H^p SPACES AND INEQUALITIES IN ERGODIC THEORY

BY

SAKIN DEMIR

Dipl., University of Yüzüncü Yil, 1991
M.S., University of Illinois, 1996

THESIS

Submitted in partial fulfillment of the requirements
for the degree of Doctor of Philosophy in Mathematics
in the Graduate College of the
University of Illinois at Urbana-Champaign, 1999

Urbana, Illinois

UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN

THE GRADUATE COLLEGE

APRIL 1999
(date)

WE HEREBY RECOMMEND THAT THE THESIS BY

SAKIN DEMIR

ENTITLED H^p SPACES AND INEQUALITIES IN ERGODIC THEORY

BE ACCEPTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR

THE DEGREE OF DOCTOR OF PHILOSOPHY

Jop Rosenblatt
Director of Thesis Research

Ph. Tondeur
Head of Department

Committee on Final Examination†

I. D. Berg
Chairperson

Joy Rosenblatt

Zhong-Jin Ruan

Jang-Mei Wu

Richard Sowers

† Required for doctor's degree but not for master's.

O-517

UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN

GRADUATE COLLEGE FORMAT APPROVAL

THIS IS TO CERTIFY THAT THE FORMAT AND QUALITY OF PRESENTATION OF THE THESIS

SUBMITTED BY SAKIN DEMIR AS ONE OF THE

REQUIREMENTS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

ARE ACCEPTABLE TO THE DEPARTMENT OF MATHEMATICS

Full Name of Department, Division or Unit

APRIL 7, 1999

Date of Approval

Departmental Representative

Abstract

This thesis combines the theory of ergodic H^p spaces, singular integral operators and the theory of Banach space-valued operators to establish a new method to study the inequalities for the operators induced by an ergodic, measure preserving transformation. This method also indicates the close connection between the classical theory of H^p spaces and ergodic H^p spaces. By means of this connection a one sided analog of a result of B. Davis [21] for martingale square function is proven for ergodic square function, and it is shown that one can prove the same result for a large class of operators in ergodic theory by the same method. As a corollary it is shown that one can find the integrability condition for the same class of operators analog to a result of D. S. Ornstein [40] for the ergodic maximal function. Furthermore, it is shown that one can extend the problems of classical H^p spaces to the ergodic H^p spaces, and as an application of this extension a number of inequalities in ergodic theory are proven for a large class of operators. Finally, it is shown in various perspectives that one can study the vector-valued inequalities in ergodic theory as in classical harmonic analysis. To study these inequalities some methods are introduced, and by means of these methods a large class of operators in ergodic theory is discussed extensively and various types of vector-valued inequalities are proven.

Dedication

To My Parents, Şevket and Döne Demir

Acknowledgments

I believe that everybody who taught me mathematics has a contribution to this work. More specifically, I would like to thank Professor Joseph Rosenblatt for being a good advisor as well as a good friend during the preparation of this work. I also would like to thank for his generous financial support granted by the National Science Foundation in the form of research assistantship.

Contents

Introduction

The behavior of some maximal functions play an interesting role in analysis. It is interesting to understand the behavior of an L_p function for which some maximal function is still in L_p. Here we consider the spaces of such functions called the H^p spaces. These spaces are of interest in harmonic analysis, in the theory of complex functions and also in martingale theory. In many cases if the related maximal function behaves well, then because of the atomic decomposition of an H^p function it is much easier to work with a function in H^p rather than a function in L_p. R. Coifman and G. Weiss [19] studied the space ergodic H^1, which they defined as the space of functions in $L_1(X)$ whose ergodic Hilbert transform is in $L_1(X)$. They proved that H^1 can be characterized in terms of maximal functions and that the dual of H^1 can be identified with the space of functions of ergodic bounded mean oscillation (EBMO). Later R. Caballero and A. de la Torre [13] found that these spaces can also be characterized in terms of atoms as in the classical theory of H^p spaces. We borrow the background materials mostly from their work. We show that these spaces and the classical H^p spaces are closely related, and present a new method of study for the behavior of operators in ergodic theory. We use our method to investigate some inequalities for certain operators. When studying L_p norm inequalities there are a lot of interpolation theorems available for

$1 < p < \infty$. Unfortunately, this is not the case for $p = 1$. For this reason it requires a completely different framework to discover the integrability of operators. Our method shows that this kind of problems can be studied when we work in ergodic $H^1(X)$ spaces. For this reason we reduce the problem to the behavior of some maximal functions, or we can simply work with an ergodic atom if needed. At the same time we introduce the close connection with the classical H^p theory. Also, we try to present almost all the proofs in detail for completeness. We first present the background materials from ergodic theory, and next we present some known results about atomic characterization of the ergodic H^p spaces, and finally we move to our main discussion and study the connection with the classical case and the behaviors of operators. Later, we prove that vector-valued inequalities can be studied in ergodic theory. We first show that our method can be extended to study these inequalities, then we show that one can also study these inequalities by means of the Calderón-Zygmund decomposition on $\mathbb{Z}$. Many other problems in the area still remain open, of course.

Chapter 1

The Fundamentals of Ergodic Theory

We will include this chapter as a preparation for our further discussion, therefore we will only state a theorem without giving a detailed proof if it is a well known result, and will refer the readers to a suitable reference for its proof.

Let us start with some basic definitions:

Let (X, β, m) be a non-atomic, complete probability space, that is a set X together with a σ-algebra β of measurable subsets of X and a countably additive non-negative set function m on β such that $m(X) = 1$ and such that β contains all subsets of sets of measure 0. We say that the sets E and F are *equal* if they agree up to a set of measure zero. The transformation $\tau : X \to X$ is *invertible* if there is a measurable transformation $\sigma : X \to X$, such that $\tau\sigma(E) = \sigma\tau(E) = E$. In this case, σ is unique and is denoted by τ^{-1}. Let $\tau : X \to X$ be a one-to-one onto map such that τ and τ^{-1} are both measurable. Then τ is called a *measure-preserving transformation* (abbreviated m.p.t.) if $m(\tau^{-1}E) = m(E)$ for all $E \in \beta$, and the systems (X, β, m, τ) are the main objects of study in ergodic theory. A set

E is called *invariant* if $\tau(E) = E$; if the only invariant sets are X and ϕ, then τ is called *ergodic.*

1.1 Mean Ergodic Theorems

The following ergodic theorems can be found in K. Petersen [43] in which one can also find a good selection of references concerning the well-known results in ergodic theory.

Theorem 1.1. (Mean Ergodic Theorem (von Neumann 1932)) *Let (X, β, m) be a σ-finite measure space, $\tau : X \to X$ a measure-preserving transformation, and $f \in L_2(X, \beta, m)$. Then there is a function $\bar{f} \in L_2(X, \beta, m)$ for which*

$$\lim_{n\to\infty} \left\| \frac{1}{n} \sum_{k=0}^{n-1} f \circ \tau^k - \bar{f} \right\|_2 = 0.$$

Note that the m.p.t. $\tau : X \to X$ determines a linear operator $U : L_2 \to L_2$ according to $(Uf)(x) = f(\tau x)$. U is an isometry, since

$$\|Uf\|_2^2 = \int |Uf(x)|^2 \, dm = \int |f(\tau x)|^2 \, dm = \|f\|_2^2.$$

U is also invertible $(U^{-1}f(x) = f(\tau^{-1}x))$, and hence U is a unitary operator (its adjoint equals its inverse). $U = U_\tau$ is called the *unitary operator induced by* τ. Then we have the following more general theorem than Theorem 1.1.

Theorem 1.2. (Mean Ergodic Theorem in Hilbert Space) *Let U be a linear contraction on a Hilbert space H (so that $\|Uf\| \leq \|f\|$ for all $f \in H$), let $M = \{f \in H : Uf = f\}$ (so M is a closed linear subspace of H), and let $P : H \to M$ be the projection of H onto M. Then* *for each $f \in H$,*

$$\frac{1}{n}\sum_{k=0}^{n-1} U^k f$$

converges in H to Pf.

Proof. Let N denote the closed linear span of $\{g - Ug : g \in H\}$; we claim that N and M are orthogonal complementary subspaces of H. It suffices to show that

$$N^{\perp} = \{h \in H :< h, n >= 0 \text{ for all } n \in N\}.$$

Let $h \in N^{\perp}$; then $< h, g - Ug >= 0$ for all $g \in H$, so

$$\begin{aligned} 0 &=< h, g > - < h, Ug > \\ &=< h, g > - < U^*h, g > \\ &=< h - U^*h, g > \end{aligned}$$

for all $g \in H$, and hence $h = U^*h$. We wish to show that $Uh = h$. Now

$$\begin{aligned} \|Uh - h\|^2 &=< Uh - h, Uh - h > \\ &= \|Uh\|^2 - < h, Uh > - < Uh, h > + \|h\|^2 \\ &= \|Uh\|^2 - < U^*h, h > - < h, U^*h > + \|h\|^2 \\ &\leq \|h\|^2 - < h, h > - < h, h > + \|h\|^2 \\ &= 0. \end{aligned}$$

Therefore $Uh = h$, so $h \in M$. We conclude that $N^{\perp} \subset M$.

Conversely, if $h \in M$, then $Uh = h$ and hence $U^*h = h$ by the same argument applied to

U^* which is also a contraction since

$$\|U^*f\|^2 = |< U^*f, U^*f >|$$

$$= |< UU^*f, f >|$$

$$\leq \|UU^*f\|\|f\|$$

$$\leq \|U^*f\|\|f\|.$$

Therefore, for any $g \in H$,

$$< h, g - Ug > = < h, g > - < h, Ug >$$

$$= < h - U^*h, g >$$

$$= 0,$$

so $h \in N^\perp$. Consequently $N^\perp = M$.

We will next show that if $f \in N$ then

$$\frac{1}{n}\sum_{k=0}^{n-1} U^k f$$

converges to zero. First, if $f = g - Ug$ for some $g \in H$, then

$$\frac{1}{n}\sum_{k=0}^{n-1} U^k f = \frac{1}{n}(g - U^n g),$$

so

$$\left\|\frac{1}{n}\sum_{k=0}^{n-1} U^k f\right\| \leq \frac{2}{n}\|g\| \to 0.$$

If $f \in N$, then there is a sequence $\{g_i\}$ in H such that $f_i = g_i - Ug_i$ converges to f. Then

$$\left\|\frac{1}{n}\sum_{k=0}^{n-1} U^k f\right\| \leq \left\|\frac{1}{n}\sum_{k=0}^{n-1} U^k (f - f_i)\right\| + \left\|\frac{1}{n}\sum_{k=0}^{n-1} U^k f_i\right\|$$

for each i. Given $\epsilon > 0$, choose i so that

$$\|f - f_i\| < \frac{1}{2}\epsilon,$$

and choose n so that

$$\left\|\frac{1}{n}\sum_{k=0}^{n-1} U^k f_i\right\| < \frac{1}{2}\epsilon;$$

then

$$\left\|\frac{1}{n}\sum_{k=0}^{n-1} U^k f\right\| < \frac{1}{2}\epsilon + \frac{1}{2}\epsilon = \epsilon.$$

Now if $f \in H$ is arbitrary, there is a unique $f_0 \in N$ such that $f = f_0 + Pf$. Then

$$\begin{aligned}\left\|\frac{1}{n}\sum_{k=0}^{n-1} U^k f - Pf\right\| &= \left\|\frac{1}{n}\sum_{k=0}^{n-1} U^k (f_0 + Pf) - Pf\right\| \\ &= \left\|\frac{1}{n}\sum_{k=0}^{n-1} U^k f_0\right\| \to 0.\end{aligned}$$

Thus the proof is complete. □

Theorem 1.3. (Mean Ergodic Theorem in Banach Space) *Let Y be a Banach space and $\tau : Y \to Y$ a continuous linear map satisfying $\|\tau^k\| \leq M$ for all $k = 1, 2, \ldots$ for some constant M. Let $y \in Y$. Then the following statements are equivalent:*

(i) $\{\frac{1}{n}\sum_{k=0}^{n-1} \tau^k y : n = 1, 2, \ldots\}$ *converges with respect to the norm on Y.*

(ii) $\{\frac{1}{n}\sum_{k=0}^{n-1} \tau^k y : n = 1, 2, \ldots\}$ *has a weak cluster point.*

(iii) τ *has a fixed point in the weakly closed convex hull of $\{\tau^k y : k = 0, 1, 2, \ldots\}$ (i.e. the intersection of all the weakly closed convex sets containing $\{\tau^k y : k = 0, 1, 2, \ldots\}$).*

Proof. (i) $\Rightarrow$ (ii): Clear.

(ii) $\Rightarrow$ (iii): Abbreviating

$$A_n y = \frac{1}{n}\sum_{k=0}^{n-1} \tau^k y,$$

suppose that a subnet $A_{n_\alpha}y$ converges weakly to $\bar{y}$. then for any continuous linear functional $\phi \in Y^*$,

$$< \tau\bar{y}, \phi > =< \bar{y}, \tau^*\phi >= \lim < A_{n_\alpha}y, \tau^*\phi >$$

$$= \lim < \tau A_{n_\alpha}y, \phi >$$

$$= \lim < A_{n_\alpha}y, \phi >$$

$$=< \bar{y}, \phi >$$

because

$$\tau A_n y - A_n y = \frac{\tau^{n+1}y - \tau y}{n} \to 0.$$

Thus $\tau\bar{y} = \bar{y}$. Of course $\bar{y}$, being the weak limit of convex combinations of $\{\tau^k y\}$, is in the weakly closed convex hull of $\{\tau^k y\}$.

(iii) $\Rightarrow$ (i): Let $\bar{y}$ be the fixed point. Since

$$\tau^k y = \tau^k\bar{y} + \tau^k(y - \bar{y}) = \bar{y} + \tau^k(y - \bar{y}),$$

we have

$$A_n y = \bar{y} + A_n(y - \bar{y}),$$

and it is enough to prove that $A_n(y - \bar{y}) \to 0$ in Y. Now $\bar{y}$ is a weak limit of convex combinations

$$S_\alpha y = \sum_{i=1}^{m_\alpha} \lambda_{i\alpha}\tau^i y, \quad \lambda_{i\alpha} \geq 0, \ \sum_i \lambda_{i\alpha} = 1.$$

This implies that

$$y - S_\alpha y = \sum_i \lambda_{i\alpha}(y - \tau^i y)$$

$$= \sum_i \lambda_{i\alpha}(I-\tau)(I+\tau+\tau^2+\cdots+\tau^{i-1})y$$

$$\in \text{range}(I-\tau),$$

so that $y-\bar{y}$ is in the weak closure of range$(I-\tau)$, hence also in $\overline{\text{range}(I-\tau)}$ (weak and strong closure coincide for subspaces-see H. L. Royden [44], p.201). Now it is easy to show that

$$A_n(y-\bar{y}) \to 0.$$

For, given $\epsilon > 0$, choose $x \in \text{range}(I-\tau)$ with

$$\|x-(y-\bar{y})\| < \epsilon.$$

Clearly, $A_n x \to 0$, since

$$A_n(I-\tau) = \frac{(\tau - \tau^{n+1})}{n}.$$

Thus for large enough n,

$$\|A_n(y-\bar{y})\| \le \|A_n x\| + \|A_n(x-(y-\bar{y}))\|$$

$$< \epsilon + M\epsilon.$$

□

Remark 1.4. "Weakly closed" can be replaced by "norm closed" in (iii) above, since in a locally convex topological vector space a convex set is strongly closed if and only if it is weakly closed (see H. L. Royden [44], p.206).

Lemma 1.5. (Kakutani-Rokhlin Lemma). *Let $\tau : X \to X$ be an ergodic, measure-preserving transformation on a non-atomic measure space (X, β, m), n a positive integers, and $\epsilon > 0$.*

Then there is a measurable set $A \subset X$ such that $A, \tau A, \ldots, \tau^{n-1}A$ are pairwise disjoint and cover X up to a set of measure less than ϵ.

We have the following question:

Problem 1.6. *Let $\tau : X \to X$ be an ergodic, measure-preserving transformation on a non-atomic measure space (X, β, m), and $\epsilon > 0$. Let k be a positive integer, then it might be interesting to determine those sequences (n_k) of integers for which there is a measurable set $A \subset X$ such that $\tau^{n_1}A, \tau^{n_2}A, \ldots, \tau^{n_k}A$ are pairwise disjoint and cover X up to a set of measure less than ϵ.*

1.2 The Ergodic Maximal Function

We will now introduce and present some properties of the ergodic maximal function using the approach of R. L. Jones [28], and also discuss the ergodic theorem of G. D. Birkhoff [4].

In ergodic theory we consider the ergodic maximal function, defined by

$$f^*(x) = \sup_n \frac{1}{n} \sum_{k=0}^{n-1} |f(\tau^k x)|,$$

which is the generalization of the Hardy-Littlewood maximal function $H_L f$ on $\mathbb{R}^n$, given by

$$H_L f(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(t)|\, dt,$$

where Q denotes an arbitrary cube in $\mathbb{R}^n$.

Later we will see that the functions $H_L f$ and f^* are closely related. If the function $H_L f$ has certain properties then one might expect to see the analog behavior for the function f^* under the corresponding assumptions. The origin of this transference argument comes from the famous work of A. P. Calderón [14] in which he proves the maximal ergodic theorem by using the analogous behavior of the Hardy-Littlewood maximal function $H_L f$.

Lemma 1.7. (The Kakutani Decomposition) *If τ is ergodic, invertible, and measure-preserving, and X is a probability space, then given B, with $m(B) > 0$, there exist disjoint sets*

$$B_i^j, \ 0 \le i \le j-1, \ 1 \le j < \infty,$$

such that

$$B = \bigcup_{j=1}^{\infty} B_0^j,$$

$$X = \bigcup_{j=1}^{\infty} \bigcup_{i=0}^{j-1} B_i^j,$$

and $\tau(B_i^j) = B_{i+1}^j$ unless $i = j-1$.

Theorem 1.8. (The Calderón-Zygmund Decomposition) *Given f in $L_1(\mathbb{R}^n)$, $f \ge 0$ and $\lambda > \|f\|_1$, there exists a collection of non-overlapping cubes $\{Q_n\}_{n=1}^{\infty}$ such that*

(i) *for a.e. $x \notin \bigcup_{n=1}^{\infty} Q_n$, $f(x) < \lambda$;*

(ii) *there exist c_1, c_2, independent of f, λ and Q_n, such that*

$$c_1 \lambda \le \frac{1}{|Q_n|} \int_{Q_n} f(x)\, dx \le c_2 \lambda.$$

The following decomposition, using f^* is similar to one given by Calderón-Zygmund using $H_L f$.

Theorem 1.9. *Given f in $L_1(X)$, $f \ge 0$, and $\lambda > \|f\|_1$, there exists a collection of disjoint sets $\{C_n\}_{n=2}^{\infty}$ such that*

(i) *for a.e. $x \notin \bigcup_{n=2}^{\infty} C_n$, $f(x) \le \lambda$;*

(ii) *for each n, $\lambda \le \frac{1}{m(C_n)} \int_{C_n} f(x)\, dm(x) \le 2\lambda$.*

Proof. Let

$$B = \{x \in X : f^*(x) \leq \lambda\}.$$

With this B, we form the Kakutani decomposition Lemma 1.7. Let

$$C_n = \bigcup_{j=1}^{n-1} B_j^n = \bigcup_{j=1}^{n-1} \tau^j B_0^n.$$

These C_n correspond to the cubes Q_n of the Calderón-Zygmund decomposition Theorem 1.8.

If $x \notin \bigcup_{n=2}^{\infty} C_n$, then $x \in B$ so that $f^*(x) \leq \lambda$, and, in particular, $f(x) \leq \lambda$. The first assertion is proved.

Now, if $x \in B_0^n$, then $f^*(x) \leq \lambda$. Since $f \geq 0$, this shows that

$$\sum_{k=1}^{n-1} f(\tau^k x) \leq n\lambda.$$

Consequently,

$$\begin{aligned}
\frac{1}{m(C_n)} \int_{C_n} f(x)\, dm(x) &= \frac{1}{(n-1)m(B_0^n)} \sum_{k=1}^{n-1} \int_{B_k^n} f(x)\, dm(x) \\
&= \frac{1}{(n-1)m(B_0^n)} \sum_{k=1}^{n-1} \int_{\tau^k B_0^n} f(x)\, dm(x) \\
&= \frac{1}{(n-1)m(B_0^n)} \int_{B_0^n} \sum_{k=1}^{n-1} f(\tau^k x)\, dm(x) \\
&\leq \frac{1}{(n-1)m(B_0^n)} \int_{B_0^n} \lambda n\, dm(x) \leq 2\lambda.
\end{aligned}$$

This proves the right-hand inequality in the second assertion of the theorem.

To prove the left-hand inequality let us replace f by $g = f - \lambda$ in order to make the notation neater. Then

$$B = \{x \in X : \sup_{n>0} \frac{1}{n} \sum_{k=0}^{n-1} g(\tau^k x) \leq 0\}.$$

We need to show that

$$\frac{1}{m(C_n)} \int_{C_n} g(x)\, dm(x) \geq 0,$$

or equivalently,

$$\int_{C_n} g(x)\, dm(x) \geq 0.$$

It is sufficient to show

$$\sum_{k=1}^{n-1} g(\tau^k x) \geq 0 \quad \text{for a.e. } x \in B_0^n,$$

since then

$$\int_{C_n} g(x)\, dm(x) = \int_{B_0^n} \sum_{k=1}^{n-1} g(\tau^k x)\, dm(x) \geq 0.$$

By construction, we have the following two facts:

(1) For $x \in B$, we have

$$\sum_{k=0}^{n} g(\tau^k x) \leq 0$$

for all n.

(2) For $x \notin B$, there exists an integer $l = l(x)$ such that

$$\sum_{k=0}^{l} g(\tau^k x) > 0,$$

and l is the smallest such integer.

We first show that if $x \in B_{n-1}^n$ (the top of column C_n), then $g(x) > 0$. This follows from the fact that $\tau(x) \in B$ and hence

$$\sum_{k=0}^{n} g(\tau^k(\tau x)) \leq 0 \quad \text{for all } n \geq 0.$$

If $g(x) \leq 0$, then we have

$$\sum_{k=0}^{l} g(\tau^k x) = g(x) + \sum_{k=1}^{l} g(\tau^k x) = g(x) + \sum_{k=0}^{l-1} g(\tau^k(\tau x)) \leq 0.$$

This is a contradiction to the definition of l. So we must have $g(x) > 0$.

More generally, for any x in the column, we see, by a similar argument, that the sum

$$g(x) + g(\tau x) + \cdots + g(\tau^l x)$$

is positive, and $T^l(x)$ is in the column.

Let $x \in B_1^n$, then

$$\sum_{k=0}^{l} g(\tau^k x) > 0.$$

Since $\tau^l(x)$ is in the column, it is either in the top or $\tau^{l+1}(x)$ is also in the column. If $\tau^l(x)$ is in the top, then we are done. Otherwise, sum from $\tau^{l+1}(x)$. Continue until we reach the top. We then have the desired result. $\square$

The following version of the Calderón-Zygmund decomposition is also interesting in its own nature, especially, when working on $\mathbb{Z}$ and transferring to the dynamical system it is quite useful in many cases.

Theorem 1.10. (The Calderón-Zygmund decomposition on $\mathbb{Z}$) *Let f be a function in $l_1(\mathbb{Z})$. Let $\lambda > 0$. Then we can write $f = g + b$ where $g \in l_2(\mathbb{Z})$, and*

(1) $\|g\|_{l_1} \leq \|f\|_{l_1}$,

(2) $\|g\|_\infty \leq 2\lambda$,

(3) $b = \sum_i b_i(x)$ *where each* b_i *satisfies:*

(a) b_i *is supported on an interval* B_i,

(b) $\sum_i b_i(j) = 0$ *for each* i,

(c) $\frac{1}{|B_i|}\sum_{j\in B_i} |b_i(j)| \leq 4\lambda$ *and* $\lambda \leq \frac{1}{|B_i|}\sum_{j\in B_i} |f(j)|$,

(d) $B_i \cap B_j = \phi$ *for* $i \neq j$.

