TOPOLOGICAL STRING THEORY AND BPS COUNTING

by

Tolga Domurcukgül

B.S., Physics, Boğaziçi University, 2020

Submitted to the Institute for Graduate Studies in
Science and Engineering in partial fulfillment of
the requirements for the degree of
Master of Science

Graduate Program in Physics
Boğaziçi University
2023

ACKNOWLEDGEMENTS

Embarking on the journey of exploring the intricate land of topological string theory and instanton counting has been a challenging and rewarding endeavor, made possible by numerous individuals' steadfast support and contributions. With heartfelt gratitude, I acknowledge their roles in shaping this thesis. My deepest appreciation goes to my thesis advisor, *Can Kozçaz.* Your unparalleled expertise, unwavering guidance, and insightful feedback have illuminated the path of this research. Your mentorship has been an invaluable asset throughout this goal. To my friends, *Berke, Deniz, Ege, Elif, Osman, Metin,* and *Mustafa*, your intellectual friendship, thought-provoking discussions, and collaborative spirit have been essential components of this journey. Your engagement enriched my understanding and sparked new dimensions of thought. I am deeply grateful to *Ilmar Gahramanov* for his insightful contributions, discussions, and guidance. Your expertise in the field has deepened the scholarly foundations of this work. To my mother and sister, your boundless love, encouragement, and sacrifices have been the bedrock of my academic voyage. Your determined belief in my potential has given me the strength to overcome challenges. In closing, this thesis stands as a testament to the collective efforts of those mentioned above. The convergence of knowledge, guidance, and friendship has shaped my academic and personal evolution, and I am profoundly thankful for this.

ABSTRACT

TOPOLOGICAL STRING THEORY AND BPS COUNTING

This thesis covers topological string theory, Seiberg-Witten theory, and instanton counting which provide insights into the interplay between geometry, topology, and quantum field theory. The target space interpretation of topological strings is explored, including its connections to physical string theory and geometric engineering. Along with that, M-theory compactification on a circle provides a target space interpretation for topological strings, leading to the discovery of new integer invariants known as Gopakumar-Vafa invariants. Seiberg-Witten theory plays a central role in understanding the dynamics of supersymmetric gauge theories. The moduli space of vacua offers a comprehensive description of the possible vacuum states of these theories, and the effective field theory approach enables the derivation of the quantum prepotential. The Seiberg-Witten solution provides exact results for various quantities in these theories and is examined in the context of elliptic curves. Then the Nekrasov partition function is used to study instantons in $\mathcal{N} = 2$ super Yang-Mills theory and the resolution of singularities and equivariant cohomology techniques are used to address non-compactness issues and subtleties arising from singularities. The role of Nekrasov factors in capturing contributions from different sectors of the instanton moduli space has been explored. The five-dimensional lift of Nekrasov partition function is introduced and studied through the connection to the Gopakumar-Vafa expansion of topological string theory.

ÖZET

TOPOLOJİK SİCİM TEORİSİ VE BPS SAYIMI

Bu tez, geometri, topoloji ve kuantum alan teorisi arasındaki etkileşimi sağlayan topolojik sicim teorisi ve Seiberg-Witten teorisi ile instanton sayımını ele almaktadır. Fiziksel sicim teorisi ve geometrik mühendislikle bağlantıları da dahil olmak üzere topolojik sicimlerin hedef uzay yorumu incelenmektedir. Bir daire üzerine M-teorisi kompaktlaştırılması, topolojik sicimler için bir hedef uzay yorumu sağlar ve Gopakumar-Vafa değişmezleri olarak bilinen yeni tamsayı değişmezlerinin keşfedilmesine yol açar. Seiberg-Witten teorisi, süpersimetrik ayar teorilerinin dinamiklerini anlamada merkezi bir rol oynar. Modül uzayı vakumları bu teorilerin olası vakum durumlarının kapsamlı bir açıklamasını sunar ve efektif alan teorisi yaklaşımı kuantum önpotansiyeli elde etme imkanı sağlar. Seiberg-Witten çözümü, bu teorilerde çeşitli büyüklükler için kesin sonuçlar sağlar ve eliptik eğriler bağlamında incelenir. Daha sonra $\mathcal{N} = 2$ süper Yang-Mills teorisindeki instantonları incelemek için Nekrasov bölüşüm fonksiyonu kullanılır ve tekilliklerin çözümü ve eşdeğer kohomoloji teknikleri, kompakt olmama sorunlarını ve tekilliklerden kaynaklanan durumları ele almak için kullanılmıştır. Beş boyutlu Nekrasov bölüşüm fonksiyonu, topolojik sicim teorisinin Gopakumar-Vafa açılımına bağlantı yoluyla tanıtılmıştır ve incelenmiştir.

TABLE OF CONTENTS

LIST OF TABLES

1. INTRODUCTION

Topological string theory and Seiberg-Witten theory are two fascinating areas of research that provide deep insights into the interplay between geometry, topology, and quantum field theory. This thesis aims to comprehensively explore various topics, including their theoretical foundations, mathematical frameworks, and applications in both physics and mathematics.

The historical background of topological string theory can be traced back to the pioneering work of Edward Witten in the late 1980s [1]. Witten's groundbreaking insights laid the foundation for this field, as he introduced the notion of topological twist and formulated the A and B models within the framework of two-dimensional $\mathcal{N} = (2, 2)$ supersymmetric theories [2]. The study of topological string theory gained further momentum with the discovery of its profound connections to physical string theory [10, 11] along with entering the branes in the picture and the discovering of the M-theory [12, 13]. The target space interpretation of topological strings was realized through the examination of M-theory compactification on a circle [24,25]. This interpretation shed light on revealing intriguing new integer invariants known as Gopakumar-Vafa invariants. The invariants introduced by Rajesh Gopakumar and Cumrun Vafa hold great significance as they provide insight into the geometric properties of Calabi-Yau threefolds and capture the spin degeneracies of the BPS (Bogomol'nyi-Prasad-Sommerfield) states coming from wrapped M2 branes. In parallel, Seiberg-Witten theory emerged as a powerful framework for understanding the dynamics of supersymmetric gauge theories [34]. This theory originated from the seminal work of Nathan Seiberg and Edward Witten in the mid-1990s, which uncovered a wealth of exact results and deep mathematical structures underlying these theories. The investigation of $\mathcal{N} = 2$ supersymmetric gauge theories and their moduli space of vacua provided a comprehensive understanding of the possible vacuum states and their behavior. The Seiberg-Witten solution, which originated from the study of elliptic curves illuminates the complex relationship between algebraic geometry and the dynamical features of the supersymmetric gauge theories. The study of instantons in Yang-Mills theory represents another essential aspect of this thesis. Instantons are nonperturbative field configurations that significantly contribute to the dynamics of gauge theories. Simon Donaldson constructed some topological invariants out of instanton moduli space and the resulting theory, the so-called Donaldson theory provides a powerful framework for studying the topology of four-dimensional manifolds. The moduli space of instantons, representing the space of all gauge field configurations satisfying the instanton equations, plays a central role in Donaldson theory, resulting in a deep understanding of the differentiable structures, characteristic classes, and the existence of various geometric invariants of four-manifolds [38]. After that, Witten introduced a new way to write a topological quantum field theory version of $\mathcal{N} = 2$ supersymmetric gauge theory through topological twisting [3]. The Donaldson-Witten theory provides perturbative techniques for calculating the Donaldson invariants. Non-perturbative techniques were later implemented in the context of Seiberg and Witten's work on $\mathcal{N} = 2$ supersymmetric Yang-Mills theory [39]. This application resulted in an unexpected relationship between Donaldson invariants and a new set of topological invariants known as Seiberg-Witten invariants. However, those interrelations between the $\mathcal{N} = 2$ supersymmetric gauge theories and the topological invariants are not the only important phenomena. The solution of the Seiberg-Witten theory requires that further instanton contributions in the prepotential and these non-perturbative corrections can be obtained by the topological twisted version of the $\mathcal{N} = 2$ super Yang-Mills theory. This theory is not the same as the original theory, but some observables are the same. In particular, the analytic prepotential is thought to be one of these observables due to its holomorphicity. Furthermore, the properly defined partition function of the twisted theory or the five-dimensional lift of the Nekrasov partition function determines not only space-time instantons but the higher derivative "gravitational" F-terms with a suitable background inducing "non-commutativity" [49, 51]. The Nekrasov partition function offers a powerful tool for studying these instantons. By employing techniques from the equivariant cohomology, significant progress has been made in understanding the dynamics of instantons and their connection to topological invariants.

The first part of this thesis focuses on topological string theory, which provides a unique perspective on the interplay between geometry and quantum field theories. We introduce the notion of two-dimensional $\mathcal{N} = (2, 2)$ supersymmetry and its relevance in topological string theory. This supersymmetry plays a crucial role in maintaining the delicate balance between bosonic and fermionic degrees of freedom in the theory. To construct topological string models, we explore the concept of topological twist, which modifies the supersymmetry transformations and yields new mathematical structures. We investigate the twisting procedure in detail, both in the general context and its specific manifestations such as the A and B models. These twisted models capture different aspects of topological string theory and offer unique perspectives on the underlying geometry. The A-model focuses on counting holomorphic curves, while the B-model explores classical geometry, such as the generalization of the Ray-Singer torsion to the loop space of Kähler manifolds. With a solid foundation in place, we probe the formalism of topological string theory. By coupling the twisted models to worldsheet gravity, we obtain topological string theories. These theories possess remarkable properties, and their free energies serve as crucial observables that encode valuable information about the underlying geometry. In particular, we explore the connection between the free energies of topological strings and the counting of holomorphic curves embedded in Calabi-Yau manifolds. These curve counts, known as Gromov-Witten invariants, provide deep insights into the interplay between algebraic geometry, topological invariants, and topological quantum field theories. In addition to the formalism of topological string theory, we investigate the target space interpretation of topological strings. We discover the profound connections between topological strings and physical string theory. One intriguing aspect is the study of M-theory compactification on a circle, which provides a target space interpretation for topological strings. Leveraging this interpretation, we reveal new integer invariants known as Gopakumar-Vafa invariants which offer valuable insights into the geometric properties of Calabi-Yau threefolds.

The second part of this thesis revolves around Seiberg-Witten theory, which plays a central role in understanding the dynamics of supersymmetric gauge theories. We begin by exploring $\mathcal{N} = 2$ supersymmetric gauge theories and the rich structure of their moduli space of vacua. This moduli space provides a comprehensive description of the possible vacuum states of these theories, capturing the behavior of the gauge fields and scalar fields. We investigate various aspects, such as the breaking of R-symmetry, the inclusion of central charges, and the establishment of a BPS (Bogomol'nyi-Prasad-Sommerfield) bound that characterizes the stability of these vacua. We explore the effective field theory approach to study deeper into the properties of $\mathcal{N} = 2$ supersymmetric gauge theories. This approach allows us to derive the quantum prepotential, which encodes the low-energy dynamics and the exact results of various observables in these theories. We investigate the concept of electric-magnetic duality, which detects dual descriptions of these theories and highlights their sophisticated connection to geometry and topology. Building upon the foundations laid in $\mathcal{N} = 2$ supersymmetric gauge theories, we explore the Seiberg-Witten solution. This solution provides exact results for various quantities in these theories, offering profound insights into their dynamics and properties. We examine the breaking of R-symmetry and its implications, uncovering the significance of singularities in the moduli space of vacua. We analyze the role of monopoles and dyons in characterizing the behavior of these theories, revealing the interplay between gauge fields and scalar fields. To provide a more concrete understanding, we scrutinize the Seiberg-Witten solution in the context of elliptic curves. This approach allows us to obtain exact results through the identification of these curves with the moduli space of vacua. By employing techniques from algebraic geometry, we disclose the rich mathematical structure underlying the Seiberg-Witten solution. This study enhances our comprehension of the connection between geometry, topology, and supersymmetric gauge theory.

The third part of this thesis will be on the Nekrasov partition function, which plays a crucial role in the study of instantons in Yang-Mills theory. We begin by investigating the concept of instantons, which are nonperturbative field configurations that contribute to the dynamics of gauge theories. We study the process of twisting $\mathcal{N} = 2$ super Yang-Mills theory, which modifies the theory and allows us to investigate instantons in a topological setting. After that, the ADHM (Atiyah-Drinfeld-Hitchin-Manin) construction of instantons is introduced, and to handle the singular and non-compact properties of instanton moduli space, we utilize methods such as the resolution of singularities and equivariant cohomology. These tools allow us to overcome non-compactness issues and address the subtleties arising from the presence of singularities. We explore equivariant localization on the instanton moduli space, a powerful technique that allows us to perform explicit computations. By integrating out the ADHM variables, we extract the non-perturbative part of the prepotential. In addition to the prepotential, we investigate the role of Nekrasov factors, which capture the poles of the partition function in terms of Young diagrams. Furthermore, we explore the intriguing concept of the five-dimensional lift, which provides a bridge between the topological string theory amplitudes and the partition function of five-dimensional $\mathcal{N} = 1$ supersymmetric gauge theory on the Ω-background. Finally, we will calculate the 5d Nekrasov partition function for a $\mathcal{N} = 1$ supersymmetric gauge theory with a gauge group of G_2, considering up to two instantons.

Throughout this thesis, our aim is to provide a comprehensive overview of topological string theory, Seiberg-Witten theory, and the Nekrasov partition function. By examining the theoretical foundations, mathematical frameworks, and applications in physics and mathematics, we strive to contribute to the ongoing research and understanding of these captivating fields.

2. TOPOLOGICAL STRING THEORY

This chapter provides a brief overview of topological string theory, starting with an introduction to the necessary tools. We summarize the concepts of two-dimensional $\mathcal{N}=(2,2)$ supersymmetry, Calabi-Yau threefolds, index theorems, and counting zero modes using simple examples. We then introduce the concept of a topological quantum field theory (TQFT) and the topological twisting of the supersymmetric non-linear sigma model, which results in the A-model and B-model. Finally, we discuss coupling these models to 2-dimensional gravity to achieve topological string theory. We will cover some parts of the references [1, 2], [4–9].

2.1. Two dimensional $\mathcal{N}=(2,2)$ supersymmetry

We begin with two-dimensional $\mathcal{N}=(2,2)$ supersymmetric quantum field theory. Firstly, we introduce the supersymmetry algebra along with the Poincaré algebra at 2 dimensions. The corresponding conserved charges for the Poincare algebra are given by H, P, M where they stand for the Hamiltonian, momentum, and angular momentum respectively. The additional conserved charges for $\mathcal{N}=(2,2)$ supersymmetry are called supercharges as

$$Q_+, Q_-, \overline{Q}_+, \overline{Q}_- \,. \tag{2.1}$$

The symbols $+$ and $-$ represent the left and right chiralities of the corresponding left and right sectors, respectively. The bar indicates complex conjugation. In addition to these symmetries, there are R-symmetries that act on supercharges. Two types of $U(1)$ R-symmetries, with corresponding conserved charges represented by F_L and F_R, will be introduced. For convenience, we can write them in a combined way

$$F_V = F_L + F_R, \quad F_A = F_L - F_R \,, \tag{2.2}$$

where F_V and F_A are called vector and axial R-symmetries. Finally, commutation and anticommutation relations of all of these charges generate the $\mathcal{N}=(2,2)$ supersymmetry algebra given by

$$
\begin{aligned}
Q_+^2 = Q_-^2 = \overline{Q}_+^2 = \overline{Q}_-^2 = 0, \\
\{\overline{Q}_+, \overline{Q}_-\} = \{Q_+, Q_-\} = 0, \\
\{Q_-, \overline{Q}_+\} = \{Q_+, \overline{Q}_-\} = 0, \\
[iM, Q_\pm] = \mp Q_\pm, \quad [iM, \overline{Q}_\pm] = \mp \overline{Q}_\pm, \\
[iF_V, Q_\pm] = -iQ_\pm, \quad [iF_V, \overline{Q}_\pm] = i\overline{Q}_\pm, \\
[iF_A, Q_\pm] = \mp iQ_\pm, \quad [iF_A, \overline{Q}_\pm] = \pm i\overline{Q}_\pm \, .
\end{aligned}
\tag{2.3}
$$

One can reach the supersymmetry algebra by the use of superspace formalism. In this formalism, we can write down supersymmetric actions on the superspace manifestly. The $\mathcal{N} = (2,2)$ superspace has four fermionic coordinates in addition to the bosonic spacetime coordinates x_0, x_1 for the two-dimensional case

$$
\theta^+, \theta^-, \overline{\theta}^+, \overline{\theta}^- \, , \tag{2.4}
$$

where these fermionic coordinates are called Grassmannian coordinates. Superfields $\mathcal{F}(x_\mu, \theta^\pm, \overline{\theta}^\pm)$ will be function taking values on the superspace. Due to the nature of the Grassmannian variable coordinates, a Taylor expansion of the superfield $\mathcal{F}$ in terms of $\theta^\pm$ and $\overline{\theta}^\pm$ contains a finite number of terms,

$$
\begin{aligned}
\mathcal{F}(x_\mu, \theta^\pm, \overline{\theta}^\pm) = f_0(x_\mu) + \theta^+ f_+(x_\mu) + \theta^- f_-(x_\mu) + \overline{\theta}^+ \overline{f}_+(x_\mu) + \\
+ \theta^+\theta^- f_{+-}(x_\mu) + \cdots
\end{aligned}
\tag{2.5}
$$

One can also write the integration measure on the superspace together with representing the spacetime coordinates (x^0, x^1) as complex coordinates $(z, \overline{z})$

$$
dz d\overline{z}\, d\theta^+ d\theta^- d\overline{\theta}^+ d\overline{\theta}^- = d^2 z d^4\theta \, . \tag{2.6}
$$

As a consequence, the action called D-terms can be written as

$$
S_D = \int d^2 z d^4\theta \, K(\mathcal{F}^i) \, , \tag{2.7}
$$

where K is a scalar function of a set of fields $\mathcal{F}$. From this construction, it can be seen that if one writes the Lagrangian density in terms of $K(\mathcal{F}^i)$ integrated over whole superspace, the action respects $\mathcal{N} = (2,2)$ supersymmetry manifestly. The infinite-dimensional representation of the supercharges can be obtained by the action of the supercharges written as differential operators acting on superfields,

$$
\begin{aligned}
\mathcal{Q}_\pm = \frac{\partial}{\partial\theta^\pm} + i\overline{\theta}^\pm \partial_\pm, \\
\overline{\mathcal{Q}}_\pm = -\frac{\partial}{\partial\overline{\theta}^\pm} - i\theta^\pm \partial_\pm \, .
\end{aligned}
\tag{2.8}
$$

$\partial_\pm$ stands for the partial derivatives with respect to $x^\pm = x^0 \pm x^1$. The superfield is acted upon by the axial $U(1)_A$ and vector $U(1)_V$ R-symmetries

$$\begin{aligned} e^{i\alpha F_V} &: \mathcal{F}\left(x^\mu, \theta^\pm, \overline{\theta}^\pm\right) \to e^{i\alpha q_V}\mathcal{F}\left(x^\mu, e^{-i\alpha}\theta^\pm, e^{i\alpha}\overline{\theta}^\pm\right), \\ e^{i\beta F_A} &: \mathcal{F}\left(x^\mu, \theta^\pm, \overline{\theta}^\pm\right) \to e^{i\beta q_A}\mathcal{F}\left(x^\mu, e^{\mp i\beta}\theta^\pm, e^{\pm i\beta}\overline{\theta}^\pm\right), \end{aligned} \tag{2.9}$$

where q_V and q_A are real numbers and they are called vector R-charge and axial R-charge of the superfield $\mathcal{F}$ respectively. Although the expansion of $\mathcal{F}$ in Equation (2.5) has 2^4 degrees of freedom, they are not independent. Constraints can be applied to $\mathcal{F}$ to reduce the degrees of freedom. This can be achieved by using a super-covariant derivative, which is another differential operator. They are given by

$$\begin{aligned} D_\pm &= \frac{\partial}{\partial\theta^\pm} - i\overline{\theta}^\pm\partial_\pm\,, \\ \overline{D}_\pm &= -\frac{\partial}{\partial\overline{\theta}^\pm} + i\theta^\pm\partial_\pm\,. \end{aligned} \tag{2.10}$$

One can notice that the super-covariant derivatives just differ from the infinite-dimensional representation of the supercharges Equation (2.10) by signs. By the use of super-covariant derivatives, we can define chiral superfield Φ as

$$\overline{D}_\pm\Phi = 0\,, \tag{2.11}$$

and for anti-chiral superfield, we have $D_\pm\overline{\Phi} = 0$. Thanks to the condition of the chiral superfield, it can be expanded

$$\Phi = \phi + \theta^\alpha\psi_\alpha + \theta^+\theta^- F\,. \tag{2.12}$$

The chiral superfield consists of scalar field ϕ, two Weyl fermions $\psi_\pm$, and non-dynamical auxiliary field F, which together form the degrees of freedom. To determine the supersymmetric transformations of these component fields, one can act supercharges on Φ and obtain the resulting supersymmetry transformations

$$\begin{aligned} \delta\phi &= \epsilon_+\psi_- - \epsilon_-\psi_+\,, \\ \delta\psi_\pm &= \pm 2i\overline{\epsilon}_\mp\partial_\pm\phi + \epsilon_\pm F\,, \\ \delta F &= -2i\overline{\epsilon}_+\partial_-\psi_+ - 2i\overline{\epsilon}_-\partial_+\psi_-\,. \end{aligned} \tag{2.13}$$

At this point, Based on the Kähler potential K of the target space X, we can now write the action for the $\mathcal{N} = (2,2)$ nonlinear sigma model as

$$S_D = S_{\text{kin}} = \int d^2z \int d^4\theta K(\Phi^i, \overline{\Phi}^{\bar{i}})\,. \tag{2.14}$$

The S_{kin} is evidently supersymmetric with $\mathcal{N} = (2,2)$. Another way to express this is through the F-term,

$$S_F = \frac{1}{2}\left(\int d^2z \int d^2\theta W\left(\Phi^i\right) + c.c.\right). \tag{2.15}$$

The function $W(\Phi^i)$, also known as the superpotential, is holomorphic with respect to Φ^i, and its complex conjugate is denoted by *c.c.*. The F-term is over half of the superspace. By utilizing the equation $\overline{Q}_\pm = \overline{D}_\pm - 2i\theta^\pm\partial_\pm$, we can demonstrate the invariance of the action under $\overline{Q}_\pm$:

$$\epsilon \int d^2zd^2\theta\overline{Q}_\pm W(\Phi^i) = \epsilon \int d^2zd^2\theta(\overline{D}_\pm - 2i\theta^\pm\partial_\pm)W(\Phi^i) = 0\,. \tag{2.16}$$

The chirality property of the fields has been utilized in the aforementioned equation. As the fields vanish at infinity, the right-hand side of the equation gives a null result. Generally, the total action comprises of the D-term and F-term. It is necessary to ensure that the two different types of $U(1)$ R-symmetries are preserved beside the invariance of the total action under the supersymmetry transformations. From a classical perspective, the integration measure of S_D and S_F change only the overall phase under the R-symmetry transformations, thus, the two terms can be analyzed separately. The D-term includes $d\theta^4$ and its superspace measure is invariant under both R-symmetry transformations according to Equation (2.9). Therefore, it can be concluded that S_{kin} is invariant under both symmetries if the integrand or Kähler potential $K(\Phi^i, \overline{\Phi}^{\bar{i}})$ also does not change under the R-symmetry transformations. The invariance of the D-term can be achieved by setting q_V and q_A of Φ^i to be zero, which gives the total vectorial and axial charge of $K(\Phi^i, \overline{\Phi}^{\bar{i}})$ as zero. In the case of the F-term, $U(1)_A$ R-charge of $d\theta^2$ vanishes and we can set q_A of Φ^i to zero. $U(1)_V$ R-charge of $d\theta^2$ is equal to minus two, and the superpotential must transform with R-symmetry vector charge $q_V = 2$ in order to preserve the $U(1)_V$ symmetry. Therefore, from a classical perspective, $U(1)_A$ is always preserved since q_A can be chosen as zero for all Φ^i. Nevertheless, the vectorial R-symmetry $U(1)_V$ is not anomalous for specific types of W. At the quantum level, a careful analysis of the zero modes and index theorems is required for a more subtle story. We can keep in mind that the K is a scalar function of chiral fields and it is defined up to the transformations

$$K(\Phi^i, \overline{\Phi}^{\bar{i}}) \mapsto K(\Phi^i, \overline{\Phi}^{\bar{i}}) + f(\Phi^i) + \overline{f}(\overline{\Phi}^{\bar{i}})\,, \tag{2.17}$$

where $f(\Phi)$ is the holomorphic function of the chiral superfields and $\overline{f}(\overline{\Phi})$ is the complex conjugate of it. We can write the corresponding action for the D-term in terms of component fields after doing the integration on superspace coordinates

$$\begin{aligned} S_{\text{kin}}\left[\Phi^i,\overline{\Phi}^{\bar{i}}\right] = \int_{\mathbb{R}^2} d^2x \Big[& -g_{i\bar{j}}\partial^\mu\phi^i\partial_\mu\overline{\phi}^{\bar{j}} + ig_{i\bar{j}}\overline{\psi}_+^{\bar{j}}\nabla_-\psi_+^i + ig_{i\bar{j}}\overline{\psi}_-^{\bar{j}}\nabla_+\psi_-^i \\ & + R_{i\bar{j}k\bar{l}}\psi_+^i\psi_-^k\overline{\psi}_-^{\bar{j}}\overline{\psi}_+^{\bar{l}} + g_{i\bar{j}}\left(F^i - \Gamma^i_{kl}\psi_+^k\psi_-^l\right)\left(\overline{F}^{\bar{j}} - \Gamma^{\bar{j}}_{\bar{k}}\overline{\psi}_-^{\bar{k}}\overline{\psi}_+^{\bar{l}}\right)\Big] , \end{aligned} \tag{2.18}$$

where $g_{i\bar{j}} = \partial_i\partial_{\bar{j}}K$, and Γ^i_{jk} and $R_{i\bar{j}k\bar{l}}$ stand for the connection coefficients and Riemann curvature tensor respectively. The ∇ is the covariant derivative of the target space X,

$$\nabla_\mu\psi^i_\pm = \partial_\mu\psi^i_\pm + \Gamma^i_{jk}\partial_\mu\phi^j\psi^k_\pm \,. \tag{2.19}$$

One can see that the kinetic terms are non-singular since the metric is positive definite. Similarly, we can write the F-term in terms of the component fields

$$\frac{1}{2}\int_{\mathbb{R}^{2|2}} W(\Phi)d^2\theta d^2x + \text{ c.c. } = \frac{1}{2}\int_{\mathbb{R}^2}\left[F^i\frac{\partial W}{\partial\phi^i} - \psi_+^i\psi_-^j\frac{\partial^2 W}{\partial\phi^i\partial\phi^j}\right]d^2x + c.c.\,, \tag{2.20}$$

and we can set the auxiliary field F by using the equation of motion

$$F^i = \Gamma^i_{dd}\psi_+^k\psi_-^l - \frac{1}{2}g^{i\bar{j}}\partial_{\bar{j}}\overline{W}\,,$$

and finally, the total action turns out to be

$$\begin{aligned} S[\phi,\psi] = \int_{\mathbb{R}^2}\Big[& -g_{i\bar{j}}\partial^\mu\phi^i\partial_\mu\overline{\phi}^{\bar{j}} + ig_{i\bar{j}}\overline{\psi}_+^{\bar{j}}\nabla_-\psi_+^i + ig_{i\bar{j}}\overline{\psi}_-^{\bar{j}}\nabla_+\psi_-^i + R_{i\bar{j}k\bar{l}}\psi_+^i\psi_-^k\overline{\psi}_-^{\bar{j}}\overline{\psi}_+^{\bar{d}} \\ & -\frac{1}{4}g^{i\bar{j}}\partial_i W\partial_{\bar{j}}\overline{W} - \frac{1}{2}\nabla_i\partial_j W\psi_+^i\psi_-^j - \frac{1}{2}\nabla_{\bar{i}}\partial_{\bar{j}}\overline{W}\overline{\psi}_-^{\bar{i}}\overline{\psi}_+^{\bar{j}}\Big]d^2x\,. \end{aligned} \tag{2.21}$$

This action is invariant under the infinitesimal supersymmetry transformations

$$\begin{aligned} \delta\phi^i &= \epsilon_+\psi_-^i - \epsilon_-\psi_+^i\,, & \delta\overline{\phi}^{\bar{i}} &= -\bar{\epsilon}_+\overline{\psi}_-^{\bar{i}} + \bar{\epsilon}_-\overline{\psi}_+^{\bar{i}}\,, \\ \delta\psi_\pm^i &= \pm 2i\bar{\epsilon}_\mp\partial_\pm\phi^i + \epsilon_\pm F^i\,, & \delta\overline{\psi}_\pm^{\bar{i}} &= \mp 2i\epsilon_\mp\partial_\pm\overline{\phi}^{\bar{i}} + \bar{\epsilon}_\pm\overline{F}^{\bar{i}}\,. \end{aligned} \tag{2.22}$$

The nonlinear sigma model action is defined with $W = 0$ and the target space X is the Kähler manifold and the coordinates of it are given by the complex scalar fields $(\phi^i,\overline{\phi}^{\bar{i}})$. The case of $W \neq 0$ is called the Landau-Ginzburg model. If we consider the general two-dimensional metric instead of the flat one, we can change $d^2x \to \sqrt{h}d^2x$, and $g_{i\bar{j}}\partial^\mu\phi^i\partial_\mu\overline{\phi}^{\bar{j}} \to g_{i\bar{j}}h^{\mu\nu}\partial_\mu\phi^i\partial_\nu\overline{\phi}^{\bar{j}}$. We also need to have a worldsheet spin connection in the fermion kinetic term since worldsheet geometry is nontrivial in which the complex plane is replaced by the general Riemann surface Σ. In that case, it is possible to write two-dimensional metrics locally as

$$h = h_{\mu\nu}(x)dx^\mu dx^\nu = e^{2\Omega}((dx^1)^2 + (dx^2)^2)\,, \tag{2.23}$$

and this means that Riemann surfaces are always conformally flat. The complex coordinates $(\phi^i,\overline{\phi}^{\bar{i}})$ which is defined on a coordinate patch U of a Kähler manifold X. The supersymmetric sigma model can be thought of as a map $\phi : \Sigma \to U$ and the worldsheet fermions are sections of the following bundle structures

$$\psi_{\pm} \in \Gamma\left(\Sigma, \phi^{\star}(T^{(1,0)}M) \otimes S_{\pm}\right) , \quad \overline{\psi}_{\pm} \in \Gamma\left(\Sigma, \phi^{\star}(T^{(0,1)}M) \otimes S_{\pm}\right) , \tag{2.24}$$

where $S_{\pm}$ are left-spin and right-spin bundles on the Riemann surface Σ and $T^{(0,1)}X$ and $T^{(1,0)}X$ denotes holomorphic and anti-holomorphic tangent bundles of Kähler manifold X. The global R-symmetries act on the chiral superfield

$$\begin{aligned} U(1)_V : \Phi\left(x^{\pm}, \theta^{\pm}, \overline{\theta}^{\pm}\right) &\mapsto e^{2iq\alpha}\Phi\left(x^{\pm}, e^{-i\alpha}\theta^{\pm}, e^{i\alpha}\overline{\theta}^{\pm}\right) , \\ U(1)_A : \Phi\left(x^{\pm}, \theta^{\pm}, \overline{\theta}^{\pm}\right) &\mapsto \Phi\left(x^{\pm}, e^{\mp i\beta}\theta^{\pm}, e^{\pm\beta}\overline{\theta}^{\pm}\right) . \end{aligned} \tag{2.25}$$

At the quantum level, classical symmetries may not be preserved due to the potential non-invariance of the path integral measure under certain transformations. Specifically, R-transformations may not preserve the measure while still preserving the classical action. To determine whether the $U(1)_V$ and $U(1)_A$ symmetries are anomalous, we can examine the case of a single massless charged Dirac fermion ψ on a torus $\mathbb{T}^2$ with a background gauge field $A \in E$ governed by the given action

$$S = i\int_{\mathbb{T}^2} \left[\overline{\psi}_+ D_z \psi_+ + \overline{\psi}_- \overline{D}_{\overline{z}} \psi_-\right] d^2 z \,, \tag{2.26}$$

where $z = x + iy$ is a local coordinate on the torus $\mathbb{T}^2$, $D_z = \partial_z + A_z$ and $\overline{D}_{\overline{z}} = \partial_{\overline{z}} + A_{\overline{z}}$ are the covariant derivatives with respect to the gauge field. This action is invariant under global transformations of the fermions classically. The quantum story needs more attention. In order to do that consider the following quantity

$$k = \int_{\mathbb{T}^2} c_1(E) = \frac{i}{2\pi}\int_{\mathbb{T}^2} F > 0 \,, \tag{2.27}$$

where $c_1(E)$ is the first Chern class of the vector bundle E. The above equation means that the gauge field A has a non-zero instanton number k and it can be written in a different form by the Atiyah-Singer index theorem

$$\dim\ker \overline{D}_{\overline{z}} - \dim\ker D_z = \int_{\mathbb{T}^2} c_1(E) = k > 0 \,, \tag{2.28}$$

where $\dim\ker D_z$ represents the number of ψ_+ zero modes and $\dim\ker \overline{D}_{\overline{z}}$ the number of ψ_- zero modes. Observe that complex conjugation operation changes the sign of the $U(1)$ charge of a fermion. Then, we have $(\psi_+)^* = \overline{\psi}_-$, and $(\psi_-)^* = \overline{\psi}_+$. Therefore, the number of ψ_- zero modes is the same as the number of $\overline{\psi}_+$ zero modes. Under this observation, the path integral measure $[\mathcal{D}\psi]$ which is nothing but the integration over all modes of the fields $\psi_{\pm}$ and $\overline{\psi}_{\pm}$, is transformed as $[\mathcal{D}\psi] \rightarrow e^{2ik\beta}[\mathcal{D}\psi]$. Note that the $U(1)_V$ transformations of all the non-zero modes cancel, that is, the vector $U(1)$ transformation $\psi_{\pm} \rightarrow e^{-i\alpha}\psi_{\pm}$ is preserved at the quantum level. On the other hand the axial $U(1)$ symmetry, $\psi_\pm \to e^{\mp i\beta}\psi_\pm$ is broken by the non-zero instanton number, i.e., $k \neq 0$. Therefore there must be some of the fermion fields must have zero modes. However, the zero modes do not contribute to the action by the definition, then the partition function on a torus $\mathbb{T}^2$ is equal to the zero. Hence we are faced with the situation that the non-trivial observables are the correlation functions rather than the path integral since the presence of the fermions is compulsory to cancel the effect of the zero-modes on the path integral. Due to the difference between the number of ψ_+ and ψ_- zero modes leading to non-zero instanton number k on $\mathbb{T}^2$, we will need the insertion of the fermions which saturate the contributions of the zero modes to have non-vanishing correlation function

$$\left\langle \psi_+(z_1)\,\psi_+(z_2)\cdots\psi_+(z_k)\,\overline{\psi}_-(w_1)\,\overline{\psi}_-(w_2)\cdots\overline{\psi}_-(w_k)\right\rangle \,, \tag{2.29}$$

where we have inserted fermions at points $\{z_1, ..., z_k, w_1, ..., w_k\} \in T^2$. Finally, it can be concluded that the anomalies in global symmetries give selection rules that restrict the types of correlation functions. Now we can move into the non-linear sigma model on a Kähler manifold X. The fermion terms in the action are given by

$$i\int \left[g_{i\bar{j}}\overline{\psi}^{\bar{j}}_-\nabla_z\psi^i_- + g_{i\bar{j}}\overline{\psi}^{\bar{j}}_+\overline{\nabla}_{\bar{z}}\psi^i_-\right] d^2z \,, \tag{2.30}$$

where the vector bundle E is replaced by the tangent bundle TX and $\gamma^\mu\nabla_\mu$ coupled to the pullback of the holomorphic tangent bundle $\phi^\star(T^{(1,0)}X)$. Then, the index theorem gives the instanton number

$$k = \int c_1(\phi^\star(T^{(1,0)}X)) = \frac{i}{2\pi}\int \mathrm{tr}(R) \,, \tag{2.31}$$

where R is the curvature 2-form which is the analog of the field strength of the gauge theory and the trace is taken with respect to the spinor indices. The relation between the difference of zero-modes and the instanton number holds for the non-linear sigma model too. If the symmetry is respected at the quantum level, we have a condition on the target space. That is, the instanton number must be zero and this gives the fact that $\mathrm{tr}(R) = 0$. The Kähler manifold X with the $\mathrm{tr}(R) = 0$ is what is called the Calabi-Yau manifold.

