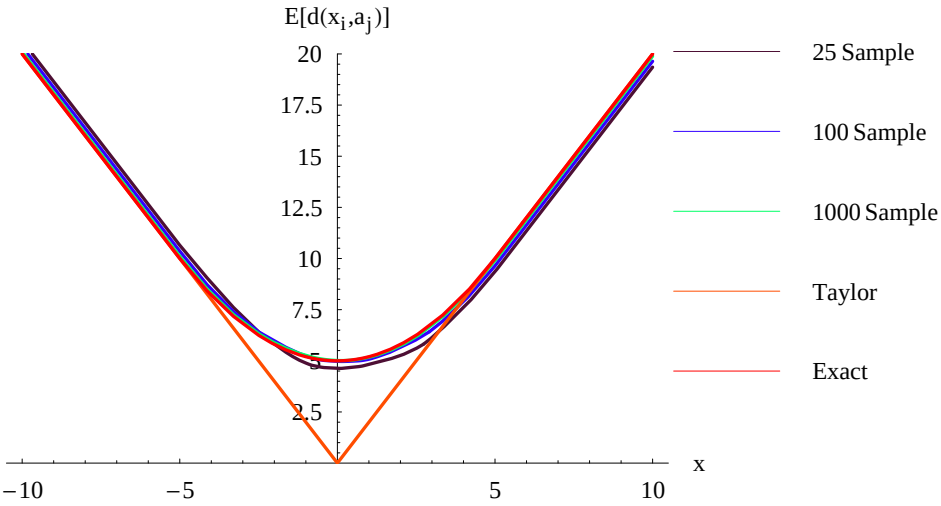



Figure 6.6. Sensitivity to sample size for symmetric distributions: Average distance vs Taylor approximation with estimated parameters

Finally, let us investigate the approximation method where the expected distance function is calculated with the Taylor approximation (Equation (4.18)) after the parameters of the given customer location sample are estimated with the unbiased

estimators (Equations (4.19) and (4.20)). In order to plot an example for this approximation, let us use the same customer location assumptions and the same samples of Figure 6.5 and Figure 6.6. In Figure 6.7 and Figure 6.8, we estimate the means of the samples and plot the approximating expected distances by using the Equation (4.18). Besides, we also plot the exact expected distance function and the Taylor's approximation function by using the original means of the customer location. In Figure 6.7, we set the $y$ coordinate variable zero and in Figure 6.8 we set $y$ coordinate variable to $x$ coordinate variable in order to see the plots from different perspectives. Notice that, $x$ coordinate is changing from -10 to 10 in both plots. As it is seen from both figures, sample approximation is very close to the Taylor's approximation where we use the original customer location parameters. Moreover, sample approximations and Taylor's approximation is very close to the exact expected distance function except through the lines $x = 0$ and $y = 0$ which are individually equals to the customer location means.

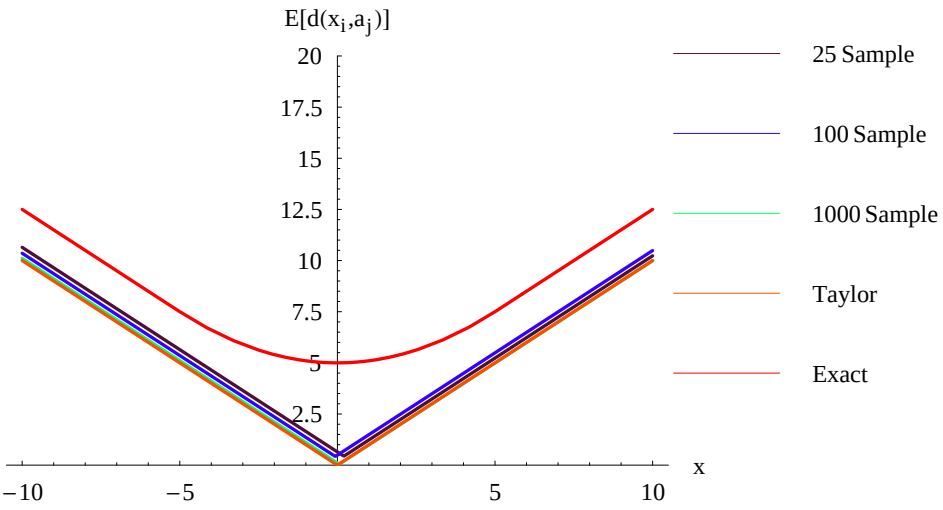



Figure 6.7. Sensitivity to sample size for symmetric distributions: Taylor approximation with exact and estimated distribution parameters for the rectilinear distance

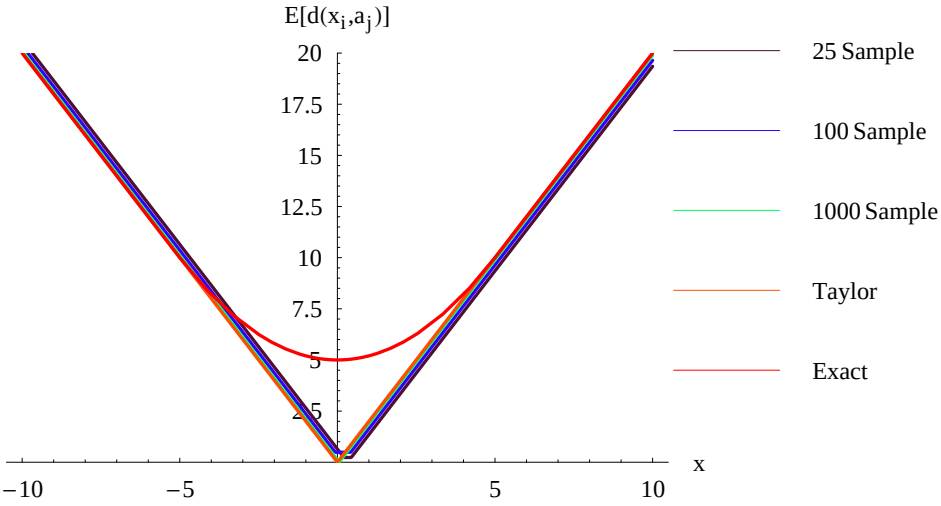



Figure 6.8. Sensitivity to sample size for symmetric distributions: Taylor approximations with exact and estimated distribution parameters

## 6.2. Approximation of $E[d(\mathbf{x_i}, \mathbf{a_j})]$ and PALA

### 6.2.1. Test Bed

As we stated earlier, Cooper (1978) developed the exact enumeration method for solving the problem. However, this algorithm requires a great amount of computation time even for moderate size problems. Therefore, Cooper (1978) applied this exact algorithm for a very small instance of PCMFWP in which the number of customers is four and the number of facilities to be located is two, $(m = 2, n = 4)$. This small problem can also be solved optimally by hand. As a result, there is no optimally solved test instance available in the literature.

It is also appropriate to claim that we were not able to find any problem set in the literature which can be directly used for testing the PCMFWP. For this reason, several test problems have been compiled from different sources to conduct the experiment. Consequently, we have gathered 42 problems in our test bed. The data for the problems 1-25 were derived by us, the data for problems 26-36 were taken from Al-Loughani (1997) and the data for problems 37-42 were taken from Sherali *et al.* (2002). However, all of these problems are derived for the deterministic CMFWP. In order to make these

data sets suitable for the PCMFWP, we have assumed that customer location means are equal to the given coordinate values in the data sets. To satisfy the assumption of our probabilistic model, the standard deviations along both dimensions are set to the same value, which is equal to the 10 per cent of the maximum range of the given coordinate values.

### 6.2.2. Test Platform and Softwares

As a test platform, we use four identical PCs. The main properties of the PCs are given in Table 6.1. Although the machines have a 64 bit processor, the operating

Table 6.1. Properties of the test platform

| | |
|---|---|
| Processor model | AMD Athlon(tm) 64 Processor 3200+ |
| Processor speed | 2.01 GHZ |
| Number of processors | 1 |
| Mainboard | nForce |
| Mainboard Buses | ISA AGP PCI i2c/SMBus |
| Chipset 1 Model | Epox Computer Co Ltd nforce 3 CPU to PCI |
| Front side bus speed | 2x1005 MHz (2010 MHz data rate) |
| Chipset 2 Model | Advanced Micro Devices (AMD) Athlon 64 |
| Front side bus speed | 2x1005 MHz (2010 MHz data rate) |
| Total memory | 1 GB DDR-SDRAM |
| Memory bus speed | 2X200 MHz (400 MHz data rate) |
| Operating System | Microsoft Windows XP/2002 Professional with SP2 |

systems on these machines are 32 bit Windows XP.

All computer programs developed for the experiments have been coded in $C^{++}$. The transportation problem in PALA heuristics have been solved with the ILOG CPLEX 9.1 (2006) by using the Network Optimizer. We prefer network solver rather than LP solver because the network solver solves the transportation problem more efficiently than the LP solver does.

Numerical computations of the double integrals, appeared in the exact expected distance functions, have been made with the Simpson's method which has retrieved from Mathc Yahoo Group (2006). Moreover, there are also some methods available in the GSL - GNU Scientific Library (2006) for evaluating the double integrals such as Monte Carlo, Miser and Vegas. However, we prefer Simpson's method since it has been used by many authors (Rao and Varma 1985, Wesolowsky 1977). While using the Simpson's method, we set the limits of the integration as $\mu_j \pm 4\sigma_j$ which has been used by Wesolowsky (1977), and Rao and Varma (1985). Furthermore, Wesolowsky (1977) claimed that the use of 80 intervals and a range of four standard deviations on either side of the mean gave accurate results in their examples. Since our test platform is fast enough, we prefer to use $200 \times 200$ intervals in order to have accurate integral evaluations.

For the symmetric bivariate normal probability distribution case, we have the closed expression for evaluating the expected distance function (Equation (3.17)). In this equation, we need to calculate the confluent hypergeometric function which is given in Equation (3.19). For calculating the function, we use GSL - GNU Scientific Library (2006).

In Section 5.1.2, we have stated that some cases of the PWP can be solved unconstrained minimization methods, and we prefer to use Simplex Method (Nelder and Mead 1965) since it does not require derivatives of the function. Moreover, we also find the simplex method in GSL - GNU Scientific Library (2006).

Finally, in the first step of the PALA algorithm, convex hull of the customer location means should be found. For computing the convex hull, *Graham Scan* algorithm (Graham 1972) has been used and the *C* code of the algorithm has been retrieved from Skiena and Revilla (2003).

### 6.2.3. Results

In order to investigate the usefulness and the effectiveness of the different expected distance approximations, we have solved all test problems 100 times with the appropriate PALA algorithms for different distance functions and different customer location probability distributions. Figure 6.9 illustrates the experiments that have been performed. Moreover, the results of these experiments are given in tables. For

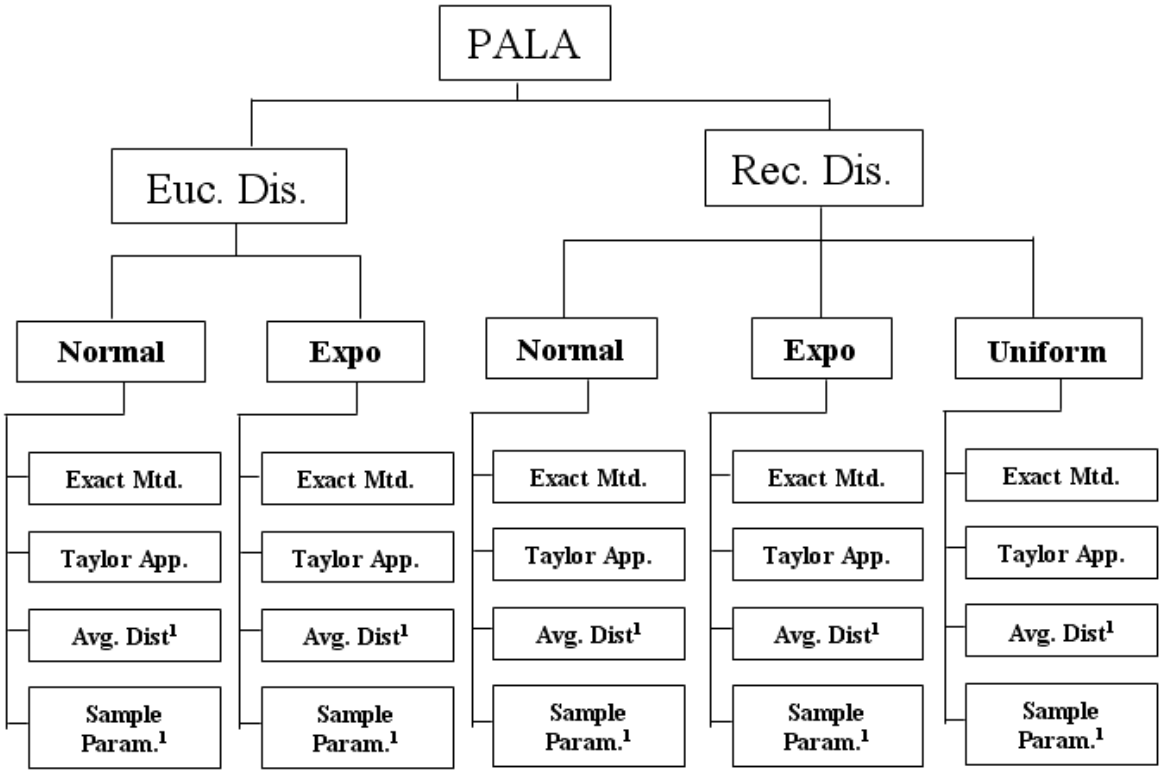



Figure 6.9. Experiments for the PALA heuristic

instance, Table 6.2 and Table 6.4 contains the results for the Euclidean distance case where customer locations follow from symmetric bivariate normal distribution and symmetric bivariate exponential distribution respectively. On the other hand, Table 6.6, Table 6.8 and Table 6.10 are prepared for the rectilinear distance case where customer locations follow from symmetric bivariate uniform, symmetric bivariate normal and symmetric bivariate exponential distribution respectively. Within each table, results of four different PALA methods, each with a different expected evaluation, are included. To illustrate, the values in "Exact Mtd" columns are obtained by running PALA with the exact expected distance evaluation while the ones in "Taylor App."

columns are calculated with the Taylor approximation with known distribution parameters. Moreover, “Sample Param.” columns represent the results for the Taylor approximation with the estimators of the distribution parameters, and “Avg. Dist. Mtd.” columns include total costs calculated with PALA using average distance to estimate the expectation. In addition, minimum, average and maximum of the values obtained running PALA runs result are introduced within the tables. Please note that the “min”, “max” and “average” are respectively the minimum, maximum and average of 100 total cost obtained by running PALA 100 times. In order to have uniformity in the randomization the same 100 initial random locations are used for each PALA family. In addition, total CPU times spent for the respective methods are presented within the Table 6.3, Table 6.5, Table 6.7, Table 6.9 and Table 6.11.

In Table 6.2, when we look at the column averages, the exact method has the smallest value among the four methods. On the other hand, average distance to sample method has very close average to the exact method with 0.28 per cent relative deviation. This negligible deviation is not a surprising result since average distance to sample function has very close values to the exact expected distance function which can be observed from Figure 6.1. In addition, per cent deviations of the Taylor approximations with known and with estimated distribution parameters are 2.133 and 1.633 respectively. These deviations are not very significant because Taylor approximation gives reasonable approximation for the exact expected distance functions. Nevertheless, related PALA results are not as close as the ones computed with PALA using average distances. When the computation times of the four methods are compared by using results presented in Table 6.3, it is seen that there is no significant deviations between them.

In the Euclidean distance - symmetric bivarite exponential distribution case, Table 6.4, exact method again has the smallest average objective value of the four methods. As anticipated, the relative deviation between results obtained using average distance to sample method and the exact method is negligible, 0.185 pre cent. Moreover, Taylor approximations with exact and estimated parameters relatively deviate from the exact method 2.281 per cent and 1.733 per cent respectively. It is interesting

that these deviations are very similar to the deviations of the Euclidean distance - symmetric bivariate normal distribution case. However, computation times are considerable higher since we do not have the closed form expression of the expected distance function. It is seen from Table 6.5 that exact method has very long computation times when compared with approximation methods. Observe also the increase in the computation time of the approximation methods. This is because of the similar reason; evaluations include double integrals.

For the rectilinear distance - symmetric bivariate uniform distribution case, when the column averages of Table 6.6 are investigated, it can be seen that the exact method has the lowest values. It is followed by the method using the average distance. Moreover, the average relative deviation between their results is 0.379 per cent. Similarly, relative deviations between the exact method and Taylor approximations with known and estimated sample parameters are 3.945 per cent and 4.264 per cent. When these deviations are compared with the ones obtained for the Euclidean distance an increase can be observed. As it can be seen from Table 6.7, the exact method has the longest computation times. Indeed, the reason of the long computation times is the numerical calculation of the double integral which appears in the expected distance formulation. However, since the probability density function of the bivariate uniform distribution is simpler than the bivariate normal and bivarite exponential probability distributions, numerical computation of the double integral becomes faster. Thus, the computation times of the exact method is not very long for the bivariate uniform distribution case.

When Table 6.8 is investigated, PALA results for the rectilinear distance – symmetric bivariate normal distribution case, the results are similar to the other cases. Clearly, the exact method has the smallest average objective function value. Furthermore, the use of average distance results 0.127 per cent relative deviation. On the other hand, the use of Taylor approximations with exact and estimated sample parameters produce respectively 3.318 per cent and 3.429 per cent relative deviations from the exact method. It can be stated that the relative deviations for the symmetric bivarite normal distribution are smaller than the symmetric bivarite uniform distribution. Furthermore, when the computation times of these two cases are compared from Table 6.9

and Table 6.7, it is clear that the computation times for symmetric bivarite normal distribution are higher than the ones of symmetric bivariate uniform distribution.

Finally, the results of the PALA runs for the rectilinear distance - symmetric bivariate exponential distribution have also similar behaviour. If the column averages of Table 6.10 are examined, it is obvious that the exact method has the minimum average objective function value among the four methods. Apparently, the use of average distance gives 0.243 per cent relative deviation from the exact method. Besides, the Taylor approximation methods with exact and with estimated parameters result in 2.841 per cent and 3.24 per cent relative deviations on the average. Furthermore, it is clear from Table 6.11 that the exact method has the largest run times. Obviously, the reason for the long computation times is due to the quadratures appeared in the expected distances formula. Other methods have also long computation times because of the objective function calculations.

Table 6.2: PALA for the Euclidean distance-normal distribution case

| | | | 100 PALA Exact Mtd. | | | 100 PALA Taylor App. | | | 100 PALA Sample Param. | | | 100 PALA Avg. Dist. Mtd. | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **P** | **m** | **n** | **Min** | **Average** | **Max** | **Min** | **Average** | **Max** | **Min** | **Average** | **Max** | **Min** | **Average** | **Max** |
| 1 | 5 | 20 | 8353.89 | 9142.87 | 10757.40 | 8388.25 | 9210.94 | 10597.00 | 8385.59 | 9193.60 | 10824.10 | 8366.43 | 9147.04 | 10748.60 |
| 2 | 5 | 25 | 10318.20 | 11296.00 | 12787.50 | 10459.90 | 11587.70 | 13400.40 | 10541.80 | 11590.70 | 13503.50 | 10317.00 | 11424.40 | 13185.70 |
| 3 | 5 | 30 | 15820.10 | 16682.40 | 17796.10 | 15929.90 | 16875.30 | 18050.60 | 15900.80 | 16875.80 | 17999.30 | 15838.10 | 16728.30 | 17803.20 |
| 4 | 5 | 40 | 22916.90 | 24805.90 | 27682.30 | 23076.80 | 25315.00 | 28044.50 | 23198.40 | 25047.00 | 27943.10 | 22923.80 | 24923.10 | 28290.20 |
| 5 | 5 | 50 | 21309.70 | 23002.80 | 25627.40 | 21390.90 | 23361.30 | 25732.70 | 21628.10 | 23455.80 | 25925.10 | 21318.50 | 23193.00 | 25642.30 |
| 6 | 5 | 100 | 35769.90 | 36826.00 | 38399.70 | 35828.20 | 37382.90 | 41185.90 | 36434.10 | 37565.00 | 39804.90 | 35814.70 | 36841.40 | 38471.70 |
| 7 | 5 | 100 | 38475.40 | 39963.00 | 42400.30 | 38624.60 | 40722.40 | 44220.80 | 38881.40 | 40705.30 | 43924.70 | 38550.70 | 39986.20 | 42602.60 |
| 8 | 5 | 100 | 41648.00 | 43813.10 | 49209.80 | 41802.20 | 44497.60 | 50077.60 | 42054.80 | 44448.00 | 49229.60 | 41679.50 | 43875.90 | 48907.60 |
| 9 | 5 | 100 | 40086.30 | 43426.10 | 53627.10 | 40371.20 | 43992.70 | 55245.40 | 40482.50 | 43915.20 | 53790.80 | 40096.70 | 43578.60 | 54883.50 |
| 10 | 5 | 100 | 40718.20 | 43306.60 | 46130.00 | 40773.10 | 43909.40 | 53493.60 | 40979.90 | 43920.30 | 47046.90 | 40706.90 | 43406.70 | 46137.90 |
| 11 | 6 | 20 | 7485.32 | 8404.33 | 9323.64 | 7777.85 | 8548.20 | 9696.00 | 7618.16 | 8498.47 | 9797.07 | 7486.49 | 8440.92 | 9489.48 |
| 12 | 6 | 25 | 9264.00 | 10341.50 | 12205.90 | 9617.92 | 10602.10 | 13157.20 | 9552.00 | 10586.30 | 12344.80 | 9386.50 | 10356.70 | 12150.60 |
| 13 | 6 | 30 | 14590.00 | 15738.00 | 17851.60 | 14650.90 | 16118.70 | 18053.70 | 14725.40 | 15875.70 | 17516.10 | 14635.60 | 15831.60 | 17832.20 |
| 14 | 7 | 20 | 6942.79 | 7744.38 | 8764.48 | 7098.19 | 7918.04 | 8702.21 | 7150.67 | 7978.74 | 8898.46 | 6943.00 | 7813.47 | 8972.11 |
| 15 | 7 | 25 | 9038.88 | 9994.53 | 12125.20 | 9439.12 | 10248.10 | 11975.00 | 9519.22 | 10268.10 | 12369.30 | 9197.81 | 10053.20 | 12120.20 |
| 16 | 7 | 30 | 13674.30 | 15094.00 | 16604.10 | 14164.50 | 15450.90 | 17292.30 | 14041.50 | 15338.60 | 16718.90 | 14005.40 | 15143.60 | 16559.00 |
| 17 | 8 | 20 | 6692.51 | 7418.39 | 8655.45 | 6896.91 | 7648.63 | 8677.64 | 6913.46 | 7638.72 | 8721.56 | 6701.02 | 7428.83 | 8654.33 |
| 18 | 8 | 25 | 8750.20 | 9466.75 | 10311.70 | 9083.88 | 9799.82 | 11107.90 | 9054.60 | 9836.19 | 11075.30 | 8763.51 | 9561.00 | 10484.90 |
| 19 | 8 | 30 | 13429.30 | 14416.10 | 16031.70 | 13620.30 | 14902.30 | 16461.00 | 13824.10 | 14793.70 | 15991.70 | 13442.20 | 14483.90 | 15923.70 |
| 20 | 9 | 20 | 6428.02 | 7049.96 | 7818.17 | 6513.37 | 7261.03 | 8015.53 | 6613.98 | 7286.31 | 8002.46 | 6427.79 | 7083.56 | 8059.59 |
| 21 | 9 | 25 | 8422.72 | 9178.62 | 10365.50 | 8603.43 | 9431.17 | 10825.90 | 8776.90 | 9549.38 | 11084.90 | 8438.67 | 9227.33 | 10998.60 |
| 22 | 9 | 30 | 12856.20 | 13927.60 | 14991.40 | 13214.50 | 14383.80 | 15890.80 | 13185.90 | 14283.00 | 15264.00 | 12874.90 | 14012.50 | 15201.70 |
| 23 | 10 | 20 | 6315.69 | 6882.69 | 7634.73 | 6436.72 | 7085.16 | 8063.18 | 6494.70 | 7136.73 | 7835.47 | 6319.32 | 6949.89 | 8048.45 |
| 24 | 10 | 25 | 8367.98 | 8858.39 | 9619.05 | 8487.91 | 9239.58 | 10373.50 | 8819.59 | 9295.47 | 10024.40 | 8370.06 | 8905.81 | 9680.66 |
| 25 | 10 | 30 | 12866.00 | 13336.60 | 14187.70 | 13135.50 | 13857.70 | 15119.70 | 13146.70 | 13786.60 | 14826.60 | 12900.70 | 13407.40 | 14622.80 |

Continued on Next Page. . .

| | | | 100 PALA Exact Mtd. | | | 100 PALA Taylor App. | | | 100 PALA Sample Param. | | | 100 PALA Avg. Dist. Mtd. | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **P** | **m** | **n** | **Min** | **Average** | **Max** | **Min** | **Average** | **Max** | **Min** | **Average** | **Max** | **Min** | **Average** | **Max** |
| 26 | 2 | 4 | 304.54 | 304.56 | 304.58 | 304.06 | 304.06 | 304.06 | 304.35 | 304.35 | 304.35 | 303.94 | 304.07 | 304.12 |
| 27 | 2 | 4 | 284.71 | 307.91 | 403.37 | 288.53 | 309.17 | 403.38 | 284.90 | 295.27 | 298.19 | 285.70 | 308.88 | 404.07 |
| 28 | 3 | 5 | 99.85 | 109.06 | 130.72 | 106.15 | 118.62 | 133.74 | 108.00 | 112.02 | 135.57 | 99.91 | 109.61 | 130.72 |
| 29 | 3 | 5 | 354.30 | 480.16 | 677.38 | 380.03 | 530.37 | 695.50 | 380.85 | 514.42 | 695.84 | 354.89 | 482.41 | 677.70 |
| 30 | 3 | 9 | 652.50 | 770.40 | 963.49 | 686.78 | 789.36 | 967.42 | 694.42 | 795.66 | 1007.24 | 654.83 | 770.14 | 963.95 |
| 31 | 3 | 9 | 1922.57 | 2111.94 | 2227.39 | 1952.54 | 2155.79 | 2343.92 | 1958.72 | 2190.16 | 2353.53 | 1924.05 | 2116.67 | 2230.35 |
| 32 | 4 | 8 | 1353.97 | 1452.71 | 1649.19 | 1406.36 | 1506.30 | 1897.69 | 1381.41 | 1487.38 | 1705.65 | 1354.92 | 1453.35 | 1648.80 |
| 33 | 5 | 15 | 12528.50 | 14907.40 | 18074.90 | 12818.10 | 15125.70 | 18414.20 | 12909.70 | 15176.80 | 17745.20 | 12535.10 | 14987.80 | 18069.70 |
| 34 | 5 | 20 | 18602.30 | 24100.40 | 32039.40 | 19191.50 | 25132.80 | 32127.70 | 18936.70 | 24318.40 | 32149.70 | 18610.70 | 24181.70 | 32040.60 |
| 35 | 5 | 20 | 2239.84 | 2764.80 | 4445.94 | 2264.79 | 2854.75 | 4630.76 | 2256.25 | 2762.32 | 4496.14 | 2239.12 | 2738.80 | 4444.97 |
| 36 | 5 | 30 | 33356.80 | 35918.20 | 43471.60 | 33606.60 | 36811.10 | 47926.50 | 33590.20 | 35713.30 | 44155.30 | 33355.00 | 35794.10 | 43506.80 |
| 37 | 5 | 10 | 5683.36 | 7046.41 | 9876.63 | 5814.62 | 7208.35 | 11389.40 | 5857.45 | 7224.33 | 10036.80 | 5690.19 | 7051.60 | 9884.26 |
| 38 | 6 | 10 | 11770.70 | 14731.10 | 19249.20 | 12000.00 | 14755.00 | 20185.50 | 11866.20 | 14868.70 | 19293.80 | 11775.80 | 14742.20 | 19249.80 |
| 39 | 7 | 10 | 10686.70 | 13902.30 | 17854.60 | 10985.20 | 14062.20 | 18166.10 | 10714.70 | 13962.40 | 17904.00 | 10698.60 | 13872.60 | 17895.70 |
| 40 | 8 | 10 | 4882.25 | 6215.56 | 10369.30 | 5002.23 | 6402.68 | 10854.20 | 4981.82 | 6253.16 | 10525.40 | 4882.07 | 6209.74 | 10378.20 |
| 41 | 9 | 10 | 8120.16 | 9666.33 | 14246.80 | 8382.71 | 9996.56 | 14001.60 | 8236.28 | 9926.34 | 14454.70 | 8125.77 | 9750.84 | 14234.30 |
| 42 | 10 | 10 | 11641.60 | 15408.00 | 21449.20 | 11955.00 | 15705.80 | 21683.70 | 11725.20 | 15341.70 | 21751.00 | 11653.30 | 15350.90 | 21466.20 |
| **Average** | | | **12976.8** | **14293.2** | **16627.2** | **13155.5** | **14598.1** | **17371.1** | **13193.1** | **14526.6** | **16892.3** | **13001.2** | **14334.0** | **16738.1** |