Remark 1.11. Note that the above imply

$$\sum_i |B_i| \leq \frac{1}{\lambda} \sum_i \|b_i\|_{l_1} \leq \frac{1}{\lambda} \|f\|_{l_1}.$$

Also, if $\lambda \geq \|f\|_\infty$, then we take $f = g$ and $b = 0$.

Proof. Find an interval I of length 2^n with n so large that

$$\frac{1}{|I|} \sum_{j \in I} |f(j)| \leq \lambda$$

and $|f(j)| \leq \lambda$ for $j \notin I$. Now divide I into two equal pieces, I_1 and I_2. Look at the average of $|f|$ over each of these pieces. If the average is more than λ on any interval, keep that interval. If the average is less than λ then divide that interval into two equal intervals, and repeat the procedure. The procedure ends with a collection of intervals, the average over which is at least λ, but because of the construction, the average is no more than 2λ. Off the selected intervals, the function is at most λ, since clearly any point where $|f|$ is more than λ would be in some selected interval, possible an interval containing only the point itself. Denote the selected intervals by $B_1, B_2, \ldots$. Now define $g(j) = f(j)$ for j not in any of the selected intervals. For j in a selected interval B_i, define

$$g(j) = \frac{1}{|B_i|} \sum_{r \in B_i} f(r).$$

Define $b(j) = f(j) - g(j)$. From the construction each of the required properties follow easily, with $b_i = b\chi_{B_i}$. □

Theorem 1.12. (The Maximal Ergodic Theorem) *If $f \in L_1(X)$, $f \geq 0$, and $\lambda > \|f\|_1$, then*

$$m\{f^* > \lambda\} \leq \frac{1}{\lambda} \int_{\{f^* > \lambda\}} f(x)\, dm(x).$$

Proof. By Theorem 1.9, we have

$$\frac{1}{m(C_n)}\int_{C_n} f(x)\,dm(x) \geq \lambda$$

or

$$\frac{1}{\lambda}\int_{C_n} f(x)\,dm(x) \geq m(C_n).$$

Since $B = \{f^* \leq \lambda\}$, we have

$$B^c = \bigcup_{n=2}^{\infty} C_n = \{f^* > \lambda\}.$$

Consequently, by summing over n, we get

$$\begin{aligned} m\{f^* > \lambda\} &= \sum_{n=2}^{\infty} m(C_n) \leq \sum_{n=2}^{\infty} \frac{1}{\lambda}\int_{C_n} f(x)\,dm(x) \\ &\leq \frac{1}{\lambda}\int_{\{f^*>\lambda\}} f(x)\,dm(x) \end{aligned}$$

as desired. □

Theorem 1.13. (The Ergodic Theorem (G. D. Birkhoff 1931)) *Let (X, β, m) be a probability space, $\tau : X \to X$ a m.p.t. and $f \in L_1(X, \beta, m)$. Then*

(1) $\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} f(\tau^k x) = \bar{f}(x)$ *exists a.e.;*

(2) $\bar{f}(\tau x) = \bar{f}(x)$ *a.e.;*

(3) $\bar{f} \in L_1$*, and in fact* $\|\bar{f}\|_1 \leq \|f\|_1$*;*

(4) *if* $A \in \beta$ *with* $\tau^{-1}A = A$*, then* $\int_A f\,dm = \int_A \bar{f}\,dm$*;*

(5) $\frac{1}{n}\sum_{k=0}^{n-1} f(\tau^k) \to \bar{f}$ *in* L_1.

Remark 1.14. If X has finite measure, the same statements hold except that (4) and (5) apply only to invariant sets of finite measure.

Proof. (1) For each $\lambda_1, \lambda_2 \in \mathbb{R}$ with $\lambda_1 < \lambda_2$, let

$$E_{\lambda_1,\lambda_2} = \left\{ x \in X : \liminf_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(\tau^k x) < \lambda_1 < \lambda_2 < \limsup_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(\tau^k x) \right\}.$$

We will show that $m(E_{\lambda_1,\lambda_2}) = 0$ for each λ_1, λ_2. Then the union over all rational λ_1, λ_2 will also have measure zero, and hence the limit exists a.e.

Now E_{λ_1,λ_2} is an invariant subset of $\{f^* > \lambda_2\}$, so by considering τ restricted to E_{λ_1,λ_2}, we see that the Maximal Ergodic Theorem implies that

$$\int_{E_{\lambda_1,\lambda_2}} f\, dm \geq \lambda_2 m(E_{\lambda_1,\lambda_2}).$$

Next we consider $-f$. Since if $x \in E_{\lambda_1,\lambda_2}$ there is an $n \geq 1$ with

$$\frac{1}{n} \sum_{k=0}^{n-1} f(\tau^k x) < \lambda_1,$$

we see that

$$E_{\lambda_1,\lambda_2} \subset \{(-f)^* > -\lambda_1\}.$$

Hence, by the Maximal Ergodic Theorem,

$$\int_{E_{\lambda_1,\lambda_2}} -f\, dm \geq -\lambda_1 m(E_{\lambda_1,\lambda_2}),$$

or

$$\int_{E_{\lambda_1,\lambda_2}} f\, dm \leq \lambda_1 m(E_{\lambda_1,\lambda_2}).$$

Thus

$$\begin{aligned} \lambda_2 m(E_{\lambda_1,\lambda_2}) &\leq \int_{E_{\lambda_1,\lambda_2}} f\, dm \\ &\leq \lambda_1 m(E_{\lambda_1,\lambda_2}) \end{aligned}$$

and, since $\lambda_1 < \lambda_2$, this is possible only if $m(E_{\lambda_1,\lambda_2}) = 0$.

(2) It is clear that $\bar{f}\tau = \bar{f}$ a.e.

(3) We use Fatou's Lemma. Since

$$\left|\frac{1}{n}\sum_{k=0}^{n-1} f\tau^k\right| \leq \frac{1}{n}\sum_{k=0}^{n-1} |f|\tau^k,$$

We have $|\bar{f}| \leq \overline{|f|}$, and thus

$$\begin{aligned}\int |\bar{f}|\, dm \leq \int \overline{|f|}\, dm &\leq \liminf_{n\to\infty} \int \frac{1}{n}\sum_{k=0}^{n-1} |f(\tau^k x)|\, dm \\ &= \int |f|\, dm < \infty.\end{aligned}$$

(4) This is usually proved by means of the Maximal Ergodic Theorem as follows; we will obtain this result as an immediate corollary of (5).

For each $n = 0, 1, 2, \ldots$ and $k = 0, \pm 1, \pm 2, \ldots$, let

$$A_{n,k} = \left\{x \in A : \frac{k}{2^n} \leq \bar{f}(x) < \frac{k+1}{2^n}\right\}.$$

The $A_{n,k}$ are invariant sets, and for each n we have

$$A = \bigcup_k A_{n,k}.$$

Fix $\epsilon > 0$, then $f^* > \frac{k}{2^n} - \epsilon$ on $A_{n,k}$, so

$$\int_{A_{n,k}} f\, dm \geq \left(\frac{k}{2^n} - \epsilon\right) m(A_{n,k}).$$

Similarly, $(-f)^* > -\frac{k+1}{2^n}$ on $A_{n,k}$, so

$$\int_{A_{n,k}} -f\, dm \geq -\frac{k+1}{2^n} m(A_{n,k}).$$

Thus

$$\left(\frac{k}{2^n} - \epsilon\right) m(A_{n,k}) \leq \int_{A_{n,k}} f\,dm \leq \frac{k+1}{2^n} m(A_{n,k}),$$

and letting $\epsilon \to 0$ gives

$$\frac{k}{2^n} m(A_{n,k}) \leq \int_{A_{n,k}} f\,dm \leq \frac{k+1}{2^n} m(A_{n,k}).$$

Since obviously

$$\frac{k}{2^n} m(A_{n,k}) \leq \int_{A_{n,k}} \bar{f}\,dm \leq \frac{k+1}{2^n} m(A_{n,k}),$$

we have that

$$\left| \int_{A_{n,k}} f\,dm - \int_{A_{n,k}} \bar{f}\,dm \right| \leq \frac{1}{2^n} m(A_{n,k}).$$

Summing over k,

$$\left| \int_A f\,dm - \int_A \bar{f}\,dm \right| \leq \frac{1}{2^n} m(A),$$

and the conclusion follows upon letting $n \to \infty$.

(5) For bounded functions the L_1 convergence would follow from the Bounded Convergence Theorem. The general case an then be proved by approximating by bounded functions (which are dense in L_1) and using (3). There is no problem in assuming that $f \geq 0$, since we may write $f = f^+ - f^-$ and deal with the two parts separately. If g is bounded and $0 \leq g \leq f$, then

$$\left\| \frac{1}{n} \sum_{k=0}^{n-1} f\tau^k - \bar{f} \right\|_1 \leq \left\| \frac{1}{n} \sum_{k=0}^{n-1} (f\tau^k - g\tau^k) \right\|_1 + \left\| \frac{1}{n} \sum_{k=0}^{n-1} g\tau^k - \bar{g} \right\|_1 + \|\bar{g} - \bar{f}\|_1.$$

By (3), the third term is less than or equal to $\|g - f\|_1$, which can be made arbitrarily small by appropriate choice of g. Similarly, the first term is also less than or equal to $\|f - g\|_1$. But once g is fixed, the second term approaches 0 as $n \to \infty$, by the Bounded Convergence Theorem. This proves (5). (Notice that it is essential here that X have finite measure.)

One can also give an alternative proof of (4) by using (5) as follows

$$\begin{aligned}\left|\int_A f\,dm - \int_A \bar{f}\,dm\right| &= \left|\int_A \left(\frac{1}{n}\sum_{k=0}^{n-1} f\tau^k - \bar{f}\right) dm\right| \\ &\leq \int \left|\frac{1}{n}\sum_{k=0}^{n-1} f\tau^k - \bar{f}\right| dm \\ &= \left\|\frac{1}{n}\sum_{k=0}^{n-1} f\tau^k - \bar{f}\right\|_{L_1(A)} \to 0.\end{aligned}$$

□

We will now present a result of D. S. Ornstein [40], but we first need a lemma.

Lemma 1.15. *For $\lambda > \|f\|_1$, we have*

$$\frac{1}{\lambda}\int_{\{f>\lambda\}} f(x)\,dm(x) \leq 2m\{f^* > \lambda\}.$$

Proof. Since we have that

$$\bigcup_{n=2}^{\infty} C_n = \{f^* > \lambda\}$$

and that the C_n are disjoint, we get

$$\begin{aligned}\sum_{n=2}^{\infty}\int_{C_n} f(x)\,dm(x) &= \int_{\bigcup_{n=2}^{\infty} C_n} f(x)\,dm(x) \\ &= \int_{\{f^*>\lambda\}} f(x)\,dm(x).\end{aligned}$$

By Theorem 1.9 and the above, we have

$$\begin{aligned}\int_{\{f^*>\lambda\}} f(x)\,dm(x) &= \sum_{n=2}^{\infty}\int_{C_n} f(x)\,dm(x) \\ &\leq \sum_{n=2}^{\infty} 2\lambda m(C_n) \\ &= 2\lambda m\{f^* > \lambda\}.\end{aligned}$$

Dividing by λ, we have

$$\frac{1}{\lambda}\int_{\{f^*>\lambda\}} f(x)\,dm(x) \le 2m\{f^* > \lambda\}.$$

Now the result follows by replacing $\{f^* > \lambda\}$, on the left side, by the smaller set, $\{f > \lambda\}$. $\square$

Now we can prove the following:

Theorem 1.16. (D. S. Ornstein 1971) *For $f \in L_1(X)$, $f \ge 0$, the maximal function f^* is integrable if and only if $f\log^+ f$ is integrable, where*

$$f(x)\log^+ f(x) = f(x)\log\max(f(x), 1).$$

Proof. By integrating the inequality of Lemma 1.15, we get

$$\begin{aligned}
\int_X f^*(x)\,dm(x) &= \int_0^\infty m\{f^* > \lambda\}\,d\lambda \\
&= \int_0^{\|f\|_1} m\{f^* > \lambda\}\,d\lambda + \int_{\|f\|_1}^\infty m\{f^* > \lambda\}\,d\lambda \\
&\ge \|f\|_1 + \int_{\|f\|_1} \frac{1}{2\lambda}\int_{\{f>\lambda\}} f(x)\,dm(x)\,d\lambda \\
&= C + \int_1^\infty \frac{1}{2\lambda}\int_{\{f>\lambda\}} f(x)\,dm(x)\,d\lambda \\
&= C + \frac{1}{2}\int_X f(x)\int_1^{f(x)} \frac{1}{\lambda}\,d\lambda\,dm(x) \\
&\ge C + \frac{1}{2}\int_X f(x)\log^+ f(x)\,dm(x).
\end{aligned}$$

Thus we get the half of the Theorem.

To prove the other half, we use the Maximal Ergodic Theorem and a truncation argument.

We write

$$f \le f\chi_{\{f>\lambda\}} + \lambda\chi_{\{f\le\lambda\}},$$

and note

$$f^* \leq (f\chi_{\{f>\lambda\}})^* + \lambda.$$

Consequently,

$$\begin{aligned}
\lambda m\{f^* > 2\lambda\} &\leq \lambda m\{(f\chi_{\{f>\lambda\}})^* > \lambda\} \\
&\leq \int_{\{(f\chi_{\{f>\lambda\}})^* > \lambda\}} f\chi_{\{f>\lambda\}}\, dm(x) \\
&\leq \int_{\{f>\lambda\}} f(x)\, dm(x).
\end{aligned}$$

So we have

$$\lambda m\{f^* > 2\lambda\} \leq \frac{1}{\lambda} \int_{\{f>\lambda\}} f(x)\, dm(x).$$

If we integrate this inequality, we see that $f^* \in L_1$ if f belongs to $L \log^+ L$. □

Remark 1.17. Note that Theorem 1.16 requires the assumption that $m(X) < \infty$. If $m(X) = \infty$, then D. S. Ornstein [40] has shown that $f \geq 0$ and $\int f(x)\, dm(x) \neq 0$ imply f^* is not integrable.

1.3 The Ergodic Hilbert Transform

The Hilbert Transform of a measurable function f, was defined as

$$Hf(x) = \lim_{n\to\infty} \sum_{k=-n}^{n}{}' \frac{f(\tau^k x)}{k}.$$

and studied extensively in M. Cotlar [20]. The prime denotes that the term with zero denominator is omitted in the summation. As in the relation between the Hardy-Littlewood maximal function and the ergodic maximal function, it was also proven in A. P. Calderón [14] by transferring from the behavior of the classical Hilbert transform that Hf is of weak type (p,p) for $1 \le p < \infty$, and of strong type (p,p) for $1 < p < \infty$. We will now present a proof of K. Petersen [42] for the a.e. existence of the ergodic Hilbert transform Hf. The proof will follow from a number of lemmas, and the first one of them is the following lemma of G. Boole [6] and L. H. Loomis [32] which will play the main role in our discussion.

Lemma 1.18. *Let $a_1, a_2, \ldots, a_n \ge 0$ and $g(s) = \sum_{i=1}^{n} \frac{a_i}{s-t_i}$. Then*

$$m\{s : g(s) > \lambda\} = m\{s : g(s) < -\lambda\} = \frac{1}{\lambda}\sum_{i=1}^{n} a_i,$$

where m denotes the Lebesgue measure on $\mathbb{R}$.

Proof. Since $g(t_i-) = -\infty$, $g(t_i+) = \infty$ and $g'(s) < 0$ for all s, there are precisely n points m_i such that $g(m_i) = \lambda$, and $t_i < m - i < t_{i+1}$, $i = 1, \ldots, n-1$, t_n, m_n. The set where $g(s) > \lambda$ thus consists of the intervals (t_i, m_i) and has total length

$$\sum_{i=1}^{n}(m_i - t_i) = \sum_{i=1}^{n} m_i - \sum_{i=1}^{n} t_i. \tag{1}$$

But the numbers m_i are the roots of the equation

$$\sum_{i=1}^{n} \frac{a_i}{s-t_i} = \lambda,$$

whose cross-multiplied form is

$$\sum_{i=1}^{n} a_i \left[\prod_{j \ne i}(s-t_i)\right] = \lambda \prod_{i=1}^{n}(s-t_i),$$

or

$$\lambda s^n - \left[\lambda \sum t_j + \sum a_i\right] s^{n-1} + \cdots = 0,$$

so that

$$\sum_{i=1}^{n} m_i = \sum_{i=1}^{n} t_i + \frac{1}{\lambda} \sum_{i=1}^{n} a_i. \tag{2}$$

The first part of the lemma follows from (1) and (2); the proof for $g(s) < -\lambda$ is almost identical. $\square$

Lemma 1.19. *There is a constant C such that if $\{a_k\} \in l_1$ and $\lambda > 0$, then*

$$\#\left\{k \in \mathbb{Z} : \left| \sideset{}{'}\sum_{i=-\infty}^{\infty} \frac{a_{k+i}}{i} \right| > \lambda \right\} \leq \frac{C}{\lambda} \sum_{i=-\infty}^{\infty} |a_i|.$$

Proof. By treating the positive and negative ones separately, we may assume that all the a_i are positive. We will count

$$A_\lambda = \left\{k : \sideset{}{'}\sum_{i=-\infty}^{\infty} \frac{a_{k+i}}{i} > \lambda \right\};$$

a similar method will apply to

$$A'_\lambda = \left\{k : \sideset{}{'}\sum_{i=-\infty}^{\infty} \frac{a_{k+i}}{i} < -\lambda \right\}.$$

Choose a finite set $A \subset A_\lambda$, and choose N so large that $A \subset [-N, N]$ and, for each $k \in A$, $\sum_{i=-N}^{\prime N} \frac{a_i}{i-k} > \lambda$. Then

$$g_k(s) = \sideset{}{'}\sum_{i=-N}^{N} \frac{a_i}{i-s} > \lambda$$

for $s = k \in A$, and hence $g_k(s) > \lambda$ for $s \in [k, k+1)$, because $g'_k(s) > 0$. If we let

$$g(s) = \sideset{}{^\lambda}\sum_{i=-N}^{N} \frac{a_i}{i-s}$$

and

$$h_k(s) = \frac{a_k}{k-s},$$

then $g = g_k + h_k$, so that for each $k \in A$

$$(k, k+1) \subset \{s : g_k(s) > \lambda\} \subset \left\{s : g(s) > \frac{\lambda}{2}\right\} \cup \left\{s : h_k(s) < -\frac{\lambda}{2}\right\}.$$

Therefore, we get

$$\begin{aligned}
\#A &= m\left(\bigcup_{k\in A}(k, k+1)\right) \\
&\leq m\left\{s : g(s) > \frac{\lambda}{2}\right\} + \sum_{k\in A} m\left\{s : h_k(s) < -\frac{\lambda}{2}\right\} \\
&\leq \frac{2C}{\lambda}\sum_{i=-N}^{N} a_i + \sum_{k\in A}\frac{2C}{\lambda}a_k \\
&\leq \frac{4C}{\lambda}\|a\|_1
\end{aligned}$$

as desired. $\square$

Lemma 1.20. *There is a constant C such that if $\{a_k\} \in l_1$ and $\lambda > 0$, then*

$$\#\left\{k \in \mathbb{Z} : \sup_{n\geq 1}\left|\sum_{i=-n}^{n}{}' \frac{a_{k+i}}{i}\right| > \lambda\right\} \leq \frac{C}{\lambda}\sum_{i=-\infty}^{\infty}|a_i|.$$

Proof. We assume as before that all the a_i are positive and drop the absolute value signs.

Let

$$A \subset \left\{k : \sup_{n\geq 1}\sum_{i=-n}^{n}{}' \frac{a_{k+i}}{i} > \lambda\right\}$$

be closed and bounded. For each $k \in A$ there is an interval of integers $I_k = [k - n_k, k + n_k]$

such that

$$\sum_{i\in I_k}{}' \frac{a_i}{i-k} > \lambda.$$

Let

$$g_k(s) = \sum_{i\in I_k}{}' \frac{a_i}{i-s}, \quad g(s) = \sum_{i=-\infty}^{\infty}{}' \frac{a_i}{i-s}, \quad h_k(s) = \sum_{i\notin I_k} \frac{a_i}{i-s}.$$

If $k \in A$, then $g_k(k) > \lambda$, so that either $g(k) > \frac{\lambda}{2}$ or $h_k(k) < -\frac{\lambda}{2}$. In the first case ($k \in A_1$), by Lemma 1.19, k falls into a single (independent of k) set of measure no more than $\frac{C}{\lambda}\|a\|_1$. To deal with the left over k's ($k \in A_2$), replace $\{I_k\}$ by a disjoint subfamily which still covers at least $\frac{1}{3}$ of A_2, by at each stage selecting an interval of maximal disjoint from the previously chosen ones. Find N such that

$$\bigcup_{k \in A_2} I_k \subset [-N, N]$$

and

$$\tilde{h}_k(k) \leq -\frac{\lambda}{2} \quad \text{for all } k \in A_2,$$

where

$$\tilde{h}_k(s) = \sum_{i \in \{-N,\ldots,N\}-I_k} \frac{a_i}{i-s}.$$

Then also $\tilde{h}_k(s) < -\frac{\lambda}{2}$ on $(k-n_k, k)$, so that we find

$$\begin{aligned}
\#A &= \#A_1 + \#A_2 \\
&\leq \frac{C}{\lambda}\|a\|_1 + 6\sum_{k \in A_2} n_k \\
&\leq \frac{C}{\lambda}\|a\|_1 + 6m\left(\bigcup_{k \in A_2}\left\{s : \tilde{h}_k(s) < -\frac{\lambda}{2}\right\}\right) \\
&\leq \frac{C}{\lambda}\|a\|_1 + 6m\left(\bigcup_{k \in A_2}\left(\left\{s : \sum_{i=-N}^{N}{}' \frac{a_i}{i-s} < -\frac{\lambda}{4}\right\} \cup \left\{s : g_k(s) > \frac{\lambda}{4}\right\}\right)\right) \\
&\leq \frac{C}{\lambda}\|a\|_1 + 6m\left\{s : \sum_{i=-N}^{N}{}' \frac{a_i}{i-s} < -\frac{\lambda}{4}\right\} + 6\sum_{k \in A_2} m\left\{s : g_k(s) > \frac{\lambda}{4}\right\} \\
&\leq \frac{C}{\lambda}\|a\|_1 + \frac{24C}{\lambda}\|a\|_1 + 6\sum_{k \in A_2}\frac{4C}{\lambda}\sum_{i \in I_k} a_i \\
&\leq \frac{49C}{\lambda}\|a\|_1.
\end{aligned}$$

$\square$

Lemma 1.21. *Let (X, β, m) be a measure space, $\tau : X \to X$ an invertible measure-preserving transformation and $f \in L_1(X, \beta, m)$. Then*

$$Hf(x) = \lim_{n\to\infty} \sum_{k=-n}^{n}{}' \frac{f(\tau^k x)}{k}$$

exists a.e. Furthermore, if $\{\tau_t : -\infty < t < \infty\}$ is a one-parameter measure-preserving flow on X, then

$$\lim_{\epsilon\to 0^+} \int_{\epsilon\le|t|\le 1/\epsilon} \frac{f(\tau_t x)}{t}\, dt$$

exists a.e., for each $f \in L_1(X, \beta, m)$.

Proof. By considering f^+ and f^- separately, we may assume that $f \geq 0$. Fix N; we will show that

$$m\left\{x : \sup_{1\le n\le N} \left| \sum_{i=-n}^{n}{}' \frac{f(\tau^i x)}{i} \right| > \lambda \right\} \leq \frac{C}{\lambda}\|f\|_1,$$

where C is a constant independent of f, λ, and N.