2.2. Topological Twist

A field theory is defined on a manifold X with a given metric h. In typical quantum field theories, the background manifold X is flat and the physical observables, such as correlation functions, depend on the relative position between the points rather than the explicit coordinate dependence. However, more general versions of these theories exist and are important for cases such as quantum gravity. One of the most fundamental quantities for computing physical observables for a quantum field theory is the partition function Z and physical operators $\mathcal{O}_i$ can be computed using

$$\langle \mathcal{O}_1 (x_1) \cdots \mathcal{O}_n (x_n) \rangle_h \,, \tag{2.32}$$

where the subscript h means that the correlation functions are computed for a fixed background metric. A topological quantum field theory (TQFT) can be classified into two types: Schwarz type and Witten-type (or cohomological type). The former has no explicit metric dependence on the classical action, making it a manifestly topological theory. The Chern-Simons theory is an example of Schwarz-type TQFTs. Our focus is on the latter, which allows for metric dependence in the classical Lagrangian and physical operators, but correlation functions remain independent of the metric. TQFT is a special type of quantum field theory with both the classical action and correlation functions being independent of the background metric or having deformation invariance. Correlation functions may have a background dependence, where operators depend on insertion points on the manifold, which has important topological implications. The Witten-type TQFT can be constructed in the following. First of all, there is a fermionic or nilpotent Noether charge operator Q. By nilpotency, we must have

$$Q^2 = 0 \,. \tag{2.33}$$

The symmetry generated by this type of operator is called sometimes BRST-symmetry, like in the case of non-abelian gauge theories. The anti-commutator of Q with the physical operators brings about the symmetry transformation as

$$\delta_\epsilon \mathcal{O}_i = i\epsilon \left\{ Q, \mathcal{O}_i \right\} \,. \tag{2.34}$$

The vacuum of the theory $|0\rangle$ must also respect this Q-symmetry or in other words, the symmetry is not spontaneously broken

$$Q|0\rangle = 0 \,. \tag{2.35}$$

The second ingredient for having Witten-type TQFTs is that it is necessary for the physical operators $\mathcal{O}_i$ to be both metric-independent and invariant under this symmetry,

$$\delta_\epsilon \mathcal{O}_i = \{Q, \mathcal{O}_i\} = 0\,. \tag{2.36}$$

This property results that the physical operators are in the equivalence class of the Q

$$\mathcal{O}_i \sim \mathcal{O}_i + \{Q, \Lambda\}\,, \tag{2.37}$$

where Λ is some operator for the underlying theory. The final property of Witten-type TQFTs is that the deformation invariance or in other words the energy-momentum tensor is Q-exact. The energy-momentum tensor is defined as

$$T_{\mu\nu} \equiv \frac{\delta S}{\delta h^{\mu\nu}} = \{Q, G_{\mu\nu}\}\,, \tag{2.38}$$

for some operator $G_{\mu\nu}$. By using these requirements, we can show that the correlation functions are independent of background metric h,

$$\begin{aligned} \frac{\delta}{\delta h^{\mu\nu}} \langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle &= \frac{\delta}{\delta h^{\mu\nu}} \left(\int D\phi \mathcal{O}_1 \cdots \mathcal{O}_n e^{\mathrm{iS}[\phi]} \right) \\ &= i \int D\phi \mathcal{O}_1 \cdots \mathcal{O}_n \frac{\delta S}{\delta h^{\mu\nu}} e^{iS[\phi]} \\ &= i \langle \mathcal{O}_1 \cdots \mathcal{O}_n \{Q, G_{\mu\nu}\} \rangle \\ &= i \langle Q \mathcal{O}_1 \cdots \mathcal{O}_n G_{\mu\nu} \rangle + i \langle \mathcal{O}_1 \cdots \mathcal{O}_n G_{\mu\nu} Q \rangle \\ &= 0\,. \end{aligned} \tag{2.39}$$

In the third equality, we have utilized the Q-exactness property of the energy-momentum tensor. The fourth one employs Equation (2.36). The last condition assumes that the vacuum respects Q-symmetry. It is worth noting that the operator Q is nilpotent, and any observable that can be expressed as $\mathcal{O}_i = \{Q, \Lambda\}$ is Q-exact, as it satisfies the nilpotency condition. Moreover, all physical observables are Q-closed, since $\{Q, \mathcal{O}_i\} = 0$. The correlation function is proven to be zero if any of the operators can be written as Q-exact,

$$\langle 0 | \mathcal{O}_{i_1} \cdots \mathcal{O}_{i_j} \{Q, \Lambda\} \mathcal{O}_{i_{j+1}} \cdots \mathcal{O}_{i_n} | 0 \rangle = \langle 0 | \mathcal{O}_{i_1} \cdots \mathcal{O}_{i_j} (Q\Lambda - \Lambda Q) \mathcal{O}_{i_{j+1}} \cdots \mathcal{O}_{i_n} | 0 \rangle, \tag{2.40}$$

and

$$\begin{aligned} \langle 0 | \mathcal{O}_{i_1} \cdots \mathcal{O}_{i_j} Q \Lambda \mathcal{O}_{i_{j+1}} \cdots \mathcal{O}_{i_n} | 0 \rangle &= \pm \langle 0 | \mathcal{O}_{i_1} \cdots Q \mathcal{O}_{i_j} \Lambda \mathcal{O}_{i_{j+1}} \cdots \mathcal{O}_{i_n} | 0 \rangle \\ &= \pm \langle 0 | Q \mathcal{O}_{i_1} \cdots \mathcal{O}_{i_j} \Lambda \mathcal{O}_{i_{j+1}} \cdots \mathcal{O}_{i_n} | 0 \rangle \\ &= 0\,. \end{aligned} \tag{2.41}$$

Thus, this shows that the correlation functions which are constructed by the Q-exact operators are trivial and The physical operators correspond uniquely to the elements in Q-cohomology,

$$H_Q = \frac{\{Q_{\text{closed operators}}\}}{\{Q_{\text{exact operators}}\}}\,. \tag{2.42}$$

Now, we can return to the supersymmetric non-linear sigma model. Getting the topological quantum field theory out of the non-linear sigma model can be attained from the topological twisting which is made by the redefinitions of some of the fields of the theory or equivalently the redefinitions of the energy-momentum tensor resulting in gauging the global R-symmetries. Recall that The non-linear sigma model was defined on a flat two-dimensional surface. However, if we consider it on a general curved Riemann surface Σ, the action may not be supersymmetric. This can be observed by examining the variation of the supersymmetric non-linear sigma model action under a supersymmetry transformation [9]

$$\delta S = \int_\Sigma \left(\nabla_\mu \epsilon_+ G^\mu_- - \nabla_\mu \epsilon_- G^\mu_+ - \nabla_\mu \bar{\epsilon}_+ \overline{G}^\mu_- + \nabla_\mu \bar{\epsilon}_- \overline{G}^\mu_+ \right) \sqrt{h} d^2 x\,. \tag{2.43}$$

Here $\epsilon_\pm$and $\bar{\epsilon}_\pm$ are the infinitesimal are spinor-valued paramters on the Riemann surface Σ. When the Riemann surface is flat, it is possible to find covariantly constant spinors $\nabla_\mu \epsilon_\pm = \nabla_\mu \bar{\epsilon}_\pm = 0$. Thus we can conclude that the Lagrangian is supersymmetric, i.e., $\delta S = 0$. On the other hand, for a curved Σ, the covariantly constant condition may not be satisfied and this breaks the supersymmetric invariance of the action. In order to resolve this problem for a general Riemann surface, the topological twisting is introduced by [1,2]. The idea is simple, recall that for the supersymmetric transformation, we have $\delta_\epsilon \equiv \epsilon Q$, then we need to change the spinorial structure of the infinitesimal parameter ϵ to the scalar since there is always covariantly constant spinor as we have mentioned and this also leads to change the spinorial nature of the supercharge Q. Before we move on to explain what we have meant above, we will change the Minkowskian signature to the Euclidean one by performing the Wick rotation $x_0 = -ix_2$. We also define complex coordinate $z = x_1 + ix_2$. Then the two-dimensional Lorentz group turns out to be the Euclidean rotation group, $SO(2)_E = U(1)_E$ with the following identification $M_E = iM$. Then, the commutation relations or Equations (2.3) becomes that

Table 2.1. $\mathbb{C}, \sqrt{K}$ and K refer to the trivial, spinor, and canonical line bundles.

	$U(1)_V$	$U(1)_A$	M_E	$\mathcal{L}$	M_E^A	$\mathcal{L}$	M_E^B	$\mathcal{L}$
Q_-	-1	1	1	$\sqrt{K}$	0	$\mathbb{C}$	2	K
$\overline{Q}_+$	1	1	-1	$\sqrt{\overline{K}}$	0	$\mathbb{C}$	0	$\mathbb{C}$
$\overline{Q}_-$	1	-1	1	$\sqrt{K}$	2	K	0	$\mathbb{C}$
Q_+	-1	-1	-1	$\sqrt{\overline{K}}$	-2	$\overline{K}$	-2	$\overline{K}$

$$\begin{aligned} [M_E, Q_\pm] &= \mp Q_\pm\,, \quad [M_E, \overline{Q}_\pm] = \mp\overline{Q}_\pm\,, \\ [F_V\,, Q_\pm] &= -Q_\pm\,, \quad [F_V\,, \overline{Q}_\pm] = \overline{Q}_\pm\,, \\ [F_A, Q_\pm] &= \mp Q_\pm\,, \quad [F_A, \overline{Q}_\pm] = \pm\overline{Q}_\pm\,. \end{aligned} \tag{2.44}$$

The topological twisting means that we redefine the Euclidean rotation by the use of mixing the diagonal part of the $U(1)_E \times U(1)_R$ with R-symmetry charges. There are two feasible ways to obtain this, namely

$$\begin{aligned} A - \text{twist} : M_E^A &= M_E + F_V\,, \\ B - \text{twist} : M_E^B &= M_E + F_A\,. \end{aligned} \tag{2.45}$$

As we have stressed there is another equivalent way to make the topological twisting which is to gauge the Lagrangian by adding either the $U(1)_V$ or the $U(1)_A$ R-symmetry current into the spin connection. The topological twisting results in a change in the flavor index to the spinor index, therefore the new fields after the twisting of the spin of the component fields are changed. Let us look at them more closely, consider chiral superfield Φ which has zero R-charges

$$\Phi = \phi + \theta^+\psi_+ + \theta^-\psi_- + \overline{\theta}^+\overline{\psi}_+ + \cdots\,. \tag{2.46}$$

The Weyl fermion ψ_+ possesses a M_E charge of -1, a $U(1)_V$ charge of $q_V = -1$, and a $U(1)_A$ charge of $q_A = -1$, based on the transformations of the superspace coordinates under the global R-symmetry group. As a result of its M_E charge being -1, the Weyl fermion ψ_+ belongs to the section of the anti-spinor bundle $\overline{S}$ over the Riemann surface Σ. After the topological twisting, such as the A-twist, its M_E' charge becomes -2, and it transforms into an anti-holomorphic one-form over Σ. Similarly, for the B-twist, it also has an M_E' charge of -2 and is an anti-holomorphic one-form. Similarly, the Weyl fermion ψ_- changes its charge properties.

Table 2.2. Effect of the topological twisting on the component fields.

	$U(1)_V$	$U(1)_A$	M_E	$\mathcal{L}$	M_E^A	$\mathcal{L}$	M_E^B	$\mathcal{L}$
ϕ	0	0	0	$\mathbb{C}$	0	$\mathbb{C}$	0	$\mathbb{C}$
ψ_-	-1	1	1	$\sqrt{K}$	0	$\mathbb{C}$	2	K
$\overline{\psi}_+$	1	1	-1	$\sqrt{\overline{K}}$	0	$\mathbb{C}$	0	$\mathbb{C}$
$\overline{\psi}_-$	1	-1	1	$\sqrt{K}$	2	K	0	$\mathbb{C}$
ψ_+	-1	-1	-1	$\sqrt{\overline{K}}$	-2	$\overline{K}$	-2	$\overline{K}$

Prior to the twist, it carries a M_E charge of $+1$, a $U(1)_V$ charge of $q_V = -1$, and a $U(1)_A$ charge of $q_A = 1$. However, after the A-twist, it now has a M'_E charge of 0 and is a scalar field in $\mathbb{C}$. The B-twist also causes changes in charge properties, as the Weyl fermion $\overline{\psi}_+$ becomes a scalar field with a zero charge under the new Euclidean rotation group M_E^A, due to its $M_E + q_A = -1 + 1 = 0$ charge. The other cases can be found by similar logic and the final field content can be seen in Table 2.2.

2.2.1. A-Model

Now we will show that after the twisting, the $\mathcal{N} = (2, 2)$ supersymmetric sigma model becomes a topological field theory. After the twisting, we observe that $\overline{Q}_+$ and Q_- become scalars. We have the option to select the fermionic or BRST symmetry charge.

$$Q_A = \overline{Q}_+ + Q_- \,. \tag{2.47}$$

and the Weyl fermions, specifically ψ_- and $\overline{\psi}_+$, transform into scalar quantities. On the other hand, ψ_+ and $\overline{\psi}_-$, which were originally part of the spinor and anti-spinor bundles, respectively, are now represented by holomorphic and anti-holomorphic one-forms. We can redefine the fields to see the new bundle structure of the component fields after the twisting

$$\begin{aligned} \chi^i &:= \psi_-^i, \quad \chi^{\bar{i}} := \overline{\psi}_+^{\bar{i}}\,, \\ \rho_z^{\bar{i}} &:= \overline{\psi}_-^{\bar{i}}, \quad \rho_{\bar{z}}^i := \psi_+^i\,. \end{aligned} \tag{2.48}$$

By introducing a $U(1)_V$ R-symmetry current into the spin connection of the original D-term, a twisted action can be obtained as

$$S_A = \int d^2z \left(g_{i\bar{j}} \left(h^{\mu\nu}\sqrt{h}\partial_\mu\phi^i\partial_\nu\phi^{\bar{j}} - i\rho_z^{\bar{j}}\nabla_{\bar{z}}\chi^i + i\rho_{\bar{z}}^i\nabla_z\chi^{\bar{j}} \right) - \frac{1}{2}R_{i\bar{k}j}\rho_{\bar{z}}^i\chi^j\rho_z^{\bar{k}}\chi^l \right) . \quad (2.49)$$

Furthermore, the supersymmetry transformation becomes:

$$\begin{aligned} &\delta\phi^i = \epsilon\chi^i, \quad \delta\overline{\phi} = \epsilon\chi^{\bar{i}}, \\ &\delta\chi^{\bar{i}} = 0, \quad \delta\rho_z^{\bar{i}} = -2i\epsilon\partial_z\overline{\phi} - \epsilon\chi^{\bar{j}}\Gamma^{\bar{i}}_{\bar{j}\bar{k}}\rho_z^{\bar{k}}, \\ &\delta\chi^i = 0, \quad \delta\rho_{\bar{z}}^i = 2\overline{\epsilon}\partial_{\bar{z}}\phi^i - \epsilon\chi^j\Gamma^i_{jk}\rho_{\bar{z}}^k, \end{aligned} \quad (2.50)$$

where we have set $\epsilon_- = \overline{\epsilon}_+ = 0$ and $\epsilon_+ = \overline{\epsilon}_- = \epsilon$. We can show that the A-model is in fact topological quantum field theory. First, we assume that the vacuum respects the fermionic/BRST symmetry. Secondly, the square of the fermionic symmetry charge Q_A is zero since $\{\overline{Q}_+, Q_-\} = 0$. Finally, it is required to show that the energy-momentum tensor can be written as a Q_A-exact form. This is true for the A-model and the A-model action can be rewritten as Q_A-exact [2],

$$S_A = \int_{\Sigma_g} d^2z\,\{Q_A, V\} + \int_{\Sigma_g} \phi^*(\omega)\,, \quad (2.51)$$

where V is given by

$$V = g_{\bar{i}j}\left(\rho_z^{\bar{i}}\partial_{\bar{z}}\phi^j + \partial_z\phi^{\bar{i}}\rho_{\bar{z}}^j\right) . \quad (2.52)$$

V is commonly referred to as the gauge fermion due to its association with the gauging of the global $U(1)_R$ symmetry. Additionally, $\int_\Sigma \phi^*(\omega)$ represents the pullback of the Kähler form of X, and its value is dependent solely on the cohomology class of ω. Hence, it respects the deformation invariance and it has no dependence on the Riemann surface metric h at all. Furthermore, it can be seen that the A-twisted non-linear sigma model depends only on the Kähler moduli of target space X since it is characterized by the cohomology class of ω which parametrizes the Kähler moduli of target space X rather than the complex structure moduli of X. Now, we can find the cohomology classes of physical operators. If we act Q_A on the most general physical operator which is constructed by the combination of the component fields, we have

$$\begin{aligned} &Q_A\left\{\mathcal{A}_{i_1,\cdots i_p\bar{j}_1\cdots\bar{j}_q}(\phi)\chi^{i_1}\cdots\chi^{i_p}\chi^{\bar{j}_1}\cdots\chi^{\bar{j}_q}\right\} = \\ &\left(\partial_k\mathcal{A}_{i_1,\cdots i_p\bar{j}_1\cdots\bar{j}_q}\chi^k + \overline{\partial}_{\bar{k}}\mathcal{A}_{i_1,\cdots i_p}\chi^k\right)\chi^{i_1}\cdots\chi^{i_p}\chi^{\bar{j}_1}\cdots\chi^{\bar{j}_q}\,. \end{aligned} \quad (2.53)$$

We can see that χ^i and $\chi^{\bar{i}}$ can be identified with dz^i and $dz^{\bar{i}}$ respectively. Therefore, Q_A acts on the physical operators as an exterior derivative d with respect to the target space X, and The Q_A-cohomology is equivalent to the de Rham cohomology of the target space X,

$$\{ \text{ physical operators } \} \simeq \{H_{Q_A}\} \ , \tag{2.54}$$

and there is a one-to-one correspondence between the physical operators or the Hilbert space of the theory and the topological structure of the target space manifold X. A more geometrical interpretation of this correspondence can be made by the following observation. There is a dual representative class of the Q_A-cohomology class which is determined by the Poincare duality. According to the Poincare duality, for every cohomology class, there is a dual homology class whose basis is given by the homology class $[D] \in H^r(X)$, where r represents the real codimension of the dual homology class. The element of the homology class D acts as a delta function on the dual element of the cohomology r-form $\mathcal{O}_r$. The operator $\mathcal{O}_r$ which is inserted at the point $x \in X$ vanishes if the point x lies outside of the dual homology cycle D, which is nothing but the definition of the delta function. In other words, the insertion points of the physical operators have a meaning of the intersection of the homology cycles. In the guidance of the above setup, the general correlation function can be constructed by means of the twisted fields,

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_s \rangle = \int \mathcal{D}\phi\mathcal{D}\rho\mathcal{D}\chi\mathcal{O}_1 \cdots \mathcal{O}_s e^{-S_A} \, . \tag{2.55}$$

The path integral receives the contribution of all possible maps $\phi : \Sigma_g \to X$ and the whole family of them is called the moduli space. The particular map can be classified by the homology class of the image of the map

$$\beta = \phi_\star[\Sigma_g] \in H_2(X, \mathbb{Z}) \, , \tag{2.56}$$

and the path integral can be rewritten in terms of summing over these topological sectors

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_s \rangle = \sum_{\beta \in H_2(X,\mathbb{Z})} \int_{\phi_\star[\Sigma_g]=\beta} \mathcal{D}\phi\mathcal{D}\rho\mathcal{D}\chi\mathcal{O}_1 \cdots \mathcal{O}_s e^{-S_A} \, . \tag{2.57}$$

Now we can turn to the anomaly problem that we faced before. The A-twisted action respects the classical $U(1)_V$ and $U(1)_A$ symmetries, however, we need to check that quantum effects do not spoil them. More precisely, the correlation functions must respect R-symmetries. Just in the case of the non-linear sigma model, the non-anomalous condition for the vector and axial symmetries can be determined by the index theorem and zero mode analysis. Consider an operator which has holomorphic and anti-holomorphic degrees given by

$$\mathcal{O}_i \to \omega_i \in H^{p_i,q_i}(X)\,. \tag{2.58}$$

The relation between the $U(1)$ charges and degrees of the particular operator is given by $q_V = -p_i + q_i$ and $q_A = p_i + q_i$. This relation comes from reading the R-symmetry charges of the twisted component fields. The vector R-symmetry is not anomalous since we can always set $q_V = 0$, then the non-vanishing condition on the correlation function gives the following constraint $\sum_{i=1}^{s} p_i = \sum_{i=1}^{s} q_i$. On the other hand, the axial R-symmetry begs the scrutiny of the fermion zero modes. The new zero mode condition can be written by looking at the differential operators for the fermions in the twisted action Equation (2.51), $k = \chi_{\text{zero modes}} - \rho_{\text{zero modes}}$ where the instanton number is given by

$$\begin{aligned} k &= \int_{\Sigma_g} \phi^\star c_1(X) + \dim_{\mathbb{C}} X(1-g) \\ &= c_1(X)\cdot\beta + \dim_{\mathbb{C}} X(1-g)\,. \end{aligned} \tag{2.59}$$

The g stands for the genus of the Riemann surface Σ_g. Recall that the axial rotation on the correlation function gives a phase factor $e^{i2k\beta}$, then for a fixed degree for the homology class $\beta = \phi_\star[\Sigma_g]$, one can write $\sum_{i=1}^{s} p_i + q_i = 2k$. Together with the condition on the vector R-symmetry, the resulting selection rule is given by

$$\sum_{i=1}^{s} p_i = \sum_{i=1}^{s} q_i = 2c_1(X)\cdot\beta + 2\dim_{\mathbb{C}} X(1-g)\,, \tag{2.60}$$

where $\dim_{\mathbb{C}}$ means the complex dimension of target space. If X is a Calabi-Yau manifold, then the first Chern class is zero, $c_1(X) = 0$ and the right-hand side turns out to be

$$\sum_{i=1}^{s} p_i = \sum_{i=1}^{s} q_i = 2\dim_{\mathbb{C}} X(1-g)\,. \tag{2.61}$$

After the derivation of the selection rule, we can compute the path integral. Note that the resulting theory is topological and it has deformation invariance. Then the value of the path integral is not sensitive to the background parameters, in other words, classical computation is exact. Therefore, the dominant contributions come from the fixed loci of the fermionic symmetry Q_A since they correspond to the solutions of the equation of motion. This type of phenomenon is called localization. The vanishing of the fermionic symmetry of twisted component fields $\delta\rho^{i}_{\bar{z}}$ and $\delta\rho^{\bar{i}}_{z}$ gives the equations

$$\partial_{\bar{z}}\phi^i = \partial_z\overline{\phi}^{\bar{i}} = 0\,. \tag{2.62}$$

These conditions give rise to the fact that the map $\phi : \Sigma_g \to X$ is holomorphic and they are called worldsheet instantons. The bosonic part of the action at the fixed loci can be expressed as

$$\begin{aligned} S_\phi &= \int_{\Sigma_g} g_{i\bar{j}}\left(\partial_z\phi^i\partial_{\bar{z}}\overline{\phi}^{\bar{j}} + \partial_{\bar{z}}\phi^i\partial_z\overline{\phi}^{\bar{j}}\right)d^2z \\ &= 2\int_{\Sigma_g} g_{ij}\partial_{\bar{z}}\phi^i\partial_z\overline{\phi}^{\bar{j}}\,d^2z + \int_{\Sigma_g}\phi^\star\omega = \int_{\Sigma_g}\phi^*\omega = \omega\cdot\beta\,, \end{aligned} \tag{2.63}$$

where we have used Equation (2.62) at the second line. It can be seen that the bosonic part of the action at fixed loci is nothing but the volume of the image of the genus g Riemann surface Σ_g. Let us return to the selection rule. If we choose the complex dimension of the Calabi-Yau manifold as three, we can see that the first non-trivial example comes from the genus zero Riemann surface which is $\mathbf{S}^2$, and the physical operators $\mathcal{O}_i$ of type $(p = 1, q = 1)$. These operators are identified with d-closed form ω_i. As we have mentioned, the Poincare dual is given by D and the resulting correlation function is just a three-point function. For $\beta = 0$ topological sector, the image of the sphere $\mathbf{S}^2$ is just a point on X and the moduli space of worldsheet instantons is nothing but the Calabi-Yau X itself. According to the geometric interpretation that we have discussed before, there is a relation between the insertion points and the intersection of the homology cycles and this relation gives the three-point function as the intersection of the three distinct divisors D_i which are the holomorphic submanifolds of X, at the trivial topological part. If we include the non-trivial topological maps $\beta \neq 0$, we can get the result

$$\langle\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\rangle = (D_1\cap D_2\cap D_3) + \sum_{\beta\in H_2(X,Z)} I_{0,3,\beta}\left(\omega_1,\omega_2,\omega_3\right)Q^\beta\,. \tag{2.64}$$

The homology class β can be expressed in terms of a basis $\{\beta_i\}$ of the second homology group $H_2(X,\mathbb{Z})$ of the Calabi-Yau threefold. This can be achieved through a mathematical construction resulting in the sum $\sum_i n_i t_i$.

$$t_i = \int_{\beta_i}\omega, \quad i = 1,\cdots,b_2(X)\,, \tag{2.65}$$

where b_i is the second Betti number of X and one can choose another dual basis for the 2-cycles $\{S_j\}$ in $H_2(X,\mathbb{R})$ in a way

$$\int_{S_j}\omega_i = \delta_{ij}\,. \tag{2.66}$$

It is possible to expand β in terms of these new dual bases $\{S_j\}$. Therefore we have

$$\int_\beta \omega_i = \sum_j n_j \int_{S_j} \omega_i = n_i\,. \tag{2.67}$$

As it was mentioned, the first term at (2.64) represents the classical intersection number of the three divisors. The second term uses the relation $Q^\beta = e^{-\omega\cdot\beta} = e^{-\sum_i n_i t_i}$. This second term is the quantum correction to the classical geometry and the coefficient $I_{0,3,\beta}\,(\omega_1,\omega_2,\omega_3)$ is given by

$$I_{0,3,C}\,(\omega_1,\omega_2,\omega_3) = \mathcal{N}_{0,\beta}\int_\beta \omega_1 \int_\beta \omega_2 \int_\beta \omega_3\,. \tag{2.68}$$

The number of holomorphic maps of a genus-zero Riemann surface, Σ, into a two-cycle of homology class β, is denoted by $\mathcal{N}_{0,\beta}$ which counts the genus-zero holomorphic curves that can be embedded into the Calabi-Yau threefold, X. These enumerative invariants are referred to as genus zero Gromov-Witten (GW) invariants and are the topological invariants of X. We can gather all these numbers through the generating function F_0 which is called the A-model prepotential

$$F_0(t_i) = \sum_{\beta\in H_2(X,\mathbb{Z})} \mathcal{N}_{0,\beta}Q^\beta\,, \tag{2.69}$$

and observe that

$$\frac{\partial F_0(t_i)}{\partial t_j} = \sum_{\beta\in H_2(X,\mathbb{Z})} \mathcal{N}_{0,\beta}\frac{\partial}{\partial t_j} e^{-\sum_i n_i t_i} = -n_j F_0(t_i)\,. \tag{2.70}$$

Then one can write that Equation (2.64) directly in terms of the A-model prepotential F_0 by using the fact that $\int_\beta \omega_j = n_j$,

$$\langle \mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\rangle = (D_1\cap D_2\cap D_3) - \frac{\partial^3 F_0}{\partial t_1\partial t_2\partial t_3}\,. \tag{2.71}$$

The A-model prepotential $F_0(t_i)$ can be found by integrating Equation (2.71) through the three-point correlation function along with the computation of the intersection product of the three divisors D_i which can be made by the classical algebraic geometry. Note that the selection rule does not forbid higher genus contribution, for instance, for $g = 1$, all correlation functions vanish except the partition function, and for $g > 1$ all of them are zero. This means that it is not possible to holomorphically embed the Riemann surface Σ_g with $g > 1$ into the Calabi-Yau threefold X while keeping the fixed metric on Σ_g. This problem can be resolved by thinking of all possible metrics on the Σ_g and this type of reasoning leads to coupling the A-model to the worldsheet gravity. Then, the selection rule is modified and we can get higher genus contributions from the generating function $F_g(t_i)$ from the topological string theory. On the other hand, the generating functional of higher genus Gromov-Witten invariants can be defined by summing over instanton sectors

$$F_g(t_i) = \sum_{\beta \in H_2(X,\mathbb{Z})} \mathcal{N}_{g,\beta} Q^{\beta} \,. \tag{2.72}$$

As a final remark, the Gromov-Witten invariants are rational numbers in general, since they include multi-covering and bubbling effects which we will discuss and return these geometric effects when we define another topological invariant, the so-called Gopakumar-Vafa invariants.

