COLLISION OF GRAVITATIONAL WAVES: AXISYMMETRIC PP WAVES

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AXISYMMETRIC PP WAVES

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ABSTRACT

COLLISION OF GRAVITATIONAL WAVES:
AXISYMMETRIC PP WAVES

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The collision of impulsive gravitational waves, electromagnetic plane waves with collinear polarization and, especially, plane fronted parallel waves (pp waves) are considered. The solution of axisymmetric pp waves is reviewed and the structures of the resulting space-times are investigated with the help of curvature invariants.

Keywords: Colliding Plane Waves, Khan-Penrose Solution, Bell-Szekeres Solution, PP Waves

ÖZ

KÜTLEÇEKİMSEL DALGALARIN ÇARPIŞMASI:
EKSEN SİMETRİK PP DALGALAR

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İtkisel kütle çekimsel dalgaların, eş-çizgisel kutuplanıma sahip elektromanyetik düzlem dalgaların ve özellikle düzlem yüzlü paralel dalgaların (pp dalgaların) çarpışması ele alındı. Eksen simetrik pp dalga için olan çözüm elde edildi ve meydana çıkan uzay-zamanların yapıları eğrilik değişmezleri ile araştırıldı.

Anahtar Kelimeler: Çarpışan Düzlem Dalgalar, Khan-Penrose Çözümü, Bell-Szekeres Çözümü, PP Dalgalar

To My Parents...

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I would like to express my deepest gratitude to Prof. Dr. Atalay Karasu for his continuous encouragement and kindness throughout this work.

I owe thanks to my family for their support and patience.

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CHAPTER 1

INTRODUCTION

In spite of the fact that gravitational waves have not been detected yet, one of the striking predictions of General Relativity is the existence of gravitational waves which can be theoretically derived by using a linearization technique. Even though this technique plays an unavoidable role in obtaining them, the amazing subject is essentially the interaction of these waves which is described by nonlinear field equations. Because of the nonlinearity, the collision of gravitational waves is one of the challenging subjects in this topic. In 1965 Penrose took the first step in his work which describes the astigmatic properties of gravitational waves and embedded the hypersurfaces in a two-dimensional section [1]. The pioneering studies of the collision problem were done in 1971 by Khan and Penrose [2], whose paper describes the collision of impulsive gravitational plane waves, and in 1970 by Szekeres [3], whose article describes the collision of gravitational plane waves. Later Bell and Szekeres [4] gave the first solution to the collision of electromagnetic waves. The other milestone in this research area that gives the solution of non-collinear gravitational wave collision was done by Nutku and Halilsoy [5]. After the Bell-Szekeres solution, there exist some studies which concern the collision between different types of waves, mainly gravitational and electromagnetic [6], neutrino [7] and null fields [8], [9]. Except for the Bell-Szekeres solution these suffer from curvature singularities which are unavoidable for the non-flat space-times in the future of the collision region. Gürses *et.al.* [10], [11], [12] have some studies which are about the same problems in the high dimensional space-times related with the string theory. In 2004 Chen *et.al.* [13] succeeded to give a formal solution to the collision of plane waves in string theory.

In this sense, plane wave symmetries are also useful for more generalized theories which deal with dilatons [14] and solitons [15]. Even though it is not mentioned as much as the plane waves, there is an upper class of them which are plane fronted and parallel, namely, pp waves. These types of waves were first suggested by Brinkmann [27]. However, pp waves remained uninvestigated until the work of Ehlers and Kundt [26]. Nowadays pp waves are widely used in string theory as well as in gravitational wave interaction in general relativity; for a concrete example, axisymmetric pp waves have a usage in the Aichelburg-Sexl ultraboost [17]. If one makes a generalization on the collision of plane waves, one needs to use symmetry groups as Gürses and Kalkanli did [16] or pp waves. Since Khan-Penrose solution is a special case of the Szekeres solution, a possible pp wave type solution must include the Szekeres solution, Nutku-Halil solution,etc...

In 1998 Ivanov succeeded to find a solution for the collision of axisymmetric pp waves [18] after some auxiliary steps [20], [21]. Actually this solution is forced to give some similarities with the Babala's solution which describes the collision of a gravitational impulsive wave and a thin plane shell of null dust [19]. Because this similarity gives us a hope in the way of finding a pp wave solution to the collision problem.

According to our motivation, reviewing the Ivanov's work [18] would be helpful for the studies about the collision of gravitational waves in the future. Because this work is an important step to solve a realistic waves which are not planar. Moreover, one can generalize the collision problems by using pp wave metrics.

In Chapter 2, the well known properties of gravitational plane waves are briefly reviewed.

In Chapter 3, first the collision problem is represented and then the most important two solutions, Khan-Penrose and Bell-Szekeres, in the collision problem are introduced.

In Chapter 4, first the main properties of PP waves are presented and then Ivanov's solution is eventually discussed.

CHAPTER 2

PLANE GRAVITATIONAL WAVES

The main purpose of this chapter is to review how gravitational waves can be interpreted theoretically. This is possible by solving the Einstein's field equations. The non-linearity structure is the main drawback in generating a wave equation. The linearization can be done in the weak energy limit by some perturbation techniques. In this subject, our the only guide is the similarities between electromagnetic theory and general relativity rather than the experimental results. Coordinate transformations are also helpful, as well as fixing or choosing gauges like Lorenz gauge. The possible solutions for the field equations must have some generalized properties that electromagnetic waves have. The polarization state of a wave and spin is some of those. These properties can be used in classifying the waves.

2.1 Linearized Field Equations

The metric of a space-time with a gravitational wave can be thought as a small perturbation about a flat space-time metric (i.e. the Minkowski metric $\eta_{\mu\nu}$);

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \qquad |h_{\mu\nu}| \ll 1 \tag{2.1}$$

where $h_{\mu\nu}$ is the metric perturbation. Like $h^2_{\mu\nu}$ terms and higher order terms can be neglected. In order to find the linearized field equations, let's begin by finding the inverse metric;

$$g_{\mu\alpha} g^{\alpha\nu} = \delta^\nu_\mu \;\Rightarrow\; g^{\mu\nu} \approx \eta^{\mu\nu} - h^{\mu\nu}. \tag{2.2}$$

Then the Christoffel symbols are

$$\begin{aligned} \Gamma^\mu_{\alpha\beta} &= \frac{1}{2} g^{\mu\nu}(g_{\nu\alpha,\beta} + g_{\nu\beta,\alpha} - g_{\alpha\beta,\nu}) \\ &\approx \frac{1}{2}(h^\mu{}_{\alpha,\beta} + h^\mu{}_{\beta,\alpha} - h_{\alpha\beta,}{}^\mu) \end{aligned} \tag{2.3}$$

where the indices are raised and lowered by $\eta_{\mu\nu}$, and the indices after the comma indicate partial differentiation with respect to those indices. The Ricci tensor can be computed by contracting the Riemann tensor

$$R_{\mu\nu} = R^{\alpha}{}_{\mu\alpha\nu} = \Gamma^{\alpha}_{\mu\nu,\alpha} - \Gamma^{\alpha}_{\mu\alpha,\nu} + \Gamma^{\alpha}_{\beta\alpha}\Gamma^{\beta}_{\mu\nu} - \Gamma^{\alpha}_{\beta\nu}\Gamma^{\beta}_{\mu\alpha}. \tag{2.4}$$

Cancelling the higher order terms, it becomes

$$R_{\mu\nu} = \Gamma^{\alpha}_{\mu\nu,\alpha} - \Gamma^{\alpha}_{\mu\alpha,\nu} \tag{2.5}$$

after using (2.3), we have

$$R_{\mu\nu} = \frac{1}{2}(h^{\alpha}{}_{\mu,\nu\alpha} + h^{\alpha}{}_{\nu,\mu\alpha} - \Box h_{\mu\nu} - h_{,\mu\nu}), \tag{2.6}$$

where $h \equiv h^{\alpha}{}_{\alpha} = \eta^{\alpha\beta}h_{\alpha\beta}$, $\Box h_{\mu\nu} = h_{\mu\nu,\alpha}{}^{\alpha}$. $\Box = \partial^{\mu}\partial_{\mu}$ is the flat space d'Alembertian operator. The Ricci scalar is

$$R = R^{\mu}{}_{\mu} = h^{\alpha\mu}{}_{,\mu\alpha} - \Box h. \tag{2.7}$$

As a result, the linearized Einstein tensor can be computed easily

$$\begin{aligned} G_{\mu\nu} &= R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R \\ &\approx \frac{1}{2}(h^{\alpha}{}_{\mu,\nu\alpha} + h^{\alpha}{}_{\nu,\mu\alpha} - \Box h_{\mu\nu} - h_{,\mu\nu} - \eta_{\mu\nu}h^{\alpha\beta}{}_{,\alpha\beta} - \eta_{\mu\nu}\Box h). \end{aligned} \tag{2.8}$$

In the weak gravitational field approximation, Riemann and Weyl tensors can also be computed in a similar way. However, let's stop linearization here for a while and define Weyl tensor which is frequently used in the collision problem and in the classification of gravitational waves. Weyl tensor is the trace free part of Riemann tensor, i.e.