Table 6.3: CPU times: Euclidean distance-normal distribution case

| P | m | n | Exact Mtd. | Taylor App. | Sample Param. | Avg. Dist. Mtd. |
|---|---|---|---|---|---|---|
| 1 | 5 | 20 | 2.30 | 1.28 | 1.00 | 1.53 |
| 2 | 5 | 25 | 2.75 | 1.46 | 1.12 | 1.75 |
| 3 | 5 | 30 | 2.89 | 1.42 | 1.28 | 2.05 |
| 4 | 5 | 40 | 4.80 | 1.96 | 1.98 | 3.10 |
| 5 | 5 | 50 | 7.16 | 2.47 | 2.37 | 4.10 |
| 6 | 5 | 100 | 11.78 | 3.99 | 4.50 | 9.34 |
| 7 | 5 | 100 | 10.64 | 4.43 | 4.69 | 8.18 |
| 8 | 5 | 100 | 11.79 | 4.24 | 4.76 | 9.07 |
| 9 | 5 | 100 | 11.92 | 4.43 | 4.87 | 7.78 |
| 10 | 5 | 100 | 10.76 | 4.49 | 4.47 | 7.56 |
| 11 | 6 | 20 | 2.43 | 1.46 | 1.16 | 1.66 |
| 12 | 6 | 25 | 3.02 | 1.96 | 1.47 | 2.11 |
| 13 | 6 | 30 | 3.77 | 1.71 | 1.75 | 2.50 |
| 14 | 7 | 20 | 2.74 | 1.77 | 1.33 | 1.91 |
| 15 | 7 | 25 | 3.18 | 2.28 | 1.64 | 2.30 |
| 16 | 7 | 30 | 3.97 | 2.05 | 1.92 | 2.98 |
| 17 | 8 | 20 | 2.93 | 2.00 | 1.50 | 2.06 |
| 18 | 8 | 25 | 3.54 | 2.75 | 1.86 | 2.55 |
| 19 | 8 | 30 | 4.60 | 2.32 | 2.37 | 3.28 |
| 20 | 9 | 20 | 2.97 | 2.35 | 1.64 | 2.17 |
| 21 | 9 | 25 | 3.71 | 3.16 | 2.09 | 2.80 |
| 22 | 9 | 30 | 4.85 | 2.66 | 2.59 | 3.45 |
| 23 | 10 | 20 | 3.19 | 2.94 | 1.89 | 2.31 |
| 24 | 10 | 25 | 3.94 | 3.25 | 2.28 | 2.97 |
| 25 | 10 | 30 | 5.07 | 2.83 | 2.69 | 3.64 |
| 26 | 2 | 4 | 0.45 | 0.45 | 0.49 | 0.34 |
| 27 | 2 | 4 | 0.36 | 0.30 | 0.31 | 0.30 |
| 28 | 3 | 5 | 0.39 | 0.31 | 0.31 | 0.30 |
| 29 | 3 | 5 | 0.53 | 0.41 | 0.42 | 0.72 |
| 30 | 3 | 9 | 0.53 | 0.67 | 0.34 | 0.34 |
| 31 | 3 | 9 | 0.58 | 0.52 | 0.34 | 0.47 |
| 32 | 4 | 8 | 0.55 | 0.39 | 0.33 | 0.39 |
| 33 | 5 | 15 | 1.44 | 0.88 | 0.69 | 0.91 |
| 34 | 5 | 20 | 1.75 | 1.52 | 1.41 | 1.08 |
| 35 | 5 | 20 | 2.13 | 2.11 | 1.02 | 1.25 |
| 36 | 5 | 30 | 3.57 | 2.13 | 1.48 | 2.09 |
| 37 | 5 | 10 | 1.08 | 0.66 | 0.50 | 0.67 |
| 38 | 6 | 10 | 1.27 | 1.03 | 0.97 | 0.83 |
| 39 | 7 | 10 | 1.42 | 1.30 | 1.27 | 0.95 |
| 40 | 8 | 10 | 1.49 | 1.06 | 0.92 | 1.03 |
| 41 | 9 | 10 | 1.64 | 1.36 | 1.31 | 1.31 |
| 42 | 10 | 10 | 1.74 | 1.41 | 1.25 | 1.19 |
| **Average** | | | **3.61** | **1.96** | **1.73** | **2.55** |

Table 6.4: PALA for the Euclidean distance-exponential distribution case

| | | | 100 PALA Exact Mtd. | | | 100 PALA Taylor App. | | | 100 PALA Sample Param. | | | 100 PALA Avg. Dist. Mtd. | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **P** | **m** | **n** | **Min** | **Average** | **Max** | **Min** | **Average** | **Max** | **Min** | **Average** | **Max** | **Min** | **Average** | **Max** |
| 1 | 5 | 20 | 8074.80 | 8907.10 | 10519.10 | 8124.95 | 8961.13 | 10346.30 | 8100.93 | 9006.32 | 10594.70 | 8080.17 | 8907.96 | 10522.00 |
| 2 | 5 | 25 | 9811.37 | 10910.20 | 13310.10 | 10031.30 | 11189.90 | 12995.10 | 10054.70 | 11132.50 | 13069.90 | 9817.30 | 11017.70 | 13314.80 |
| 3 | 5 | 30 | 15241.90 | 16099.60 | 17261.50 | 15375.40 | 16327.20 | 17538.90 | 15334.10 | 16287.40 | 17852.40 | 15249.60 | 16116.00 | 17268.50 |
| 4 | 5 | 40 | 22165.20 | 24113.10 | 26956.00 | 22325.60 | 24605.90 | 27360.90 | 22418.10 | 24415.20 | 27206.30 | 22173.70 | 24264.20 | 27615.00 |
| 5 | 5 | 50 | 20519.50 | 22330.20 | 24974.90 | 20612.70 | 22641.60 | 25052.60 | 20821.40 | 22689.50 | 24798.30 | 20525.40 | 22441.90 | 24980.70 |
| 6 | 5 | 100 | 34242.90 | 35375.60 | 37415.20 | 34313.50 | 35915.70 | 39889.40 | 34580.20 | 36005.40 | 38324.20 | 34255.80 | 35388.90 | 37426.60 |
| 7 | 5 | 100 | 36564.60 | 38139.00 | 40680.40 | 36737.10 | 38982.00 | 42666.90 | 37098.90 | 39022.80 | 41961.20 | 36591.80 | 38156.00 | 40755.10 |
| 8 | 5 | 100 | 39950.50 | 42155.10 | 47768.80 | 40159.70 | 42913.20 | 48640.60 | 40249.00 | 42972.20 | 48124.10 | 39961.70 | 42144.10 | 47820.50 |
| 9 | 5 | 100 | 38293.10 | 41884.60 | 53410.90 | 38614.00 | 42384.00 | 53850.10 | 39030.30 | 42543.30 | 52717.10 | 38344.80 | 41854.30 | 52598.50 |
| 10 | 5 | 100 | 39068.70 | 41700.30 | 44440.60 | 39177.30 | 42372.80 | 52205.30 | 39651.10 | 42479.00 | 52316.30 | 39074.40 | 41745.60 | 44474.70 |
| 11 | 6 | 20 | 7157.30 | 8101.21 | 9081.00 | 7477.89 | 8265.47 | 9431.28 | 7295.95 | 8221.06 | 9529.70 | 7163.39 | 8127.06 | 9222.49 |
| 12 | 6 | 25 | 8764.06 | 9877.30 | 11779.00 | 9156.10 | 10171.30 | 12743.20 | 9124.30 | 10131.40 | 12026.00 | 8888.63 | 9899.40 | 11781.50 |
| 13 | 6 | 30 | 13967.00 | 15165.90 | 17204.90 | 14044.80 | 15540.70 | 17488.50 | 14060.40 | 15347.70 | 17267.40 | 13978.90 | 15187.40 | 17072.80 |
| 14 | 7 | 20 | 6571.17 | 7417.68 | 8517.47 | 6769.29 | 7611.44 | 8433.89 | 6772.94 | 7631.74 | 8631.15 | 6576.31 | 7434.57 | 8521.12 |
| 15 | 7 | 25 | 8495.33 | 9513.08 | 11688.40 | 8970.75 | 9800.92 | 11538.00 | 8964.68 | 9822.35 | 11855.70 | 8481.38 | 9521.96 | 11432.30 |
| 16 | 7 | 30 | 12980.70 | 14444.80 | 15975.00 | 13515.60 | 14845.20 | 16717.80 | 13237.80 | 14742.20 | 16083.10 | 12990.30 | 14483.80 | 16075.90 |
| 17 | 8 | 20 | 6311.06 | 7082.51 | 8352.79 | 6555.02 | 7324.40 | 8375.73 | 6533.03 | 7316.48 | 8309.93 | 6320.24 | 7138.97 | 8354.56 |
| 18 | 8 | 25 | 8205.65 | 8989.43 | 9967.49 | 8596.70 | 9326.52 | 10623.10 | 8554.75 | 9285.88 | 10568.30 | 8215.66 | 9020.46 | 10159.80 |
| 19 | 8 | 30 | 12658.60 | 13765.70 | 15327.60 | 12911.40 | 14269.30 | 15868.70 | 12958.10 | 14166.00 | 15926.10 | 12670.50 | 13803.90 | 15413.50 |
| 20 | 9 | 20 | 6015.21 | 6679.40 | 7518.23 | 6128.82 | 6915.26 | 7699.76 | 6246.78 | 6910.40 | 7817.20 | 6025.60 | 6698.04 | 7610.60 |
| 21 | 9 | 25 | 7799.89 | 8662.11 | 9833.83 | 8061.21 | 8934.62 | 10365.50 | 8112.42 | 8923.36 | 10462.90 | 7805.62 | 8648.90 | 9911.32 |
| 22 | 9 | 30 | 12061.60 | 13237.30 | 14371.80 | 12511.50 | 13725.10 | 15249.10 | 12371.10 | 13654.90 | 15185.40 | 12073.40 | 13234.30 | 14780.90 |
| 23 | 10 | 20 | 5895.39 | 6517.74 | 7452.69 | 6048.88 | 6726.99 | 7726.28 | 6128.09 | 6733.61 | 8011.57 | 5907.01 | 6503.08 | 7194.22 |
| 24 | 10 | 25 | 7764.07 | 8304.96 | 9334.06 | 7926.84 | 8738.48 | 9865.99 | 8178.53 | 8672.78 | 9420.58 | 7767.80 | 8325.04 | 9351.27 |
| 25 | 10 | 30 | 12025.40 | 12599.30 | 13896.50 | 12420.60 | 13171.70 | 14461.70 | 12411.80 | 13098.70 | 13970.30 | 12078.00 | 12623.60 | 14117.40 |

Continued on Next Page...

| | | | 100 PALA Exact Mtd. | | | 100 PALA Taylor App. | | | 100 PALA Sample Param. | | | 100 PALA Avg. Dist. Mtd. | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **P** | **m** | **n** | **Min** | **Average** | **Max** | **Min** | **Average** | **Max** | **Min** | **Average** | **Max** | **Min** | **Average** | **Max** |
| 26 | 2 | 4 | 297.34 | 297.42 | 297.46 | 297.33 | 297.33 | 297.33 | 299.40 | 299.40 | 299.40 | 298.24 | 298.53 | 298.66 |
| 27 | 2 | 4 | 276.98 | 300.46 | 399.07 | 280.63 | 301.77 | 398.50 | 289.31 | 289.31 | 289.31 | 277.24 | 300.52 | 399.35 |
| 28 | 3 | 5 | 91.57 | 106.25 | 217.43 | 99.73 | 111.83 | 126.32 | 101.32 | 112.37 | 124.44 | 91.62 | 101.94 | 122.06 |
| 29 | 3 | 5 | 327.83 | 462.63 | 648.54 | 360.79 | 508.30 | 670.77 | 356.87 | 489.43 | 661.96 | 328.16 | 462.56 | 651.29 |
| 30 | 3 | 9 | 610.10 | 732.58 | 921.77 | 655.93 | 759.21 | 932.31 | 647.79 | 765.81 | 973.38 | 611.18 | 731.24 | 922.57 |
| 31 | 3 | 9 | 1826.91 | 1997.08 | 2095.72 | 1867.97 | 2062.52 | 2239.02 | 1876.76 | 2071.99 | 2202.50 | 1826.84 | 1999.14 | 2096.90 |
| 32 | 4 | 8 | 1279.37 | 1389.67 | 1576.76 | 1338.68 | 1449.81 | 1830.67 | 1340.29 | 1434.64 | 1648.11 | 1282.20 | 1392.92 | 1585.73 |
| 33 | 5 | 15 | 12041.40 | 14644.50 | 17973.20 | 12412.50 | 14727.90 | 17980.30 | 12466.10 | 14689.40 | 17804.20 | 12060.20 | 14639.40 | 17984.80 |
| 34 | 5 | 20 | 17960.50 | 23659.80 | 31657.20 | 18615.90 | 24646.30 | 31771.20 | 18114.50 | 23501.00 | 31754.30 | 17963.40 | 23630.60 | 31659.00 |
| 35 | 5 | 20 | 2152.66 | 2635.60 | 4252.24 | 2184.88 | 2761.46 | 4490.88 | 2170.63 | 2674.15 | 4312.55 | 2156.96 | 2705.84 | 4256.17 |
| 36 | 5 | 30 | 32592.70 | 34955.30 | 41978.90 | 32890.70 | 35873.40 | 46798.80 | 32869.80 | 34966.90 | 43816.40 | 32603.90 | 35069.60 | 44228.10 |
| 37 | 5 | 10 | 5415.43 | 6773.01 | 9627.77 | 5608.46 | 6975.08 | 10981.60 | 5535.75 | 6913.94 | 11637.60 | 5421.38 | 6780.91 | 9632.62 |
| 38 | 6 | 10 | 11406.10 | 14351.90 | 18901.90 | 11608.30 | 14347.40 | 19895.50 | 11479.90 | 14471.30 | 18337.50 | 11414.90 | 14420.10 | 18910.50 |
| 39 | 7 | 10 | 10308.80 | 13447.30 | 17419.20 | 10615.30 | 13648.80 | 17836.90 | 10341.90 | 13621.20 | 17550.20 | 10314.50 | 13499.40 | 17424.80 |
| 40 | 8 | 10 | 4586.66 | 5974.78 | 10088.30 | 4756.42 | 6137.48 | 10537.80 | 4667.88 | 5859.66 | 10382.80 | 4595.13 | 5928.54 | 10093.20 |
| 41 | 9 | 10 | 7699.35 | 9302.28 | 13753.20 | 8020.14 | 9625.89 | 13606.60 | 7828.14 | 9497.45 | 13706.90 | 7703.40 | 9321.24 | 13761.80 |
| 42 | 10 | 10 | 11210.00 | 15008.80 | 21018.40 | 11550.30 | 15302.10 | 21328.50 | 11289.00 | 15157.80 | 21205.50 | 11215.10 | 15112.00 | 20923.70 |
| **Average** | | | **12397.3** | **13762.2** | **16187.3** | **12612.6** | **14076.1** | **16877.6** | **12618.2** | **14000.7** | **16636.6** | **12409.1** | **13787.7** | **16255.4** |

Table 6.5: CPU times: Euclidean distance-exponential distribution case

| P | m | n | Exact Mtd. | Taylor App. | Sample Param. | Avg. Dist. Mtd. |
|---|---|---|---|---|---|---|
| 1 | 5 | 20 | 1298.17 | 70.57 | 72.93 | 74.17 |
| 2 | 5 | 25 | 1545.22 | 84.41 | 84.10 | 84.73 |
| 3 | 5 | 30 | 1789.57 | 101.75 | 103.49 | 106.67 |
| 4 | 5 | 40 | 3044.21 | 144.16 | 156.63 | 153.59 |
| 5 | 5 | 50 | 3969.01 | 188.45 | 211.76 | 228.75 |
| 6 | 5 | 100 | 8359.39 | 381.78 | 462.81 | 546.44 |
| 7 | 5 | 100 | 7360.34 | 396.52 | 418.52 | 467.26 |
| 8 | 5 | 100 | 7675.34 | 360.78 | 424.26 | 499.35 |
| 9 | 5 | 100 | 7190.23 | 362.07 | 411.55 | 435.41 |
| 10 | 5 | 100 | 6636.86 | 362.90 | 381.12 | 402.96 |
| 11 | 6 | 20 | 1386.80 | 73.34 | 75.52 | 78.00 |
| 12 | 6 | 25 | 1690.00 | 98.98 | 94.62 | 98.39 |
| 13 | 6 | 30 | 2126.41 | 108.26 | 121.88 | 115.06 |
| 14 | 7 | 20 | 1532.52 | 76.67 | 78.88 | 84.17 |
| 15 | 7 | 25 | 1645.70 | 99.04 | 95.15 | 97.39 |
| 16 | 7 | 30 | 2238.80 | 110.97 | 120.46 | 121.23 |
| 17 | 8 | 20 | 1500.78 | 77.65 | 82.26 | 81.59 |
| 18 | 8 | 25 | 1853.02 | 100.90 | 102.01 | 104.00 |
| 19 | 8 | 30 | 2437.08 | 113.14 | 123.76 | 130.23 |
| 20 | 9 | 20 | 1529.83 | 78.84 | 83.07 | 82.80 |
| 21 | 9 | 25 | 1835.24 | 103.09 | 108.54 | 107.51 |
| 22 | 9 | 30 | 2515.58 | 119.25 | 133.92 | 139.09 |
| 23 | 10 | 20 | 1647.72 | 75.16 | 86.04 | 93.06 |
| 24 | 10 | 25 | 1925.97 | 104.47 | 109.08 | 108.78 |
| 25 | 10 | 30 | 2546.99 | 118.55 | 127.89 | 133.12 |
| 26 | 2 | 4 | 67.71 | 5.97 | 5.83 | 5.83 |
| 27 | 2 | 4 | 59.43 | 8.20 | 9.44 | 8.08 |
| 28 | 3 | 5 | 122.27 | 14.59 | 13.98 | 14.09 |
| 29 | 3 | 5 | 117.12 | 12.77 | 12.58 | 14.06 |
| 30 | 3 | 9 | 211.91 | 22.65 | 20.25 | 20.78 |
| 31 | 3 | 9 | 277.50 | 23.69 | 26.49 | 25.38 |
| 32 | 4 | 8 | 240.77 | 24.13 | 22.12 | 22.94 |
| 33 | 5 | 15 | 613.79 | 53.06 | 49.08 | 51.19 |
| 34 | 5 | 20 | 721.82 | 56.94 | 64.16 | 58.06 |
| 35 | 5 | 20 | 873.18 | 67.16 | 60.33 | 60.88 |
| 36 | 5 | 30 | 1705.59 | 118.87 | 115.97 | 114.16 |
| 37 | 5 | 10 | 403.43 | 36.92 | 32.15 | 33.24 |
| 38 | 6 | 10 | 488.89 | 40.46 | 40.07 | 40.31 |
| 39 | 7 | 10 | 553.86 | 45.65 | 44.16 | 44.88 |
| 40 | 8 | 10 | 541.98 | 47.69 | 48.38 | 44.94 |
| 41 | 9 | 10 | 607.75 | 47.76 | 46.24 | 48.66 |
| 42 | 10 | 10 | 628.67 | 53.13 | 50.97 | 48.92 |
| **Average** | | | **2036.1** | **109.3** | **117.4** | **124.5** |

Table 6.6: PALA for the rectilinear distance-uniform distribution case

| | | | 100 PALA Exact Mtd. | | | 100 PALA Taylor App. | | | 100 PALA Sample Param. | | | 100 PALA Avg. Dist. Mtd. | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **P** | **m** | **n** | **Min** | **Average** | **Max** | **Min** | **Average** | **Max** | **Min** | **Average** | **Max** | **Min** | **Average** | **Max** |
| 1 | 5 | 20 | 10725.60 | 11671.80 | 13408.50 | 11202.50 | 12305.70 | 13791.90 | 11281.60 | 12399.60 | 16020.10 | 10747.50 | 11734.80 | 13366.40 |
| 2 | 5 | 25 | 13435.60 | 14766.60 | 17244.60 | 13842.50 | 15533.40 | 19394.30 | 13702.30 | 15439.40 | 19162.70 | 13459.10 | 14780.80 | 17602.80 |
| 3 | 5 | 30 | 20325.40 | 21454.20 | 23067.50 | 20830.40 | 22133.40 | 24346.20 | 20834.50 | 22290.10 | 24014.90 | 20459.90 | 21507.30 | 22926.20 |
| 4 | 5 | 40 | 29456.60 | 31787.00 | 34716.60 | 29882.10 | 32975.10 | 36943.50 | 29860.70 | 32954.90 | 35881.50 | 29481.20 | 31807.60 | 34954.40 |
| 5 | 5 | 50 | 27511.50 | 29715.40 | 33053.00 | 27807.60 | 30551.50 | 33996.50 | 28453.50 | 30669.40 | 33863.20 | 27535.10 | 29903.80 | 33081.90 |
| 6 | 5 | 100 | 46420.40 | 47779.10 | 49232.10 | 46932.90 | 48364.30 | 50594.30 | 46992.90 | 48596.60 | 51654.70 | 46471.50 | 48001.80 | 49653.50 |
| 7 | 5 | 100 | 50242.40 | 52095.50 | 54968.80 | 50849.00 | 52967.40 | 56395.00 | 50860.20 | 53102.60 | 56341.60 | 50386.50 | 52318.80 | 55246.50 |
| 8 | 5 | 100 | 53881.20 | 56760.90 | 62986.10 | 54719.20 | 58008.60 | 65101.00 | 54454.40 | 57785.00 | 64073.00 | 53895.30 | 56685.10 | 62875.00 |
| 9 | 5 | 100 | 51906.30 | 56016.30 | 67760.10 | 52385.60 | 57164.00 | 70560.60 | 52222.60 | 57144.70 | 70339.20 | 51955.20 | 55968.60 | 67831.10 |
| 10 | 5 | 100 | 52922.60 | 56153.20 | 67626.40 | 53796.60 | 56877.20 | 69894.30 | 53538.60 | 57231.80 | 72160.50 | 52948.60 | 56216.30 | 66545.00 |
| 11 | 6 | 20 | 9679.31 | 10811.60 | 12021.90 | 10205.10 | 11410.80 | 12737.60 | 10255.90 | 11550.20 | 13229.50 | 9704.65 | 10842.30 | 12256.70 |
| 12 | 6 | 25 | 12285.60 | 13442.30 | 15672.00 | 12572.80 | 14060.50 | 17197.70 | 12646.20 | 14091.80 | 15739.50 | 12301.90 | 13470.40 | 15689.10 |
| 13 | 6 | 30 | 18910.60 | 20330.10 | 23229.80 | 19819.20 | 21393.60 | 23752.80 | 19915.80 | 21483.50 | 23837.90 | 19024.40 | 20502.30 | 23244.00 |
| 14 | 7 | 20 | 9078.16 | 10074.40 | 11335.60 | 9422.07 | 10528.10 | 11930.10 | 9433.78 | 10610.50 | 12369.30 | 9107.63 | 10184.20 | 11353.80 |
| 15 | 7 | 25 | 11862.80 | 13019.40 | 15634.60 | 12097.40 | 13624.90 | 15934.60 | 12421.10 | 13745.60 | 16347.80 | 11928.90 | 13058.20 | 15668.60 |
| 16 | 7 | 30 | 18300.20 | 19670.80 | 21169.60 | 18569.00 | 20387.80 | 22580.30 | 18959.20 | 20601.40 | 22655.80 | 18345.30 | 19683.90 | 21198.90 |
| 17 | 8 | 20 | 8777.26 | 9708.39 | 10732.70 | 9052.87 | 10184.20 | 11628.40 | 9133.76 | 10235.30 | 11773.40 | 8825.18 | 9732.40 | 10895.90 |
| 18 | 8 | 25 | 11705.40 | 12537.40 | 13897.50 | 12064.40 | 13105.00 | 15174.90 | 12159.80 | 13212.50 | 15252.90 | 11587.90 | 12605.60 | 14199.90 |
| 19 | 8 | 30 | 17655.90 | 18816.20 | 20711.10 | 18325.40 | 19645.10 | 21856.30 | 18310.10 | 19753.90 | 22716.90 | 17676.50 | 18882.10 | 21388.40 |
| 20 | 9 | 20 | 8482.36 | 9234.35 | 10033.10 | 8618.39 | 9865.86 | 11741.90 | 8800.20 | 9891.97 | 12162.20 | 8552.18 | 9278.03 | 10184.90 |
| 21 | 9 | 25 | 11205.90 | 12047.40 | 13668.10 | 11741.90 | 12657.00 | 14199.50 | 11774.60 | 12770.00 | 14160.90 | 11257.40 | 12130.80 | 13711.10 |
| 22 | 9 | 30 | 16974.30 | 18288.30 | 19856.40 | 17501.70 | 19210.40 | 21880.80 | 17401.60 | 19301.20 | 21948.70 | 17333.60 | 18373.90 | 19636.60 |
| 23 | 10 | 20 | 8361.66 | 9071.84 | 10099.30 | 8587.17 | 9668.60 | 11106.60 | 8609.58 | 9662.60 | 11303.90 | 8408.82 | 9127.30 | 10057.80 |
| 24 | 10 | 25 | 11205.90 | 11732.70 | 12988.70 | 11466.10 | 12284.10 | 13970.80 | 11558.90 | 12433.10 | 14131.80 | 11280.50 | 11818.50 | 13128.10 |
| 25 | 10 | 30 | 16994.10 | 17597.20 | 19149.80 | 17381.40 | 18521.80 | 20598.50 | 17562.80 | 18637.80 | 20632.10 | 17063.60 | 17708.10 | 18769.40 |