2.2.2. B-Model

The fermionic/BRST symmetry charge can be expressed as a scalar when considering the B-twist. Specifically, recall that $\overline{Q}_+$ and $\overline{Q}_-$ also become scalars in this context

$$Q_B = \overline{Q}_+ + \overline{Q}_- \,. \tag{2.73}$$

The fermions $\overline{\psi}_+$ and $\overline{\psi}_-$ transform into scalar fields, while the spinor and anti-spinor fields ψ_- and ψ_+ become holomorphic and anti-holomorphic one-forms according to Table 2.1. Similarly for the A-model, the fields are relabeled to clearly observe the effect of the B-twist.

$$\begin{aligned} \rho^i_z &= \psi^i_-, \quad \rho^i_{\bar{z}} = \psi^i_+, \\ \eta^{\bar{i}} &= -\left(\overline{\psi}^{\bar{i}}_+ + \overline{\psi}^{\bar{i}}_-\right), \\ g^{i\bar{j}}\theta_j &= \overline{\psi}^{\bar{i}}_- - \overline{\psi}^{\bar{i}}_+ \,, \end{aligned} \tag{2.74}$$

and the new supersymmetry transformation can be found by setting $\epsilon_+ = \epsilon_- = 0$ and $\overline{\epsilon}_+ = \overline{\epsilon}_- = \overline{\epsilon}$,

$$\begin{aligned} \delta\rho^i_\mu &= \pm 2i\overline{\epsilon}\partial_\mu \phi^i, \\ \delta\phi^i &= 0, \quad \delta\theta_i = 0, \\ \delta\overline{\phi}^{\bar{i}} &= \overline{\epsilon}\eta^{\bar{i}}, \quad \delta\eta^{\bar{i}} = 0 \,. \end{aligned} \tag{2.75}$$

In order to determine if the B model meets the necessary conditions for being a topological quantum field theory, certain criteria must be considered as before. Firstly, it must possess a fermionic symmetry Q_B invariant vacuum. Secondly, the fermionic symmetry charge Q_B squares to zero, due to all supercharges being nilpotent and $\{\overline{Q}_+, \overline{Q}_-\} = 0$. Finally, the energy-momentum tensor for the B-model can be expressed as Q_B-exact, as the action is Q_B-exact except for an additional term,

$$S_B = \int_\Sigma d^2z \, \{Q_B, V\} + \int_\Sigma W \,, \tag{2.76}$$

where V and W are given by

$$V = g_{i\bar{j}} \left(\rho^i_z \partial_{\bar{z}} \phi^{\bar{j}} + \rho^i_{\bar{z}} \partial_z \phi^{\bar{j}} \right) , \tag{2.77}$$

$$W = \left(-\theta_i D\rho^i - \frac{i}{2} R_{\widetilde{i\bar{i}j}_j j} g^{kj} \rho^i \rho^j \eta^{\bar{i}} \theta_k \right) . \tag{2.78}$$

D is defined as the exterior derivative on genus g Riemann surface and W has no dependence on the worldsheet metric g. These give rise to the fact that the energy-momentum tensor for the B-model is Q_B-exact. According to the [2], the effect of the transformation of the Kähler metric on the W can be rewritten as $\delta W = \{Q_B, \cdots\}$, then the action is independent of the Kähler moduli of X in contrast to the A-model. However, the B-model depends on the complex structure moduli of X since the infinitesimal symmetry transformations depend on it.

The correlation functions are determined from $\phi^i, \overline{\phi}, \eta^{\bar{i}}$ and θ_i and the Hilbert space of theory is identified with the Q_B-cohomology. Then, there is a correspondence between the geometric data of the Calabi-Yau threefold X and the twisted component fields with the following identifications

$$\begin{aligned} \eta^{\bar{i}} &\longleftrightarrow d\overline{z}^{\bar{i}} \,, \\ \theta_i &\longleftrightarrow \frac{\partial}{\partial z^i} \,. \end{aligned} \tag{2.79}$$

The most general physical operator is constructed out of the component fields

$$\mathcal{B}^{j_1 \cdots j_q}_{i_1 \cdots i_p}(\phi) \eta^{\bar{i}_1} \cdots \eta^{\bar{i}_p} \theta_{j_1} \cdots \theta_{j_q} \,. \tag{2.80}$$

One can show that the above physical operator is Q_B-closed and it acts an anti-holomorphic p-form which takes the values in $\wedge^q TX$

$$\mathcal{B}^{j_p \cdots j_q}_{i_1 \cdots i_p}(\phi) d\overline{z}^{\bar{i}_1} \cdots d\overline{z}^{\bar{i}_p} \frac{\partial}{\partial z^{j_1}} \cdots \frac{\partial}{\partial z^{j_q}} \,. \tag{2.81}$$

$\wedge^q TX$ denotes the q^{th} exterior power of the holomorphic tangent bundle T_X and the above physical operator is $\overline{\partial}$ closed. Therefore, we can conclude that the the Hilbert space of the physical operators is given the Dolbeault cohomology group of X

$$\{\text{physical operators }\} \simeq \bigoplus_{p,q=0}^{n} H^{0,p}\left(X, \wedge^{q} TX\right) , \tag{2.82}$$

where n is the dimension of the X. If we go back to the correlation function, the most general correlation function for the operators $\{\mathcal{O}_i\}$ is given by inserting the twisted fields through the path integral

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_s \rangle = \int \mathcal{D}\phi \mathcal{D}\eta \mathcal{D}\theta \mathcal{O}_1 \cdots \mathcal{O}_s e^{-S_B} . \tag{2.83}$$

Similarly for the A-model, the B-model satisfies the conditions for being topological quantum field theory and it is exact in the sense of being theory is independent of the background parameters and we can make the exact computations at the classical level. Therefore, we can compute the path integral at loci of Q_B and they are given by setting the $\delta\rho = 0$ which results in

$$\partial_\mu \phi^i = 0 . \tag{2.84}$$

The solution of the above equation gives the constant function and the whole space of maps from the Riemann surface to the Calabi-Yau X is nothing but the constant map $\phi : \Sigma_g \to X$. Therefore, we can see that the path integral is equivalent to an integral over X. This observation gives rise that the A-model and the B-model are different, the fixed loci of the B-model is just the Calabi-Yau manifold X while the fixed loci of the A-model is the moduli space of all holomorphic maps as we have seen. Hence, the computations at the B-model is simpler than the A-model. We can elobarate again the $U(1)$ R-symmetries, the $U(1)_V$ gives $\sum_{i=1}^{s} p_i = \sum_{i=1}^{s} q_i$. and $U(1)_A$ anomaly gives $\sum_{i=1}^{l} p_i + q_i = 2\text{dim}_{\mathbb{C}} X(1-g)$ differently, then the selection rule for the B-model turns out to be

$$\sum_{i=1}^{s} p_i = \sum_{i=1}^{s} q_i = 2\text{dim}_{\mathbb{C}} X(1-g) . \tag{2.85}$$

So unlike the A-model, there is no Chern class term that determines the underlying target space X. As an example, we can consider $g = 0$ along with the selection rule for s physical operators $\mathcal{O}_i(i = 1, \cdots, s)$. One can choose X to be a Calabi-Yau threefold,$\text{dim}_{\mathbb{C}} X = 3$. This condition gives that the line bundle or the corresponding cohomology group $H^{3,0}(X)$ is trivial and there exist a nowhere vanishing closed holomorphic $(3,0)$-form $\Omega = \sum \Omega_{ijk} dz^i \wedge dz^j \wedge dz^k$ globally. Consider the physical operator which corresponds to the complex structure deformations of the moduli space of the Calabi-Yau threefold X. The infinitesimal deformation of the complex structure of

Calabi-Yau X is determined by the the Beltrami differentials, $\mu_a \in H^{0,1}(X, TX)$ and they are given as

$$\mu_a = (\mu_a)^k_{\bar{j}} \, d\bar{z}^{\bar{j}} \frac{\partial}{\partial z^k} \,. \tag{2.86}$$

Thanks to the selection rule and the above mathematical machinery, the three-point function can be attained from the θ zero modes in the path integral [1]

$$\begin{aligned} \langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \rangle &= \int_X \langle \mu_1 \wedge \mu_2 \wedge \mu_3, \Omega \rangle \wedge \Omega \\ &= \int_X (\mu_1)^i_{\bar{i}} \, (\mu_2)^j_{\bar{j}} \, (\mu_3)^k_{\bar{k}} \, \Omega_{ijk} d\bar{z}^{\bar{i}} d\bar{z}^{\bar{j}} d\bar{z}^k \wedge \Omega \,. \end{aligned} \tag{2.87}$$

It is clear that the $\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \rangle$ does depend only on the complex structure moduli instead of the Kähler moduli. Just in the case of the A-model, there is a connection between the three-point function and the prepotential of the $B-$model F_0. As it understood that there are two distinct moduli spaces for Calabi-Yau manifolds: the Kähler moduli space $\mathcal{M}_K$ and the complex structure moduli space $\mathcal{M}_K$. The latter $\mathcal{M}_C$ is the space of all possible complex deformations on the Calabi-Yau manifold. To be more clear, a point on $\mathcal{M}_C$ represents a fixed complex structure on X and at that particular point where nowhere vanishing holomorphic form Ω sits. The variation of complex structure has the line bundle $\mathcal{L}$ structure since as we move along the $\mathcal{M}_C$, the holomorphicity of Ω changes point by point and at a different point, we have a new holomorphic form $\widetilde{\Omega}$. Therefore, this type of construction is nothing but the line bundle and this line bundle is characterized by complex structure deformations. Note that all different Ω's still be closed due to the fact that the exterior derivative that we introduced is independent of the complex structure deformations of the Calabi-Yau. The metric on this line bundle $\mathcal{L}$ is given by

$$h_{\mathcal{L}} = ||\Omega||^2 = i \int_X \Omega \wedge \overline{\Omega} \,, \tag{2.88}$$

where $\Omega \wedge \overline{\Omega}$ is a $(3,3)$ form and its integral along the X does not give zero globally. Note that Ω is defined up to a nonzero scaling function, it is possible to multiply Ω with a globally non-zero holomorphic function e^f. This gives the fact that the metric transform as $h_{\mathcal{L}} \to \left||e^f\right||^2 h_{\mathcal{L}}$ and this type of transformation can be achieved by introducing the Kähler potential

$$K = -\log ||\Omega||^2 = -\log \int_X \Omega \wedge \overline{\Omega} \,. \tag{2.89}$$

Note that K is transformed as $K \to K - f - \overline{f}$ and then by this Kähler potential, we can construct a globally well-defined Kähler metric

$$g_{a\bar{b}} = \frac{\partial^2 K}{\partial t_a \partial \bar{t}_b}\,, \tag{2.90}$$

on $\mathcal{M}_C$. Therefore the complex structure moduli space $\mathcal{M}_C$ is Kähler manifold. This type of relationship is sometimes called special geometry [9, 11].

Now we can turn to another mathematical construction which is called a variation of the Hodge structure. According to the Hodge decomposition theorem, the vector space $H^3(X, \mathbb{C})$ contains $H^{3,0}(X)$ and from this, the change in the Ω by moving along the complex structure moduli space $\mathcal{M}_C$ can be understood as the vector bundle. To be more precise, the vector spaces $H^3(X, \mathbb{C})$ which sit at the points of $\mathcal{M}_C$ has a vector bundle structure $\mathcal{V}$ on $\mathcal{M}_X$ and this vector bundle is called Hodge bundle. The Hodge bundle also includes the line bundle $\mathcal{L}$ as a sub-bundle. Then the variation of the Hodge structure can be stated as the variation of the line bundle $\mathcal{L}$ which is a sub-bundle of $\mathcal{V}$, or equivalently variation of the Hodge decomposition on the Calabi-Yau threefold X. This can be accomplished by using a special type of coordinates, the so-called period, on the line bundle $\mathcal{L}$. Similarly for the line bundle that we have defined above, there is also a Hermitian metric on the vector $\mathcal{V}$ which is given by the following inner product

$$(\mu, \nu) = i \int_X \mu \wedge \overline{\nu}\,, \tag{2.91}$$

where $\mu, \nu \in H^3(X, \mathbb{C})$. Apart from that, one can write the symplectic-basis of the by the following integral of three-forms α_a, β^b where a, b runs over from 1 to $h^3(X)/2 = h^{2,1}(X) + 1$. h^i is the i^{th} Hodge number of the Calabi-Yau threefold X and we have used the Hodge decomposition theorem. These three forms satisfy

$$(\alpha_a, \alpha_b) = (\beta^a, \beta^b) = 0, \quad \text{and} \quad (\alpha_a, \beta^b) = i\delta_a^b\,, \tag{2.92}$$

and these basis are invariant under the symplectic transformations $\mathrm{Sp}\,(2h^{2,1} + 2, \mathbb{R})$. One can also introduce the Poincaré duals of α_a, β^b given by A^a, B_b. Then the periods can be written as

$$q^a = \int_{A^a} \Omega, \quad p_b = \int_{B_b} \Omega\,. \tag{2.93}$$

This gives the fact that it is possible to expand Ω in terms of the symplectic basis,

$$\Omega = q^a \alpha_a - p_b \beta^b\,. \tag{2.94}$$

The periods (q^a, p_b) depend on the complex structure of the moduli space of X and they are functions defined on $\mathcal{M}_C$. They are not quite independent from each other since the dimension of the complex structure moduli space $\mathcal{M}_C$ is $h^{2,1}$ rather than the $2h^{2,1}+2$ that is the correct degrees of freedom for the symplectic group $\mathrm{Sp}\,(2h^{2,1}+2, \mathbb{R})$. Therefore, the p_a's are nontrivial functions of q^a or vice versa. Furthermore, q^a's are called complex projective coordinates at the complex geometry, and by using this fact we can eliminate one of them. Let us choose q', to introduce the inhomogeneous ones

$$t_a = \frac{q^a}{q'}, \quad a = 1, \cdots, h^{2,1}\,, \tag{2.95}$$

where we resummed to $h^{2,1}$ instead of $h^{2,1}+1$. Hence the periods are the coordinates on the line bundle $\mathcal{L}$ from the variation of the Hodge structures. In order to express the three-point function through the B-model prepotential F_0, one can use Griffith transversality relation from the algebraic geometry

$$\int_X \Omega \wedge \frac{\partial \Omega}{\partial p^a} = 0\,. \tag{2.96}$$

This can be shown by taking the first order of variation of Ω

$$\begin{aligned} \frac{\partial \Omega}{\partial t_a} &= (3,0) \text{ form } + (2,1) \text{ form} \\ &= k_a \Omega + \chi_a\,. \end{aligned} \tag{2.97}$$

The only surviving term χ_a is in the type of $(2,1)$ at the leading order since the variation of a holomorphic $(1,0)$-form dx brings along $(1,0)$ and $(0,1)$ pieces and k_a are some holomorphic functions on $\mathcal{M}_C$. Furthermore, one can introduce the prepotential F_0 as

$$F_0 := \frac{1}{2} p_a q^a\,, \tag{2.98}$$

then we can write $p_c = \partial F_0 / \partial q^c$ and this gives rise to F_0 being homogeneous of degree 2 function with respect to the q^c. Moreover, it is not hard to see that there is a one-to-one correspondence between the χ_a and the Beltrami differential μ_a if we contract the Beltrami differentials with the nowhere vanishing holomorphic $(3,0)$-form Ω,

$$\mu_a = (\mu_a)^k_{\bar{j}}\, d\bar{z}^{\bar{j}} \frac{\partial}{\partial z^k} \longleftrightarrow \chi_a = (\mu_a)^k_{\bar{j}}\, \Omega_{kmn} d\bar{z}^{\bar{j}} \wedge dz^m \wedge dz^n\,. \tag{2.99}$$

To find the relation between the prepotential and the three-point function, consider the term

$$\int_X \Omega \wedge \frac{\partial^3 \Omega}{\partial t_1 \partial t_2 \partial t_3}\,, \tag{2.100}$$

and by using the fact that Equation (2.99), the above expression can be written as

$$\frac{\partial^3 \Omega}{\partial t_1 \partial t_2 \partial t_3} = (3,0)\text{form} + (2,1) \text{ form } + (1,2) \text{ form } + (0,3) \text{ form }\,. \tag{2.101}$$

Therefore, $(0,3)$ form is nothing but $(\mu_1)^i_{\bar{i}}\,(\mu_2)^j_{\bar{j}}\,(\mu_3)^k_{\bar{k}}\,\Omega_{ijk}d\bar{z}^{\bar{i}}d\bar{z}^{j}d\bar{z}^{k}$ since it contains the 3 anti-holomorphic coordinates. Furthermore, there is no other term rather than this term that contributes to the integral, since the integral is taken over the Calabi-Yau threefold X whose complex dimension is three. Consequently, this term is equal to the three-point function for the B-model. Apart from this observation, one can show that by using the Equation (2.94) and the $F_0 = p_a q^a/2$ the above expression can also be rewritten as

$$\int_X \Omega \wedge \frac{\partial^3 \Omega}{\partial t_1 \partial t_2 \partial t_3} = \frac{\partial^3 F_0}{\partial t_1 \partial t_2 \partial t_3}\,. \tag{2.102}$$

On that account, the three-point function of the B-model is equivalent to the third derivative of the F_0 with respect to the inhomogeneous coordinates t^a

$$C_{123} = \langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \rangle = \frac{\partial^3 F_0}{\partial t_1 \partial t_2 \partial t_3}\,. \tag{2.103}$$

Note that there is an important difference between the A and B-models according to the above construction. In the B-model there are no infinitely many worldsheet instanton corrections to the three-point function or the three-point function can be expressed as purely classical geometric data contrary to the A-model since F_0 encodes the all information about them. Although the computations at the B-model is plainer than the A-model, the A-model is much more interesting than its counterpart since it includes the quantum corrections in the classical geometry.

2.3. Topological String Theory

So far, we have only been interested in fixed two-dimensional background field theories. On the other hand, we can take the background to be dynamical rather than fixed and it plays a huge role in the resulting physics. The ordinary string theory has such a property where the worldsheet metric enters the path integral of the string theory and one has to consider not only the integration of the path integral over the maps ϕ^i to the target space and the fermions but also a path integral over the degrees of freedom of the worldsheet metric $h_{\mu\nu}$. Furthermore, We have also seen that $\mathcal{N} = (2,2)$ supersymmetric nonlinear sigma model along with the twisted versions have not enough information about the geometry of target spacetimes since the selection rules constrained all correlation functions in the sense that only some of them are non-vanishing, namely partition function at $g = 1$ and the three-point function at $g = 0$. Therefore, in order to resolve this issue, the A-model and the B-model should be coupled with two-dimensional gravity. The resulting theory is called $\mathcal{N} = (2, 2)$ topological string theory which is an analog of the usual string theories. Consider a general quantum field theory but its background metric is dynamical, or quantum gravity, and one needs to take into account the fact that the path integral is integrated over the space of all metrics. Gravity can be considered as some sort of gauge theory and the idea of the Faddeev-Popov mechanism is valid as we will see. There are at least two possible considerations when gravity is quantum mechanically dynamical. Firstly, the action or the Lagrangian of the theory must be written in a covariant way, which means that ordinary derivatives turn out to the covariant derivatives and spacetime volume factor changes by

$$\int d^d x \longrightarrow \int d^d x \sqrt{\det h}\,, \tag{2.104}$$

and it is necessary to include the Einstein-Hilbert term in the action, along with some diffeomorphism invariant terms. To proceed, the next step involves integrating the path integral of the final theory over all possible background metrics.

For our purpose, we will not be interested in covariance-related issues, then our main concern is integrating the path integral measure. Similarly, for the gauge theories, diffeomorphism invariance puts a volume factor in front of the path integral as diffeomorphisms are redundant. Therefore, it is important to ensure that each configuration is counted only once. The path integral is written as

$$\frac{1}{\mathrm{Vol}(G)} \int \mathcal{D}h_{\mu\nu} Z[h] \stackrel{?}{=} Z[h^0_{\mu\nu}]\,, \tag{2.105}$$

where $h^0_{\mu\nu}$ stands for the arbitrary background metric and $\mathrm{Vol}(G)$ is the volume of the gauge transformations, in our case, they are diffeomorphisms. On the other hand, this construction can be problematic since there can be large diffeomorphisms resulting that there can be some metric configuration that cannot be obtained by the continuous transformations. Secondly, there can be anomalies leading to breaking the validity of the conclusion that all the configurations in a gauge group orbit are equivalent. Furthermore, conformal anomalies can also occur. For instance, in the usual bosonic string theory, the two-dimensional non-linear sigma model has an additional conformal symmetry and this makes the energy-momentum tensor traceless. This case should be taken into consideration when evaluating the path integral. Accordingly, we can conclude that there are two types of symmetry factors associated with two-dimensional diffeomorphisms and conformal symmetry. Firstly one has to integrate over all conformally equivalent metrics and after that integrate over the quotient space which is known as the Teichmüller space. Note that the two-dimensional conformal group is large, however, Teichmüller space is finite-dimensional.

Before we move on to the effect of the topological twisting on these issues, we will first study them alone and we will begin with the first one, a conformal anomaly from an algebraic point of view. As we have mentioned, there are four different conserved charges for the sigma model on the plane, the first conserved Noether current, an integrated version of the Hamiltonian, associated with global translations on $\mathbb{C}$ is the energy-momentum tensor $T_{\mu\nu}$. According to the conformal field theory, the energy-momentum tensor satisfies the traceless condition $T_{z\overline{z}} = T_{\overline{z}z} = 0$, and it is also conserved $\partial_\mu T^\mu_\nu = 0$. These two conditions give the fact that T_{zz} and $T_{\overline{zz}}$ depends on z and $\overline{z}$ respectively, in other words $T_{zz} \equiv T(z)$ is holomorphic while $T_{\overline{zz}} \equiv \overline{T}(\overline{z})$ is anti-holomorphic. In two dimensions, the mode expansion of the energy-momentum tensor T_{zz} and its anti-holomorphic counterpart can be made by the Laurent expansion

$$T(z) = \sum L_m z^{-m-2}, \quad \overline{T}(\overline{z}) = \sum \overline{L}_m \overline{z}^{-m-2}\,, \tag{2.106}$$

where L_m are the generators of Virasoro algebra. They satisfy the following commutation relations, central extension of the Witt algebra of the conformal field theory

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n}\,. \tag{2.107}$$

c is called a central charge and it is a quantum mechanical effect. There is a relation between the central charge and the conformal anomaly as we will see. As we have remarked above, the conformal symmetry puts a constraint on the energy-momentum tensor

$$\frac{\delta S}{\delta h^{\mu\nu}} = T_{\mu\nu} = 0\,. \tag{2.108}$$

Furthermore, this restriction must act as an operator on the Hilbert space of the conformal field theory after the quantization, or in other words, the physical states $|\psi\rangle$ must satisfy

$$L_m|\psi\rangle = 0 \quad \forall m \in \mathbb{Z}\,. \tag{2.109}$$

It can be seen that this condition cannot be satisfied due to the central charge term, the right-hand-side gives zero trivially while the central charge term gives non-zero contribution $\delta_{m+n}|\psi\rangle \neq 0$. Thus the central charge must be zero in order not to have a conformal anomaly. This type of conformal anomaly can be gotten rid of by means of choosing the dimension of the target spacetime for the sting theory. Now, we can return to the topological twisted theories. As we have discussed that there are further the global R-symmetries of the $\mathcal{N}=(2,2)$ theory. The same consideration above can be made for the global $U(1)$ R-symmetries. The corresponding Noether current for the global $U(1)$ R-symmetries J_μ. Similarly, due to the conservation law, J_z is holomorphic while $J_{\overline{z}}$ is anti-holomorphic. This can be also understood from the closed string theory point of view in which the left-moving and the right-moving modes are decoupled. Then, we can separate the global $U(1)$ R-symmetries as the left- and right-moving terms with the corresponding Noether charges F_L and F_R. Similarly, we can make the expansion as

$$J(z) = \sum J_m z^{-m-1}\,, \tag{2.110}$$

with the corresponding Virasoro Algebras

$$\begin{aligned} [L_m, L_n] &= (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n}\,, \\ [L_m, J_n] &= -nJ_{m+n}\,, \\ [J_m, J_n] &= \frac{c}{3}m\delta_{m+n}\,. \end{aligned} \tag{2.111}$$

The central charge c breaks the classical commutation relations in the J- and the L-commutators. The conserved current for R-symmetries can be obtained as a contour integral,

$$J \equiv F_L = \oint_{z=0} J(z)dz = 2\pi i J_0\,. \tag{2.112}$$

By using this and the similar for the right movers' mode, we can calculate the new Lorentz conserved charges by the topological twisting which we have introduced

$$\begin{aligned} M_A^E &= M + F_V = F_L + F_R\,, \\ M_B^E &= M + F_A = F_L - F_R\,, \end{aligned} \tag{2.113}$$

where M is the generator of Lorentz transformations given by $M = 2\pi i(L_0 - \overline{L}_0)$ from the two dimensional conformal field theory. Note that this conserved charge is actually associated with the translation invariance. However, it can also be interpreted as the rotations at z-plane since z-plane is equivalent to the cylinder under the compactification of the time direction, where time direction is simply $\mathbf{S}^1$. Combining M with topological twisting results in the fact that the energy-momentum tensor is modified by

$$\begin{aligned} &\text{A-model}: \quad T \to \widetilde{T} = T_A = T + \frac{1}{2}\partial J \quad ,\overline{T} \to \widetilde{\overline{T}} = \overline{T}_A = \overline{T} - \frac{1}{2}\overline{\partial J}\,, \\ &B\text{-model}: \quad T \to \widetilde{T} = \overline{T}_B = T - \frac{1}{2}\partial J \quad ,\overline{T} \to \widetilde{\overline{T}} = \overline{T}_B = \overline{T} - \frac{1}{2}\overline{\partial J}\,. \end{aligned} \tag{2.114}$$

From this perspective, it can be also seen that the spin of the field is changed and the new energy-momentum tensors are still conserved, $\overline{\partial}\widetilde{T} = 0$ due to the holomorphicity. Furthermore, the Virasoro algebra is modified and they are given by the new modes $\widetilde{L}_m$

$$[\widetilde{L}_m, \widetilde{L}_n] = (m-n)\widetilde{L}_{m+n}\,. \tag{2.115}$$

Finally, we can conclude that the topological twisting mechanism causes the new Virasoro algebra to be modified and the new central charge is zero. The topological string theory is free of conformal anomaly and it is defined on any dimension of the target manifold.

The second step involves the understanding of the integration of the path integral over all possible metrics. As in the case of the string theory, the integration of all possible metrics is reduced to the conformal equivalence metrics. As we have mentioned, this can be understood that conformal equivalence classes are mapped to the possible complex structures of the Riemann surfaces Σ_g with a fixed genus. Therefore the moduli space of all possible metrics is nothing but the moduli space of complex structures on the Riemann surface Σ_g and it is denoted as $\mathcal{M}_g$. The dimension of $\mathcal{M}_g$ can be computed by the Riemann-Roch theorem and for $g = 0$, it is zero-dimensional, and for $g = 1$ the complex dimension of the moduli space is one-dimensional. The corresponding moduli space is the fundamental domain of the torus. For genus is greater than one, the dimension of the moduli space of complex structures on the Riemann surface Σ_g is given by

$$\dim_{\mathbb{C}}\mathcal{M}_g = 3g - 3\,. \tag{2.116}$$

After we gave a little summary of what we need for the effects of coupling two-dimensional gravity to the twisted sigma models, we can return to our original issue. The bosonic string theory is a two-dimensional conformal field theory with a fermionic BRST current j_{BRST}, which has a spin 1 with the associated charge

$$Q_{BRST} = \oint_{\Sigma} j_{BRST} d^2 z, \quad \text{with} \quad Q_{BRST}^2 = 0\,. \tag{2.117}$$

This type of BRST symmetry is called ghost symmetry which is generated by Q_{BRST} and it results in a ghost number anomaly $3g-3$ on the Riemann surface Σ_g. Moreover, in addition to the field X^{μ} on the Riemann surface, there exists a spin 2 antighost field b with the charge under the ghost symmetry is -1 with its zero modes satisfies $b_0^2 = 0$. The energy-momentum tensor for the ghost and the matter part of the conformal field theory is given by the anti-ghost field b

$$T(z) = \{Q_{BRST}, b(z)\}\,. \tag{2.118}$$

Note that we only discussed the left-mover sector, the right-mover part is straightforward, and all of the physical quantities are replaced by the barred ones. Note that we used the same symbols for the fields b and their corresponding zero modes b_0. The more precise statement is that the energy-momentum tensor is $T(z) = \{Q_{BRST}, b_0(z)\}$.Just like the topological field theory that we have discussed, the physical operators of the bosonic string theory are determined by the cohomology group of Q_{BRST}. As we have formulated, The moduli space of such a complex curve is a complex $3g-3$ dimensional space $\mathcal{M}_g$. The infinitesimal transformations on the complex structure are associated by with Beltrami differentials $\mu_{\bar{z}}^{z}$ which are nothing but the tangent vectors on this Riemann surface with genus g. The space of Beltrami differentials have a complex dimension $3g-3$ for $g > 1$. Now, one can define the following holomorphic and anti-holomorphic measure on the $\mathcal{M}_g$ by using the fact that b is has spin 2

$$\int_{\Sigma} b(\mu_k) \equiv \int_{\Sigma} b_{zz} (\mu_k)_{\bar{z}}^{z}\,. \tag{2.119}$$

Therefore, the free energy of the bosonic string theory at genus $g > 1$ is defined by

$$F_g = \int_{\mathcal{M}_g} \langle \prod_{k=1}^{3g-3} (\int_{\Sigma} b(\mu_k))(\int_{\Sigma} \bar{b}(\bar{\mu}_k)) \rangle_{\Sigma_g}\,, \tag{2.120}$$

where $\langle \cdots \rangle_{\Sigma_g}$ stands for the conformal field theory correlators. Note that the integration is meaningful since the dimension of the $\mathcal{M}_g$ and the total number of $b(\mu_k)$ which is integrated over the $\mathcal{M}_g$ are matched. Nevertheless, the ghost current must have an anomaly to have nontrivial F_g since there are $3g-3$ anti-ghost b fields which create a ghost anomaly in the definition of F_g. This anomaly must be canceled by a total $3g-3$ ghost charge and these $3g-3$ units of ghost charge are already included in the path-integral measure of the CFT correlator at Equation (2.120). Thus the whole construction of F_g makes sense and the total ghost charge is zero. Now, we can use the machinery above to discuss the twisted non-linear sigma model. Recall that we constructed the conserved currents from the Noether charges that we mentioned before, there are four(left-mover) conserved currents

$$T, J, G^+, G^- \,, \tag{2.121}$$

where the conserved currents are energy-momentum tensor, $U(1)$ R-symmetry current and $G^\pm$ are the two supercurrents with charge ± 1 under J and their spins are given by $2, 1$ and $\frac{3}{2}$, respectively. Recall that the energy-momentum tensor can be twisted by the amount of

$$T \to \widetilde{T} = T - \frac{1}{2}\partial J \,, \tag{2.122}$$

for the B-twist. We discussed that $\widetilde{T}$ satisfies the Virasoro algebra and twisting from T to $\widetilde{T}$ changes the spin of every field by $-1/2$ its global $U(1)$ R-symmetry charge. Then, according to the twisted energy-momentum tensor, the spin of supercurrent G^+ becomes 1 while the spin of G^- turns out to be 2. Remind that twisted energy-momentum tensor has no central charge, similar to the bosonic string theory in which the central charge of the underlying conformal field theory is canceled by the contributions of the ghost fields. By motivating with these observations, it is natural to make the following identifications between the bosonic string theory and the twisted $\mathcal{N}=(2,2)$ supersymmetric non-linear sigma model

$$\begin{aligned} T &\to \widetilde{T} \,, \\ j_{BRST} &\to G^+ \,, \\ b &\to G^- \,, \\ j_{ghost} &\to J \,. \end{aligned} \tag{2.123}$$

Since these two algebra is isomorphic, then the free energy of topological string theory can be defined as

$$F_g = \int_{\mathcal{M}_g} \langle \prod_{k=1}^{3g-3} (\int_\Sigma G^-(\mu_k))(\int_\Sigma \overline{G}^-(\overline{\mu}_k)) \rangle_{\Sigma_g} \,. \tag{2.124}$$

To determine whether vector and axial anomalies occur in the given definition, we must first express the measure in terms of the supercurrents explicitly as

$$\int_\Sigma G(\mu_k) \equiv \int_\Sigma G_{zz}(\mu_k)^z_{\bar{z}}, \qquad \int_\Sigma \overline{G}(\overline{\mu}_k) \equiv \int_\Sigma G_{\overline{zz}}(\overline{\mu}_k)^{\bar{z}}_z\,, \tag{2.125}$$

where G_{zz} and $G_{\overline{zz}}$ are the corresponding supercurrents for the supercharges Q_+ and $\overline{Q}_-$, respectively. They also generate the holomorphic and anti-holomorphic piece of the energy-momentum tensor

$$T_{zz} = \{Q_B, G_{zz}\}\,, \quad T_{\overline{zz}} = \{Q_B, G_{\overline{zz}}\}\,. \tag{2.126}$$

Therefore, we can read their vector and axial charge from the supercharges Q_+ and $\overline{Q}_-$ and both of them have axial charge is -1, while Q_+'s vector charge is -1 and $\overline{Q}_-$'s vector charge is 1. Then the insertion of G_{zz} and $G_{\overline{zz}}$ into the path integral gives a net zero vector charge, so F_g is free of the vector R-symmetry anomaly. On the other hand, the total axial charge is non-zero since the axial charge of the F_g is $(3-3g, 3-3g)$ with respect to the holomorphic and anti-holomorphic sector, total there are $6-6g$ axial charges. It can be seen that the ghost charge is mapped to the axial charge and similarly for the bosonic string, in order not to have trivially zero free energy of the topological string theory, the measure of F_g has the right number of anomalies which cancels the axial charge anomaly. Then, we find that the selection rule for the B-model is modified

$$\sum_{i=1}^s p_i = \sum_{,i=1}^s q_i = 6(g-1) - 2\mathrm{dim}_{\mathbb{C}}X(g-1)\,. \tag{2.127}$$

It can be seen that if we choose the target space that has a complex dimension of three the axial charge of the measure is zero for any g, and hence the B-model partition function or free energy is non-trivial for every genus $g>1$. On the other hand, the selection rule for the A-model becomes

$$\sum_{i=1}^s p_i = \sum_{,i=1}^s q_i = 6(g-1) - 2\mathrm{dim}_{\mathbb{C}}X(g-1) + 2\int_{\Sigma_g} \phi^\star(c_1(X))\,. \tag{2.128}$$

If we choose the target space as the Calabi-Yau threefold, $c_1(X)=0$ and $\mathrm{dim}_{\mathbb{C}}X=3$, we find that exactly the same condition at $g>1$. For $g=0$ and $g=1$ cases that need special care which we shall discuss later. We can also write the F_1 for the B-model given by

$$F_1 = \frac{1}{2}\int \frac{d^2\tau}{\tau_2}\,\mathrm{Tr}(-1)^F F_L F_R q^{H_L}\overline{q}^{H_R}\,, \tag{2.129}$$

where $q = e^{2\pi i\tau}$. The integration over the fundamental domain is the moduli space of $\mathbb{T}^2$

and (H_L, H_R) denote the Hamiltonian of the left and right moving sectors, respectively. The form of F_1 can be deduced by the insertion of the one physical operator $\mathcal{O}$ which has axial charge $(1,1)$ in order to cancel the $(-1,-1)$ axial charge from the insertion of $(G_{zz}, G_{\overline{zz}})$. The B-model free energy for higher genus $F_{g\geq 2}$ can be determined by the "holomorphic anomaly equation" which we will not discuss [11]. The story for the A-model free energy for the $g = 0, 1$ and $g \geq 2$ is more subtle since the worldsheet instantons complicate their calculations of them.