For fixed x and K, let $a_k = f(\tau^k x)$ and

$$a_k^K = \begin{cases} a_k, & \text{if } |k| \leq K + N, \\ 0, & \text{if } |k| > K + N, \end{cases}$$

so that $\{a_k^K\} \in l_1$. For each $j \in \mathbb{Z}$, let

$$G_j(x) = \sup_{1\le n\le N} \left| \sum_{k=-n}^{n}{}' \frac{a_{k+j}}{k} \right|, \quad \text{and} \quad G_j^K(x) = \sup_{1\le n\le N} \left| \sum_{k=-n}^{n}{}' \frac{a_{k+j}^K}{k} \right|.$$

Then

$$\begin{aligned} G_j(x) &= \sup_{1\le n\le N} \left| \sum_{k=-n}^{n}{}' \frac{a_{k+j}^K}{k} + \frac{a_{k+j} - a_{k+j}^K}{k} \right| \\ &\leq G_j^K(x) + \sup_{1\le n\le N} \left| \sum_{k=-n}^{n}{}' \frac{a_{k+j} - a_{k+j}^K}{k} \right|, \end{aligned}$$

so that $G_j(x) \le G_j^K(x)$ for $|j| \le K$.

Now let $E = \{x : G_0(x) > \lambda\}$, so that $\{x : G_j(x) > \lambda\} = \tau^{-j}E$. Let $\bar{E} = \{(x,j) : G_j^K(x) > \lambda\}$. Then, if $\#$ continues to denote the counting measure on $\mathbb{Z}$,

$$\begin{aligned} m \times \#(\bar{E}) &= \int_X \#\{j : G_j^K(x) > \lambda\}\, dm(x) \\ &\le \int_X \frac{C}{\lambda} \sum_{j=-\infty}^{\infty} |a_j^K|\, dm \\ &\le \int_X \frac{C}{\lambda} \sum_{-(K+N)}^{K+N} |a_j|\, dm \\ &\le \frac{C}{\lambda}[2(K+N)+1]\|f\|_1, \end{aligned}$$

and also

$$\begin{aligned} m \times \#(\bar{E}) &\ge \sum_{j=-K}^{K} m\{x : G_j^K(x) > \lambda\} \\ &\ge \sum_{j=-K}^{K} m\{x : G_j(x) > \lambda\} \\ &= \sum_{j=-K}^{K} m(\tau^{-j}E) \\ &= (2K+1)m(E). \end{aligned}$$

Thus, we have

$$m(E) \le \frac{C}{\lambda} \frac{2(K+N)+1}{2K+1} \|f\|_1,$$

and now let $K \to \infty$ to get the desired result.

A similar argument can be used to establish the continuous-parameter part of the theorem; alternatively, one may consider time ϵ maps and approximate. $\square$

Chapter 2

Ergodic H^p Spaces

In this chapter we will discuss some properties of ergodic H^p spaces based on the characterization by ergodic atoms analog to the classical case. Most of the materials that will be presented here can be found in R. Caballero and A. de la Torre [13]. Later it will also be clear in our main discussion that these spaces and the classical H^p spaces are closely related.

We will always assume in our discussions that (X, β, m) is a non-atomic probability space and $\tau : X \to X$ is an invertible, measure-preserving, ergodic transformation.

2.1 Ergodic Rectangles and Atoms

In this section we will present the generalization of a real atom to ergodic theory, and will also discuss some properties of the ergodic rectangles.

Definition 2.1. *Let $B \subset X$ be a set of positive measure such that for some $k \geq 1$ we have*

$$\tau^i B \cap \tau^j B = \phi, \;\; i \neq j, \;\; 0 \leq i, j \leq k-1 \; .$$

Then the set

$$R = \bigcup_{i=0}^{k-1} \tau^i B$$

will be called an ergodic rectangle with base B and length k.

Although the decomposition theorem that we will present after the following lemma is not a well-known result, in some cases it provides a good method to study the behavior of operators in ergodic theory, for instance see F. J. Martin-Reyes [33].

Lemma 2.2. *Let (X, β, m) be a σ-finite measure space and let $\tau : X \to X$ be a measure preserving transformation. Then, for every measurable set $Y \subset X$ there exists a countable family $\{B_i \in \mathbb{Z}^+\}$ of measurable sets such that*

(a) $Y = \bigcup_{i=0}^{\infty} B_i$.

(b) $B_i \cap B_j = \phi$ *if* $i \neq j$.

(c) $\tau^{-1} B_i \cap B_i = \phi$ *if* $i \geq 1$.

(d) $\tau^{-1} A = A$ *for every subset* $A \subset B_0$.

Proof. Let assume that $m(Y) < \infty$. Let $B_1 = \phi$ and let us choose by induction sets B_i with $i > 1$. Suppose that $B_1, B_2, \ldots, B_n$ $n \geq 1$ have been chosen. We define

$$\gamma_{n+1} = \left\{ B \in \beta : B \subset Y,\ m(B) > 0,\ B \cap \left(\bigcup_{i=1}^{n} B_i \right) = \phi,\ B \cap \tau^{-1} B = \phi \right\}.$$

If $\gamma_{n+1} = \phi$ we choose $B_{n+1} = \phi$; otherwise we choose a set $B_{n+1} \in \gamma_{n+1}$ such that

$$\sup\{m(B) : B \in \gamma_{n+1}\} < 2m(B_{n+1}).$$

Finally, let $B_0 = Y - \bigcup_{i=1}^{\infty} B_i$. It is evident that $\{B_i\}_{i=0}^{\infty}$ satisfies (a)-(c). In order to prove that B_0 satisfies (d) we observe that in the choice of the sets B_i we have one of the following situations:

(i) $\gamma_{m_0} = \phi$ for some $m_0 \geq 2$,

(ii) $\gamma_m \neq \phi$ for every $m \geq 2$.

Assume we are in case (i). Let A be any measurable subset of B_0. It is clear that

$$(A - \tau^{-1}A) \cap (\bigcup_{i=1}^{m_0-1} B_i) = \phi$$

and

$$(A - \tau^{-1}A) \cap \tau^{-1}(A - \tau^{-1}A) = \phi.$$

Then $m(A - \tau^{-1}A) = 0$ by (i) and since τ preserves the measure m and $m(A) < \infty$ we get that $A = \tau^{-1}A$.

We now assume that $\gamma_m \neq \phi$ for every $m \geq 2$. The first thing we notice is that $\lim m(B_i) = 0$ since

$$\sum_{i=1}^{\infty} m(B_i) = m(\bigcup_{i=1}^{\infty} B_i) = m(Y) < \infty.$$

Let A be any measurable subset of B_0 and let $D = A - \tau^{-1}A$. If $m(D) > 0$, then there exists $n_0 \geq 2$ such that $0 < 3m(B_{n_0}) < m(D)$ and then $3m(B_{n_0}) < \sup\{m(B) : B \in \gamma_{n_0}\}$ since $D \in \gamma_{n_0}$. That contradicts the choice of B_{n_0}. Hence $m(D) = 0$ and consequently $\tau^{-1}(A) = A$. This finishes the proof when $m(Y) < \infty$.

Now let Y be a measurable subset of X. Since X is σ-finite we have that

$$Y = \bigcup_{n=0}^{\infty} Y_n, \quad Y_n \cap Y_m = \phi \text{ if } n \neq m \text{ and } m(Y_n) < \infty.$$

For each Y_n we have, by what we have already shown,

$$Y_n = \bigcup_{i=0}^{\infty} B_i^n$$

where $\{B_i^n\}_{i=1}^{\infty}$ is a sequence with the properties (b)-(d) in our Lemma. Then we get the decomposition if

$$B_0 = \bigcup_{n=0}^{\infty} B_0^n$$

and the countable family $\{B_i^n : i \geq 1,\, n \in \mathbb{Z}^+\}$ is obtained after renumbering the collection $\{B_j\}_{j=1}^{\infty}$. □

Theorem 2.3. *Let (X, β, m) be a σ-finite measure space, and let Y be a measurable set of X and k be a nonnegative integer. Then, there exists a countable family $\{B_i : i \in \mathbb{Z}^+\}$ of sets of finite measure such that*

(i) $Y = \bigcup_{i=0}^{\infty} B_i$.

(ii) $B_i \cap B_j = \phi$ *if* $i \neq j$.

(iii) *For every i, B_i is the base of an ergodic rectangle of length $1 + s(i)$ with $0 \leq s(i) \leq k$ and such that if $s(i) < k$, then $\tau^{1+s(i)}A = A$ for every measurable set $A \subset B_i$.*

(iv) *Consequently, for every measurable set $A \subset B_i$,*

$$\sum_{j=0}^{k} \chi_{\tau^j A} \leq C(i) \sum_{j=0}^{s(i)} \chi_{\tau^j A} = C(i)\chi_{\bigcup_{j=0}^{s(i)} \tau^j A} \leq 2 \sum_{j=0}^{k} \chi_{\tau^j A}$$

where $C(i)$ is the least integer not smaller than $(k+1)(1+s(i))^{-1}$.

Proof. We shall begin by assuming that $m(Y)$ is finite. If $k = 0$ there is nothing to prove. So, let k be a positive integer. For every h, $1 \leq h \leq k$, let $\{B_{i,h}\}_{i=1}^{\infty}$ be the sequence given by Lemma 2.2 for the set Y and the transformation τ^{-h}. Then

$$Y = \bigcap_{h=1}^{k} \Big(\bigcup_{i=0}^{\infty} B_{i,h}\Big) = \bigcup_{t \in J} B_t,$$

where J is the set of the mappings from $\{1, \ldots, k\}$ into $\mathbb{Z}^+$ and $B_t = \bigcap_{t \in J} B_t$. It is clear that the collection $\{B_t : t \in J\}$ is countable, disjoint family of sets of finite measure. On the other hand, if $\{h : t(h) = 0\} = \phi$ we have that B_t is the base of a rectangle of length $k+1$. If $\{h : t(h) = 0\} \neq \phi$, it is clear that B_t is the base of a rectangle of length $s(t)+1$, where $s(t) + 1 = \min\{h : t(h) = 0\}$ (observe that $s(t) < k$)). In such a case we get that $\tau^{s(t)+1}A = A$ for any $A \subset B_t$ since B_t is a subset of $B_{0,s(t)+1}$. Once these properties have been established we easily deduce (iv).

Now let Y be any measurable subset of X. Since X is σ-finite, we get that

$$Y = \bigcup_{n=0}^{\infty} Y_n, \quad Y_n \cap Y - m = \phi \text{ if } n \neq m \text{ and } m(Y_n) < \infty.$$

For every Y_n we consider the decomposition $Y_n = \bigcup_{i=0}^{\infty} B_i^n$ given in our Theorem in the case of finite measure. It is evident that the family $\{B_i^n : i \in \mathbb{Z}^+, n \in \mathbb{Z}^+\}$ satisfies the required conditions. □

The following corollaries of Theorem 2.3 are the generalization of Lemma 2.2 in E. Atencia and A. de la Torre [2].

Corollary 2.4. *Let (X, β, m) be a σ-finite non-atomic measure space and τ an ergodic, invertible measure preserving transformation from X onto itself. Let Y be a subset of X. Then for every nonnegative integer k the set Y can be written as a countable union of bases of ergodic rectangles of length $k+1$.*

Proof. Let $X = \bigcup_{i=0}^{\infty} B_i$ be the decomposition given by Theorem 2.3 for X and k. To prove the corollary it will suffice to show that $s(i) = k$ for every B_i of positive measure. If $s(i) < k$, the set $A \cup \tau A \cup \cdots \cup \tau^{s(i)}A$ is invariant for every subset A of B_i. Hence

$$m(A \cup \tau A \cup \cdots \cup \tau^{s(t)}A) = 0$$

or

$$m(X - A \cup \tau A \cup \cdots \cup \tau^{s(i)} A) = 0.$$

However both conditions are impossible for a subset A of B_i such that $m(A) < (s(i)+1)^{-1} m(B_i)$. Therefore $s(i) = k$ if $m(B_i) > 0$. □

Corollary 2.5. *Let (X, β, m) be a σ-finite measure space with $m(X) = \infty$ and let τ be an ergodic, invertible measure preserving transformation from X onto itself. Let Y be a measurable subset of X. Then for every nonnegative integer k the set Y can be written as a countable union of bases of ergodic rectangles of length $k+1$.*

Proof. We follow the proof of Corollary 2.4 taking $A = B_i$. Then, since $m(B_i) > 0$ we have

$$m(X - B_i \cup \cdots \cup \tau^{s(i)} B_i) = 0.$$

But this implies that $m(X) < \infty$ because $m(B_i) < \infty$. Consequently, $s(i) = k$ if $m(B_i) > 0$. □

Definition 2.6. *Let p,q be real numbers, $1/2 < p \leq 1 \leq q \leq \infty$, $p < q$. A real-valued function a is a (p,q) atom if either*

(1) *a is zero outside an ergodic rectangle $R = \bigcup_{i=0}^{k-1} \tau^i B$ and satisfies*

(a) $\sum_{j=0}^{k-1} a(\tau^j x) = 0$ *for* $x \in B$,

(b) $\frac{1}{k} \sum_{j=0}^{k-1} |a(\tau^j x)|^q \leq m(R)^{-q/p}$, $x \in B$, $q < \infty$,

$\|a\|_\infty \leq m(R)^{-1/p}$ *if* $q = \infty$,

or

(2) $a \in L_q(X)$, $\|a\|_q \leq 1$.

We will say that an atom is of type 1 *if it satisfies* (1) *and of type* 2 *if it satisfies* (2).

The following definition will also be needed in our discussion:

Definition 2.7. *We say that a function $A : \mathbb{Z} \to \mathbb{R}$ is a (p, q) atom in the integers if its support is contained in an interval $(l, l+1, \ldots, l+k-1)$ and*

(a) $\sum_n A(n) = 0$,

(b) $\frac{1}{k}\sum_{i=0}^{k-1} |A(l+i)|^q \leq k^{-q/p}$.

Proposition 2.8. (b) *implies that*

$$\|a\|_q \leq m(R)^{(1/q)-(1/p)}.$$

Proof. Since

$$\begin{aligned}
\|a\|_q^q &= \int_X |a|^q\,dx = \int_R |a|^q\,dx \\
&= \int_{\bigcup_{j=0}^{k-1} \tau^j B} |a|^q\,dx = \int_B \sum_{j=0}^{k-1} |a(\tau^j x)|^q\,dx \\
&= \int_B kk^{-1}\sum_{j=0}^{k-1} |a(\tau^j x)|^q\,dx \leq \int_B km(R)^{-q/p}\,dx \qquad \text{by (b)} \\
&= m(B)km(R)^{-q/p} = m(R)m(R)^{-q/p} \\
&= m(R)^{1-q/p}.
\end{aligned}$$

Hence, we have

$$\|a\|_q \leq m(R)^{(1/q)-(1/p)}$$

as desired.

□

Definition 2.9. *For p, q as before, $H^{p,q}$ is the set of functions f that can be written as*

$$f = \sum_{i=1}^{\infty} c_i\,a_i$$

where a_i are (p,q) atoms,

$$\sum_{i=1}^{\infty} |c_i|^p < \infty,$$

and the convergence is in the L_p-metric.

We introduce a metric in $H^{p,q}(X)$ by

$$d_{p,q}(h_1, h_2) = |h_1 - h_2|_{p,q}$$

where

$$|h|_{p,q} = \{\inf \sum_{i=1}^{\infty} |c_i|^p : h = \sum_{i=1}^{\infty} |c_i|^p < \infty\}.$$

It is clear that $H^{p,\infty} \subset H^{p,q} \subset H^{p,r}$ if $r \leq q \leq \infty$, the inclusion being continuous.

We will later show that the converse also holds by using a characterization of the $H^{p,q}$ spaces in terms of a maximal operator.

Remark 2.10. In analogy with the classical case, $H^1(X)$ can also be defined as those functions in L_1, whose ergodic Hilbert Transform is in L_1. With the natural norm

$$\|f\|_{H^1} = \|f\|_1 + \|Hf\|_1,$$

$H^1(X)$ becomes a Banach space and this norm and the norm defined before are equivalent as can easily be seen by duality.

2.2 The Maximal Operator

Let $\psi \in L_1(\mathbb{Z})$ and $f \in L_1(X)$. The *convolution* of f and ψ is defined as

$$(f * \psi)(x) = \sum_{m \in \mathbb{Z}} f(\tau^{-m}x)\, \psi(m).$$

Obviously, we have

$$\|f * \psi\|_1 \leq \|f\|_1 \, \|\psi\|_1.$$

Definition 2.11. *Let ϕ be a nonnegative $C^\infty(\mathbb{R})$ function with support in $(-1,1)$ and $L > 0$. For $f \in L_1(X)$ we define*

$$M(L,\phi)f(x) = \sup_{|i|<n<L} |(f * \phi_n)(\tau^i x)|$$

and

$$M(\phi)f(x) = \lim_{L\to\infty} M(L,\phi)f(x)$$

where

$$\phi_n(m) = \frac{1}{n}\phi(\frac{m}{n}), \quad m \in \mathbb{Z}.$$

We also define

$$M(L)f(x) = \sup M(L,\phi)f(x)(A(\phi))^{-1}$$

where the sup is taken over all C^∞ functions with support in $(-1,1)$ and where $A(\phi)$ is the normalizing factor

$$A(\phi) = \|\phi\|_\infty + \|\phi'\|_\infty.$$

Finally,

$$Mf(x) = \lim_{L\to\infty} M(L)f(x).$$

Remark 2.12. One can prove that $Mf(x)$ is dominated by the ergodic maximal operator, and therefore $Mf(x)$ is of weak type $(1,1)$ and bounded in any L_q, $1 < q$.

We will next show that atoms behave well with respect to the maximal operator.

Proposition 2.13. *Let a be a (p,q) atom. Then $Ma \in L_p$, and $\|Ma\|_p \le C$, where C is an absolute constant.*

Proof. When a is a type 2 atom we have

$$\begin{aligned}\|Ma\|_p \le \|Ma\|_q &\le C\|a\|_q \\ &\le C \quad \text{for } q > 1\end{aligned}$$

since our space has measure one.

If $q = 1$ we have

$$\|Ma\|_p \le C\|a\|_1 \le C$$

by Kolmogorov's inequality.

If a be a type 1 atom, then we will use a transfer principle.

Let $A : \mathbb{Z} \to \mathbb{R}$ is a (p,q) atom with support in $(l, l+1, \ldots, l+k-1)$. We will prove that

$$\sum_{-\infty}^{\infty} |MA(n)|^p \le C(p,q)$$

where C depends only on p and q and M is the maximal operator defined above.

First of all, since M commutes with translations, we can assume that $l = 0$. Now by Hölder's inequality we have

$$\begin{aligned}\sum_{m=-4k}^{4k} |MA(m)|^p &\le (8k+1)\left(\frac{1}{8k+1}\sum_{m=-4k}^{4k} |MA(m)|^q\right)^{p/q} \\ &\le C(p,q)(8k+1)^{1-(p/q)}\left(\sum_{m=-\infty}^{\infty} |A(m)|^q\right)^{p/q} \\ &\le C(p,q)(8k+1)^{1-(p/q)}k^{-(1-(p/q))} \\ &\le C'(p,q).\end{aligned}$$

Let us now fix $|m| > 4k$, and let us fix n, L, ϕ and $|i| < n < L$. Then

$$(A * \phi_n)(m+i) = \sum_j A(m+i-j)\phi_n(j) = \sum_{j=0}^{k-1} A(j)\phi_n(m+i-j)$$
$$= \sum_{j=0}^{k-1} A(j)\left(\frac{1}{n}\left(\phi\left(\frac{m+i-j}{n}\right) - \phi\left(\frac{m+i}{n}\right)\right)\right).$$

Now the sum is zero unless $n > |m/4|$, for if $n \leq |m/4|$ then $|i| < |m|/4$, $j < k < |m|/4$, and thus $|m+i-j| > |m| - |m/2| > n$. Now by Mean Value Theorem we have

$$|(A * \phi_n)(m+i)| \leq \frac{1}{n^2} \sum_{j=1}^{k-1} |A(j)| A(\phi) j$$

and by Hölder's inequality we have

$$\sum_{j=0}^{k-1} |A(j)| \leq \left(\sum_{j=0}^{k-1} |A(j)|^q\right)^{1/q} k^{1-1/q}$$
$$= k^{1/q}\left(\frac{1}{k}\sum_{j=0}^{k-1} |A(j)|^q\right)^{1/q} k^{1-1/q}$$
$$\leq k k^{-1/p}$$
$$= k^{1-(1/p)}.$$

Combining this result with the previous inequality we see that

$$|(A * \phi_n)(m+1)| \leq \frac{CA(\phi)}{m^2} k^{2-(1/p)}.$$

This means that

$$|MA(m)| \leq \frac{C}{m^2} k^{2-(1/p)} \qquad \text{if } |m| > 4k$$

and

$$\sum_{|m|>4k} |MA(m)|^p \leq C k^{2p-1} \sum_{|m|>4k} \frac{1}{m^{2p}} \leq C' \qquad \text{if } p > 1/2.$$

Now let a be a (p,q) atom with support in

$$R = \bigcup_{i=0}^{k-1} \tau^i B$$

and let us consider the function $a_x(n) = a(\tau^n x)$ for each $x \in X$. Since the orbit of x enters B infinitely many times, let us call y_i the points of the orbit that belong to B. Then for each n $a_x(n) = a(\tau^l y_i)$ for some i, $0 \le l \le k$. It is then clear that we can write $a_x(n) = \sum_{i\in\mathbb{Z}} m(B)^{-1/p} A_{i,x}(n)$ where $A_{i,x}(n) = m(B)^{1/p} a(\tau^n x)$ are (p,q) atoms in the integers with support in $(l_i, \dots, l_i + k - 1)$ with $y_i = \tau^{l_i} x \in B$. Therefore for $N > L > k$

$$\begin{aligned}
\int_X |M(L)a(x)|^p\,dx &= \int_X \frac{1}{2N+1} \sum_{n=-N}^{N} |M(L)a_x(n)|^p\,dx \\
&= \int_X \frac{1}{2N+1} \sum_{n=-N}^{N} \sum_i |M(L)m(B)^{-1/p} A_{i,x}(n)|^p\,dx.
\end{aligned}$$

Now the atoms $A_{i,x}$ whose support does not cut the interval $(-3N, 3N)$ do not contribute anything to $M(L)$ since it is easy to see that

$$M(L)A_{i,x}(n) = 0 \qquad \text{if } n \notin (-3N, 3N);$$

since $k < L < N$, we can thus restrict our attention to those atoms whose support is contained in $(-4N, 4N)$.

Now we have

$$\begin{aligned}
\int_X |M(L)a(x)|^p\,dx &\le \int_X \frac{1}{2N+1} \sum_{n=-N}^{N} \sum_{\mathrm{supp} A_i \in (-4N,4N)} |M(L)m(B)^{-1/p} A_{i,x}(n)|^p\,dx \\
&\le \frac{1}{2N+1} m(B)^{-1} \int_X \sum_{\mathrm{supp} A_i \in (-4N,4N)} \sum_{-\infty}^{\infty} |M(L)A_{i,x}(n)|^p\,dx \\
&\le \frac{m(R)^{-1}}{2N+1} \int_X k \sum_{\mathrm{supp} A_i \subset (-4N,4N)} C\,dx \\
&= \frac{m(R)^{-1}}{2N+1} \int_X k\{\text{number of } A_i\text{'s with } \mathrm{supp} A_i \in (-4N,4N)\}
\end{aligned}$$

$$= \frac{m(R)^{-1}}{2N+1} \int_X \sum_{i=-4N}^{4N} \chi_R(\tau^i x)\, dx$$
$$= C \frac{8N+1}{2N+1} m(R)^{-1} \int_X \chi_R$$
$$= C \frac{8N+1}{2N+1}.$$

Letting N and L go to infinity, we are done. □

Remark 2.14. This proposition implies that if $f \in H^{p,q}$ then $Mf \in L_p$ and

$$\int |Mf|^p \leq C \sum |c_i|^p.$$

This means that any $H^{p,q}$ function has a maximal function in L_p.