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| | | | 100 PALA Exact Mtd. | | | 100 PALA Taylor App. | | | 100 PALA Sample Param. | | | 100 PALA Avg. Dist. Mtd. | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **P** | **m** | **n** | **Min** | **Average** | **Max** | **Min** | **Average** | **Max** | **Min** | **Average** | **Max** | **Min** | **Average** | **Max** |
| 26 | 2 | 4 | 395.26 | 395.26 | 395.26 | 416.15 | 416.15 | 416.15 | 413.34 | 413.34 | 413.34 | 395.56 | 395.56 | 395.56 |
| 27 | 2 | 4 | 353.36 | 396.40 | 496.70 | 375.27 | 398.14 | 538.60 | 374.26 | 397.04 | 536.99 | 354.01 | 393.00 | 496.65 |
| 28 | 3 | 5 | 134.28 | 148.70 | 173.01 | 138.83 | 160.66 | 307.62 | 139.77 | 162.02 | 310.23 | 134.72 | 149.57 | 173.33 |
| 29 | 3 | 5 | 484.78 | 664.37 | 915.12 | 514.70 | 752.98 | 959.06 | 515.57 | 753.12 | 955.37 | 485.99 | 666.46 | 916.10 |
| 30 | 3 | 9 | 867.45 | 1003.41 | 1241.35 | 896.63 | 1014.46 | 1830.68 | 887.99 | 1024.43 | 1823.00 | 871.43 | 1006.49 | 1244.96 |
| 31 | 3 | 9 | 2485.62 | 2803.69 | 3028.84 | 2585.14 | 2871.36 | 3178.48 | 2574.15 | 2877.82 | 3191.21 | 2487.05 | 2806.30 | 3032.03 |
| 32 | 4 | 8 | 1766.81 | 1928.60 | 2172.77 | 1784.51 | 2039.35 | 2762.99 | 1776.55 | 2044.83 | 2720.20 | 1767.91 | 1930.34 | 2174.05 |
| 33 | 5 | 15 | 16252.50 | 19539.80 | 22830.30 | 16746.20 | 20500.10 | 23631.60 | 16733.50 | 20599.00 | 23561.00 | 16271.10 | 19554.20 | 22844.80 |
| 34 | 5 | 20 | 23764.20 | 30817.60 | 38171.50 | 24166.30 | 32852.80 | 42874.80 | 24226.50 | 32846.20 | 42822.50 | 23769.50 | 31483.90 | 41887.20 |
| 35 | 5 | 20 | 2866.71 | 3474.77 | 5745.97 | 2983.40 | 4008.60 | 6078.68 | 2985.81 | 3948.23 | 6024.67 | 2870.20 | 3546.30 | 5263.02 |
| 36 | 5 | 30 | 43457.20 | 46599.50 | 55528.60 | 44372.60 | 49744.30 | 65117.30 | 44308.20 | 49657.70 | 60626.40 | 43512.90 | 46740.90 | 57885.20 |
| 37 | 5 | 10 | 7177.00 | 8994.71 | 12923.80 | 7260.42 | 9455.34 | 14347.00 | 7315.95 | 9494.03 | 14470.60 | 7205.50 | 9094.50 | 15372.60 |
| 38 | 6 | 10 | 15117.60 | 18434.20 | 23460.70 | 16596.10 | 19710.20 | 24028.90 | 16656.30 | 19623.80 | 24037.30 | 15135.90 | 18514.90 | 23520.60 |
| 39 | 7 | 10 | 13738.80 | 17386.30 | 22570.30 | 14462.70 | 18344.40 | 22979.20 | 14757.50 | 18401.90 | 23044.70 | 13756.40 | 17467.40 | 22617.80 |
| 40 | 8 | 10 | 6451.69 | 8027.56 | 13036.30 | 7075.92 | 8760.43 | 13447.90 | 7012.19 | 8867.93 | 13411.80 | 6480.65 | 7977.41 | 13147.90 |
| 41 | 9 | 10 | 10510.30 | 12653.40 | 18379.60 | 11191.40 | 13247.40 | 17033.40 | 11294.30 | 13318.50 | 17740.80 | 10545.00 | 12621.10 | 18439.50 |
| 42 | 10 | 10 | 15633.60 | 19949.40 | 27527.30 | 16168.30 | 20776.00 | 30554.00 | 16138.90 | 20945.40 | 30377.70 | 15626.80 | 20074.30 | 27160.30 |
| **Average** | | | **16899.1** | **18519.0** | **21496.7** | **17319.2** | **19249.5** | **22795.1** | **17363.2** | **19308.8** | **22805.8** | **16938.3** | **18589.2** | **21715.2** |

Table 6.7: CPU times: rectilinear distance-uniform distribution case

| P | m | n | Exact Mtd. | Taylor App. | Sample Param. | Avg. Dist. Mtd. |
|---|---|---|---|---|---|---|
| 1 | 5 | 20 | 123.52 | 15.30 | 15.42 | 17.37 |
| 2 | 5 | 25 | 144.53 | 19.22 | 18.77 | 19.45 |
| 3 | 5 | 30 | 180.70 | 23.99 | 22.57 | 23.43 |
| 4 | 5 | 40 | 305.70 | 33.16 | 33.92 | 36.02 |
| 5 | 5 | 50 | 416.47 | 44.37 | 41.72 | 46.30 |
| 6 | 5 | 100 | 1044.52 | 115.19 | 102.43 | 113.34 |
| 7 | 5 | 100 | 922.21 | 91.71 | 86.32 | 106.60 |
| 8 | 5 | 100 | 953.04 | 90.07 | 92.76 | 108.31 |
| 9 | 5 | 100 | 871.33 | 90.42 | 89.05 | 100.08 |
| 10 | 5 | 100 | 836.69 | 85.24 | 82.19 | 94.56 |
| 11 | 6 | 20 | 143.61 | 16.36 | 15.93 | 16.81 |
| 12 | 6 | 25 | 184.90 | 19.55 | 19.50 | 21.75 |
| 13 | 6 | 30 | 252.10 | 24.31 | 23.73 | 26.61 |
| 14 | 7 | 20 | 170.52 | 17.03 | 16.79 | 17.48 |
| 15 | 7 | 25 | 225.66 | 20.63 | 20.35 | 22.80 |
| 16 | 7 | 30 | 292.54 | 26.46 | 25.07 | 29.47 |
| 17 | 8 | 20 | 199.65 | 17.28 | 17.23 | 19.57 |
| 18 | 8 | 25 | 247.17 | 22.13 | 21.12 | 22.68 |
| 19 | 8 | 30 | 354.77 | 27.53 | 26.08 | 30.17 |
| 20 | 9 | 20 | 224.60 | 18.57 | 18.94 | 20.19 |
| 21 | 9 | 25 | 294.61 | 22.80 | 21.45 | 24.19 |
| 22 | 9 | 30 | 385.53 | 27.48 | 27.33 | 31.79 |
| 23 | 10 | 20 | 259.03 | 18.86 | 19.47 | 21.89 |
| 24 | 10 | 25 | 330.40 | 22.93 | 21.59 | 24.91 |
| 25 | 10 | 30 | 434.44 | 27.95 | 27.41 | 32.54 |
| 26 | 2 | 4 | 3.33 | 1.72 | 1.73 | 1.72 |
| 27 | 2 | 4 | 4.51 | 2.64 | 2.60 | 2.02 |
| 28 | 3 | 5 | 12.57 | 3.36 | 3.35 | 4.36 |
| 29 | 3 | 5 | 12.02 | 3.63 | 3.60 | 3.45 |
| 30 | 3 | 9 | 20.96 | 5.82 | 5.41 | 5.67 |
| 31 | 3 | 9 | 23.85 | 6.10 | 5.72 | 5.89 |
| 32 | 4 | 8 | 25.02 | 5.68 | 5.52 | 5.67 |
| 33 | 5 | 15 | 78.83 | 11.51 | 11.62 | 12.46 |
| 34 | 5 | 20 | 96.18 | 14.14 | 13.47 | 15.18 |
| 35 | 5 | 20 | 106.61 | 14.23 | 14.31 | 15.46 |
| 36 | 5 | 30 | 205.37 | 22.74 | 22.93 | 26.42 |
| 37 | 5 | 10 | 47.51 | 8.29 | 8.22 | 7.94 |
| 38 | 6 | 10 | 64.02 | 9.84 | 9.51 | 10.47 |
| 39 | 7 | 10 | 77.97 | 10.39 | 10.31 | 11.88 |
| 40 | 8 | 10 | 86.62 | 10.87 | 10.76 | 13.69 |
| 41 | 9 | 10 | 96.04 | 10.98 | 10.88 | 12.46 |
| 42 | 10 | 10 | 100.97 | 11.43 | 11.86 | 12.11 |
| **Average** | | | **258.6** | **26.0** | **25.2** | **28.5** |

Table 6.8: PALA for the rectilinear distance-normal distribution case

| | | | 100 PALA Exact Mtd. | | | 100 PALA Taylor App. | | | 100 PALA Sample Param. | | | 100 PALA Avg. Dist. Mtd. | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **P** | **m** | **n** | **Min** | **Average** | **Max** | **Min** | **Average** | **Max** | **Min** | **Average** | **Max** | **Min** | **Average** | **Max** |
| 1 | 5 | 20 | 10602.30 | 11548.90 | 13507.40 | 10940.60 | 12059.10 | 13627.90 | 10931.40 | 12039.00 | 13844.10 | 10601.30 | 11565.60 | 13522.10 |
| 2 | 5 | 25 | 13026.70 | 14496.50 | 17243.60 | 13356.00 | 15114.60 | 18934.40 | 13385.90 | 15170.30 | 18358.40 | 13054.90 | 14493.10 | 17260.50 |
| 3 | 5 | 30 | 19980.40 | 21039.20 | 22742.90 | 20392.70 | 21670.80 | 23872.30 | 20465.80 | 21599.00 | 23180.30 | 20010.30 | 21078.40 | 22699.90 |
| 4 | 5 | 40 | 29027.70 | 31476.00 | 34491.30 | 29329.00 | 32348.30 | 36353.30 | 29326.30 | 32239.40 | 35193.30 | 29040.70 | 31344.10 | 34273.90 |
| 5 | 5 | 50 | 27010.00 | 29317.30 | 32818.30 | 27243.00 | 30001.40 | 33554.90 | 27210.10 | 30075.10 | 33423.10 | 27026.60 | 29365.50 | 32848.30 |
| 6 | 5 | 100 | 45504.40 | 46835.10 | 48550.60 | 45912.90 | 47274.00 | 49388.90 | 46111.40 | 47672.00 | 50803.30 | 45567.10 | 46865.10 | 48520.90 |
| 7 | 5 | 100 | 48864.40 | 50829.70 | 53829.70 | 49361.30 | 51624.50 | 55097.40 | 49184.70 | 51730.50 | 55198.10 | 49010.60 | 50990.90 | 54313.60 |
| 8 | 5 | 100 | 52874.90 | 55807.20 | 62735.40 | 53541.60 | 56865.20 | 64166.60 | 53498.90 | 56938.40 | 63444.80 | 52891.70 | 56026.40 | 62728.40 |
| 9 | 5 | 100 | 50620.20 | 54968.00 | 67142.40 | 51055.30 | 55947.30 | 69246.70 | 51005.70 | 55961.70 | 69770.40 | 50636.20 | 55164.60 | 67249.60 |
| 10 | 5 | 100 | 51992.00 | 55457.70 | 66973.00 | 52707.50 | 55765.70 | 69065.90 | 52728.30 | 55840.40 | 69002.00 | 52005.00 | 55209.50 | 65929.20 |
| 11 | 6 | 20 | 9458.52 | 10653.40 | 12249.50 | 9917.07 | 11120.60 | 12452.90 | 9962.19 | 11151.20 | 13396.60 | 9477.58 | 10641.50 | 11859.10 |
| 12 | 6 | 25 | 11815.80 | 13110.50 | 15423.00 | 12047.50 | 13589.10 | 16774.80 | 12106.30 | 13711.20 | 16723.50 | 11846.00 | 13075.30 | 15439.80 |
| 13 | 6 | 30 | 18503.10 | 19985.10 | 22814.60 | 19310.40 | 20872.90 | 23205.40 | 19153.00 | 20809.60 | 23235.00 | 18607.20 | 20046.30 | 22837.30 |
| 14 | 7 | 20 | 8775.58 | 9820.16 | 11012.70 | 9054.08 | 10203.40 | 11627.10 | 9033.73 | 10202.40 | 11650.30 | 8784.65 | 9857.31 | 11080.00 |
| 15 | 7 | 25 | 11308.60 | 12599.90 | 15327.40 | 11556.60 | 13127.10 | 15467.60 | 11576.80 | 13231.30 | 16147.20 | 11476.80 | 12666.90 | 15420.10 |
| 16 | 7 | 30 | 17769.10 | 19184.40 | 20734.70 | 18027.60 | 19834.10 | 22089.10 | 18138.70 | 19896.90 | 22516.90 | 17813.80 | 19192.80 | 21011.70 |
| 17 | 8 | 20 | 8481.02 | 9412.63 | 10524.00 | 8692.32 | 9841.05 | 11284.10 | 9003.04 | 9866.69 | 11119.60 | 8484.80 | 9377.96 | 10589.10 |
| 18 | 8 | 25 | 11106.40 | 12097.90 | 13532.40 | 11515.00 | 12564.30 | 14681.20 | 11517.10 | 12656.60 | 14646.70 | 11106.90 | 12117.50 | 13550.30 |
| 19 | 8 | 30 | 17039.80 | 18319.20 | 20299.50 | 17718.80 | 19026.90 | 21221.90 | 17818.40 | 19198.00 | 21657.10 | 17072.10 | 18364.60 | 20296.90 |
| 20 | 9 | 20 | 8139.30 | 8939.19 | 9941.81 | 8209.55 | 9503.27 | 11405.20 | 8340.05 | 9554.67 | 11902.60 | 8095.80 | 8931.31 | 9866.23 |
| 21 | 9 | 25 | 10687.60 | 11606.60 | 13263.40 | 11230.90 | 12091.10 | 13679.10 | 11241.50 | 12068.60 | 13392.00 | 10695.60 | 11617.50 | 13308.90 |
| 22 | 9 | 30 | 16487.10 | 17702.60 | 19129.50 | 16823.80 | 18545.30 | 21223.90 | 16883.50 | 18671.40 | 21119.50 | 16371.00 | 17885.10 | 19413.50 |
| 23 | 10 | 20 | 8004.40 | 8782.78 | 9781.35 | 8180.38 | 9283.81 | 10796.50 | 8239.36 | 9312.89 | 10573.00 | 8018.30 | 8831.58 | 9800.34 |
| 24 | 10 | 25 | 10631.00 | 11235.40 | 12716.00 | 10842.00 | 11701.20 | 13470.50 | 10934.70 | 11772.10 | 13581.70 | 10698.90 | 11238.40 | 12597.90 |
| 25 | 10 | 30 | 16307.50 | 17023.00 | 18599.70 | 16693.10 | 17819.50 | 19988.00 | 16913.60 | 18056.20 | 20325.00 | 16292.70 | 17118.40 | 18655.80 |

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| | | | 100 PALA Exact Mtd. | | | 100 PALA Taylor App. | | | 100 PALA Sample Param. | | | 100 PALA Avg. Dist. Mtd. | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **P** | **m** | **n** | **Min** | **Average** | **Max** | **Min** | **Average** | **Max** | **Min** | **Average** | **Max** | **Min** | **Average** | **Max** |
| 26 | 2 | 4 | 393.35 | 393.35 | 393.35 | 408.99 | 408.99 | 408.99 | 407.09 | 407.09 | 407.09 | 393.93 | 393.93 | 393.93 |
| 27 | 2 | 4 | 348.69 | 390.91 | 497.84 | 365.15 | 388.18 | 529.63 | 370.53 | 403.74 | 529.52 | 349.60 | 392.54 | 498.56 |
| 28 | 3 | 5 | 126.59 | 140.79 | 165.07 | 129.83 | 151.49 | 296.82 | 129.27 | 149.50 | 295.76 | 126.84 | 140.78 | 165.25 |
| 29 | 3 | 5 | 460.91 | 644.83 | 894.92 | 485.92 | 717.53 | 934.09 | 491.14 | 720.63 | 925.99 | 462.32 | 634.48 | 897.48 |
| 30 | 3 | 9 | 830.03 | 972.36 | 1212.03 | 855.71 | 971.56 | 1730.46 | 857.83 | 967.62 | 1747.72 | 834.88 | 978.33 | 1214.06 |
| 31 | 3 | 9 | 2412.44 | 2691.82 | 2872.96 | 2475.53 | 2755.92 | 3046.27 | 2518.34 | 2791.82 | 3079.62 | 2414.22 | 2696.62 | 2876.02 |
| 32 | 4 | 8 | 1735.19 | 1884.18 | 2122.21 | 1751.02 | 1974.55 | 2697.08 | 1755.19 | 1974.26 | 2777.12 | 1737.37 | 1887.31 | 2126.43 |
| 33 | 5 | 15 | 15944.60 | 19301.60 | 22695.20 | 16273.50 | 20084.00 | 23337.60 | 16249.70 | 20021.20 | 23345.40 | 15955.90 | 19296.70 | 22706.70 |
| 34 | 5 | 20 | 23347.10 | 30755.00 | 37955.60 | 23734.10 | 32476.10 | 42516.10 | 23594.70 | 32386.70 | 42953.30 | 23387.80 | 31058.00 | 41733.50 |
| 35 | 5 | 20 | 2804.57 | 3416.29 | 5739.19 | 2894.64 | 3899.44 | 5920.80 | 2881.27 | 3785.60 | 5892.59 | 2808.77 | 3380.22 | 5303.08 |
| 36 | 5 | 30 | 43033.40 | 46297.50 | 55544.60 | 43696.10 | 48882.70 | 63734.20 | 43866.60 | 48597.80 | 58285.40 | 43061.00 | 46180.30 | 55215.90 |
| 37 | 5 | 10 | 6937.23 | 8788.75 | 12728.30 | 7007.96 | 9186.76 | 13857.90 | 7027.27 | 9135.78 | 13828.10 | 6959.44 | 8723.94 | 12756.80 |
| 38 | 6 | 10 | 14951.70 | 18207.80 | 23043.50 | 16135.30 | 19236.60 | 23500.00 | 16015.60 | 19246.00 | 23692.50 | 14980.10 | 18302.70 | 23073.10 |
| 39 | 7 | 10 | 13539.60 | 17146.70 | 22151.30 | 14088.60 | 17872.10 | 22535.10 | 14253.80 | 17702.50 | 22602.20 | 13564.90 | 17208.20 | 22285.90 |
| 40 | 8 | 10 | 6216.36 | 7731.29 | 12776.10 | 6745.32 | 8422.65 | 13101.70 | 6671.32 | 8305.03 | 12761.60 | 6210.77 | 7795.44 | 13458.00 |
| 41 | 9 | 10 | 10198.10 | 12312.50 | 18057.80 | 10717.20 | 12776.70 | 16285.20 | 10673.40 | 12768.60 | 16045.40 | 10214.90 | 12210.70 | 18078.70 |
| 42 | 10 | 10 | 15207.40 | 19704.10 | 27327.50 | 15842.90 | 20349.30 | 30126.50 | 15870.30 | 20403.30 | 28851.50 | 15408.20 | 19653.30 | 26896.00 |
| **Average** | | | **16488.2** | **18167.3** | **21227.7** | **16839.7** | **18770.2** | **22315.1** | **16865.3** | **18790.3** | **22195.8** | **16513.3** | **18190.5** | **21303.6** |

Table 6.9: CPU times: rectilinear distance-normal distribution case

| **P** | **m** | **n** | bf Exact Mtd. | **Taylor App.** | **Sample Param.** | **Avg. Dist. Mtd.** |
|---|---|---|---|---|---|---|
| 1 | 5 | 20 | 311.39 | 58.01 | 58.69 | 62.19 |
| 2 | 5 | 25 | 349.40 | 73.03 | 71.69 | 75.07 |
| 3 | 5 | 30 | 461.29 | 91.97 | 87.83 | 92.62 |
| 4 | 5 | 40 | 714.11 | 138.79 | 135.30 | 147.39 |
| 5 | 5 | 50 | 1015.28 | 184.66 | 166.20 | 193.83 |
| 6 | 5 | 100 | 2549.97 | 456.82 | 407.18 | 479.99 |
| 7 | 5 | 100 | 2187.65 | 357.68 | 335.56 | 406.91 |
| 8 | 5 | 100 | 2334.44 | 350.89 | 354.92 | 396.91 |
| 9 | 5 | 100 | 2225.02 | 354.09 | 343.22 | 383.78 |
| 10 | 5 | 100 | 1865.66 | 332.80 | 320.24 | 351.22 |
| 11 | 6 | 20 | 340.61 | 63.49 | 60.78 | 66.64 |
| 12 | 6 | 25 | 437.77 | 76.55 | 77.18 | 81.58 |
| 13 | 6 | 30 | 619.48 | 95.50 | 93.75 | 106.22 |
| 14 | 7 | 20 | 399.86 | 65.26 | 68.79 | 69.16 |
| 15 | 7 | 25 | 524.02 | 79.14 | 83.61 | 84.63 |
| 16 | 7 | 30 | 674.86 | 99.47 | 107.94 | 113.08 |
| 17 | 8 | 20 | 470.08 | 66.06 | 67.50 | 76.32 |
| 18 | 8 | 25 | 565.58 | 84.98 | 82.02 | 86.26 |
| 19 | 8 | 30 | 796.34 | 105.65 | 102.76 | 113.43 |
| 20 | 9 | 20 | 508.89 | 70.15 | 68.72 | 79.26 |
| 21 | 9 | 25 | 633.04 | 85.71 | 84.51 | 89.96 |
| 22 | 9 | 30 | 908.99 | 105.54 | 105.26 | 114.97 |
| 23 | 10 | 20 | 564.93 | 72.03 | 72.43 | 80.03 |
| 24 | 10 | 25 | 715.27 | 87.87 | 85.46 | 96.31 |
| 25 | 10 | 30 | 936.13 | 107.90 | 107.23 | 122.22 |
| 26 | 2 | 4 | 9.63 | 5.06 | 5.08 | 5.11 |
| 27 | 2 | 4 | 15.52 | 9.34 | 6.74 | 6.75 |
| 28 | 3 | 5 | 38.78 | 12.26 | 13.22 | 12.88 |
| 29 | 3 | 5 | 33.20 | 12.80 | 12.78 | 12.21 |
| 30 | 3 | 9 | 59.23 | 20.39 | 20.04 | 19.57 |
| 31 | 3 | 9 | 80.02 | 21.84 | 21.73 | 24.43 |
| 32 | 4 | 8 | 66.19 | 21.11 | 21.18 | 19.99 |
| 33 | 5 | 15 | 203.54 | 43.75 | 44.29 | 47.52 |
| 34 | 5 | 20 | 243.79 | 52.86 | 53.99 | 53.43 |
| 35 | 5 | 20 | 276.22 | 52.96 | 54.27 | 57.90 |
| 36 | 5 | 30 | 478.84 | 86.21 | 88.48 | 100.04 |
| 37 | 5 | 10 | 116.31 | 31.13 | 31.34 | 29.61 |
| 38 | 6 | 10 | 155.17 | 36.81 | 35.21 | 36.93 |
| 39 | 7 | 10 | 185.82 | 38.25 | 38.18 | 40.66 |
| 40 | 8 | 10 | 205.23 | 41.32 | 41.74 | 43.55 |
| 41 | 9 | 10 | 224.90 | 41.12 | 41.82 | 44.87 |
| 42 | 10 | 10 | 237.23 | 43.17 | 42.21 | 49.40 |
| **Average** | | | **612.8** | **100.8** | **98.1** | **108.9** |

Table 6.10: PALA for the rectilinear distance-exponential distribution case

| | | | 100 PALA Exact Mtd. | | | 100 PALA Taylor App. | | | 100 PALA Sample Param. | | | 100 PALA Avg. Dist. Mtd. | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **P** | **m** | **n** | **Min** | **Average** | **Max** | **Min** | **Average** | **Max** | **Min** | **Average** | **Max** | **Min** | **Average** | **Max** |
| 1 | 5 | 20 | 10186.70 | 11117.00 | 13111.50 | 10422.50 | 11561.90 | 13220.60 | 10490.70 | 11628.10 | 13375.30 | 10217.70 | 11184.70 | 13026.30 |
| 2 | 5 | 25 | 12242.80 | 13790.90 | 16635.50 | 12502.00 | 14326.00 | 18081.20 | 12450.10 | 14333.70 | 17931.60 | 12250.70 | 13858.10 | 17274.00 |
| 3 | 5 | 30 | 19121.20 | 20155.10 | 21912.90 | 19434.00 | 20674.70 | 22889.70 | 19458.10 | 20723.90 | 22336.60 | 19135.40 | 20196.90 | 22072.00 |
| 4 | 5 | 40 | 27878.40 | 30342.70 | 33252.80 | 28110.90 | 31044.30 | 35083.90 | 28129.20 | 30897.70 | 34075.90 | 27892.50 | 30397.70 | 33802.80 |
| 5 | 5 | 50 | 25839.80 | 28240.70 | 31941.10 | 26019.40 | 28790.30 | 32484.30 | 26373.30 | 28907.40 | 32429.70 | 25885.90 | 28388.20 | 31956.80 |
| 6 | 5 | 100 | 43383.70 | 44588.80 | 46250.60 | 43634.10 | 44926.90 | 46935.90 | 43783.10 | 45236.10 | 48025.10 | 43393.80 | 44684.40 | 46553.80 |
| 7 | 5 | 100 | 46087.30 | 48225.50 | 51360.80 | 46467.50 | 48875.60 | 52402.30 | 46451.90 | 48942.20 | 52889.30 | 45874.40 | 48249.90 | 51400.40 |
| 8 | 5 | 100 | 50514.90 | 53410.30 | 60344.90 | 50926.70 | 54338.00 | 61795.40 | 50984.60 | 54396.90 | 67028.50 | 50610.50 | 53305.90 | 59841.70 |
| 9 | 5 | 100 | 47989.00 | 52642.60 | 66104.10 | 48379.80 | 53363.30 | 66575.40 | 48448.50 | 53540.70 | 67706.90 | 48077.10 | 52593.00 | 66147.20 |
| 10 | 5 | 100 | 49743.00 | 52974.80 | 64927.30 | 50293.00 | 53324.30 | 66937.30 | 50243.30 | 53317.50 | 62882.00 | 49756.60 | 52898.70 | 56593.00 |
| 11 | 6 | 20 | 8972.84 | 10189.30 | 11533.80 | 9364.11 | 10572.90 | 11907.90 | 9251.59 | 10660.30 | 12384.00 | 8984.44 | 10205.30 | 11541.90 |
| 12 | 6 | 25 | 10944.80 | 12330.10 | 14821.00 | 11124.10 | 12733.60 | 15962.50 | 11176.90 | 12812.90 | 15992.60 | 10903.40 | 12325.70 | 14837.60 |
| 13 | 6 | 30 | 17641.40 | 19124.70 | 21889.20 | 18236.90 | 19814.30 | 22115.40 | 18285.20 | 19853.90 | 23115.30 | 17661.30 | 19103.20 | 22138.00 |
| 14 | 7 | 20 | 8202.71 | 9299.06 | 10667.00 | 8418.76 | 9614.49 | 11057.20 | 8480.56 | 9701.27 | 11098.60 | 8212.05 | 9300.60 | 10717.80 |
| 15 | 7 | 25 | 10445.70 | 11811.50 | 13915.60 | 10619.20 | 12237.80 | 14610.50 | 10616.90 | 12290.30 | 15321.20 | 10461.60 | 11812.90 | 14607.10 |
| 16 | 7 | 30 | 16721.70 | 18210.90 | 19705.90 | 16922.00 | 18733.60 | 21077.30 | 17217.60 | 18872.20 | 21345.60 | 16650.40 | 18153.40 | 20160.70 |
| 17 | 8 | 20 | 7903.08 | 8873.34 | 10016.00 | 8057.34 | 9227.66 | 10669.80 | 8276.78 | 9371.68 | 10716.10 | 7916.91 | 8891.31 | 10015.90 |
| 18 | 8 | 25 | 10217.20 | 11246.10 | 12805.70 | 10526.60 | 11624.10 | 13798.20 | 10607.50 | 11797.80 | 13859.40 | 10252.60 | 11263.00 | 12883.40 |
| 19 | 8 | 30 | 15883.10 | 17243.90 | 19882.50 | 16538.60 | 17854.40 | 20044.30 | 16648.00 | 18117.00 | 21149.50 | 15897.30 | 17255.10 | 19052.00 |
| 20 | 9 | 20 | 7510.81 | 8378.46 | 9866.86 | 7516.98 | 8865.90 | 10793.80 | 7725.91 | 8958.89 | 11205.20 | 7533.20 | 8396.40 | 9256.73 |
| 21 | 9 | 25 | 9767.66 | 10730.70 | 12454.00 | 10252.20 | 11119.40 | 12757.80 | 10036.90 | 11192.70 | 12646.00 | 9745.09 | 10728.10 | 12427.70 |
| 22 | 9 | 30 | 15346.60 | 16651.80 | 18322.40 | 15572.30 | 17318.10 | 19989.60 | 15477.30 | 17543.70 | 20170.00 | 15343.60 | 16705.40 | 18327.60 |
| 23 | 10 | 20 | 7361.10 | 8166.34 | 9245.38 | 7470.67 | 8615.72 | 10205.90 | 7517.06 | 8644.39 | 10264.60 | 7380.03 | 8223.77 | 9640.73 |
| 24 | 10 | 25 | 9663.79 | 10312.10 | 11760.50 | 9801.81 | 10706.80 | 12565.90 | 9963.03 | 10861.50 | 12690.10 | 9719.49 | 10340.80 | 11147.70 |
| 25 | 10 | 30 | 15010.20 | 15775.30 | 17598.70 | 15425.60 | 16547.60 | 18819.50 | 15203.40 | 16762.40 | 18956.80 | 15014.40 | 15841.10 | 18492.30 |