$$C^{a}_{bad} = 0. \tag{2.9}$$

The Weyl tensor can be generated by subtracting the trace parts from the original Riemann tensor. So in an n dimensional space-time this trace-free tensor looks like

$$C_{\rho\sigma\mu\nu} = R_{\rho\sigma\mu\nu} - \frac{2}{n-2}(g_{\rho[\mu}R_{\nu]\sigma} - g_{\sigma[\mu}R_{\nu]\rho}) + \frac{2}{(n-1)(n-2)}g_{\rho[\mu}g_{\nu]\sigma}R. \tag{2.10}$$

Here the brackets denotes antisymmetricity. The Weyl tensor has the same mathematical properties as the Riemann tensor, which are

$$C_{\rho\sigma\mu\nu} = C_{[\rho\sigma][\mu\nu]} \tag{2.11}$$

$$C_{\rho\sigma\mu\nu} = C_{\mu\nu\rho\sigma} \tag{2.12}$$

$$C_{\rho[\sigma\mu\nu]} = 0. \tag{2.13}$$

The Weyl tensor can give a relationship between two distinct looking spacetimes. If two space-times $\overline{g}_{\mu\nu}$ and $g_{\mu\nu}$ are said to be conformally related,then they must satisfy

$$\overline{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}, \tag{2.14}$$

where $\Omega(x)$ is a non-zero differentiable arbitrary function. In fact conformal mapping which describes a function conserving angles between lines is actually defined in complex analysis. The conformally related metrics have the same Weyl tensor that is

$$\overline{C}^{\rho}_{\sigma\mu\nu} = C^{\rho}_{\sigma\mu\nu}. \tag{2.15}$$

If $\Omega = 1$, then the metrics are called conformally invariant. As it can be seen, the Weyl tensors are conformally invariant objects which are frequently used in the collision problems to seek the conformal relations with the Minkowski metric (i.e. conformally flatness). This implies that in a conformally flat metric, Weyl tensor vanishes which is often used in the subsequent chapters. The symmetries of the Weyl tensor can also be used when classifying the space-times. One of them is the Petrov classification. Briefly in the Petrov classification one should investigate the multiplicity of the principle null directions (k^{μ}) which satisfy

$$k^{\mu}k_{\mu} = 0 \tag{2.16}$$

by using the equations in the below.

$$k_{[\rho}C_{\kappa]\lambda\mu[\nu}k_{\sigma]}k^{\lambda}k^{\mu} = 0 \tag{2.17}$$

$$C_{\kappa\lambda\mu[\nu}k_{\sigma]}k^{\lambda}k^{\mu} = 0 \tag{2.18}$$

$$C_{\kappa\lambda\mu[\nu}k_{\sigma]}k^{\mu} = 0 \tag{2.19}$$

$$C_{\kappa\lambda\mu\nu}k^{\mu} = 0 \tag{2.20}$$

2.2 Gauge Transformations

In electromagnetic theory, wave solutions can be found by using gauge transformation of the vector potential. In our case, let's begin with the coordinate transformation

$$x^{\mu} \to x'^{\mu} = x^{\mu} + \varepsilon\xi^{\mu}(x), \tag{2.21}$$

where ε is a small dimensionless parameter and ξ^μ is an arbitrary vector field. Then our metric would be transformed according to

$$g_{\mu\nu}(x) = \frac{\partial x'^\alpha}{\partial x^\mu}\frac{\partial x'^\beta}{\partial x^\nu} g'_{\alpha\beta}(x). \tag{2.22}$$

The gauge transformation of the metric perturbation can be easily found as

$$h_{\mu\nu} \to h'_{\mu\nu}(x) = h_{\mu\nu}(x) - 2\xi_{(\mu,\nu)}, \tag{2.23}$$

where $\xi_{(\mu,\nu)} = \frac{1}{2}(\xi_{\mu,\nu} + \xi_{\nu,\mu})$. Similarly, it can be shown that the Riemann tensor, the Ricci tensor and the Ricci scalar are invariant under this transformation.
Define

$$\psi_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h, \tag{2.24}$$

which transforms like

$$\psi_{\mu\nu} \to \psi'_{\mu\nu} = \psi_{\mu\nu} - \xi_{\mu,\nu} - \xi_{\nu,\mu} - \eta_{\mu\nu}\Box\xi, \tag{2.25}$$

then plug $h_{\mu\nu}$ into Ricci tensor, Ricci scalar and Einstein tensor which are given by (2.6), (2.7) and (2.8), respectively. Then they become

$$\begin{aligned} R_{\mu\nu} &= \tfrac{1}{2}(\psi^\alpha{}_{\mu,\nu\alpha} + \psi^\alpha{}_{\nu,\mu\alpha} - \Box h_{\mu\nu}), \\ R &= \psi^{\alpha\beta}{}_{,\alpha\beta} - \tfrac{1}{2}\Box h, \\ G_{\mu\nu} &= \tfrac{1}{2}(\psi^\alpha{}_{\mu,\nu\alpha} + \psi^\alpha{}_{\nu,\mu\alpha} - \Box\psi_{\mu\nu} - \eta_{\mu\nu}\psi^{\alpha\beta}{}_{,\alpha\beta}). \end{aligned} \tag{2.26}$$

Assume that we are working in the harmonic coordinate system which has the property

$$g^{\mu\nu}\Gamma^\alpha_{\mu\nu} = 0. \tag{2.27}$$

When metric is linearized, this gives

$$h^\mu{}_{\nu,\mu} - \frac{1}{2}h_{,\nu} = 0. \tag{2.28}$$

By using (2.24), (2.28) can be written shortly

$$\psi^\mu{}_{\nu,\mu} = 0. \tag{2.29}$$

This condition is called the Lorentz gauge, which is the analogue to the one in the electromagnetic theory. Then from (2.25), this condition becomes

$$\Box\xi_\nu = \psi^\mu{}_{\nu,\mu} = 0. \tag{2.30}$$

The Einstein tensor reduces to

$$G_{\mu\nu} = -\frac{1}{2}\Box\psi_{\mu\nu}. \tag{2.31}$$

So in the vacuum we have

$$\Box\psi_{\mu\nu} = 0, \tag{2.32}$$

and also

$$\Box\psi = 0. \tag{2.33}$$

Eventually, combining (2.24) and (2.32), one gets

$$\Box h_{\mu\nu} = 0. \tag{2.34}$$

This is the equation that must be solved to get wave solutions.

2.3 Linearized Plane Gravitational Waves

One can achieve to get the wave equation by using gauge transformations as we did in the previous section. But we must solve (2.34) in order to get a generic gravitational metric. Fortunately, we are familiar to this problem from the electromagnetic theory. Suppose that

$$h_{\mu\nu} = A_{\mu\nu}e^{ik_\alpha z^\alpha} \tag{2.35}$$

satisfies (2.34), where $A_{\mu\nu}$ is called the polarization tensor and k is the wave vector whose components are constant for plane waves. If one inserts (2.35) into (2.34), then one gets

$$k_\mu k^\mu = 0. \tag{2.36}$$

which shows that wave is propagating at the speed of light. If (2.35) is put into (2.28), then one finds

$$k_\mu A^\mu{}_\nu = \frac{1}{2}k_\nu A^\mu{}_\mu. \tag{2.37}$$

This shows that $A_{\mu\nu}$ is symmetric. This reduces the linearly independent components from ten to six. Similarly (2.30) can be solved as

$$\xi_\mu = B_\mu e^{ik_\lambda x^\lambda}, \tag{2.38}$$

where ξ_μ and B_μ are constant coefficients. Then ξ_μ can be chosen such that we can write $A_{\mu\nu}$ such as

$$A'_{\mu\nu} = A_{\mu\nu} + k_\mu B_\nu + k_\nu B_\mu. \tag{2.39}$$

This reduces the number of linearly independent components from six to two, since B_μ has four components. For example, assume that our wave propagates in the z-axis, i.e.

$$k^2 = k^3 \equiv 0, \qquad k^1 = k^0 \equiv k. \tag{2.40}$$

By using (2.37)

$$\begin{aligned} A_{12} + A_{02} &= A_{13} + A_{03} = 0, \\ A_{11} + A_{01} &= \tfrac{1}{2}(A_{11} + A_{22} + A_{33} - A_{00}), \\ A_{01} + A_{00} &= -\tfrac{1}{2}(A_{11} + A_{22} + A_{33} - A_{00}). \end{aligned} \tag{2.41}$$

Then

$$A_{03} = -A_{13}, \tag{2.42}$$

$$A_{02} = -A_{12}, \tag{2.43}$$

$$A_{01} = -\frac{1}{2}(A_{00} + A_{11}), \tag{2.44}$$

$$A_{22} = -A_{33}. \tag{2.45}$$

B_μ stated in (2.39) can be arranged such that $A'_{\mu\nu} = 0$ except for A_{22}, A_{23}, A_{32}, A_{33}; that is

$$A_{\mu\nu} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & A_{22} & A_{23} \\ 0 & 0 & A_{23} & -A_{22} \end{bmatrix}. \tag{2.46}$$

2.4 Polarization States

The metric we found in the previous section can be written without cross terms in the form

$$ds^2 = dt^2 - dz^2 - [1 - h_{22}(t-z)]dy^2 - [1 + h_{22}(t-z)]dx^2 \tag{2.47}$$

which is called h_{22}-*wave* because metric does not have any perturbation term other than h_{22}. Assume that, we have a group of test particles around a circle in (x, y)-plane. When $h_{22} > 0$, test particles begin to squeeze in y-coordinate and to stretch in x-coordinate. The circle is deformed to be an ellipse in shape. If $h_{22} < 0$, then the test particles would act vice versa. Because of this transverse characteristic, this state of wave is known as "+ polarization". At first, we have assumed that $h_{23} = 0$, now