In the following we will show that one can also prove the special case of Proposition 2.13 using the Hilbert transform characterization of ergodic H^1 spaces as in Remark 2.10.

Proposition 2.15. *Let a be a $(1, \infty)$ atom. Then a is in H^1 and has norm less than an absolute constant C.*

Proof. We see by the definition of an atom that a is in L_1, and has norm less than 1. It is enough then to study the L_1 norm of Ha.

Let the support of a be an ergodic rectangle $R = \bigcup_{i=0}^{k-1} \tau^i B$. We observe that for fix x, the function of n defined by

$$n \to a_x(n) = a(\tau^n x) m(B)$$

is of the form

$$a_x(n) = \sum_i a_i(n),$$

where each of the a_i, is a discrete $(1, \infty)$ atom. We also observe that

$$Ha_x(n) = Ha(\tau^n x)m(B),$$

where in one case H stands as the ergodic Hilbert transform in X, and in the other as the Hilbert transform in the integers.

We fix now integers $M > 2N > k$. We have

$$\sum_{|n|<M} |H_N a(\tau^n x)| < \frac{1}{m(B)} \sum_i \sum_{|n|<M} |H_N a_i(n)|,$$

where

$$H_N f(x) = \sum_{k=-N}^{N}{}' \frac{f(\tau^k x)}{k}.$$

Now if the support of a_i, does not intersect $(-2M, 2M)$, then $H_N a_i(n) = 0$, for all n, $|n| < M$. Therefore we can restrict our attention to those a_i, whose support is contained in $(-3M, 3M)$. On the other hand by using that the Hilbert transform on $\mathbb{R}$ is bounded in L_p and that for x away from the support of a real atom a, the Hilbert transform of a behaves like $\frac{1}{x}$, one obtains that Ha is in L_1, and has norm less than an absolute constant. Using exactly the same argument as for the real line one obtains

$$\sum |Ha(n)| \leq C$$

for an atom a on $\mathbb{Z}$. Now combining this fact with the above observations, we get

$$\begin{aligned} \sum_{|n|<M} |H_N a(\tau^n x)| &< C\frac{1}{m(B)}\{\# \text{ of } a_i \text{ with support in } (-3M, 3M)\} \\ &\leq C\frac{1}{m(R)} \sum_{|n|<3M} \chi_R(\tau^n x). \end{aligned}$$

Hence, we have

$$\int_X |H_N a(x)|\,dx = \int_X \frac{1}{2M+1} \sum_{|n|<M} |H_N a(\tau^n x)|\,dx$$
$$\leq C \int_X \frac{1}{(2M+1)m(R)} \sum_{|n|<3M} \chi_R(\tau^n x)\,dx$$
$$\leq 3C.$$

Since N is arbitrarily large, we get the result as desired. $\square$

We will next prove the converse of Proposition 2.13, namely if f is an L_p function whose maximal function Mf is in L_p then f is in $H^{p,\infty}$.

We first need the following lemmas:

Lemma 2.16. *Let $O \subset X$ be a set of positive measure such that the subset of $\mathbb{Z}$ defined by $O^x = \{n \in \mathbb{Z} : \tau^n x \in O\}$ does not contain an interval of finite length. Then O can be written as a disjoint union of ergodic rectangles R_i of length i.*

Proof. One just defines $R_i = \{x \in O : l(O_0^x) = i\}$, i.e. the set of points in O such that the interval of O^x that contains the origin has length i. It is clear that $O = \bigcup_{i=1}^{\infty} R_i$ and that the R_i are pairwise disjoint. To see that R_i is a rectangle one just observes that $R_i = \bigcup_{j=0}^{i-1} \tau^j B_i$, where B_i is the set of points x in O such that $\tau^{-1}x \notin O$, $\tau^j x \in O$ for $j = 1, 2, \ldots, i-1$, and $\tau^i x \notin O$. $\square$

Lemma 2.17. *Let I be an interval in $\mathbb{R}$ of the form $[a, b]$, $a, b \in \mathbb{Z}$. Then there exist a finite number of C^∞ functions $\{\Psi_j(x)\}$ such that*

(1) $\sum_j \Psi_j(x) = 1$ *for* $x \in I$; *supp* $\Psi_j \subset [a - 1/2, b + 1/2]$,

(2) $\|\Psi_j'\|_\infty \leq C|supp\,\Psi_j|^{-1}$,

(3) $d(supp\,\Psi_j, \mathbb{R} - \bar{I}) \sim |supp\,\Psi_j|$; $\bar{I} - [a-1, b+1]$,

(4) $\sum_{n\in\mathbb{Z}} \Psi_j(n) \geq C|supp\,\Psi_j|$.

This is just an updated version to the integers of a smooth partition of characteristic function of an interval.

Proof. Let $N = b - a$. If $N \leq 3$, we do it with just one function since it is clear that one can always construct a C^∞ function identically 1 on $[a, b]$ with support in $[a - 1/2, b + 1/2]$ and satisfying (2), (3) and (4).

If $N \geq 3$, we consider an interval $I_1 = [a_1, b_1]$, a_1, $b_1 \in \mathbb{Z}$, $b_1 - a_1 = [N/3]$ and situated in the middle of I. We take a C^∞ function ξ_1 satisfying

$$\xi_1(x) = 1,\ x \in I_1,\ 0 \leq \xi_1 \leq 1,$$

$$\operatorname{supp}\xi_1 \subset \big[a_1 - [N/6], b - 1 + [N/6]\big],$$

$$\|\xi_1'\|_\infty \leq 6/N.$$

From now on we consider only what is left on the right-hand side of I_1 and proceed on the left side in the same way. Let $J = [b_1, b]$. We cut it in half, consider $\big[b_1, b_1 + [|J|/2]\big] = [b_1, b_2]$ and construct ξ_2 such that $0 \leq \xi_2 \leq 1$, $\xi_2 \in C^\infty$ and

$$\operatorname{supp}\xi_2 \subset [b_1 - |J|/4, b_2 + |J|/4],\ \|\xi_2'\|_\infty \leq 4/|J|,$$

$$\xi_2(x) = 1,\ \ x \in [b_1, b_2].$$

One then repeats the process on $[b_2, b]$ until one gets an interval $[b_k, b]$ of length less than 4, in which we define ξ_{k+1} as above, identically 1 on $[b_k, b]$, with support in $[b_k - 1/2, b + 1/2]$, and $||\xi'_{k+1}||_\infty \leq 2$. Now it is clear that if we define

$$\Psi_j = \frac{\xi_j}{\sum_j \xi_j}$$

we have the family of functions satisfying (1), (2), (3), and (4). $\square$

Now we are ready to prove the following result of R. Caballero and A. de la Torre [13].

Theorem 2.18. *Let f be an L_p function such that Mf is in L_p. Then f can be written as $f(x) = \sum_i c_i a_i(x)$ where the a_i are (p, ∞) atoms and*

$$\sum_i |c_i|^p \leq C||Mf||_p^p.$$

Proof. For each $\lambda > 0$ consider the set

$$O(\lambda) = \{x \in X : Mf(x) > \lambda\}.$$

Let $\lambda_0 = \inf\{\lambda > 0 : m(O_\lambda < 1\}$. If $\lambda_0 \neq 0$, we consider the sequence $\lambda_k = 2^k\lambda_0,\ k = 0, 1, \ldots$ Let $k \neq 0$. Then $m(O(\lambda_k)) < 1$ and since τ is ergodic we can use Lemma 2.16 and write

$$O(\lambda_k) = \bigcup_i R_i^k$$

where the R_i^k are disjoint rectangles of length i; we now write for k fixed

$$f = \sum_i f\chi_{R_i^k} + f(1 - \chi_{O(\lambda_k)}).$$

For each i fixed, we use Lemma 2.17 on the interval $[0, l(i)]$ ($l(i) + 1$ =length of R_i) and we call $\{\Psi_{i,j}^k\}$ the corresponding partition of unity. Let B_i^k be the base of R_i^k and write

$$m_{i,j}^k(\tau^m x) = \frac{\sum_n f(\tau^n x)\,\Psi_{i,j}^k(n)}{\sum_n \Psi_{i,j}^k(n)}, \quad x \in B_i^k,\ m \in [0, l(i)],$$

$$m_{i,j}^k(\tau^m x) = 0 \ \text{ if } \tau^m x \notin R_i^k.$$

Each $\Psi_{i,j}^k$ can be used to define a function on X, which we will call by the same name, as

$$\Psi_{i,j}^k(\tau^n x) = \Psi_{i,j}^k(n) \text{ for } x \in B_i^k,\ 0 \le n \le l(i),$$

and zero otherwise. It is clear that $\sum_j \Psi_{i,j}^k = \chi_{R_i^k}$. Now for k fixed we may write

$$\sum_i \sum_j (f - m_{i,j}^k)\, \Psi_{i,j}^k + \sum_{i,j} m_{i,j}^k\, \Psi_{i,j}^k + f(1 - \chi_{O(\lambda_k)}) = b_k + g_k$$

where

$$b_k = \sum_i \sum_j (f - m_{i,j}^k)\, \Psi_{i,j}^k,$$

$$g_k = \sum_{i,j} m_{i,j}^k\, \Psi_{i,j}^k + f(1 - \chi_{O(\lambda_k)}).$$

Let $(a-h, a+h)$ be the smallest interval containing the support of $\Psi = \Psi_{i,j}^k$ with $a, h \in \mathbb{Z}$. Let N be a fixed number (independent of f, i, j, k) such that $(a - Nh, a + Nh)$ intersects $\mathbb{R} - (-1, l(i) + 1)$. Then let

$$\phi(s) = Nh\Psi(a - sNh).$$

Now we see that

$$\|\phi\|_\infty \le Nh, \quad \text{and} \quad \|\phi'\|_\infty \le (Nh)^2 \|\Psi'\|_\infty \le Ch$$

and

$$\begin{aligned}(f * \phi_{Nh})(\tau^a x) &= \sum_n f(\tau^{a-n} x)\, \phi_{Nh}(n) \\ &= \sum_n f(\tau^{a-n})\, \Psi(a - n) \\ &= \sum_n f(\tau^n x)\, \Psi(n).\end{aligned}$$

Remembering the definition of our maximal operator, we have

$$\Big|\sum_n f(\tau^n x)\,\Psi(n)\Big| \le CMf(\tau^{a+1}x)\,h$$

provided $|l| < Nh$. Now because of property (3) in Lemma 2.17 we choose l such that $\tau^{a+l}(x) \notin O(\lambda_k)$ and by property (4)

$$m_{i,j}^k \le CMf(x') \text{ with } x' \in X - O(\lambda_k).$$

Therefore

$$m_{i,j}^k \le C\lambda_k.$$

On the other hand, if $x \notin O(\lambda_k)$ then

$$f(x) \le Mf(x) < \lambda_k,$$

and we obtain

$$g_k(x) \le C\lambda_k.$$

Since we are assuming that $Mf \in L_p$, it is clear that $m(O(\lambda_k)) \to 0$ as $\lambda_k \to \infty$, and therefore, since b_k has support in $O(\lambda_k)$, we have

$$f(x) = \lim_{k\to\infty} g_k,$$

and defining

$$b_0(x) = f(x)$$

we have

$$f(x) = \sum_{k=0}^{\infty}(g_{k+1} - g_k) = \sum_{k=0}^{\infty}(b_k - b_{k+1}).$$

Now we observe that $O(\lambda_{k+1}) \subset O(\lambda_k)$ and therefore we have

$$R_j^{k+1} \subset O(\lambda_k) = \bigcup_i R_i^k;$$

so if we write

$$R_j^{k+1} = \bigcup_i (R_j^{k+1} \cap R_i^k)$$

we obtain R_i^{k+1} as a disjoint union of $R_{i,j}^{k+1} = R_j^{k+1} \cap R_i^k$, where each $R_{i,j}^{k+1}$ is a rectangle with base $B_{i,j}^{k+1} = B_j^{k+1} \cap R_i^k$, and length that of R_j^{k+1}. If we write

$$a_{i,j}^k = (f - m_{i,j}^k)\, \Psi_{i,j}^k$$

then

$$b_k = \sum_i b_{i,k} \text{ with } b_{i,k} = \sum_j a_{i,j}^k.$$

Let us fix i and write

$$A_{i,k} = b_{i,k} - \sum_j b_{j,k+1}\, \chi_{R_{i,j}^{k+1}},$$

the sum extended over all j such that $R_{i,j}^{k+1} \neq \phi$. From the above observation on the decompositions of $O(\lambda_{k+1})$ and $O(\lambda_k)$ it is clear that

$$b_k - b_{k+1} = \sum_i A_{i,k} = g_{k+1} - g_k.$$

But since the $A_{i,k}$ have disjoint supports for k fixed, it follows that for any x

$$|A_{i,k}(x)| = |g_{k+1}(x) - g_k(x)|$$

$$\leq C\lambda_k.$$

Also, from the definition of $m_{i,j}^k$ it follows that

$$\sum_{i=0}^{i-1} a_{i,j}(\tau^l x) = 0, \;\; x \in B_i^k.$$

These last two observations imply that

$$\tilde{A}_{i,k} = \left(C\lambda_k m(R_i^k)^{1/p}\right)^{-1} A_{i,k}$$

is a (p,∞) atom for $k = 1, 2, \dots$

For $k = 0$ we define $A_0 = b_0 - b_1 = g_1 - g_0$, which implies

$$|A_0(x)| \lesssim C\lambda_0,$$

so that

$$\tilde{A}_0 = (C\lambda_0)^{-1} A_0$$

is a (p,∞) atom of type 2 we may then write

$$\begin{aligned} f &= \sum_{k=0}^{\infty} (b_k - b_{k+1}) \\ &= C\lambda_0 \tilde{A}_0 + \sum_{i,k} C\lambda_k \big(m(R_{i,k})\big)^{1/p} \tilde{A}_{i,k} \end{aligned}$$

where $\tilde{A}_0$, $\tilde{A}_{i,k}$ are (p,∞) atoms, while the sum of the p-th powers of the coefficients is dominated by

$$\begin{aligned} C\sum_k \lambda_k^p m\big(O(\lambda_k)\big) &= C\sum_k \lambda_k^p \int_{\{x: Mf > \lambda_k\}} 1\, dx \\ &= \int_X \sum_{k=0}^{\lambda_k < Mf(x)} \lambda_k^p\, dx \\ &\leq C \int_X |Mf(x)|^p\, dx. \end{aligned}$$

If $\lambda_0 = 0$, we choose $\lambda_k = 2^k$, $k \in \mathbb{Z}$. then for each k, $m\big(O(\lambda_k)\big) < 1$ and we may write as above

$$f = b_k + g_k$$

and

$$f(x) = \sum_{k=-\infty}^{\infty} (g_{k+1} - g_k)(x) = \sum_{k=-\infty}^{\infty} (b_k - b_{k+1})(x).$$

From this equality we proceed as in the case $\lambda_0 \neq 0$. $\square$

Corollary 2.19. *We have*

$$H^{p,q} = H^{p,\infty} = \{f \in L_p : Mf \in L_p\}.$$

Proof. By Theorem 2.18 we see that the set $\{f \in L_p : Mf \in L_p\}$ is contained in $H^{p,\infty} \subset H^{p,q}$ and the metric induced by Mf is equivalent to that of $H^{p,\infty}$ so the result follows. $\square$

Combining Corollary 2.19 with Remark 2.12 and Remark 2.10, we also find the following corollaries.

Corollary 2.20. *For all $1 \leq p \leq \infty$, there is a constant C such that*

$$\|f\|_{H^p} \leq C\|f^*\|_p$$

for all $f \in L_p(X)$.

Corollary 2.21. *There is a constant C such that*

$$\|f\|_1 + \|Hf\|_1 \leq C\|f^*\|_1$$

for all $f \in L_1(X)$.

2.3 Dual Spaces

In this section we will show that the dual of H^1 is the space of functions of ergodic bounded mean oscillation (EBMO), while the dual of $H^p(p < 1)$ is trivial.

For $p = 1$ the result was first proved by R. Coifman and G. Weiss [19], while for $p < 1$, in a different setting, P. Muhly [34] proved that $(H^p)^*$ is trivial.

Definition 2.22. *Let f be an integrable function and let*

$$f^{\#}(x) = \sup_n \frac{1}{n} \sum_{i=0}^{n-1} |f(\tau^i x) - T_n f(x)|$$

where

$$T_n f(x) = \frac{1}{n} \sum_{i=0}^{n-1} f(\tau^i x).$$

A function is said to belong to EBMO *iff $f^{\#}$ is bounded, and* EBMO *norm is denoted by*

$$\|f\|_{\mathrm{EBMO}} = \|f\|_1 + \|f^{\#}\|_\infty.$$

Next we will show that any EBMO function is a linear functional in H^1.

Proposition 2.23. *Let $f \in$ EBMO. Then for any $h \in H^1$ of the form $h = \sum_{i=1}^{N} \lambda_i a_i$,*

$$< f, h > \equiv \int fh$$

satisfies

$$| < f, h > | \leq \|f\|_{\mathrm{EBMO}} \sum_{i=1}^{N} |\lambda_i| \leq \|f\|_{\mathrm{EBMO}} \, \|h\|_{H^1}.$$

Therefore f induces a linear functional in H^1 with norm at most $\|f\|_{\mathrm{EBMO}}$.

Proof. It is enough to show that if a is a $(1, \infty)$ atom, then

$$\left| \int fa \right| \leq \|f\|_{\mathrm{EBMO}}.$$

If a is a type 2 atom then

$$\left| \int fa \right| \leq \int |f| \leq \|f\|_{\mathrm{EBMO}}$$

since $\|a\|_\infty \leq 1$.

If a is a type 1 atom supported in the rectangle $R = \bigcup_{i=0}^{k-1} \tau^i B$, then

$$\begin{aligned}
\left|\int_X f(x)\,a(x)\right| = \left|\int_R f(x)\,a(x)\right| &= \left|\int_B \sum_{i=0}^{k-1} a(\tau^i x)\, f(\tau^i x)\right| \\
&= \left|\int_B \sum_{i=0}^{k-1} a(\tau^i x)\,(f(\tau^i x) - T_n f(x))\right| \\
&\leq \int_B \sum_{i=0}^{k-1} |a(\tau^i x)|\,|f(\tau^i x) - T_n f(x)| \\
&\leq km(R)^{-1} \int_B k^{-1} \sum_{i=0}^{k-1} |f(\tau^i x) - T_n f(x)| \\
&\leq km(R)^{-1} m(B) \|f\|_{\mathrm{EBMO}} = \|f\|_{\mathrm{EBMO}}.
\end{aligned}$$

It follows that EBMO $\subset (H^1)^*$. For $p < 1$ it is trivial that any constant produces a continuous linear functional on H^p.

Now let L be an element of $(H^p)^*$. Fix any $q, 1 < q < \infty$. Then if h is in L_q, we have

$$\|Mh\|_q \leq C_q \|h\|_q$$

and therefore

$$\left(\int |Mh|^p\right)^{1/p} \leq \|Mh\|_q \leq C\|h\|_q,$$

and this means that L defines a linear functional in L_q, and so it can be represented by a function f in $L_{q'} \subset L_1$. Let now a be a (p, ∞) atom of type 1. Then

$$|< L, a >| = \left|\int_R fa\right| \leq \|L\|.$$

Let us now fix an ergodic rectangle $R = \bigcup_{i=0}^{k-1} \tau^i B$, and let us write, for any $y \in R$,

$$T_k f(y) = \frac{1}{k} \sum_{i=0}^{k-1} f(\tau^i x) \quad \text{where } x \in B,\ y = \tau^j x,\ 0 \leq j \leq k-1.$$

We observe that

$$\|f - T_k f\|_{L_1(R)} = \sup_{\|g\chi_R\|_\infty \le 1} \left| \int_R (f - T_k f)\, g \right|.$$

But

$$\left| \int_R (f - T_k f)\, g \right| = \left| \int_R (f - T_k f)\, (g - T_k g) \right|$$

since

$$\int_{\mathbb{R}} (f - T_k f)\, g = \int_B \sum_{i=0}^{k-1} (f - T_k f)\, T_k g\, (\tau^i x),$$

but $T_k g(\tau^i x)$ is independent of i and

$$\sum_{i=0}^{k-1} (f - T_k f)\, (\tau^i x) = T_k f(x) - T_k f(x) = 0.$$

The same argument shows that

$$\int_R (T_k f)\, (g - T_k g) = 0$$

and therefore

$$\begin{aligned} \left| \int_R (f - T_k f)\, g \right| &= \left| \int_R f\, (g - T_k g) \right| \\ &= \left| \int_R f \frac{g - T_k g}{(km(B))^{1/p}} (km(B))^{1/p} \right| \\ &\le 2(km(B))^{1/p} \left| \int_R fa \right| \end{aligned}$$

where

$$a = \frac{g - T_k g}{2(km(B))^{1/p}} \chi_R,$$

which is a (p, ∞) atom. Therefore

$$\|f - T_k f\|_{L_1(R)} \le 2m(R)^{1/p} \|L\|.$$

This can be written as

$$\frac{1}{m(B)}\int_B \frac{1}{k}\sum_{i=0}^{k-1}|f(\tau^i x)-T_k f(x)| \le 2m(R)^{(1/p)-1}\|L\|.$$

Let now $A \subset B$, $m(A)>0$. It is obvious that A is the base of the rectangle $\tilde{R}=\bigcup_{i=0}^{k-1}\tau^i A$ and so

$$\frac{1}{m(A)}\int_A \frac{1}{k}\sum_{i=0}^{k-1}|f(\tau^i x)-T_k f| \le 2\|L\|\, m(\tilde{R})^{(1/p)-1}.$$

Since A is arbitrary, this implies that for almost all x in B we have

$$\frac{1}{k}\sum_{i=0}^{k-1}|f(\tau^i x)-T_k f| \le 2\|L\|\, m(R)^{(1/p)-1}.$$

If $p=1$, this means that

$$\frac{1}{k}\sum_{i=0}^{k-1}|f(\tau^i x)-T_k f| \le 2\|L\| \quad \text{a.e.}$$

If $p<1$, by choosing small rectangles one gets

$$\frac{1}{k}\sum_{i=0}^{k-1}|f(\tau^i x)-T_k f| = 0 \quad \text{a.e.},$$

i.e. f is constant on orbits, and since T is ergodic, f is constant.

So we have shown that for $p<1$, f must be constant, while for $p=1$ we have $f \in \mathrm{EBMO}$ and

$$\|f^{\#}\| \le 2\|L\|.$$

Furthermore, since any function g such that $\|g\|_\infty \le 1$ is a $(1,\infty)$ atom, we have $|\int fg| \le \|L\|$ for any such g. Therefore $\|f\|_1 \le \|L\|$ and finally

$$\|f\|_{\mathrm{EBMO}} = \|f\|_1 + \|f^{\#}\|_\infty \le 3\|L\|.$$

□

2.4 Interpolation

In this section we will state and prove two interpolation theorems from R. Caballero and A. de la Torre [13] between H^p, $1/2 < p \leq 1$ and L_q, $q > 1$. For this we first need the following definition:

Definition 2.24. *Let T be a sublinear operator. We will say that T is of weak type (H^p, p), $p \leq 1$ if*

$$m\{x : |Tf(x)| > \lambda\} \leq \left(\frac{M}{\lambda}\right)^p |f|_{p,\infty},$$

and we will say that T is of weak type (p,p), $1 \leq p$ if

$$m\{x : |Tf(x)| > \lambda\} \leq \left(\frac{N}{\lambda}\right)^p \|f\|_p^p,$$

where M, N are constants and $\lambda > 0$.

Theorem 2.25. *Let T be a sublinear operator of weak type (H^{p_1}, p_1) and (p_2, p_2) with $1/2 < p_1 \leq p_2$. Then T is bounded from H^p into L_p, $p_1 < p \leq 1$.*

Proof. It is enough to show that $\|Ta\|_p < C$ for any (p, ∞) atom.