Continued on Next Page...

| | | | 100 PALA Exact Mtd. | | | 100 PALA Taylor App. | | | 100 PALA Sample Param. | | | 100 PALA Avg. Dist. Mtd. | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **P** | **m** | **n** | **Min** | **Average** | **Max** | **Min** | **Average** | **Max** | **Min** | **Average** | **Max** | **Min** | **Average** | **Max** |
| 26 | 2 | 4 | 385.11 | 385.11 | 385.11 | 395.39 | 395.39 | 395.39 | 391.21 | 391.21 | 391.21 | 385.40 | 385.40 | 385.40 |
| 27 | 2 | 4 | 336.20 | 377.76 | 490.56 | 347.49 | 370.59 | 512.53 | 348.46 | 371.32 | 511.75 | 336.55 | 377.86 | 490.66 |
| 28 | 3 | 5 | 113.14 | 126.82 | 150.17 | 115.29 | 136.46 | 277.94 | 115.40 | 137.31 | 275.90 | 113.39 | 128.64 | 153.69 |
| 29 | 3 | 5 | 420.31 | 601.28 | 861.96 | 438.68 | 658.50 | 889.55 | 440.63 | 658.07 | 878.62 | 420.93 | 602.13 | 862.63 |
| 30 | 3 | 9 | 765.63 | 919.04 | 1152.91 | 787.98 | 899.32 | 1565.54 | 777.33 | 915.92 | 1566.69 | 768.50 | 919.33 | 1154.68 |
| 31 | 3 | 9 | 2250.11 | 2516.54 | 2679.65 | 2283.66 | 2555.61 | 2821.07 | 2289.13 | 2573.99 | 2863.61 | 2252.49 | 2514.31 | 2680.88 |
| 32 | 4 | 8 | 1641.32 | 1792.53 | 2019.33 | 1654.49 | 1858.38 | 2553.31 | 1650.07 | 1873.78 | 2512.92 | 1644.91 | 1813.86 | 2113.18 |
| 33 | 5 | 15 | 15189.00 | 18771.10 | 22227.60 | 15367.20 | 19267.70 | 22777.20 | 15437.30 | 19260.40 | 22822.30 | 15207.60 | 18735.20 | 22269.70 |
| 34 | 5 | 20 | 22414.40 | 29763.60 | 37488.70 | 22769.90 | 31557.00 | 41577.20 | 22964.50 | 31234.10 | 41541.20 | 22444.50 | 30461.40 | 41189.90 |
| 35 | 5 | 20 | 2674.71 | 3293.42 | 5644.01 | 2730.26 | 3699.48 | 5680.90 | 2751.59 | 3645.26 | 5605.52 | 2683.71 | 3430.15 | 5645.46 |
| 36 | 5 | 30 | 41889.60 | 44990.70 | 53946.60 | 42279.30 | 47113.00 | 61186.60 | 42159.30 | 47265.20 | 56600.80 | 41720.00 | 45409.60 | 56457.70 |
| 37 | 5 | 10 | 6513.39 | 8398.69 | 12781.30 | 6568.34 | 8702.65 | 13378.50 | 6617.39 | 8746.32 | 13400.60 | 6526.18 | 8501.40 | 12634.60 |
| 38 | 6 | 10 | 14424.50 | 17673.10 | 24477.70 | 15271.10 | 18361.50 | 22535.30 | 15246.20 | 18361.60 | 22551.50 | 14435.20 | 17692.30 | 22241.70 |
| 39 | 7 | 10 | 12986.30 | 16476.30 | 21527.40 | 13375.90 | 17005.30 | 21765.00 | 13694.40 | 17047.10 | 21920.60 | 12992.50 | 16468.80 | 21540.20 |
| 40 | 8 | 10 | 5782.73 | 7358.15 | 12256.30 | 6181.28 | 7838.96 | 12488.90 | 6230.60 | 7934.08 | 12444.40 | 5794.00 | 7377.58 | 12263.50 |
| 41 | 9 | 10 | 9594.62 | 11604.80 | 17357.10 | 9900.93 | 11959.80 | 15009.30 | 10049.00 | 12091.60 | 16500.80 | 9606.63 | 11657.30 | 17381.60 |
| 42 | 10 | 10 | 14593.30 | 19148.60 | 26699.50 | 15187.90 | 19520.40 | 29276.60 | 15132.40 | 19741.90 | 27547.40 | 14797.90 | 19019.00 | 26414.70 |
| **Average** | | | **15632.2** | **17334.0** | **20535.0** | **15897.4** | **17826.5** | **21368.4** | **15942.7** | **17895.6** | **21417.4** | **15640.5** | **17376.1** | **20471.3** |

Table 6.11: CPU times: rectilinear distance-exponential distribution case

| P | m | n | Exact Mtd. | Taylor App. | Sample Param. | Avg. Dist. Mtd. |
|---|---|---|---|---|---|---|
| 1 | 5 | 20 | 225.79 | 46.80 | 46.09 | 49.83 |
| 2 | 5 | 25 | 265.81 | 58.78 | 55.48 | 58.60 |
| 3 | 5 | 30 | 336.23 | 73.06 | 69.39 | 73.11 |
| 4 | 5 | 40 | 531.29 | 101.58 | 101.47 | 107.55 |
| 5 | 5 | 50 | 737.93 | 138.91 | 135.90 | 146.54 |
| 6 | 5 | 100 | 1909.14 | 351.41 | 312.99 | 367.02 |
| 7 | 5 | 100 | 1611.13 | 280.45 | 276.62 | 308.29 |
| 8 | 5 | 100 | 1738.75 | 275.31 | 284.71 | 344.07 |
| 9 | 5 | 100 | 1529.78 | 276.14 | 264.51 | 305.80 |
| 10 | 5 | 100 | 1391.45 | 260.81 | 258.65 | 268.69 |
| 11 | 6 | 20 | 248.54 | 49.24 | 48.87 | 51.84 |
| 12 | 6 | 25 | 312.57 | 59.74 | 58.94 | 64.27 |
| 13 | 6 | 30 | 417.95 | 74.38 | 72.51 | 85.10 |
| 14 | 7 | 20 | 298.78 | 51.92 | 51.23 | 56.23 |
| 15 | 7 | 25 | 378.51 | 62.80 | 61.30 | 67.25 |
| 16 | 7 | 30 | 495.37 | 79.03 | 77.87 | 86.91 |
| 17 | 8 | 20 | 339.64 | 52.09 | 52.06 | 57.59 |
| 18 | 8 | 25 | 410.59 | 67.59 | 63.89 | 68.83 |
| 19 | 8 | 30 | 576.70 | 83.97 | 78.94 | 89.71 |
| 20 | 9 | 20 | 373.03 | 55.63 | 55.37 | 58.38 |
| 21 | 9 | 25 | 469.15 | 67.80 | 65.34 | 69.44 |
| 22 | 9 | 30 | 646.34 | 83.78 | 79.90 | 89.35 |
| 23 | 10 | 20 | 425.92 | 57.84 | 58.72 | 62.54 |
| 24 | 10 | 25 | 530.74 | 69.86 | 63.75 | 74.26 |
| 25 | 10 | 30 | 727.21 | 85.39 | 80.86 | 96.12 |
| 26 | 2 | 4 | 6.78 | 4.03 | 4.03 | 4.31 |
| 27 | 2 | 4 | 10.33 | 7.52 | 7.48 | 5.36 |
| 28 | 3 | 5 | 26.73 | 9.78 | 9.73 | 10.18 |
| 29 | 3 | 5 | 23.64 | 10.34 | 10.20 | 9.63 |
| 30 | 3 | 9 | 38.23 | 16.50 | 15.64 | 15.74 |
| 31 | 3 | 9 | 49.98 | 17.66 | 17.38 | 17.52 |
| 32 | 4 | 8 | 48.34 | 17.02 | 16.52 | 15.23 |
| 33 | 5 | 15 | 143.72 | 35.11 | 34.92 | 36.96 |
| 34 | 5 | 20 | 184.40 | 41.95 | 43.12 | 40.61 |
| 35 | 5 | 20 | 194.36 | 42.50 | 41.64 | 44.15 |
| 36 | 5 | 30 | 348.79 | 68.24 | 70.25 | 71.71 |
| 37 | 5 | 10 | 85.28 | 24.80 | 25.38 | 22.99 |
| 38 | 6 | 10 | 111.66 | 29.72 | 28.20 | 28.87 |
| 39 | 7 | 10 | 139.33 | 30.50 | 30.30 | 31.32 |
| 40 | 8 | 10 | 150.95 | 32.31 | 32.11 | 32.53 |
| 41 | 9 | 10 | 163.39 | 32.85 | 32.30 | 34.54 |
| 42 | 10 | 10 | 172.25 | 35.93 | 33.70 | 35.54 |
| **Average** | | | **448.2** | **79.1** | **76.9** | **84.9** |

### 6.2.4. Effects of the Sample Size

In the previous section, the effects of the expected distance approximations on the PALA heuristic has been examined. Previously, we assume that we have 100 sample data for each customer location. In this situation, we solve the PCMFWP using the average distance and Taylor approximation with estimated parameters. In fact, the size of the available samples can be varied. Moreover, big sample sizes may incredibly improve the results. Thus, it can be logical to collect more data. On the contrary, the size of the sample may not affect the results. Hence, collecting some extra data can be superfluous. Therefore, inspecting the effects of the sample size is worthwhile.

In order to inspect the effects of the sample size on the solution of PCMFWP with PALA algorithm, 25, 50, 100, 250, 500 and 1000 samples are generated. Moreover, we will only consider the Euclidean distance and bivariate symmetric normal distribution since similar conclusions can be obtained for other distance function and probability distributions. In the following, the results of the Taylor approximation with estimated parameters are given in Table 6.12 and Table 6.13 and the results with the average distance sample are given in Table 6.14 and Table 6.15. Increasing the sample sizes improves the results as expected. Specifically, for the Taylor approximation with estimated sample parameters the per cent average improvement between 25 and 1000 sample case is 1.488 per cent. Similarly, for the average distance sample case the improvement is 0.8241 per cent.

Table 6.12: The effect of the sample size: Taylor Approximation with estimated parameters (1)

| | | | 25 SAMPLE | | | 50 SAMPLE | | | 100 SAMPLE | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| **P** | **m** | **n** | **Min** | **Average** | **Max** | **Min** | **Average** | **Max** | **Min** | **Average** | **Max** |
| 1 | 5 | 20 | 8496.38 | 9292.1 | 10616.6 | 8378.86 | 9231.48 | 10838.5 | 8385.59 | 9193.6 | 10824.1 |
| 2 | 5 | 25 | 10487.1 | 11600.2 | 13166 | 10524 | 11613.9 | 13386.1 | 10541.8 | 11590.7 | 13503.5 |
| 3 | 5 | 30 | 16029.5 | 17042.6 | 18658.2 | 15940.7 | 16895.6 | 18005.8 | 15900.8 | 16875.8 | 17999.3 |
| 4 | 5 | 40 | 23402.9 | 25419 | 28221.9 | 23203.8 | 25077 | 28229.1 | 23198.4 | 25047 | 27943.1 |
| 5 | 5 | 50 | 21817.6 | 23655.2 | 25533.4 | 21595.6 | 23493.5 | 25682.4 | 21628.1 | 23455.8 | 25925.1 |
| 6 | 5 | 100 | 36270.8 | 37998.9 | 40478.7 | 36397.9 | 37849.1 | 40406.8 | 36434.1 | 37565 | 39804.9 |
| 7 | 5 | 100 | 39107.4 | 41233.9 | 44296.2 | 39149.5 | 41016.6 | 44075.5 | 38881.4 | 40705.3 | 43924.7 |
| 8 | 5 | 100 | 42369.3 | 44988 | 50033.6 | 42397.4 | 44601.9 | 49826.2 | 42054.8 | 44448 | 49229.6 |
| 9 | 5 | 100 | 41335.6 | 44419.3 | 54020.2 | 40871.8 | 44145.8 | 53829.4 | 40482.5 | 43915.2 | 53790.8 |
| 10 | 5 | 100 | 41357.6 | 44273.2 | 55332.4 | 41094.1 | 43960.9 | 47321.9 | 40979.9 | 43920.3 | 47046.9 |
| 11 | 6 | 20 | 7621.86 | 8595.24 | 9746.77 | 7631.51 | 8506.13 | 9248.35 | 7618.16 | 8498.47 | 9797.07 |
| 12 | 6 | 25 | 9649.72 | 10725.2 | 12763 | 9557.87 | 10547.3 | 12383.5 | 9552 | 10586.3 | 12344.8 |
| 13 | 6 | 30 | 14939.6 | 16142.8 | 18207.6 | 14770.8 | 15971 | 18229 | 14725.4 | 15875.7 | 17516.1 |
| 14 | 7 | 20 | 7213.82 | 8095.21 | 9031.28 | 7210.08 | 8025.73 | 9024.49 | 7150.67 | 7978.74 | 8898.46 |
| 15 | 7 | 25 | 9366.24 | 10359.9 | 12860.3 | 9453.25 | 10260.2 | 12493.8 | 9519.22 | 10268.1 | 12369.3 |
| 16 | 7 | 30 | 14343.3 | 15589.5 | 17311.9 | 14211.5 | 15402.3 | 17614 | 14041.5 | 15338.6 | 16718.9 |
| 17 | 8 | 20 | 7053.54 | 7728.67 | 8664.22 | 6975.82 | 7685.37 | 8780.5 | 6913.46 | 7638.72 | 8721.56 |
| 18 | 8 | 25 | 9117.32 | 10014.1 | 10860.6 | 9162.71 | 9859.68 | 10892.9 | 9054.6 | 9836.19 | 11075.3 |
| 19 | 8 | 30 | 14015.3 | 14955 | 16396 | 14004.7 | 14878 | 16291.4 | 13824.1 | 14793.7 | 15991.7 |
| 20 | 9 | 20 | 6754.36 | 7417.57 | 8315.29 | 6749.71 | 7370.65 | 8063.98 | 6613.98 | 7286.31 | 8002.46 |
| 21 | 9 | 25 | 8864.13 | 9773.85 | 11469.3 | 8764.86 | 9612.28 | 11156.3 | 8776.9 | 9549.38 | 11084.9 |
| 22 | 9 | 30 | 13405.2 | 14494.4 | 15655.6 | 13233.8 | 14374.7 | 16260.4 | 13185.9 | 14283 | 15264 |
| 23 | 10 | 20 | 6669.13 | 7291.32 | 8268.16 | 6555.1 | 7177.89 | 7929.95 | 6494.7 | 7136.73 | 7835.47 |
| 24 | 10 | 25 | 8870.75 | 9471.69 | 11013.8 | 8838.19 | 9349.99 | 10141.7 | 8819.59 | 9295.47 | 10024.4 |

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| | | | 25 SAMPLE | | | 50 SAMPLE | | | 100 SAMPLE | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| **P** | **m** | **n** | **Min** | **Average** | **Max** | **Min** | **Average** | **Max** | **Min** | **Average** | **Max** |
| 25 | 10 | 30 | 13383.3 | 14094.3 | 15675.9 | 13248 | 13880.8 | 14785 | 13146.7 | 13786.6 | 14826.6 |
| 26 | 2 | 4 | 307.335 | 307.335 | 307.335 | 305.478 | 305.478 | 305.478 | 304.352 | 304.352 | 304.352 |
| 27 | 2 | 4 | 285.97 | 308.996 | 403.367 | 284.688 | 296.653 | 300.028 | 284.901 | 295.269 | 298.194 |
| 28 | 3 | 5 | 110.68 | 126.608 | 140.145 | 108.604 | 119.865 | 137.271 | 107.995 | 112.02 | 135.572 |
| 29 | 3 | 5 | 375.988 | 521.944 | 693.744 | 378.439 | 508.227 | 692.221 | 380.854 | 514.423 | 695.837 |
| 30 | 3 | 9 | 703.111 | 797.183 | 1004.15 | 701.711 | 793.593 | 1000.72 | 694.42 | 795.656 | 1007.24 |
| 31 | 3 | 9 | 1971.08 | 2215.65 | 2376.04 | 1956.42 | 2162.76 | 2298.69 | 1958.72 | 2190.16 | 2353.53 |
| 32 | 4 | 8 | 1414.5 | 1513.55 | 1748.32 | 1389.82 | 1486.73 | 1718.35 | 1381.41 | 1487.38 | 1705.65 |
| 33 | 5 | 15 | 13078.4 | 15084.6 | 18099 | 12854.7 | 15102.4 | 17469.3 | 12909.7 | 15176.8 | 17745.2 |
| 34 | 5 | 20 | 19963.2 | 24459.5 | 32331.2 | 18954.1 | 24254 | 32197.7 | 18936.7 | 24318.4 | 32149.7 |
| 35 | 5 | 20 | 2255.92 | 2697.82 | 4222.95 | 2254.59 | 2727.87 | 4230.13 | 2256.25 | 2762.32 | 4496.14 |
| 36 | 5 | 30 | 33884.8 | 36390.5 | 44201.3 | 33640.9 | 35926.9 | 43299.6 | 33590.2 | 35713.3 | 44155.3 |
| 37 | 5 | 10 | 5989.97 | 7331.84 | 10121.5 | 5874.45 | 7259.76 | 11304.5 | 5857.45 | 7224.33 | 10036.8 |
| 38 | 6 | 10 | 11936.1 | 15059.3 | 19344 | 11875.6 | 14925.8 | 19316.9 | 11866.2 | 14868.7 | 19293.8 |
| 39 | 7 | 10 | 11053.5 | 14212.4 | 17972.9 | 10730.5 | 13987.5 | 17931.6 | 10714.7 | 13962.4 | 17904 |
| 40 | 8 | 10 | 5066.9 | 6428.42 | 10574.1 | 4991.11 | 6357.71 | 10566.8 | 4981.82 | 6253.16 | 10525.4 |
| 41 | 9 | 10 | 8343.57 | 10214.1 | 14564.4 | 8238.16 | 9922.08 | 14423.7 | 8236.28 | 9926.34 | 14454.7 |
| 42 | 10 | 10 | 11935.4 | 15839.9 | 21536.2 | 11724.4 | 15623.6 | 21569 | 11725.2 | 15341.7 | 21751 |
| **Average** | | | **13348.1** | **14718.6** | **17291.4** | **13242.5** | **14584.5** | **16991.9** | **13193.1** | **14526.6** | **16892.3** |

Table 6.13: Effects of the sample size: Taylor Approximation with estimated parameters (2)

| | | | 250 SAMPLE | | | 500 SAMPLE | | | 1000 SAMPLE | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| **P** | **m** | **n** | **Min** | **Average** | **Max** | **Min** | **Average** | **Max** | **Min** | **Average** | **Max** |
| 1 | 5 | 20 | 8389.78 | 9187.01 | 10803.9 | 8379.9 | 9174.49 | 10797.9 | 8382.33 | 9151.09 | 10801 |
| 2 | 5 | 25 | 10529.1 | 11576.5 | 14020.9 | 10519.4 | 11546.5 | 13486.6 | 10515.6 | 11535.6 | 13318.9 |

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| | | | 250 SAMPLE | | | 500 SAMPLE | | | 1000 SAMPLE | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| **P** | **m** | **n** | **Min** | **Average** | **Max** | **Min** | **Average** | **Max** | **Min** | **Average** | **Max** |
| 3 | 5 | 30 | 15886.2 | 16806.1 | 17941.7 | 15878.4 | 16819.3 | 17930.7 | 15871.1 | 16836.4 | 17959.4 |
| 4 | 5 | 40 | 23056.4 | 25090.2 | 28079.3 | 23063 | 25069.2 | 27911.1 | 23074 | 25040.1 | 27896.1 |
| 5 | 5 | 50 | 21595.6 | 23367.6 | 25703.3 | 21567.3 | 23347.6 | 25750.3 | 21574.6 | 23380.4 | 25750.2 |
| 6 | 5 | 100 | 36240.4 | 37486.3 | 40704.7 | 36118.6 | 37394.2 | 39864.1 | 36121.2 | 37374.7 | 38899.6 |
| 7 | 5 | 100 | 38844.5 | 40681.7 | 43460.2 | 38860.6 | 40618.8 | 43430.7 | 38831.2 | 40644.6 | 43397.3 |
| 8 | 5 | 100 | 42028.8 | 44368.6 | 49579.4 | 42001.3 | 44371.4 | 49585.3 | 41886.2 | 44275.3 | 49347.6 |
| 9 | 5 | 100 | 40494.6 | 43881.4 | 53740.9 | 41016.1 | 44006 | 53723.5 | 40701.8 | 44052.8 | 53702 |
| 10 | 5 | 100 | 41039.4 | 43845.7 | 48005.9 | 40988.6 | 43909.3 | 50716 | 40970.6 | 43964.4 | 50426.7 |
| 11 | 6 | 20 | 7608.73 | 8484.94 | 9761.61 | 7600.55 | 8479.84 | 9757.72 | 7600.63 | 8461.96 | 9755 |
| 12 | 6 | 25 | 9565.86 | 10539.5 | 12380 | 9568.47 | 10588.7 | 12511.5 | 9455.75 | 10551.3 | 12513.1 |
| 13 | 6 | 30 | 14689.1 | 15888.9 | 17757.3 | 14669.7 | 15859.3 | 17577.1 | 14676.1 | 15867 | 17782.1 |
| 14 | 7 | 20 | 7148.3 | 7924.08 | 8905.69 | 7135.59 | 7923.97 | 8900.33 | 7133.11 | 7927.74 | 8898.51 |
| 15 | 7 | 25 | 9462.98 | 10229.8 | 12324 | 9423.01 | 10207.4 | 12266.7 | 9417 | 10225 | 12292 |
| 16 | 7 | 30 | 13932.1 | 15334.9 | 16733.8 | 13905.8 | 15303.6 | 16604.7 | 13924.6 | 15345.8 | 16618 |
| 17 | 8 | 20 | 6907.85 | 7655.27 | 8768.56 | 6891.91 | 7646.98 | 8767.33 | 6895.57 | 7653.98 | 8779.5 |
| 18 | 8 | 25 | 9089.69 | 9809.54 | 10774.3 | 9078.57 | 9773.8 | 10693.6 | 9075.53 | 9794.37 | 10747.9 |
| 19 | 8 | 30 | 13773.6 | 14773.2 | 16219.1 | 13733.6 | 14747.6 | 16612.7 | 13724.5 | 14762.7 | 16208.7 |
| 20 | 9 | 20 | 6666.44 | 7291.03 | 7957.27 | 6734.99 | 7294.94 | 8015.4 | 6744.88 | 7274.44 | 7861.4 |
| 21 | 9 | 25 | 8742.1 | 9496.83 | 10538.2 | 8784.76 | 9488.61 | 10460.6 | 8727.38 | 9504.17 | 10472.1 |
| 22 | 9 | 30 | 13196.5 | 14260.2 | 15224.4 | 13183.7 | 14241.7 | 15196.1 | 13179.4 | 14228.9 | 15177.2 |
| 23 | 10 | 20 | 6522.71 | 7130.81 | 7926.88 | 6506.11 | 7122.52 | 7883.77 | 6495.8 | 7131.08 | 7871.66 |
| 24 | 10 | 25 | 8787.4 | 9248.86 | 9998.04 | 8749.61 | 9230.21 | 9968.38 | 8738.08 | 9215.5 | 10045.5 |
| 25 | 10 | 30 | 13171.8 | 13767.1 | 14867.3 | 13159.8 | 13720.1 | 14925.1 | 13179 | 13715.7 | 14830.9 |
| 26 | 2 | 4 | 304.213 | 304.213 | 304.213 | 304.474 | 304.474 | 304.474 | 304.143 | 304.143 | 304.143 |
| 27 | 2 | 4 | 296.333 | 319.771 | 402.87 | 295.776 | 319.326 | 402.82 | 296.358 | 296.358 | 296.358 |
| 28 | 3 | 5 | 106.215 | 116.724 | 135.329 | 106.216 | 116.949 | 134.732 | 105.904 | 116.627 | 132.773 |
| 29 | 3 | 5 | 381.934 | 520.562 | 684.213 | 377.664 | 518.403 | 683.666 | 376.226 | 517.968 | 683.335 |

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| | | | 250 SAMPLE | | | 500 SAMPLE | | | 1000 SAMPLE | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| **P** | **m** | **n** | **Min** | **Average** | **Max** | **Min** | **Average** | **Max** | **Min** | **Average** | **Max** |
| 30 | 3 | 9 | 700.515 | 792.921 | 957.996 | 694.235 | 793.173 | 1002.36 | 688.541 | 791.198 | 1002.01 |
| 31 | 3 | 9 | 1958.6 | 2173.38 | 2316.4 | 1955.69 | 2163.96 | 2304.1 | 1957.35 | 2160.86 | 2298.87 |
| 32 | 4 | 8 | 1377.05 | 1485.84 | 1677.59 | 1376.85 | 1488.99 | 1678.43 | 1379.65 | 1488.17 | 1674.65 |
| 33 | 5 | 15 | 12746.8 | 15122.1 | 18169.5 | 12798.9 | 15118.2 | 18105.8 | 12814.9 | 15014.3 | 18279.3 |
| 34 | 5 | 20 | 18758.9 | 24234.9 | 32161.2 | 18786.2 | 24128.3 | 32142.8 | 18771.8 | 24071.1 | 32142.1 |
| 35 | 5 | 20 | 2255.57 | 2771.4 | 4493.88 | 2254.58 | 2768.7 | 4485.65 | 2252.67 | 2767.96 | 4489.23 |
| 36 | 5 | 30 | 33628.6 | 35774.6 | 44135.9 | 33563.8 | 35706.6 | 43326.9 | 33569.9 | 35819.3 | 44119.2 |
| 37 | 5 | 10 | 5839.2 | 7187.88 | 9952.78 | 5773.89 | 7118.99 | 9915.2 | 5777.71 | 7164.83 | 11805.2 |
| 38 | 6 | 10 | 11841.4 | 14934.8 | 19265.2 | 11820.8 | 14935.5 | 19260.8 | 11823.4 | 14926.4 | 19347.5 |
| 39 | 7 | 10 | 10845.3 | 13970.3 | 17888.7 | 10720.2 | 13968.5 | 17887.3 | 10719.7 | 14003.8 | 17891.8 |
| 40 | 8 | 10 | 4945.43 | 6228.98 | 10535.9 | 4939.01 | 6220.43 | 10588.4 | 5007.77 | 6244.54 | 10555.6 |
| 41 | 9 | 10 | 8192.45 | 9871.64 | 14031.2 | 8176.63 | 9847.89 | 14044.1 | 8193.3 | 9806.14 | 14038.8 |
| 42 | 10 | 10 | 11780.6 | 15441.1 | 21538.2 | 11726.7 | 15448.8 | 21507.5 | 11725.5 | 15573.4 | 21563.3 |
| **Average** | | | **13174.5** | **14509.0** | **16924.5** | **13171.2** | **14496.5** | **16931.2** | **13158.6** | **14499.6** | **16951.8** |