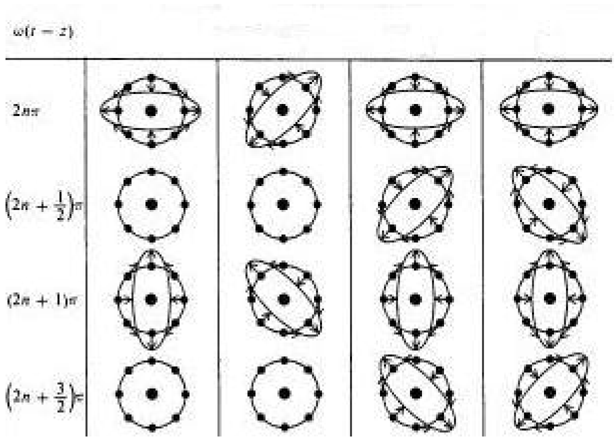

Figure 2.1: Polarization of a plane gravitational wave is shown. In the first column +-polarization, in the second column ×-polarization, in the third and fourth columns right and left polarizations, where they can be linearly represented by +-polarization and ×-polarization, are illustrated.

further assuming $h_{23} \neq 0$ and $h_{22} = 0$, the metric becomes

$$ds^2 = dt^2 - dz^2 - dy^2 + 2h_{23}(t-z)dydx - dx^2. \tag{2.48}$$

After making a transformation of the

$$y \to y' = \frac{1}{\sqrt{2}}(y+x), \quad z \to z' = \frac{1}{\sqrt{2}}(-y+x), \tag{2.49}$$

which means a rotation through 45° in the (y, z)-plane. The line element turns to be a familiar form

$$ds^2 = dt^2 - dz^2 - [1 - h_{23}(t-z)]dy^2 - [1 + h_{23}(t-z)]dx^2. \tag{2.50}$$

This is called h_{23}-*wave*. Since we have rotated + polarization, now we have a × polarization. Both + and × polarizations are linear polarizations. In general, if we have neither $h_{22} = 0$ nor $h_{23} = 0$, then this wave would be a combination of +-polarization and ×-polarization (see Figure 2.1). Although here polarization is found by using matrix identities, these results can also be achieved by geodesic deviation.

The classical radiation field of a spin-S particle is always invariant under a rotation of $\frac{360°}{S}$ about its propagation direction. As a result, we can immediately draw the conclusion that the graviton, the messenger particle of gravitational field, is a spin-2 particle [25].

It can be asked whether waves can have polarization states different than plane polarizations. In this sense the Petrov classification is more helpful when plane symmetry

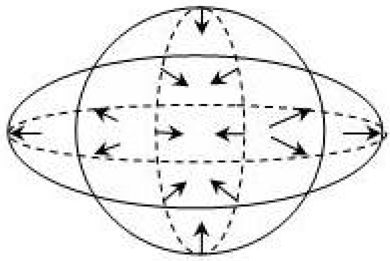

Figure 2.2: One of the non-plane polarizations can be in a spherical form.

does not exist. For example in type-N fields has a characteristic like + polarization and type-III is the same as × polarization. But type-D field, which is another type of Petrov classification, defines a new polarization state that does not exist in plane polarization. In a type-D field test particles move in a spherical and an ellipsoidal form (see Figure 2.2).

2.5 Exact Plane Gravitational Waves

h_{22}-wave metric can be written in the following form

$$ds^2 = 2dudv - f^2(u)dx^2 - g^2(u)dy^2 \tag{2.51}$$

where $f^2(u) = 1 - h_{22}(u)$, $g^2(u) = 1 + h_{22}(u)$. The plane wave metric admits a five-parameter group of symmetries and there is a two-parameter Abelian subgroup of symmetries acting like planar translations in the spacelike 2-surfaces. Here our main purpose is to see whether vacuum field solutions are satisfied or not. In order to see them once again, it is needed to find non-vanishing Christoffel symbols, Riemann tensor, Ricci tensor and Ricci scalar which are

$$\begin{aligned}
&\Gamma^1_{22} = ff', \quad \Gamma^1_{33} = gg', \quad \Gamma^2_{02} = \tfrac{f'}{f}, \quad \Gamma^3_{03} = \tfrac{g'}{g};\\
&R_{0202} = ff', \quad R_{0303} = gg';\\
&R_{00} = -\tfrac{f''}{f} - \tfrac{g''}{g};\\
&R = 0,
\end{aligned} \tag{2.52}$$

respectively. Here, prime denotes partial derivative with respective to u. Hence, it can be easily seen that the only field equation is

$$\frac{f''}{f} + \frac{g''}{g} = 0. \tag{2.53}$$

The solution of this implies that in this metric we have two travelling waves moving in opposite directions. This type of metric is called *Rosen form* which shows that it is possible to interpret the space-time as the collision of two gravitational waves. Surely, the collision of gravitational waves are the exact solutions of Einstein field equations. Instead of seeing this property in field equations, there is another possibility that it can be seen in the metric from the beginning. After using the coordinate transformations

$$\bar{u} = u, \qquad \bar{v} = v + \frac{1}{2}y^2 f f' + \frac{1}{2}x^2 g g', \qquad Y = fy, \qquad X = gx; \tag{2.54}$$

the Rosen type line element becomes

$$ds^2 = d\bar{u}d\bar{v} + h(\bar{u})(Y^2 - X^2)d\bar{u}^2 - dX^2 - dY^2 \tag{2.55}$$

which is called *Brinkmann form*. In this type, the amplitude of gravitational wave can be also observed. Although the nonlinearity of Einstein field equations doesn't allow any superposition principle in general relativity, these solutions revealed a limited superposition principle in that two plane waves moving in the same direction can be superposed simply by adding their corresponding h functions. Here, it is worth noting that scattering angle is not important while considering collision of gravitational waves, since one can always find a class of observers who consider the collision to be head on. Hence, it is sufficient to work in a coordinate system in which the waves appear to collide head on.

2.6 The Class of Plane Waves

In the previous section, the Brinkmann form metric was obtained. The term $h(\bar{u})(X^2 - Y^2)$ can be generalized to $H(\bar{u}, X, Y)$

$$ds^2 = 2d\bar{u}d\bar{v} + H(\bar{u}, X, Y)d\bar{u}^2 - dX^2 - dY^2. \tag{2.56}$$

This general type of waves are known as *pp waves* and they will be discussed in Chapter 3 in detail. Plane waves, the special case of pp waves, can be defined with the line element in the form

$$ds^2 = 2d\bar{u}d\bar{v} + (h_{11}X^2 + 2h_{12}XY + h_{22}Y^2)d\bar{u}^2 - dX^2 - dY^2 \tag{2.57}$$

where h_{ij} are functions of $\bar{u}$ only. Observe that plane wave metric has the function $H(\bar{u}, X, Y)$ quadratic with X and Y coordinate. This line element describes a gravitational wave in a vacuum space-time if $h_{22} = -h_{11}$. In addition, if h_{12} is proportional to h_{11} then the gravitational wave will have constant linear polarization such as

$$H = h(\bar{u})(\cos\alpha(X^2 - Y^2) + 2\sin\alpha XY) \tag{2.58}$$

If one wants to align the polarization with the x-axis, then one should set $\alpha = 0$. So the metric becomes

$$ds^2 = 2d\bar{u}d\bar{v} + (h_{11}(X^2 - Y^2))d\bar{u}^2 - dX^2 - dY^2 \tag{2.59}$$

which is already obtained in the previous section as Brinkmann form. Similarly, pure electromagnetic waves can be also defined in the same manner

$$ds^2 = 2d\bar{u}d\bar{v} + (h_{11}(X^2 + Y^2))d\bar{u}^2 - dX^2 - dY^2. \tag{2.60}$$

There is another type of metric which is called *Szekeres line element* and it is widely used in the collision of gravitational waves.

$$ds^2 = 2e^{-M}dudv - e^{-U}(e^V \cosh W dx^2 - 2\sinh W dxdy + e^{-V}\cosh W dy^2) \tag{2.61}$$

where U, V, M, W are functions of u only. W describes the alignment of the linear polarization. Szekeres showed that Rosen type metric can be transformed into a form given in (2.61).

Aligned polarized gravitational waves can be further classified by using the known simple functions in order to investigate the properties more easily. For example, impulsive gravitational plane waves have $h(u) \equiv \delta(u)$. The geometry of the space-time where gravitational wave passes through is changed in a moment and returns back to its old flat geometry. Although impulse function is an idealized function, it must have some finite thickness in real life. Because of this reason, impulsive gravitational waves are sometimes called *sandwich waves*.

Another example can be *step waves* which has $h(u) \equiv \theta(u)$. Then for gravitational waves the metric would be

$$ds^2 = 2d\bar{u}d\bar{v} + a^2\theta(\bar{u})(X^2 - Y^2)d\bar{u}^2 - dX^2 - dY^2 \tag{2.62}$$

where a is the amplitude of the wave. For electromagnetic waves it is

$$ds^2 = 2d\bar{u}d\bar{v} + a^2\theta(\bar{u})(X^2 + Y^2)d\bar{u}^2 - dX^2 - dY^2. \tag{2.63}$$

In the Rosen form they are

$$ds^2 = 2dudv + \cos^2 au\, dx^2 - \cosh^2 au\, dy^2, \tag{2.64}$$

$$ds^2 = 2dudv + \cos^2 au\, (dx^2 + dy^2), \tag{2.65}$$

respectively.