2.4. Target Space Interpretation and Gopakumar-Vafa Invariants

In this part, we will give a different approach to the topological string theory. All we have done so far is construct the topological string theory from two-dimensional field theories on Riemann surface Σ, however, it is also possible to find the topological string theory amplitudes from the target space where the two-dimensional field theory lives on. This type of interpretation lies on the superstring theory compactifed on the ten-dimensional background which is a type of Calabi-Yau threefold is fibered over the four-dimensional spacetime. There are five different string theories that are related by dualities and according to their low energy particle content, they are called Type I, Type II-A, Type II-B, Heterotic $SO(32)$ and Heterotic $E_8 \times E_8$. These theories live in ten-dimensional spacetime and eleven-dimensional M-theory gives a unifying perspective in a way that all of these theories can be seen as special configurations of space of all possible and consistent string theories.

In particular, we will be interested in Type II-A and Type II-B. They are both closed string theories with left-moving and right-moving sectors with an additional supersymmetry $\mathcal{N} = (1,1)$ and $\mathcal{N} = (2,0)$ on their worldsheet. They are similar to the bosonic string in a way that they can be considered as all possible maps $\Sigma \to M^{10}$ of Riemann surfaces to some ten-dimensional Poincare invariant spacetime M^{10}. The excitations of the string correspond to particle states which live in ten-dimensional Poincare invariant spacetime M^{10}, and there are infinitely many particle states and each excitation is determined by the mass of the string. Apart from the infinitely many particle species which sit on the string theory spectrum unlike the usual field theory, there is a further crucial difference between them. One of the excitations of the string theory, so-called dilaton ϕ, determines the interaction between the strings in a way that $e^{\langle\phi\rangle} = \lambda$, where λ is the string coupling constant and $\langle\cdots\rangle$ stands for the vacuum expectation value of the dilaton. Now, we can give a more precise meaning of compactification. Consider the following manifold which is the product of a four-dimensional Minkowski space with a six-dimensional compact internal manifold which is general Calabi–Yau threefold

$$\mathbb{R}^{3,1} \times X\,. \tag{2.130}$$

Then we can make the Kaluza-Klein ansatz for the ten-dimensional background metric as

$$ds^2 \;= \eta_{\mu\nu}(x)dx^{\mu}dx^{\nu} + g_{ij}(y)\,dy^i dy^j\,, \tag{2.131}$$

where $\eta_{\mu\nu}, \mu, \nu = 0, \ldots, 3$ is a four-dimensional Minkowski metric and $g_{ij}, i, j = 1 \ldots 3$ is the Calabi-Yau metric. Part of the four-dimensional fields arise as variations around this background metric of the Calabi–Yau threefold, if we assume that the internal compact manifold X is small compared with the four-dimensional spacetime and the total spacetime can be seen as macroscopically four-dimensional. The geometric data of the Calabi-Yau threefold encodes the particle content as well as the interactions between them as we will see. For simplicity, we can consider a single massless scalar field ϕ in ten dimensions that satisfy the following Laplace's equation:

$$\nabla_{(10)}\Phi(x^{\mu}, y^i) = 0\,, \tag{2.132}$$

where x^{μ} are coordinates on four-dimensional spacetime and y^i are coordinates on the internal space. We can separate the Laplacian operator as a four-dimensional part and a six-dimensional part, we can write

$$(\nabla_{(4)} + m^2)\Phi(x) = 0\,, \tag{2.133}$$

with the assumption we made $\Phi(x, y) = \Phi_{(4)}(x)\Phi_{(6)}(y)$, it can be seen that the above equation is nothing but the Laplace equation on the internal space and $\Phi_{(4)}$ is an eigenfunction with corresponding eigenvalue m^2. Moreover, four dimensional observer interprets Equation (2.133) as a field equation for $\Phi_{(4)}$ with the mass m. The Laplacian operator scales as $1/R^2$, where R determines the size of the internal space, then it is possible to ignore the effect of the massive modes when we take the internal space as sufficiently small. On the other hand, there are also massless modes which are survived after taking the internal space is not large as the four-dimensional spacetime and these modes are the zero modes of the Laplacian operator of the internal space X. The other types of particles, gauge fields, fermions, and gravity can be analyzed by the same method, they are zero modes of certain differential operators on the internal space.

Now, we can be more specific, and consider ten-dimensional Type II-A supergravity, which is a low energy limit of Type II-A string theory, or in other words, it contains the massless modes of the Type II-A string theory. This theory has a maximal supersymmetry in ten spacetime dimensions, and it has two different sectors according to the particle species the NS-NS sector and R-R sector. The spectrum of Type II-A supergravity consists of two gravitinos of opposite chirality as well as the bosonic fields which are the dilaton Φ, the ten-dimensional metric G_{MN}, and the two-form B_2 in the NS-NS sector. the R-R sector of the Type II-A supergravity has the one- and three-forms C_1, C_3. Calabi-Yau compactification of Type II-A supergravity to a four-dimensional theory can be made via the ansatz Equation (2.131). The two ten-dimensional gravitinos can be decomposed into covariantly constant spinor times some four-dimensional spinor by using the fact that the Calabi-Yau threefold has one covariantly constant spinor. The resulting theory has two gravitinos in four dimensions and it is the $\mathcal{N} = 2$ supergravity theory in four spacetime dimensions. The other bosonic fields can be expanded in terms of harmonic forms of the Calabi-Yau threefold X and the geometric deformations of the internal metric, which is the moduli space of X, together with the expansion of the fields of the Type II-A supergravity can be gathered into a gravity multiplet(graviton and $U(1)$ gauge field graviphoton), $h^{(1,1)}$ vector multiplets(vector field and complex scalars) and $h^{(2,1)} + 1$ hypermultiplets(complex scalars and dilaton). Hence, since it is known that the zero modes of the differential operators on the Calabi-Yau X are in one-to-one correspondence with the harmonic forms on X, the full field content is the massless four dimensional $\mathcal{N} = 2$ supergravity multiplets. Type II-B compactifications on Calabi-Yau spaces can be studied similarly. Type II-B supergravity has also $\mathcal{N} = 2$ supersymmetry with two chiral gravitinos.

Table 2.3. The number of massless modes of Type II-A(B) compactifications.

	vector	hyper	gravity
Type II-A on X	$h^{1,1}(X)$	$h^{2,1}(X)+1$	1
Type II-B on X	$h^{2,1}(X)$	$h^{1,1}(X)+1$	1

Type II-B supergravity possesses the same NS-NS fields as Type IIA and contrary to Type II-A it consists of even forms, the axion C_0, a two-form C_2, and a four-form C_4 in the R-R sector. As in the case of Type II-A compactifications on the Calabi-Yau threefold, we use the ansatz Equation (2.131) for the ten-dimensional background. However, there is an essential difference, although there is a gravity multiplet again, the Calabi-Yau compactification of Type II-B has $h^{(2,1)}$ vector multiplets(vector field and complex scalars) and $h^{(2,1)}+1$ hypermultiplets(complex scalars and dilaton). To summarize, if we compactify Type-IIA/B on X, the resulting low energy descriptions are given by $\mathcal{N}=2$ four-dimensional supergravity theories. The massless field content of them can be classified as $(h^{1,1}/h^{2,1})$ vector multiplets $(h^{2,1}+1/h^{1,1}+1)$ hypermultiplets and 1 gravity multiplet for both of them. Note that the number of vector multiplets matches the dimension of Kähler structure moduli and complex structure moduli of the Calabi-Yau threefold X and four-dimensional Type II-A(B) supergravity vector multiplets parametrize the Kähler structure moduli(complex structure moduli).

Now, we can go back to the relation between the topological string amplitudes and the Calabi-Yau compactifications of Type-IIA/B string theory. The resulting theory, $\mathcal{N}=2$ supergravity can be constructed from $\mathcal{N}=2$ superspace action. Firstly, the four-dimensional theory has $\mathcal{N}=2$ supersymmetry and has the superspace coordinates $d^8\theta$ which leads to D-terms which we have seen in the two-dimensional supersymmetric non-linear sigma model. Similarly, there are two different F-terms and the superspace coordinates of them are $d^4\theta$ and $d^4\overline{\theta}$. Furthermore, all different supermultiplets have different F-term contributions to the effective action of the four-dimensional $\mathcal{N}=2$ supergravity. The reason for this is that the Calabi-Yau has two different moduli: the Kähler structure moduli and the complex structure moduli, they are decoupled at the classical level. However, there can be loop corrections due to the interactions of the strings, and these loop corrections can be obtained by the topological string theory amplitudes. Now we can understand the topological string theory amplitudes from a perspective of four-dimensional $\mathcal{N} = 2$ supergravity multiples. In order to accomplish this task, it is necessary to have the appropriate equipment for the construction of these multiplets in the superfield formalism. In the superfield formalism, there are two superfields for this content, Weyl and matter chiral superfields.

$$\begin{aligned} W_{\mu\nu}(x,\theta) &= F_{\mu\nu} - \frac{1}{2} R_{\mu\nu\rho\sigma}\epsilon_{\alpha\beta}\theta^{\alpha}\sigma_{\lambda\rho}\theta^{\beta} + \cdots , \\ \Phi^I(x,\theta) &= X^I + \cdots \quad , \end{aligned} \tag{2.134}$$

where $\alpha, \beta = 1, 2$, $I = 0, 1, \cdots, h^{(1,1)}$ and $F_{\mu\nu}$ is a self-dual field strength of graviphoton, which is the lowest component of gravity multiplet and $R_{\mu\nu\rho\sigma}$ is the Riemann tensor. X^I is the scalar field that parametrizes the Kähler structure moduli or complex structure moduli of the Calabi-Yau threefold X. We can construct the scalar chiral superfield W^2 from Weyl chiral superfield and collect into a single holomorphic degree two functions that are generalized pre-potential

$$F(\Phi^I, W^2) \equiv \sum_{g=0}^{\infty} \widetilde{F}_g(\Phi^I) W^{2g} , \tag{2.135}$$

where $\widetilde{F}_g(X^I)$ is is homogeneous of degree $2-2g$ function. After performing the θ integration, one can write the bosonic part of the effective action of the four-dimensional $\mathcal{N} = 2$ supergravity in terms of components

$$\begin{aligned} S_{eff} &= \int d^4x d^4\theta F(\Phi^I, W^2) \\ &= \int d^4x \frac{\partial^2 \widetilde{F}_0}{\partial X^I \partial X^J} F_+^I \wedge F_+^I + \sum_{g=1}^{\infty} \int d^4x \widetilde{F}_g(X^I) R_+^2 F_+^{2g-2} + \cdots , \end{aligned} \tag{2.135}$$

where F_+ is the self-dual part of graviphoton $F_+ = F + \star F$ and R_+^2 is the contraction of the self-dual part of the Riemann tensor. Let us consider only the Type II-A string theory in the following. As in the case of the projective coordinates which are introduced in the B-model, we can make the relabeling the scalar fields in terms of the complexified Kähler class for the A-model

$$t_i = \int_{\beta_i} \omega + iB_2 = X^i/X^0 \quad i = 1, \cdots, h^{(1,1)} . \tag{2.136}$$

The field $B_2 \in H^2(X, \mathbb{R})$ is nothing but the NS two-form, and the complexification of the Kähler form is natural from the Type II-A string theory point of view. It can be shown that the unspecified couplings $\widetilde{F}_g$ is nothing but the A-model topological string theory genus g free energy F_g [10,11]. Then we can write Equation (2.135) in terms of topological string amplitudes

$$S_{eff} = \int d^4x \frac{\partial^2 F_0}{\partial t_i \partial t_j} F_+^i \wedge F_+^j + \sum_{g=1}^{\infty} \int d^4x F_g(t_i) R_+^2 F_+^{2g-2} + \cdots . \tag{2.137}$$

The first term is the three-level(classical) action associated with the kinetic term of the graviphoton field. If we compare with the gauge field kinetic term, we see that the gauge coupling constant can be determined by the genus g topological free energy or the prepotential

$$\tau_{ij} = \frac{\partial^2 F_0}{\partial t_i \partial t_j} . \tag{2.138}$$

The second term $F_g(t_i)R_+^2 F_+^{2g-2}$ corresponds to the higher genus gravitational corrections to the vector multiplets and this coupling is also exact nonperturbatively in a sense that it has no dilaton(string coupling) dependence thanks to the fact that dilation belongs to the hypermultiplet rather than the vector multiplet. Accordingly, the topological string theory computes the strength of scattering amplitude between 2 gravitons and $2g-2$ graviphotons. If we turn on a self-dual, constant graviphoton field strength by giving the vacuum expectation value $\lambda = \langle F_+ \rangle$, we can write the gravitational corrections to the F-term as

$$\left(\sum_{g=0}^{\infty} F_g(t_i) F_+^{2g-2} \right) R_+^2 = \left(\sum_{g=0}^{\infty} F_g(t_i) \lambda^{2g-2} \right) R_+^2 = F(t_i, \lambda) R_+^2 , \tag{2.139}$$

where we have defined the total free energy as

$$F(t_i, \lambda) = \sum_{g=0}^{\infty} \lambda^{2g-2} F_g(t_i) . \tag{2.140}$$

We rewritten the whole coupling in terms of genus expansion of the total free energy. All we have to do is compute the full free energy through the target space physics. This type of computation is given by the celebrated work of Gopakumar and Vafa [24,25]. The coupling which is given by Equation (2.140) can be interpreted employing integrating out of some solitonic states of the string theory and the resulting theory gives the effective action which we have discussed. These solitonic states are solutions of the supergravity equations of motion, charged under the R-R fields and they resemble the black hole solutions or higher dimensional generalizations of them, i.e. black-brane solutions [13]. Furthermore, these states are also equivalent to some other objects which are called Dp-branes which is a $p+1$ dimensional wall and open strings can be attached with their endpoints on it [12]. The reason for Dp brane being solitonic is that their masses are proportional to the inverse string coupling, $M_p \propto 1/\lambda l_s^{p+1}$, where l_s is the string length. These solitonic objects are hidden in the perturbative expansion of the string theory since their masses are very large compared with the other modes of the string theory which we have discussed early. When we go to the strongly coupled regime or non-perturbative regime, $\lambda \gg 1$, they are quite light, the perturbative description of the string theory breaks down and a new picture emerges. These solitonic states are quite special in the sense that their masses are equal to their charges, or in other words, they satisfy the BPS property. From a supersymmetric point of view, the BPS states are the short multiplets of the corresponding supersymmetry algebra and they preserved half of the supersymmetry. As long as the supersymmetry is not broken their masses/charges are protected from the various quantum effects. This means that at generic points in the moduli space of couplings of the underlying supersymmetric theory, they are stable and still have degrees of freedom of the underlying theory. A general Dp brane which wraps a p-cycle in a general spacetime gives a particle that has zero spatial extensions and the mass of this particle is determined by the integration of the tension of Dp brane along the p-cycle. If Dp brane is BPS or supersymmetric, the mass saturates the BPS bound and the mass of the particle is equal to its charge. Let us consider Type II-A string theory. Type II-A string theory has Dp branes, where p is an even number, as well as the magnetic dual of them since they are charged under the R-R gauge fields C_1, C_3. This picture can also be enlarged to what is called M-theory [13]. M-theory is the non-perturbative completion of Type II-A string theory. This can be obtained by the dimensional reduction of eleven-dimensional supergravity, which is the low energy limit of the M-theory, to the ten-dimensional supergravity or equivalently, compactification of the M-theory on circle $\mathbf{S}^1$ leads to Type II-A string theory. The corresponding map between these two different theories is given by the relation between the radius of the eleventh compact dimension, $\mathbf{S}^1$, and the coupling constant of the Type II-A string theory

$$R_{11} = \lambda^{2/3} l_p \,, \tag{2.141}$$

where l_p is the eleven-dimensional Planck length and when λ is small, the radius is small then we have the perturbative string theory description. However, when λ is large and this gives the fact that the circle is decompactified, then we are in the M-theory regime. M-theory has also solitonic objects like string theory and they are the soliton-like solutions of the eleven-dimensional supergravity, namely M2 brane and its magnetic dual M5 brane. We can also relate the M-theory branes with Type II-A branes by matching their tensions/masses via Equation (2.141). For instance, the M2 brane which wraps the M-theory circle $\mathbf{S}^1$ gives the D0 branes as the Kaluza-Klein modes of the M2 brane, and the M2 brane which does not wrap the M-theory circle $\mathbf{S}^1$ corresponds to the D2 brane. It can be seen that the M-theory unifies different objects into a single one. The above construction can also hold when we compactify the Type II-A string theory on the Calavi-Yau threefold X and this configuration is equivalent to the compactification of M-theory on $X \times \mathbf{S}^1$. As we have mentioned, wrapped Dp branes around p-cycles give the particles. In our case, the bound state of the wrapped D2 brane around the two-cycle of the Calabi-Yau threefold with D0 branes turns out to be a charged particle in four dimensions. Hence, from a four-dimensional point of view, the effects of the solitonic states on the graviphoton field strength can be integrated out and this gives the effective action which involves the coupling between the graviphoton and the gravity. This type of computation is very similar to the Schwinger computation which we will discuss in the following. The Schwinger computation is the evaluation of the one-loop free energy by integrating out the scalar fields which are charged under the self-dual $U(1)$ gauge field in two dimensions. The Schwinger computation has one subtle difference compared with our case, that is our background is supersymmetric while the Schwinger computation has no supersymmetry at all and there are no gravitons too in the original Schwinger model. However, they are equivalent due to the fact that the existing hypermultiplets come from the wrapped D2 branes. These hypermultiplets have a crucial effect when we take into account the gravitational corrections since their presence of cancels the contribution of the gravity R_+^2 to the coupling $R_+^2 F_+^{2g-2}$. In other words, the integrating out of the charged particles with a constant graviphoton field strength in the supersymmetric background with gravity turns out to be non-supersymmetric Schwinger computation. As we have discussed, these BPS particles come in spin multiples, and their spin content is given by the little group of massive representations of the five-dimensional spatial rotation group $SO(4) = SU(2)_L \times SU(2)_R$. Then we can make a conjecture that the BPS content of these particles is given by

$$[(\frac{1}{2},0)\oplus 2(0,0)]\otimes \sum_{j_L,j_R} N_C^{(j_L,j_R)}(j_L,j_R)\,. \tag{2.142}$$

$N_C^{(j_L,j_R)}$ are the BPS degeneracies of various supersymmetric multiplets which come from the wrapped branes. Due to the self-duality condition for the graviphoton field strength, only the particles that have left spin couple to the constant graviphoton field, then we can sum over right spins by supersymmetric index

$$\sum_{j_R} N_C^{(j_L,j_R)}(-1)^{2j_R}(2j_R+1) = N_C^{(j_L)}\,. \tag{2.143}$$

This index does not depend on the complex structure deformations of the Calabi-Yau threefold X. The relevant branes which wrap the 2-cycles, $\beta \in H_2(X,\mathbb{Z})$ are D2 branes in addition to D0-branes since we think of the compactification of Type II-A string theory where we have taken the size of the internal manifold is larger than the string length and the other D-branes are much heavier than the D2 and D0 branes due to the fact that $M_p \propto 1/\lambda l_s^{p+1}$. Since these particles are BPS, this means that their masses are equal to their charges

$$m = e = \frac{1}{\lambda}(t_i + 2i\pi n)\,, \tag{2.144}$$

where e is the charge of the particle. The first term $t_i = \int_{\beta_i}\omega + iB_2$ is the volume of 2-cycle which is wrapped by the D2 brane and the second one is the number of D0 branes which are bounded to the D2 brane. From the M-theory perspective, the first one is the volume of the 2-cycle wrapped by the M2 brane and the second term corresponds to the momentum of the M2 brane in the compactified circle direction. The integrating out at one loop of one charged particle which has left and right spins (J_L, J_R) is given by Schwinger one-loop free energy formula [24, 25]

$$F_{\text{Schwinger}(J_L,J_R)} = \int_\epsilon^\infty \frac{ds}{s}\frac{\mathrm{Tr}_{(j_L,j_R)}(-1)^{\sigma_L+\sigma_R}e^{-s(\beta.\omega+2\pi i n)}e^{-2s(\sigma_L+\sigma_R)\lambda}}{(2\sinh(s\lambda/2))^2}\,. \tag{2.145}$$

The trace is taken over the left and right spins and $\sigma_{L/R}$ is the Lie-algebra representations of the $SU(2)_{L/R}$ respectively. The integration variable s is the parameter of the loop that we consider.

We have all gatherings to write the gravitational corrections in a simple compact form. Before doing that let us summarize the all arguments we have used so far.

The coupling $R_+^2 F_+^{2g-2}$ does not receive any stringy corrections, since dilaton does not sit in the vector multiplet. So we can make calculations at strong coupling. At the strong coupling, the main contributions come from the lightest states which are D-branes. They give rise to BPS states which correspond to the charged particles in four dimensions. The relevant configuration is the D2 branes which wrapped around cycles, $\beta = \phi_\star[\Sigma_g] \in H_2(X,\mathbb{Z})$, bound to D0 branes and this is identical to the wrapped M2 branes when M-theory compactified on a circle. The integration of these BPS states at one loop gives the amplitude that we want to compute. All we need is to integrate out these charged particles which come from wrapped D2/D0 branes or M2 branes on various cycles on the Calabi-Yau threefold, at one loop, in the presence of the constant graviphoton background. To get full free energy, we have to sum over all possible cycles, momentum, and left BPS states $F_{\rm GV}(t_i,\lambda) = \sum_{\beta\in H_2(X,\mathbb{Z})}\sum_{n\in\mathbb{Z}}\sum_{j_L} N_\beta^{j_L} F_{\text{Schwinger}(J_L,0)}$

$$F_{\rm GV}(t_i,\lambda) = \sum_{\beta\in H_2(X,\mathbb{Z})}\sum_{n\in\mathbb{Z}}\sum_{j_L} N_\beta^{j_L}\int_\epsilon^\infty \frac{ds}{s}\frac{\mathrm{Tr}_{j_L}(-1)^{\sigma_L} e^{-s(\beta.\omega+2\pi i n)} e^{-2s\sigma_L\lambda}}{(2\sinh(s\lambda/2))^2}\,. \tag{2.146}$$

The sum over n can calculated by the Poisson resummation formula

$$\sum_n e^{-2\pi i n s} = \sum_k \delta(s-k)\,, \tag{2.147}$$

and this gives

$$F_{\rm GV}(t_i,\lambda) = \sum_{\beta\in H_2(X,\mathbb{Z})}\sum_{k=1}\sum_{j_L} N_\beta^{j_L}\frac{\mathrm{Tr}_{j_L}(-1)^{\sigma_L} e^{-k\beta.\omega} e^{-2k\sigma_L\lambda}}{(2\sinh(k\lambda/2))^2}\,. \tag{2.148}$$

It is possible to define an another convenient basis for $SU(2)_L$ representation

$$I_r = \left(2(0) + (\frac{1}{2})\right)^{\otimes r}\,. \tag{2.149}$$

Then we have the following identity which transforms the left-spin content to the above basis

$$\sum_{j_L} N_\beta^{j_L}[j_L] = \sum_{r=0}^\infty n_\beta^r I_r\,, \tag{2.150}$$

where n_β^r are integers. We can write F in terms of these integers

$$F_{\rm GV}(t_i,\lambda) = \sum_{\beta\in H_2(X,\mathbb{Z})}\sum_{k=1}^\infty\sum_{r=0}^\infty n_\beta^r\frac{\mathrm{Tr}_{I_r}(-1)^{\sigma_L} e^{-k\beta.\omega} e^{-2k\sigma_L\lambda}}{(2\sinh(k\lambda/2))^2}\,, \tag{2.151}$$

and we can evaluate the trace with respect to the new basis

$$\mathrm{Tr}_{I_r}(-1)^{\sigma_L} e^{-2k\sigma_L\lambda} = \left(\mathrm{Tr}_{I_1}(-1)^{\sigma_L} e^{-2k\sigma_L\lambda}\right)^r = \left(2\sinh(k\lambda/2)\right)^{2r}\,.$$

Finally, the full free energy reads

$$F_{\rm GV}(t_i,\lambda) = \sum_{\beta\in H_2(X,\mathbb{Z})}\sum_{k=1}^\infty\sum_{r=0}^\infty \frac{n_\beta^r}{k}\left(2\sinh\frac{k\lambda}{2}\right)^{2r-2} Q^{k\beta}\,, \tag{2.152}$$

where $Q^\beta = e^{-\omega\cdot\beta}$ as in the case of the Gromov-Witten invariants. The integers n^r_β are called the Gopakumar-Vafa invariants of Calabi-Yau threefolds. Due to the supersymmetric index which we have introduced, these integers do not depend on complex structure deformations of the Calabi-Yau threefold. Note that Equation (2.152) does not include contributions coming from constant maps, F_g for $g \geq 2$ as well as the $g = 0, 1$ ones. The former is independent of the Kähler class while this is not the case for the latter. Let us begin with the former. Then we separate the whole sum as $F(t_i, \lambda) = \hat{F}(t_i, \lambda) + F_{\mathrm{GV}}(t_i, \lambda)$, where $\hat{F}(t_i, \lambda)$ denotes the constant map terms. The genus $g \geq 2$ amplitudes can be written as a Hodge integral over the moduli space of genus g curves [27]

$$\mathcal{N}_{g,0} = (-1)^g \frac{\chi(X)}{2} \int_{\mathcal{M}_g} c^3_{g-1}, \qquad g \geq 2\,, \tag{2.153}$$

where $\mathcal{N}_{g,0}$ indicates the Gromov-Witten invariant associated with the constant maps for the A-model topological string theory. The $\chi(X)$ is the Euler characteristic of the Calabi-Yau manifold X and c_{g-1} stands for the $(g-1)^{\mathrm{th}}$ Chern class of the Hodge bundle that is defined as the g-dimensional holomorphic vector bundle over the genus g Riemann surface $g > 1$. The $g \geq 2$ constant map contributions can be found from the computation of the integral of Chern class of the Hodge bundle [11], [24, 27]

$$\int_{\mathcal{M}_g} c^3_{g-1} = \frac{B_g}{2g(2g-2)} \frac{B_{g-1}}{(2g-2)!}\,, \tag{2.154}$$

where B_g are the Bernoulli numbers. Therefore the constant map amplitudes whose genus is greater than one are given by

$$\hat{F}_g(t_i = 0) = \frac{(-1)^g \chi(X) B_g B_{g-1}}{4g(2g-2)(2g-2)!}, \qquad g \geq 2\,. \tag{2.155}$$

The above formula can be rewritten differently from the underlying physical perspective. These amplitudes can be understood from the unwrapped D2-branes since it does not depend on the topological sector. Then the only contribution comes from the momentum modes associated with the M-theory circle, given by $m = 2\pi i n/\lambda$. By the use the Schwinger-type formula [9], [24, 25]. We have

$$\int \frac{ds}{s} \frac{e^{-sm}}{[2\sin(s\lambda/2)]^2} = \sum \left(\frac{\lambda}{m}\right)^{2g-2} \chi_g\,. \tag{2.156}$$

χ_g is the Euler characteristic of moduli space of genus g Riemann surfaces and it is equal to $(-1)^{g-1} B_g / 2g(2g-2)$. Hence if sum all momentum modes, we will recover the Equation (2.155) in terms of unwrapped D2-branes, [9], [24, 25].

It was mentioned that the genus g constant map amplitudes do not depend on t. On the other hand, the remaining constant map terms $\hat{F}_0$ and $\hat{F}_1$ depend on the Kähler class parameters nevertheless there are no wrapped D2-branes so the origin of this dependence comes from massless particles. This behavior can be explained in terms of the logarithmic expansion of genus-zero amplitude on the conifold point [26]. They are given as in the following,

$$\begin{aligned} \hat{F}_0(t_i) &= \frac{1}{6} C_{ijk} t^i t^j t^k + P_2(t)\,, \\ \hat{F}_1(t_i) &= -\frac{1}{24} c_2^i t_i + \text{const.}\,, \end{aligned} \tag{2.157}$$

where C_{ijk} represents the classical intersection number which comes through the dependence of the A-model prepotential considered at Equation (2.71) before. Additional term $P_2(t)$ is an ambiguous function and c_2 is the second Chern class of Calabi-Yau threefold X. After considering the contributions of both wrapped and unwrapped D2-branes on the topological string free energy, in addition to the contributions from massless modes, the total A-model topological string theory amplitude can be written as

$$\begin{aligned} F(t_i, \lambda) = &\frac{1}{\lambda^2}\left[\frac{1}{6} C_{ijk} t_i t_j t_k + P_2(t)\right] - \frac{1}{24} c_2^i t_i + \text{ const.} \\ &- \sum_{g>1} \frac{\chi(X) B_g B_{g-1}}{4g(2g-2)(2g-2)!} \lambda^{2g-2} + \sum_{\beta \in H_2(X,\mathbb{Z})} \sum_{k=1}^{\infty} \sum_{r=0}^{\infty} \frac{n_\beta^r}{k} \left(2 \sinh \frac{k\lambda}{2}\right)^{2r-2} Q^{k\beta}\,. \end{aligned} \tag{2.158}$$

The summation over k represents the multi-covering effects that Gromov-Witten invariants suffice. We can compute the Gromov-Witten invariants from Gopakumar-Vafa invariants by expanding free energy in powers of λ and collecting the terms with power λ^{2g-2} gives the corresponding Gromov-Witten invariants. Let us consider for simplicity, the genus zero free energy. The total $F_0(t_i)$ is found by expanding the string coupling expansion, then we have $\sinh(k\lambda/2) \approx k\lambda/2$. We can see that the $F_0(t_i)$ corresponds to the the Gopakumar-Vafa invariant with $r = 0$ together with the constant map contribution as

$$F_0(t_i) = \frac{1}{6} C_{ijk} t_i t_j t_k + P_2(t) + \sum_{\beta \in H_2(X,\mathbb{Z})} \sum_{k>1}^{\infty} \frac{n_\beta^0}{k^3} Q^{k\beta}. \tag{2.159}$$

If we compare the above sum with Equation (2.69), we can realize that the sum of genus zero Gopakumar-Vafa invariants comes with a multicovering effect through the k dependence. Thus, the expansion Equation (2.159) has already included the rational nature of Gromov-Witten invariants. The more proper relation between the Gromov-Witten invariants and the Gopakumar-Vafa invariants can be explained in the following manner. Consider the map $\phi : \mathbf{S}^2 \to \mathbf{S}^2$ with a holomorphic curve $\beta \in H_2(X, \mathbb{Z})$. We can think of all map $\phi : \mathbf{S}^2 \to X$ with also given by the curves $k\beta$, so the Gopakumar-Vafa formalism gives this correction naturally. Therefore in order to count the actual number of primitive curves, we should subtract the corresponding Gromov-Witten invariants which come from lower degree $d < k$. The second geometric effect which we should consider is the bubbling effect. Consider again, a map from Riemann surface to Calabi-Yau threefold $\phi : \Sigma_g \to X$, if we glue Σ_g to very small Riemann surface genus h, we find another Riemann surface whose genus is $g' = g + h$. Therefore, in order to find actual curves at genus g', we should take into account the fact that all "primitive" curves at genus g contribute to $g' > g$.