Table 6.14: The effect of the sample size: Average distance (1)

| | | | 25 SAMPLE | | | 50 SAMPLE | | | 100 SAMPLE | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| **P** | **m** | **n** | **Min** | **Average** | **Max** | **Min** | **Average** | **Max** | **Min** | **Average** | **Max** |
| 1 | 5 | 20 | 8418.84 | 9236.03 | 10368.4 | 8389.28 | 9217.51 | 10755.9 | 8366.43 | 9147.04 | 10748.6 |
| 2 | 5 | 25 | 10389.4 | 11479.1 | 13219.2 | 10328.9 | 11427.5 | 13182.6 | 10317 | 11424.4 | 13185.7 |
| 3 | 5 | 30 | 15876.4 | 16831.2 | 17731.7 | 15857.5 | 16731.3 | 17824.1 | 15838.1 | 16728.3 | 17803.2 |
| 4 | 5 | 40 | 22955.8 | 25016.3 | 27863.8 | 22960.9 | 24895.2 | 27648.9 | 22923.8 | 24923.1 | 28290.2 |
| 5 | 5 | 50 | 21633.2 | 23404.7 | 25834.7 | 21350.4 | 23319 | 25812.1 | 21318.5 | 23193 | 25642.3 |
| 6 | 5 | 100 | 36070.4 | 37065 | 38453.1 | 35898.7 | 36952.3 | 38184.8 | 35814.7 | 36841.4 | 38471.7 |
| 7 | 5 | 100 | 38837.1 | 40453.4 | 43666.5 | 38560.4 | 40244.2 | 43538.3 | 38550.7 | 39986.2 | 42602.6 |
| 8 | 5 | 100 | 41861.4 | 44014 | 49129.5 | 41740.7 | 43851.3 | 47956.4 | 41679.5 | 43875.9 | 48907.6 |

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| | | | 25 SAMPLE | | | 50 SAMPLE | | | 100 SAMPLE | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| **P** | **m** | **n** | **Min** | **Average** | **Max** | **Min** | **Average** | **Max** | **Min** | **Average** | **Max** |
| 9 | 5 | 100 | 40183.5 | 43633.5 | 53787.6 | 40155.7 | 43566 | 53662.2 | 40096.7 | 43578.6 | 54883.5 |
| 10 | 5 | 100 | 40868 | 43582.6 | 46498.3 | 40723.5 | 43452.7 | 46326.9 | 40706.9 | 43406.7 | 46137.9 |
| 11 | 6 | 20 | 7516.19 | 8502.08 | 9618.4 | 7492.05 | 8457.56 | 9398.44 | 7486.49 | 8440.92 | 9489.48 |
| 12 | 6 | 25 | 9453.98 | 10529.2 | 12367.4 | 9408.01 | 10417.8 | 12233.5 | 9386.5 | 10356.7 | 12150.6 |
| 13 | 6 | 30 | 14759.5 | 15834.1 | 17593.6 | 14647.6 | 15808.8 | 17911.3 | 14635.6 | 15831.6 | 17832.2 |
| 14 | 7 | 20 | 6970.82 | 7882.18 | 8898.38 | 6949.84 | 7848.74 | 8868.3 | 6943 | 7813.47 | 8972.11 |
| 15 | 7 | 25 | 9306.56 | 10150.8 | 11643 | 9161.83 | 10056.7 | 12125.1 | 9197.81 | 10053.2 | 12120.2 |
| 16 | 7 | 30 | 13808.5 | 15243.1 | 16933.5 | 13738.5 | 15172.7 | 17567.3 | 14005.4 | 15143.6 | 16559 |
| 17 | 8 | 20 | 6755.48 | 7537.46 | 8554.04 | 6730.74 | 7513.54 | 8381.31 | 6701.02 | 7428.83 | 8654.33 |
| 18 | 8 | 25 | 8826.07 | 9646.6 | 10631.1 | 8781.01 | 9577.95 | 10573 | 8763.51 | 9561 | 10484.9 |
| 19 | 8 | 30 | 13518.4 | 14657.2 | 16164 | 13447 | 14502.2 | 16049.7 | 13442.2 | 14483.9 | 15923.7 |
| 20 | 9 | 20 | 6488.72 | 7175.47 | 7884.86 | 6469.28 | 7140.63 | 8106.65 | 6427.79 | 7083.56 | 8059.59 |
| 21 | 9 | 25 | 8523.33 | 9407.74 | 11164.9 | 8461.28 | 9262.05 | 10493.6 | 8438.67 | 9227.33 | 10998.6 |
| 22 | 9 | 30 | 12967.3 | 14043.6 | 15208.5 | 12900.8 | 14037 | 16056.6 | 12874.9 | 14012.5 | 15201.7 |
| 23 | 10 | 20 | 6404.67 | 7041.2 | 7942.56 | 6336.34 | 6996.53 | 7818.94 | 6319.32 | 6949.89 | 8048.45 |
| 24 | 10 | 25 | 8491.14 | 9016.66 | 9776.69 | 8430.88 | 8950.64 | 9743.59 | 8370.06 | 8905.81 | 9680.66 |
| 25 | 10 | 30 | 12962.6 | 13513.6 | 14389.3 | 12921.4 | 13438.9 | 14366.7 | 12900.7 | 13407.4 | 14622.8 |
| 26 | 2 | 4 | 306.079 | 306.614 | 306.745 | 305.387 | 305.738 | 305.888 | 303.944 | 304.068 | 304.12 |
| 27 | 2 | 4 | 284.995 | 309.506 | 404.17 | 286.031 | 309.914 | 404.27 | 285.702 | 308.882 | 404.065 |
| 28 | 3 | 5 | 102.503 | 111.763 | 132.262 | 99.9562 | 110.497 | 130.753 | 99.9059 | 109.605 | 130.715 |
| 29 | 3 | 5 | 355.46 | 504.625 | 683.308 | 355.013 | 483.631 | 681.614 | 354.886 | 482.405 | 677.703 |
| 30 | 3 | 9 | 658.688 | 766.201 | 977.397 | 655.199 | 768.61 | 969.804 | 654.831 | 770.144 | 963.946 |
| 31 | 3 | 9 | 1941.21 | 2150.43 | 2281.68 | 1943.63 | 2129.56 | 2245.02 | 1924.05 | 2116.67 | 2230.35 |
| 32 | 4 | 8 | 1369.47 | 1464.9 | 1668.11 | 1361.02 | 1456.13 | 1653.62 | 1354.92 | 1453.35 | 1648.8 |
| 33 | 5 | 15 | 12668 | 14877.9 | 18071.2 | 12542.9 | 14950.9 | 18082.3 | 12535.1 | 14987.8 | 18069.7 |
| 34 | 5 | 20 | 18613.1 | 24073 | 32114.7 | 18611 | 24035.4 | 32045.2 | 18610.7 | 24181.7 | 32040.6 |
| 35 | 5 | 20 | 2248.57 | 2670.75 | 4018.36 | 2239.96 | 2673.35 | 3974.1 | 2239.12 | 2738.8 | 4444.97 |

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| | | | 25 SAMPLE | | | 50 SAMPLE | | | 100 SAMPLE | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| **P** | **m** | **n** | **Min** | **Average** | **Max** | **Min** | **Average** | **Max** | **Min** | **Average** | **Max** |
| 36 | 5 | 30 | 33463.8 | 36038.6 | 44093.7 | 33366.6 | 35864.1 | 42953.8 | 33355 | 35794.1 | 43506.8 |
| 37 | 5 | 10 | 5728.78 | 7083.43 | 9895.22 | 5696.32 | 7058.57 | 9886.25 | 5690.19 | 7051.6 | 9884.26 |
| 38 | 6 | 10 | 11866.6 | 14905.4 | 19279 | 11796.9 | 14827.7 | 19254.3 | 11775.8 | 14742.2 | 19249.8 |
| 39 | 7 | 10 | 10781.6 | 14057.2 | 17989.9 | 10715.1 | 14019.1 | 17985.9 | 10698.6 | 13872.6 | 17895.7 |
| 40 | 8 | 10 | 4927.29 | 6277.9 | 10409.4 | 4903.31 | 6234.97 | 10405.1 | 4882.07 | 6209.74 | 10378.2 |
| 41 | 9 | 10 | 8173.71 | 9921.82 | 14326.8 | 8139.86 | 9725.49 | 14260.7 | 8125.77 | 9750.84 | 14234.3 |
| 42 | 10 | 10 | 11719.3 | 15645.1 | 21466.1 | 11678.5 | 15445.8 | 21472 | 11653.3 | 15350.9 | 21466.2 |
| **Average** | | | **13072.8** | **14430.0** | **16748.6** | **13012.8** | **14362.8** | **16697.5** | **13001.2** | **14334.0** | **16738.1** |

Table 6.15: The effect of the sample size: Average Distance (2)

| | | | 250 SAMPLE | | | 500 SAMPLE | | | 1000 SAMPLE | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| **P** | **m** | **n** | **Min** | **Average** | **Max** | **Min** | **Average** | **Max** | **Min** | **Average** | **Max** |
| 1 | 5 | 20 | 8349.19 | 9135.77 | 10746 | 8346.43 | 9124.13 | 10745.1 | 8346.31 | 9135.76 | 10744.6 |
| 2 | 5 | 25 | 10316.9 | 11395.3 | 13701.5 | 10314.2 | 11405.9 | 13699.5 | 10311.6 | 11373 | 13933.6 |
| 3 | 5 | 30 | 15821.2 | 16680.4 | 17793.9 | 15819.8 | 16669.3 | 17790 | 15814.9 | 16679.6 | 17789.3 |
| 4 | 5 | 40 | 22905.7 | 24876.3 | 27670.1 | 22918.3 | 24874.2 | 27816.3 | 22903.7 | 24861.3 | 27668.7 |
| 5 | 5 | 50 | 21305.9 | 23166.7 | 25727.9 | 21307.4 | 23135.5 | 25713.4 | 21305.1 | 23159.5 | 25711.8 |
| 6 | 5 | 100 | 35823.3 | 36944.3 | 38411.8 | 35775.2 | 36862.2 | 38280.3 | 35775.1 | 36798.4 | 38392.2 |
| 7 | 5 | 100 | 38496.9 | 40049.8 | 43181.6 | 38481.6 | 39972.9 | 42413 | 38475.4 | 40007.2 | 43088.2 |
| 8 | 5 | 100 | 41672.3 | 43796.8 | 49209 | 41639.6 | 43873.8 | 49206.1 | 41639.9 | 43831 | 49205.3 |
| 9 | 5 | 100 | 40089.3 | 43573.6 | 54844.7 | 40085.8 | 43471.8 | 53608.6 | 40084.5 | 43478.9 | 53613.9 |
| 10 | 5 | 100 | 40703.1 | 43344.7 | 46117.4 | 40700.3 | 43293.3 | 46115.9 | 40699.6 | 43325.7 | 46138.5 |
| 11 | 6 | 20 | 7480.61 | 8434.5 | 9361.29 | 7479.12 | 8404.99 | 9357.39 | 7478.18 | 8410.62 | 9314.67 |
| 12 | 6 | 25 | 9265.12 | 10301.2 | 12202.7 | 9261.36 | 10319.1 | 12200.9 | 9268.73 | 10333.2 | 12140 |
| 13 | 6 | 30 | 14588.3 | 15823.1 | 17819.6 | 14585.4 | 15795.3 | 17814.5 | 14584.6 | 15761.6 | 17813.4 |
| 14 | 7 | 20 | 6941.03 | 7786.25 | 8851.49 | 6938.05 | 7753.85 | 8751.41 | 6936.97 | 7779.33 | 8848.54 |

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| | | | 250 SAMPLE | | | 500 SAMPLE | | | 1000 SAMPLE | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| **P** | **m** | **n** | **Min** | **Average** | **Max** | **Min** | **Average** | **Max** | **Min** | **Average** | **Max** |
| 15 | 7 | 25 | 9013.63 | 10016.1 | 12114.4 | 9158.95 | 10026.6 | 12116.3 | 9136.11 | 10044 | 12113.8 |
| 16 | 7 | 30 | 13679.6 | 15145.1 | 16462.1 | 13673.9 | 15115.3 | 16504.2 | 13673.4 | 15131.6 | 16457.6 |
| 17 | 8 | 20 | 6692.88 | 7460.57 | 8649.33 | 6688.12 | 7432.8 | 8650.13 | 6687.09 | 7434.08 | 8649.09 |
| 18 | 8 | 25 | 8753.49 | 9521.62 | 10478.3 | 8749.91 | 9520.98 | 10455.1 | 8748.16 | 9504.85 | 10470.9 |
| 19 | 8 | 30 | 13432.5 | 14504.4 | 15848.8 | 13427.1 | 14488.8 | 16081.3 | 13426.1 | 14482.5 | 16026.3 |
| 20 | 9 | 20 | 6425.88 | 7071.65 | 7757.27 | 6422.39 | 7081.23 | 7875.12 | 6423.91 | 7069.36 | 7813.06 |
| 21 | 9 | 25 | 8429.87 | 9200.92 | 10991.2 | 8425.73 | 9201.73 | 10986.4 | 8422.63 | 9213.19 | 10983.6 |
| 22 | 9 | 30 | 12861.9 | 13989.2 | 15486.4 | 12854.8 | 13942.9 | 15155.4 | 12854.7 | 13932.4 | 14988.1 |
| 23 | 10 | 20 | 6316.47 | 6940.13 | 7805.69 | 6312.95 | 6919.39 | 7799.45 | 6311.91 | 6910.27 | 7798.56 |
| 24 | 10 | 25 | 8366.9 | 8865.29 | 9674.03 | 8361.79 | 8868.92 | 9870.79 | 8359.13 | 8867.04 | 9869.88 |
| 25 | 10 | 30 | 12870.8 | 13363.1 | 14636.6 | 12865 | 13348 | 14632 | 12864.1 | 13363.4 | 14630.7 |
| 26 | 2 | 4 | 303.971 | 304.067 | 304.069 | 304.03 | 304.118 | 304.149 | 303.893 | 303.937 | 303.951 |
| 27 | 2 | 4 | 284.53 | 307.837 | 403.66 | 284.78 | 308.042 | 403.609 | 284.528 | 307.949 | 403.18 |
| 28 | 3 | 5 | 100.192 | 108.904 | 130.792 | 99.9862 | 108.688 | 130.671 | 100.018 | 108.81 | 130.659 |
| 29 | 3 | 5 | 354.301 | 481.746 | 676.58 | 354.187 | 490.212 | 676.595 | 354.165 | 490.238 | 676.499 |
| 30 | 3 | 9 | 653.601 | 770.421 | 962.062 | 652.624 | 770.685 | 961.918 | 652.439 | 769.667 | 961.671 |
| 31 | 3 | 9 | 1923.54 | 2114.3 | 2227.22 | 1922.77 | 2112.98 | 2226.86 | 1922.07 | 2112.65 | 2225.95 |
| 32 | 4 | 8 | 1353.53 | 1447.99 | 1624.93 | 1353.22 | 1450.58 | 1624.3 | 1353.01 | 1451.84 | 1648.07 |
| 33 | 5 | 15 | 12538.9 | 14930.7 | 18075.5 | 12531.8 | 14958.3 | 18066.7 | 12529.2 | 14928.4 | 18065.9 |
| 34 | 5 | 20 | 18605.7 | 24076.9 | 32044.7 | 18601.5 | 23946.4 | 32040.8 | 18600.9 | 24080.1 | 32041.2 |
| 35 | 5 | 20 | 2237.36 | 2744.39 | 4446.59 | 2237.27 | 2741.08 | 4444.21 | 2236.67 | 2752.43 | 4444.03 |
| 36 | 5 | 30 | 33355.9 | 35600.9 | 43497.6 | 33347.7 | 35814.4 | 43450.5 | 33347.7 | 35759.1 | 43450.9 |
| 37 | 5 | 10 | 5684.94 | 7045.24 | 9878.54 | 5682.15 | 7034.89 | 9847.62 | 5681.63 | 7040.1 | 9876.01 |
| 38 | 6 | 10 | 11770.1 | 14791.5 | 19242 | 11762.9 | 14825 | 19240.2 | 11763.5 | 14832.6 | 19240.2 |
| 39 | 7 | 10 | 10695.3 | 13891.5 | 17826.2 | 10689.4 | 13910.4 | 17818.4 | 10690 | 13908.8 | 17820.7 |
| 40 | 8 | 10 | 4880.08 | 6257.11 | 10366 | 4878.08 | 6230.58 | 10361.5 | 4877.16 | 6237.56 | 10360.7 |
| 41 | 9 | 10 | 8120.31 | 9664.03 | 14229.7 | 8116.57 | 9734.33 | 14222.8 | 8115.34 | 9726.12 | 14219.6 |

Continued on Next Page...

| | | | 250 SAMPLE | | | 500 SAMPLE | | | 1000 SAMPLE | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| **P** | **m** | **n** | **Min** | **Average** | **Max** | **Min** | **Average** | **Max** | **Min** | **Average** | **Max** |
| 42 | 10 | 10 | 11648.2 | 15463.4 | 21451.2 | 11643 | 15484.9 | 21448.8 | 11642 | 15369.3 | 21447.9 |
| **Average** | | | **12978.9** | **14318.8** | **16736.4** | **12977.5** | **14312.6** | **16688.3** | **12975.9** | **14311.1** | **16702.9** |

## 6.3. Solution of PCMFWP With Discrete Approximations

### 6.3.1. Computational Setup

We have introduced three discrete approximations for solving the PCMFWP in Section 5.2: *Cellular heuristic*, *Grids on Means heuristic* and *Mean DLAP heuristic*. Prior to this, we have also proposed several methods to calculate expected distances. Hence, each of these discrete approximation methods can be adopted to solve PCMFWP with different distance norms and probability distribution assumptions.

In order to examine the quality of the discrete approximations, we use the same test bed and computational environment; they are introduced in Section 6.2.1 and Section 6.2.2. As stated in Section 6.2.1, the data for problems 1-25 are generated during this research. However, data for problems 26-36 are taken from Al-Loughani (1997) and data for problems 37-42 are obtained from Sherali *et al.* (2002).

Since discrete approximations require candidate points found with PALA runs, the same softwares which are introduced in Section 6.2.2 are used by default. Discrete approximations also includes a mixed integer programming problem (APCMFWP). In order to solve APCMFWP, concert technology of the ILOG CPLEX (2006) is used. In addition, the run times of the APCMFWP can be very long depending on the behaviour of the mixed integer problem. For this reason, we set a time limit, 7200 seconds, for solving the APCMFWP and reported best solution found.

### 6.3.2. Results

In order to analyze the discrete approximations, we perform the experiments for only the Euclidean distance - symmetric bivariate normal distribution case. Figure 6.10 illustrates the experiments which have been performed. Moreover, the results of these experiments are given in Table 6.16 to Table 6.21. Table 6.16 is on the Cellular heuristic, Table 6.18 shows the results of the Grids On Means heuristic and Table 6.20 reports the results of the Mean DLAP heuristic. In addition, CPU times are also listed

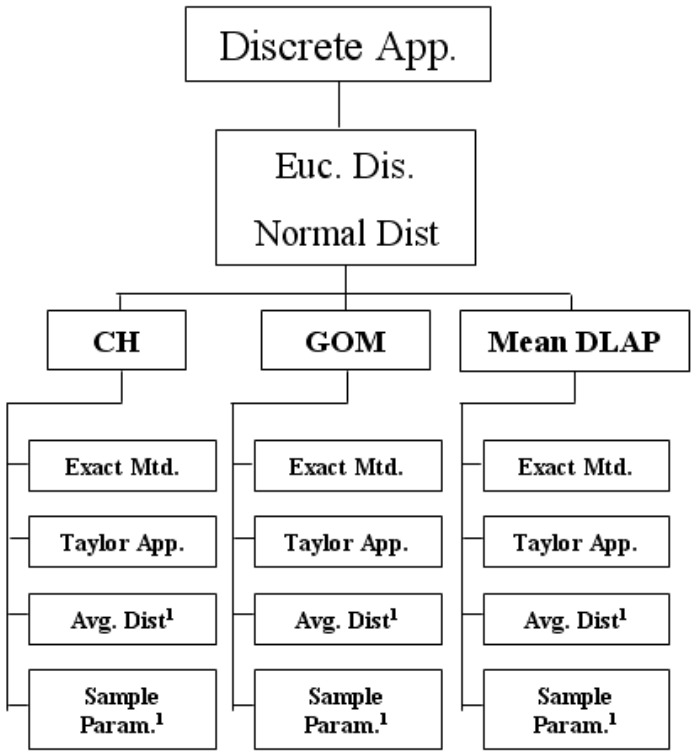


Figure 6.10. Experiments for the discrete approximations

in Table 6.17, Table 6.19 and Table 6.21. It is necessary to state that the number of PALA runs for producing candidate points in each discrete approximation is 100. Besides, in Cellular heuristic, we construct 25 cells ($h = 5$, $v = 5$) by dividing the smallest rectangle which covers all candidate points. Moreover, each discrete approximation is solved with different expected distance calculation methods but the final objective values are evaluated with the exact expected distance function for comparing the results. As anticipated, these methods are the exact calculation method, the Taylor approximation by using the known distribution parameters, the Taylor approximation method by using the estimations of the means and the standard deviations of the given sample and the average distance to sample method. Respectively, headings of these methods in each table are "Exact Mtd", "Taylor App.", "Sample Param." and "Avg. Dist. Mtd.". Now, let us interpret the results of the computational experiments.

In Table 6.16, it is seen that some problems could not be solved to optimaly owing to the time limits on APCMFWP. When the time limit is over, the solver terminates the solution procedure and reports the best integer solution found. Therefore, these problems are omitted from the column averages. When we compare the column

averages of Table 6.16, it is obvious that the exact solution method has the minimum value. Moreover, average distance to sample method follows the exact method and the relative deviation between these two methods is 0.027 per cent in the average. As it is emphasized many times, this small gap is not a surprising since average distance is a succesful estimator of the expected distance. On the other hand, average relative percent deviations of the Taylor approximations with exact and estimated parameters from the exact method are 2.05 per cent and 5.319 per cent respectively. Remark that, Taylor approximation with estimated parameters has considerable deviation from the exact method. On the other hand, Taylor approximation with exact parameters has nearly one third deviation of the Taylor approximation with estimated sample parameters. From the computation times point of view, Taylor approximation with estimated parameters has the minimum average CPU time which can be seen from Table 6.17. Additionaly, exact method and its variant using average distance have nearly the same CPU times on the average and the Taylor approximation with exact parameters has the highest average.

It is conceivable that the number of optimally solved problems is decreasing in Grids on Means heuristic method since the number of cells are increasing. Obviously, when the number of cells are increasing, it is expected to obtain more candidate points. Moreover, Figure 5.1 and 5.2 illustrate this perception. Therefore, the run time of the APCMFWP is increasing. In Table 6.18 and 6.19, it is seen that 16 problems are not optimaly solved by the Grids on Means heuristic. When the column averages of the optimally solved problems are compared, it is again seen that the exact method has the minimum average value. It is followed by the average distance to sample method with 0.179 per cent relative deviation. Besides, the relative per cent deviations of the Taylor approximations with exact estimated sample parameters are respectively 2.949 and 6.859. Apparently, the percent deviation of Taylor approximation with estimated parameters is considerably high. For the computation times, based on the results presented in Table 6.19. It is possible to say that Taylor approximation with estimated parameters and average distances result in shorth CPU times. On the contrary, the use of exact method and the Taylor approximation with exact parameters have very long computation times compared to other methods.

In Mean DLAP heuristic, the number of the optimally solved problems is higher than the other methods since the number of candidate points is lower than the others. It can be verified from Table 6.20 that only six of the 42 problems could not be solved optimally. When the column averages of this table is examined, it is again possible to observe that the exact method and the average distance to sample method have the lowest average objective values. Similar to other discrete approximations, the average relative per cent deviation between these two methods is 0.0228 per cent; they are almost equally accurate. Moreover, Taylor approximations with exact parameters and estimated parameters have respectively 2.391 and 5.255 per cent average relative deviations from the exact method. Notice that, deviations from the exact method are parallel in all discrete approximations. In addition, when the computation times are inspected from Table 6.21, it is clear that Taylor approximation with estimated parameters is more efficient than the other methods. On the other hand, the average CPU times of the remaining methods are close to each other. Here, it is convenient to state that running times are strongly depending on the solution time of the APCMFWP. Since APCMFWP is a mixed integer problem, its computation time is not only depending on the size of it but also depending on the structure of the problem. Hence, it is not surprising to have different computation times for the same problem.

For a better comparison of the approximation based on their effect on the optimal solution, some of the test problems should be discarded; which are not solved optimally. These are problems 3, 5-10, 13, 18-25. It is observed that the results of the three approximations are very close to each other. Moreover, in some problems, Grids on Means heuristic seems to have lower objective values than the ones of other two approximations.