CHAPTER 3

COLLIDING PLANE GRAVITATIONAL WAVES

The non-linearity of Einstein's field equations is the messenger of new features of gravitational waves. One of these is that waves do not possibly superpose. While further investigating this, we can use the simple wave structures and the collision of known waves (i.e. electromagnetic waves).

Firstly, the concrete framework is given in order to seek general solutions. One of the exact solutions are investigated by using this framework and the other solution is investigated without it in order to show how hard it is to deal with the field equations.

3.1 Collision of Plane Waves

The main reason of using the Rosen coordinates is to represent the collision problem schematically. After disregarding the spacelike coordinates which are x and y coordinates, two gravitational wave can be represented as in the Figure (3.1). Beside, omitting the spacelike coordinates provides a visualization on the collision subject. For instance, according to timelike observers, they see a collision as a head-on collision. We can understand and visualize what that observer sees in this sense.

Before the collision, the space-time is flat (i.e. Minkowski metric describes the region I ($u < 0$, $v < 0$)). The region II ($u > 0$, $v < 0$) and region III ($u < 0$, $v > 0$) represent the approaching gravitational waves. Those are flat except at the boundaries where Lichnerowicz conditions or O'Brien-Synge conditions must be satisfied. According to the Lichnerowicz conditions, metric function $g_{\mu\nu}$ must have continuous derivatives up to first order across the boundary but higher order derivatives need not to be continuous. Weaker than the Lichnerowicz conditions O'Brien-Synge conditions tell that the null hypersurface with $x^0 = constant$ and $g_{00} = 0$ must have the components $g_{\mu\nu}$, $g^{ij}g_{ij,0}$, $g^{i0}g_{ij,0}$ (i, j=1,2,3) continuous across the null boundary. Here the

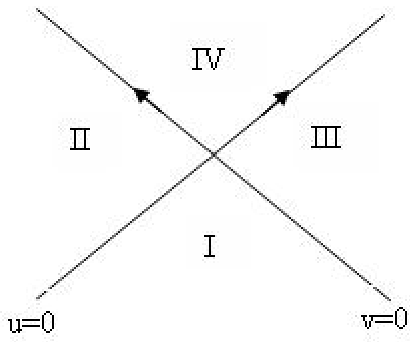

Figure 3.1: Although this picture represents impulsive gravitational waves, the plane waves can be represented in a similar way. Space-time can be divided into four regions. The spacelike coordinates are omitted and only null coordinates are used. Because of the division into four regions, approaching waves carry information about the initial data.

main assumption is that in interaction region IV ($u > 0$, $v > 0$) the metric is a function of null boundaries (i.e. u and v) and does not vary with spacelike components. The reason of making such an assumption is to use the characteristic initial value problem which tells that if the main equations hold everywhere especially on a hypersurface, then the contracted Bianchi identities ensure that conditions hold everywhere [23]. Characteristic here means a special attribute specifies the null hypersurfaces are interested. This technique guarantees that the solution is unique if the initial conditions are well set. There is no need to solve the field equations for the regions other than region IV. So if one wants to obtain the solution for the collision problem, the first job must be to seek a solution to the Rosen type metric which can be written in the form

$$ds^2 = 2e^{-M} dudv - g_{ij} dx^i dx^j \tag{3.1}$$

where $M = M(u, v)$; $g_{ij} = g_{ij}(u, v)$; $i, j = 2, 3$. Briefly one should find a way to describe the function g_{ij}. As one can expect, the solution sought is given in (2.61).
That the Rosen type metric has two null coordinates mentioned before is the starting point of this investigation. Because of having null coordinates, it is appropriate to pass to the null tetrad formalism which has a wide range of usage in general relativity, especially in gravitational radiation and black holes. In null tetrad formalism, the main idea is to write a metric by suitable vector fields whose length is zero (i.e. null

vectors)

$$l^\mu l_\mu = n^\mu n_\mu = m^\mu m_\mu = \bar{m}^\mu \bar{m}_\mu = 0, \tag{3.2}$$

where $\bar{m}$ is the complex conjugate of m. These null vectors have normalization conditions which are

$$l^\mu n_\mu = 1, \qquad\qquad m^\mu \bar{m}_\mu = -1. \tag{3.3}$$

After this definition, one can write a metric as

$$g_{\mu\nu} = l_\mu n_\nu + l_\nu n_\mu - m_\mu \bar{m}_\nu - m_\nu \bar{m}_\mu. \tag{3.4}$$

Then according to (3.1), let the null vector fields be

$$A l_\mu = u_{,\mu} \quad \text{and} \quad B n_\mu = v_{,\mu} \tag{3.5}$$

where $A = A(u,v)$ and $B = B(u,v)$ are integrating factors. The other complex null vectors can be defined as

$$m^\mu = D^2(u,v)\delta^\mu_2 + D^3(u,v)\delta^\mu_3. \tag{3.6}$$

After using the field equations in Newman-Penrose formalism, one can find that

$$M = \ln(AB). \tag{3.7}$$

Then if commutation relations, scale invariance and the uniqueness theorem of ordinary differential equations are used, the metric becomes

$$ds^2 = 2e^{-M} du dv - (D_i \bar{D}_j + \bar{D}_i D_j) dx^i dx^j \tag{3.8}$$

where

$$D^2 = \frac{1}{\sqrt{2}} e^{\frac{U-V}{2}} (\cosh(\frac{W}{2}) + i \sinh(\frac{W}{2})) \tag{3.9}$$

$$D^3 = \frac{1}{\sqrt{2}} e^{\frac{U+V}{2}} (\sinh(\frac{W}{2}) + i \cosh(\frac{W}{2})) \tag{3.10}$$

and $U = -\ln(\det g_{ij})$, W is a function of u and v and defines a spatial rotation, $V = V(u,v)$.

Initial conditions of the Szekeres metric are

$$U = V = M = W = 0 \qquad\qquad \text{in region I;} \tag{3.11}$$

$$V = V(u,0),\ M = M(u,0),\ W = W(u,0),\ U = -\ln(f(u) + \frac{1}{2}) \text{ in region II;} \tag{3.12}$$

$$V = V(0,v),\ M = M(0,v),\ W = W(0,v),\ U = -\ln(g(v) + \frac{1}{2}) \text{ in region III,} \tag{3.13}$$

where according to one of the field equations $e^{-U} = f(u) + g(v)$.

These are the techniques used in the collision problem while seeking an exact solution. Now we can give some of them.

3.2 Colliding Impulsive Plane Gravitational Waves

As mentioned in the previous chapter an impulsive gravitational plane wave

$$ds^2 = 2d\bar{u}d\bar{v} + \delta(\bar{u})(X^2 - Y^2)du^2 - dX^2 - dY^2 \tag{3.14}$$

can be transformed into Rosen form as

$$ds^2 = 2L(u)dudv - F^2(u)dx^2 - G^2(u)dy^2 \tag{3.15}$$

by changing the coordinates as

$$\bar{u} = u, \qquad \bar{v} = v + \frac{1}{2}y^2FF' + \frac{1}{2}z^2GG', \qquad Y = Gy, \qquad X = Fx; \tag{3.16}$$

where $L = 1$, $F = 1 + u\theta(u)$, $G = 1 - u\theta(u)$.

It is possible to think that if a travelling wave is described by the functions of u in that direction, then another travelling wave can be described by the functions of another orthogonal direction v in a perpendicular direction [2]. Since perpendicularity is relative, such a reference frame can always be found in which the observer thinks the interaction is a head on collision. Here the problem is to find a solution of the Einstein equations for the region after the collision. Because of nonlinearity, the resulting waves are not simply sums or products of the instant waves. It may be useful to start with the generalized Rosen form in order to find the explicit solution of Einstein's vacuum equations representing a collision between two impulsive gravitational waves

$$ds^2 = 2L(u,v)dudv - F^2(u,v)dx^2 - G^2(u,v)dy^2. \tag{3.17}$$

The non-vanishing components of the Ricci tensor, Ricci scalar and Einstein tensor are

$$
\begin{aligned}
R_{00} &= \frac{\left[\frac{(2F_uL_u-LF_{uu})}{F}+\frac{(2G_uL_u-LG_{uu})}{G}\right]}{L}, && (3.18)\\
R_{01} &= -\frac{F_{uv}}{F}-\frac{G_{uv}}{G}+2\frac{(L_uL_v-LL_{uv})}{L^2}, && (3.19)\\
R_{11} &= \frac{\left[\frac{(2F_vL_v-LF_{vv})}{F}+\frac{(2G_vL_v-LG_{vv})}{G}\right]}{L}, && (3.20)\\
R_{22} &= \frac{F(G_vF_u+F_vG_u+2GF_{uv})}{GL^2}, && (3.21)\\
R_{33} &= \frac{G(G_vF_u+F_vG_u+2FG_{uv})}{FL^2}, && (3.22)\\
R &= -\frac{(2(-FGL_vL_u+L^2(G_vF_u+F_vG_u+2GF_{uv}+2FG_{uv})+FGLL_{uv}))}{FGL^3}, && (3.23)\\
G_{00} &= \frac{\left[\frac{(2F_uL_u-LF_{uu})}{F}+\frac{(2G_uL_u-LG_{uu})}{G}\right]}{L}, && (3.24)\\
G_{01} &= \frac{G_vF_u+F_vG_u+GF_{uv}+FG_{uv}}{FG}, && (3.25)\\
G_{11} &= \frac{\left[\frac{(2F_vL_v-LF_{vv})}{F}+\frac{(2G_vL_v-LG_{vv})}{G}\right]}{L}, && (3.26)\\
G_{22} &= \frac{2F^2L^2G_{uv}+2F^2G(LL_{uv}-L_uL_v)}{GL^4}, && (3.27)\\
G_{33} &= \frac{2G^2L^2F_{uv}+2G^2F(LL_{uv}-L_uL_v)}{FL^4}. && (3.28)
\end{aligned}
$$