The total free energy can be expanded on another basis, sometimes called BPS state expansion of Gopakumar-Vafa formalism, by computing the trace with respect to the spins of the fields directly

$$F(Q,q) = \sum_{\beta \in H_2(X,\mathbb{Z})} \sum_{k=1}^{\infty} \sum_{j_L} N_\beta^{(j_L)} (-1)^{2j_L} \left(\frac{q^{-2kj_L} + ... + q^{+2kj_L}}{k(q^{k/2} - q^{-k/2})^2} \right) Q^{k\beta}, \quad q = e^{i\lambda} \,. \tag{2.160}$$

As we have mentioned, $N_\beta^{(j_L,j_R)}$'s are not invariants for general Calabi-Yau threefolds. However, there are some special types of them which are called non-compact Calabi-Yau threefolds which do not admit any complex structure deformations. Then, we have the freedom to tune the full spin content $N_C^{(j_L,j_R)}$ [28, 63].

We can reach the same conclusion from a more physical setting, if M-theory is compactified on non-compact toric Calabi-Yau threefold X times the circle $\mathbf{S}^1$, we get five-dimensional $\mathcal{N} = 1$ supersymmetric gauge theory. This theory has an additional $U(1)_{\mathcal{R}} \subset SU(2)_{\mathcal{R}}$ R-symmetry at 5d which preserves supersymmetry in four dimensions. This will be more clear when we study the Nekrasov partition function. This allows us to turn on the non-self dual part of constant graviphoton field strength $F = F_1 dx^1 \wedge dx^2 + F_2 dx^3 \wedge dx^4$. We can write a refined version of free energy as

$$F_{\text{Schwinger}(J_L,J_R)} = \sum_{\beta\in H_2(X,\mathbb{Z})}\sum_{n\in\mathbb{Z}}\sum_{j_L,j_R} N_\beta^{(j_L,j_R)} \int_\epsilon^\infty \frac{ds}{s}\frac{\text{Tr}_{j_L,j_R}(-1)^{\sigma_L+\sigma_R} e^{-s(T_\beta+2\pi ni/\lambda)} e^{-2se(\sigma_L F_+ + \sigma_R F_-)}}{(2\sinh(seF_1+/2))(2\sinh(seF_2/2))}. \tag{2.161}$$

We can follow the same steps where we calculate the unrefined free energy, one can obtain

$$\begin{aligned} F(Q,q,t) = & \sum_{\beta\in H_2(X,\mathbb{Z})}\sum_{k=1}^{\infty}\sum_{j_L,j_R} \frac{N_\beta^{(j_L,j_R)}(-1)^{2j_L+2j_R}}{k(t^{k/2}-t^{-k/2})(q^{k/2}-q^{-k/2})} Q^{k\beta} \\ & \times \left((tq)^{-kj_L}+\cdots+(tq)^{+kj_L}\right)\left(\left(\frac{t}{q}\right)^{-kj_L}+\cdots+\left(\frac{t}{q}\right)^{+kj_L}\right). \end{aligned} \tag{2.162}$$

where $q = e^{F_1}$, $t = e^{F_2}$. It can be seen that the parameters $\sqrt{qt}$ and $\sqrt{\frac{q}{t}}$ couple with $SU(2)_L$ and $SU(2)_R$ spin and the full spin content information can be acquired by computing the full free energy and extracting the degeneracies of BPS states $N_\beta^{(j_L,j_R)}$'s from the Equation (2.162).

3. SEIBERG-WITTEN THEORY

In this chapter, we will introduce $\mathcal{N} = 2$ supersymmetric gauge theory and its moduli space of vacua, which describes the possible vacuum states of the theory. The central charges and the establishment of the BPS bound, which characterizes the stability of these vacua will be discussed. We will explore the effective field theory approach to $\mathcal{N} = 2$ supersymmetric gauge theories, providing a description of their low-energy dynamics. Discusses the breaking of R-symmetry and its implications for the theory is also introduced as well as the meaning of singularities in the moduli space of vacua and their significance in the theory's dynamics. We will focus on the significance of monopoles and dyons in comprehending the behavior of $\mathcal{N} = 2$ supersymmetric gauge theories. The exact solution will be highlighted from elliptic curves via the Seiberg-Witten approach, providing valuable insights into the dynamics of these theories. Overall, this chapter presents a comprehensive analysis of the Seiberg-Witten theory, including its mathematical underpinnings and practical applications in comprehending the dynamics and characteristics of $\mathcal{N} = 2$ supersymmetric gauge theories [32–35].

3.1. $\mathcal{N} = 2$ Supersymmetric Gauge Theory and Moduli Space of Vacua

In this section, we shall give a review of the four-dimensional $\mathcal{N} = 2$ supersymmetric gauge theories and how they can be studied in terms of Seiberg-Witten theory. In order to achieve this, we start the central extension of the 4d $\mathcal{N} = 2$ super-Poincare algebra which is an extension of the 4d Poincare algebra $(P_\mu, M_{\mu\nu})$ by two supercharges $Q, \widetilde{Q}$

$$\begin{aligned} \{Q^I_\alpha, \widetilde{Q}_{\dot\beta J}\} &= 2\sigma^\mu_{\alpha\dot\beta} P_\mu \delta^I_J \,, \\ \{Q^I_l, Q^J_\beta\} &= 2\sqrt{2}\epsilon_{\alpha\beta} Z^{IJ} \,, \\ \{\widetilde{Q}_{\dot\alpha I}, \widetilde{Q}_{\dot\beta J}\} &= 2\sqrt{2}\epsilon_{\dot\alpha\dot\beta} Z^*_{IJ} \,, \end{aligned} \tag{3.1}$$

where $\sigma^\mu = \{1, \sigma^i\}$ are the four-dimensional chiral representation of the Poincare algebra with $\alpha, \beta = 1, 2$, ϵ^{IJ} is the Levi-Civita tensor and P_μ corresponds to the four-dimensional translation symmetry. The superscripts I, J denote the $SU(2)_{\mathcal{R}}$ symmetry indices and Z denotes the central charge which extends the ordinary supersymmetry algebra. Let us ignore the central extension for the moment. We can construct a four-dimensional $\mathcal{N}=2$ supersymmetric Lagrangian with irreducible representations of the $\mathcal{N}=2$ supersymmetry algebra of the gauge theory with a gauge group G. According to this fact, there are two distinct supermultiplets which are the vector multiplet and the hypermultiplet. The vector multiplet $V_{\mathcal{N}=2}$ includes one complex scalar ϕ, complex Weyl gaugino $(\lambda_\alpha, \widetilde{\lambda}_{\dot\alpha})$, Weyl fermions $(\psi_\alpha, \widetilde{\psi}_{\dot\alpha})$ and the field strength of the gauge group G. Since all these fields are in the same multiplet with the gauge field A, they must be in the adjoint representation of the gauge group G, in order to be compatible with gauge invariance. The above fields can be obtained from the supersymmetry algebra, one can start with the scalar field ϕ which satisfies

$$Q^I|\phi\rangle = 0, \quad I = 1, 2\,. \tag{3.2}$$

Notice that complex scalar is annihilated by the supercharges in the chiral representation. The other fields are obtained by acting with anti-chiral supercharges $\widetilde{Q}^I|\phi\rangle \propto \lambda$ and further action of $\widetilde{Q}^I$ give the field strength of the gauge group G. The scalar field ϕ has charge two under the $U(1)_{\mathcal{R}}$ symmetry and λ^J_α, A_μ have charge one-half and zero under the $U(1)_{\mathcal{R}}$ symmetry, respectively. On the other hand, the hypermultiplet $H_{\mathcal{N}=2}$ has two complex scalars $(q, \widetilde{q})$ and Weyl fermions $(\psi_q\,, \widetilde{\psi}^\dagger_q)$. The scalars are annihilated by the supercharges which have different chiralities,

$$Q^1|q\rangle = \widetilde{Q}^1|q\rangle = Q^1|\widetilde{q}\rangle = \widetilde{Q}^1|\widetilde{q}\rangle = 0\,. \tag{3.3}$$

Analogously, we can act with the other supercharges to get the supersymmetric descendants and in this case, the single action of supercharges gives the Weyl fermions. Although the hypermultiplet complex scalars do not transform under the $U(1)_{\mathcal{R}}$ symmetry, $(q\,, \widetilde{q}^\dagger)$ transform as doublet under $SU(2)_{\mathcal{R}}$. The $U(1)_{\mathcal{R}}$ charges of the hypermultiplet fermions is $\pm 1/2$ and they are neutral under $SU(2)_{\mathcal{R}}$ symmetry.

After the construction of the field content of the four-dimensional $\mathcal{N}=2$ supersymmetry algebra for the gauge theories, we can write the corresponding Lagrangian by using the superfield formalism. Similarly for the $\mathcal{N}=(2,2)$ supersymmetry, the extension of the spacetime coordinates to superspace with additional Grassmannian coordinates $(\theta^{\alpha}, \overline{\theta}_{\dot{\alpha}})$ gives what we needed for this goal. The relevant superfield for our purpose is the $\mathcal{N}=1$ the real vector superfield V since it is possible to write down the whole vector multiplet which contains the complex scalar, Weyl fermion λ_{α} and a vector field A_{μ} in a simple way. The real vector superfield V is given by

$$V = -i\theta\sigma^{\mu}\overline{\theta}A_{\mu} + i\theta\theta\overline{\theta}\overline{\lambda} - i\overline{\theta}\overline{\theta}\theta\lambda + \frac{1}{2}\theta\theta\overline{\theta}\overline{\theta}D\,, \tag{3.4}$$

where D is an auxiliary field. The matter part of the four-dimensional $\mathcal{N}=1$ is given by chiral multiplet with the corresponding chiral superfield $\mathbf{Q}$

$$\mathbf{Q}^{A} = q^{A} + i\psi_{q}^{A}\theta + F^{A}\theta\theta\,, \tag{3.5}$$

where A denotes the flavor index and F^{A} is an auxiliary field. F and D can be eliminated by the equations of motion. We can write an $\mathcal{N}=2$ vector multiplet in terms of the $\mathcal{N}=1$ vector superfield and the $\mathcal{N}=1$ chiral superfield:

$$\begin{array}{ccccccl} & \swarrow\!\!\nearrow & \lambda & \longleftrightarrow & A_{\mu} & & \mathcal{N}=1 \text{ vector multiplet} \\ \phi & \longleftrightarrow & \psi & \swarrow\!\!\nearrow & & & \mathcal{N}=1 \text{ chiral multiplet} \end{array} \Longrightarrow V_{\mathcal{N}=2} \rightarrow (V,\Phi)_{\mathcal{N}=1}\,, \tag{3.6}$$

where the chiral superfield $\Phi_{\mathcal{N}=1}$ includes the complex scalar and Weyl Fermion (ϕ, ψ). The decomposition of $\mathcal{N}=2$ and $\mathcal{N}=1$ for the vector multiplet, $\mathcal{N}=2$ hypermultiplet can be rewritten by means of two $\mathcal{N}=1$ chiral superfields

$$\begin{array}{ccccccl} & \swarrow\!\!\nearrow & \mathbf{Q} & \longleftrightarrow & \psi_{q} & & \mathcal{N}=1 \text{ chiral multiplet} \\ \widetilde{\psi}_{q}^{\dagger} & \longleftrightarrow & \widetilde{\mathbf{Q}}^{\dagger} & \swarrow\!\!\nearrow & & & \mathcal{N}=1 \text{ antichiral multiplet} \end{array} \Longrightarrow H_{\mathcal{N}=2} \rightarrow (\mathbf{Q}, \widetilde{\mathbf{Q}})_{\mathcal{N}=1}\,. \tag{3.7}$$

Just in the case of the supersymmetric non-linear sigma model, the whole superspace Lagrangian can be expressed in terms of the superpotential for the chiral superfields and Kähler potential which corresponds to the interaction between chiral superfields and the real vector superfield V

$$\mathcal{L}_{\mathcal{N}=2} = \mathcal{L}_{\mathcal{N}=2}^{\text{SYM}} + \mathcal{L}_{\mathcal{N}=2}^{\text{matter}}\,. \tag{3.8}$$

The action consists of two terms: the first term is associated with the four-dimensional $\mathcal{N}=2$ vector multiplets, while the second term corresponds to the $\mathcal{N}=2$ hypermultiplets and their coupling with the vector multiplet. One can express the action in a gauge invariant way as the real vector superfield V does not explicitly include the gauge field strength $F_{\mu\nu}$. This can be achieved by rewriting the vector multiplet V into a gaugino superfield W_{α}, similar to the Weyl superfield for supergravity:

$$W_{\alpha} = -i\lambda_{\alpha} + iF_{\mu\nu}(\sigma^{\mu\nu}\theta)_{\alpha} + D\theta_{\alpha} + \theta\theta(\sigma^{\mu}\partial_{\mu}\overline{\lambda}_{\alpha})\,. \tag{3.9}$$

The lowest component of Equation (3.9) is the Weyl gaugino and one can find the gaugino superfield from the real vector superfield V via

$$W_\alpha = -\frac{1}{4}\overline{D}^2 D_\alpha V \,. \tag{3.10}$$

It can be seen that according to Equation (3.6) $\mathcal{N}=2$ vector multiplet has two pieces, namely an $\mathcal{N}=1$ gaugino superfield W_α and an $\mathcal{N}=1$ chiral superfield Φ. Then we can write down $\mathcal{N}=2$ supersymmetric Yang-Mills Lagrangian in the superspace as

$$\mathcal{L}^{\text{SYM}}_{\mathcal{N}=2} = \frac{1}{8\pi i}\int d^2\theta \text{Tr}\left(\tau W_\alpha W^\alpha + \text{ c.c.}\right) + \frac{\text{Im}(\tau)}{4\pi}\int d^2\theta d^2\overline{\theta}\text{Tr}\left(\Phi^\dagger e^{\text{adj}(V)}\Phi\right). \tag{3.11}$$

The meaning of the τ will be clear after doing the superspace integrals and *c.c.* means that complex conjugation. The first term gives the kinetic Yang-Mills term for the $\mathcal{N}=1$ as we will see and the second term corresponds to the $\mathcal{N}=1$ kinetic term for the chiral superfield coupled with the vector superfield. Note that the chiral superfield is in the adjoint representation of the gauge group G. We can perform the superspace integrals and expand the above Lagrangian in terms of component fields

$$\begin{aligned}\mathcal{L}^{\text{SYM}}_{\mathcal{N}=2} = \ & \frac{1}{g^2_{\text{YM}}}\text{Tr}\Big(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + g^2_{\text{YM}}\frac{\theta}{32\pi^2}F_{\mu\nu}\widetilde{F}^{\mu\nu} + (\nabla_\mu\phi)^\dagger\nabla^\mu\phi - \frac{1}{2}[\phi^\dagger\,,\,\phi]^2 \\ & - i\,\lambda\sigma^\mu\nabla_\mu\overline{\lambda} - i\,\overline{\psi}\overline{\sigma}^\mu\nabla_\mu\psi - i\sqrt{2}\,[\lambda,\psi]\,\phi^\dagger - i\sqrt{2}\,[\overline{\lambda},\overline{\psi}]\,\phi\Big)\,,\end{aligned} \tag{3.12}$$

where $\widetilde{F}^{\mu\nu} = \varepsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}/2$, it is the Hodge dual operation in four dimensions. Note we have eliminated the auxiliary fields D and F by using the equations of motion. The gauge covariant derivative is denoted by $\nabla_\mu = \partial_\mu\lambda - ig[A_\mu,\lambda]$. It is easy to see that τ is nothing but the complexified gauge coupling which includes the Yang-Mills coupling and the theta angle,

$$\tau = \frac{4\pi i}{g^2_{\text{YM}}} + \frac{\theta_{\text{YM}}}{2\pi}\,. \tag{3.13}$$

The $\mathcal{N}=2$ supersymmetric Lagrangian for the hypermultiplets has the kinetic terms for $(\mathbf{Q},\widetilde{\mathbf{Q}})$ of (3.7) and the coupling with the real vector superfield V in the representation of the gauge group G,rep$_G$,and if we add the corresponding masses,

$$\mathcal{L}^{\text{matter}}_{\mathcal{N}=2} = \int d^2\theta d^2\overline{\theta}\left[\mathbf{Q}^\dagger e^{\text{rep}_G(V)}\mathbf{Q} + \widetilde{\mathbf{Q}}^\dagger e^{\overline{\text{rep}}_G}\widetilde{\mathbf{Q}}\right] + \int d^2\theta\left(\widetilde{\mathbf{Q}}\text{r}_G(\Phi)\mathbf{Q} + \mu\widetilde{\mathbf{Q}}\mathbf{Q}\right) + \text{ c.c.}\,. \tag{3.14}$$

The $\mathcal{N}=2$ supersymmetry can also be acquired by using $\mathcal{N}=2$ superspace notation in a more manifest way. To do so, one can introduce further anticommuting Grassmannian coordinates $\widetilde{\theta}_\alpha, \overline{\widetilde{\theta}}_{\dot{\alpha}}$ in addition to the anticommuting Grassmannian variables of $\mathcal{N}=1$ supersymmetry, $\theta_\alpha, \overline{\theta}_{\dot{\alpha}}$. Then, we can write a new chiral superfield in terms of $\mathcal{N}=2$

$$\Psi = \Phi(\widetilde{y}, \theta) + \sqrt{2}\widetilde{\theta}^{\alpha} W_{\alpha}(\widetilde{y}, \theta) + \widetilde{\theta}^{\alpha}\widetilde{\theta}_{\alpha} G(\widetilde{y}, \theta)\,, \tag{3.15}$$

where $\widetilde{y}^{\mu} = x^{\mu} + i\theta\sigma\overline{\theta} + i\widetilde{\theta}\sigma\overline{\widetilde{\theta}} = y^{\mu} + i\widetilde{\theta}\sigma\overline{\widetilde{\theta}}$ and $(\widetilde{y}, \theta)$ is given by

$$G(\widetilde{y}, \theta) = -\frac{1}{2}\int d^2\overline{\theta}\left[\Phi(\widetilde{y} - i\theta\sigma\overline{\theta}, \theta, \overline{\theta})\right]^{\dagger} e^{\left[-2gV(\widetilde{y} - i\theta\sigma\overline{\theta}, \theta, \overline{\theta})\right]}\,. \tag{3.16}$$

$\Phi(x, \theta, \overline{\theta})$ is chiral superfield, $\Phi(y, \theta)$ is the transformed chiral superfield and $W(y, \theta)$ is the gaugino superfield which we have discussed before. Then the action of the $\mathcal{N} = 2$ supersymmetric Yang-Mills theory in the $\mathcal{N} = 2$ superspace formalism reads

$$S^{\mathrm{SYM}}_{\mathcal{N}=2} = \mathrm{Im}\left[\frac{\tau}{32\pi}\int d^4x d^2\widetilde{\theta}\,\mathrm{Tr}\Psi^2\right]\,, \tag{3.17}$$

analogously with $\mathcal{N} = (2, 2)$ supersymmetric non-linear sigma model, we can define more general supersymmetric action in the $\mathcal{N} = 2$ superspace language as

$$S^{\mathrm{SYM}}_{\mathcal{N}=2} = \mathrm{Im}\left[\frac{\tau}{16\pi}\int d^4x d^2\theta d^2\widetilde{\theta}\mathcal{F}(\Psi)\right]\,, \tag{3.18}$$

where $\mathcal{F}(\Psi)$ is so-called the $\mathcal{N} = 2$ prepotential and it has dependence on Ψ solely. This type of holomorphic dependence of prepotential has great importance for examining the dynamics of $\mathcal{N} = 2$ supersymmetric gauge theories. Note that if we compare Equation (3.17) with Equation (3.18), we find that the classical prepotential of the $\mathcal{N} = 2$ supersymmetric Yang-Mills action as

$$\mathcal{F}(\Psi) \equiv \mathcal{F}_{\mathrm{class}}(\Psi) = \frac{1}{2}\,\mathrm{Tr}\{\tau\Psi^2\}\,. \tag{3.19}$$

The prepotential is referred to as a classical one since once we take into account the quantum corrections to the effective actions which are constructed from the microscopic gauge theory, the general form of $\mathcal{F}$ changes by means of the perturbative and nonperturbative contributions as we shall see. The general action of the $\mathcal{N} = 2$ supersymmetric Yang-Mills theory that we have discussed above can be rewritten in terms of $\mathcal{N} = 1$ supersymmetry by integrating out the second Grassmannian coordinate $\widetilde{\theta}$

$$S^{\mathrm{SYM}}_{\mathcal{N}=2} = \frac{1}{16\pi}\,\mathrm{Im}\int d^4x\left[\int d^2\theta\frac{\partial\mathcal{F}(\Phi)}{\partial\Phi^i\partial\Phi^j}W^i_{\alpha}W^{j\alpha} + \int d^2\theta d^2\widetilde{\theta}\left(\Phi^{+}e^{-2gV}\right)^i\frac{\partial\mathcal{F}(\Phi)}{\partial\Phi^i}\right]\,. \tag{3.20}$$

3.1.1. Moduli Space of Vacua of $\mathcal{N} = 2$ Supersymmetry

In this section, we begin to discuss one of the most important concepts of the supersymmetric field theories, the moduli space of the vacua of the corresponding supersymmetric gauge theory. The moduli space of the vacua can be found from the scalar potential $V(\phi, q, \widetilde{q})$. The general scalar potential, including hypermultiplet contributions as well, can be found from the Lagrangian Equation (3.8) by using the classical values of the other fields, in other words, one needs to use equations of motion for the auxiliary fields and eliminate them. Therefore, the classical scalar potential V is a sum of squares of the auxiliary fields,

$$V(\phi, q, \widetilde{q}) = \frac{1}{2}\mathrm{Tr}\left(D^2\right) + F^\dagger F\,, \tag{3.21}$$

as we emphasized above, the auxiliary fields D and F depend on the physical scalar fields $\phi, q, \widetilde{q}$ through the equations of motion. The moduli space of supersymmetric vacua is the zero loci of the scalar potential,

$$\mathcal{M}^{classical} = \{\phi, q, \widetilde{q}\,|V(\phi, q, \widetilde{q}) = 0\}\ , \tag{3.22}$$

where the vacuum(classically) is defined as the minima of the scalar potential similar to the nonsupersymmetric quantum field theories. Since $V(\phi, q, \widetilde{q})$ includes positive definite pieces, the necessary and the sufficient condition for supersymmetric vacua can be rephrased as

$$D(\phi, q, \widetilde{q}) = F(\phi, q, \widetilde{q}) = 0\,. \tag{3.23}$$

As an example, consider the $\mathcal{N} = 2$ supersymmetric Yang-Mills theory with the gauge group is given by $G = SU(N)$ and with N_f matter fields $H^A_{\mathcal{N}=2}$, $A = 1, \ldots, N_f$ in the fundamental representation of $SU(N)$ with the generators T^i, where $i = 1, \cdots, N^2-1$. The condition for the auxiliary field D is given by

$$D(\phi, q, \widetilde{q}) = \frac{1}{g^2_{\mathrm{YM}}}\left[\phi, \phi^\dagger\right] + \left(q_A(q^B)^\dagger - (\widetilde{q}_A)^\dagger \widetilde{q}^B\right)(T^i)^A_B = 0\,, \tag{3.24}$$

along with F-term conditions. The supersymmetric moduli space of the classical vacua can be found by the following equations

$$\begin{aligned}
&\left[\phi, \phi^\dagger\right] = 0\,,\\
&\left(q_A(q^B)^\dagger - (\widetilde{q}_A)^\dagger \widetilde{q}^B\right)\left(T^i\right)^A_B = 0\,,\\
&q_A \widetilde{q}^B\left(T^i\right)^A_B = 0\,,\\
&\phi q_A = \widetilde{q}^A \phi^\dagger = 0\,,\\
&\phi^\dagger q_A = \widetilde{q}^A \phi = 0\,.
\end{aligned} \tag{3.25}$$

Note that ϕ transforms under adjoint representation of the gauge group $SU(N)$ and q_A ($\widetilde{q}^A$) are the scalar components of the matter part which is given by the hypermultiplets. The solutions of these equations are characterized by three different conditions and according to that they are called Coulomb Branch, Higgs Branch in the literature.

The Coulomb Branch (CB) $\mathcal{M}_{CB}$ is defined in a way that scalar fields that sit in the hypermultiplet q and $\widetilde{q}$ have zero classical value, or from a quantum field theoretical language, they have no vacuum expectation value(VEV). Therefore we are left with the equation for the general gauge group

$$\mathcal{M}_{\rm CB} = \left\{ \ [\phi, \phi^\dagger] = 0 \ \right\} \Big/ \left\{ G - \text{gauge transformation} \right\}. \tag{3.26}$$

This equation has non-trivial solutions when scalar fields commute, resulting in ϕ belonging to the Cartan subalgebra of the gauge group with the Lie-algebra g modulo gauge transformations. The number of independent commuting fields determines the set of solutions, which is equivalent to the rank of the gauge group. This determines the dimension of the moduli space $\mathcal{M}_{CB}$ of the Coulomb Branch as

$$\dim_{\mathbb{C}}(\mathcal{M}_{CB}) = \text{rank}(G)\,, \tag{3.27}$$

and the corresponding coordinates for the $\mathcal{M}_{CB}$ are given by the VEVs of the gauge invariant combinations of the set of fields in the Coulomb Branch. If we return to our particular case, $G = SU(N)$, the coordinates of the moduli space are given by the chiral Coulomb Branch fields, $\text{Tr}(\phi^k)$, for $k = 2, \ldots, N$, as well as the anti-chiral counterparts, are equal to the $\text{Tr}(\overline{\phi}^k)$. As we have mentioned, the non-trivial solutions admit VEVs and they give rise to break the gauge group G to the maximal torus subgroup $U(1)^r$, where $r = \text{rank}(G)$. This means that the low-energy effective behavior of the Coulomb Branch is determined by the $\mathcal{N} = 2$ supersymmetric gauge theory whose gauge group is given by $U(1)^r$. The Higgs branch (HB) of the moduli space is defined by the condition in which the hypermultiplet scalars q and $\widetilde{q}$ have non-trivial VEV and the vector multiplet scalar ϕ has a zero VEV. Therefore, we have:

$$\mathcal{M}_{\rm HB} = \left\{ \begin{array}{c} (q_A q^{\dagger\, B} - \widetilde{q}^{\dagger}_A \widetilde{q}^B)(T^i)^A_B = 0 \\ (q_a \widetilde{q}^b)(T^i)^a_b = 0 \end{array} \right\} \Big/ \left\{ G - \text{gauge transformation} \right\}, \tag{3.28}$$

with the complex dimensions of the Higgs branch of the moduli space

$$\dim_{\mathbb{C}}(\mathcal{M}_{\rm HB}) = 2(n_H - n_V)\,, \tag{3.29}$$

where n_H and n_V denote the number of hypermultiplets and vector multiplets that respectively in the Higgs phase and note that the Higgs branch (HB) of the moduli space also corresponds to gauge inequivalent vacua. The Mixed Branch is the phase of theory where vector multiplet and hypermultiplet scalars have non-zero VEV. We will not discuss them further.

3.1.2. Adding Central Charges and BPS bound

In the previous section, we ignored the central extension of the supersymmetry algebra, however, the central charges play an important role in the dynamics of the supersymmetry field theories since they correspond to the BPS states which are somewhat very special as we will see. The extended supersymmetry algebra which is obtained by adding the central charges can be given as

$$\begin{aligned} \left\{Q^I_\alpha, \widetilde{Q}_{\dot{\alpha}J}\right\} &= 2\sigma^\mu_{\alpha\dot{\alpha}} P_\mu \delta^I_J\,, \\ \left\{Q^I_\alpha, Q^J_\beta\right\} &= 2\sqrt{2}\epsilon_{\alpha\beta} Z^{IJ}\,, \\ \left\{\widetilde{Q}_{\dot{\alpha}I}, \widetilde{Q}_{\dot{\beta}J}\right\} &= 2\sqrt{2}\epsilon_{\alpha\dot{\beta}} Z^*_{IJ}\,, \end{aligned} \tag{3.30}$$

where the Z^{IJ} is called the central charge matrix and it is purely antisymmetric. By definition, it commutes with the other observables of the theory and for that reason, the central charges can be thought of as the additional conserved charges. If the number of supersymmetry $\mathcal{N}$ is an even number, the central charge matrix can be made as the skew-diagonalized matrix. Then we can write that the central charge matrix as $Z = \epsilon \otimes D$, where D stands for the $\mathcal{N}/2$ dimensional diagonal matrix and the ϵ is the two-dimensional Levi-Civita tensor. Therefore, the supersymmetry algebra in terms of this decomposition turns out to be

$$\begin{aligned} \left\{Q^I_\alpha, \widetilde{Q}_{\dot{\alpha}J}\right\} &= 2\sigma^\mu_{\alpha\dot{\beta}} P_\mu \delta^I_J\,, \\ \left\{Q^a_\alpha, Q^b_\beta\right\} &= 2\sqrt{2}\epsilon_{\alpha\beta}\epsilon^{ab} Z\,, \\ \left\{\widetilde{Q}_{\dot{\alpha}a}, \widetilde{Q}_{\dot{\alpha}b}\right\} &= 2\sqrt{2}\epsilon_{\dot{\alpha}\dot{\alpha}}\epsilon_{ab} Z^*\,, \end{aligned} \tag{3.31}$$

where Z is a complex number and it commutes with the other observables. Let us choose the rest frame to find an irreducible representation of the above algebra, $P_\mu = (M, 0, 0, 0)$. This causes the Lorentz symmetry group $SO(3,1)$ to break the spatial rotation group $SO(3)$ at four dimensions. Therefore, we can define creation and annihilation operators $\mathcal{A}$ and $\mathcal{B}$ which is a general linear combination of the supersymmetric generators Q^1 and Q^2:

$$\mathcal{A}^I_\alpha = \frac{1}{2}\left(Q^1_\alpha + e^{i\varphi}\epsilon_{\alpha\beta}\left(Q^2_\beta\right)^\dagger\right)^I, \quad \mathcal{B}^I_\alpha = \frac{1}{2}\left(Q^1_\alpha - e^{-i\varphi}\epsilon_{\alpha\beta}\left(Q^2_\beta\right)^\dagger\right)^I, \tag{3.32}$$

where $e^{i\varphi}$ is just a phase. Since we separate the dotted and undotted spinor indices, the only non-trivial commutator is given by

$$\begin{aligned} \left\{\mathcal{A}^I_\alpha, \mathcal{A}^{\dagger J}_\gamma\right\} &= \frac{1}{4}\left\{(Q^1_\alpha + e^{i\varphi}\epsilon_{\alpha\beta}\,(Q^2_\beta)^\dagger)^J, ((Q^1)^\dagger_\gamma + e^{-i\varphi}\epsilon_{\gamma\delta}Q^2_\delta)^J\right\} \\ &= \delta_{\alpha\gamma}\delta^{IJ}\left(M + \sqrt{2}\,\mathrm{Re}\left(e^{-i\varphi}Z_J\right)\right) . \end{aligned} \tag{3.33}$$

We have used the following identity, $\epsilon_{\alpha\beta}\epsilon_{\gamma\delta} = \delta_{\alpha\gamma}\delta_{\beta\delta} - \delta_{\alpha\delta}\delta_{\beta\gamma}$, in the second line. The same form can be deduced from the other supercharge $\mathcal{B}$,

$$\left\{\mathcal{B}^I_l, \mathcal{B}^{\dagger J}_\gamma\right\} = \delta_{\alpha\gamma}\delta^{IJ}\left(M - \sqrt{2}\,\mathrm{Re}\left(e^{-i\varphi}Z_J\right)\right) . \tag{3.35}$$

Furthermore, let us consider an operator A with the following property $\{A, A^\dagger\} = c$ where c is a positive definite constant. If we choose a state $|\psi\rangle$ which the operator A acts on, we have that

$$\left|A^\dagger|\psi\rangle\right|^2 + |A|\psi\rangle|^2 = \langle\psi|AA^\dagger|\psi\rangle + \langle\psi|A^\dagger A|\psi\rangle = c\langle\psi|\psi\rangle\,, \tag{3.36}$$

this gives the fact that $c \geq 0$. Now we can identify the operator A with $\mathcal{A}$. This results from the following relation

$$M \geq \mathrm{Re}(e^{-i\varphi}Z)\,. \tag{3.37}$$

Since the phase term is nothing but the rotation of the Z in the complex plane can be concluded that

$$M \geq \sqrt{2}|Z|\,. \tag{3.38}$$

This relation between the central charges and the masses is the celebrated Bogomol'nyi-Prasad-Sommerfield bound or BPS bound. The BPS bound is one of the most important properties of the supersymmetric field theories since their mass-charge relation is protected by supersymmetry. There are two distinct cases for the unitary irreducible representations of the supersymmetry algebra in the context of the BPS bound. They are massless and massive representations. For the sake of an argument, let us consider the generic case with $\mathcal{N}$ supersymmetry. One can start with the supersymmetric vacuum with the assumption of no supersymmetry breaking at all. The supersymmetric vacuum, $|\Omega\rangle$, is annihilated by the $\mathcal{A}^I_\alpha$ and we can find the higher level supersymmetric states by acting with the creation operators $\mathcal{A}^{\dagger J}_\alpha$. The number of states that are acquired by acting creation operators are $\binom{2\mathcal{N}}{n}$, where the n denotes the oscillator level of particular excitation. Therefore, total number of states is given by the $\sum_{n=0}^{2\mathcal{N}}$ $\binom{2\mathcal{N}}{n}$ which splits into bosonic and fermionic excitations. Recall that the central charge matrix is skew-diagonalized and there are two different creation annihilation operators according to the definition we made at Equation (3.32). If the bound saturates, we have the fact that all Z_r's are equal to $\sqrt{2}M$, where $0 \leq r \leq \mathcal{N}/2$, and due to mass-charge saturation, $2r$ of the $\mathcal{B}$-oscillators are zero and we are left with $2\mathcal{N} - 2r$ times creation and annihilation operators. Therefore, the number of massive states that saturate the bound is equal to $2^{2\mathcal{N}-2r}$. When r is equal to the $\mathcal{N}/2$, the corresponding multiplet is called short BPS multiplet. In the remaining case, $0 < r < \mathcal{N}/2$, are called as intermediate or long BPS multiplets. The short multiplets are special in the sense that they are free of quantum corrections as long as the supersymmetry is broken.