Table 6.16: Cellular heuristic: Euclidean distance and normal distribution

| P | m | n | Exact Mtd. | Taylor App. | Sample Param. | Avg. Dist. Mtd. |
|---|---|---|---|---|---|---|
| 1 | 5 | 20 | 8355.90 | 8390.53 | 8385.59 | 8349.77 |
| 2 | 5 | 25 | 10318.20 | 10459.90 | 10541.80 | 10316.60 |
| 3 | 5 | 30 | 15818.00 | 15931.70 | 16112.80 | 15828.00 |
| 4 | 5 | 40 | 22918.60 | 23183.80 | 23347.00 | 22913.00 |
| 5 | 5 | 50 | 21317.30 | 21390.90 | 21628.10 | 21318.90 |
| 6 | 5 | 100 | 35846.60[1] | 35790.60[1] | 36469.90[1] | 35866.10[1] |
| 7 | 5 | 100 | 38530.60[1] | 39857.90[1] | 39891.00[1] | 38358.00[1] |
| 8 | 5 | 100 | 42064.00[1] | 42637.70[1] | 42514.60[1] | 41724.20[1] |
| 9 | 5 | 100 | 40232.80[1] | 40243.60[1] | 40482.50[1] | 40243.30[1] |
| 10 | 5 | 100 | 40719.00[1] | 40773.10[1] | 41938.80[1] | 40708.50[1] |
| 11 | 6 | 20 | 7427.65 | 7701.86 | 7550.01 | 7426.45 |
| 12 | 6 | 25 | 9187.40 | 9636.15 | 9466.19 | 9191.07 |
| 13 | 6 | 30 | 14706.10 | 14882.70 | 14759.20 | 14638.40 |
| 14 | 7 | 20 | 6738.33 | 6870.30 | 6926.47 | 6737.89 |
| 15 | 7 | 25 | 8830.59 | 9299.49 | 9197.54 | 8840.24 |
| 16 | 7 | 30 | 13733.70 | 13905.10 | 14118.70 | 13748.80 |
| 17 | 8 | 20 | 6350.03 | 6502.54 | 6492.75 | 6355.68 |
| 18 | 8 | 25 | 8442.95 | 8902.12 | 8705.83 | 8480.27 |
| 19 | 8 | 30 | 13393.30 | 13479.60 | 13866.70 | 13405.50 |
| 20 | 9 | 20 | 6109.26 | 6190.60 | 6325.98 | 6117.59 |
| 21 | 9 | 25 | 8187.85 | 8487.96[1] | 8402.78 | 8200.61 |
| 22 | 9 | 30 | 12845.50 | 13321.60 | 13378.60 | 12821.30 |
| 23 | 10 | 20 | 5914.27 | 5999.12 | 6151.86 | 5923.62 |
| 24 | 10 | 25 | 8152.50[1] | 8631.10[1] | 8387.67 | 8211.07[1] |
| 25 | 10 | 30 | 12525.70 | 13328.30[1] | 13205.40 | 12510.60 |
| 26 | 2 | 4 | 304.74 | 304.06 | 304.35 | 303.88 |
| 27 | 2 | 4 | 285.75 | 288.53 | 298.19 | 295.19 |
| 28 | 3 | 5 | 99.90 | 102.98 | 102.48 | 99.89 |
| 29 | 3 | 5 | 354.36 | 380.03 | 501.40 | 354.92 |
| 30 | 3 | 9 | 652.52 | 656.36 | 691.63 | 654.92 |
| 31 | 3 | 9 | 1922.59 | 1952.54 | 1958.72 | 1924.68 |
| 32 | 4 | 8 | 1370.67 | 1406.08 | 1440.18 | 1378.76 |
| 33 | 5 | 15 | 12567.80 | 12818.10 | 13545.60 | 12701.20 |
| 34 | 5 | 20 | 18603.40 | 19798.80 | 19433.50 | 18613.20 |
| 35 | 5 | 20 | 2240.40 | 2264.79 | 2768.70 | 2239.24 |
| 36 | 5 | 30 | 33364.00 | 33606.60 | 33800.10 | 33356.40 |
| 37 | 5 | 10 | 5683.50 | 5823.43 | 6435.47 | 5691.94 |
| 38 | 6 | 10 | 11810.10 | 12014.20 | 14425.70 | 11801.90 |
| 39 | 7 | 10 | 10794.40 | 10994.60 | 10833.40 | 10700.30 |
| 40 | 8 | 10 | 4883.12 | 5041.96 | 7363.76 | 4882.72 |
| 41 | 9 | 10 | 8194.16 | 8476.97 | 10742.00 | 8204.17 |
| 42 | 10 | 10 | 11644.80 | 11710.00 | 12454.30 | 11652.60 |
| **Average** | | | **9328.9** | **9520.2** | **9825.1** | **9331.4** |

Table 6.17: CPU times: Cellular heuristic

| P | m | n | Exact Mtd. | Taylor App. | Sample Param. | Avg. Dist. Mtd. |
|---|---|---|---|---|---|---|
| 1 | 5 | 20 | 6.92 | 4.83 | 4.53 | 6.19 |
| 2 | 5 | 25 | 10.23 | 8.70 | 6.52 | 8.14 |
| 3 | 5 | 30 | 282.88 | 38.39 | 63.31 | 275.63 |
| 4 | 5 | 40 | 44.66 | 30.72 | 35.30 | 35.19 |
| 5 | 5 | 50 | 345.59 | 903.27 | 432.45 | 413.09 |
| 6 | 5 | 100 | 7252.17[1] | 7262.24[1] | 7266.23[1] | 7346.89[1] |
| 7 | 5 | 100 | 7264.84[1] | 7294.53[1] | 7230.44[1] | 7223.04[1] |
| 8 | 5 | 100 | 7291.34[1] | 7243.13[1] | 7272.00[1] | 7258.99[1] |
| 9 | 5 | 100 | 7275.71[1] | 7277.14[1] | 7236.03[1] | 7267.88[1] |
| 10 | 5 | 100 | 7251.80[1] | 7278.64[1] | 7328.54[1] | 7249.32[1] |
| 11 | 6 | 20 | 7.02 | 4.98 | 6.23 | 7.53 |
| 12 | 6 | 25 | 29.28 | 23.36 | 22.48 | 34.64 |
| 13 | 6 | 30 | 51.23 | 51.91 | 43.00 | 122.65 |
| 14 | 7 | 20 | 10.61 | 10.11 | 9.28 | 6.48 |
| 15 | 7 | 25 | 32.81 | 31.94 | 26.87 | 32.67 |
| 16 | 7 | 30 | 55.26 | 65.05 | 25.23 | 45.25 |
| 17 | 8 | 20 | 10.75 | 11.94 | 9.95 | 11.97 |
| 18 | 8 | 25 | 116.20 | 548.23 | 34.03 | 115.04 |
| 19 | 8 | 30 | 159.03 | 211.13 | 62.25 | 119.83 |
| 20 | 9 | 20 | 42.87 | 53.44 | 18.13 | 24.64 |
| 21 | 9 | 25 | 3148.98 | 7242.33[1] | 184.18 | 5381.21 |
| 22 | 9 | 30 | 190.01 | 117.52 | 128.00 | 280.32 |
| 23 | 10 | 20 | 41.42 | 477.99 | 19.90 | 29.97 |
| 24 | 10 | 25 | 7403.53[1] | 7373.71[1] | 91.36 | 7565.12[1] |
| 25 | 10 | 30 | 4938.83 | 7449.47[1] | 179.75 | 6351.14 |
| 26 | 2 | 4 | 0.49 | 0.49 | 0.55 | 0.34 |
| 27 | 2 | 4 | 0.41 | 0.38 | 0.33 | 0.44 |
| 28 | 3 | 5 | 0.39 | 0.34 | 0.33 | 0.51 |
| 29 | 3 | 5 | 0.55 | 0.66 | 0.48 | 0.66 |
| 30 | 3 | 9 | 0.63 | 0.77 | 0.50 | 0.72 |
| 31 | 3 | 9 | 0.67 | 0.56 | 0.44 | 0.73 |
| 32 | 4 | 8 | 0.73 | 0.55 | 0.62 | 0.81 |
| 33 | 5 | 15 | 3.36 | 1.73 | 1.39 | 1.92 |
| 34 | 5 | 20 | 3.00 | 2.39 | 2.65 | 2.46 |
| 35 | 5 | 20 | 4.27 | 3.94 | 2.62 | 3.52 |
| 36 | 5 | 30 | 10.94 | 10.20 | 7.51 | 10.33 |
| 37 | 5 | 10 | 1.17 | 0.95 | 0.83 | 1.09 |
| 38 | 6 | 10 | 1.69 | 1.28 | 1.53 | 1.45 |
| 39 | 7 | 10 | 1.94 | 1.72 | 1.67 | 1.82 |
| 40 | 8 | 10 | 1.78 | 1.39 | 1.76 | 1.56 |
| 41 | 9 | 10 | 2.45 | 1.87 | 2.91 | 2.35 |
| 42 | 10 | 10 | 2.11 | 1.75 | 1.68 | 1.89 |

Continued on Next Page. . .

[1]This problem could not be solved optimaly within 2 hours and omitted from the average.

| P | m | n | Exact Mtd. | Taylor App. | Sample Param. | Avg. Dist. Mtd. |
|---|---|---|---|---|---|---|
| **Average** | | | **43.3** | **77.2** | **28.7** | **47.1** |

Table 6.18: Grids on Means heuristic: Euclidean distance and normal distribution

| P | m | n | Exact Mtd. | Taylor App. | Sample Param. | Avg. Dist. Mtd. |
|---|---|---|---|---|---|---|
| 1 | 5 | 20 | 8354.57 | 8390.53 | 8385.59 | 8349.77 |
| 2 | 5 | 25 | 10318.30 | 10459.90 | 10541.80 | 10316.60 |
| 3 | 5 | 30 | 15820.00[1] | 15873.00 | 16112.80 | 15828.10 |
| 4 | 5 | 40 | 22916.60 | 23127.90 | 23347.00 | 22913.00 |
| 5 | 5 | 50 | 21309.50[1] | 21390.90[1] | 21628.10 | 21318.90 |
| 6 | 5 | 100 | 35770.00[1] | 35842.90[1] | 36113.30[1] | 35836.40[1] |
| 7 | 5 | 100 | 38287.60[1] | 38634.50[1] | 39145.10[1] | 38370.60[1] |
| 8 | 5 | 100 | 41884.10[1] | 41905.60[1] | 42473.70[1] | 41586.20[1] |
| 9 | 5 | 100 | 40086.30[1] | 40373.30[1] | 41332.20[1] | 40273.60[1] |
| 10 | 5 | 100 | 41813.40[1] | 41136.10[1] | 41862.40[1] | 41145.50[1] |
| 11 | 6 | 20 | 7427.50 | 7492.39 | 7550.01 | 7426.45 |
| 12 | 6 | 25 | 9186.99 | 9544.91 | 9466.19 | 9191.10 |
| 13 | 6 | 30 | 14581.70[1] | 14650.90 | 14759.20 | 14638.50 |
| 14 | 7 | 20 | 6737.74 | 6874.13 | 6926.47 | 6737.89 |
| 15 | 7 | 25 | 8829.84 | 9118.88 | 9197.54 | 8840.24 |
| 16 | 7 | 30 | 13684.20 | 13913.90 | 14118.70 | 13748.80 |
| 17 | 8 | 20 | 6350.00 | 6457.86 | 6492.75 | 6355.68 |
| 18 | 8 | 25 | 8423.77[1] | 8715.71 | 8705.83 | 8480.27 |
| 19 | 8 | 30 | 13320.50[1] | 13570.10[1] | 13866.70 | 13405.80 |
| 20 | 9 | 20 | 6109.17[1] | 6329.46 | 6325.98 | 6117.59 |
| 21 | 9 | 25 | 8180.77[1] | 8393.07[1] | 8402.78 | 8200.61 |
| 22 | 9 | 30 | 12725.80[1] | 13048.30 | 13378.60 | 12821.30 |
| 23 | 10 | 20 | 5914.26[1] | 6049.10[1] | 6151.86 | 5923.62 |
| 24 | 10 | 25 | 8162.81[1] | 8610.28[1] | 8387.67 | 8211.07[1] |
| 25 | 10 | 30 | 12520.70[1] | 12960.90[1] | 13205.40 | 12510.60 |
| 26 | 2 | 4 | 304.74 | 304.06 | 304.35 | 303.88 |
| 27 | 2 | 4 | 284.96 | 288.45 | 298.19 | 295.24 |
| 28 | 3 | 5 | 99.90 | 106.15 | 298.19 | 99.89 |
| 29 | 3 | 5 | 354.10 | 380.03 | 501.40 | 354.92 |
| 30 | 3 | 9 | 652.51 | 718.38 | 691.63 | 654.92 |
| 31 | 3 | 9 | 1922.59 | 1952.54 | 1958.72 | 1924.68 |
| 32 | 4 | 8 | 1354.30 | 1393.32 | 1440.18 | 1378.76 |
| 33 | 5 | 15 | 12528.60 | 12918.60 | 13545.60 | 12701.20 |
| 34 | 5 | 20 | 18602.90 | 19523.70 | 19433.50 | 18613.70 |
| 35 | 5 | 20 | 2239.99 | 2264.79 | 2768.70 | 2239.24 |
| 36 | 5 | 30 | 33242.70 | 34978.00 | 33800.10 | 33356.40 |
| 37 | 5 | 10 | 5683.32 | 5967.31 | 6435.47 | 5691.94 |

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[1]This problem could not be solved optimaly within 2 hours and omitted from the average.

| P | m | n | Exact Mtd. | Taylor App. | Sample Param. | Avg. Dist. Mtd. |
|---|---|---|---|---|---|---|
| 38 | 6 | 10 | 11779.40 | 12000.00 | 14425.70 | 11801.90 |
| 39 | 7 | 10 | 10766.00 | 10994.60 | 10833.40 | 10700.30 |
| 40 | 8 | 10 | 4882.96 | 5023.75 | 7363.76 | 4882.72 |
| 41 | 9 | 10 | 8194.09 | 8745.08 | 10742.00 | 8204.17 |
| 42 | 10 | 10 | 11644.70 | 11844.10 | 12454.30 | 11652.60 |
| **Average** | | | **8397.8** | **8645.5** | **8973.9** | **8412.9** |

Table 6.19: CPU times: Grids on Means heuristic

| P | m | n | Exact Mtd. | Taylor App. | Sample Param. | Avg. Dist. Mtd. |
|---|---|---|---|---|---|---|
| 1 | 5 | 20 | 97.21 | 51.38 | 2.25 | 2.82 |
| 2 | 5 | 25 | 69.09 | 53.17 | 3.26 | 3.24 |
| 3 | 5 | 30 | 7271.86[1] | 815.77 | 72.24 | 57.27 |
| 4 | 5 | 40 | 910.06 | 560.76 | 14.51 | 13.83 |
| 5 | 5 | 50 | 7251.83[1] | 7227.14[1] | 206.51 | 188.28 |
| 6 | 5 | 100 | 7309.35[1] | 7328.75[1] | 7249.62[1] | 7244.35[1] |
| 7 | 5 | 100 | 7276.94[1] | 7274.74[1] | 7252.70[1] | 7226.06[1] |
| 8 | 5 | 100 | 7309.29[1] | 7277.62[1] | 7243.49[1] | 7265.93[1] |
| 9 | 5 | 100 | 7256.32[1] | 7282.08[1] | 7248.57[1] | 7283.71[1] |
| 10 | 5 | 100 | 7268.02[1] | 7268.58[1] | 7278.44[1] | 7246.08[1] |
| 11 | 6 | 20 | 93.61 | 46.09 | 2.52 | 3.70 |
| 12 | 6 | 25 | 633.91 | 149.93 | 7.06 | 8.66 |
| 13 | 6 | 30 | 7244.64[1] | 864.20 | 16.69 | 42.85 |
| 14 | 7 | 20 | 241.68 | 94.06 | 4.32 | 4.87 |
| 15 | 7 | 25 | 1879.31 | 461.00 | 13.09 | 9.45 |
| 16 | 7 | 30 | 2985.98 | 1163.58 | 10.29 | 18.34 |
| 17 | 8 | 20 | 553.13 | 167.45 | 5.63 | 8.26 |
| 18 | 8 | 25 | 7251.59[1] | 542.61 | 16.96 | 29.79 |
| 19 | 8 | 30 | 7263.88[1] | 7202.75[1] | 23.67 | 44.31 |
| 20 | 9 | 20 | 7275.95[1] | 2413.67 | 8.49 | 16.19 |
| 21 | 9 | 25 | 7272.23[1] | 7274.10[1] | 102.06 | 907.75 |
| 22 | 9 | 30 | 7219.25[1] | 2037.90 | 52.64 | 93.75 |
| 23 | 10 | 20 | 7336.63[1] | 7205.38[1] | 10.75 | 20.95 |
| 24 | 10 | 25 | 7354.21[1] | 7339.67[1] | 55.15 | 7408.33[1] |
| 25 | 10 | 30 | 7244.73[1] | 7283.95[1] | 82.68 | 2128.60 |
| 26 | 2 | 4 | 0.48 | 0.48 | 0.45 | 0.33 |
| 27 | 2 | 4 | 0.37 | 0.38 | 0.33 | 0.41 |
| 28 | 3 | 5 | 0.44 | 0.38 | 0.33 | 0.36 |
| 29 | 3 | 5 | 0.61 | 0.47 | 0.50 | 0.52 |
| 30 | 3 | 9 | 0.81 | 0.78 | 0.45 | 0.46 |
| 31 | 3 | 9 | 0.76 | 0.69 | 0.37 | 0.57 |
| 32 | 4 | 8 | 0.80 | 0.61 | 0.58 | 0.69 |
| 33 | 5 | 15 | 7.47 | 4.59 | 1.20 | 1.55 |

Continued on Next Page...

[1]This problem could not be solved optimaly within 2 hours and omitted from the average.

| P | m | n | Exact Mtd. | Taylor App. | Sample Param. | Avg. Dist. Mtd. |
|---|---|---|---|---|---|---|
| 34 | 5 | 20 | 5.70 | 4.80 | 1.98 | 1.47 |
| 35 | 5 | 20 | 7.26 | 4.84 | 1.62 | 2.13 |
| 36 | 5 | 30 | 54.36 | 44.01 | 4.76 | 6.17 |
| 37 | 5 | 10 | 1.64 | 1.36 | 0.75 | 0.93 |
| 38 | 6 | 10 | 2.32 | 1.77 | 1.64 | 1.32 |
| 39 | 7 | 10 | 3.54 | 2.89 | 1.65 | 1.60 |
| 40 | 8 | 10 | 2.98 | 1.55 | 2.01 | 1.49 |
| 41 | 9 | 10 | 5.46 | 4.53 | 3.17 | 2.01 |
| 42 | 10 | 10 | 2.88 | 2.16 | 1.65 | 1.71 |
| **Average** | | | **290.8** | **108.6** | **3.3** | **3.7** |

Table 6.20: Mean DLAP heuristic: Euclidean distance and normal distribution

| P | m | n | Exact Mtd. | Taylor App. | Sample Param. | Avg. Dist. Mtd. |
|---|---|---|---|---|---|---|
| 1 | 5 | 20 | 8355.90 | 8390.53 | 8385.59 | 8349.77 |
| 2 | 5 | 25 | 10318.20 | 10477.50 | 10541.80 | 10316.60 |
| 3 | 5 | 30 | 15818.00 | 15931.70 | 16112.80 | 15828.10 |
| 4 | 5 | 40 | 22918.60 | 23183.80 | 23347.00 | 22913.00 |
| 5 | 5 | 50 | 21317.30 | 21390.90 | 21628.10 | 21318.90 |
| 6 | 5 | 100 | 35814.80[1] | 35857.10[1] | 36113.30[1] | 35836.40[1] |
| 7 | 5 | 100 | 38303.00[1] | 38743.00[1] | 39145.10[1] | 38370.60[1] |
| 8 | 5 | 100 | 41799.50[1] | 42603.20[1] | 42473.70[1] | 41586.20[1] |
| 9 | 5 | 100 | 40232.80[1] | 40371.20[1] | 41332.20[1] | 40273.60[1] |
| 10 | 5 | 100 | 40719.00[1] | 41679.70[1] | 41862.40[1] | 41134.20[1] |
| 11 | 6 | 20 | 7427.65 | 7701.86 | 7550.01 | 7426.45 |
| 12 | 6 | 25 | 9187.57 | 9636.15 | 9466.19 | 9191.10 |
| 13 | 6 | 30 | 14706.10 | 14694.30 | 14759.20 | 14638.50 |
| 14 | 7 | 20 | 6738.33 | 6870.30 | 6926.47 | 6737.89 |
| 15 | 7 | 25 | 8830.59 | 9299.49 | 9197.54 | 8840.24 |
| 16 | 7 | 30 | 13733.70 | 13866.50 | 14118.70 | 13748.80 |
| 17 | 8 | 20 | 6350.03 | 6502.54 | 6492.75 | 6355.68 |
| 18 | 8 | 25 | 8442.95 | 9090.01 | 8705.83 | 8480.27 |
| 19 | 8 | 30 | 13393.30 | 13479.60 | 13866.70 | 13405.80 |
| 20 | 9 | 20 | 6109.26 | 6190.60 | 6325.98 | 6117.59 |
| 21 | 9 | 25 | 8187.85 | 8487.96 | 8402.78 | 8200.61 |
| 22 | 9 | 30 | 12845.50 | 13493.60 | 13378.60 | 12821.30 |
| 23 | 10 | 20 | 5914.27 | 5999.12 | 6151.86 | 5923.62 |
| 24 | 10 | 25 | 8152.50[1] | 8362.66[1] | 8387.67 | 8211.07[1] |
| 25 | 10 | 30 | 12525.70 | 13495.40 | 13205.40 | 12510.60 |
| 26 | 2 | 4 | 304.66 | 304.06 | 304.35 | 303.88 |
| 27 | 2 | 4 | 285.75 | 288.53 | 298.19 | 295.26 |
| 28 | 3 | 5 | 99.90 | 102.98 | 102.48 | 99.89 |
| 29 | 3 | 5 | 354.36 | 380.03 | 501.40 | 354.92 |

Continued on Next Page...

[1]This problem could not be solved optimaly within 2 hours and omitted from the average.

| P | m | n | Exact Mtd. | Taylor App. | Sample Param. | Avg. Dist. Mtd. |
|---|---|---|---|---|---|---|
| 30 | 3 | 9 | 652.52 | 656.36 | 691.63 | 654.92 |
| 31 | 3 | 9 | 1922.59 | 1952.54 | 1958.72 | 1924.68 |
| 32 | 4 | 8 | 1370.67 | 1406.08 | 1440.18 | 1378.76 |
| 33 | 5 | 15 | 12567.80 | 12983.70 | 13545.60 | 12701.20 |
| 34 | 5 | 20 | 18611.50 | 19798.80 | 19433.50 | 18614.50 |
| 35 | 5 | 20 | 2240.40 | 2264.79 | 2768.70 | 2239.24 |
| 36 | 5 | 30 | 33364.00 | 33606.60 | 33800.10 | 33356.40 |
| 37 | 5 | 10 | 5683.50 | 5823.43 | 6435.47 | 5691.94 |
| 38 | 6 | 10 | 11810.10 | 12014.20 | 14425.70 | 11801.90 |
| 39 | 7 | 10 | 10794.40 | 10994.60 | 10833.40 | 10700.30 |
| 40 | 8 | 10 | 4883.12 | 5041.96 | 7363.76 | 4882.72 |
| 41 | 9 | 10 | 8194.16 | 8476.97 | 10742.00 | 8204.17 |
| 42 | 10 | 10 | 11644.80 | 11710.00 | 12454.30 | 11652.60 |
| **Average** | | | **9386.3** | **9610.8** | **9879.5** | **9388.4** |

Table 6.21: CPU times: Mean DLAP heuristic

| P | m | n | Exact Mtd. | Taylor App. | Sample Param. | Avg. Dist. Mtd. |
|---|---|---|---|---|---|---|
| 1 | 5 | 20 | 1.56 | 1.59 | 0.70 | 1.19 |
| 2 | 5 | 25 | 1.69 | 2.34 | 2.48 | 1.58 |
| 3 | 5 | 30 | 64.11 | 47.62 | 70.93 | 70.11 |
| 4 | 5 | 40 | 14.14 | 8.65 | 12.55 | 10.95 |
| 5 | 5 | 50 | 134.98 | 368.42 | 203.61 | 184.59 |
| 6[1] | 5 | 100 | 7258.90[1] | 7243.23[1] | 7242.26[1] | 7251.27[1] |
| 7[1] | 5 | 100 | 7242.47[1] | 7269.22[1] | 7245.65[1] | 7248.17[1] |
| 8[1] | 5 | 100 | 7220.54[1] | 7246.98[1] | 7280.51[1] | 7255.75[1] |
| 9[1] | 5 | 100 | 7375.00[1] | 7252.73[1] | 7253.08[1] | 7254.85[1] |
| 10[1] | 5 | 100 | 7234.95[1] | 7239.63[1] | 7251.97[1] | 7255.99[1] |
| 11 | 6 | 20 | 2.12 | 1.42 | 1.59 | 1.56 |
| 12 | 6 | 25 | 6.44 | 9.11 | 5.66 | 6.64 |
| 13 | 6 | 30 | 18.36 | 11.41 | 15.33 | 32.23 |
| 14 | 7 | 20 | 2.98 | 1.99 | 3.11 | 2.37 |
| 15 | 7 | 25 | 8.58 | 3.36 | 8.41 | 7.28 |
| 16 | 7 | 30 | 13.50 | 20.22 | 8.41 | 9.80 |
| 17 | 8 | 20 | 4.55 | 3.19 | 2.95 | 4.89 |
| 18 | 8 | 25 | 18.87 | 31.53 | 14.15 | 27.37 |
| 19 | 8 | 30 | 55.93 | 53.72 | 21.40 | 32.35 |
| 20 | 9 | 20 | 11.28 | 27.19 | 7.45 | 9.34 |
| 21 | 9 | 25 | 788.27 | 1438.59 | 56.82 | 911.61 |
| 22 | 9 | 30 | 71.67 | 38.33 | 50.51 | 65.38 |
| 23 | 10 | 20 | 14.56 | 203.48 | 5.59 | 12.44 |
| 24 | 10 | 25 | 7409.96[1] | 7247.86[1] | 38.56 | 7450.71[1] |
| 25 | 10 | 30 | 1120.31 | 943.71 | 76.43 | 1888.31 |

Continued on Next Page...

[1]This problem could not be solved optimaly within 2 hours and omitted from the average.

| P | m | n | Exact Mtd. | Taylor App. | Sample Param. | Avg. Dist. Mtd. |
|---|---|---|---|---|---|---|
| 26 | 2 | 4 | 0.06 | 0.08 | 0.05 | 0.00 |
| 27 | 2 | 4 | 0.14 | 0.11 | 0.02 | 0.03 |
| 28 | 3 | 5 | 0.11 | 0.09 | 0.08 | 0.02 |
| 29 | 3 | 5 | 0.11 | 0.14 | 0.05 | 0.02 |
| 30 | 3 | 9 | 0.16 | 0.15 | 0.08 | 0.09 |
| 31 | 3 | 9 | 0.13 | 0.12 | 0.06 | 0.09 |
| 32 | 4 | 8 | 0.20 | 0.14 | 0.11 | 0.14 |
| 33 | 5 | 15 | 0.62 | 0.54 | 0.31 | 0.59 |
| 34 | 5 | 20 | 0.44 | 0.43 | 0.50 | 0.39 |
| 35 | 5 | 20 | 0.70 | 0.68 | 0.72 | 0.62 |
| 36 | 5 | 30 | 4.13 | 3.71 | 3.25 | 4.07 |
| 37 | 5 | 10 | 0.19 | 0.26 | 0.11 | 0.45 |
| 38 | 6 | 10 | 0.28 | 0.26 | 0.28 | 0.22 |
| 39 | 7 | 10 | 0.31 | 0.26 | 0.19 | 0.22 |
| 40 | 8 | 10 | 0.25 | 0.26 | 0.41 | 0.19 |
| 41 | 9 | 10 | 0.44 | 0.34 | 0.56 | 0.34 |
| 42 | 10 | 10 | 0.28 | 0.31 | 0.25 | 0.30 |
| **Average** | | | **65.6** | **89.5** | **16.0** | **91.3** |

### 6.3.3. Effects of the Final PALA Runs

The results of the discrete approximations are given in Table 6.16 to Table 6.21. While solving the problems with the discrete algorithms, namely Algorithm 2, Algorithm 3 and Algorithm 4, PALA is applied for the final improvements after APCMFWP is solved. However, we do not know the effects of these final PALA runs. In order to show the effects of these final PALA runs, we give the results of the discrete approximations before and after PALA algorithm is applied. In Table 6.22 to Table 6.24 results are given for all discrete approximations. Columns "B. PALA" include objective values before PALA runs and "A. PALA" report objective values after PALA runs. Average per cent impovements are also given at the last row of the tables. When the results are inspected, it is clear that the final PALA runs remarkably improve the objective functions for all cases. To illustrate, the overall average improvement for the Cellular heuristic is 3.032 per cent. This value becomes 2.387 per cent for the Grids on Means heuristic and 2.885 per cent for the Mean DLAP heuristic. Therefore, a final PALA run at the end of the discrete algorithms increases the quality of the results.

[1]This problem could not be solved optimaly within 2 hours and omitted from the average.