From now on we won't use the comma indicating partial differentiation. Using (3.21), (3.22) and (3.25) the relationship between the functions F and G can be found

$$GF_{uv} = FG_{uv}. \tag{3.29}$$

If (3.19) is used for investigating the effect of the function L to the functions F and G, then

$$\frac{G_{uv}}{G} = \frac{F_{uv}}{F} = (\ln L^{-1})_{uv}. \tag{3.30}$$

On the other hand, (3.25) implies that

$$(GF)_{uv} = 0. \tag{3.31}$$

This equation is particularly important because the structure of the solution is described here. The solution of (3.31) is

$$GF = c_1 + c_2h_1(u) + c_3h_2(v), \tag{3.32}$$

where c_1, c_2, c_3 are real constants but for the sake of simplicity they can be taken as $c_1 = c_2 = c_3 = 1$, and because of the symmetry of the metric the algebraic type of the functions h_1 and h_2 must be the same. Once again using (3.25) with the help of R_{22} and R_{33}, some series of equations can be found

$$\begin{aligned} (G_v F)_u &= -(G_u F)_v && (3.33) \\ (F_v G)_u &= -(F_u G)_v. && (3.34) \end{aligned}$$

Then putting (3.32) into (3.34), the relationship between F and H can be obtained in a different manner

$$(\ln H)_u + (\ln H)_v = (\ln F)_u + (\ln F)_v, \tag{3.35}$$

where $H = 1 + h(u) + h(v)$. If the solution in (3.31) is put into (3.30), then it becomes

$$-2(\ln F)_{uv} = (\ln H)_v (\ln F)_u + (\ln H)_u (\ln F)_v. \tag{3.36}$$

After using these last two equations and assuming H is the simplest function that satisfies them, it can be obtained that

$$F = \frac{(1 - u^2\theta(u) - v^2\theta(v))}{G}. \tag{3.37}$$

Let's write this solution again into the metric and solve the Einstein and Ricci tensors once again. Then the resulting components of the tensors are

$$\begin{aligned} &R_{00} = G_{00} = 2\frac{t^4 L G_u^2 + G^2((-1+v^2)L + ut^2 L_u)}{t^2 G^2 L}, \\ &R_{01} = \frac{2uv}{t^4} - 2\frac{G_v G_u}{G^2} + \frac{L_v L_u - L L_{uv}}{L^2}, \\ &R_{11} = G_{11} = 2\frac{t^4 L G_v^2 + G^2((-1+u^2)L + vtL_v)}{t^2 G^2 L}, \\ &R_{33} = R_{22} = 2\frac{t^2 G_u G_v + G(uG_v + vG_u - t^2 G_{uv})}{G^4 L}, \\ &G_{22} = \frac{2(uG_v + vG_u - t^2 G_{uv}) + (\frac{G(-2uvL^2 - t^4 L_u L_v + t^4 L L_{uv})}{-t^2 L^2})}{G^3 L}, \\ &G_{33} = \frac{2(t^2 G_u G_v + G(uG_v + vG_u - t^2 G_{uv}))}{L} - \frac{t^2 G^2 (\frac{2uv}{t^4} - \frac{2G_u G_v}{G^2})}{L} + \frac{t^2 G^2 (\frac{L_u L_v - L L_{uv}}{L^2})}{L}. \end{aligned} \tag{3.38}$$

Although they seem very complicated, there are only 4 linearly independent equations exist which are

$$\begin{aligned} \frac{t^2 G_u G_v}{G^2} + \frac{uG_v}{G} + \frac{vG_u}{G} - \frac{t^2 G_{uv}}{G} &= 0, \\ \frac{2uv}{t^4} - 2\frac{G_v G_u}{G^2} + \frac{L_v L_u - L L_{uv}}{L^2} &= 0, \\ \frac{G_u^2}{G^2} - \frac{1-v^2}{t^4} + \frac{uL_u}{t^2 L} &= 0, \\ \frac{G_v^2}{G^2} - \frac{1-u^2}{t^4} + \frac{vL_v}{t^2 L} &= 0, \end{aligned} \tag{3.39}$$

where $t = \sqrt{1 - u^2 - v^2}$. First two equations imply with the help of (3.22) that

$$\frac{uv}{t^4} = \frac{G_u G_v}{G^2}. \tag{3.40}$$

This equation can be simply integrated if it is separated like

$$(\ln G_1)_u = -\frac{\sqrt{1-v^2}}{t^2} \quad \text{and} \quad (\ln G_1)_v = -\frac{-uv}{\sqrt{1-v^2}t^2}, \tag{3.41}$$

or

$$(\ln G_2)_u = -\frac{-uv}{\sqrt{1-u^2}t^2} \quad \text{and} \quad (\ln G_2)_v = -\frac{\sqrt{1-u^2}}{t^2}. \tag{3.42}$$

Since these equations are particular solutions, it can be written in the form $\ln G = \ln G_1 \ln G_2$. Thus, the solution is

$$G = \sqrt{\left(\frac{\sqrt{1-v^2}+u}{\sqrt{1-v^2}-u}\right)\left(\frac{\sqrt{1-u^2}+v}{\sqrt{1-u^2}-v}\right)}. \tag{3.43}$$

After plugging this solution to the metric, the resulting Ricci and Einstein tensors are

$$\begin{aligned}
R_{00} &= G_{00} = -2u\frac{uv^2+2v\sqrt{1-u^2}\sqrt{1-v^2})L}{(1-u^2)t^4L} + \frac{L_u}{t^2L}, \\
R_{01} &= 2\frac{-1+v^2+u^2(1-2v^2)-uv\sqrt{1-u^2}\sqrt{1-v^2}}{\sqrt{1-u^2}\sqrt{1-v^2}(1-u^2-v^2)^2} - (\ln L)_{uv}, \\
R_{11} &= G_{11} = \frac{2v(u(2\sqrt{1-u^2}-2v^2\sqrt{1-u^2}+uv\sqrt{1-v^2})L}{\sqrt{(1-v^2)^3}t^4L} - \frac{L_v}{t^2L}, \\
G_{22} &= \frac{t^2(\frac{2(-1+v^2+u^2(1-2v^2)-uv\sqrt{1-u^2}\sqrt{1-v^2})}{\sqrt{1-u^2}\sqrt{1-v^2}t^4}-(\ln L)_{uv})}{\left(\frac{\sqrt{1-u^2}-v}{\sqrt{1-u^2}+v}\right)\left(\frac{\sqrt{1-v^2}-u}{u+\sqrt{1-v^2}}\right)L}, \\
G_{33} &= \frac{t^2(\frac{2(-1+v^2+u^2(1-2v^2)-uv\sqrt{1-u^2}\sqrt{1-v^2})}{\sqrt{1-u^2}\sqrt{1-v^2}t^4}-(\ln L)_{uv})}{\left(\frac{\sqrt{1-u^2}+v}{\sqrt{1-u^2}-v}\right)\left(\frac{(u+\sqrt{1-v^2})}{(\sqrt{1-v^2}-u)}\right)L}.
\end{aligned} \tag{3.44}$$

Finally using R_{00} and R_{11}, the function L can be computed as

$$L = \frac{(1-u^2-v^2)^{3/2}}{\sqrt{1-u^2}\sqrt{1-v^2}(uv+\sqrt{1-u^2}\sqrt{1-v^2})}. \tag{3.45}$$

Thus, the metric is

$$\begin{aligned}
ds^2 &= 2\frac{(1-u^2-v^2)^{\frac{3}{2}}}{\sqrt{1-u^2}\sqrt{1-v^2}(uv+\sqrt{1-u^2}\sqrt{1-v^2})}dudv \\
&\quad -(1-u^2-v^2)(\frac{(1-u\sqrt{1-v^2}-v\sqrt{1-u^2})}{(1+u\sqrt{1-v^2}+v\sqrt{1-u^2})}dx^2 \\
&\quad +\frac{(1+u\sqrt{1-v^2}+v\sqrt{1-u^2})}{(1-u\sqrt{1-v^2}-v\sqrt{1-u^2})}dy^2).
\end{aligned} \tag{3.46}$$

If it is assumed to be $u = 0$ for $v < 0$ and $v = 0$ for $u < 0$, then the metric would describe the region below $u^2 + v^2 = 1, u = 1, v = 1$. There are two types of singularity in this space-time. One of them is a curvature singularity at which curvature scalar invariants blow up at $u^2 + v^2 = 1$ (see Figure 3.2). This singularity is due to the focusing effect of waves. The other one is a coordinate singularity which