Recall that one can usually determine the corresponding Hilbert spaces, coupling constants, and masses as well as their quantum-corrected versions when the coupling constant is small by the machinery of perturbation theory. When we move along the space of coupling constants to the strongly coupled regime, it is generally not possible to do that since the perturbation theory breaks down. On the other hand, the short BPS multiplets remain BPS since they cannot disintegrate into some other states which leads to violation of the saturation of the BPS bound. In other words, if there are no further states which can mix the short multiplets into the long ones as the coupling constant varies, the mass-charge relation is protected. The BPS states can still live in the strongly coupled regime of the moduli space of the coupling constants and they are non-perturbative objects. Let us return to the massless case. We can choose a particular frame for the massless states as $P_\mu = (-E, 0, 0, E)$. The supersymmetry algebra turns out to be

$$\{Q^I_\alpha, \widetilde{Q}_{J\dot{\alpha}}\} = 2 \begin{pmatrix} E & 0 \\ 0 & 0 \end{pmatrix} \delta^I_J\,, \tag{3.39}$$

where the other (anti-)commutators are zero. By using this, we can see that $Q^I_2, \widetilde{Q}^I_{\dot{2}}$ must be zero since they anti-commute. We can define the creation and annihilation operators out of $\mathcal{N}$ surviving supercharges

$$\mathcal{A}^I = \frac{Q^I_1}{2\sqrt{E}}, \quad \mathcal{A}^{\dagger I} = \frac{\widetilde{Q}^I_{\dot{1}}}{2\sqrt{E}}\,. \tag{3.40}$$

The dimension of the massless representation is simply $2^{\mathcal{N}}$ which coincides with the dimension of the Hilbert space of massive short multiplets. Hence, they respect the BPS bound trivially.

To sum up, the BPS states are very special their quantum numbers are constrained by the supersymmetry algebra as we have seen. Their masses are completely determined by their charges as well as the other parameters of the supersymmetric field theories. The mass-charge relation is nonperturbatively exact unless supersymmetry is broken or the short multiplets become long multiplets.

3.1.3. Effective Field Theory approach to $\mathcal{N} = 2$ Theories

In this section, we will look at the dynamical behavior of the $\mathcal{N} = 2$ supersymmetric gauge theories through the Coulomb branch effective field theory. We have introduced that the microscopic (UV) gauge group is broken to the $U(1)^r$, where r is the rank of the gauge group. The relevant degrees of the freedom of theory in terms of low energy counterpart is given by the r times $\mathcal{N} = 2$ Abelian vector multiplets and the Coulomb Branch of the moduli space is parametrized by the complex scalar fields, $U(1)^r$ gauge fields and the supersymmetric fermionic partners. After the Higgsing of the non-Abelian vector multiplets, we have the Abelian vector multiplets which are the Cartan generators of the gauge group G with a rank r. Remind that the non-Abelian vector multiplet scalar admits a VEV $\langle\phi\rangle$. If the gauge group is $G = SU(r+1)$, modulo gauge transformations, the VEV of the ϕ is given by diagonal entries of the $r \times r$ matrix

$$\langle\phi\rangle = \text{diag}\left(a^1, a^2, \ldots, a^{r+1} = -\sum_{i=1}^{r} a^i\right) . \tag{3.41}$$

The values a^i lead W-bosons to gain mass by the Higgs mechanism $m_W \sim |a^i - a^j|$ and their masses are determined by $m_M \sim |a^i|/g_i^2$ in whiche g_i stands for the Abelian gauge theory couplings in the IR. Note that we are interested in the behavior of a field theory at energies lower than some momentum cutoff Λ, that is the characteristic energy scale of the theory in the UV. This can be done by the Wilsonian approach to finding effective action which governs the low energy dynamics of the theory with the lower cutoff Λ. The idea of getting the Wilsonian effective action is to integrate out higher-momentum modes, $k > \Lambda$, to have the effective field theory. Then the resulting theory is insensitive to the microscopic physics. This is a very hard task in general, however, $\mathcal{N} = 2$ supersymmetry has stronger constraints on the form of the effective field theory Lagrangian on the Coulomb Branch due to holomorphicity for instance. The corresponding Wilsonian effective action is given by [34]

$$S_{\text{eff}} = \frac{1}{4\pi}\text{Im}\int d^4x \left[\int d^4\theta \frac{\partial \mathcal{F}(\Phi)}{\partial \Phi^i}\overline{\Phi}^i + \int d^2\theta \frac{1}{2}\frac{\partial^2 \mathcal{F}(\Phi)}{\partial \Phi^i \partial \Phi^j} W^i_\alpha W^{j\alpha}\right], \quad i,j = 1,...,r\,. \tag{3.42}$$

Φ^i and W^i_α are chiral and gaugino superfields in the $\mathcal{N}=2$ Abelian vector multiplets and they attain a VEV through the a^i, i.e., $\phi^i \to a^i$ As we shall see that the moduli space of Coulomb Branch is nothing but the Kähler manifold. To see this, let us denote the scalar fields with the corresponding VEVs, we can write the second term of Equation (3.42):

$$\text{Im}\left(\frac{\partial^2 \mathcal{F}(a)}{\partial a^i \partial a^j} da^i d\overline{a}^{\bar{j}}\right)\,. \tag{3.43}$$

Furthermore, we can expand the first term of the Wilsonian effective action

$$\frac{1}{4\pi}\,\text{Im}\int d^4x \left[\frac{\partial^2 \mathcal{F}(a)}{\partial a^i \partial a^j}\nabla_\mu \phi^i \nabla^\mu \overline{\phi}^{\bar{j}} - i\frac{\partial^2 \mathcal{F}(a)}{\partial a^i \partial a^j}\overline{\psi}^i \overline{\sigma}^\mu \nabla_\mu \overline{\psi}^j + \dots\right]. \tag{3.44}$$

Recall that this has the same form of the Kähler potential in the case of the non-linear sigma model. Then the second derivative of the prepotential gives a metric on the Kähler manifold

$$ds^2_{\mathcal{M}_{CB}} = g_{i\bar{j}} da^i da^{\bar{j}} = \partial_i \partial_{\bar{j}} K = \text{Im}\left(\frac{\partial^2 \mathcal{F}(a)}{\partial a^i \partial a^j}\right). \tag{3.45}$$

Additionally, together with some physical requirements, such as unitarity necessitates the scalar kinetic term of to be positive, thus the prepotential $\mathcal{F}(a)$ is constrained such that the sigma model metric (3.45) is positive-definite. Therefore, we have a natural metric structure on the Coulomb Branch moduli space $\mathcal{M}_{\text{CB}}$ which are parametrized by the scalar fields and it is governed by the Kähler potential analogously the Kähler manifold. If we expand the second term on the RHS of (3.42) we will find that the standard kinetic term for the $U(1)^r$ gauge fields as

$$\mathcal{L}_{U(1)^r} = \frac{1}{16\pi}\,\text{Im}\left(\frac{\partial^2 \mathcal{F}(a)}{\partial a^i \partial a^j}\right)\left[F^i_{\mu\nu}\widetilde{F}^{\mu\nu j} + F^i_{\mu\nu}F^{\mu\nu j}\right] \quad i,j = 1,...,r\,. \tag{3.46}$$

We can also identify the coupling constants of the $U(1)^r$ gauge fields with the help of prepotential $\mathcal{F}(a)$ since it determines the field-dependent complexified gauge coupling in the low energy regime(IR). Recall that the complexified gauge coupling is given by

$$\tau^{IR} = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2_{\text{YM}}}\,, \tag{3.47}$$

and we find that

$$\tau^{IR}_{ij}(a) = \frac{\partial^2 \mathcal{F}(a)}{\partial a^i \partial a^j}\,. \tag{3.48}$$

We can see that the imaginary part of complexified low energy gauge coupling is the same as the moduli space of the Coulomb Branch metric Equation (3.83) and similarly positive definite property of the Coulomb Branch metric forces coupling constants of the abelian low energy gauge theory to be positive. Thus we have a well-defined and consistent map between them. To summarize the main points above, we can deduce that the unitarity of the gauge theory requires the moduli space metric to be positive definite and the moduli space of the Coulomb Branch being the Kähler manifold gives the gauge couplings must be a positive quantity.

Note that so far the vacuum expectation values, a^i and $a^{\bar{i}}$ are taken as coordinates on the moduli space of the Coulomb Branch. However, this is not quite true since it is defined in a way that the whole vacua of the underlying supersymmetric gauge theory must consist of gauge inequivalent pieces, and the coordinates a^i and $a^{\bar{i}}$ are not gauge invariant. This can be seen from the simplest case, i.e., the gauge group $G = SU(2)$. As we discussed in the previous section, these non-zero vacuum expectation values break the gauge group of the $\mathcal{N} = 2$ super Yang-Mills theory as $SU(2) \to U(1)$ and there is a corresponding potential term in the $\mathcal{N} = 2$ super Yang-Mills theory

$$V(\phi) = \frac{1}{2g_{\text{YM}}^2} \operatorname{Tr} \left[\phi, \phi^\dagger\right]^2 , \tag{3.49}$$

and the classical vacua are determined by the zeroes of the above potential. This condition does not necessarily require that ϕ vanish and the necessary and sufficient condition is that ϕ and $\phi^\dagger$ commute. For the gauge group $SU(2)$, we have three generators T^i, $i = 1, 2, 3$ and they are given by the Pauli matrices $T^i = \frac{1}{2}\sigma^i$. We can identify ϕ with $\frac{1}{2}a\sigma_3$ by performing a gauge rotation without loss of generality, where a is a complex Coulomb Branch parameter. Due to the presence of a nonzero VEV in the direction of the third generator, there will be two abelian vector multiplets which give rise to massive $W^\pm$ bosons and their masses are given by the Coulomb Branch parameter.

In the remaining direction, we have massless $U(1)$ gauge theory. However, there is also discrete gauge rotation around the unbroken directions of $SU(2)$ which results from a transformation on the Coulomb Branch parameter, i.e. a to $-a$. Thus, due to this ambiguity, the correct gauge invariant quantity that parametrizes gauge inequivalent vacua is $a^2/2$ rather than a. Then, we can write

$$u \equiv \left\langle \mathrm{Tr}\, \phi^2 \right\rangle \,, \tag{3.50}$$

where the trace is taken over Lie-algebra indices and, classically we have $u = a^2/2$. Hence, the parameter u is the complex coordinate on the moduli space of the $SU(2)$ $\mathcal{N} = 2$ gauge theory. The generalization of the above case to a higher-rank gauge group is straightforward. The complex coordinates on the moduli space of the $SU(N)$ $\mathcal{N} = 2$ gauge theory is given by

$$u^i \equiv \mathrm{Tr} \left\langle \phi^{i+1} \right\rangle , \quad i = 1, \ldots, r \,. \tag{3.51}$$

The complex coordinates u^i are gauge invariant operators which parametrize $\mathcal{M}_{CB}$ and r is the rank of the gauge group, $SU(r+1)$. Similarly, the Coulomb Branch parameters $\{a^i\}$ are not gauge invariant since they do change under the Weyl group S_{r+1} of $G = SU(r+1)$. Note that for $SU(2)$ case, Weyl group is just a reflection symmetry.

Accordingly, the moduli space of vacua, $\mathcal{M}_{CB}$, can be thought of as the differentiable manifold with an inherited Kähler structure, and the coordinates $\{a^i\}$ are only defined on it locally, and as we have stressed that they are not free from the gauge transformations. Furthermore, there is an additional ambiguity that we did not discuss so far, they also respect duality transformations from one patch of $\mathcal{M}_{CB}$ to the other patch as we shall see in a moment. This redundancy comes from the non-unique property of the microscopic theory in the IR and there is a family of effective field theories which connect by the electric-magnetic duality. Furthermore, this duality is an exact nonperturbative duality, since elementary degrees of freedom of the effective Lagrangian in one duality frame transform to the solitonic states of the other equivalent low energy effective theory in the IR.

3.1.4. Electric-Magnetic Duality

We have seen that the effective field theory on the Coulomb Branch is given by the Lagrangian (3.42) for the Abelian vector multiplets. As we have discussed, there are many but equivalent low-energy effective Lagrangians, and their fundamental degrees of freedom change from one duality frame to another by the electric-magnetic duality. In our case, it is always possible to find a particular weakly coupled effective field theory on $\mathcal{M}_{\rm CB}$ locally. Nevertheless, the global story is different since the Coulomb Branch effective field theory enjoys non-trivial duality transformations and one can determine the whole supersymmetric vacua by gluing the local patches of $\mathcal{M}_{\rm CB}$ via duality transformations. To find the explicit duality transformations, we can focus on the gauge field terms in the low-energy effective action. Let us consider only $U(1)$ gauge theory for simplicity, however, we shall discuss the supersymmetric version of the higher-rank case later. We can rewrite the kinetic and theta term of Abelian gauge theory as a more compact form

$$S_{U(1)} = -\frac{1}{32\pi}\int \operatorname{Im}\tau(F+i\widetilde{F})^2\,, \tag{3.52}$$

where $\widetilde{F}$ corresponds the the Hodge dual of F, i.e. $\widetilde{F}^{\mu\nu} = \varepsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}/2$ and we suppressed the integration variables in the action. We can expand the corresponding Lagrangian of the gauge term by using the fact that $(\widetilde{F})^2 = -F^2$

$$\begin{aligned} -\frac{1}{32\pi}\int \operatorname{Im}\tau(F+i\widetilde{F})^2 &= \frac{1}{16\pi}\operatorname{Im}\tau\left(F^2 + iF\widetilde{F}\right) \\ &= \frac{1}{4g_{\rm YM}^2}F_{\mu\nu}F^{\mu\nu} + \frac{\theta}{32\pi^2}F_{\mu\nu}\widetilde{F}^{\mu\nu}\,. \end{aligned} \tag{3.53}$$

We find the usual form of the kinetic and theta term of $U(1)$ gauge field without surprise. Firstly, duality affects not only the classical Lagrangian but also the path integral of theory. The path integral of the Abelian gauge field together with the theta term is given by

$$\begin{aligned} Z &= \int \mathcal{D}A_\mu \exp\left[\int d^4x\left(\frac{1}{4g_{\rm YM}^2}F_{\mu\nu}F^{\mu\nu} + \frac{\theta}{32\pi^2}F_{\mu\nu}\widetilde{F}^{\mu\nu}\right)\right] \\ &= \int \mathcal{D}A_\mu \exp\left[-\frac{1}{32\pi}\int \operatorname{Im}\tau(F+i\widetilde{F})^2)\right]. \end{aligned} \tag{3.54}$$

In order to perform the duality transformation, it is more convenient to consider the field strength as a fundamental variable of the classical action and the path integral, rather than the gauge field A_μ itself. However, a constraint in the form of the Bianchi identity $dF = 0$ needs to be imposed on the action. This can be achieved by adding a Lagrangian multiplier to the original action. The dualized $U(1)$ vector field $A_{D\mu}$ with the field strength $F_{D\mu\nu} = \partial_\mu A_{D\nu} - \partial_\nu A_{D\mu}$ corresponds to the Lagrange multiplier field. The Lagrange multiplier can be introduced by coupling the vector field to a magnetic monopole, which satisfies the constraint that is taken into account as an equation of motion for the dual theory

$$\epsilon^{0\rho\mu\nu}\partial_\rho F_{\mu\nu} = 8\pi\delta^{(3)}(x - x_0)\,, \tag{3.55}$$

where x_0 is the location of the monopole. Hence, the Lagrange multiplier term can be written as

$$\begin{aligned}
\frac{1}{8\pi}\int A_{D\mu}\epsilon^{\mu\nu\rho\sigma}\partial_\nu F_{\rho\sigma} &= -\frac{1}{8\pi}\int \epsilon^{\mu\nu\rho\sigma}\partial_\nu A_{D\mu}F_{\rho\sigma} \\
&= \frac{1}{16\pi}\int \epsilon^{\mu\nu\rho\sigma}\left(\partial_\mu A_{D\nu} - \partial_\nu A_{D\mu}\right)F_{\rho\sigma} \\
&= \frac{1}{16\pi}\int \epsilon^{\mu\nu\rho\sigma}F_{D\mu\nu}F_{\rho\sigma} \\
&= \frac{1}{8\pi}\int \widetilde{F}_D F\,.
\end{aligned} \tag{3.56}$$

Note that in the first line, we eliminate the boundary term and in the second line, we have used the antisymmetry property of the Levi-Civita tensor. Now we can write the term $\widetilde{F}_D F$ as

$$\frac{1}{8\pi}\int \widetilde{F}_D F = \frac{1}{16\pi}\ln\int (F_D + i\widetilde{F}_D)(F + i\widetilde{F})\,, \tag{3.57}$$

where we have used that $F_D F = \widetilde{F}_D\widetilde{F}$. This can be seen from the fact that the Hodge star property $\star^2 = 1$. Then, if we add the Lagrange multiplier term to the action $S_{U(1)}$ and complete the square, we find that

$$\begin{aligned}
&\frac{1}{32\pi}\operatorname{Im}\int\left(-\tau(F + i\widetilde{F})^2 + 2(F_D + i\widetilde{F}_D)(F + i\widetilde{F})\right) = \\
&= \frac{1}{32\pi}\operatorname{lm}\int\left(\left(\sqrt{\tau}(F + i\widetilde{F}) - \frac{1}{\sqrt{\tau}}(F_D + i\widetilde{F}_D)\right)^2 - \frac{1}{\tau}(F_D + i\widetilde{F}_D)^2\right)\,.
\end{aligned} \tag{3.58}$$

Note that the first term is nothing but the shift of the field strength $F_{\mu\nu}$ and the dual field strength can be considered as a dummy variable in the path integral, then we can relabel it

$$Z = \int[\mathcal{D}F][\mathcal{D}A_D]\exp\left[\frac{1}{32\pi}\operatorname{lm}\int\left(\sqrt{\tau}(F + i\widetilde{F}) - \frac{1}{\sqrt{\tau}}(F_D + i\widetilde{F}_D)\right)^2 - \frac{1}{\tau}(F_D + i\widetilde{F}_D)^2\right]$$

$$\begin{aligned}
Z &= \int[\mathcal{D}F_{new}]\exp\left[\frac{1}{32\pi}\operatorname{lm}\int(F_{new})^2\right]\int[\mathcal{D}A_D]\exp\left[-\frac{1}{32}\int\operatorname{Im}\frac{1}{\tau}(F_D + i\widetilde{F}_D)^2\right] \\
&= \mathcal{N}\int[\mathcal{D}A_D]\exp\left[-\frac{1}{32\pi}\int\operatorname{Im}\frac{1}{\tau}(F_D + i\widetilde{F}_D)^2\right]\,.
\end{aligned} \tag{3.59}$$

We integrated out the original field strength part of the integral by relabeling the integration variable of the path integral due to it being quadratic in field strength and the resulting is just a normalization constant for the remaining part since it is independent of the dual theory. Now if we look at the integrand of the resulting path integral closely, the dual action is given by

$$S^D_{U(1)} = \frac{1}{32\pi}\int \mathrm{Im}\,\frac{1}{\tau}(F_D + i\widetilde{F}_D)^2\,. \tag{3.60}$$

Hence, we have a new dual theory in terms of the Lagrange multiplier gauge field having the same functional form as the original gauge field with one crucial difference. As we have discussed, the duality maps a gauge field that is coupled with electric charges to a dual gauge field that is coupled with magnetic monopoles. If we define a dual complex coupling constant as $\tau_D = -1/\tau$, we can write

$$\tau \to \tau_D = -\frac{1}{\tau}\,. \tag{3.61}$$

Then, the electric-magnetic duality simply exchanges the dynamics of the theory in a way that the weakly coupled description becomes the strongly coupled one, this is the reason why electric-magnetic is a special form of S-duality, i.e., strong-weak duality. The above construction can be made for $\mathcal{N}=2$ supersymmetry supersymmetric gauge theories. Since the scalar complex field a_i is in the same supermultiplet as the gauge field A_i, they also respect electric-magnetic duality. Note that $\mathrm{Im}\left(\tau^{IR}_{ij}\right)$ is defined locally, and the global structure of the classical vacua is not free of perturbative and non-perturbative quantum effects. These quantum contributions may affect the classical moduli space. We will discuss these relations which are between the geometric properties of $\mathcal{M}_{CB}$ and the dynamical properties of the $\mathcal{N}=2$ supersymmetric gauge theories in terms of the Seiberg-Witten solution. If we define another coordinate as

$$a_D = \frac{\partial \mathcal{F}(a)}{\partial a}\,, \tag{3.62}$$

then the metric on the Coulomb Branch moduli space becomes

$$ds^2 = \mathrm{Im}\, da_D d\bar{a} = -\frac{i}{2}\left(da_D d\bar{a} - da d\bar{a}_D\right)\,. \tag{3.63}$$

The coordinates a and a_D are holomorphic functions of $u \equiv \mathrm{Tr}\,(\Phi^2)$ on $\mathcal{M}_{\mathrm{CB}}$ and we can write the more general duality transformation by $SL(2,\mathbb{Z})$ acting on (a, a_D) as

$$\begin{pmatrix} a_D \\ a \end{pmatrix} \to \begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} a_D \\ a \end{pmatrix}, \tag{3.64}$$

where $A,\ B,\ C,\ D \in \mathbb{Z}$ and the determinant of the matrix equal to one. The group $SL(2,\mathbb{Z})$ is so-called the special linear group of rank 2 over the integers. There are two generators of $SL(2,\mathbb{Z})$, which are given by the S and T matrices. One can obtain any element of $SL(2,\mathbb{Z})$ with a finite operation of S and T matrices. These building blocks for $SL(2,\mathbb{Z})$ transformations given by

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} . \tag{3.65}$$

The $SL(2,\mathbb{Z})$ acts on the complex plane:

$$z \to \frac{az+b}{cz+d} , \tag{3.66}$$

which is nothing but the Möbius transformation. Returning to our discussion of duality we can see the action of $SL(2,\mathbb{Z})$ on our theory. The above transformation does not act upon the metric (3.63) and it acts on the effective coupling τ^{IR} as

$$\tau^{IR} \to \frac{A\tau^{IR}+B}{C\tau^{IR}+D} , \tag{3.67}$$

where we have used

$$\tau^{IR} = \frac{\partial a_D}{\partial a} . \tag{3.68}$$

The duality transformation which we have discussed above is the same as the $SL(2,\mathbb{Z})$ transformation, in other words, the duality group is given by $SL(2,\mathbb{Z})$. One can be skeptical about why the duality group is given by $SL(2,\mathbb{Z})$ instead of $SL(2,\mathbb{R})$. The first argument comes from the theta term effective coupling. In order to have well-defined transformation law on the theta term, which is the integer shift on theta term $\theta \to \theta + 2\pi B$, the corresponding parameter must be an integer. The other reason which is related to the former but more concrete one is the Dirac quantization condition which is the quantization of the electromagnetic charges of elementary particle states. In order to have a well-defined duality transformation on the path integral such a transformation must respect the quantization of quantum numbers of corresponding states. Therefore, the full duality group is reduced by means of the quantum effects.

Let us return to the discussion of how other fields belong to the $\mathcal{N}=2$ vector multiplet with respect to the duality group. The coordinates which are defined in different patches of $\mathcal{M}_{\text{CB}}$,

$$(a, a_D) , \tag{3.69}$$

are sometimes called special coordinates just in the case of periods in the B-model. The reason for this will become clear as we will examine the geometric meaning when we study the Seiberg-Witten curve.

Recall that the coordinate a stands for the scalar component in an $\mathcal{N}=2$ vector multiple. The other coordinate a_D is the magnetic dual of a which sits on the dual $\mathcal{N}=2$ vector multiplet. One can also have another but equivalent picture of the dynamics of the gauge theory in terms of a_D by using the prepotential $\mathcal{F}$ via Equation (3.62). Then the prepotential has central importance for capturing all ingredients to determine the dynamical behavior of the underlying theory. Recall that the local structure of the moduli space of the Coulomb Branch of the effective field theory can be encoded through the pair of coordinates (a, a_D). Moreover, the global structure of $\mathcal{M}_{CB}$ needs a more sophisticated consideration since there can be singular points where such a local description cannot be enlarged to the global one as we will see. The gauge coupling can be determined via the relation $g_{\mathrm{YM}}^{-2} \sim \mathrm{Im}\,\frac{\partial a_D}{\partial a}$. According to this relation, there are some regions where the effective field theory behaves weakly and one can go to another region in which the effective field theory is written in terms of another doublet (a', a'_D) may become strongly interacting. All of these descriptions can be provided with the $SL(2,\mathbb{Z})$ duality group transformation properties. As a final remark, from a mathematical perspective, the coordinates (a, a_D) are the holomorphic section of an $SL(2,\mathbb{Z})$ bundle over the moduli space of the Coulomb Branch and duality transformations can be thought of as the transition functions of $SL(2,\mathbb{Z})$. On the other hand, global duality transformations can suffer from monodromies associated with the singular points of $\mathcal{M}_{CB}$. We can also generalize the above story to a higher rank-r gauge theory. The corresponding $\mathcal{M}_{\mathrm{CB}}$ becomes an r complex dimensional space and the dual also respect $Sp(2r,\mathbb{Z})$ and the special coordinates $\vec{a}$ and $\vec{a}_D$ transforms under $Sp(2r,\mathbb{Z})$ as an r-dimensional vector.

3.1.5. The Quantum Prepotential

Recall that the $\mathcal{N}=2$ Wilsonian low energy effective action in $\mathcal{N}=1$ superspace notation is given by

$$\frac{1}{4\pi}\,\mathrm{Im}\left[\int d^4\theta\Phi^\dagger\frac{\partial\mathcal{F}(\Phi)}{\partial\Phi}+\int d^2\theta\frac{1}{2}\frac{\partial^2\mathcal{F}(\Phi)}{\partial\Phi^2}W_\alpha W^\alpha\right]. \tag{3.70}$$

The holomorphic function $\mathcal{F}$ is called the prepotential and it is purely classical. it controls the kinetic terms of the low-energy Abelian theory. However, there can be perturbative quantum corrections to the classical prepotential as well as the non-perturbative contributions, such as instantons. Note that when we write the low energy effective action, there are no massive degrees of freedom in the IR since all of them are integrated out. However, the resulting Wilsonian effective action still includes interactions between the low energy massless fields and off-shell massive states since the massive modes that are integrated out are on-shell. Hence, we can conclude that $\mathcal{F}_{\text{class}}$ receives from both perturbative and non-perturbative quantum corrections. This is the stage where $\mathcal{N}=2$ supersymmetry helps us due to the non-renormalization theorem [33].

The β function for the coupling constant τ is exact at the one-loop level since the higher order loops are zero due to the fermionic compensation of the bosonic loop integrals. The relation between the effective coupling constant is given by $\tau(\Phi) = \partial^2\mathcal{F}/\partial\Phi^2$ and if we find that the form of one-loop correction to the effective coupling, we can determine the corresponding perturbative contribution to $\mathcal{F}$. However, we will take an alternative approach to this by using the global $U(1)_{\mathcal{R}}$-symmetry anomaly since we want to find the prepotential directly. We will need the form of effective coupling when we consider the nonperturbative instanton corrections. Recall that there is an additional $U(1)_{\mathcal{R}}$ symmetry of the microscopic theory and it is possible that this global symmetry is broken by the chiral anomaly due to the quantum effects which come from the loop integrals. We will not derive this and only quote the resulting anomaly relation for the $\mathcal{N}=2$ $SU(2)$ gauge theory is given by

$$\partial_\mu J_5^\mu = -\frac{1}{2\pi^2} F_{\mu\nu}\widetilde{F}^{\mu\nu}\,. \tag{3.71}$$

Furthermore, the $\mathcal{N}=1$ chiral superfield Φ transforms under the global $U(1)_{\mathcal{R}}$, $\Phi \to e^{2i\alpha}\Phi$ and this leads that the effective Lagrangian has the additional counterterm in order to be anomaly free:

$$\delta\mathcal{L}_{\text{eff}} = -\frac{\alpha}{4\pi^2} F\widetilde{F}\,. \tag{3.72}$$

We see that the above counterterm to the $\mathcal{L}_{\text{eff}}$ has a quadratic dependence on the field strength. Thus the relevant effect comes from only the kinetic term and the theta term in the $\mathcal{L}_{U(1)}$. Then, under the global $U(1)_{\mathcal{R}}$-symmetry we have the following relation

$$\frac{1}{16\pi}\,\text{Im}\left[\frac{\partial^2\mathcal{F}\left(e^{2i\alpha}\Phi\right)}{\partial\Phi\partial\Phi}(-F^2 + iF\widetilde{F})\right] = \frac{1}{16\pi}\,\text{Im}\left[\frac{\partial^2\mathcal{F}(\Phi)}{\partial\Phi\partial\Phi}(-F^2 + iF\widetilde{F})\right] - \frac{\alpha}{4\pi^2}F\widetilde{F}\,. \tag{3.73}$$

This gives the following differential equation

$$\frac{\partial^2 \mathcal{F}\left(e^{2i\alpha}\Phi\right)}{\partial\Phi\partial\Phi} = \frac{\partial^2 \mathcal{F}(\Phi)}{\partial\Phi\partial\Phi} - \frac{4\alpha}{\pi}\,. \tag{3.74}$$

If we take the derivative with respect to α at the point $\alpha = 0$, we find that the following equation

$$\frac{\partial^3 \mathcal{F}}{\partial\Phi^3} = \frac{2i}{\pi\Phi}\,. \tag{3.75}$$

The integration with respect to the chiral superfield Φ and setting the non-trivial VEV to the scalar component of it, i.e. $\phi \to a$ gives that the one-loop correction to the prepotential

$$\mathcal{F}_{\text{1-loop}}(\Lambda, a) = \frac{i}{2\pi} a^2 \log\left(\frac{a^2}{\Lambda^2}\right)\,, \tag{3.76}$$

where Λ denotes the dynamical energy scale which comes from the dimensional transmutation of $SU(2)$ gauge theory. Although the prepotential is one-loop exact in perturbation theory due to the supersymmetry, there are further non-perturbative corrections due to instanton effects [33]. In order to see this, we should incorporate one-loop running complexified gauge coupling. It is given by [33]

$$\tau = -\frac{i}{\pi}\log\left(\frac{a}{\Lambda}\right)\,. \tag{3.77}$$

We can also write the dual counterpart which is written terms of the dual coordinate a_D

$$\tau_D = -\frac{i}{\pi}\log\left(\frac{a_D}{\Lambda}\right)\,. \tag{3.78}$$

The nonperturbative part of the prepotential can be found as follows. The general instanton contribution which sits in the theta term in component field expansion of our supersymmetric gauge theory is given by $e^{-8\pi^2 k/g_{\text{YM}}^2}$, where k denotes the number of instantons and g is the gauge coupling. We will give a more detailed discussion about the instantons in the next chapter.