If a is a type 1 atom with support in $R = \bigcup_{i=0}^{k-1} \tau^i B$, then it follows that

$$\|a\|_{p_2} \leq M(R)^{(1/p_2)-(1/p)}$$

and

$$\left(\frac{1}{k}\sum_{i=0}^{k-1} |a(\tau^i x)|^{p_2}\right)^{1/p_2} \leq m(R)^{-1/p},$$

which means that

$$b = m(R)^{(1/p)-(1/p_1)} a$$

is a (p_1, p_2) atom since

$$\left(\frac{1}{k}\sum_{i=0}^{k-1}|b(\tau^i x)|^{p_2}\right)^{1/p_2} \leq m(R)^{-1/p_1}.$$

Since $H^{p_1,p_2} = H^{p_1,\infty} = H^{p_1}$ with equivalent norms it follows that $a \in H^{p_1}$ and

$$|a|_{p_1} \leq m(R)^{((1/p_1)-(1/p))p_1} = m(R)^{1-(p_1/p)}.$$

Therefore we know that

$$m\{x : |Ta(x)| > \lambda\} \leq \left(\frac{M_1}{\lambda}\right)^{p_1} |a|_{p_1} \leq \left(\frac{M_1}{\lambda}\right)^{p_1} m(R)^{1-(p-1/p)}$$

and

$$m\{x : |Ta(x)| > \lambda\} \leq \left(\frac{M_2}{\lambda}\right)^{p_2} \int |a|^{p_2} \leq \left(\frac{M_2}{\lambda}\right)^{p_2} m(R)^{1-(p_2/p)}.$$

From these two estimates one obtains a bound for $\|Ta\|_p^p$ in a usual way: we fix a number D and write

$$\begin{aligned}\int |Ta|^p &= p\int_0^{Dm(R)^{-1/p}} \lambda^{p-1} m\{x : |Ta(x)| > \lambda\}\, d\lambda + \\ &\quad + p\int_{Dm(R)^{-1/p}}^{\infty} \lambda^{p-1} m\{x : |Ta(x)| > \lambda\}\, d\lambda \\ &\leq \int_0^{Dm(R)^{-1/p}} \lambda^{p-1-p_1} M_1^{p_1} m(R)^{1-(p_1/p)}\, d\lambda + \\ &\quad + p\int_{Dm(R)^{-1/p}}^{\infty} M_2^{p_2} m(R)^{1-(p_2/p)}\, d\lambda.\end{aligned}$$

Since $p_1 < p < p_2$, the last expression is bounded by

$$\begin{aligned}&\frac{p}{p-p_1} M_1^{p_1} m(R)^{1-(p_1/p)} (Dm(R)^{-1/p})^{p-p_1} + \\ &\quad + \frac{p}{p_2 - p} M_2^{p_1} m(R)^{1-(p_2/p)} (Dm(R)^{-1/p})^{p-p_2} \\ &= \frac{p}{p-p_1} M_1^{p_1} + \frac{p}{p_2-p} M_2^{p_2} D^{p-p_2}.\end{aligned}$$

Taking $D = (M_2^{p_2} M_1^{-p_1})^{1/(p_2-p_1)}$, we obtain

$$\|Ta\|_p \leq \left(\frac{p}{p-p_1} + \frac{p}{p_2-p} \right)^{1/p} M_1^t M_2^{1-t}$$

with

$$t = \frac{p_1(p_2-p)}{p(p_2-p_1)}.$$

If a is a type 2 atom, then obviously $a \in H^{p_1}$ with $|a|_{p_1} \leq 1$ and $a \in L_{p_2}$ with $\|a\|_{p_2} \leq 1$, and we may write

$$\begin{aligned}
\int |Ta|^p &= p\int_0^D \lambda^{p-1} m\{x : |Ta| > \lambda\}\, d\lambda + p\int_D^\infty \lambda^{p-1} m\{x : |Ta| > \lambda\}\, d\lambda \\
&\leq p\int_0^D \lambda^{p-1-p_1} M_1^{p_1}\, d\lambda + p\int_D^\infty \lambda^{p-1-p_2} M_2^{p_2}\, d\lambda \\
&\leq \frac{p}{p-p_1} M_1^{p_1} D^{p-p_1} + \frac{p}{p_2-p} M_2^{p_2} D^{p-p_2}.
\end{aligned}$$

Choosing D as before, we have the same bound for $\|Ta\|_p$, and this gives the desired result.

$\square$

To prove the other interpolation theorem we will need a lemma but first a definition:

Definition 2.26. *For $p > 1$, let us fix p_0, $1 < p_0 < p$, and let us define the operator*

$$\Lambda_{p_0}(f) = (|f|^{p_0 *})^{1/p_0}$$

where

$$g^*(x) = \sup_k \frac{1}{k} \sum_{i=0}^{k-1} |g(\tau^i x)|.$$

Since $p/p_0 > 1$, we have

$$\int |\Lambda_{p_0}(f)|^p = \int (|f|^{p_0 *})^{p/p_0} \leq C_{p/p_0} \int |f|^p,$$

which means that Λ_{p_0} is a bounded operator in L_p, and the set

$$O(\lambda) = \{x : \Lambda_{p_0}(f)(x) > \lambda\}$$

has measure less than 1 if $\lambda > C_{p/p_0}\|f\|_p$.

Lemma 2.27. *Let $f \in L_p$ and $\lambda > C_{p/p_0}\|f\|_p$. Then*

$$O(\lambda) = \{x : \Lambda_{p_0}f(x) > \lambda\} = \bigcup R_j$$

where the R_j are ergodic rectangles, pairwise disjoint, and for each $x \in B_j$ (the base of R_j) we have

$$\left(\frac{1}{j}\sum_{i=0}^{j-1} |f(\tau^i x)|^{p_0}\right)^{1/p_0} \leq 2\lambda.$$

Proof. Let us define

$$B_j = \{x \in O(\lambda) : \tau^{-1}x \notin O(\lambda),\, x \in O(\lambda), \ldots, \tau^{j-1}x \in O(\lambda),\, \tau^j x \notin O(\lambda)\},$$

and it follows that $O(\lambda) = \bigcup R_j$ with $R_j = \bigcup_{i=0}^{j-1} \tau^i B_j$. Finally

$$\begin{aligned} \frac{1}{j}\sum_{i=0}^{j-1} |f(\tau^i x)|^{p_0} &\leq \frac{2}{j+1}\sum_{i=-1}^{j-1} |f(\tau^i x)|^{p_0} \\ &\leq 2 f^{p_0 *}(\tau^{-1}x) \\ &\leq 2\lambda^{p_0} \end{aligned}$$

since $\tau^{-1}x \notin O(\lambda)$. □

Theorem 2.28. *Let T be a sublinear operator of weak type $(H^1, 1)$ and (p_2, p_2), $1 < p_2 < \infty$. Then T is bounded in L_p, $1 < p < p_2$.*

For $p_2 = \infty$, the result holds assuming that T is bounded in L_∞.

Proof. We will prove the theorem only in the case $p_2 < \infty$, since the other case is similar.

Let f be a function in L_p, $1 < p < p_2$. We choose p_0, $1 < p_0 < p$, and we consider the operator Λ_{p_0}. For each $\lambda > C_{p/p_0}\|f\|_p$, by Lemma 2.27 we can write

$$O(\lambda) = \{x : \Lambda_{p_0} f(x) > \lambda\} = \bigcup R_j.$$

For each $y \in R_j$, we define

$$(T_j f)(y) = \frac{1}{j}\sum_{i=0}^{j-1} f(\tau^i x)$$

where $x \in B_j$, $y = \tau^l x$, $0 \le l \le j-1$. We may then write

$$f = \sum_j (f - T_j f)\chi_{R_j} + \sum_j (T_j f)\chi_{R_j} + f(1 - \chi_{O(\lambda)}) \equiv b_\lambda + g_\lambda$$

where

$$b_\lambda = \sum_j (f - T_j f)\chi_{R_j}.$$

Since

$$\begin{aligned} |(T_j)(y)| &\le \left(\frac{1}{j}\sum_{i=0}^{j-1} |f(\tau^i x)|^{p_0}\right)^{1/p_0} \\ &\le 2\lambda, \end{aligned}$$

we have for each j, and for each $x \in B_j$,

$$\begin{aligned} \left(\frac{1}{j}\sum_{i=0}^{j-1} |f(\tau^i x) - T_j f(\tau^i x)|^{p_0}\right)^{1/p_0} &\le \left(\frac{1}{j}\sum_{i=0}^{j-1} |f(\tau^i x)|^{p_0}\right)^{1/p_0} + |T_j f(x)| \\ &\le 4\lambda. \end{aligned}$$

Therefore the function

$$a_j = \frac{1}{4\lambda m(R_j)}(f - T_j f)\chi_{R_j}$$

is a $(1, p_0)$ atom supported in the rectangle R_j. This means that we can write b_λ as an H^{1,p_0} function, since

$$b_\lambda = \sum_j 4\lambda m(R_j) a_j$$

with norm bounded by $4\lambda \sum m(R_j) = 4\lambda m(O(\lambda))$. Since $H^{1,p_0} = H^1$ with equivalent norms, we have $b_\lambda \in H^1$ and $|b|_{H^1} \leq C\lambda m(O(\lambda))$. On the other hand, g_λ is in L_{p_2} since

$$g_\lambda = \sum_j (T_j f)\chi_{R_j} + f(1 - \chi_{O(\lambda)});$$

so if $y \in O(\lambda)$ we have $g_\lambda(y) = T_j f(y)$ for some j, and then $|g_\lambda(y)| \leq 2\lambda$, while if $y \notin O(\lambda)$, then

$$|g_\lambda(y)| = |f(y)| \leq (\Lambda_{p_0} f)(y) \leq \lambda.$$

Let us consider a constant L larger than $C_{p/p_0}\|f\|_p$. Then we have

$$\begin{aligned}
\int |Tf|^p \leq p \int_0^L & \lambda^{p-1} m\{x : |Tf(x)| > \lambda\}\, d\lambda + \\
& + p \int_L^\infty \lambda^{p-1} m\{x : |Tg_\lambda(x)| > \lambda/2\}\, d\lambda + \\
& + p \int_L^\infty \lambda^{p-1} m\{x : |Tg_\lambda(x)| > \lambda/2\}\, d\lambda \\
& = I_1 + I_2 + I_3.
\end{aligned}$$

Note that if f is in L_p then

$$a = \frac{f}{\|f\|_p}$$

is a $(1, p)$ atom, and therefore f belongs to H^1 with norm bounded by $C\|f\|_p$. Therefore

$$I_1 \leq C_p \int_0^L \lambda^{p-2} \|f\|_p \, d\lambda = C\frac{p}{p-1}\|f\|_p L^{p-1}.$$

On the other hand, since $b_\lambda \in H^1$ with norm bounded by $C\lambda m(O(\lambda))$ we get

$$
\begin{aligned}
I_3 &\le C_p \int_L^\infty \lambda^{p-2} \lambda m(O(\lambda))\, d\lambda \\
&\le C_p \int_0^\infty \lambda^{p-1} m\{x : \Lambda_{p_0} f > \lambda\}\, d\lambda \\
&= C \int (\Lambda_{p_0} f)^p \le C \cdot C_{p/p_0}^p \int |f|^p \\
&= C' \|f\|_p^p.
\end{aligned}
$$

In order to estimate I_2 we use the fact that T is of weak type (p_2, p_2), and we have

$$
\begin{aligned}
I_2 &\le C_p \int_L^\infty \lambda^{p-p_2-1} \int |g|^{p_2}\, dx\, d\lambda \\
&= C_p \int_L^\infty \lambda^{p-p_2-1} \left(\int_{O(\lambda)} |g|^{p_2}\, dx + \int_{X-O(\lambda)} |g|^{p_2}\, dx \right) d\lambda \\
&\le C_p \int_0^\infty \lambda^{p-p_2-1} (2\lambda)^{p_2} m(O(\lambda))\, d\lambda + \\
&\quad + C_p \int_L^\infty \lambda^{p-p_2-1} \int_{\{\lambda_{p_0} f \le \lambda\}} |f|^{p_2}\, dx\, d\lambda \\
&\le C \cdot 2^{p_2} p \int_0^\infty \lambda^{p-1} m\{x : \Lambda_{P-0} f > \lambda\}\, d\lambda + \\
&\quad + C_p \int |f|^{p_2} \int_{\Lambda_{p_0} f}^\infty \lambda^{p-p_2-1}\, d\lambda\, dx \\
&\le C \cdot 2^{p_2} \int (\Lambda_{p_0} f)^p\, dx + C \frac{p}{p_2 - p} \int |f|^{p_2} (\Lambda_{p_0} f)^{p-p_2}\, dx \\
&\le \left(C \cdot 2^{p_2} + C \frac{p}{p_2 - p}\right) \int (\Lambda_{p_0} f)^p\, dx \\
&\le C' C_{p/p_0}^p \int |f|^p\, dx.
\end{aligned}
$$

Choosing now $L = 2C_{p/p_0} \|f\|_p$, we have obtained

$$\int |Tf|^p \le C \|f\|_p^p$$

as desired. $\square$

Chapter 3

Real and Ergodic Operators

This chapter can be considered as the body of the thesis since almost all of the main results and the methods are included in this chapter. However, it is also clear from our presentation that the importance of the other chapters can not be ignored. Actually, they can be thought of the complement of our main discussion. We will here first focus on the study of certain inequalities on $\mathbb{R}$, and then we will extend our result to a general ergodic, measure preserving dynamical system. One can see from our discussion that the method that we will present here can be used to study the inequalites for a large class of operators in ergodic theory as well as in harmonic analysis. The method of this chapter will also indicate the close connection between the classical H^p spaces and the ergodic H^p spaces. It is a well known problem to investigate if an operator maps H^1 to L_1, and this kind of problems have been studied by many authors such as E. M. Stein, A. P. Calderón and J. L. Rubio de Francia in harmonic analysis and also studied by D. L. Burkholder, R. Gundy and B. Davis in martingale theory. Here we also show that the same problem can be studied in ergodic theory. Finally, we will show that one can study the vector-valued inequalities in ergodic theory. Note also that any result of this chapter is a new result if it is not cited.

3.1 Singular Integral Operators

Let B be a Banach space and $p < \infty$, and let f be a B-valued (strongly) measurable function defined in $\mathbb{R}$. Then the Bochner-Lebesgue space $L_B^p = L_B^p(\mathbb{R})$ is defined as

$$L_B^p = \{f : \|f\|_{L_B^p} < \infty\},$$

where

$$\|f\|_{L_B^p} = \left(\int_{\mathbb{R}} \|f(x)\|_B^p \, dx\right)^{1/p}.$$

When B is the scalar field, we simply write L_p and $\|\cdot\|_p$. We also define the space $WL_B^p = \text{weak} - L_B^p$ as the space of all B-valued functions f such that

$$\|f\|_{WL_B^p} = \sup_{\lambda>0} \lambda \big(m\{x \in \mathbb{R} : \|f(x)\|_B > \lambda\}\big)^{1/p} < \infty.$$

When we replace Lebesgue measure by $w(x)\,dx$ for some positive weight w in $\mathbb{R}$ we denote the corresponding spaces by $L_B^p(w)$ and $WL_B^p(w)$. When $p = \infty$, we write

$$L^\infty(B) = \{f : \|f\|_{L^\infty(B)} < \infty\},$$

where

$$\|f\|_{L^\infty(B)} = \operatorname{ess\,sup} \|f\|_B$$

and the space of all compactly supported members of L_B^∞ will be denoted by $L_c^\infty(B)$.

For a locally integrable B-valued function f, we define the maximal functions

$$M_r f(x) = \sup_{x \in I} \left(\frac{1}{|I|} \int_I \|f(y)\|_B^r \, dy \right)^{1/r}, \quad 1 \leq r \leq \infty,$$

and

$$f^{\#}(x) = \sup_{x \in I} \frac{1}{|I|} \int_I \|f(y) - f_I\|_B \, dy,$$

where I denotes an arbitrary interval in $\mathbb{R}$ and

$$f_I = \frac{1}{|I|} \int_I f(t) \, dt$$

which is an element of B.

Note that $f^{\#}$ is the sharp maximal function in the classical case when $B = \mathbb{R}$ and $\|\cdot\|_B = |\cdot|$, $M_1 f$ is the Hardy-Littlewood maximal function and $M_\infty f$ is the constant function with value in $\|f\|_{L^\infty(B)}$. We define the space

$$\mathrm{BMO}(B) = \{ f \in L^1_{\mathrm{loc}.B} : \|f\|_{\mathrm{BMO}(B)} = \|f^{\#}\|_\infty < \infty \}.$$

Remark 3.1. Given a B-valued function f, we obtain a nonnegative function $\|f\|_B$ defined by

$$\|f\|_B(x) = \|f(x)\|_B,$$

and it is also important to point out that

$$(\|f\|_B)^{\#}(x) \leq 2 f^{\#}(x)$$

and

$$\|(\|f\|_B)\|_{\mathrm{BMO}} \leq 2 \|f\|_{\mathrm{BMO}(B)}.$$

As usual a B-atom is a function $a \in L^\infty(B)$ supported in an interval I and such that

$$\|a(x)\|_B \leq \frac{1}{|I|}, \quad \int_I a(x)\,dx = 0$$

and the space $H^1_B(\mathbb{R})$ is the space of functions in $L^1_B(\mathbb{R})$ such that

$$f(x) = \sum_j \lambda_j a_j(x); \quad (\lambda_j) \in l_1,$$

where a_j are B-atoms with $\|f\|_{H^1_B} = \inf \sum_j |\lambda_j|$.

For simplicity, we will only consider $(1,\infty)$-atoms, since this will be enough for our discussion.

Remark 3.2. Let $B \in$ UMD. Then by Theorem 2.6 in J. L. Rubio de Francia *et al* [46] we also have

$$H^1_B(\mathbb{R}) = \{f \in L^1_B(\mathbb{R}) : Hf \in L^1_B(\mathbb{R})\}$$

and

$$\|f\|_{H^1_B} \sim \|f\|_{L^1_B} + \|Hf\|_{L^1_B},$$

where Hf denotes the Hilbert transform of f on $\mathbb{R}$.

We will consider here those kernels $K(x)$ which are strongly measurable and defined in $\mathbb{R}$ with the values in the space $L(A,B)$ of all bounded linear operators from A to B, with respect to the operator norm $\|\cdot\| = \|\cdot\|_{L(A,B)}$. We will also assume that $\|K(x)\|$ is locally integrable away from the origin, and that one of the following conditions is satisfied:

Definition 3.3. *Given $1 \leq r \leq \infty$, we say that K satisfies the condition (D_r), and write $K \in (D_r)$, if there exists a sequence (c_k) such that*

$$\sum_{k=1}^{\infty} c_k = D_r(K) < \infty$$

and, for all $k \geq 1$ and $y \in \mathbb{R}$,

$$\left(\int_{S_k(|y|)} \|K(x-y) - K(x)\|^r \, dx \right)^{1/r} \leq c_k |S_k(|y|)|^{-1/r'},$$

where

$$S_k(|y|) = \{x : 2^k|y| < |x| \leq 2^{k+1}|y|\}.$$

When $r = \infty$, this must be understood in the usual way, it is also easy to see that $K \in (D_\infty)$ if

$$\|K(x-y) - K(x)\| \leq C \frac{|y|}{|x|^{n+1}},$$

whenever $|x| > 2|y|$. Also, $K \in (D_1)$ is the Hörmander condition

$$\int_{|x|>2|y|} \|K(x-y) - K(x)\| \, dx \leq D_1(K) < \infty,$$

where $D_1(K)$ does not depend on $y \in \mathbb{R}$.

Definition 3.4. *A linear operator T mapping A-valued functions into B-valued functions is called a singular integral operator of convolution type if the following conditions are satisfied:*

(i) *T is a bounded operator from $L_A^q(\mathbb{R})$ to $L_B^q(\mathbb{R})$ for some q, $1 \leq q \leq \infty$.*

(ii) *There exists a kernel $K \in (D_1)$ such that*

$$Tf(x) = \int K(x-y) \cdot f(y) \, dy$$

for every $f \in L_A^q$ with compact support and for a.e. $x \notin supp(f)$.

The proof of the following lemma can be found in J. L. Rubio de Francia *et al* [46].

Lemma 3.5. *Let T be a singular integral operator. Then T can be extended to an operator defined in all L^p_A, $1 \le p < \infty$, and satisfying*

(a) $\|Tf\|_{L^p_B} \le C_p \|f\|_{L^p_A}, \quad 1 < p < \infty,$

(b) $\|Tf\|_{WL^1_B} \le C\|f\|_{L^1_B} \le C\|f\|_{L^1_A},$

(c) $\|Tf\|_{L^1_B} \le C\|f\|_{H^1_A},$

(d) $\|Tf\|_{\mathrm{BMO}(B)} \le C\|f\|_{L^\infty(A)}, \quad f \in L^\infty_c(A).$

Remark 3.6. By a well known interpolation theorem it is clear that (c) and (d) together imply (a) in Lemma 3.5.

3.2 Inequalities in Ergodic Theory

In this section we will study the inequalities in ergodic theory and harmonic analysis. We will first need the following lemma.

Lemma 3.7. *Now let us define the kernel operator $K : \mathbb{R} \to l_2(\mathbb{Z}^+)$ by*

$$K(x) = \left(\frac{1}{2^{k+1}} \chi_{[0,2^{k+1}]}(x) - \frac{1}{2^k} \chi_{[0,2^k]}(x) : k = 1, 2, 3, \dots \right).$$

Then, $K \in (D_1)$, i.e. K satisfies the Hörmander condition.

Proof. We first need to evaluate $K(x-y) - K(x)$ for $|x| > 2|y|$. Let us first consider the case $x > 0$, $y > 0$. We have

$$K(x-y) - K(x) = \Phi_{k+1}(x, y) - \Phi_k(x, y),$$

where

$$\Phi_k(x,y) = \frac{1}{2^k}\left[\chi_{[0,2^k]}(x-y) - \chi_{[0,2^k]}(x)\right].$$

So it suffices to evaluate $\Phi_k(x,y)$ in the estimate to get the Hörmander condition. Since $x > 2y$, we have

$$\Phi_k(x,y) = \frac{1}{2^k}\left[\chi_{[y,y+2^k]}(x) - \chi_{[y,2^k]}(x)\right].$$

Now, if $x \in [y, y+2^k] \cap [y, 2^k]$, then obviously, $\Phi_k(x,y) = 0$. But we have $[y, 2^k] \subset [y, y+2^k]$, so we can assume that $x \in [y, y+2^k] - [y, 2^k]$. Hence, we only need to consider the interval $[2^k, y+2^k]$. Thus, we get

$$\Phi_k(x,y) = \frac{1}{2^k}\chi_{[2^k,y+2^k]}(x).$$

Now, suppose that $x \leq 0$, $y \leq 0$, then we have $y > x$ since $|x| > 2|y|$. In this case we have $\Phi_k(x,y) = 0$, since $x \notin [y, y+2^k] \cup [y, 2^k]$. Also, same is true for $x \leq 0$, $y \geq 0$ since this implies that $x < y$. If $x \geq 0$, $y \leq 0$, then we have the same situation as in the first case. Thus, we need only to evaluate

$$\Phi_k(x,y) = \frac{1}{2^k}\chi_{[2^k,y+2^k]}(x).$$

We have

$$\begin{aligned}
\int_{x>2y} \|\Phi_k(x,y)\|_{l_2(\mathbb{Z}^+)}\, dx &= \int_{x>2y} \left(\sum_{k=1}^{\infty} \left|\frac{1}{2^k}\chi_{[2^k,y+2^k]}(x)\right|^2\right)^{1/2} dx \\
&\leq \int_{x>2y} \left(\sum_{k=1}^{\infty} \frac{1}{2^{2k}}\chi_{[2^k,y+2^k]}\right)^{1/2} dx \\
&\leq \int_{\mathbb{R}} \sum_{k=1}^{\infty}{}' \frac{1}{2^k}\chi_{[2^k,y+2^k]}\, dx \\
&= y\sum_{k=1}^{\infty}{}' \frac{1}{2^k} \leq C,
\end{aligned}$$

where prime denotes that the term in the summation is omitted when $2^k < y$. This shows that $K \in (D_1)$ as desired. $\square$

Remark 3.8. The same method can easily be applied when the sequence (2^k) is replaced with a lacunary sequence (n_k).