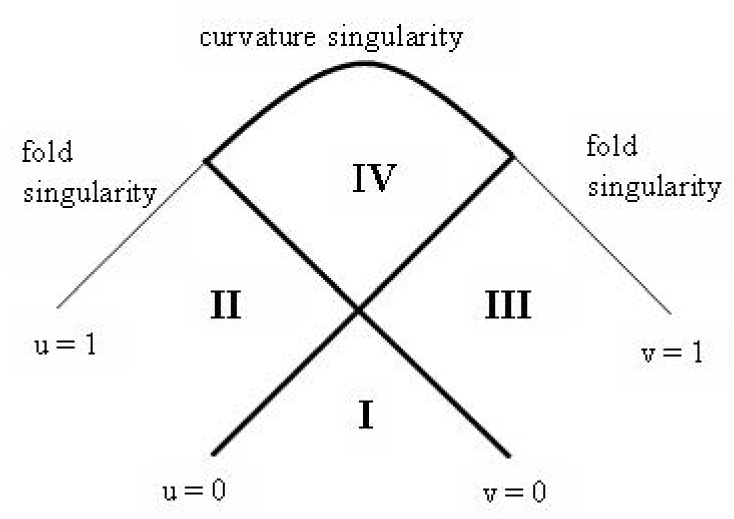

Figure 3.2: The singularity structure of the Khan-Penrose solution. The interiors of regions I, II, III are flat. However, there is a discontinuity at the boundaries. Unlike these three regions, the fourth region is curved.

can be removed by some appropriate coordinate transformations. It can be concluded that after collision plane waves would focus on each other and give up their properties. So it can be understood that the space-time is not flat anymore and the Weyl tensor is not equal to zero in anywhere of interaction region.

One can be anxious about the behaviour of a light ray, which has a null geodesic, near the boundaries for example at $v = -1$. So it can be thought that this particle can escape from the singularity but this is not the case. Because when this particle propagates through $v = -1$ hypersurface, after a time it would come across with the curvature singularity before it reaches the caustic two-surfaces (i.e. fold singularity) particle trajectory by focusing on curvature singularity. Any particle can never reach fold singularity. So one can make an inference that the curvature singularity is not avoidable.

Now one can ask why the general framework of the previous section is not used in this section. However, the Szekeres solution also includes the one that Khan-Penrose did or in other words Khan-Penrose solution is a special form of Szekeres solution[3]. Thus, one can alternatively solve the collision problem of impulsive gravitational waves by using the techniques discussed in the previous section.

3.3 Colliding Plane Electromagnetic Waves

As mentioned in the previous chapter an electromagnetic step plane wave has the metric of the form

$$ds^2 = 2d\bar{u}d\bar{v} + a^2\theta(\bar{u})(X^2+Y^2)d\bar{u}^2 - dX^2 - dY^2 \tag{3.47}$$

which can be transformed to

$$ds^2 = \begin{cases} 2dudv + \cos^2 au\,(dx^2+dy^2), & v<0, \quad u\geq 0 \\ 2dudv + \cos^2 bv\,(dx^2+dy^2), & u<0, \quad v\geq 0 \end{cases} \tag{3.48}$$

by using the coordinate transformation

$$\begin{aligned} &X = ax + by, \quad Y = ex + cy, \\ &\bar{v} = v + \tfrac{1}{2}(aa' + ee')x^2 + \tfrac{1}{2}(ba' + ab' + ec' + ce')xy + \tfrac{1}{2}(bb' + cc')y^2. \end{aligned} \tag{3.49}$$

Alternatively, the linearly polarized waves can be described by the Szekeres line element. As we found previously, the interaction region can be represented as

$$ds^2 = 2e^{-M}dudv - e^{-U}(e^V dx^2 + e^{-V}dy^2) \tag{3.50}$$

where U, V, M are functions of u and v.

The space-time we are searching is not vacuum anymore. So it has a non-vanishing energy momentum tensor

$$\begin{aligned} T_{\mu\nu} &= \frac{g^{\gamma\alpha}}{4\pi}\begin{bmatrix} F_{0\alpha}F_{0\beta} & 0 & 0 & 0 \\ 0 & F_{1\alpha}F_{1\beta} & 0 & 0 \\ 0 & 0 & F_{2\alpha}F_{2\beta} - \frac{1}{4}g_{22}F_{\alpha\beta}F^{\alpha\beta} & 0 \\ 0 & 0 & 0 & F_{3\alpha}F_{3\beta} - \frac{1}{4}g_{33}F_{\alpha\beta}F^{\alpha\beta} \end{bmatrix} \\ &= \frac{1}{4\pi}\begin{bmatrix} e^{U+V}A_u^2 & 0 & 0 & 0 \\ 0 & e^{U+V}A_v^2 & 0 & 0 \\ 0 & 0 & -A_uA_v & 0 \\ 0 & 0 & 0 & A_uA_v \end{bmatrix}, \end{aligned} \tag{3.51}$$

where A is the vector potential used in the electromagnetic theory and $F_{\mu\nu}$ is the electromagnetic field tensor.

The non-vanishing components of the Ricci tensor, the Ricci scalar and the Einstein tensor are, respectively,

$$
\begin{aligned}
R_{00} &= U_{uu} + M_u U_u - \frac{1}{2}(U_u^2 + V_u^2), & (3.52)\\
R_{01} &= U_{uv} + M_{uv} - \frac{1}{2}(U_u U_v + V_u V_v), & (3.53)\\
R_{11} &= U_{vv} + M_v U_v - \frac{1}{2}(U_v^2 + V_v^2,) & (3.54)\\
R_{22} &= e^{M-U+V}[U_{uv} - V_{uv} - U_u U_v + \frac{1}{2}(U_u V_v + U_v V_u)], & (3.55)\\
R_{33} &= e^{M-U-V}[U_{uv} + V_{uv} - U_u U_v - \frac{1}{2}(U_u V_v + U_v V_u)], & (3.56)\\
R &= e^{M}(2M_{uv} + 4U_{uv} + 3U_u U_v - V_u V_v), & (3.57)
\end{aligned}
$$

$$
\begin{aligned}
G_{00} &= U_{uu} + M_u U_u - \frac{1}{2}(U_u^2 + V_u^2), & (3.58)\\
G_{01} &= U_u U_v - U_{uv}, & (3.59)\\
G_{11} &= U_{vv} + M_v U_v - \frac{1}{2}(U_v^2 + V_v^2), & (3.60)\\
G_{22} &= e^{-U-V}(-U_{uv} - V_{uv} - M_{uv} + \frac{(U_v + V_u)(U_u + V_v)}{2}), & (3.61)\\
G_{33} &= e^{-U-V}(-U_{uv} + V_{uv} - M_{uv} + \frac{(U_v - V_v)(U_u - V_u)}{2}). & (3.62)
\end{aligned}
$$

Differentiating (3.58) and using (3.61) or (3.62) with the help of (3.53), it can be found that

$$
\begin{aligned}
2V_{uv} - U_u V_v - U_v V_u &= 4e^{U+V} A_u A_v, & (3.63)\\
V_u A_v + V_v A_u &= -2A_{uv}. & (3.64)
\end{aligned}
$$

From (3.59)

$$U = -\log[f(u) + g(v)] \tag{3.65}$$

Using (3.48) as initial values of boundary conditions

$$U = \begin{cases} -2\log\cos au, & v<0 \\ -2\log\cos bv, & u<0. \end{cases} \tag{3.66}$$

It is necessary to put some constant in order not to get zero at the boundaries:

$$e^{-U} = \begin{cases} 1, & u \le 0, \quad v \le 0 \\ \frac{1}{2} + f(u), & u \ge 0, \quad v \le 0 \\ \frac{1}{2} + g(v), & v \ge 0, \quad u \le 0 \\ f(u) + g(v), & u \ge 0, \quad v \ge 0. \end{cases} \tag{3.67}$$

According to the assumption previously made, we can find f and g as

$$\begin{aligned} f(u) &= \frac{1}{2} - \sin^2 au, && (3.68) \\ g(v) &= \frac{1}{2} - \sin^2 bv. && (3.69) \end{aligned}$$

So U becomes

$$U = -\log\cos(au + bv) - \log\cos(au - bv). \tag{3.70}$$

After getting U, V can be guessed similarly using (3.63), (3.64) and then it is easy to obtain the vector potential A:

$$\begin{aligned} V &= \log\cos(au + bv) - \log\cos(au - bv), && (3.71) \\ A &= \sin(au - bv). && (3.72) \end{aligned}$$

Now the only unknown function is M. After plugging U into the (3.58), M becomes

$$\begin{aligned} M_u &= -\frac{f_{uu}}{f_u} + \frac{f_u}{2(f+g)} - \frac{(f+g)}{2f_u}[V_u^2 + 4e^{U+V}A_u^2], && (3.73) \\ M_v &= -\frac{g_{vv}}{g_v} + \frac{g_v}{2(f+g)} - \frac{(f+g)}{2g_v}[V_v^2 + 4e^{U+V}A_v^2]. && (3.74) \end{aligned}$$

The first two terms can be reduced as

$$e^{-M} = \frac{f_u g_v}{\sqrt{f+g}} e^{-S}, \tag{3.75}$$

where S is a function satisfying the third term of (3.73) and (3.74). Then (3.73) and (3.74) become (after a transformation $u \to f$ and $v \to g$)

$$S_f = -(f+g)[V_f^2 + 4e^{U+V}A_f^2], \tag{3.76}$$

$$S_g = -(f+g)[V_g^2 + 4e^{U+V}A_g^2]. \tag{3.77}$$

Submitting the U, V and A into (3.76) and (3.77), then integrating S can be found as

$$S = \frac{1}{2}[-log(f+g) + log(\frac{1}{2}+f) + log(\frac{1}{2}+g) + log(\frac{1}{2}-f) + log(\frac{1}{2}-g)] + c, \tag{3.78}$$

where c is an integration constant. Now it's time to get M from (3.75)

$$e^{-M} = \frac{cf_u g_v}{\sqrt{\frac{1}{4} - f^2}\sqrt{\frac{1}{4} - g^2}} = 4abc. \tag{3.79}$$