We can use the one-loop complexified gauge coupling Equation (3.77) in order to show that the k-instanton part can be rewritten as

$$e^{-8\pi^2 k/g_{\text{YM}}^2} = \left(\frac{\Lambda}{a}\right)^{4k}\,. \tag{3.79}$$

In order to have non-broken $U(1)_{\mathcal{R}}$ symmetry, $U(1)_R$ charge of Λ must be 2 since $\Phi \to e^{2i\alpha}\Phi$. Then the prepotential transforms under $U(1)_{\mathcal{R}}$ symmetry as $\mathcal{F} \to e^{4i\alpha}\mathcal{F}$ due to the one-loop form of it or in other words, the instanton correction term has a quadratic dependence on a. If we sum up all of these perturbative and non-perturbative corrections to the classical prepotential, we find that

$$\mathcal{F}(\Lambda, a) = \mathcal{F}_{\text{class}} + \frac{i}{2\pi} a^2 \log\left(\frac{a^2}{\Lambda^2}\right) + \sum_{k=1}^{\infty} \mathcal{F}_k \left(\frac{\Lambda}{a}\right)^{4k} a^2 \,. \tag{3.80}$$

The supersymmetry ensures that the instanton coefficients $\mathcal{F}_k$ are independent of the fields. Therefore, determining these coefficients is necessary to obtain the complete quantum prepotential, and note that we ignored the negative k contributions since the prepotential is a holomorphic function depending on instantons rather than anti-instantons. The first coefficient $\mathcal{F}_1$ was calculated in [33] and it is not an easy task to find the higher order coefficients. Note that if one can find the all coefficients, the theory is completely solved and the Wilsonian low-energy effective action can be computed exactly. One way of getting higher order instanton coefficients can be made by the celebrated work of Seiberg and Witten [34, 35] and we will discuss their method at the end of this chapter. Besides, there is a more direct way of getting higher order instantons, which is the instanton counting method. Nekrasov has been successful to write the exact form the instanton partition functions of the $\mathcal{N} = 2$ supersymmetric gauge theories by placing them in the Ω background in order to determine the exact form of the Seiberg-Witten prepotential [51]. We shall discuss his approach in the next chapter. Let us return to the determination of the prepotential via Seiberg-Witten theory. This approach requires understanding the global structure of the moduli space of vacua in terms of Coulomb Branch moduli $a(u)$, where u is the complex coordinate of $\mathcal{M}_{CB}$ as we have stressed out before. Note that if we can determine the prepotential, we can also find the dual theory which is expressed in terms of $a_D(u)$ through the definition of $a_D = \partial\mathcal{F}/\partial a$. Before we close this section, we can make a comment on the prepotential of higher-rank gauge groups. The higher rank gauge groups can be classified in terms of root vector $\vec{\alpha}$, and the one-loop prepotential is given by

$$\mathcal{F}_{\text{1-loop}}(\Lambda, \vec{a}) = \frac{i}{4\pi} \sum_{\alpha>0} (\vec{\alpha}\cdot\vec{a})^2 \log\left(\frac{(\vec{\alpha}\cdot\vec{a})^2}{\Lambda^2}\right) , \tag{3.81}$$

and the nonperturbative part can be found by the same reasoning above which is by reading the global R-symmetry charges of the theory.

3.2. Seiberg-Witten Solution in $\mathcal{N}=2$ Supersymmetric Gauge Theories

In order to understand the effects of monodromies around the various singular points of the moduli space of vacua of $SU(2)$ supersymmetric gauge theory, we first need to analyze the property of these singularities as well as their physical interpretations of them. Note that the moduli space can have nontrivial global topology due to the existence of singularities. Firstly, consider the large VEV of the scalar field limit which corresponds to $a \to \infty$. This type of limit is sometimes called the semiclassical or weak coupling limit as it will be clear later on. In this semiclassical limit, the dominant contributions to the holomorphic prepotential $\mathcal{F}$ come from the perturbative one-loop correction since nonperturbative effects are suppressed due to the presence of a^{2-4k} at the nonpertubative part of Equation (3.80). Thus at large a, the prepotential becomes

$$\mathcal{F}_{\text{semi-cl}}(\Lambda, a) = \frac{i}{2\pi} a^2 \log\left(\frac{a^2}{\Lambda^2}\right) . \tag{3.82}$$

The metric obtained by $\mathcal{F}_{\text{semi-cl}}(a)$ is perturbatively one-loop exact. In the semiclassical limit, we can determine the metric on the Coulomb Branch moduli space by using the relation

$$\begin{aligned} ds^2_{\text{semi-cl}} &= \operatorname{Im} \frac{i}{\pi}\left(2\log\left|\frac{a}{\Lambda}\right| + 3\right) da d\bar{a} \\ &= \left(\frac{2}{\pi}\log\left|\frac{a}{\Lambda}\right| + \frac{3}{\pi}\right) da d\bar{a} \, . \end{aligned} \tag{3.83}$$

Note that, the metric must be positive definite and it can be seen that the metric in the semi-classical or weak coupling region is positive when the VEV $|a|$ is large. However, when the $|a|$ is smaller and smaller, the metric which we found above turns out to be negative. Then, we can conclude that the metric which is the weakly coupled region of $\mathcal{M}_{CB}$ cannot be the well-defined entire moduli space. To see why the above reasoning fails, the following argument can be given. Note that $\mathcal{F}$ is a holomorphic function then the metric, $\operatorname{Im} \partial^2\mathcal{F}/\partial a^2$ is a harmonic function. On the other hand, harmonic functions cannot have a global minimum. Therefore, the globally defined metric cannot be positive definite. But we know that the metric must be positive definite and this gives a clue that the moduli space has a highly nontrivial global structure and there must exist some singular points on $\mathcal{M}_{CB}$.

3.2.1. Breaking of R-symmetry

Note that we have discussed the R-symmetry of gauge theory is broken due to Higgsing of the underlying microscopic theory and quantum corrections. From classical perspective, the global R-symmetry of $SU(N)$ gauge theory is given by $SU(2)_{\mathcal{R}}\times U(1)_{\mathcal{R}}$ where $SU(2)_{\mathcal{R}}$ corresponds to the transformation of (ψ, λ) in the $\mathcal{N}=1$ chiral multiplet and $U(1)_{\mathcal{R}}$ rotates the superspace coordinates. Due to presence of quantum effects, $U(1)_{\mathcal{R}}$ is broken to a discrete subgroup by means of the addition of counterterms to the classical Lagrangian. Here, we shall give an alternative method to this issue. We can compute the corresponding anomaly for the gauge group $SU(N)$ by using the index theorem like in the case of the supersymmetric non-linear sigma model. Each $2N$ instanton comes with the zero-modes for each left-handed fermion in the adjoint representation of the vector multiplet. To compute the correlation function, one needs to consider integration over the fermionic collective coordinates of these zero-modes. Therefore, for a nontrivial correlator, there must be enough fermion insertions on the path integral to soak the corresponding zero-modes. Then we have

$$G(x_1, \cdots, x_{2N}, y_1, \cdots, y_{2N}) = \langle \lambda(x_1)\cdots\lambda(x_{2N})\psi(y_1)\cdots\psi(y_{2N}))\rangle \,. \tag{3.84}$$

Under the global $U(1)_{\mathcal{R}}$, the correlator transforms by recalling that $(\lambda, \psi) \to e^{i\alpha}(\lambda, \psi)$

$$G \to e^{i4\alpha N} G \,. \tag{3.85}$$

It can be seen that $U(1)_{\mathcal{R}}$ is broken to the discrete group $\mathbb{Z}_{4N}$. Then the corresponding discrete subgroup is denoted by $e^{2\pi i\alpha}$, where $\alpha = n/4N \,, n = 1, \cdots 4N$. The full global symmetry group is given by $SU(2)_{\mathcal{R}}\times\mathbb{Z}_{4N}$. Furthermore, note that $n = 2N$ corresponding term is nothing but the center of $SU(2)_{\mathcal{R}}$, which commutes with all other elements of $SU(2)_{\mathcal{R}}$ and the center is also an element of discrete subgroup $\mathbb{Z}_{4N}$. So we need to mod out the center. Then the true symmetry group is given by $(SU(2)_{\mathcal{R}} \times \mathbb{Z}_{4N})/\mathbb{Z}_2$. Recall that Higgsing the microscopic theory also leads to the further broken effect of the above global R-symmetry group. The field ϕ^2 has charge 4 under $\mathbb{Z}_{4N}$ since $\phi \to e^{2i\pi\alpha}\phi$. Hence ϕ^2 transforms under the discrete subgroup as $\phi^2 \to e^{2\pi i n/N}\phi^2$. The remaining invariance preserved only where n takes the values, $n = N, 2N, 3N, 4N$. If there is non-zero VEV for ϕ or we are in the Higgs vacuum, then $\mathbb{Z}_{4N}$ is further broken to $\mathbb{Z}_4$. Therefore, the residual global symmetry acts on ϕ^2 as $\mathbb{Z}_2 : \phi^2 \to -\phi^2$ and the surviving R-symmetry group for the $\mathcal{N} = 2$ supersymmetric $SU(2)$ gauge theory is given by

$$\frac{SU(2)_{\mathcal{R}} \times \mathbb{Z}_4}{\mathbb{Z}_2} . \tag{3.86}$$

3.2.2. The Meaning of Singularities

We have seen that u represents a good coordinate for inequivalent vacua of the Coulomb Branch in $\mathcal{N} = 2$ supersymmetric gauge theory. Let us choose the gauge group G as $SU(2)$. Note that there can be regions on the $\mathcal{M}_{CB}$, semi-classically, one has an unbroken $U(1)$ gauge symmetry since it is enhanced to $SU(2)$. This enhancement appears due to the fact that the Higgsing parameter goes to zero, i.e. $u = 0$. On the other hand, in order to find Wilsonian effective Lagrangian description of the theory, we need to expand the Lagrangian around any point $u = u_0$. This low-energy effective Lagrangian can be determined by integrating out all massive modes above some cut-off ($\Lambda \approx a$). In our case, the relevant massive modes are the charged massive gauge bosons $W^{\pm}$. The resulting low-energy effective action only contains the neutral gauge multiplet at the semi-classical region. However, the origin point is problematic since the enhanced gauge symmetry leads the $W^{\pm}$ bosons to become massless. Therefore, from a semi-classical point of view, the Wilsonian effective action is valid only neighborhood of $u = 0$ on the moduli space of vacua. This gives the fact that $u = 0$ is a singular point. This $SU(2)$ enhancement point can be shifted via quantum correction as we will see. The charged W-bosons must be added separately to obtain well-defined Wilsonian low-energy effective action around the origin $u = 0$. As we have discussed, the other singular point $u = \infty$ for large a is associated with a weak coupling region. All in all, these singular points and the corresponding monodromies on $\mathcal{M}_{CB}$ are correlated with the properties of the supersymmetric multiplet which becomes massless.

3.2.3. Monopoles and Dyons

Note that we have analyzed the moduli space in terms of classical considerations with possible quantum effects of singularities on $\mathcal{M}_{CB}$. We can argue that the $U(1)$

gauge symmetry is enhanced to the unbroken nonabelian symmetry at some particular point $u = \langle \mathrm{Tr}\, \phi^2 \rangle \neq 0$ where massive fields become massless in addition to the W-bosons being restored at $u = 0$. In that scenario, perturbative or nonperturbative corrections would change the point in which the gauge symmetry is enhanced to $u \neq 0$. Whereas Seiberg and Witten [34, 35] concluded that such a situation cannot exist in $\mathcal{N} = 2$ supersymmetric theory that is, resulting enhancement cannot be achieved from the low energy behavior of the field theory. The massive modes which become massless at some non-zero u are not elementary gauge fields but composite objects which cannot be coupled to elementary multiplets in the local description. We will elaborate on this type of argumentation in the following. It was discussed that the presence of breaking of the $U(1)_{\mathcal{R}}$ to the $\mathbb{Z}_2$ leads to the singularities of moduli space coming in pairs around the point $u = 0$ and let us assume for a moment that the singularities arise from extra massless gauge bosons. These extra massless gauge bosons require that the IR fixed point must correspond to conformal field theory. There are no dimensionful parameters for any conformal field theory, then the operator $\mathrm{Tr}\, \phi^2$ must have dimension zero if the theory is conformal. On the other hand, there is only one dimension zero operator for any unitary quantum field theory, which is nothing but the identity operator. Furthermore, the identity operator is even under the broken $\mathbb{Z}_2$ symmetry while $\mathrm{Tr}\, \phi^2$ is odd, therefore two operators have different behavior. This is the reason why the full global symmetry of microscopic theory is just $\mathcal{N} = 2$ supersymmetry rather than a superconformal one. Moreover, there is also a related argument that such an IR fixed point cannot be represented by conformal symmetry. That is, the full superconformal algebra respects the $U(1)_{\mathcal{R}}$ and there is no way to break via instanton effect. In other words, the global chiral symmetry would not be anomalous for superconformal field theory, but it is broken by the $F\widetilde{F}$ as we have seen in $\mathcal{N} = 2$ supersymmetry. Thus, we are forced to conclude that the singularities come from massive spin $s \leq 1/2$ fields which become massless at particular points: $u \neq 0$ on the moduli space of vacua. Since we are interested in $\mathcal{N} = 2$ global supersymmetry, we ignored higher spins $s \leq 2$, such as supergravity. $\mathcal{N} = 2$ supersymmetry highly restricts the leftover possibilities of such a supermultiplet, therefore, the particles with spin $s \leq 1/2$ can be represented by hypermultiplets. Nevertheless, our original theory is the pure $\mathcal{N} = 2$ supersymmetric gauge theory, there is no such a hypermultiplet that we can consider as an elementary field in the Lagrangian. Then, the remaining possibility is that they are composite non-perturbative objects which are ignored in the semi-classical regime as their masses are huge. The only such hypermultiplets which respect the $\mathcal{N}=2$ supersymmetry are monopoles or more generally dyons carrying both electric and magnetic charges. These non-perturbative degrees of freedom are nothing but the BPS states for the $\mathcal{N}=2$ supersymmetry algebra. They saturate the BPS bound on the massive multiplets $M \geq \sqrt{2}|Z|$, where Z is the corresponding central charge. Let us discuss the monopole solution first. The BPS bound for the monopole solution [32] is

$$M_m \geq \sqrt{2} n_m \frac{4\pi}{g_{\mathrm{YM}}^2} a \,, \tag{3.87}$$

where $n_m \in \mathbb{Z}$ is so-called the winding number and a is the non-trivial vacuum value for the adjoint scalar in the supermultiplet. We can see that the mass of the monopole depends on the coupling constant g_{YM}^2. When the theory is strongly coupled, the monopoles become light and turn out to be elementary degrees of freedom. The other way around, in the strongly coupled regime, $g_{\mathrm{YM}} \gg 1$, the monopole is very heavy and it can be integrated out from the local Lagrangian description. We can also write the BPS bound formula for the dyon [32] by using the

$$\begin{aligned} Q_{ele} &= -\frac{1}{a g_{\mathrm{YM}}} \int d^3x \partial_i (F^{a0i} A^a) = g n_e \,, \\ Q_{mag} &= -\frac{1}{a g_{\mathrm{YM}}} \int d^3x \partial_i (\widetilde{F}^{a0i} A^a) = \frac{4\pi}{g} n_m \,, \end{aligned} \tag{3.88}$$

and the extended supersymmetry algebra Equation (3.31). Then the central charge for the dyons can be written as

$$Z = n_e a + \frac{4\pi i}{g_{\mathrm{YM}}^2} n_m a \,. \tag{3.89}$$

We can also add the $\theta F \widetilde{F}$ term which leads to the change in the electric charge of the dyon, then the electric charge becomes $Q_{ele} = g_{\mathrm{YM}} n_e + (\theta g_{\mathrm{YM}}^2 / 8\pi^2) Q_{mag}$. At the end of the day, the central charge of the dyon is given

$$Z_{class} = n_e a + n_m \tau_{\mathrm{class}} a \,, \tag{3.90}$$

where we replaced the term $\theta/2\pi + 4\pi i / g_{\mathrm{YM}}^2$ with τ_{class}. We should keep in mind that the complex gauge coupling constant is classical at this point. Moreover, it is possible to write an exact central charge which is free from quantum corrections. This can be carried out by coupling magnetic charge or central charge without any electric degrees of freedom to the vector multiplet in the dual description and the central charge is written in terms of dual parameter, i.e., $Z = n_m a_D$. Then, we have

$$M = \sqrt{2}|Z|, \qquad Z = a n_e + a_D n_m \,. \tag{3.91}$$

It can be seen that the central charge in this form respects an electric-magnetic duality in which exchanging a with a_D and n_e with n_m leaves the spectrum that does not alter. Thus the saturation of the bound demonstrates that the central charges or masses of BPS states are exact at all points in the moduli space of vacua.

3.2.4. Exploring Monodromies in the Moduli Space of Vacua

Recall that the global properties of the moduli space of vacua are determined by the nontrivial transformations of the coordinates a and a_D under monodromies associated with singular points. This behavior plays a crucial role in governing the complete form of the prepotential of the $\mathcal{N} = 2$ supersymmetric gauge theory. We discussed that the vacuum expectation value of the adjoint scalar goes to infinity in the semiclassical regime. This can be explained in the following. Let us take the gauge group as $SU(2)$ and consider the Higgs scale a where the gauge symmetry is reduced to the $U(1)$. If the Higgs value a is very large, this corresponds that the Higgs scale is larger than the strong coupling energy scale, $\Lambda_{<a}$, i.e., $a \gg \Lambda = \Lambda_{<a}$. Therefore, the theory is asymptotically free at energies higher than a. On the other hand, the scale which is smaller than the Higgs scale $\Lambda_{<a}$, the theory has abelian gauge symmetry, resulting in the coupling either does not change with respect to the energy scale or the theory is free in the IR. Then, the energy scale $\Lambda_{>a}$ corresponds to the strongly coupled realm and this leads to the fact that $a \to \infty$ is in tune with the coupling being small. Under this argument we will simply analyze the various monodromy transformations.

Firstly, we need to find an expression for dual variables in order to see the effects of the monodromy transformations on the $(a(u), a_D(u))$. The moduli space is in the form of a complex plane before adding the point at the infinity [34, 35]. Let us start with the monodromy around the infinity corresponding to the limit $a \gg \Lambda$. The reason why we begin the point at infinity is that we have already a local expression for the low-energy effective action. In this limit, u is determined by the classical value $u = a^2/2$ and the holomorphic prepotential is computed from the Equation (3.82). This gives rise to the calculation of the dual variable as

$$a_D = \frac{\partial \mathcal{F}(\Lambda, a)}{\partial a} = \frac{2ia}{\pi} \log\left(\frac{a}{\Lambda}\right) + \frac{ia}{\pi} . \tag{3.92}$$

The correspondent monodromy to the large value of u can be expressed by selecting a closed loop on the complex plane that covers both the origin and the point at infinity. Choosing the counterclockwise contour on this closed loop results in the monodromy transformation given by $u \to e^{2\pi i}u$. Then a transforms as $a \to e^{i\pi}a = -a$ and the dual coordinate a_D undergoes the following transformation

$$a_D \to -\frac{2ia}{\pi} \log\left(\frac{a}{\Lambda}\right) - \frac{ia}{\pi} + 2a = -a_D + 2a , \tag{3.93}$$

and the monodromy transformations can be written in compact form as

$$\begin{pmatrix} a_D \\ a \end{pmatrix} \to \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a_D \\ a \end{pmatrix} . \tag{3.94}$$

Hence, the associated monodromy matrix for the singularity at infinity is given by M_∞:

$$M_\infty = PT^{-2} = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix} , \tag{3.95}$$

where P corresponds to the negative of the identity element in $SL(2, \mathbb{Z})$ and T is the one of the basis elements of the $SL(2, \mathbb{Z})$ group as it is introduced before Equation (3.103). We know that there is a classical singular point located at $u = \infty$ with a corresponding monodromy matrix M_∞. However, it is important to consider whether there could be other singular points besides infinity. Recall that the a_D can also be rewritten in terms of u by substituting the classical value of a, then we can write

$$a_D = -\frac{i}{\pi}\sqrt{2u}\left(\log \frac{2u}{\Lambda^2} + 1\right) . \tag{3.96}$$

As we can see, the point $u = \infty$ is a branch point of a_D due to the fact that it is a logarithmic singularity. Furthermore, since the moduli space has $\mathbb{Z}_2$ global symmetry, the other singularities(if they exist) come in pairs around $u = 0$, then we have $\pm u_0$. The presence of $\mathbb{Z}_2$ global symmetry gives rise to the fact that there are two fixed points on $\mathcal{M}_{CB}$ they are $u = 0$ and $u = \infty$. If there are no further singular points rather than $u = \infty$, the other one would have to be located at $u = 0$. However, if this is the case then the moduli space metric in the semiclassical limit is globally well-defined, and a is a good global coordinate. We know that this is not true since the globally defined metric is not positive-definite. Let us assume that there are at least three singularities at $u_0, -u_0$, and ∞. We know that the counterclockwise loop which covers both finite monodromies should give the monodromy at infinity with the help of contour deformation, resulting in $M_\infty = M_{u_0} M_{-u_0}$. As we discussed, quantum effects can shift the classical singular point $u = 0$. classically, the $SU(2)$ gauge symmetry is recovered at $u = 0$ and the gluons become massless since their masses are determined by a. If we take into account the quantum contributions, the Higgs value a can be smaller than the cut-off scale. In other words, we do not take the semi-classical limit and the theory turns out to be strongly coupled. Therefore, $a = 0$ does not correlate with $u = 0$ on the quantum moduli space. Recall that the masses of dyons are given by $M = \sqrt{2}|Z| = \sqrt{2}|n_m a_D + n_e a|$ and zero value of a as well as the $a_D = \frac{4\pi i}{g^2_{\text{YM}}} a \to a_D = 0$ cause that dyons become massless and we ignore the θ-term at this point. Nevertheless, BPS states cannot be affected by perturbative or non-perturbative corrections and they are still present at the strongly coupled region of $\mathcal{M}_{CB}$. Then, we can conclude that the singularity at $u = 0$ in the classical moduli does not exist in the quantum moduli space.

Now we have three singular points on the exact moduli space of vacua and our next task is to compute the remaining monodromies associated with the points $u = \pm u_0$. This can be made by coupling the dyonic states to $\mathcal{N} = 2$ low energy effective field theory. Notice that we identified the points u_0 with Λ^2 from the classical relation, $u \approx \Lambda$, $u = \frac{1}{2}a^2 \to u_0 = \frac{1}{2}\Lambda^2$. Remind that we start with the pure $SU(2)$ gauge theory without hypermultiplets. Then the only hypermultiplets that we will couple to vector multiplet in IR are the monopoles or dyons. The coupling can be made by leveraging the duality property of our theory and we can go to the dual perspective in which the monopoles become the elementary degrees of freedom. This brings about an abelian $\mathcal{N} = 2$ gauge theory coupled with the monopole hypermultiplet, which provides an equivalent low-energy description. One can compute the monodromies in order to write Lagrangian of the low energy effective theory. Let us take a point u_0 on the moduli space in which the masses of the monopoles become zero, we have

$a_D(u_0) = 0$ and we can move to the dual description in which we integrate out the massive monopoles around this point. The resulting theory is given by the low-energy effective theory which is weakly coupled $U(1)$ gauge theory. Then we can use the corresponding one-loop running gauge coupling

$$\tau_D = -\frac{i}{\pi} \log\left(\frac{a_D}{\Lambda}\right) . \tag{3.97}$$

Furthermore, from $\tau = da_D/da$, we obtain that

$$a = a_0 + \frac{i}{\pi} a_D \log\left(\frac{a_D}{\Lambda}\right) - \frac{i}{\pi} a_D , \tag{3.98}$$

where a_0 is some non-zero constant. If this constant is zero, then all electric objects turn out to be massless states at u_0 and this contradicts the assumption that we have abelian $\mathcal{N} = 2$ gauge theory coupled with monopole hypermultiplet. We can expand the dual coordinate around the u_0 at the linear order, then $a_D \approx c_0\,(u - u_0)$ in which c_0 is some complex constant. If we plug this into a, then we have

$$a \approx a_0 + \frac{i}{\pi} c_0\,(u - u_0) \log\,(u - u_0) + \ldots , \tag{3.99}$$

where the higher order terms in u are suppressed due to the fact that u is very small. Now we can write the monodromy by choosing a counterclockwise loop around $u_0 : (u - u_0) \to e^{2\pi i}\,(u - u_0)$, the (a_D, a) transforms under the monodromy as $a_D \to a_D, a \to a - 2a_D$. Then the monodromy matrix becomes

$$M_{u_0} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} . \tag{3.100}$$

Apart from that the monodromies are related by the monodromy at infinity by $M_\infty = M_{u_0} M_{-u_0}$, the monodromy matrix associated with the point $u = -u_0$ can be found from

$$M_{-u_0} = M_{u_0}^{-1} M_\infty = \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix} . \tag{3.101}$$

Notice that the order of matrices matters since the monodromy matrices do not commute. On the other hand, this is not an ambiguity since the monodromies depend on the representation of the fundamental group on the moduli space. The correct way of defining them comes with choosing a base point P. For instance, it is possible to start with the M_{-u_0} due to the presence of $\mathbb{Z}_2$ global symmetry and this leads to the fact that $M_\infty = M_{-u_0} M_{u_0}$ corresponds to choosing the other base point, given by $-P$. Let us write the general form of monodromy matrix for a (n_m, n_e) dyon as [32]

$$\begin{pmatrix} a_D \\ a \end{pmatrix} \to \begin{pmatrix} 1+2n_en_m & 2n_e^2 \\ -2n_m^2 & 1-2n_en_m \end{pmatrix} \begin{pmatrix} a_D \\ a \end{pmatrix}. \tag{3.102}$$

Observe that monodromy at u_0 corresponds to the point in which the monopole becomes massless since the winding number is one and $n_e = 0$ while the M_{-u_0} monodromy corresponds to the $(n_e, n_m) = (1, -1)$ massless dyon. The monodromy at infinity does not correspond to any massless state. As we mentioned above, one can also begin with the base point at $-P$, and the monodromy matrix $M_{-u_0} = M_\infty M_{u_0}^{-1}$ gives a massless $(1, 1)$ dyon. This shows that these points are related to each other by global symmetry. See for a more detailed discussion about the effect of global $\mathbb{Z}_2$ on the moduli space of vacua [32]. Before we move on to the Seiberg-Witten solution, we necessitate discussing the one issue that is swept under the rug. Note that we made a minimal assumption that we have at least three singularities which are given by $u = \infty, +1, -1$ and the question is whether there are further strongly coupled singular points. Assume that there are k singularities at $u_1, u_2, \ldots u_p$ and semiclassical singularity at infinity. Then we know that the monodromy at infinity is given by the product of the monodromies corresponding to the singularities $M_\infty = M_{u_1} M_{u_2} \ldots M_{u_k}$. It is known for $k > 2$ there is no such solution to this factorization problem, therefore we have three singularities.

3.2.5. Exact Solution from Elliptic Curves via Seiberg-Witten

Based on our calculations of the monodromies, we can now obtain the low-energy effective action by computing the $a(u)$ and $a_D(u)$. We find that there are three singular points u_0 and $-u_0$ as $u = 1$ and $u = -1$ with the monodromies

$$M_\infty = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}, \quad M_1 = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}, \quad M_{-1} = \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix}. \tag{3.103}$$

Seiberg and Witten constructed a family of elliptic curves in order to solve the model [34,35]. Recall that the metric must be a positive definite quantity and the monodromy matrices are the elements of the subgroup $\Gamma(2)$ of $SL(2, \mathbb{Z})$. Seiberg and Witten defined an elliptic curve E_u at each point u on the moduli space or in other words, the elliptic curve is in the form of fibration over the base space which is the moduli space of vacua. The quantities $(a_D(u), a(u))$ are the holomorphic section of the bundle $SL(2, \mathbb{Z})$ and the elliptic curve for $SU(2)$ gauge theory is given by

$$y^2 = (x-1)(x+1)(x-u)\,, \tag{3.104}$$

where (x, y) are the complex numbers. This elliptic curve is sometimes called the Seiberg-Witten curve. As we will see, the elliptic curve E_u is a torus with two non-trivial cycles and these 2-cycles shrink to zero sizes where the singularities occur. Therefore the base space has some punctures associated with the elliptic curve. Let us return to the strongly coupled singularities. We determined $(a_D(u), a(u))$

$$u \approx \infty : \begin{cases} a \cong \sqrt{2u}\,, \\ a_D \approx i \\ frac\sqrt{2u}\pi\{\log\ u\,, \end{cases} \qquad u = 1 : \begin{cases} a_D \approx\ c_0(u-1)\,, \\ a \approx a_0 + \frac{i}{\pi}\ a_D \log\ a_D\,. \end{cases} \tag{3.105}$$

The metric on the moduli space is proportional to the imaginary part of the complex coupling constant as

$$\tau = \frac{\partial a_D}{\partial a} = \frac{da_D/du}{da/du}\,. \tag{3.106}$$

The coupling constant gains meaning in terms of the complex structure modulus of the elliptic curve as we shall see. The elliptic curve in the above can be expressed as genus one Riemann surface. This can be understood in terms of the following. One can draw a circle c_1 on a torus by translating it along the other independent circle c_2. These circles can be exhibited as the line segments l_1 and l_2. Therefore we have two different parts of the torus which are attached by the l_1 and l_2 where those line segments are the branch cuts. A loop that encircles one of the two cuts on a torus creates a cycle on the torus. Another loop that goes around the points 1 and u results in another cycle. When two branch points come together, they cause one of those cycles on the curve to shrink down to zero size. As a result, the curve becomes singular. In the family of curves represented by E_u, the first cycle decreases in size as u goes to infinity, while the latter cycle shrinks when u approaches 1. Thus we can see that at certain points on the u-plane, a vanishing cycle appears as a singularity in the curve. Now we can give an understanding of a_D and a in terms of this construction. Let us take two independent cycles γ_1 and γ_2, normalized such that their intersection number is one.

$$(\gamma_1, \gamma_2) = 1\,, \tag{3.107}$$

and they are elements of the the first homology group of E_u. Recall that every cycle has a dual form λ which belongs to the first cohomology group and this relation can be represented by

$$\gamma \rightarrow \oint_{\gamma} \lambda \,, \tag{3.108}$$

where λ is called a Seiberg-Witten meromorphic differential. We can choose the following two holomorphic differentials basis for the cohomology group

$$\lambda_1 = \frac{dx}{y} \,, \qquad \lambda_2 = \frac{x dx}{y} \,. \tag{3.109}$$

Moreover, we can write by the use of pairing

$$b_1 = \oint_{\gamma_1} \lambda_1 \,, \quad b_2 = \oint_{\gamma_2} \lambda_1 \,. \tag{3.110}$$

At this point, we can see that the elliptic curve can be determined by the following parameter

$$\tau_u = b_1/b_2 \,, \qquad \text{with} \quad \text{Im}(\tau_u) > 0 \,. \tag{3.111}$$

The section pair of the $SL(2,\mathbb{Z})$ bundle can be written in terms of the independent cycles and the Seiberg-Witten differential

$$a_D = \oint_{\gamma_1} \lambda \,, \qquad a = \oint_{\gamma_2} \lambda \,, \tag{3.112}$$

where λ is in the form of $\lambda = a_1(u)\lambda_1 + a_2(u)\lambda_2$. We can also take the differentiation with respect to u:

$$\frac{da_D}{du} = \oint_{\gamma_1} \frac{d\lambda}{du} \,, \qquad \frac{da}{du} = \oint_{\gamma_2} \frac{d\lambda}{du} \,. \tag{3.113}$$

Note that there λ is defined up to some function which can be fixed by using the condition $\text{Im}\,\tau > 0$ for the metric on $\mathcal{M}_{CB}$ in Equation (3.108). Let us consider the following generalization:

$$\frac{d\lambda}{du} = f(u)\frac{dx}{y} \,, \tag{3.114}$$

where $f(u)$ is the function of only u. Now we know that the torus has a parameter $\tau_u = b_1/b_2$ in which the imaginary part of it is greater than zero. Then, we can write

$$\tau = \frac{da_D/du}{da/du} = \frac{\oint_{\gamma_1} f(u)\lambda_1}{\oint_{\gamma_2} f(u)\lambda_1} = \frac{b_1}{b_2} = \tau_u \,. \tag{3.115}$$

It is evident that for the metric to be positive definite, $\text{Im}\,\tau > 0$, we must also have $\text{Im}\,\tau_u > 0$, which is already satisfied. Therefore, the function has no effect on the mapping between the complex u-plane and the coupling constant. The value of the function $f(u)$ can be fixed by the behavior of the theory near the singularities on the u-plane, and the correct choice is given by $f(u) = -\sqrt{2}/4\pi$ [34,35]. Then we can compute the Seiberg-Witten differential by integrating the Equation (3.114) along with the replacement y by the algebraic equation for the elliptic curve

$$\lambda = -\frac{\sqrt{2}}{4\pi}\frac{dx}{\sqrt{(x^2-1)}}\int\frac{du}{\sqrt{(x-u)}} = \frac{\sqrt{2}}{2\pi}\frac{dx\sqrt{x-u}}{\sqrt{x^2-1}}\,. \tag{3.116}$$