Table 6.22: Effects of final PALA runs: Cellular Heuristic

| | | | Exact Mtd. | | Taylor App. | | Sample Param. | | Avg. Dist. Mtd. | |
|---|---|---|---|---|---|---|---|---|---|---|
| **P** | **m** | **n** | **B.PALA** | **A. PALA** | **B.PALA** | **A. PALA** | **B.PALA** | **A. PALA** | **B.PALA** | **A. PALA** |
| 1 | 5 | 20 | 8733.81 | 8355.90 | 8822.85 | 8390.53 | 8719.04 | 8385.59 | 8732.84 | 8349.77 |
| 2 | 5 | 25 | 10562.90 | 10318.20 | 10906.10 | 10459.90 | 10567.70 | 10541.80 | 10569.00 | 10316.60 |
| 3 | 5 | 30 | 16267.00 | 15818.00 | 16374.00 | 15931.70 | 16725.60 | 16112.80 | 16275.30 | 15828.00 |
| 4 | 5 | 40 | 23307.90 | 22918.60 | 23342.70 | 23183.80 | 23347.00 | 23408.80 | 23329.90 | 22913.00 |
| 5 | 5 | 50 | 21791.90 | 21317.30 | 21925.30 | 21390.90 | 22090.10 | 21628.10 | 21790.90 | 21318.90 |
| 6 | 5 | 100 | 36241.40[1] | 35846.60[1] | 36442.00[1] | 35790.60[1] | 36469.90[1] | 36499.60[1] | 36176.00[1] | 35866.10[1] |
| 7 | 5 | 100 | 38975.50[1] | 38530.60[1] | 40219.00[1] | 39857.90[1] | 40256.00[1] | 39891.00[1] | 38694.00[1] | 38358.00[1] |
| 8 | 5 | 100 | 42457.60[1] | 42064.00[1] | 42867.60[1] | 42637.70[1] | 42742.90[1] | 42514.60[1] | 42621.40[1] | 41724.20[1] |
| 9 | 5 | 100 | 40549.00[1] | 40232.80[1] | 41483.40[1] | 40243.60[1] | 41224.80[1] | 40482.50[1] | 40529.40[1] | 40243.30[1] |
| 10 | 5 | 100 | 41125.30[1] | 40719.00[1] | 41544.70[1] | 40773.10[1] | 42042.30[1] | 41938.80[1] | 41229.20[1] | 40708.50[1] |
| 11 | 6 | 20 | 7685.06 | 7427.65 | 7892.63 | 7701.86 | 7687.51 | 7550.01 | 7648.89 | 7426.45 |
| 12 | 6 | 25 | 9449.85 | 9187.40 | 9963.58 | 9636.15 | 9698.25 | 9466.19 | 9492.53 | 9191.07 |
| 13 | 6 | 30 | 14997.10 | 14706.10 | 15500.80 | 14882.70 | 15203.70 | 14759.20 | 15132.20 | 14638.40 |
| 14 | 7 | 20 | 6933.94 | 6738.33 | 6986.86 | 6870.30 | 6926.76 | 6926.47 | 6926.76 | 6737.89 |
| 15 | 7 | 25 | 8981.59 | 8830.59 | 9722.49 | 9299.49 | 9197.54 | 9262.72 | 8997.95 | 8840.24 |
| 16 | 7 | 30 | 13987.00 | 13733.70 | 14435.00 | 13905.10 | 14163.20 | 14118.70 | 14009.40 | 13748.80 |
| 17 | 8 | 20 | 6475.31 | 6350.03 | 6529.88 | 6502.54 | 6492.75 | 6597.21 | 6492.75 | 6355.68 |
| 18 | 8 | 25 | 8603.65 | 8442.95 | 9281.95 | 8902.12 | 8705.83 | 8775.52 | 8651.92 | 8480.27 |
| 19 | 8 | 30 | 13624.90 | 13393.30 | 13911.70 | 13479.60 | 14016.60 | 13866.70 | 13622.20 | 13405.50 |
| 20 | 9 | 20 | 6210.31 | 6109.26 | 6210.58 | 6190.60 | 6325.98 | 6372.08 | 6228.08 | 6117.59 |
| 21 | 9 | 25 | 8377.24 | 8187.85 | 9065.02[1] | 8487.96[1] | 8402.78 | 8636.05 | 8366.06 | 8200.61 |
| 22 | 9 | 30 | 12977.90 | 12845.50 | 13902.70 | 13321.60 | 13687.90 | 13378.60 | 12958.10 | 12821.30 |
| 23 | 10 | 20 | 5960.91 | 5914.27 | 5999.12 | 6049.17 | 6151.86 | 6309.98 | 6001.20 | 5923.62 |
| 24 | 10 | 25 | 8286.36[1] | 8152.50[1] | 8951.62[1] | 8631.10[1] | 8387.67 | 8541.04 | 8299.36[1] | 8211.07[1] |
| 25 | 10 | 30 | 12748.50 | 12525.70 | 14000.40[1] | 13328.30[1] | 13461.00 | 13205.40 | 12753.40 | 12510.60 |

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| | | | Exact Mtd. | | Taylor App. | | Sample Param. | | Avg. Dist. Mtd. | |
|---|---|---|---|---|---|---|---|---|---|---|
| **P** | **m** | **n** | **B.PALA** | **A. PALA** | **B.PALA** | **A. PALA** | **B.PALA** | **A. PALA** | **B.PALA** | **A. PALA** |
| 26 | 2 | 4 | 313.73 | 304.74 | 313.73 | 304.06 | 313.41 | 304.35 | 313.41 | 303.88 |
| 27 | 2 | 4 | 300.28 | 285.75 | 300.28 | 288.53 | 301.81 | 298.19 | 300.33 | 295.19 |
| 28 | 3 | 5 | 102.98 | 99.90 | 102.98 | 106.15 | 102.48 | 108.00 | 102.48 | 99.89 |
| 29 | 3 | 5 | 357.62 | 354.36 | 423.62 | 380.03 | 559.32 | 501.40 | 360.06 | 354.92 |
| 30 | 3 | 9 | 656.36 | 652.52 | 656.36 | 686.78 | 691.63 | 711.80 | 660.27 | 654.92 |
| 31 | 3 | 9 | 1948.69 | 1922.59 | 2080.39 | 1952.54 | 2023.05 | 1958.72 | 1971.32 | 1924.68 |
| 32 | 4 | 8 | 1416.35 | 1370.67 | 1449.90 | 1406.08 | 1463.72 | 1440.18 | 1421.25 | 1378.76 |
| 33 | 5 | 15 | 13276.90 | 12567.80 | 14169.00 | 12818.10 | 13796.70 | 13545.60 | 13129.20 | 12701.20 |
| 34 | 5 | 20 | 18867.70 | 18603.40 | 20080.30 | 19798.80 | 23528.50 | 19433.50 | 18758.20 | 18613.20 |
| 35 | 5 | 20 | 2319.92 | 2240.40 | 2434.97 | 2264.79 | 2768.70 | 2791.92 | 2308.58 | 2239.24 |
| 36 | 5 | 30 | 33723.70 | 33364.00 | 34333.50 | 33606.60 | 40236.40 | 33800.10 | 33863.40 | 33356.40 |
| 37 | 5 | 10 | 5754.35 | 5683.50 | 6108.95 | 5823.43 | 6435.47 | 6703.02 | 5771.76 | 5691.94 |
| 38 | 6 | 10 | 12209.40 | 11810.10 | 12209.40 | 12014.20 | 14804.10 | 14425.70 | 12334.90 | 11801.90 |
| 39 | 7 | 10 | 11198.40 | 10794.40 | 11240.20 | 10994.60 | 12167.90 | 10833.40 | 11337.20 | 10700.30 |
| 40 | 8 | 10 | 5048.70 | 4883.12 | 5696.04 | 5041.96 | 7363.76 | 7437.55 | 5004.99 | 4882.72 |
| 41 | 9 | 10 | 8281.59 | 8194.16 | 9051.68 | 8476.97 | 10742.00 | 11057.60 | 8316.60 | 8204.17 |
| 42 | 10 | 10 | 11710.00 | 11644.80 | 11710.00 | 11955.00 | 13511.00 | 12454.30 | 11801.00 | 11652.60 |
| **Average** | | | **9530.52** | **9328.92** | **9825.57** | **9529.90** | **10309.33** | **9860.76** | **9547.50** | **9331.44** |
| **Dev. %** | | | **2.16** | | **3.10** | | **4.55** | | **2.32** | |

[1]This problem could not be solved optimaly within 2 hours and omitted from the average.

Table 6.23: Effects of final PALA run: Grids on Means Heuristic

| | | | Exact Mtd. | | Taylor App. | | Sample Param. | | Avg. Dist. Mtd. | |
|---|---|---|---|---|---|---|---|---|---|---|
| **P** | **m** | **n** | **B.PALA** | **A. PALA** | **B.PALA** | **A. PALA** | **B.PALA** | **A. PALA** | **B.PALA** | **A. PALA** |
| 1 | 5 | 20 | 8354.99 | 8354.57 | 8439.38 | 8390.53 | 8719.04 | 8385.59 | 8732.84 | 8349.77 |
| 2 | 5 | 25 | 10318.70 | 10318.30 | 10527.50 | 10459.90 | 10567.70 | 10541.80 | 10569.00 | 10316.60 |
| 3 | 5 | 30 | 15863.90[1] | 15820.00[1] | 15889.00 | 15873.00 | 16725.60 | 16112.80 | 16330.20 | 15828.10 |
| 4 | 5 | 40 | 22917.70 | 22916.60 | 23541.50 | 23127.90 | 23347.00 | 23408.80 | 23329.90 | 22913.00 |
| 5 | 5 | 50 | 21312.30[1] | 21309.50[1] | 21687.80[1] | 21390.90[1] | 22090.10 | 21628.10 | 21790.90 | 21318.90 |
| 6 | 5 | 100 | 35817.90[1] | 35770.00[1] | 36391.40[1] | 35842.90[1] | 36113.30[1] | 36273.50[1] | 36205.80[1] | 35836.40[1] |
| 7 | 5 | 100 | 38304.50[1] | 38287.60[1] | 39163.70[1] | 38634.50[1] | 39557.10[1] | 39145.10[1] | 38704.50[1] | 38370.60[1] |
| 8 | 5 | 100 | 42119.90[1] | 41884.10[1] | 42499.80[1] | 41905.60[1] | 43201.00[1] | 42473.70[1] | 41798.60[1] | 41586.20[1] |
| 9 | 5 | 100 | 40113.40[1] | 40086.30[1] | 41307.10[1] | 40373.30[1] | 42109.00[1] | 41332.20[1] | 40547.70[1] | 40273.60[1] |
| 10 | 5 | 100 | 41818.30[1] | 41813.40[1] | 41535.20[1] | 41136.10[1] | 41862.40[1] | 42109.40[1] | 41503.80[1] | 41145.50[1] |
| 11 | 6 | 20 | 7459.49 | 7427.50 | 7681.60 | 7492.39 | 7687.51 | 7550.01 | 7648.89 | 7426.45 |
| 12 | 6 | 25 | 9204.99 | 9186.99 | 9549.74 | 9544.91 | 9698.25 | 9466.19 | 9551.86 | 9191.10 |
| 13 | 6 | 30 | 14640.90[1] | 14581.70[1] | 14801.30 | 14650.90 | 15203.70 | 14759.20 | 15160.60 | 14638.50 |
| 14 | 7 | 20 | 6740.23 | 6737.74 | 6945.39 | 6874.13 | 6926.76 | 6926.47 | 6926.76 | 6737.89 |
| 15 | 7 | 25 | 8835.67 | 8829.84 | 9252.88 | 9118.88 | 9197.54 | 9262.72 | 8997.95 | 8840.24 |
| 16 | 7 | 30 | 13692.40 | 13684.20 | 13913.90 | 13997.00 | 14163.20 | 14118.70 | 14009.40 | 13748.80 |
| 17 | 8 | 20 | 6354.39 | 6350.00 | 6808.87 | 6457.86 | 6492.75 | 6597.21 | 6492.75 | 6355.68 |
| 18 | 8 | 25 | 8428.38[1] | 8423.77[1] | 8740.62 | 8715.71 | 8705.83 | 8775.52 | 8651.92 | 8480.27 |
| 19 | 8 | 30 | 13328.60[1] | 13320.50[1] | 13768.70[1] | 13570.10[1] | 14016.60 | 13866.70 | 13685.00 | 13405.80 |
| 20 | 9 | 20 | 6118.92[1] | 6109.17[1] | 6518.62 | 6329.46 | 6325.98 | 6372.08 | 6228.08 | 6117.59 |
| 21 | 9 | 25 | 8199.77[1] | 8180.77[1] | 8617.90[1] | 8393.07[1] | 8402.78 | 8636.05 | 8366.06 | 8200.61 |
| 22 | 9 | 30 | 12742.40[1] | 12725.80[1] | 13166.00 | 13048.30 | 13687.90 | 13378.60 | 12958.10 | 12821.30 |
| 23 | 10 | 20 | 5922.47[1] | 5914.26[1] | 6392.07[1] | 6049.10[1] | 6151.86 | 6309.98 | 6001.20 | 5923.62 |
| 24 | 10 | 25 | 8179.76[1] | 8162.81[1] | 8652.82[1] | 8610.28[1] | 8387.67 | 8541.04 | 8299.36[1] | 8211.07[1] |
| 25 | 10 | 30 | 12527.80[1] | 12520.70[1] | 13090.10[1] | 12960.90[1] | 13461.00 | 13205.40 | 12753.40 | 12510.60 |

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| | | | Exact Mtd. | | Taylor App. | | Sample Param. | | Avg. Dist. Mtd. | |
|---|---|---|---|---|---|---|---|---|---|---|
| **P** | **m** | **n** | **B.PALA** | **A. PALA** | **B.PALA** | **A. PALA** | **B.PALA** | **A. PALA** | **B.PALA** | **A. PALA** |
| 26 | 2 | 4 | 313.73 | 304.74 | 304.06 | 304.06 | 313.41 | 304.35 | 313.41 | 303.88 |
| 27 | 2 | 4 | 292.25 | 284.96 | 288.45 | 288.53 | 301.81 | 298.19 | 301.28 | 295.24 |
| 28 | 3 | 5 | 101.54 | 99.90 | 106.15 | 106.15 | 301.81 | 298.19 | 102.48 | 99.89 |
| 29 | 3 | 5 | 354.10 | 354.33 | 384.45 | 380.03 | 559.32 | 501.40 | 360.06 | 354.92 |
| 30 | 3 | 9 | 652.99 | 652.51 | 718.38 | 729.66 | 691.63 | 711.80 | 660.27 | 654.92 |
| 31 | 3 | 9 | 1923.42 | 1922.59 | 2033.58 | 1952.54 | 2023.05 | 1958.72 | 1971.32 | 1924.68 |
| 32 | 4 | 8 | 1355.14 | 1354.30 | 1422.66 | 1393.32 | 1463.72 | 1440.18 | 1421.25 | 1378.76 |
| 33 | 5 | 15 | 12546.30 | 12528.60 | 12936.60 | 12918.60 | 13796.70 | 13545.60 | 13129.20 | 12701.20 |
| 34 | 5 | 20 | 18606.30 | 18602.90 | 19523.70 | 19820.30 | 23528.50 | 19433.50 | 18759.60 | 18613.70 |
| 35 | 5 | 20 | 2243.14 | 2239.99 | 2337.75 | 2264.79 | 2768.70 | 2791.92 | 2308.58 | 2239.24 |
| 36 | 5 | 30 | 33326.70 | 33242.70 | 35356.40 | 34978.00 | 40236.40 | 33800.10 | 33863.40 | 33356.40 |
| 37 | 5 | 10 | 5694.24 | 5683.32 | 5986.98 | 5967.31 | 6435.47 | 6703.02 | 5771.76 | 5691.94 |
| 38 | 6 | 10 | 11923.10 | 11779.40 | 12195.30 | 12000.00 | 14804.10 | 14425.70 | 12334.90 | 11801.90 |
| 39 | 7 | 10 | 10945.40 | 10766.00 | 11455.90 | 10994.60 | 12167.90 | 10833.40 | 11337.20 | 10700.30 |
| 40 | 8 | 10 | 4903.06 | 4882.96 | 5104.80 | 5023.75 | 7363.76 | 7437.55 | 5004.99 | 4882.72 |
| 41 | 9 | 10 | 8208.43 | 8194.09 | 9065.95 | 8745.08 | 10742.00 | 11057.60 | 8316.60 | 8204.17 |
| 42 | 10 | 10 | 11663.70 | 11644.70 | 11844.10 | 11964.60 | 13511.00 | 12454.30 | 11801.00 | 11652.60 |
| **Average** | | | **8420.5** | **8397.8** | **8758.7** | **8665.2** | **9531.1** | **9009.7** | **8616.0** | **8412.9** |
| **Dev. %** | | | **0.27** | | **1.08** | | **5.79** | | **2.41** | |

[1]This problem could not be solved optimaly within 2 hours and omitted from the average.

Table 6.24: Effects of final PALA run: Mean DLAP Heuristic

| | | | Exact Mtd. | | Taylor App. | | Sample Param. | | Avg. Dist. Mtd. | |
|---|---|---|---|---|---|---|---|---|---|---|
| **P** | **m** | **n** | **B.PALA** | **A. PALA** | **B.PALA** | **A. PALA** | **B.PALA** | **A. PALA** | **B.PALA** | **A. PALA** |
| 1 | 5 | 20 | 8733.81 | 8355.90 | 8822.85 | 8390.53 | 8719.04 | 8385.59 | 8732.84 | 8349.77 |
| 2 | 5 | 25 | 10562.90 | 10318.20 | 10720.50 | 10477.50 | 10567.70 | 10541.80 | 10569.00 | 10316.60 |
| 3 | 5 | 30 | 16311.70 | 15818.00 | 16413.00 | 15931.70 | 16725.60 | 16112.80 | 16330.20 | 15828.10 |
| 4 | 5 | 40 | 23307.90 | 22918.60 | 23342.70 | 23183.80 | 23347.00 | 23408.80 | 23329.90 | 22913.00 |
| 5 | 5 | 50 | 21791.90 | 21317.30 | 21925.30 | 21390.90 | 22090.10 | 21628.10 | 21790.90 | 21318.90 |
| 6 | 5 | 100 | 36223.70[1] | 35814.80[1] | 36539.00[1] | 35857.10[1] | 36113.30[1] | 36273.50[1] | 36205.80[1] | 35836.40[1] |
| 7 | 5 | 100 | 38606.10[1] | 38303.00[1] | 40268.10[1] | 38743.00[1] | 39557.10[1] | 39145.10[1] | 38704.50[1] | 38370.60[1] |
| 8 | 5 | 100 | 42249.30[1] | 41799.50[1] | 43687.70[1] | 42603.20[1] | 43201.00[1] | 42473.70[1] | 41798.60[1] | 41586.20[1] |
| 9 | 5 | 100 | 40573.50[1] | 40232.80[1] | 41814.60[1] | 40371.20[1] | 42109.00[1] | 41332.20[1] | 40547.70[1] | 40273.60[1] |
| 10 | 5 | 100 | 41125.30[1] | 40719.00[1] | 42452.00[1] | 41679.70[1] | 41862.40[1] | 42109.40[1] | 41497.20[1] | 41134.20[1] |
| 11 | 6 | 20 | 7685.06 | 7427.65 | 7892.63 | 7701.86 | 7687.51 | 7550.01 | 7648.89 | 7426.45 |
| 12 | 6 | 25 | 9482.09 | 9187.57 | 9793.61 | 9636.15 | 9698.25 | 9466.19 | 9551.86 | 9191.10 |
| 13 | 6 | 30 | 14997.10 | 14706.10 | 15318.20 | 14694.30 | 15203.70 | 14759.20 | 15160.60 | 14638.50 |
| 14 | 7 | 20 | 6933.94 | 6738.33 | 6986.86 | 6870.30 | 6926.76 | 6926.47 | 6926.76 | 6737.89 |
| 15 | 7 | 25 | 8981.59 | 8830.59 | 9464.16 | 9299.49 | 9197.54 | 9262.72 | 8997.95 | 8840.24 |
| 16 | 7 | 30 | 13987.00 | 13733.70 | 14375.30 | 13866.50 | 14163.20 | 14118.70 | 14009.40 | 13748.80 |
| 17 | 8 | 20 | 6475.31 | 6350.03 | 6529.88 | 6502.54 | 6492.75 | 6597.21 | 6492.75 | 6355.68 |
| 18 | 8 | 25 | 8603.65 | 8442.95 | 9195.55 | 9090.01 | 8705.83 | 8775.52 | 8651.92 | 8480.27 |
| 19 | 8 | 30 | 13629.00 | 13393.30 | 13842.40 | 13479.60 | 14016.60 | 13866.70 | 13685.00 | 13405.80 |
| 20 | 9 | 20 | 6210.31 | 6109.26 | 6216.15 | 6190.60 | 6325.98 | 6372.08 | 6228.08 | 6117.59 |
| 21 | 9 | 25 | 8377.24 | 8187.85 | 8918.12 | 8487.96 | 8402.78 | 8636.05 | 8366.06 | 8200.61 |
| 22 | 9 | 30 | 12977.90 | 12845.50 | 13872.90 | 13493.60 | 13687.90 | 13378.60 | 12958.10 | 12821.30 |
| 23 | 10 | 20 | 5960.91 | 5914.27 | 5999.12 | 6049.17 | 6151.86 | 6309.98 | 6001.20 | 5923.62 |
| 24 | 10 | 25 | 8287.50 | 8152.50 | 8666.85 | 8362.66 | 8387.67 | 8541.04 | 8299.36 | 8211.07 |
| 25 | 10 | 30 | 12748.50[1] | 12525.70[1] | 14028.20[1] | 13495.40[1] | 13461.00 | 13205.40 | 12753.40 | 12510.60 |

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| | | | Exact Mtd. | | Taylor App. | | Sample Param. | | Avg. Dist. Mtd. | |
|---|---|---|---|---|---|---|---|---|---|---|
| **P** | **m** | **n** | **B.PALA** | **A. PALA** | **B.PALA** | **A. PALA** | **B.PALA** | **A. PALA** | **B.PALA** | **A. PALA** |
| 26 | 2 | 4 | 313.73 | 304.66 | 313.73 | 304.06 | 313.41 | 304.35 | 313.41 | 303.88 |
| 27 | 2 | 4 | 300.28 | 285.75 | 300.28 | 288.53 | 301.81 | 298.19 | 301.81 | 295.26 |
| 28 | 3 | 5 | 102.98 | 99.90 | 102.98 | 106.15 | 102.48 | 108.00 | 102.48 | 99.89 |
| 29 | 3 | 5 | 357.62 | 354.36 | 423.62 | 380.03 | 559.32 | 501.40 | 360.06 | 354.92 |
| 30 | 3 | 9 | 656.36 | 652.52 | 656.36 | 686.78 | 691.63 | 711.80 | 660.27 | 654.92 |
| 31 | 3 | 9 | 1948.69 | 1922.59 | 2080.39 | 1952.54 | 2023.05 | 1958.72 | 1971.32 | 1924.68 |
| 32 | 4 | 8 | 1416.35 | 1370.67 | 1423.94 | 1406.08 | 1463.72 | 1440.18 | 1421.25 | 1378.76 |
| 33 | 5 | 15 | 13276.90 | 12567.80 | 14062.90 | 12983.70 | 13796.70 | 13545.60 | 13129.40 | 12701.20 |
| 34 | 5 | 20 | 18870.00 | 18611.50 | 19982.30 | 19798.80 | 23528.50 | 19433.50 | 18761.40 | 18614.50 |
| 35 | 5 | 20 | 2319.92 | 2240.40 | 2434.97 | 2264.79 | 2768.70 | 2791.92 | 2308.58 | 2239.24 |
| 36 | 5 | 30 | 33723.70 | 33364.00 | 34333.50 | 33606.60 | 40236.40 | 33800.10 | 33863.40 | 33356.40 |
| 37 | 5 | 10 | 5754.35 | 5683.50 | 6108.95 | 5823.43 | 6435.47 | 6703.02 | 5771.76 | 5691.94 |
| 38 | 6 | 10 | 12209.40 | 11810.10 | 12209.40 | 12014.20 | 14804.10 | 14425.70 | 12334.90 | 11801.90 |
| 39 | 7 | 10 | 11198.40 | 10794.40 | 11240.20 | 10994.60 | 12167.90 | 10833.40 | 11337.20 | 10700.30 |
| 40 | 8 | 10 | 5048.70 | 4883.12 | 5696.04 | 5041.96 | 7363.76 | 7437.55 | 5004.99 | 4882.72 |
| 41 | 9 | 10 | 8281.59 | 8194.16 | 9051.68 | 8476.97 | 10742.00 | 11057.60 | 8316.60 | 8204.17 |
| 42 | 10 | 10 | 11710.00 | 11644.80 | 11710.00 | 11955.00 | 13511.00 | 12454.30 | 11801.00 | 11652.60 |
| **Average** | | | **9590.2** | **9386.3** | **9883.0** | **9619.9** | **10343.9** | **9919.6** | **9609.6** | **9388.4** |
| **Dev. %** | | | **2.17** | | **2.74** | | **4.27** | | **2.36** | |

[1]This problem could not be solved optimaly within 2 hours and omitted from the average.

### 6.3.4. Comparison of the Multi-Start PALA and Discrete Approximations

It is desirable to compare PALA heuristic and discrete approximation methods. Clearly, PALA algorithm is the basis of the discrete approximations. As emphasized many times, discrete approximations use the candidate points which are obtained with the multi-start PALA. Therefore, it is expected from discrete approximations to find better results than the multi-start PALA algorithm (MPALA). Otherwise, the effort for solving the costly APCMFWP is meaningless. In order to compare the methods, we give the results of the multi-start PALA algorithm and the Cellular heuristics in Table 6.25 for the Euclidean distance and bivariate normal distribution. Moreover, we only compare the results where expected distances are calculated with the exact method. Remark that, comparisons for other cases can be made by inspecting related tables. It is obvious from the column averages of Table 6.25 that Cellular heuristic is better than the minimum obtained with MPALA. Therefore, we can claim that discrete approximations considerably improve the solution quality of PCMFWP.