The integration constant can be chosen as

$$c = \frac{1}{4ab}. \tag{3.80}$$

This means that

$$M = 0. \tag{3.81}$$

Finally the metric in the interaction region is

$$ds^2 = 2dudv - \cos^2(au - bv)dx^2 - \cos^2(au + bv)dy^2. \tag{3.82}$$

After obtaining the metric in the interaction region, the geometry of that region can be further investigated. The non-vanishing components of the Riemann tensor and the Weyl tensor are

$$\begin{aligned}
R_{0202} &= e^{2V} A_u^2 - \frac{1}{2} e^{-U+V} (V_{uu} - U_u V_u), && (3.83)\\
R_{0212} &= e^{2V} A_u A_v, && (3.84)\\
R_{0303} &= A_u^2 - \frac{1}{2} e^{-U-V} (V_{uu} - U_u V_u), && (3.85)\\
R_{1212} &= e^{2V} A_v^2 - \frac{1}{2} e^{-U+V} (V_{vv} - U_v V_v), && (3.86)\\
R_{1313} &= A_v^2 - \frac{1}{2} e^{-U-V} (V_{vv} - U_v V_v), && (3.87)\\
R_{0313} &= -A_u A_v; && (3.88)
\end{aligned}$$

$$\begin{aligned}
C_{0202} &= -\frac{1}{2} e^{-U+V} (V_{uu} - U_u V_u), && (3.89)\\
C_{0303} &= \frac{1}{2} e^{-U-V} (V_{uu} - U_u V_u), && (3.90)\\
C_{1212} &= -\frac{1}{2} e^{-U+V} (V_{vv} - U_v V_v), && (3.91)\\
C_{1313} &= \frac{1}{2} e^{-U-V} (V_{vv} - U_v V_v). && (3.92)
\end{aligned}$$

If Weyl tensor is calculated using the equations (3.71) and (3.70), then the solutions in the interior of the interaction region gives a zero valued Weyl tensor which implies that interaction region is conformally flat. But at the boundaries of the interaction region, where $u = 0$ or $v = 0$ (i.e. null boundaries), the solution is impulsive which means that colliding step wave like electromagnetic waves create impulsive gravitational waves. The reason of this creation is that gravitational waves disturb the Weyl tensor in the same way in a flat region. In addition to this unexpected result, there is no curvature singularity in the interaction region different than the Khan-Penrose solution. This can be seen by looking at the scalar invariants. One of the scalar invariants used for determining the structure of space-time is the Riemann scalar invariant $R^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta}$ which is constant for the solution at hand. This information leads us to look at the other scalar invariants, but one of the singularities can be easily seen as $u = \pi/2a$ and $v = \pi/2b$. This type of singularity can be removed by the coordinate transformations which are given by

$$\acute{u} = \cos(au + bv)\cosh(cy), \tag{3.93}$$

$$\acute{v} = \cos(au + bv)\sinh(cy), \tag{3.94}$$

$$\acute{x} = \cos(au - bv)\cos(cx), \tag{3.95}$$

$$\acute{y} = \cos(au - bv)\sin(cx), \tag{3.96}$$

where $c = \sqrt{2ab}$ [4]. There can be some other singularities which can appear in the Weyl tensor. These can also be removed with another appropriate coordinate transformations.

CHAPTER 4

PP WAVES

In this chapter, first the derivation of pp waves and their unique properties are discussed. Then the only solution of collision of pp waves are given.

4.1 The Class of PP Waves

In the first chapter the derivation of generic gravitational plane waves, whose line element can be thought as a perturbation of flat Minkowski metric, were given as

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}. \tag{4.1}$$

After getting the Einstein field equations, the wave equation is found. Then using the propagation property of plane waves and light-cone coordinates the metric is found to be

$$ds^2 = 2dUdV + (\delta_{ij} + h_{ij}(U))dx^i dx^j. \tag{4.2}$$

If one forgets that the perturbation is small, then the equation (4.2) is simply

$$ds^2 = 2dUdV + g_{ij}(U)dx^i dx^j. \tag{4.3}$$

Here, it is worth noting that there is a Killing vector which is

$$\xi = \partial_v. \tag{4.4}$$

Let ξ be a null vector field that satisfies

$$\nabla_\mu \xi^\nu = 0 \tag{4.5}$$

or equivalently,

$$\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0, \tag{4.6}$$

$$\nabla_\mu \xi_\nu - \nabla_\nu \xi_\mu = 0. \tag{4.7}$$

So it can be found that

$$\xi_\mu = g_{\mu\nu} \tag{4.8}$$

where the null property implies that

$$\xi_v = g_{vv} = 0. \tag{4.9}$$

The Killing equation states that the line element is independent of v, whereas (4.7) implies

$$\xi_\mu = g_{v\mu} = \partial_\mu u. \tag{4.10}$$

Then the metric can be written as [28]

$$ds^2 = 2dudv + P(u, x^c)du^2 + 2R_a(u, x^c)dx^a du + S_{ab}(u, x^c)dx^a dx^b. \tag{4.11}$$

The special case of this metric is having $S_{ab}(u, x^c) = \delta_{ab}$ which is

$$ds^2 = 2dudv + P(u, x^c)du^2 + 2R_a(u, x^c)dx^a du. \tag{4.12}$$

This type of metric describes plane fronted waves with parallel rays or shortly pp waves. Here plane fronted means that the wave is non-twisting and non-expanding. Parallelism is satisfied with the parallel null vector. If bivectors exist in a normal hyperbolic V_4, the metric can be written in the metric form ofBrinkmann's definition of pp waves [26]

$$ds^2 = 2dudv + H(u, x, y)du^2 + dx^2 + dy^2. \tag{4.13}$$

The only surviving Ricci tensor component is

$$R_{11} = -\frac{1}{2}(H_{xx} + H_{yy}). \tag{4.14}$$

The Ricci scalar is therefore equal to zero and the linearly independent non-vanishing Weyl tensor components are

$$C_{1341} = \frac{1}{2}H_{xy} \tag{4.15}$$
$$C_{1331} = \frac{1}{2}(H_{xx} + H_{yy}). \tag{4.16}$$

Brinkmann also proved [27] an interesting property that vacuum fields (i.e. pp waves) can be mapped conformally on each other. This raises to another surprising property of the pp waves; namely, having the superposition principle although Einstein field equations are non-linear. They can propagate through each other without any effect.

They also pass through a flat space-time without giving any change [29]. Non-flat spaces can also be interpreted as the pp waves if they admit a constant bivector [26]. Another conclusion that can be derived is that pp waves can not be pictured like the plane waves. Because the pp wave metric has a component which includes the spacelike coordinates. Recall that the figures in the previous chapters are obtained by suppressing the spacelike coordinates.

The plane waves which are discussed in chapter 1 are so defined that they are homogeneous pp waves: the amplitude is constant everywhere on the wave. A general plane wave metric is a very special subclass of pp waves which has $R_a(u, x^b) = 0$ and $P(u, x^c)$ quadratic in x^c. Then we can write P as

$$P = P_{ab}x^a x^b. \tag{4.17}$$

As a result pp wave metric becomes like this

$$ds^2 = 2dudv + 2P_{ab}x^a x^b du^2 - dx^2. \tag{4.18}$$

4.2 Colliding Axisymmetric PP Waves

The metric of plane-fronted waves with parallel rays (i.e.pp waves) can be written in cylindrical coordinates

$$ds^2 = 2dud\bar{v} + 2Hdu^2 - dP^2 - P^2 d\Phi^2, \tag{4.19}$$

where $H = H(u, P, \Phi)$ [20]. Here our first aim is to get a metric without non-diagonal terms. In order to do that, the following equations must be satisfied:

$$\begin{aligned}
&\bar{v} = v + \bar{v}_1(u, r, \varphi), && \text{for } g_{uv} = 1, && (4.20)\\
&P_v = \Phi_v = 0, && \text{for } g_{vv} = 0, && (4.21)\\
&P_r P_\varphi + P^2 \Phi_r \Phi_\varphi = 0, && \text{for } g_{r\varphi} = 0, && (4.22)\\
&\bar{v}_{1r} = P_u P_r, && \text{for } g_{ur} = 0, && (4.23)\\
&\bar{v}_{1\varphi} = P^2 \Phi_u \Phi_\varphi, && \text{for } g_{u\varphi} = 0. && (4.24)
\end{aligned}$$