Remark 3.9. Let $\phi_n(x) = \frac{1}{n}\chi_{[0,n]}(x)$, and $n_1 \leq n_2 \leq \dots$ be a sequence of positive integers. Let us define

$$S_{\mathbb{R}} f = \left(\sum_{k=1}^{\infty} |(\phi_{n_{k+1}} - \phi_{n_k}) * f|^2 \right)^{1/2}.$$

By applying transfer principle to Theorem 4.6 in R. L. Jones *et al* [29], we see that there exists a constant C_p such that

$$\|S_{\mathbb{R}} f\|_p \leq C_p \|f\|_p$$

for all $f \in L_p(\mathbb{R})$, $1 < p < \infty$, and for any increasing sequence (n_k) of positive integers.

Now, let us define

$$Tf(x) = \int K(x-y) \cdot f(y)\, dy.$$

Then, by Lemma 3.7 and Remark 3.9 we see that T is an l_2-valued singular integral operator, and we have

$$\begin{aligned} \|Tf(x)\|_{l_2(\mathbb{Z}^+)} &= \left(\sum_{k=1}^{\infty} |(\phi_{n_{k+1}} - \phi_{n_k}) * f(x)|^2 \right)^{1/2} \\ &= S_{\mathbb{R}} f(x). \end{aligned}$$

Also, we have by Remark 3.1 that

$$\|S_{\mathbb{R}} f\|_{\mathrm{BMO}} \leq 2\|Tf\|_{\mathrm{BMO}(l_2(\mathbb{Z}^+))}.$$

Combining our discussion with Lemma 3.5, we find the following theorem:

Theorem 3.10. *Let (n_k) be a lacunary sequence. Then we have*

(a) $\|S_{\mathbb{R}}f\|_p \le C\|f\|_p, \quad 1 < p < \infty,$

(b) $m\{x : |S_{\mathbb{R}}f(x)| > \lambda\} \le \frac{C}{\lambda}\|f\|_1,$

(c) $\|S_{\mathbb{R}}f\|_1 \le C\|f\|_{H^1}, \quad f \in L_1(\mathbb{R}),$

(d) $\|S_{\mathbb{R}}f\|_{\mathrm{BMO}} \le C\|f\|_\infty, \quad f \in L_c^\infty(\mathbb{R}).$

Let now $A_n f = \frac{1}{n}\sum_{k=1}^{n} f\circ\tau^k$ denote the usual average in ergodic theory, let $n_1 \le n_2 \le \dots$ and

$$Sf = \left(\sum_{k=1}^{\infty} |A_{n_{k+1}}f - A_{n_k}f|^2\right)^{1/2}.$$

Then, by applying Calderón transfer principle to Theorem 3.10 we find the following corollary with the same constants:

Corollary 3.11. *Let (n_k) be a lacunary sequence. Then we have*

(a) $\|Sf\|_p \le C\|f\|_p, \quad 1 < p < \infty,$

(b) $m\{x : |Sf(x)| > \lambda\} \le \frac{C}{\lambda}\|f\|_1,$

(c) $\|Sf\|_1 \le C\|f\|_{H^1}, \quad f \in L_1(X),$

(d) $\|Sf\|_{\mathrm{EBMO}} \le C\|f\|_\infty, \quad f \in L_c^\infty(X).$

Now, our next result follows from Corollary 3.11 and Corollary 2.20.

Corollary 3.12. *Let (n_k) be a lacunary sequence and $f \in L_1(X)$. Then, there exists a constant C such that*

$$\|Sf\|_1 \le C\|f^*\|_1,$$

where f^* denotes the ergodic maximal function.

We apply Theorem 1.16 to Corollary 3.12 to find the following result:

Corollary 3.13. *Let (n_k) be a lacunary sequence and $f \in L_1(X)$, $f \geq 0$. Then Sf is integrable if $f \log^+ f$ is integrable, where*

$$f(x) \log^+ f(x) = f(x) \log \max(f(x), 1).$$

Let now $f \in L_1(\mathbb{R})$ and define

$$Nf(x) = \sup_k |\phi_{n_k} * f(x)|$$

where (n_k) is a sequence of integers and $\phi_n(x) = \frac{1}{n}\chi_{[0,n]}(x)$.

It is clear that Nf can be viewed as a linear operator taking the real-valued function $f(x)$ into the $l_\infty(\mathbb{Z}^+)$-valued function $(\phi_{n_k} * f(x))_{k=1}^\infty$. It is also quite simple to see that Nf is a bounded operator from $L_\infty(\mathbb{R})$ to itself. On the other hand in order to show that its kernel $K(x) = (\phi_{n_k}(x))_{k=1}^\infty \in (D_1)$, it is necessary and sufficient that

$$\int_{|x|>2|y|} \sup_k |\phi_{n_k}(x-y) - \phi_{n_k}(x)|\, dx \leq C \tag{Z}$$

for some constant C which does not depend on $y \in \mathbb{R}$.

The condition (Z) is due to F. Zo [54].

Lemma 3.14. *The kernel operator $K(x) = (\phi_{n_k}(x))_{k=1}^\infty$ satisfies the condition (Z) when the sequence (n_k) is lacunary.*

Proof. As in the proof of Lemma 3.7, it suffices to consider the case $x > 0$, $y > 0$ and evaluate

$$\frac{1}{n_k}\chi_{[n_k, y+n_k]}(x).$$

Since the sequence (n_k) is lacunary, a similar argument gives a constant C so that the condition (Z) holds for the operator K. $\square$

We find the following theorem as a consequence of Lemma 3.5 and Lemma 3.14.

Theorem 3.15. *Let (n_k) be a lacunary sequence. Then the following inequalities hold:*

(a) $\|Nf\|_p \leq C\|f\|_p, \quad 1 < p < \infty,$

(b) $m\{x : |Nf(x)| > \lambda\} \leq \frac{C}{\lambda}\|f\|_1,$

(c) $\|Nf\|_1 \leq C\|f\|_{H^1}, \quad f \in L_1(\mathbb{R}),$

(d) $\|Nf\|_{\mathrm{BMO}} \leq C\|f\|_\infty, \quad f \in L_c^\infty(\mathbb{R}).$

Let (n_k) be a sequence of integers and $f \in L_1(X)$. Define

$$f_*(x) = \sup_k \frac{1}{n_k}\Big|\sum_{i=1}^{n_k} f(\tau^i x)\Big|.$$

Then applying Calderón transfer principle to Theorem 3.15 we find the following result:

Corollary 3.16. *Let (n_k) be a lacunary sequence. Then the following inequalities hold:*

(a) $\|f_*\|_p \leq C\|f\|_p, \quad 1 < p < \infty,$

(b) $m\{x : |f_*(x)| > \lambda\} \leq \frac{C}{\lambda}\|f\|_1,$

(c) $\|f_*\|_1 \leq C\|f\|_{H^1}, \quad f \in L_1(X),$

(d) $\|f_*\|_{\mathrm{EBMO}} \leq C\|f\|_\infty, \quad f \in L_c^\infty(X).$

Let $2 < s$ and define

$$V_{\mathbb{R}}f = \left(\sum_{k=1}^{\infty} |(\phi_{n_{k+1}} - \phi_{n_k}) * f|^s\right)^{1/s}.$$

We will apply our method to $V_{\mathbb{R}}f$ and investigate the analogous behaviors of the operator

$$Vf = \left(\sum_{k=1}^{\infty} |A_{n_{k+1}}f - A_{n_k}f|^s \right)^{1/s}.$$

Let us define the kernel operator $K : \mathbb{R} \to l_s(\mathbb{Z}^+)$ as

$$K(x) = (\Delta_k * f(x))_{k=1}^{\infty},$$

where $\Delta_k(x) = \phi_{n_{k+1}}(x) - \phi_{n_k}(x)$. It is known from R. L. Jones *et al* [29] that Vf is a bounded operator from $L_p(X)$ to $L_p(X)$, $1 < p < \infty$, for an increasing sequence (n_k) of positive integers, we see by transfer principle that $V_{\mathbb{R}}f$ is also bounded from $L_p(\mathbb{R})$ to $L_p(\mathbb{R})$, $1 < p < \infty$. Let us consider

$$Tf(x) = \int K(x-y) \cdot f(y)\, dy.$$

Then we have

$$\|Tf(x)\|_{l_s(\mathbb{Z}^+)} = V_{\mathbb{R}}f(x)$$

and

$$\|V_{\mathbb{R}}f\|_{\mathrm{BMO}} \leq 2\|Tf\|_{\mathrm{BMO}(l_s(\mathbb{Z}^+))}.$$

On the other hand, to see that Tf is an $l_s(\mathbb{Z}^+)$-valued singular integral operator, it suffices to show that $K \in (D_1)$. But a similar method to the proof of Lemma 3.7 shows that this is indeed true when the sequence (n_k) is lacunary.

Let us apply our discussion to Lemma 3.5 to find the following theorem:

Theorem 3.17. *Let (n_k) be a lacunary sequence. Then we have*

(a) $\|V_{\mathbb{R}}f\|_p \leq C\|f\|_p, \quad 1 < p < \infty,$

(b) $m\{x : |V_{\mathbb{R}}f(x)| > \lambda\} \leq \frac{C}{\lambda}\|f\|_1$,

(c) $\|V_{\mathbb{R}}f\|_1 \leq C\|f\|_{H^1}, \quad f \in L_1(\mathbb{R})$,

(d) $\|V_{\mathbb{R}}f\|_{\mathrm{BMO}} \leq C\|f\|_\infty, \quad f \in L_c^\infty(\mathbb{R})$.

Now we apply transfer principle to Theorem 3.17 to find the following:

Corollary 3.18. *Let (n_k) be a lacunary sequence. Then we have*

(a) $\|Vf\|_p \leq C\|f\|_p, \quad 1 < p < \infty$,

(b) $m\{x : |Vf(x)| > \lambda\} \leq \frac{C}{\lambda}\|f\|_1$,

(c) $\|Vf\|_1 \leq C\|f\|_{H^1}, \quad f \in L_1(X)$,

(d) $\|Vf\|_{\mathrm{EBMO}} \leq C\|f\|_\infty, \quad f \in L_c^\infty(X)$.

Remark 3.19. It is shown in R. L. Jones *et al* [29] that for any increasing sequence (n_k) of positive integers the operator Vf satisfies a weak type $(1,1)$ and a strong type (p,p) inequality for $1 < p < \infty$. Here our method also proves those results in Corollary 3.18 (a) and (b) in a different perspective when the sequence (n_k) is lacunary. It might be interesting if any of the above results holds for Vf when $s < 2$.

Let Mf denote the maximal operator in the characterization of ergodic $H^p(X)$. Then as a consequence of our previous discussion we find the following corollary:

Corollary 3.20. *Let (n_k) be a lacunary sequence, and $f \in L_1(X)$. Then there exist some constants C_i, $i = 1, \ldots, 4$ such that*

(a) $\|Sf\|_1 \leq C_1\|Mf\|_1$,

(b) $\|f_*\|_1 \leq C_2 \|Mf\|_1$,

(c) $\|Vf\|_1 \leq C_3 \|Mf\|_1$,

(d) $\|Vf\|_1 \leq C_4 \|f^*\|_1$.

Our next result shows the importance of lacunarity condition in our discussion.

Theorem 3.21. *There exists a B-valued convolution operator*

$$Tf(x) = \int K(x-y) \cdot f(y)\, dy$$

such that $\|Tf\|_B$ *maps* L_∞ *to BMO, and* L_p *to* L_p, $1 < p < \infty$, *but the kernel operator* K *does not satisfy the Hörmander condition.*

Proof. Let us consider the kernel operator $K : \mathbb{R} \to l_2(\mathbb{Z}^+)$ given by

$$K(x) = (\phi_{n_{k+1}}(x) - \phi_{n_k}(x))_{k=1}^{\infty},$$

where (n_k) is an increasing sequence of positive integers. Now let us define

$$Tf(x) = \int K(x-y) \cdot f(y)\, dy.$$

Then we have as before

$$\|Tf(x)\|_{l_2(\mathbb{Z}^+)} = S_{\mathbb{R}} f(x).$$

We see by applying transfer principle to Theorem 4.4 and Theorem 4.6 in R. L. Jones *et al* [29] that

(a) $\|S_{\mathbb{R}} f\|_{\mathrm{BMO}} \leq C_1 \|f\|_\infty, \quad f \in L_\infty(\mathbb{R})$,

(b) $\|S_{\mathbb{R}} f\|_p \leq C_2 \|f\|_p, \quad 1 < p < \infty$, for some constants C_1 and C_2. But the proof of Lemma 3.7 shows that lacunarity condition on the sequence (n_k) is needed for the kernel operator K to satisfy Hörmander condition, and this completes our proof. $\square$

Remark 3.22. It is known by an interpolation theorem that if an operator T maps $H^1(\mathbb{R})$ to $L_1(\mathbb{R})$ and $L_\infty(\mathbb{R})$ to BMO, then T maps $L_p(\mathbb{R})$ to $L_p(\mathbb{R})$, $1 < p < \infty$. This fact also shows the importance of Hörmander condition, and thus the importance of lacunarity in our discussion.

Note also that in the proof of Theorem 3.21 the operator $S_{\mathbb{R}}f$ can be replaced with the variation operator $V_{\mathbb{R}}f$ for $s \geq 2$ with the suitable modification on the construction of Tf.

Let (n_k) be an increasing sequence of positive integers, and consider the operator Tf in Theorem 3.21, then it might be interesting to know if there exists a constant C such that

$$\|Tf\|_{\mathrm{BMO}(l_2(\mathbb{Z}^+))} \leq C\|f\|_\infty, \quad f \in L_\infty(\mathbb{R}).$$

Remark 3.23. Let us now consider the square function

$$Sf(x) = \left(\sum_{k=1}^{\infty} |A_{k+1}f(x) - A_kf(x)|^2\right)^{1/2}.$$

Then it is easy to see that Sf does not map $H^1(X)$ to $L_1(X)$. See for example the estimates in R. L. Jones [28] about the relation between the square function Sf and the ergodic maximal function f^*.

Clearly, the proof of Theorem 3.21 shows that the square function Sf maps L_∞ to BMO, and L_p to L_p, $1 < p < \infty$. Thus, we see that Hörmander condition does not hold for the sequence $(n_k) = (k)$ since Sf does not map H^1 to L_1 in the sense of Theorem 3.21.

Let $(c_k) \in l_\infty(\mathbb{Z})$ with $\|c_k\|_{l_\infty} \leq 1$ and consider the kernel operator $K : \mathbb{R} \to \mathbb{R}$ defined as

$$K(x) = \sum_{k=1}^{\infty} c_k(\phi_{n_{k+1}}(x) - \phi_{n_k}(x)).$$

Then we have the following lemma:

Lemma 3.24. *Let (n_k) be a lacunary sequence, then the kernel operator K satisfies the Hörmander condition.*

Proof. It suffices to show that for some constant C

$$\int_{|x|>2|y|} |K(x-y) - K(x)|\, dx \leq C.$$

We need to evaluate $K(x-y) - K(x)$ for $|x| > 2|y|$. Let us first consider the case $x > 0$, $y > 0$. We have

$$|K(x-y) - K(x)| \leq \sum_{k=1}^{\infty} |\Phi_{k+1}(x,y) - \Phi_k(x,y)|,$$

where

$$\Phi_k(x,y) = \frac{1}{n_k}\left[\chi_{[0,n_k]}(x-y) - \chi_{[0,n_k]}(x)\right].$$

So we only need to evaluate $\Phi_k(x,y)$ in the estimate to get the Hörmander condition. Since $x > 2y$, we have

$$\Phi_k(x,y) = \frac{1}{n_k}\left[\chi_{[y,y+n_k]}(x) - \chi_{[y,n_k]}(x)\right].$$

Now, if $x \in [y, y+n_k] \cap [y, n_k]$, then we have $\Phi_k(x,y) = 0$. But we have $[y, n_k] \subset [y, y+n_k]$, so we can assume that $x \in [y, y+n_k] - [y, n_k]$. Hence, it suffices to consider the interval $[n_k, y+n_k]$. Thus, we get

$$\Phi_k(x,y) = \frac{1}{n_k}\chi_{[n_k,y+n_k]}(x).$$

Now, suppose that $x \leq 0$, $y \leq 0$, then we have $y > x$ since $|x| > 2|y|$. In this case we have $\Phi_k(x, y) = 0$, since $x \notin [y, y + n_k] \cup [y, n_k]$. Also, the same is true for $x \leq 0$, $y \geq 0$ since this implies that $x < y$. If $x \geq 0$, $y \leq 0$, then we have the same situation as in the first case. Thus, it suffices to evaluate

$$\sum_{k=1}^{\infty} \Phi_k(x, y) = \sum_{k=1}^{\infty} \frac{1}{n_k} \chi_{[n_k, y+n_k]}(x).$$

On the other hand, since $\frac{n_{k+1}}{n_k} \geq \beta > 1$, there exists a constant $C(\beta)$ such that

$$\sum_{k=1}^{\infty}{}' \frac{1}{n_k} \leq \frac{C(\beta)}{y},$$

where prime denotes that the term in the summation is omitted when $n_k < y$. Thus we have

$$\begin{aligned}
\int_{x>2y} |K(x-y) - K(x)|\, dx &\leq 2 \int_{x>2y} \sum_{k=1}^{\infty} \frac{1}{n_k} \chi_{[n_k, y+n_k]}(x)\, dx \\
&\leq 2 \sum_{k=1}^{\infty}{}' \frac{1}{n_k} \int_{\mathbb{R}} \chi_{[n_k, y+n_k]}(x)\, dx \\
&= 2 \sum_{k=1}^{\infty}{}' \frac{1}{n_k} y \\
&\leq 2C(\beta),
\end{aligned}$$

and our proof is complete. □

Let us now define the operator

$$Tf(x) = \int K(x-y) \cdot f(y)\, dy,$$

where K is the kernel operator given in Lemma 3.24. Then it is clear that Tf is an $\mathbb{R}$-valued convolution operator, and that

$$\|Tf(x)\|_{\mathbb{R}} = |Tf(x)| = G_{\mathbb{R}} f(x),$$

where

$$G_{\mathbb{R}}f(x) = \left| \sum_{k=1}^{\infty} c_k(\phi_{n_{k+1}} - \phi_{n_k}) * f(x) \right|.$$

We also know that the operator $G_{\mathbb{R}}f$ satisfies a strong type $(2,2)$ inequality which can be seen either by Cotlar's lemma or by a Fourier transform argument. Therefore we see that T^*f is a singular integral operator of convolution type.

Thus we find the following result as an application of Lemma 3.5.

Theorem 3.25. *Let (n_k) be a lacunary sequence. Then we have*

(a) $\|G_{\mathbb{R}}f\|_p \leq C\|f\|_p, \quad 1 < p < \infty,$

(b) $m\{x : |G_{\mathbb{R}}f(x)| > \lambda\} \leq \frac{C}{\lambda}\|f\|_1,$

(c) $\|G_{\mathbb{R}}f\|_1 \leq C\|f\|_{H^1}, \quad f \in L_1(\mathbb{R}),$

(d) $\|G_{\mathbb{R}}f\|_{\mathrm{BMO}} \leq C\|f\|_\infty, \quad f \in L_c^\infty(\mathbb{R}).$

Let us define

$$Gf(x) = \left| \sum_{k=1}^{\infty} c_k(A_{n_{k+1}}f(x) - A_{n_k}f(x)) \right|.$$

Then we apply Calderón transfer principle to Theorem 3.25 to find the following:

Corollary 3.26. *Let (n_k) be a lacunary sequence. Then we have*

(a) $\|Gf\|_p \leq C\|f\|_p, \quad 1 < p < \infty,$

(b) $m\{x : |Gf(x)| > \lambda\} \leq \frac{C}{\lambda}\|f\|_1,$

(c) $\|Gf\|_1 \leq C\|f\|_{H^1}, \quad f \in L_1(X),$

(d) $\|Gf\|_{\mathrm{EBMO}} \leq C\|f\|_\infty, \quad f \in L_c^\infty(X),$

(e) $\|Gf\|_1 \leq C\|Mf\|_1, \quad f \in L_1(X),$

(f) $\|Gf\|_1 \leq C\|f^*\|_1, \quad f \in L_1(X).$

It might also be interesting to investigate the behavior of the maximal operator

$$G^*f(x) = \sup_\alpha \left| \sum_{k=1}^{\alpha} c_k(A_{n_{k+1}}f(x) - A_{n_k}f(x)) \right|.$$

In the following we will show that our method can also be used to study this type of maximal operator.

Let us consider the kernel operator $K : \mathbb{R} \to l_\infty(\mathbb{Z}^+)$ defined as

$$K(x) = \left(\sum_{k=1}^{\alpha} c_k(\phi_{n_{k+1}}(x) - \phi_{n_k}(x)) \right)_{\alpha=1}^{\infty}.$$

Then we have the following:

Lemma 3.27. *Let (n_k) be a lacunary sequence, then the kernel operator K satisfies the Hörmander condition.*

Proof. Since K is an $l_\infty(\mathbb{Z}^+)$-valued kernel, we need to show that the condition (Z) holds for K. To do this it suffices to show that for some constant C

$$\int_{|x|>2|y|} \sup_\alpha |k_\alpha(x-y) - k_\alpha(x)|\, dx \leq C,$$

where

$$k_\alpha(x) = \sum_{k=1}^{\alpha} c_k(\phi_{n_{k+1}}(x) - \phi_{n_k}(x)).$$

We also know as before that we only need to consider the function

$$\Phi_k(x,y) = \frac{1}{n_k}\chi_{[n_k, y+n_k]}(x)$$

in the summation for $x > 0$, $y > 0$.

Also, since $\frac{n_{k+1}}{n_k} \geq \beta > 1$, there exists a constant $C(\beta)$ such that

$$\sup_{\alpha} \sum_{k=1}^{\alpha}{}' \frac{1}{n_k} \leq \frac{C(\beta)}{y}.$$

Then we have

$$\begin{aligned}
\int_{x>2y} \sup_{\alpha} |k_\alpha(x-y) - k_\alpha(x)|\, dx &\leq 2 \sup_{\alpha} \int_{x>2y} \sum_{k=1}^{\alpha} \frac{1}{n_k} \chi_{[n_k, y+n_k]}(x)\, dx \\
&\leq 2 \sup_{\alpha} \sum_{k=1}^{\alpha}{}' \frac{1}{n_k} \int_{\mathbb{R}} \chi_{[n_k, y+n_k]}(x)\, dx \\
&= 2 \sup_{\alpha} \sum_{k=1}^{\alpha}{}' \frac{1}{n_k} y \\
&\leq 2C(\beta),
\end{aligned}$$

and this completes our proof. □

We now define the operator

$$T^* f(x) = \int K(x-y) \cdot f(y)\, dy,$$

where K is the kernel operator given in Lemma 3.27. Then it is clear that $T^* f$ is an $l_\infty(\mathbb{Z}^+)$-valued convolution operator, and that

$$\|T^* f(x)\|_{l_\infty(\mathbb{Z}^+)} = G^*_{\mathbb{R}} f(x),$$

where

$$G^*_{\mathbb{R}} f(x) = \sup_{\alpha} \left| \sum_{k=1}^{\alpha} c_k (\phi_{n_{k+1}} - \phi_{n_k}) * f(x) \right|.$$

We also know that the operator $G^*_{\mathbb{R}} f$ satisfies a strong type $(2,2)$ inequality. Therefore we see that $T^* f$ is a singular integral operator of convolution type.