Table 6.25: Comparison of MPALA and Cellular Heuristic

| | | | **100 PALA Exact Mtd.** | | | **Cellular** |
|---|---|---|---|---|---|---|
| **P** | **m** | **n** | **Min** | **Average** | **Max** | **Heuristic** |
| 1 | 5 | 20 | 8353.89 | 9142.87 | 10757.40 | 8355.90 |
| 2 | 5 | 25 | 10318.20 | 11296.00 | 12787.50 | 10318.20 |
| 3 | 5 | 30 | 15820.10 | 16682.40 | 17796.10 | 15818.00 |
| 4 | 5 | 40 | 22916.90 | 24805.90 | 27682.30 | 22918.60 |
| 5 | 5 | 50 | 21309.70 | 23002.80 | 25627.40 | 21317.30 |
| 6 | 5 | 100 | 35769.90 | 36826.00 | 38399.70 | 35846.60[1] |
| 7 | 5 | 100 | 38475.40 | 39963.00 | 42400.30 | 38530.60[1] |
| 8 | 5 | 100 | 41648.00 | 43813.10 | 49209.80 | 42064.00[1] |
| 9 | 5 | 100 | 40086.30 | 43426.10 | 53627.10 | 40232.80[1] |
| 10 | 5 | 100 | 40718.20 | 43306.60 | 46130.00 | 40719.00[1] |
| 11 | 6 | 20 | 7485.32 | 8404.33 | 9323.64 | 7427.65 |
| 12 | 6 | 25 | 9264.00 | 10341.50 | 12205.90 | 9187.40 |
| 13 | 6 | 30 | 14590.00 | 15738.00 | 17851.60 | 14706.10 |
| 14 | 7 | 20 | 6942.79 | 7744.38 | 8764.48 | 6738.33 |
| 15 | 7 | 25 | 9038.88 | 9994.53 | 12125.20 | 8830.59 |
| 16 | 7 | 30 | 13674.30 | 15094.00 | 16604.10 | 13733.70 |
| 17 | 8 | 20 | 6692.51 | 7418.39 | 8655.45 | 6350.03 |
| 18 | 8 | 25 | 8750.20 | 9466.75 | 10311.70 | 8442.95 |
| 19 | 8 | 30 | 13429.30 | 14416.10 | 16031.70 | 13393.30 |
| 20 | 9 | 20 | 6428.02 | 7049.96 | 7818.17 | 6109.26 |

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| | | | 100 PALA Exact Mtd. | | | Cellular |
|---|---|---|---|---|---|---|
| **P** | **m** | **n** | **Min** | **Average** | **Max** | **Heuristic** |
| 21 | 9 | 25 | 8422.72 | 9178.62 | 10365.50 | 8187.85 |
| 22 | 9 | 30 | 12856.20 | 13927.60 | 14991.40 | 12845.50 |
| 23 | 10 | 20 | 6315.69 | 6882.69 | 7634.73 | 5914.27 |
| 24 | 10 | 25 | 8367.98 | 8858.39 | 9619.05 | 8152.50[1] |
| 25 | 10 | 30 | 12866.00 | 13336.60 | 14187.70 | 12525.70 |
| 26 | 2 | 4 | 304.54 | 304.56 | 304.58 | 304.74 |
| 27 | 2 | 4 | 284.71 | 307.91 | 403.37 | 285.75 |
| 28 | 3 | 5 | 99.85 | 109.06 | 130.72 | 99.90 |
| 29 | 3 | 5 | 354.30 | 480.16 | 677.38 | 354.36 |
| 30 | 3 | 9 | 652.50 | 770.40 | 963.49 | 652.52 |
| 31 | 3 | 9 | 1922.57 | 2111.94 | 2227.39 | 1922.59 |
| 32 | 4 | 8 | 1353.97 | 1452.71 | 1649.19 | 1370.67 |
| 33 | 5 | 15 | 12528.50 | 14907.40 | 18074.90 | 12567.80 |
| 34 | 5 | 20 | 18602.30 | 24100.40 | 32039.40 | 18603.40 |
| 35 | 5 | 20 | 2239.84 | 2764.80 | 4445.94 | 2240.40 |
| 36 | 5 | 30 | 33356.80 | 35918.20 | 43471.60 | 33364.00 |
| 37 | 5 | 10 | 5683.36 | 7046.41 | 9876.63 | 5683.50 |
| 38 | 6 | 10 | 11770.70 | 14731.10 | 19249.20 | 11810.10 |
| 39 | 7 | 10 | 10686.70 | 13902.30 | 17854.60 | 10794.40 |
| 40 | 8 | 10 | 4882.25 | 6215.56 | 10369.30 | 4883.12 |
| 41 | 9 | 10 | 8120.16 | 9666.33 | 14246.80 | 8194.16 |
| 42 | 10 | 10 | 11641.60 | 15408.00 | 21449.20 | 11644.80 |
| **Average** | | | **9443.32** | **10670.02** | **12748.77** | **9386.02** |

### 6.3.5. Comparison of the discrete approximations

Eventually, it is worthwhile to compare the results of discrete approximations and PALA heuristic in order to comment on their qualities. For this reason, we compare the average results of 100 PALA and the results of the three discrete approximations in Table 6.26 to Table 6.29. The differences between these four tables are due to expected distance calculation methods. Namely, exact calculation of the expected distance is used in Table 6.26, Taylor approximation is used in Table 6.27, Taylor approximation with estimated sample parameters is used in Table 6.28 and average distance to sample is used in Table 6.29. Moreover, the headings "100 PALA Avg" denote the average objective function value of the 100 PALA runs, "CH" denotes the Cellular heuristic, "GOM" denotes the grids on means heuristic and "MDLAP" denotes the Mean DLAP

[1]This problem could not be solved optimaly within 2 hours and omitted from the average.

heuristic. It is also necessary to state that the number of PALA runs for producing candidate points in each discrete approximation is 100. Besides, in Cellular heuristic, we construct 25 cells ($h = 5$, $v = 5$) by dividing the smallest rectangle which covers all candidate points.

In Table 6.26, when the column averages are investigated, Grids on Means heuristic has the lowest average objective function value. This result is expected because the number of the candidate points in Grids on Means heuristic is remarkably higher than the remaining two discrete approximations. On the other hand, Cellular heuristic and Mean DLAP heuristic have almost the same average objective value. The interpretation of this result is that the APCMFWP locates facilities on the customer means. Furthermore, the average of the 100 PALA runs is remarkably higher than the discrete approximations. The relative deviations of Cellular heuristic, Grids on Means heuristic and Mean DLAP heuristic from PALA heuristic are repectively 14.07, 14.18 and 14.06 per cents.

When Taylor approximation is used, Table 6.27, Cellular have the smallest average objective value and the average of Mean DLAP heuristic is very close to it. On the other hand, average objective value of Grids on Means heuristic is moderately higher than the other discrete approximations. This result can be unexpected since the resolution of the Grids on Means is higher than the Cellular Heuristic. However, the final costs are calculated with the exact expected distance function. Therefore, the objective function of the Grids on Means heuristic can be higher than the Cellular heuristics. In addition, discrete approximations have better results than the PALA heuristics. To illustrate, relative deviations of Cellular heuristic, Grids on Means heuristic and Mean DLAP heuristic from PALA heuristic are repectively 12.90, 12.72 and 12.81 per cents.

In Table 6.28, Cellular heuristic and Mean DLAP heuristic has the lowest average objective value. Besides, the average objective value of Grids on Means heuristic is slightly higher than the other discrete approximations. On the other hand, the average objective value of the PALA runs is significantly higher than average objective value of the discrete approximations. Besides, the relative deviations between the average

objective value of the PALA heuristic and the discrete approximations is approximately 8.8 per cent.

In the last comparison table, Table 6.29, all three discrete approximations have the same average objective value and they all have better averages than the PALA heuristic. Moreover, the relative deviations of the discrete approximations from the PALA heuristic is approximately 12.31 per cent.

Table 6.26: Discrete approximations and PALA: exact case

| P | m | n | 100 PALA Avg. | CH | GOM | MDLAP |
|---|---|---|---|---|---|---|
| 1 | 5 | 20 | 9142.87 | 8355.90 | 8354.57 | 8355.90 |
| 2 | 5 | 25 | 11296.00 | 10318.20 | 10318.30 | 10318.20 |
| 3 | 5 | 30 | 16682.40 | 15818.00 | 15820.00[1] | 15818.00 |
| 4 | 5 | 40 | 24805.90 | 22918.60 | 22916.60 | 22918.60 |
| 5 | 5 | 50 | 23002.80 | 21317.30 | 21309.50[1] | 21317.30 |
| 6 | 5 | 100 | 36826.00 | 35846.60[1] | 35770.00[1] | 35814.80[1] |
| 7 | 5 | 100 | 39963.00 | 38530.60[1] | 38287.60[1] | 38303.00[1] |
| 8 | 5 | 100 | 43813.10 | 42064.00[1] | 41884.10[1] | 41799.50[1] |
| 9 | 5 | 100 | 43426.10 | 40232.80[1] | 40086.30[1] | 40232.80[1] |
| 10 | 5 | 100 | 43306.60 | 40719.00[1] | 41813.40[1] | 40719.00[1] |
| 11 | 6 | 20 | 8404.33 | 7427.65 | 7427.50 | 7427.65 |
| 12 | 6 | 25 | 10341.50 | 9187.40 | 9186.99 | 9187.57 |
| 13 | 6 | 30 | 15738.00 | 14706.10 | 14581.70[1] | 14706.10 |
| 14 | 7 | 20 | 7744.38 | 6738.33 | 6737.74 | 6738.33 |
| 15 | 7 | 25 | 9994.53 | 8830.59 | 8829.84 | 8830.59 |
| 16 | 7 | 30 | 15094.00 | 13733.70 | 13684.20 | 13733.70 |
| 17 | 8 | 20 | 7418.39 | 6350.03 | 6350.00 | 6350.03 |
| 18 | 8 | 25 | 9466.75 | 8442.95 | 8423.77[1] | 8442.95 |
| 19 | 8 | 30 | 14416.10 | 13393.30 | 13320.50[1] | 13393.30 |
| 20 | 9 | 20 | 7049.96 | 6109.26 | 6109.17[1] | 6109.26 |
| 21 | 9 | 25 | 9178.62 | 8187.85 | 8180.77[1] | 8187.85 |
| 22 | 9 | 30 | 13927.60 | 12845.50 | 12725.80[1] | 12845.50 |
| 23 | 10 | 20 | 6882.69 | 5914.27 | 5914.26[1] | 5914.27 |
| 24 | 10 | 25 | 8858.39 | 8152.50[1] | 8162.81[1] | 8152.50[1] |
| 25 | 10 | 30 | 13336.60 | 12525.70 | 12520.70[1] | 12525.70 |
| 26 | 2 | 4 | 304.56 | 304.74 | 304.74 | 304.66 |
| 27 | 2 | 4 | 307.91 | 285.75 | 284.96 | 285.75 |
| 28 | 3 | 5 | 109.06 | 99.90 | 99.90 | 99.90 |
| 29 | 3 | 5 | 480.16 | 354.36 | 354.10 | 354.36 |
| 30 | 3 | 9 | 770.40 | 652.52 | 652.51 | 652.52 |
| 31 | 3 | 9 | 2111.94 | 1922.59 | 1922.59 | 1922.59 |
| 32 | 4 | 8 | 1452.71 | 1370.67 | 1354.30 | 1370.67 |
| 33 | 5 | 15 | 14907.40 | 12567.80 | 12528.60 | 12567.80 |

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| P | m | n | 100 PALA Avg. | CH | GOM | MDLAP |
|---|---|---|---|---|---|---|
| 34 | 5 | 20 | 24100.40 | 18603.40 | 18602.90 | 18611.50 |
| 35 | 5 | 20 | 2764.80 | 2240.40 | 2239.99 | 2240.40 |
| 36 | 5 | 30 | 35918.20 | 33364.00 | 33242.70 | 33364.00 |
| 37 | 5 | 10 | 7046.41 | 5683.50 | 5683.32 | 5683.50 |
| 38 | 6 | 10 | 14731.10 | 11810.10 | 11779.40 | 11810.10 |
| 39 | 7 | 10 | 13902.30 | 10794.40 | 10766.00 | 10794.40 |
| 40 | 8 | 10 | 6215.56 | 4883.12 | 4882.96 | 4883.12 |
| 41 | 9 | 10 | 9666.33 | 8194.16 | 8194.09 | 8194.16 |
| 42 | 10 | 10 | 15408.00 | 11644.80 | 11644.70 | 11644.80 |
| **Average** | | | **9786.1** | **8409.1** | **8397.8** | **8409.4** |

Table 6.27: Discrete approximations and PALA: Taylor Approximation case

| P | m | n | 100 PALA Avg. | CH | GOM | MDLAP |
|---|---|---|---|---|---|---|
| 1 | 5 | 20 | 9210.94 | 8390.53 | 8390.53 | 8390.53 |
| 2 | 5 | 25 | 11587.70 | 10459.90 | 10459.90 | 10477.50 |
| 3 | 5 | 30 | 16875.30 | 15931.70 | 15873.00 | 15931.70 |
| 4 | 5 | 40 | 25315.00 | 23183.80 | 23127.90 | 23183.80 |
| 5 | 5 | 50 | 23361.30 | 21390.90 | 21390.90[1] | 21390.90 |
| 6 | 5 | 100 | 37382.90 | 35790.60[1] | 35842.90[1] | 35857.10[1] |
| 7 | 5 | 100 | 40722.40 | 39857.90[1] | 38634.50[1] | 38743.00[1] |
| 8 | 5 | 100 | 44497.60 | 42637.70[1] | 41905.60[1] | 42603.20[1] |
| 9 | 5 | 100 | 43992.70 | 40243.60[1] | 40373.30[1] | 40371.20[1] |
| 10 | 5 | 100 | 43909.40 | 40773.10[1] | 41136.10[1] | 41679.70[1] |
| 11 | 6 | 20 | 8548.20 | 7701.86 | 7492.39 | 7701.86 |
| 12 | 6 | 25 | 10602.10 | 9636.15 | 9544.91 | 9636.15 |
| 13 | 6 | 30 | 16118.70 | 14882.70 | 14650.90 | 14694.30 |
| 14 | 7 | 20 | 7918.04 | 6870.30 | 6874.13 | 6870.30 |
| 15 | 7 | 25 | 10248.10 | 9299.49 | 9118.88 | 9299.49 |
| 16 | 7 | 30 | 15450.90 | 13905.10 | 13913.90 | 13866.50 |
| 17 | 8 | 20 | 7648.63 | 6502.54 | 6457.86 | 6502.54 |
| 18 | 8 | 25 | 9799.82 | 8902.12 | 8715.71 | 9090.01 |
| 19 | 8 | 30 | 14902.30 | 13479.60 | 13570.10[1] | 13479.60 |
| 20 | 9 | 20 | 7261.03 | 6190.60 | 6329.46 | 6190.60 |
| 21 | 9 | 25 | 9431.17 | 8487.96[1] | 8393.07[1] | 8487.96 |
| 22 | 9 | 30 | 14383.80 | 13321.60 | 13048.30 | 13493.60 |
| 23 | 10 | 20 | 7085.16 | 5999.12 | 6049.10[1] | 5999.12 |
| 24 | 10 | 25 | 9239.58 | 8631.10[1] | 8610.28[1] | 8362.66[1] |
| 25 | 10 | 30 | 13857.70 | 13328.30[1] | 12960.90[1] | 13495.40 |
| 26 | 2 | 4 | 304.06 | 304.06 | 304.06 | 304.06 |
| 27 | 2 | 4 | 309.17 | 288.53 | 288.45 | 288.53 |
| 28 | 3 | 5 | 118.62 | 102.98 | 106.15 | 102.98 |
| 29 | 3 | 5 | 530.37 | 380.03 | 380.03 | 380.03 |

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[1]This problem could not be solved optimaly within 2 hours and omitted from the average.

| P | m | n | 100 PALA Avg. | CH | GOM | MDLAP |
|---|---|---|---|---|---|---|
| 30 | 3 | 9 | 789.36 | 656.36 | 718.38 | 656.36 |
| 31 | 3 | 9 | 2155.79 | 1952.54 | 1952.54 | 1952.54 |
| 32 | 4 | 8 | 1506.30 | 1406.08 | 1393.32 | 1406.08 |
| 33 | 5 | 15 | 15125.70 | 12818.10 | 12918.60 | 12983.70 |
| 34 | 5 | 20 | 25132.80 | 19798.80 | 19523.70 | 19798.80 |
| 35 | 5 | 20 | 2854.75 | 2264.79 | 2264.79 | 2264.79 |
| 36 | 5 | 30 | 36811.10 | 33606.60 | 34978.00 | 33606.60 |
| 37 | 5 | 10 | 7208.35 | 5823.43 | 5967.31 | 5823.43 |
| 38 | 6 | 10 | 14755.00 | 12014.20 | 12000.00 | 12014.20 |
| 39 | 7 | 10 | 14062.20 | 10994.60 | 10994.60 | 10994.60 |
| 40 | 8 | 10 | 6402.68 | 5041.96 | 5023.75 | 5041.96 |
| 41 | 9 | 10 | 9996.56 | 8476.97 | 8745.08 | 8476.97 |
| 42 | 10 | 10 | 15705.80 | 11710.00 | 11844.10 | 11710.00 |
| **Average** | | | **10475.38** | **9123.18** | **9141.96** | **9133.37** |

Table 6.28: Discrete approximations and PALA: sample parameters

| P | m | n | 100 PALA Avg. | CH | GOM | MDLAP |
|---|---|---|---|---|---|---|
| 1 | 5 | 20 | 9193.60 | 8385.59 | 8385.59 | 8385.59 |
| 2 | 5 | 25 | 11590.70 | 10541.80 | 10541.80 | 10541.80 |
| 3 | 5 | 30 | 16875.80 | 16112.80 | 16112.80 | 16112.80 |
| 4 | 5 | 40 | 25047.00 | 23347.00 | 23347.00 | 23347.00 |
| 5 | 5 | 50 | 23455.80 | 21628.10 | 21628.10 | 21628.10 |
| 6 | 5 | 100 | 37565.00 | 36469.90[1] | 36113.30[1] | 36113.30[1] |
| 7 | 5 | 100 | 40705.30 | 39891.00[1] | 39145.10[1] | 39145.10[1] |
| 8 | 5 | 100 | 44448.00 | 42514.60[1] | 42473.70[1] | 42473.70[1] |
| 9 | 5 | 100 | 43915.20 | 40482.50[1] | 41332.20[1] | 41332.20[1] |
| 10 | 5 | 100 | 43920.30 | 41938.80[1] | 41862.40[1] | 41862.40[1] |
| 11 | 6 | 20 | 8498.47 | 7550.01 | 7550.01 | 7550.01 |
| 12 | 6 | 25 | 10586.30 | 9466.19 | 9466.19 | 9466.19 |
| 13 | 6 | 30 | 15875.70 | 14759.20 | 14759.20 | 14759.20 |
| 14 | 7 | 20 | 7978.74 | 6926.47 | 6926.47 | 6926.47 |
| 15 | 7 | 25 | 10268.10 | 9197.54 | 9197.54 | 9197.54 |
| 16 | 7 | 30 | 15338.60 | 14118.70 | 14118.70 | 14118.70 |
| 17 | 8 | 20 | 7638.72 | 6492.75 | 6492.75 | 6492.75 |
| 18 | 8 | 25 | 9836.19 | 8705.83 | 8705.83 | 8705.83 |
| 19 | 8 | 30 | 14793.70 | 13866.70 | 13866.70 | 13866.70 |
| 20 | 9 | 20 | 7286.31 | 6325.98 | 6325.98 | 6325.98 |
| 21 | 9 | 25 | 9549.38 | 8402.78 | 8402.78 | 8402.78 |
| 22 | 9 | 30 | 14283.00 | 13378.60 | 13378.60 | 13378.60 |
| 23 | 10 | 20 | 7136.73 | 6151.86 | 6151.86 | 6151.86 |
| 24 | 10 | 25 | 9295.47 | 8387.67 | 8387.67 | 8387.67 |
| 25 | 10 | 30 | 13786.60 | 13205.40 | 13205.40 | 13205.40 |

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[1]This problem could not be solved optimaly within 2 hours and omitted from the average.

| P | m | n | 100 PALA Avg. | CH | GOM | MDLAP |
|---|---|---|---|---|---|---|
| 26 | 2 | 4 | 304.35 | 304.35 | 304.35 | 304.35 |
| 27 | 2 | 4 | 295.27 | 298.19 | 298.19 | 298.19 |
| 28 | 3 | 5 | 112.02 | 102.48 | 298.19 | 102.48 |
| 29 | 3 | 5 | 514.42 | 501.40 | 501.40 | 501.40 |
| 30 | 3 | 9 | 795.66 | 691.63 | 691.63 | 691.63 |
| 31 | 3 | 9 | 2190.16 | 1958.72 | 1958.72 | 1958.72 |
| 32 | 4 | 8 | 1487.38 | 1440.18 | 1440.18 | 1440.18 |
| 33 | 5 | 15 | 15176.80 | 13545.60 | 13545.60 | 13545.60 |
| 34 | 5 | 20 | 24318.40 | 19433.50 | 19433.50 | 19433.50 |
| 35 | 5 | 20 | 2762.32 | 2768.70 | 2768.70 | 2768.70 |
| 36 | 5 | 30 | 35713.30 | 33800.10 | 33800.10 | 33800.10 |
| 37 | 5 | 10 | 7224.33 | 6435.47 | 6435.47 | 6435.47 |
| 38 | 6 | 10 | 14868.70 | 14425.70 | 14425.70 | 14425.70 |
| 39 | 7 | 10 | 13962.40 | 10833.40 | 10833.40 | 10833.40 |
| 40 | 8 | 10 | 6253.16 | 7363.76 | 7363.76 | 7363.76 |
| 41 | 9 | 10 | 9926.34 | 10742.00 | 10742.00 | 10742.00 |
| 42 | 10 | 10 | 15341.70 | 12454.30 | 12454.30 | 12454.30 |
| **Average** | | | **10798.96** | **9839.20** | **9844.49** | **9839.20** |

Table 6.29: Discrete approximations and PALA: average distance

| P | m | n | 100 PALA Avg. | CH | GOM | MDLAP |
|---|---|---|---|---|---|---|
| 1 | 5 | 20 | 9147.04 | 8349.77 | 8349.77 | 8349.77 |
| 2 | 5 | 25 | 11424.40 | 10316.60 | 10316.60 | 10316.60 |
| 3 | 5 | 30 | 16728.30 | 15828.00 | 15828.10 | 15828.10 |
| 4 | 5 | 40 | 24923.10 | 22913.00 | 22913.00 | 22913.00 |
| 5 | 5 | 50 | 23193.00 | 21318.90 | 21318.90 | 21318.90 |
| 6[1] | 5 | 100 | 36841.40 | 35866.10[1] | 35836.40[1] | 35836.40[1] |
| 7[1] | 5 | 100 | 39986.20 | 38358.00[1] | 38370.60[1] | 38370.60[1] |
| 8[1] | 5 | 100 | 43875.90 | 41724.20[1] | 41586.20[1] | 41586.20[1] |
| 9[1] | 5 | 100 | 43578.60 | 40243.30[1] | 40273.60[1] | 40273.60[1] |
| 10[1] | 5 | 100 | 43406.70 | 40708.50[1] | 41145.50[1] | 41134.20[1] |
| 11 | 6 | 20 | 8440.92 | 7426.45 | 7426.45 | 7426.45 |
| 12 | 6 | 25 | 10356.70 | 9191.07 | 9191.10 | 9191.10 |
| 13 | 6 | 30 | 15831.60 | 14638.40 | 14638.50 | 14638.50 |
| 14 | 7 | 20 | 7813.47 | 6737.89 | 6737.89 | 6737.89 |
| 15 | 7 | 25 | 10053.20 | 8840.24 | 8840.24 | 8840.24 |
| 16 | 7 | 30 | 15143.60 | 13748.80 | 13748.80 | 13748.80 |
| 17 | 8 | 20 | 7428.83 | 6355.68 | 6355.68 | 6355.68 |
| 18 | 8 | 25 | 9561.00 | 8480.27 | 8480.27 | 8480.27 |
| 19 | 8 | 30 | 14483.90 | 13405.50 | 13405.80 | 13405.80 |
| 20 | 9 | 20 | 7083.56 | 6117.59 | 6117.59 | 6117.59 |
| 21 | 9 | 25 | 9227.33 | 8200.61 | 8200.61 | 8200.61 |

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[1]This problem could not be solved optimaly within 2 hours and omitted from the average.

| P | m | n | 100 PALA Avg. | CH | GOM | MDLAP |
|---|---|---|---|---|---|---|
| 22 | 9 | 30 | 14012.50 | 12821.30 | 12821.30 | 12821.30 |
| 23 | 10 | 20 | 6949.89 | 5923.62 | 5923.62 | 5923.62 |
| 24 | 10 | 25 | 8905.81[1] | 8211.07[1] | 8211.07[1] | 8211.07[1] |
| 25 | 10 | 30 | 13407.40 | 12510.60 | 12510.60 | 12510.60 |
| 26 | 2 | 4 | 304.07 | 303.88 | 303.88 | 303.88 |
| 27 | 2 | 4 | 308.88 | 295.19 | 295.24 | 295.26 |
| 28 | 3 | 5 | 109.61 | 99.89 | 99.89 | 99.89 |
| 29 | 3 | 5 | 482.41 | 354.92 | 354.92 | 354.92 |
| 30 | 3 | 9 | 770.14 | 654.92 | 654.92 | 654.92 |
| 31 | 3 | 9 | 2116.67 | 1924.68 | 1924.68 | 1924.68 |
| 32 | 4 | 8 | 1453.35 | 1378.76 | 1378.76 | 1378.76 |
| 33 | 5 | 15 | 14987.80 | 12701.20 | 12701.20 | 12701.20 |
| 34 | 5 | 20 | 24181.70 | 18613.20 | 18613.70 | 18614.50 |
| 35 | 5 | 20 | 2738.80 | 2239.24 | 2239.24 | 2239.24 |
| 36 | 5 | 30 | 35794.10 | 33356.40 | 33356.40 | 33356.40 |
| 37 | 5 | 10 | 7051.60 | 5691.94 | 5691.94 | 5691.94 |
| 38 | 6 | 10 | 14742.20 | 11801.90 | 11801.90 | 11801.90 |
| 39 | 7 | 10 | 13872.60 | 10700.30 | 10700.30 | 10700.30 |
| 40 | 8 | 10 | 6209.74 | 4882.72 | 4882.72 | 4882.72 |
| 41 | 9 | 10 | 9750.84 | 8204.17 | 8204.17 | 8204.17 |
| 42 | 10 | 10 | 15350.90 | 11652.60 | 11652.60 | 11652.60 |
| **Average** | | | **10706.53** | **9388.34** | **9388.37** | **9388.39** |

[1]This problem could not be solved optimaly within 2 hours and omitted from the average.

# 7. CONCLUSIONS

Throughout the study, we attempt to solve PCMFWP with different distance norms and probability density function assumptions for the customer locations. In this study, we consider the general case and reports for only Euclidean and rectilinear distances. Customer locations are assumed to follow bivariate symmetric normal and exponential distributions for the Euclidean distance case. The same probability density functions and the bivariate symmetric uniform distribution are also used in the rectilinear distance case.

Solving the PCMFWP is a difficult task because the objective function of the problem is neither a convex function nor a concave function (Cooper 1978). It is found in the literature survey that the only way for solving PCMFWP to globally optimum is the enumeration of the extreme points of the transportation polytope (Cooper 1978). However, this enumeration method is very inefficient. Therefore, some heuristics are developed and tested on some test instances. No appropriate test bed is found for the PCMFWP in the literature. Thus, some test problems are gathered for the deterministic location-allocation problems and the required parameters from the original data are generated. Therefore, the global optimum solutions of the test problems are not known.

Probabilistic version of the Cooper's ALA type two phase algorithm is developed in the first method. The convergence of the PALA seems to be very fast based on our experiments. However, the computation times of the expected distances are very long because they require evaluation of quadratures. Therefore, Taylor approximation is developed in order to calculate the expected distances. Moreover, the probability distributions of the customer locations are sometimes not known exactly but some samples are available. Thus, average distance to sample method and the Taylor approximation with estimated sample parameters are developed. For all distance norms and probability distribution assumptions, PALA algorithms are tested by using these expected distance approximations. When the exact method is compared with the approximation

methods, the results are very satisfactory and the deviations are tolerable. The second group solution methods are the discrete approximations. Cellular heuristic, Grids on Means Heuristic and Mean DLAP heuristic are developed as the second group solution methods. All test problems with these discrete approximations for the Euclidean distance - bivariate symmetric normal distribution case are solved. The results show that discrete approximations improve the solutions. Moreover, the multi-start PALA algorithm is compared with the discrete approximations. It is found that discrete approximations can find better solutions than the multi-start PALA algorithm. However, computation times of the discrete algorithms are very long owing to the MILP. A time limit on the computation times of the APCMFWP is set while the problems are solved with discrete approximations. Therefore, some test problems are not solved within the time limits. However, it is possible to solve APCMFWP approximately with the Lagrangean heuristics. Thus, big instances of the PCMFWP can be solved with the discrete approximations.

The PALA algorithm and the discrete approximations do not guarantee the global solutions. As it is stated in the paragraphs above, allocations of the global optimum solution can be found by enumerating the extreme points of the transportation polytope. Afterwards, the global optimum location points of the facilities can be easily found by the Weiszfeld type iterative equations. Based on this information, it is possible to develop an efficient global optimization algorithm which cleverly searches the extreme points of the transportation polytope and evaluates the optimal facility locations. As a result, it is thought that spanning tree property can be useful to achieve this clever search. Furthermore, many neighborhood search strategies can be developed by using the spanning tree property.

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