First integrating (4.23) and then differentiating it with respect to φ in order to obtain a similar left hand side with 4.24) and compare them. In this way, it can be found that $P^2\Phi_u\Phi_\varphi = 0$ since $P_\varphi = 0$. The simplest assumption we can make about Φ is that $\Phi_u = 0$ and $\Phi_\varphi = constant$. This leads to

$$\Phi = \varphi. \tag{4.25}$$

Lastly to obtain $g_{uu} = 0$, the equation that must be satisfied is

$$2H = -2\bar{v}_{1u} + P_u^2. \tag{4.26}$$

This means

$$H = H(u, P). \tag{4.27}$$

Using (4.26) and (4.27),

$$P_{uu} = -H_P \tag{4.28}$$

can be easily found. So the metric becomes

$$ds^2 = 2dudv - Q^2 dr^2 - P^2 d\varphi^2. \tag{4.29}$$

The non-vanishing components of the Ricci and the Einstein tensor are

$$\begin{aligned}
R_{00} &= -\tfrac{Q_{uu}}{Q} - \tfrac{P_{uu}}{P},\\
R_{01} &= -\tfrac{Q_{uv}}{Q} - \tfrac{P_{uv}}{P},\\
R_{11} &= -\tfrac{Q_{vv}}{Q} - \tfrac{P_{vv}}{P},\\
R_{02} &= -\tfrac{1}{P}(P_{ur} - \tfrac{Q_u P_r}{Q}),\\
R_{12} &= -\tfrac{1}{P}(P_{vr} - \tfrac{Q_v P_r}{Q}),\\
R_{22} &= \tfrac{(P_r Q_r + Q^2(Q_v P_u + Q_u P_v) + 2Q^2 P Q_{uv} - Q P_{rr})}{PQ},\\
R_{33} &= P(2P_{uv} + \tfrac{Q_v P_u + Q_u P_v}{Q}) + \tfrac{P Q_r P_r}{Q^3} - \tfrac{P P_{rr}}{Q^2}),
\end{aligned} \tag{4.30}$$

$$\begin{aligned}
G_{00} &= -\tfrac{Q_{uu}}{Q} - \tfrac{P_{uu}}{P},\\
G_{01} &= \tfrac{P_r Q_r + Q(-P_{rr} + Q(Q_v P_u + Q_u P_v + Q P_{uv} + Q_{uv} P))}{PQ^3},\\
G_{11} &= -\tfrac{Q_{vv}}{Q} - \tfrac{P_{vv}}{P},\\
G_{20} &= \tfrac{Q_u P_r - Q P_{ur}}{PQ},\\
G_{21} &= \tfrac{Q_v P_r - Q P_{vr}}{PQ},\\
G_{22} &= -\tfrac{2Q^2 P_{uv}}{P},\\
G_{33} &= -\tfrac{2P^2 Q_{uv}}{Q}.
\end{aligned} \tag{4.31}$$

These equations can be written shortly [18]

$$PQ_{uu} + QP_{uu} = 0, \tag{4.32}$$

$$PQ_{uu} + QP_{uu} = 0, \tag{4.33}$$

$$QP_{ur} + Q_u P_r = 0, \tag{4.34}$$

$$QP_{vr} + Q_v P_r = 0, \tag{4.35}$$

$$Q_{uv} = P_{uv} = 0, \tag{4.36}$$

$$Q^2(Q_v P_u + Q_u P_v) - QP_{rr} + Q_r P_r = 0. \tag{4.37}$$

When $P_r = Q_r = 0$, these equations are equivalent to the ones for the plane wave equations. If they are not equal to zero, (4.35) and (4.36) help to find Q

$$Q = H(r)P_r \tag{4.38}$$

where $H(r)$ is an arbitrary function. This function can be found using initial conditions. If Minkowski space-time is considered, it can be found that $H(r) = 1$, which gives

$$Q = P_r. \tag{4.39}$$

With plugging (4.39) into (4.32) and (4.33), one gets

$$(P_{uu}P)_r = (P_{vv}P)_r = 0. \tag{4.40}$$

If the metric is equal to the Szekeres metric, this equation can also be written in the form

$$\left(e^{-U}\right)_{uv} = 0. \tag{4.41}$$

where $P = e^{-\frac{U+V}{2}}$ and $Q = e^{\frac{-U+V}{2}}$. If (4.39) is put into (4.37), then

$$(P_u P_v)_r = 0. \tag{4.42}$$

There are two cases to consider in (4.40): One is to assume $P_{uu} = P_{vv} = 0$. Then the solution is

$$P = b_0(r) - b_1(r)u - b_2(r)v. \tag{4.43}$$

The other one is to take $P_{uu} \neq 0$ and $P_{vv} \neq 0$. Then it reduces to

$$\left(\frac{P_{uu}}{P_{vv}}\right)_r = 0, \tag{4.44}$$

or

$$(P_{uu}P_{vv})_r = 0. \tag{4.45}$$

These pair of equations can be written alternatively as

$$P_{uur}P_{vv} = 0, \tag{4.46}$$

$$P_{uu}P_{vvr} = 0. \tag{4.47}$$

The only possibility is $P_{uur} = P_{vvr} = 0$. This implies $P_r = Q = 0$ from (4.39). This is a solution for the hypersurface of this space-time.

Here Ivanov mentions that there is another possibility that $P_{vv} = 0$ and $P_{uu} = 0$, but this contradicts with the assumption given before (4.44) and (4.45), thus it can be neither $P_{uu} = 0$ nor $P_{vv} = 0$. Another objection can be about the amplitudes of the waves which are found by (4.46) and (4.47). The result implies that if one wave comes from null direction u with amplitude $b_1(r)$, then the other wave has $\frac{a}{b_1(r)} = b_2(r)$, where a is an arbitrary constant.

The metric in the equation (4.29) represents the interaction region which can be also generalized to the regions where, the instant waves present.

$$ds^2 = 2du_i d\bar{v}_i + 2H_i(P)\delta(u_i)du_i^2 - P_r^2 dr^2 - P^2 d\varphi^2. \tag{4.48}$$

where $u_1 = u$, $u_2 = v$, $\bar{v}_1 = v + \acute{v}_1(u, P)$, $\bar{v}_2 = u + \acute{v}_2(v, P)$, $(H_i(r))_r = b_i(r)$.

$$P = r - bu\theta(u) - \frac{a}{b}v\theta(v) \tag{4.49}$$

$$Q = 1 - b_r u\theta(u) + \frac{ab_r}{b^2}v\theta(v) \tag{4.50}$$

Actually this solution is similar to Babala's solution [19]. However, there is a problem on the boundary such as

$$P_{uu} \neq 0 \;\; \text{and} \;\; P_{vv} \neq 0. \tag{4.51}$$

As mentioned in (4.41), this metric is also a solution to the Szekeres metric in cylindrical coordinates when $M = 0$,

$$ds^2 = 2dudv - e^{-U}(e^V dr^2 + e^{-V} d\varphi^2). \tag{4.52}$$

So the boundary conditions are expected to be satisfied by this solution.

It is time to look at the geometry of the interaction region by examining the Riemann and the Weyl tensors and one scalar invariant:

$$\begin{aligned} R_{0202} &= QQ_{uu}, \\ R_{0303} &= PP_{uu}, \\ R_{1212} &= QQ_{vv}, \\ R_{1313} &= PP_{vv}, \end{aligned} \tag{4.53}$$

$$\begin{aligned} C_{0202} &= \tfrac{1}{2}Q(Q_{uu} - \tfrac{QP_{uu}}{P}), \\ C_{0303} &= \tfrac{1}{2}P(P_{uu} - \tfrac{PQ_{uu}}{Q}), \\ C_{1212} &= \tfrac{1}{2}Q(Q_{vv} - \tfrac{QP_{vv}}{P}), \\ C_{1313} &= \tfrac{1}{2}P(P_{vv} - \tfrac{PQ_{vv}}{Q}), \end{aligned} \tag{4.54}$$

$$I = R^2_{\mu\nu\alpha\beta} = 8\frac{P_{uu}P_{vv}}{P^2} + 8\frac{Q_{uu}Q_{vv}}{Q^2}. \tag{4.55}$$

First of all if $P = 0$, then this is an unavoidable singularity in other words it is the curvature singularity. If $P \neq 0$, then all the components of the Riemann tensor are zero except at the boundaries. So the components of Weyl tensor and scalar invariant are. This fact implies that the space-time is conformally flat except the boundaries. Moreover, the Weyl tensor supports that the approaching waves are impulsive since it is not equal to zero.

CHAPTER 5

CONCLUSION

In this work, collision of impulsive gravitational waves and collision of step electromagnetic waves are examined. Although the impulsive gravitational wave solution does not exlicitly exist, i believe that i did solve it as Khan-Penrose did by using only field equations. The structures of both impulsive gravitational wave collision and step electromagnetic wave collision are studied. Even though the former has a non-flat interaction region, the latter has a flat interaction region. Another interesting property of collision of electromagnetic waves is that they produce gravitational waves at the boundaries after the collision.

In this thesis, the collision of diagonalized axisymmetric pp waves, which are generalized plane waves, are reviewed. In this solution even though the collision doesn't take place between two equivalent gravitational waves, they are found to have impulsive characteristics. It is shown that the resulting region is conformally flat but does no longer have the pp wave space-time feature anymore. This is a surprising result since one expects them to pass through without seeing each other. It is also mentioned that there are some curvature singularities across on the boundaries similar to the Bell-Szekeres solution. However, the boundaries are problematic because of not being satisfied by the field equations. In spite of having drawbacks, it can be expected from this solution to cover some of the other plane wave solutions.

We need to study more on this subject in order to clarify the problematic parts. However, before that, more work and knowledge are strongly needed in order to cover the pp waves rather than using plane wave solutions or symmetries. Because Szekeres framework is valid only for the plane waves. One should generalize this framework further for pp waves. Another problematic part is the visualization of collision of pp waves in four dimensional space-time.

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