Now by Lemma 3.5 we find the following:

Theorem 3.28. *Let (n_k) be a lacunary sequence. Then we have*

(a) $\|G^*_{\mathbb{R}}f\|_p \le C\|f\|_p, \quad 1 < p < \infty,$

(b) $m\{x : |G^*_{\mathbb{R}}f(x)| > \lambda\} \le \frac{C}{\lambda}\|f\|_1,$

(c) $\|G^*_{\mathbb{R}}f\|_1 \le C\|f\|_{H^1}, \quad f \in L_1(\mathbb{R}),$

(d) $\|G^*_{\mathbb{R}}f\|_{\mathrm{BMO}} \le C\|f\|_\infty, \quad f \in L^\infty_c(\mathbb{R}).$

Then we apply Calderón transfer principle to Theorem 3.28 to find the following:

Corollary 3.29. *Let (n_k) be a lacunary sequence. Then we have*

(a) $\|G^*f\|_p \le C\|f\|_p, \quad 1 < p < \infty,$

(b) $m\{x : |G^*f(x)| > \lambda\} \le \frac{C}{\lambda}\|f\|_1,$

(c) $\|G^*f\|_1 \le C\|f\|_{H^1}, \quad f \in L_1(X),$

(d) $\|G^*f\|_{\mathrm{EBMO}} \le C\|f\|_\infty, \quad f \in L^\infty_c(X),$

(e) $\|G^*f\|_1 \le C\|Mf\|_1, \quad f \in L_1(X),$

(f) $\|G^*f\|_1 \le C\|f^*\|_1, \quad f \in L_1(X).$

Remark 3.30. It is clear that in Corollary 3.13 the square function Sf can be replaced with any of the above operators.

Let us define the maximal operator

$$\Delta^*f(x) = \sup_\alpha |A_{n_{\alpha+1}}f(x) - A_{n_\alpha}f(x)|,$$

it is then clear that we find this operator by applying transfer principle to

$$\Delta^*_{\mathbb{R}}f(x) = \sup_\alpha |(\phi_{n_{\alpha+1}} - \phi_{n_\alpha}) * f(x)|.$$

In this case our kernel operator $K : \mathbb{R} \to l_\infty(\mathbb{Z}^+)$ is given by

$$K(x) = (k_\alpha(x))_{\alpha=1}^\infty$$

where

$$k_\alpha(x) = \phi_{n_{\alpha+1}}(x) - \phi_{n_\alpha}(x).$$

Then, the kernel operator K satisfies the condition (Z) as in our previous discussion. So by this observation almost all the operators that we discussed before can be replaced with $\Delta_{\mathbb{R}}^* f$ and $\Delta^* f$ in real and ergodic cases respectively. Thus, we have the following results:

Theorem 3.31. *Let (n_k) be a lacunary sequence. Then we have*

(a) $\|\Delta_{\mathbb{R}}^* f\|_p \leq C\|f\|_p, \quad 1 < p < \infty,$

(b) $m\{x : |\Delta_{\mathbb{R}}^* f(x)| > \lambda\} \leq \frac{C}{\lambda}\|f\|_1,$

(c) $\|\Delta_{\mathbb{R}}^* f\|_1 \leq C\|f\|_{H^1}, \quad f \in L_1(\mathbb{R}),$

(d) $\|\Delta_{\mathbb{R}}^* f\|_{\mathrm{BMO}} \leq C\|f\|_\infty, \quad f \in L_c^\infty(\mathbb{R}).$

Corollary 3.32. *Let (n_k) be a lacunary sequence. Then we have*

(a) $\|\Delta^* f\|_p \leq C\|f\|_p, \quad 1 < p < \infty,$

(b) $m\{x : |\Delta^* f(x)| > \lambda\} \leq \frac{C}{\lambda}\|f\|_1,$

(c) $\|\Delta^* f\|_1 \leq C\|f\|_{H^1}, \quad f \in L_1(X),$

(d) $\|\Delta^* f\|_{\mathrm{EBMO}} \leq C\|f\|_\infty, \quad f \in L_c^\infty(X).$

3.3 Vector-valued Inequalities

Vector-valued inequalities had a lot of interest in harmonic analysis, and have been studied by many authors. Of course, then it is reasonable to ask if one has these kind of inequalities in ergodic theory. In this section we answer this question affirmatively. We first show that an extension of our previous method is still available to study these kind of inequalities in ergodic theory. Later we prove by working with ergodic square function Sf that one can also use the Calderón-Zygmund decomposition on $\mathbb{Z}$ to investigate these inequalities without lacunarity condition on the sequence (n_k).

Let T be a singular integral operator, then as in J. L. Rubio de Francia *et al* [46] we can define a new operator $\tilde{T}$ mapping $l_r(A)$-valued functions into $l_r(B)$-valued functions, where r is fixed, $1 < r < \infty$ as

$$\tilde{T}(f_1, f_2, \ldots, f_j, \ldots) = (Tf_1, Tf_2, \ldots, Tf_j, \ldots).$$

Then $\tilde{T}$ is bounded from $L^r_{l_r(A)}$ to $L^r_{l_r(B)}$ and the kernel associated to it is

$$\tilde{K}(x) = K(x) \otimes \mathrm{Id},$$

so that

$$\|\tilde{K}\| = \|K\|$$

and

$$D_r(\tilde{K}) = D_r(K).$$

Also, $\tilde{T}$ is a new singular integral operator.

Let H denote the Hilbert transform on $\mathbb{R}$, and let us combine our last observation with Lemma 3.5 to find the following theorem:

Theorem 3.33. *Let (n_k) be a lacunary sequence. Then the following inequalities hold, for all $1 < r < \infty$,*

(a) $\|(\sum_j (S_{\mathbb{R}} f_j)^r)^{1/r}\|_p \leq C_{p,r}\|(\sum_j |f_j|^r)^{1/r}\|_p$, $\quad 1 < p < \infty$,

(b) $m\{x : (\sum_j (S_{\mathbb{R}} f_j(x))^r)^{1/r} > \lambda\} \leq \frac{C_r}{\lambda}\int (\sum_j |f_j(x)|^r)^{1/r}\, dx$,

(c) $\|(\sum_j (S_{\mathbb{R}} f_j)^r)^{1/r}\|_1 \leq C_r\|(\sum_j |f_j|^r)^{1/r}\|_1 + C_r\|(\sum_j (Hf_j)^r)^{1/r}\|_1$,

(d) $\|(\sum_j (S_{\mathbb{R}} f_j)^r)^{1/r}\|_{\mathrm{BMO}} \leq C_r\|(\sum_j |f_j|^r)^{1/r}\|_\infty$.

Similarly we also have the following theorems:

Theorem 3.34. *Let (n_k) be a lacunary sequence, and $2 < s$. Then the following inequalities hold, for all $1 < r < \infty$,*

(a) $\|(\sum_j (V_{\mathbb{R}} f_j)^r)^{1/r}\|_p \leq C_{p,r}\|(\sum_j |f_j|^r)^{1/r}\|_p$, $\quad 1 < p < \infty$,

(b) $m\{x : (\sum_j (V_{\mathbb{R}} f_j(x))^r)^{1/r} > \lambda\} \leq \frac{C_r}{\lambda}\int (\sum_j |f_j(x)|^r)^{1/r}\, dx$,

(c) $\|(\sum_j (V_{\mathbb{R}} f_j)^r)^{1/r}\|_1 \leq C_r\|(\sum_j |f_j|^r)^{1/r}\|_1 + C_r\|(\sum_j (Hf_j)^r)^{1/r}\|_1$,

(d) $\|(\sum_j (V_{\mathbb{R}} f_j)^r)^{1/r}\|_{\mathrm{BMO}} \leq C_r\|(\sum_j |f_j|^r)^{1/r}\|_\infty$.

Theorem 3.35. *Let (n_k) be a lacunary sequence. Then the following inequalities hold, for all $1 < r < \infty$,*

(a) $\|(\sum_j (G_{\mathbb{R}} f_j)^r)^{1/r}\|_p \leq C_{p,r}\|(\sum_j |f_j|^r)^{1/r}\|_p$, $\quad 1 < p < \infty$,

(b) $m\{x : (\sum_j (G_{\mathbb{R}} f_j(x))^r)^{1/r} > \lambda\} \leq \frac{C_r}{\lambda}\int (\sum_j |f_j(x)|^r)^{1/r}\, dx$,

(c) $\|(\sum_j (G_{\mathbb{R}} f_j)^r)^{1/r}\|_1 \leq C_r\|(\sum_j |f_j|^r)^{1/r}\|_1 + C_r\|(\sum_j (Hf_j)^r)^{1/r}\|_1$,

(d) $\|(\sum_j (G_{\mathbb{R}} f_j)^r)^{1/r}\|_{\mathrm{BMO}} \leq C_r\|(\sum_j |f_j|^r)^{1/r}\|_\infty$.

Theorem 3.36. *Let (n_k) be a lacunary sequence. Then the following inequalities hold, for all $1 < r < \infty$,*

(a) $\big\|\big(\sum_j (G^*_{\mathbb{R}} f_j)^r\big)^{1/r}\big\|_p \leq C_{p,r}\big\|\big(\sum_j |f_j|^r\big)^{1/r}\big\|_p, \quad 1 < p < \infty,$

(b) $m\{x : \big(\sum_j (G^*_{\mathbb{R}} f_j(x))^r\big)^{1/r} > \lambda\} \leq \frac{C_r}{\lambda} \int \big(\sum_j |f_j(x)|^r\big)^{1/r}\, dx,$

(c) $\big\|\big(\sum_j (G^*_{\mathbb{R}} f_j)^r\big)^{1/r}\big\|_1 \leq C_r\big\|\big(\sum_j |f_j|^r\big)^{1/r}\big\|_1 + C_r\big\|\big(\sum_j (Hf_j)^r\big)^{1/r}\big\|_1,$

(d) $\big\|\big(\sum_j (G^*_{\mathbb{R}} f_j)^r\big)^{1/r}\big\|_{\mathrm{BMO}} \leq C_r\big\|\big(\sum_j |f_j|^r\big)^{1/r}\big\|_\infty.$

Theorem 3.37. *Let (n_k) be a lacunary sequence. Then the following inequalities hold, for all $1 < r < \infty$,*

(a) $\big\|\big(\sum_j (Nf_j)^r\big)^{1/r}\big\|_p \leq C_{p,r}\big\|\big(\sum_j |f_j|^r\big)^{1/r}\big\|_p, \quad 1 < p < \infty,$

(b) $m\{x : \big(\sum_j (Nf_j(x))^r\big)^{1/r} > \lambda\} \leq \frac{C_r}{\lambda} \int \big(\sum_j |f_j(x)|^r\big)^{1/r}\, dx,$

(c) $\big\|\big(\sum_j (Nf_j)^r\big)^{1/r}\big\|_1 \leq C_r\big\|\big(\sum_j |f_j|^r\big)^{1/r}\big\|_1 + C_r\big\|\big(\sum_j (Hf_j)^r\big)^{1/r}\big\|_1,$

(d) $\big\|\big(\sum_j (Nf_j)^r\big)^{1/r}\big\|_{\mathrm{BMO}} \leq C_r\big\|\big(\sum_j |f_j|^r\big)^{1/r}\big\|_\infty.$

Let now H denote the ergodic Hilbert transform, then we apply transfer principle to the above results to find the following corollaries:

Corollary 3.38. *Let (n_k) be a lacunary sequence. Then the following inequalities hold, for all $1 < r < \infty$,*

(a) $\big\|\big(\sum_j (Sf_j)^r\big)^{1/r}\big\|_p \leq C_{p,r}\big\|\big(\sum_j |f_j|^r\big)^{1/r}\big\|_p, \quad 1 < p < \infty,$

(b) $m\{x : \big(\sum_j (Sf_j(x))^r\big)^{1/r} > \lambda\} \leq \frac{C_r}{\lambda} \int \big(\sum_j |f_j(x)|^r\big)^{1/r}\, dx,$

(c) $\big\|\big(\sum_j (Sf_j)^r\big)^{1/r}\big\|_1 \leq C_r\big\|\big(\sum_j |f_j|^r\big)^{1/r}\big\|_1 + C_r\big\|\big(\sum_j (Hf_j)^r\big)^{1/r}\big\|_1,$

(d) $\left\|\left(\sum_j (Sf_j)^r\right)^{1/r}\right\|_{\mathrm{EBMO}} \leq C_r \left\|\left(\sum_j |f_j|^r\right)^{1/r}\right\|_\infty$.

Corollary 3.39. *Let (n_k) be a lacunary sequence, and $2 < s$. Then the following inequalities hold, for all $1 < r < \infty$,*

(a) $\left\|\left(\sum_j (Vf_j)^r\right)^{1/r}\right\|_p \leq C_{p,r} \left\|\left(\sum_j |f_j|^r\right)^{1/r}\right\|_p, \quad 1 < p < \infty,$

(b) $m\{x : \left(\sum_j (Vf_j(x))^r\right)^{1/r} > \lambda\} \leq \frac{C_r}{\lambda} \int \left(\sum_j |f_j(x)|^r\right)^{1/r} dx,$

(c) $\left\|\left(\sum_j (Vf_j)^r\right)^{1/r}\right\|_1 \leq C_r \left\|\left(\sum_j |f_j|^r\right)^{1/r}\right\|_1 + C_r \left\|\left(\sum_j (Hf_j)^r\right)^{1/r}\right\|_1,$

(d) $\left\|\left(\sum_j (Vf_j)^r\right)^{1/r}\right\|_{\mathrm{EBMO}} \leq C_r \left\|\left(\sum_j |f_j|^r\right)^{1/r}\right\|_\infty$.

Corollary 3.40. *Let (n_k) be a lacunary sequence. Then the following inequalities hold, for all $1 < r < \infty$,*

(a) $\left\|\left(\sum_j (Gf_j)^r\right)^{1/r}\right\|_p \leq C_{p,r} \left\|\left(\sum_j |f_j|^r\right)^{1/r}\right\|_p, \quad 1 < p < \infty,$

(b) $m\{x : \left(\sum_j (Gf_j(x))^r\right)^{1/r} > \lambda\} \leq \frac{C_r}{\lambda} \int \left(\sum_j |f_j(x)|^r\right)^{1/r} dx,$

(c) $\left\|\left(\sum_j (Gf_j)^r\right)^{1/r}\right\|_1 \leq C_r \left\|\left(\sum_j |f_j|^r\right)^{1/r}\right\|_1 + C_r \left\|\left(\sum_j (Hf_j)^r\right)^{1/r}\right\|_1,$

(d) $\left\|\left(\sum_j (Gf_j)^r\right)^{1/r}\right\|_{\mathrm{EBMO}} \leq C_r \left\|\left(\sum_j |f_j|^r\right)^{1/r}\right\|_\infty$.

Corollary 3.41. *Let (n_k) be a lacunary sequence. Then the following inequalities hold, for all $1 < r < \infty$,*

(a) $\left\|\left(\sum_j (G^*f_j)^r\right)^{1/r}\right\|_p \leq C_{p,r} \left\|\left(\sum_j |f_j|^r\right)^{1/r}\right\|_p, \quad 1 < p < \infty,$

(b) $m\{x : \left(\sum_j (G^*f_j(x))^r\right)^{1/r} > \lambda\} \leq \frac{C_r}{\lambda} \int \left(\sum_j |f_j(x)|^r\right)^{1/r} dx,$

(c) $\left\|\left(\sum_j (G^*f_j)^r\right)^{1/r}\right\|_1 \leq C_r \left\|\left(\sum_j |f_j|^r\right)^{1/r}\right\|_1 + C_r \left\|\left(\sum_j (Hf_j)^r\right)^{1/r}\right\|_1,$

(d) $\left\|\left(\sum_j (G^*f_j)^r\right)^{1/r}\right\|_{\mathrm{EBMO}} \leq C_r \left\|\left(\sum_j |f_j|^r\right)^{1/r}\right\|_\infty$.

Corollary 3.42. *Let (n_k) be a lacunary sequence. Then the following inequalities hold, for all $1 < r < \infty$,*

(a) $\left\|\left(\sum_j (f_{j*})^r\right)^{1/r}\right\|_p \leq C_{p,r}\left\|\left(\sum_j |f_j|^r\right)^{1/r}\right\|_p, \quad 1 < p < \infty,$

(b) $m\{x : \left(\sum_j (f_{j*}(x))^r\right)^{1/r} > \lambda\} \leq \frac{C_r}{\lambda}\int \left(\sum_j |f_j(x)|^r\right)^{1/r} dx,$

(c) $\left\|\left(\sum_j (f_{j*})^r\right)^{1/r}\right\|_1 \leq C_r\left\|\left(\sum_j |f_j|^r\right)^{1/r}\right\|_1 + C_r\left\|\left(\sum_j (Hf_j)^r\right)^{1/r}\right\|_1,$

(d) $\left\|\left(\sum_j (f_{j*})^r\right)^{1/r}\right\|_{\mathrm{EBMO}} \leq C_r\left\|\left(\sum_j |f_j|^r\right)^{1/r}\right\|_\infty.$

Theorem 3.43. *Let (n_k) be a lacunary sequence. Then the following inequalities hold, for all $1 < r < \infty$,*

(a) $\left\|\left(\sum_j (\Delta^*_{\mathbb{R}} f_j)^r\right)^{1/r}\right\|_p \leq C_{p,r}\left\|\left(\sum_j |f_j|^r\right)^{1/r}\right\|_p, \quad 1 < p < \infty,$

(b) $m\{x : \left(\sum_j (\Delta^*_{\mathbb{R}} f_j(x))^r\right)^{1/r} > \lambda\} \leq \frac{C_r}{\lambda}\int \left(\sum_j |f_j(x)|^r\right)^{1/r} dx,$

(c) $\left\|\left(\sum_j (\Delta^*_{\mathbb{R}} f_j)^r\right)^{1/r}\right\|_1 \leq C_r\left\|\left(\sum_j |f_j|^r\right)^{1/r}\right\|_1 + C_r\left\|\left(\sum_j (Hf_j)^r\right)^{1/r}\right\|_1,$

(d) $\left\|\left(\sum_j (\Delta^*_{\mathbb{R}} f_j)^r\right)^{1/r}\right\|_{\mathrm{BMO}} \leq C_r\left\|\left(\sum_j |f_j|^r\right)^{1/r}\right\|_\infty.$

Corollary 3.44. *Let (n_k) be a lacunary sequence. Then the following inequalities hold, for all $1 < r < \infty$,*

(a) $\left\|\left(\sum_j (\Delta^* f_j)^r\right)^{1/r}\right\|_p \leq C_{p,r}\left\|\left(\sum_j |f_j|^r\right)^{1/r}\right\|_p, \quad 1 < p < \infty,$

(b) $m\{x : \left(\sum_j (\Delta^* f_j(x))^r\right)^{1/r} > \lambda\} \leq \frac{C_r}{\lambda}\int \left(\sum_j |f_j(x)|^r\right)^{1/r} dx,$

(c) $\left\|\left(\sum_j (\Delta^* f_j)^r\right)^{1/r}\right\|_1 \leq C_r\left\|\left(\sum_j |f_j|^r\right)^{1/r}\right\|_1 + C_r\left\|\left(\sum_j (Hf_j)^r\right)^{1/r}\right\|_1,$

(d) $\left\|\left(\sum_j (\Delta^* f_j)^r\right)^{1/r}\right\|_{\mathrm{EBMO}} \leq C_r\left\|\left(\sum_j |f_j|^r\right)^{1/r}\right\|_\infty.$

Remark 3.45. It might be interesting if any of the above vector-valued inequalities holds without lacunarity condition. I conjecture that (c) might not hold for all increasing sequences. But one might consider this problem for (a), (b) and (d). It is also an interesting problem if one can reverse the vector-valued inequalities.

In the following we will consider the above problems for ergodic square function Sf with $r = p$.

Lemma 3.46. *Let (n_k) be a lacunary sequence with no non-trivial common divisor. Then there exists a constant C such that*

$$C \leq \sum_{k=1}^{\infty} |\hat{\phi}_{n_{k+1}}(x) - \hat{\phi}_{n_k}(x)|^2.$$

Proof. The argument of the proof of Theorem 2.13 in R. L. Jones and J. Rosenblatt [30] can be used to prove this result since

$$|\hat{\phi}_{n_{k+1}}(x) - \hat{\phi}_{n_k}(x)| = \left| \frac{\sin(n_{k+1}x)}{n_{k+1}x} - \frac{\sin(n_k x)}{n_k x} \right|.$$

□

We now can prove the following result:

Theorem 3.47. *Let (n_k) be a lacunary sequence with no non-trivial common divisor. Then there exists a constant C such that*

$$\|\Big(\sum_j |f_j|^2\Big)^{1/2}\|_2 \leq C \|\Big(\sum_j (S_{\mathbb{R}} f_j)^2\Big)^{1/2}\|_2,$$

for all $f_j \in L_2^0(\mathbb{R})$, $j = 1, 2, \ldots$

Proof. We use Plancherel Theorem and Lemma 3.46 to get

$$
\begin{aligned}
\Big\|\Big(\sum_j |f_j|^2\Big)^{1/2}\Big\|_2^2 &= \int \sum_j |f_j(x)|^2\,dx = \sum_j \int |f_j(x)|^2\,dx = \sum_j \int |\hat{f}_j(x)|^2\,dx \\
&\le C \sum_j \int \sum_{k=1}^{\infty} |\hat{\phi}_{n_{k+1}}(x) - \hat{\phi}_{n_k}(x)|^2 |\hat{f}_j(x)|^2\,dx \\
&= C \sum_j \int \sum_{k=1}^{\infty} |\widehat{\phi_{n_{k+1}} * f_j}(x) - \widehat{\phi_{n_k} * f_j}(x)|^2\,dx \\
&= C \sum_j \int \sum_{k=1}^{\infty} |(\phi_{n_{k+1}} - \phi_{n_k}) * f_j(x)|^2\,dx \\
&= C \int \sum_j \sum_{k=1}^{\infty} |(\phi_{n_{k+1}} - \phi_{n_k}) * f_j(x)|^2\,dx \\
&= C \Big\|\Big(\sum_j (S_{\mathbb{R}} f_j)^2\Big)^{1/2}\Big\|_2^2,
\end{aligned}
$$

and this proves our theorem. □

Then we have the following result as a corollary of Theorem 3.47.

Corollary 3.48. *Let (n_k) be a lacunary sequence with no non-trivial common divisor. Then there exists a constant C such that*

$$
\Big\|\Big(\sum_j |f_j|^2\Big)^{1/2}\Big\|_2 \le C_p \Big\|\Big(\sum_j (Sf_j)^2\Big)^{1/2}\Big\|_2,
$$

for all $f_j \in L_2^0(X)$, $j = 1, 2, \ldots$

The following lemma is Theorem 3.7 in R. L. Jones and J. Rosenblatt [30].

Lemma 3.49. *Let (n_k) be a lacunary sequence with no non-trivial common divisor. Then, there exists a constant C_p such that*

$$
\|f\|_p \le C_p \|Sf\|_p,
$$

for all $f \in L_p^0(X)$, $1 < p < \infty$.

We can then prove the following more general form of Corollary 3.48.

Theorem 3.50. *Let (n_k) be a lacunary sequence with no non-trivial common divisor, and $1 < p < \infty$. Then, there exists a constant C_p such that*

$$\Big\|\Big(\sum_j |f_j|^p\Big)^{1/p}\Big\|_p \leq C_p \Big\|\Big(\sum_j (Sf_j)^p\Big)^{1/p}\Big\|_p,$$

for all $f_j \in L_p^0(X)$, $j = 1, 2, \ldots$

Proof. By Lemma 3.49 we have

$$\begin{aligned}
\Big\|\Big(\sum_j |f_j|^p\Big)^{1/p}\Big\|_p^p &= \int \sum_j |f_j(x)|^p \, dx \\
&= \sum_j \int |f_j(x)|^p \, dx \\
&\leq C_p \sum_j \int |Sf_j(x)|^p \, dx \\
&= C_p \int \sum_j |Sf_j(x)|^p \, dx \\
&= C_p \Big\|\Big(\sum_j |Sf_j|^p\Big)^{1/p}\Big\|_p^p,
\end{aligned}$$

as desired. □

We will next show that vector-valued strong type (p, p) inequalities might hold for our variation operator Vf without lacunarity condition on the sequence (n_k). We will first need the following remark:

Remark 3.51. Let $2 \leq s$ and consider the operator

$$Vf = \left(\sum_{k=1}^{\infty} |A_{n_{k+1}} f - A_{n_k} f|^s\right)^{1/s}.$$

Then by the results of R. L. Jones *et al* [29] we know that for any increasing sequence (n_k) of positive integers there exists a constant C_p such that

$$\|Vf\|_p \le C_p\|f\|_p,$$

for all $f \in L_p(X)$, $1 < p < \infty$.

We now have the following theorem:

Theorem 3.52. *Let (n_k) be an increasing sequence of positive integers. Then there exists a constant C_p such that*

$$\Big\|\Big(\sum_j (Vf_j)^p\Big)^{1/p}\Big\|_p \le C_p\Big\|\Big(\sum_j |f_j|^p\Big)^{1/p}\Big\|_p, \quad 1 < p < \infty,$$

for all $f_j \in L_p(X)$, $j = 1, 2, \ldots$

Proof. By using Remark 3.51 we find

$$\begin{aligned}\Big\|\Big(\sum_j (Vf_j)^p\Big)^{1/p}\Big\|_p^p &= \int \sum_j (Vf_j(x))^p\,dx\\ &= \sum_j \int (Vf_j(x))^p\,dx\\ &\le C_p \sum_j \int |f_j(x)|^p\,dx\\ &= C_p \int \sum_j |f_j(x)|^p\,dx\\ &= C_p\Big\|\Big(\sum_j |f_j|^p\Big)^{1/p}\Big\|_p^p,\end{aligned}$$

and this proves the result as desired. □

In particular, we have the following corollary:

Corollary 3.53. *Let (n_k) be an increasing sequence of positive integers. Then there exists a constant C_p such that*

$$\Big\|\big(\sum_j (Sf_j)^p\big)^{1/p}\Big\|_p \leq C_p \Big\|\big(\sum_j |f_j|^p\big)^{1/p}\Big\|_p, \quad 1 < p < \infty,$$

for all $f_j \in L_p(X)$, $j = 1, 2, \ldots$

Remark 3.54. It might be interesting to investigate the vector-valued inequalities for all $1 < r < \infty$.

The argument of our next result gives some insight to study the vector-valued weak type inequalities without lacunarity condition.

Theorem 3.55. *Let (n_k) be an increasing sequence of positive integers, and $2 \leq r < \infty$. Then there exists a constant C such that*

$$m\left\{x : \left(\sum_j \big(Sf_j(x)\big)^r\right)^{1/r} > \lambda\right\} \leq \frac{C}{\lambda}\sum_j \|f_j\|_1$$

for all $f_j \in L_1(X)$, $j = 1, 2, \ldots$

Proof. To prove this result we will use the Calderón-Zygmund decomposition on $\mathbb{Z}$, and transfer by Calderón transfer principle to the dynamical system. Let f_j be a function in $l_1(\mathbb{Z})$, and $\lambda > 0$. Then we can write $f_j = g_j + b_j$ where $g_j \in l_2(\mathbb{Z})$,

(1) $\|g_j\|_{l_1} \leq \|f_j\|_{l_1}$,

(2) $\|g_j\|_\infty \leq 2\lambda$,

(3) $b_j = \sum_i b_j^i(x)$ where each b_j^i satisfies:

  (a) b_j^i is supported on an interval B_j^i,

  (b) $\sum_i b_j^i(l) = 0$ for each i,

(c) $\frac{1}{|B_j^i|}\sum_{l\in B_j^i}|b_j^i(l)| \leq 4\lambda$ and $\lambda \leq \frac{1}{|B_j^i|}\sum_{l\in B_j^i}|f_j(l)|$,

(d) $B_j^i \cap B_j^k = \phi$ for $i \neq k$.

Note also that the above imply

$$\sum_i |B_j^i| \leq \frac{1}{\lambda}\sum_i \|b_j^i\|_{l_1} \leq \frac{1}{\lambda}\|f_j\|_{l_1}.$$

Now let $\tilde{B}_j^i$ denote an interval of length $5|B_j^i|$ and with the same center as B_j^i. Let $\tilde{B}_j = \cup_i \tilde{B}_j^i$, and $\tilde{B} = \cup_j \tilde{B}_j$. Let $l \notin \tilde{B}$. We have

$$\begin{aligned} Sb_j(l)^2 &= \sum_{k=1}^{\infty} |A_{n_{k+1}} b_j(l) - A_{n_k} b_j(l)|^2 \\ &= \sum_{k=1}^{\infty} \left| A_{n_{k+1}}\left(\sum_i b_j^i(l)\right) - A_{n_k}\left(\sum_i b_j^i(l)\right)\right|^2 \\ &= \sum_{k=1}^{\infty} \left| \sum_i (A_{n_{k+1}} b_j^i(l) - A_{n_k} b_j^i(l))\right|^2. \end{aligned}$$

Note that for any i for which the average includes all the points in B_j^i, the average is 0 by (b). Thus for each fixed k, $A_{n_{k+1}} b_j^i(l) - A_{n_k} b_j^i(l)$ is non-zero only if $l+1 \in B_j^i$, i.e. at least the average starts in B_j^i. The first possibility, starting in B_j^i, is excluded since $l \notin \tilde{B}_j$. Hence for each fixed k and l, $A_{n_{k+1}} b_j^i(l) - A_{n_k} b_j^i(l) \neq 0$ for at most 2 values of i, an ending value for $A_{n_{k+1}} b_j^i(l)$ and an ending value for $A_{n_k} b_j^i(l)$. Thus we know

$$\begin{aligned} Sb_j(l)^2 &\leq 2\sum_{k=1}^{\infty}\sum_i |A_{n_{k+1}} b_j^i(l) - A_{n_k} b_j^i(l)|^2 \\ &= 2\sum_i Sb_j^i(l)^2. \end{aligned}$$

Then we have

$$\begin{aligned} \#\left\{l : \left(\sum_j (Sb_j(l))^r\right)^{1/r} > \lambda\right\} &\leq \#\left\{l : \sum_j (Sb_j(l))^2 > \lambda^2\right\} \\ &= \#\left\{l : l \notin \tilde{B}, \sum_j (Sb_j(l))^2 > \lambda^2\right\} + \end{aligned}$$

$$+\#\left\{l : l \in \tilde{B}, \sum_j \left(Sb_j(l)\right)^2 > \lambda^2\right\}$$

$$\le \frac{1}{\lambda^2}\sum_{l\notin\tilde{B}}\sum_j \left(Sb_j(l)\right)^2 + |\tilde{B}|.$$

We also have

$$\frac{1}{\lambda^2}\sum_{l\notin\tilde{B}} Sb_j(l)^2 \le \frac{1}{\lambda^2}\sum_{l\notin\tilde{B}} 2\sum_i Sb_j^i(l)^2$$

$$\le 2\sum_i \frac{1}{\lambda^2}\sum_{l\notin\tilde{B}_j^i} Sb_j^i(l)^2.$$

Now let fix j, then we claim that for each i we have

$$\frac{1}{\lambda^2}\sum_{l\notin\tilde{B}_j^i} Sb_j^i(l)^2 \le C|B_j^i|.$$

We know that translation by an integer is measure preserving, thus we can assume, without loss of generality, that $B_j^i = [0, N-1]$ where $|B_j^i| = N$. Note that since we only need to consider $l \in \tilde{B_j^i}^c$, we do not need to consider $l \in (-2N, 3N)$, and since we are only looking at forward averages, we only need to consider $l \in (-\infty, -2N]$. To have a non-zero value of $A_{n_{k+1}}b_j^i(l)$ we must reach the support of b_j^i. Hence, we must have $n_{k+1} + l \ge 0$. Thus, $n_{k+1} \ge |l|$. But we might have $n_k + l \ge 0$ or $n_k + l < 0$ for that particular value of k. Let $n(l)$ be the smallest integer such that $n_{n(l)+1} \ge |l|$. Then $n_{n(l)} + l < 0$ and so we have arranged $b_j^i(l+r) = 0$ for all $r = 1, \ldots, n_{n(l)}$.

$$Sb_j^i(l)^2 = \sum_{n_{k+1}\ge|l|} |A_{n_{k+1}}b_j^i(l) - A_{n_k}b_j^i(l)|^2 \le \left(\sum_{n_{k+1}\ge|l|} |A_{n_{k+1}}b_j^i(l) - A_{n_k}b_j^i(l)|\right)^2$$

$$\le \left(\sum_{n_{k+1}\ge|l|}\left\{\left|\left(\frac{1}{n_{k+1}} - \frac{1}{n_k}\right)\sum_{r=1}^{n_k} b_j^i(l+r)\right| + \frac{1}{n_{k+1}}\sum_{r=n_k+1}^{n_{k+1}} |b_j^i(l+r)|\right\}\right)^2$$

$$\le \left(\sum_{n_{k+1}\ge|l|}\left(\frac{1}{n_k} - \frac{1}{n_{k+1}}\right)\sum_{r=1}^{n_k}|b_j^i(l+r)| + \sum_{n_{k+1}\ge|l|}\frac{1}{n_{k+1}}\sum_{r=n_k+1}^{n_{k+1}}|b_j^i(l+r)|\right)^2$$

$$
\begin{aligned}
&\leq 2\Bigg(\sum_{n_{k+1}\geq|l|}\left(\frac{1}{n_k}-\frac{1}{n_{k+}}\right)\sum_{r=1}^{n_k}|b_j^i(l+r)|\Bigg)^2+2\Bigg(\sum_{n_{k+1}\geq|l|}\frac{1}{n_{k+1}}\sum_{r=n_k+1}^{n_{k+1}}|b_j^i(l+r)|\Bigg)^2\\
&\leq 2\Bigg(\sum_{k=n(l)+1}^{\infty}\left(\frac{1}{n_k}-\frac{1}{n_{k+1}}\right)\sum_{r=0}^{N-1}|b_j^i(r)|\Bigg)^2+2\Bigg(\sum_{k=n(l)}^{\infty}\frac{1}{n_{k+1}}\sum_{r=n_k+1}^{n_{k+1}}|b_j^i(l+r)|\Bigg)^2\\
&\leq 2\Bigg(\frac{1}{n_{n(l)+1}}N\frac{1}{N}\sum_{r=0}^{N-1}|b_j^i(r)|\Bigg)^2+2\Bigg(\sum_{k=n(l)}^{\infty}\frac{1}{n_{n(l)+1}}\sum_{r=n_k+1}^{n_{k+1}}|b_j^i(l+r)|\Bigg)^2\\
&\leq 32\left(\frac{1}{n_{n(l)+1}}N\lambda\right)^2+32\Bigg(\frac{1}{n_{n(l)+1}}N\frac{1}{N}\sum_{r=0}^{N-1}|b_j^i(r)|\Bigg)^2\\
&\leq 32\left(\frac{1}{n_{n(l)+1}}N\lambda\right)^2+32\left(\frac{1}{n_{n(l)+1}}N\lambda\right)^2\\
&\leq 64\left(\frac{1}{n_{n(l)+1}}N\lambda\right)^2.
\end{aligned}
$$

So we have

$$
\begin{aligned}
\sum_{l\notin\tilde{B}_j^i} Sb_j^i(l)^2 &\leq \sum_{l\leq -2N} 64\left(\frac{1}{n_{n(l)+1}}N\lambda\right)^2\\
&\leq \sum_{l\leq -2N} 64\left(\frac{1}{l}N\lambda\right)^2\\
&\leq 64N^2\lambda^2\sum_{l\leq -2N}\frac{1}{l^2}\\
&\leq 64\lambda^2 N\\
&= 64\lambda^2|B_j^i|
\end{aligned}
$$

and this proves our claim.

Now we use our claim to get

$$
\begin{aligned}
\#\left\{l:\Big(\sum_j (Sb_j(l))^r\Big)^{1/r}>\frac{\lambda}{2}\right\} &\leq \#\left\{l:\sum_j (Sb_j(l))^2>\left(\frac{\lambda}{2}\right)^2\right\}\\
&\leq \frac{4}{\lambda^2}\sum_{l\notin\tilde{B}_j}\sum_j Sb_j((l))^2+|\tilde{B}|
\end{aligned}
$$

$$\leq \frac{4}{\lambda^2} \sum_i \sum_{l \notin \tilde{B}_j^i} \sum_j Sb_j^i((l))^2 + |\tilde{B}|$$

$$\leq C \sum_j \sum_i |B_j^i| + |\tilde{B}|$$

$$\leq \frac{C+1}{\lambda} \sum_j \|f_j\|_{l_1},$$

since we also have

$$|\tilde{B}| \leq \sum_j \sum_i |B_j^i| \leq \frac{1}{\lambda} \sum_j \|f_j\|_{l_1}.$$

On the other hand we have

$$\#\left\{l : \left(\sum_j (Sg_j(l))^r\right)^{1/r} > \frac{\lambda}{2}\right\} \leq \frac{C}{\lambda^2} \sum_l \left(\sum_j (Sg_j(l))^r\right)^{2/r}$$

$$\leq \frac{C}{\lambda^2} \sum_l \sum_j (Sg_j(l))^2$$

$$\leq \frac{C}{\lambda^2} \sum_j \sum_l |g_j(l)|^2$$

$$\leq \frac{C}{\lambda} \sum_j \|g_j\|_{l_1}$$

$$\leq \frac{C}{\lambda} \sum_j \|f_j\|_{l_1}.$$

Thus we have

$$\#\left\{l : \left(\sum_j (Sf_j(l))^r\right)^{1/r} > \lambda\right\} \leq \frac{C}{\lambda} \sum_j \|f_j\|_{l_1}.$$

Now we apply the Calderón transfer principle to get the desired result with the same constant.

$\square$

Bibliography

[1] M. A. Akcoglu, R. L. Jones and P. O. Schwartz, *Variation in probability, ergodic theory and analysis*, Illinois J. Math. 42 1 (1998) 154-177.

[2] E. Atencia and A. de la Torre, *A dominated ergodic estimate for L_p spaces with weights*, Studia Math. 74 (1982) 35-47.

[3] A. Benedek, A. P. Calderón and R. Panzone, *Convolution operators on Banach space valued functions*, Proc. Nat. Acad. Sci. USA 48 (1962) 356-365.

[4] G. D. Birkhoff, *Proof of the ergodic theorem*, Proc. Nat. Acad. Sci. USA 17 (1931) 656-660.

[5] R. Boas, *Isomorphism between H^p and L^p*, Amer. J. Math. 77 (1955) 655-656.

[6] G. Boole, *On the comparision of trancendents with certain applications to the theory of definite integrals*, Philos. Trans. Roy. Soc. London 147 (1957) 745-803.

[7] J. Bourgain, *On the radial variation of bounded analytic functions on the disc*, Duke Math. J. 69 (1993) 671-682.

[8] J. Bourgain, *Pointwise ergodic theorems for arithmetic sets*, Publ. Math. Institut Hautes Etudes Scientifiques, 69 (1989) 5-45.

[9] J. Bourgain, *On the maximal ergodic theorem for certain subsets of the integers*, Israel J. Math. 61 1 (1988) 39-72.

[10] D. L. Burkholder, *Martingale transforms*, Ann. Math. Statist. 37 (1966) 1494-1504.

[11] D. L. Burkholder and R. F. Gundy, *Extrapolation and interpolation of quasi-linear operators on martingales*, Acta Math. 124 (1970) 249-304.

[12] D. L. Burkholder, R. F. Gundy and M. Silverstein, *A maximal function characterization of the class H^p*, Trans. AMS 157 (1971) 137-153.

[13] R. Caballero and A. de la Torre, *An atomic theory of ergodic H^p spaces*, Studida Math. 82 (1985) 39-69.

[14] A. P. Calderón, *Ergodic theory and translation-invariant operators*, Proc. Nat. Acad. Sci. USA. 59 (1968) 349-353.

[15] A. Carbery, *A version of Cotlar's lemma for L^p spaces and some applications*, Harmonic Analysis and Operator Theory (Caracas 1994), Contemporary Math. 189 (1995) 117-134.

[16] L. Carleson, *Two remarks on H^1 and BMO*, Analse Harmonique d'Orsay 164 (1975) 1-11.

[17] R. Coifman, *Characterizations of Fourier transforms of Hardy spaces*, Proc. Nat. Acad. Sci. USA 71 10 (1974) 4133-4134.

[18] R. Coifman, P. W. Jones, J. L. Rubio de Francia, *On a constuctive decomposition of BMO functions and factorizations of A_p weights*, Proc. AMS 87 (1983) 675-676.

[19] R. Coifman and G. Weiss, *Maximal functions and H^p spaces defined by ergodic transformations*, Proc. Nat. Acad. Sci. USA 70 (1973) 1761-1763.

[20] M. Cotlar, *A unified theory of Hilbert transforms and ergodic theorems*, Revista Mat. Cuyana, 1 (1955) 105-167.

[21] B. Davis, *On the integrability of the martingale square function*, Israel J. Math. 8 (1970) 187-190.

[22] A. de la Torre, *Ergodic H^1 spaces*, Bol. Soc. Mat. Mexicana 22 (1977) 10-22.

[23] J. Diestel, A. H. Jarchow and A. Tonge, *Absolutly Summing Operators*, Cambridge Studies in Advance Mathematics, 43 Cambridge University Press, New York, 1995.

[24] N. Dunford and J. Schwartz, *Linear operators*, I, Interscience, New York and London, 1964.

[25] J. Duoandikoetxea and J. L. Rubio de Francia, *Maximal and singular integral operators via Fourier transform estimates*, Invent. Math. 84 (1986) 541-561.

[26] C. Fefferman and E. M. Stein, *H^p Spaces of several variables*, Acta Math. 129 (1972) 137-193.

[27] A. Garsia, *Martingale inequalities*, Seminar Notes on Recent Progress, Benjamin Reading, 1973.

[28] R. L. Jones, *Inequalities for the ergodic maximal function*, Studia Math. 60 (1977) 111-129.

[29] R. L. Jones, R. Kaufman, J. Rosenblatt, and M. Wierdl, *Oscillation in ergodic theory*, Ergod. Th. & Dynam. Sys. 18 (1998) 889-935.

[30] R. L. Jones and J. Rosenblatt, *Reverse norm inequalities in ergodic theory* , priprint, 22 pages.

[31] Y. Katznelson, *An Inroduction to Harmonic Analysis*, Second Ed. Dover Pub. Inc. New York, 1976.

[32] L. H. Loomis, *A note on the Hilbert transform*, Bull. AMS 52 (1946) 1082-1086.

[33] F. J. Martin-Reyes, *Inequalities for the ergodic maximal function and convergence of the averages in weighted L^p-spaces*, Trans. AMS 296 1 (1986) 61-82.

[34] P. Muhly, *Ergodic Hardy spaces and duality*, Michigan Math J. 25 (1978) 317-323.

[35] P. Muhly, *Isometries of Ergodic Hardy spaces*, Israel Journal of Math. 36 1 (1980) 50-74.

[36] P. Muhly, *Function algebras and flows I*, Acta Sci. Math. (Szeged) 35 (1973) 111-121.

[37] P. Muhly, *Function algebras and flows II*, Ark. Mat. 11 (1973) 203-213.

[38] P. Muhly, *Function algebras and flows III*, Math. Z. 136 (1974) 253-260.

[39] P. Muhly, *Function algebras and flows IV*, Trans. AMS 203 (1975) 55-66.

[40] D. S. Ornstein, *A remark on the Birkhoff ergodic theorem*, Illinois J. Math. 15 (1971) 77-79.

[41] K. Petersen, *A constraction of ergodic BMO functions*, Proc. AMS 79 4 (1980) 549-555.

[42] K. Petersen, *Another proof of the existence of the ergodic Hilbert transform*, Proc. AMS 88 1 (1983) 39-43.

[43] K. Petersen, *Ergodic Theory*, Cambridge University Press, New York, 1983

[44] H. L. Royden, *Real Analysis*, Macmillan Pub. Company, New York, 1968.

[45] J. Rosenblatt and M. Wierdl, *Fourier Analysis and Almost Everywhere Convergence*, in the Proceedings of the Conference on Ergodic Theory and its Connections with Harmonic Analysis, Alexandria, Egypt, Cambridge University Press, 1994, pages 3-151.

[46] J. L. Rubio de Francia, F. J. Ruiz and J. L. Torrea, *Calderón-Zygmund theory for operator-valued kernels*, Adv. Math. 62 (1986) 7-48.

[47] W. Rudin, *The radial variation of analytic functions*, Duke Math. J. 22 (1955) 235-242.

[48] S. Sawyer, *Maximal inequalities of weak type*, Ann. of Math. 2 84 (1966) 157-174.

[49] E. M. Stein, *On limits of sequences of operators*, Ann. of Math. 2 74 (1961) 140-170.

[50] E. M. Stein, *Singular Integrals and Differentiablity Properties of Functions*, Princeton University Press, Princeton, N.J., 1970.

[51] E. M. Stein, *The development of square functions in the work of A. Zygmund*, Bull. AMS 7 (1982) 359-376.

[52] E. M. Stein, *Topics in Harmonic Analysis Related to Littlewood-Paley Theory*, Princeton University Press, Princeton, NJ, 1970.

[53] E. M. Stein, *Harmonic Analysis*, Real-variable Methods, Orthogonality, and Oscillatory Integrals, Princeton Mathematical Series, 43, Princeton University Press, Princeton, N.J., 1993.

[54] F. Zo, *A note on approximation of the identity*, Studia Math. 55 (1976) 111-122.

[55] A. Zygmund, *Trigonometric Series*, Second Ed. Vol. I&II, Cambridge University Press, New York, 1988.

Curriculum Vitae

Sakin Demir

Personal Information

Born on May 5, 1970, in Gürün, Sivas, Turkey.

Academic Degrees

Ph.D., Mathematics, University of Illinois, Jan. 1996-May 1999 (expected).

M.S., Mathematics, University of Illinois, Jan. 1994-Jan. 1996.

B.S., Mathematics, Highest Honors, University of Yüzüncü Yil, Van, Turkey, Sept. 1987-June 1991, ranked first in the university with a cummulative GPA of 90.45 out of 100.

Honors and Scholarships

Scholarship with Highest Honors, Turkish Ministry of National Education.

Ranked second in Turkey with a score of 85 out of 100.

> **Graduate study in math,** University of Illinois, Jan. 1994-Jan. 1998.
>
> **Intensive English study,** University of Illinois, June 1993-Jan. 1994,
>
> Middle East Technical University, Ankara, Turkey, Jan. 1993-June 1994.

Research Assistant, NSF grant, University of Illinois, Jan. 1998-Present.

Research Assistant, NSF grant, University of Illinois, Jan. 1997-Aug. 1997.

Teaching Experience

Teaching Assistant, University of Illinois, Jan. 1999- March 1999.

Teaching Assistant, University of Illinois, Aug. 1997-Dec. 1997.

Teaching Assistant, University of Illinois, Jan. 1997-May 1997.

Teaching Assistant, University of Illinois, Aug. 1996-Dec. 1996.

Math Teacher, Van Business and Trade High School, Van, Turkey, May 1992-Jan. 1993.

Math Teacher, Çayiralan High School, Yozgat, Turkey, Nov. 1991-May 1992.

Study Beyond Last Earned Degree

Graduate study in math, University of Yüzüncü Yil, Van, Turkey, Jan. 1992-June 1993.

Teaching Certificate, University of Yüzüncü Yil, Van, Turkey, Sept. 1988-June 1991.

Professional Affiliations

American Mathematical Society (since 1994).

The Honor Society of Phi Kappa Phi (recent).

Publications

Summability of Norms in Ergodic Theory, submitted, 7 pages.

Inequalities for Ergodic Operators, submitted, 22 pages.

Vector-valued Inequalities in Ergodic Theory, submitted, 27 pages.

Weighted Inequalities in Ergodic Theory, in preparation.