Now we can calculate a and a_D. The former can be found by identifying γ_2 with the circle c_1 which corresponds to the line traversing from -1 to 1 along the branch cut. This approach enables us to integrate λ along this cut

$$a(u) = \oint_{\gamma_2}\lambda = \frac{\sqrt{2}}{2\pi}\oint_{c_1}\frac{dx\,\sqrt{x-u}}{\sqrt{x^2-1}} = \frac{\sqrt{2}}{\pi}\int_{-1}^{1}\frac{dx\,\sqrt{x-u}}{\sqrt{x^2-1}}\,. \tag{3.117}$$

Note that the integral over c_1 is contributed from both segments $(-1,1)$ and $(1,-1)$ and they have equal contributions. Similarly, for $a_D(u)$, we can identify the other independent cycle γ_1 with the loop surrounding the cut from 1 to u

$$a_D(u) = \frac{\sqrt{2}}{\pi}\int_{1}^{u}\frac{dx\,\sqrt{x-u}}{\sqrt{x^2-1}}\,. \tag{3.118}$$

The next step is to assess the accuracy of the behavior at the monodromies by examining the derived expressions for $a(u)$ and $a_D(u)$. We can start by exploring the monodromy at infinity, where we are in the regime of large u. By calculating $a(u)$ in this limit, we observe the emergence of the anticipated semiclassical relationship

$$a(u) = \frac{\sqrt{2}}{\pi}\int_{-1}^{1}\frac{dx\,\sqrt{x-u}}{\sqrt{x^2-1}} \approx \frac{\sqrt{2u}}{\pi}\int_{-1}^{1}\frac{dx}{\sqrt{x^2-1}} = \sqrt{2u}\,. \tag{3.119}$$

As we have expected, we find that $u = \frac{1}{2}a^2$ from the obtained expression. Furthermore, we can proceed to compute $a_D(u)$ in the regime of large u

$$a_D(u) = \frac{\sqrt{2}}{\pi}\int_{1}^{u}\frac{dx\,\sqrt{x-u}}{\sqrt{x^2-1}} = \frac{\sqrt{2u}}{\pi}\int_{1/u}^{1}\frac{dz\sqrt{z-1}}{\sqrt{z^2-1/u^2}}\,, \tag{3.120}$$

where we made the redefinition for the integral variable as $x = uz$. This integral has a logarithmic divergence when we take the limit $u \to \infty$ near $z = 0$. In this region, the main contribution comes from the lower bound of the integral, then the numerator becomes $\sqrt{-1}$ and the denominator turns out to be z. Therefore, the above integral can be expressed as a logarithmic function

$$a_D(u) \approx \frac{i\sqrt{2u}}{\pi}\log u\,. \tag{3.121}$$

This logarithmic behavior gives the asymptotic freedom property of $\mathcal{N} = 2$ gauge theory in the semiclassical limit. Moreover, if we take the monodromy transformation $u \to e^{2\pi i}u$, which encloses the singularity at infinity, we observe the following transformations for our expressions a and a_D: a transforms to $-a$, and a_D transforms to

$-a_D + 2a$. Remarkably, these transformations are precisely the expected monodromy transformation. Let us consider the finite monodromy at the point $u = 1$. First of all, $a(u)$ becomes when $u = 1$

$$a(u=1) = \frac{\sqrt{2}}{\pi}\int_{-1}^{1}\frac{dx\sqrt{x-1}}{\sqrt{x^2-1}} = \frac{\sqrt{2}}{\pi}\int_{-1}^{1}\frac{dx}{\sqrt{x+1}} = \frac{4}{\pi}\,. \tag{3.122}$$

The expression for $a(u)$ around the singular point $u = 0$ can be obtained by the first-order term of the expansion of $a(u)$ as it would be a more sufficient approximation for our purpose. To accomplish this, we resorted to differentiating with respect to u and subsequently evaluating the limit as u approaches one

$$\frac{da}{du} = -\frac{\sqrt{2}}{2\pi}\int_{-1}^{1}\frac{dx}{\sqrt{(x^2-1)\,(x-u)}}\,. \tag{3.123}$$

Notice that the integral is logarithmically divergent when $u \to 1$ through $(x-1)$ factor in the denominator. If we evaluate the integral by putting $x = 1$, we find that

$$\begin{aligned}\left.\frac{da}{du}\right|_{x=1} &= -\frac{1}{2\pi}\int\left.\frac{dx}{\sqrt{(x-1)(x-u)}}\right|_{x=1}\\ &= -\frac{1}{2\pi}\log(2x-1-u+2\sqrt{x-1}\sqrt{x-u})\Big|_{x=1}\\ &= -\frac{1}{2\pi}\log(1-u)\,.\end{aligned} \tag{3.124}$$

Finally, we can write the expansion near the point $u = 1$

$$a = \frac{4}{\pi} - \frac{1}{2\pi}(u-1)\log(u-1) + \dots\,, \tag{3.125}$$

additionally, the dual coordinate a_D can be calculated at the point $u = 1$. Observe that in the limit as $u \to 1$ the integral is divergent which is originated from the $z - u^{-1}$ term at the denominator. Then, we can write the first term as $z + u^{-1} \approx 2u$ since $u \to 1, z \to 1 \mapsto z \approx u$, we have

$$\begin{aligned}a_D &= \frac{\sqrt{2u}}{\pi}\int_{1/u}^{1}\frac{dz\sqrt{z-1}}{\sqrt{z^2-1/u^2}} = \frac{\sqrt{2u}}{\pi}\int_{1/u}^{1}\frac{dz\sqrt{z-1}}{\sqrt{z+u^{-1}}\sqrt{z-u^{-1}}}\\ &\approx \frac{1}{\pi}\int_{1/u}^{1}\frac{dz\sqrt{z-1}}{\sqrt{z-1/u}}\,.\end{aligned} \tag{3.126}$$

The second line of Equation (3.126) can be evaluated and in the limit $u \to 1$, we finally reach the expression for a_D as [34]

$$a_D = \frac{i}{2}\left(1-\frac{1}{u}\right) \approx \frac{i}{2}(u-1)\,. \tag{3.127}$$

We can conclude that by encircling the singularity located at $u = 1$ through the loop $u \to e^{2\pi i u}$, the transformations $a_D \to a_D$ and $a \to a - 2a_D$ accurately give the monodromy transformations which we discussed before.

The higher rank generalization of the above solution as well as the addition of matter hypermultiplet can be made similarly, however, the Seiberg-Witten curve becomes more sophisticated [32,36]. Furthermore, the Seiberg-Witten geometry can also be studied in terms of the geometry of Calabi-Yau threefolds through string theory construction, by means of the singularities of Calabi-Yau threefolds corresponding to the simply laced Lie groups of ADE-type [37]. On the dual side of this framework, via mirror symmetry, the Seiberg-Witten curve has genuine geometric meaning rather than being an auxiliary Riemann surface.

4. NEKRASOV PARTITION FUNCTION

In this chapter, we will study $\mathcal{N} = 2$ supersymmetric gauge theories in terms of instanton counting method [51]. Recall that the Seiberg-Witten solution can be obtained via some auxiliary Riemann surface and this elliptic curve determines the dynamics of low energy of $\mathcal{N} = 2$ supersymmetric gauge theory from monodromies on the moduli of Coulomb Branch. Here we shall calculate the instanton contributions to the prepotential through supersymmetric localization or equivariant localization [3], [47, 48], [51–54]. We will twist the Lagrangian of $\mathcal{N} = 2$ supersymmetric gauge theory just like what we did with the supersymmetric non-linear sigma model before. Similarly, we redefine the quantum numbers of the Lorentz group or supercharges via the R-symmetry of our gauge theory. This gives the topological quantum field theory. This topological twisted gauge theory or Donaldson theory can be used for computing certain topological invariants of underlying manifold through the partition function being localized on the instantons. In a more general situation with the Abelian gauge group, the corresponding solution is the instanton equation with the matter fields, called as the Seiberg-Witten monopole equation. By using localization of the path integral to instanton contribution, the computation of path integral turns out to the integration of the moduli space of instantons. From a more precise mathematical point of view, the partition function computes the integral of the equivariant Euler class over the moduli space. Before we move on to the above problem, we will review the instantons in the usual gauge theory.

4.1. Instantons of Yang-Mills Theory

The Yang-Mills theory can be constructed from a principal bundle with the structure group G. Similarly, for the affine connection for the Riemannian manifold, there is a connection that is Lie-algebra valued for the G-bundle along with $\mathfrak{g} = Lie(G)$. The corresponding curvature two-form for the connection A is given by

$$F = dA + A \wedge A \,, \tag{4.1}$$

where F is the corresponding connection. The gauge transformations lead to the following transformations on the Lie algebra valued connection and curvature two-form

$$A \longmapsto gAg^{-1} + gdg^{-1} \quad \text{for} \quad g \in G \,. \tag{4.2}$$

Note that F naturally sits in the adjoint representation of the gauge group G since $F \longmapsto gFg^{-1}$. One can write the action of the Yang-Mills theory:

$$S_{\text{YM}}[A] = \frac{1}{g_{\text{YM}}^2} \int \text{Tr}\left(F \wedge \star F\right) \,, \tag{4.3}$$

where $\star$ is the Hodge star operator on the base space. Equation of motion for the action (4.3) gives

$$\nabla \star F = 0 \,, \tag{4.4}$$

where $\nabla = d + A$ is the covariant derivative for the corresponding gauge group. The differential Equation (4.4) is second order at the connection. There are many solutions for this partial differential equation. however, we will focus on the following particular solution which is first order at the field strength. Let us take the dimension of the base space as $d = 4$ and recall that the square of the Hodge star operation is equal to one. This gives the fact it is possible to separate the gauge field into the self-dual and anti-self-dual parts, then we can write the curvature of the G-bundle as

$$F_{\pm} := \frac{1}{2}\left(F \pm \star F\right) \qquad \Longrightarrow \qquad \star F_{\pm} = \pm F_{\pm} \,, \tag{4.5}$$

where $F_{\pm}$ stands for the anti-self-dual or self-dual part. We will be interested in the self-dual part since it is associated with the positive instanton number. Then the self-dual condition gives

$$F_{+} = 0 \qquad \Longleftrightarrow \qquad \star F = -F \,. \tag{4.6}$$

The corresponding connection for the self-dual curvature is called the self-dual connection. Note that the self-dual part also solves the equation of motion, we can see that in the following way

$$\nabla \star F = -\nabla F = 0 \,. \tag{4.7}$$

The equation of motion is satisfied regardless of whether the curvature is self-dual, as Bianchi's identity (i.e., $\nabla F = 0$) holds. Therefore, the self-dual part of F solves the equation of motion. Then we can write the Yang-Mills action as in the following

$$
\begin{aligned}
S_{\mathrm{YM}}[A] &= \frac{1}{2g_{\mathrm{YM}}^2}\int_{\mathcal{X}} \mathrm{Tr}\,(F-\star F)\wedge\star(F-\star F) + \frac{1}{g_{\mathrm{YM}}^2}\int_{\mathcal{X}} \mathrm{Tr}(F\wedge F) \\
&= \frac{2}{g_{\mathrm{YM}}^2}\int_{\mathcal{X}} \mathrm{Tr}\,(F_-\wedge\star F_-) + \frac{8\pi^2 k}{g_{\mathrm{YM}}^2} \geq \frac{8\pi^2|k|}{g_{\mathrm{YM}}^2}\,,
\end{aligned}
\tag{4.8}
$$

where we denoted the topological charge of the field configuration as $k\in\mathbb{Z}$, so-called the instanton number or winding number. The instanton number is also equal to the integral of the second Chern class

$$
k = c_2[\mathcal{X}] = \frac{1}{8\pi^2}\int_{\mathcal{X}} \mathrm{Tr}\,(F\wedge F)\;, \tag{4.9}
$$

where $\mathcal{X}$ is the base of the principal bundle. Note that the instanton number k is positive since we picked the self-dual part of the curvature. On the other hand, the anti-self-dual field configuration gives the anti-instantons which have negative instanton number. Recall that we can also add another topological term to the Yang-Mills action, i.e., θ-term given by

$$
S_\theta[A] = -\frac{i\theta}{8\pi^2}\int_{\mathcal{X}} \mathrm{Tr}(F\wedge F)\,. \tag{4.10}
$$

Since this term is not dynamical and it has no effect on the equation of motion, the instanton solution does not change

$$
\begin{aligned}
S_{\mathrm{tot}}[A] &= S_{\mathrm{YM}}[A] + S_\theta[A] \\
&= \frac{2}{g_{\mathrm{YM}}^2}\int_{\mathcal{X}} \mathrm{Tr}(F_-\wedge\star F_-) + \left(\frac{1}{g_{\mathrm{YM}}^2} - \frac{i\theta}{8\pi^2}\right)\int_{\mathcal{X}} \mathrm{Tr}(F\wedge F)\,.
\end{aligned}
\tag{4.11}
$$

Therefore we can write the total action as being the sum of k-instanton part and the fluctuation part, $A = A_{\mathrm{inst}}^{(k)} + \delta A$, we have

$$
S_{\mathrm{tot}}[A] = \left(\frac{8\pi^2}{g_{\mathrm{YM}}^2} - i\theta\right)|k| + \int d^4x[(\text{terms quadratic in }\delta A) + (\text{higher terms})]\,. \tag{4.12}
$$

The first relevant term is the quadratic at the fluctuated gauge connection. We can also introduce the instanton fugacity parameter as

$$
\mathfrak{q} := \exp\left(2\pi i\tau\right) = \exp\left(-\frac{8\pi^2}{g_{\mathrm{YM}}^2} + i\theta\right)\,, \tag{4.13}
$$

where τ is the complex gauge coupling constant that we introduced before. Now we can write the partition function of the instanton solution as

$$
\begin{aligned}
Z[A] &= \int [\mathcal{D}A] e^{-S_{\mathrm{tot}[A]}} \\
&= \sum_{k=0}^{\infty} \mathfrak{q}^k \int_{\mathcal{M}_k^G} [\mathcal{D}A_{\mathrm{inst}}^{(k)}] \int [\mathcal{D}\delta A] e^{-S_{\mathrm{fluc}}[\delta A]}\,.
\end{aligned}
\tag{4.14}
$$

It can be seen that the total gauge field configuration has two different parts the instanton part and the fluctuation part. The path integral measure associated with the instantons $\mathcal{D}A^{(k)}_{\text{inst}}$ can be rephrased in terms of moduli space of instantons with the gauge group G, i.e. $\mathcal{M}^G_k$ and instanton moduli space contains all possible instanton configurations with the instanton number k.

4.2. Twisting $\mathcal{N}=2$ Super Yang-Mills Theory

Now we can turn our discussion into $\mathcal{N}=2$ supersymmetric gauge theory and how it can be written as a topological field theory after the topological twisting. As it was discussed, the topological twisting means the redefinition of the Lorentz group structure. For $d=4$, $\mathcal{X}=\mathbb{R}^4\cong\mathbb{C}^2$, the Lorentz group can be written as $SO(4)=SU(2)_L\times SU(2)_R$. The relevant R-symmetry group is the $SU(2)_{\mathcal{R}}$ for our construction. Recall that the $\mathcal{N}=2$ supersymmetric gauge theory has eight following conserved supercharges and they transform under $SU(2)_L\times SU(2)_R\times SU(2)_{\mathcal{R}}$. The topological twist leads to bosons and fermions to transform under the subgroup $SU(2)_L\times SU(2)_D\subset SU(2)_L\times SU(2)_R\times SU(2)_{\mathcal{R}}$ in a similar way, where D denotes the diagonal subgroup of $SU(2)_R\times SU(2)_{\mathcal{R}}$. Note that under $SU(2)_L\times SU(2)_D$ transformations, the two Weyl fermions transform as a four-vector, $\psi_\mu\in(1/2,1/2)$ transform as a singlet, $\overline{\eta}\in(0,0)$ transform as a scalar and $\overline{\chi}^+_{\mu\nu}\in(0,1)$ transforms as a self-dual tensor. After the twist, the new spins of the fields (j_L,j_D) are given by the representation of the $SU(2)_L\times SU(2)_D$ and the superspace coordinates $\theta\to\theta_\mu,\overline{\theta}^+_{\mu\nu},\overline{\theta}$ and the chiral superfield Φ have a new expansion together with the other fields

$$\begin{aligned}\textbf{Fermions}&:\psi_\mu,\chi^+_{\mu\nu},\eta\,,\\ \textbf{Superspace}&:\theta^\mu,\overline{\theta}^+_{\mu\nu},\overline{\theta}\,,\\ \textbf{Superfield}&:\Phi=\phi+\theta^\mu\psi_\mu+\theta^\mu\theta^\nu F_{\mu\nu}+\dots\,,\\ \textbf{Supercharges}&:\mathcal{Q},Q^+_{\mu\nu},G_\mu\,.\end{aligned}\tag{4.15}$$

The supercharge $\mathcal{Q}=\epsilon^{\dot{\alpha}I}\overline{Q}_{\dot{\alpha}I}$ is a scalar and can be used as the topological BRST charge. The $\mathcal{N}=2$ super-Yang-Mills action turns out to be

$$S^{\mathcal{N}=2}_{twisted}=\int_{\mathbb{R}^4\cong\mathbb{C}^2}\text{Tr}(F\wedge F)+\left\{\mathcal{Q},\phi\partial_\mu\psi^\mu+F_{\mu\nu}\chi^{\mu\nu}+\eta\left[\overline{\phi},\phi\right]\right\}.\tag{4.16}$$

This theory is called Donaldson-Witten theory [39]. The first term of the above action is topological and the latter is written as an exact term with respect to the twisted supercharge. Note that the full action is invariant under the BRST transformation of

$\mathcal{Q}$ since the first term is topological and the second term is $\mathcal{Q}$-exact. Similarly with the twisted $\mathcal{N}=2$ sigma model, the domain of the whole path integral of Donaldson-Witten theory is finite-dimensional. Twisted supercharges have a special property, they can be represented as closed Kähler forms for defining hyperkähler manifolds [3]. The form of this twisted supercharge can also be enlarged via Ω-deformation as follows. The $\mathcal{N}=2$ supersymmetric gauge theory can be obtained from the dimensional reduction of the $\mathcal{N}=1$ Super-Yang-Mills theory, living on the $\mathbb{R}^2 \times \mathbb{T}^2$ with the following Ω-deformed metric [52]

$$ds^2_{deformed} = \mathrm{vol}(\mathbb{T}^2) dz d\overline{z} + \sum_{i=1}^{4} \left(dx^i + \Omega^i_j x^j dz + \overline{\Omega}^i_j x^j d\overline{z} \right)^2 , \tag{4.17}$$

where $x^1, \ldots, x^4$ are coordinates on Euclidean space and $z, \overline{z}$ are complex conjugate coordinates on torus. The Ω^i_j and $\overline{\Omega}^i_j$ correspond to the non-commutative deformation of $\mathbb{R}^4$ and given by

$$\Omega^i_j = \begin{pmatrix} 0 & \epsilon_1 & 0 & 0 \\ -\epsilon_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \epsilon_2 \\ 0 & 0 & -\epsilon_2 & 0 \end{pmatrix} , \tag{4.18}$$

where ϵ_1 and ϵ_2 are deformation parameters. Furthermore, the Ω-deformation can be defined in a way that the background $\mathbb{R}^4 \times \mathbb{T}^2$ can be thought of as a non-trivial vector bundle as in the form of $\left(\mathbb{R}^4 \times \mathbb{R}^2\right)/\mathbb{Z}_2$. The $\mathrm{Re}\,\Omega$ and $\mathrm{Im}\,\Omega$ are generators of the quotient of $\mathbb{Z}_2$. Geometrically, the parameters describe the rotations in two non-intersecting complex planes, (z_1, z_2) about their axes. This action is identified as the fixed points of the assoicated vector field $V(E) \in SO(4)$ where the parameters ϵ_1, ϵ_2 denote these fixed points. Note that the deformation can break all supersymmetry of the $\mathcal{N}=2$ theory due to non-trivial vector bundle structure. However, the Ω matrices can be chosen in such a way that the Donaldson-Witten supercharge has an additional deformation and this deformed supercharge preserves the topological structure of Donaldson-Witten theory [42, 51]. The deformed supercharge is called an equivariant supercharge

$$\mathcal{Q}_E = \mathcal{Q} + E_a V^a_{\mu\nu} x^\mu G^\nu , \tag{4.19}$$

where the vector fields $V^a_{\mu\nu} = -V^a_{\nu\mu}$ are the generators of the corresponding Euclidean rotation group $SO(4)$ with the associated symmetry parameters E_a along with

$$V(E) = -\bar{z}_1\partial_{z_1} + z_1\partial_{\bar{z}_1} - \bar{z}_2\partial_{z_2} + z_2\partial_{\bar{z}_2}\,. \tag{4.20}$$

z_1, z_2 are the coordinates of $\mathbb{C}^2 \cong \mathbb{R}^4$. One can construct the equivariant p-form with the help of the supercharges

$$\Omega(E) = \sum_p \Omega_p(E) = \sum_p \sum_{i_1,\ldots,i_p} \frac{1}{p!}\Omega_{i_1,\ldots,i_p} dx^{i_1} \wedge \ldots dx^{i_p}\,. \tag{4.21}$$

$\Omega(E)$ transforms under the spatial rotations as $R\Omega(E) = \Omega(R^{-1}ER)$, where $R \in SO(4)$ and the equivariant exterior derivative is given by

$$D := d + \iota_{V(E)}\,, \tag{4.22}$$

and D can play the same role as the scalar supercharge or BRST operator. $\iota_{V(E)}$ is the operator used for the contraction with the vector field $V(E) = E_a V^a_{\mu\nu} x^\mu G^\nu$. The exterior equivariant derivative acts on the equivariant $p-$form as $D\Omega_p = d\Omega_p + \iota_{V(E)}\Omega_p$. It can be seen that the action of D on a p-form gives $(p+1)$-form $d\Omega_p$ and a $(p-1)$-form $\iota_{V(E)}\Omega_p$. As we have mentioned the topological observables can be obtained from the equivariant form and exterior derivative, however in the $\mathbb{R}^4$ background, all topological observables are trivial if one can start with the $\Omega(E)$ and D, see more detailed discussion [51]. To achieve non-trivial correlation functions, we can use the subgroup of the spatial rotational group of $\mathbb{R}^4$, $E^a \in U(2)_\omega \subset SO(4)$, where $U(2)_\omega$ is the stability group with respect to the Kähler form

$$\omega = dx^1 \wedge dx^2 + dx^3 \wedge dx^4\,, \tag{4.23}$$

and it is closed, $d\omega = 0$. The corresponding Lie derivative which is generated by the vector field is given by $V(E)$, $\mathcal{L}_{V(E)} := d\iota_{V(E)} + \iota_{V(E)}d$. By using this relation, the equivariance property can be stated in terms of the vanishing of the Lie derivative of ω, as we will see in a moment, then we have

$$\mathcal{L}_{V(E)}\omega = d(\iota_{V(E)}\omega) + \iota_{V(E)}d\omega = 0\,. \tag{4.24}$$

Since ω is closed, this gives the fact that we can write $\iota_{V(E)}\omega = d\mu(E)$, where $\mu(E)$ is the so-called moment map, so we have

$$D(\omega - \mu(E)) = 0\,, \tag{4.25}$$

which results that $\omega - \mu(E)$ is equivariantly closed. One can write the moment map $\mu(E)$ in terms of 't Hooft symbols $\eta^a_{\mu\nu}$, $h(x) \equiv \mu^0 = \delta_{\mu\nu}x^\mu x^\nu$ and $\mu^a = \sum_{\mu<\nu}\eta^a_{\mu\nu}x^\mu x^\nu$. Like in the case of the vector field $V(E)$, the Kähler form ω can be rewritten in terms of complex coordinates, $\omega = dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2$. All of these constructions give rise to fact that the moduli space of instantons have a hyperKähler structure inherited from the isometries of $\mathbb{R}^4$. By using the above reparameterization, the Hamiltonian associated with the vector field $V(E)$ can be expressed in terms of a real moment map $H = \mu_{\mathbb{R}}(E) = \epsilon_1|z_1|^2 + \epsilon_2|z_2|^2$ where $\mu_{\mathbb{R}}(E)$ is given by

$$\mu_{\mathbb{R}}(E) = \frac{1}{2}(\epsilon_1 + \epsilon_2)\mu^0(z, \bar{z}) + \frac{1}{2}(\epsilon_1 - \epsilon_2)\mu^3(z, \bar{z})\,. \tag{4.26}$$

The complex moment map is given as $\mu_{\mathbb{C}} = \mu^1 + i\mu^2$. By using the above Kähler form and the Hamiltonian, the topological observables or in other words, $\mathcal{Q}_E$-closed observables can be determined by the following partition function [51]

$$Z(a, \epsilon_1, \epsilon_2) = \left\langle \exp\left\{ \frac{1}{(2\pi i)^2} \int_{\mathbb{R}^4 \cong \mathbb{C}^2} \left[\omega \wedge \mathrm{Tr}(\phi F + \frac{1}{2}\psi \wedge \psi) - \frac{1}{2} H\, \mathrm{Tr}(F \wedge F) \right] \right\} \right\rangle_a , \tag{4.27}$$

where a denotes the vacuum expectation value of scalar component of the $\mathcal{N} = 2$ vector multiplet. We summarized the main idea about the equivariant localization of the twisted $\mathcal{N} = 2$. The significance of topological twisting is that it allows us to compute the certain sub-sectors of the original $\mathcal{N} = 2$ supersymmetric Yang-Mills theory. In particular, the twisted version encodes all the instanton corrections to the prepotential $\mathcal{F}(a, \Lambda)$. Apart from that, higher gravitational F-term corrections to the prepotential can be found from the five-dimensional lift of Equation (4.27) from the Ω-background, $\mathbb{C}^2_{\epsilon_1\epsilon_2} \cong \mathbb{R}^4_{\epsilon_1\epsilon_2}$. This last statement has a counterpart in Gopakumar-Vafa picture of the topological string theory. The topological string theory computes the coupling $R_+^2 F_+^{2g-2}$ in the $\mathcal{N} = 2$ supergravity theory, so there is a map between these two cases. The relation between two different pictures is the core part of this thesis and we will elaborate on it in the last part of this chapter.

4.3. ADHM Construction of Instantons

In this part, we will see that the moduli space of instanton can be obtained by the ADHM construction [47]. Atiyah, Drinfeld, Hitchin and Manin observed that the instanton moduli space can be written as hyperKähler the quotient of a vector space, so-called ADHM data. Note that we wrote the pure gauge theory in a flat background while the ADHM construction of the moduli space of instantons is made for the four-sphere $\mathbf{S}^4$. This can be resolved by adding a point to $\mathbb{R}^4 \cong \mathbb{C}^2$ at infinity, $\mathbf{S}^4 = \mathbb{R}^4 \cup \infty$.

The relevant action for the instantons is finite, so we are interested in the vacuum in which $F = 0$ at the infinity or the boundary of $\mathbb{R}^4$. This gives the constraint on the connection such that A is a pure gauge, then $A \to gdg^{-1}$ at the boundary. Furthermore, there is also rigid gauge transformation at infinity, that is the global gauge symmetry. This residual transformation is called the framing of instanton with the gauge group G, rotating the instantons at infinity. Let us start with the $SU(2)$ gauge theory with one instanton in order to illuminate the properties of the moduli space before we construct it in terms of ADHM variables. The $k = 1$ instanton solution in $SU(2)$ gauge theory is given by:

$$A_\mu = \frac{\rho^2 (x-X)_\nu}{(x-X)^2((x-X)^2+\rho^2)}\,\overline{\eta}^i_{\mu\nu}\,(g\sigma^i g^{-1})\,, \tag{4.28}$$

where σ^i, $i = 1, 2, 3$ are the Pauli matrices which corresponds to the $Lie(SU(2))$ and $\overline{\eta}^i$ are three 4×4 anti-self-dual 't Hooft matrices. The other parameters, x, X and ρ so-called collective coordinates and they parameterize the instanton moduli space. There are three different types of collective parameters for the $SU(2)$ one instanton solution. The first one is the translation of the one instanton and there are four of them, X_μ. The single instanton is localized around the point $x_\mu = X_\mu$, where $x_\mu \in \mathbb{R}^4$. The second parameter is a scale of the instanton which is denoted by ρ. The reason why it is called a scale comes from the fact that rescaling of x and X in the above solution does not alter. The remaining collective coordinates are the three global gauge transformations $g \in SU(2)$ which act on instanton configuration at infinity.

Note that there is a singularity which is located at $x_\mu = X_\mu$ and it is related to the redundancy of our description, then it is not physical. In the $\rho \to 0$ limit, the gauge connection turns out to be zero for every part of $\mathbb{R}^4$ and the instanton shrinks to zero size. This type of singularity is called a small instanton singularity. The story for multi-instantons or instantons on generic four-manifolds is more subtle compared to the one-instanton configuration however it is possible to one out of k-instanton can be made shrunken to zero sizes.

All in all, we have 8 collective coordinates for one instanton with gauge group $SU(2)$. Higher rank gauge groups can be studied similarly by embedding the one instanton $SU(2)$ solution in the $SU(N)$ gauge group. For instance, we can embed one instanton solution in the upper left-hand corner of an $N \times N$ matrix for $SU(N)$ gauge theory. Furthermore, the global part of $SU(N)$ gauge symmetry rotates this particular solution into another embedding. Then there are $N^2 - 1$ rotations and we need to mod out $(N-2)^2$ rotations since our one instanton solutions sit in the 2×2 block of $N \times N$ matrix and it is not affected by $(N-2)^2$ rotations. Hence, the number of remaining parameters is given by $4N - 5$. If we add the parameter that controls the size of the instanton and 4 positions, we find that there are $4N$ collective coordinates for the $k = 1$ solution. The multi-instanton solution can be rearranged in the following. We can construct such a solution by taking $k-1$-instantons which are separated and add them as corrections to the Equation (4.6). It seems that this operation is linear one, however, the resulting equations are non-linear. As a result, there will be $4Nk$ parameters, in other words, the dimension of the instanton moduli space is given by [43]

$$\dim \mathcal{M}_{N,k} = 4Nk\,. \tag{4.29}$$

One can also compute the numbers of collective coordinates by adding fermions in the instanton background and using the state of the art of the index theorems. The above argument can be made in a more constructive way by studying the zero modes of self-dual connections and computing the metric of the instanton moduli space from them. We will not discuss this, see [40]. Let us illustrate the $SU(2)$ case to see the form of the one-instanton moduli space. The metric component associated with the spacetime translations and scale symmetry is proportional to the finite instanton action. The remaining one which corresponds to the $SU(2)$ gauge rotations at the infinity has ρ dependence and goes with ρ^2. This statement can be explained by the incorporation of $SU(2)$ gauge rotations and scale symmetry in $\mathbb{R}^+$. This results in $\mathbb{R}^4 = \mathbf{S}^3 \cup \mathbb{R}^+$, where we have used the fact that $SU(2)$ gauge group manifold is in the form of $\mathbf{S}^3$. Furthermore, recall that the gauge field undergoes a transformation in the adjoint representation of $SU(2)$. Additionally, it remains untouched under the discrete subgroup of $SU(2)$, which is represented by $\mathbb{Z}_2 \subset SU(2)$. As a result, we obtain $\mathbf{S}^3/\mathbb{Z}_2$. Together with position of single instanton $\mathbb{R}^4$, the moduli space of a one instanton for $SU(2)$ gauge theory is given by

$$\mathcal{M}_{2,1} \cong \mathbb{R}^4 \times \mathbb{R}^4/\mathbb{Z}_2\,, \tag{4.30}$$

where we combined the $\mathbb{R}^+$ and $\mathbf{S}^3/\mathbb{Z}_2$. We can see that small instanton singularity is nothing but the orbifold singularity, $\mathbb{R}^4/\mathbb{Z}_2$. Note that this moduli space is non-compact and singular due to the presence of $\mathbb{R}^4$ and the small size of instantons, respectively. The multi-instanton configuration has also non-compactness and singular behaviors thus some sort of regularization scheme should come into play in order to evaluate the partition function whose domain is given by instanton moduli space.

4.3.1. ADHM Variables

Before we discuss ADHM construction, let us review some aspects of the hyperKähler quotient. The Kähler manifold $\mathbf{X}$ is a complex manifold together with a hermitian metric g and two-form or Kähler form $\omega(\cdot,\cdot) = g(I,\cdot)$ which is closed and the complex structure I, is covariantly constant $\nabla I = 0$, with respect to the Levi-Civita connection ∇. Along the same lines, a hyperKähler manifold $\mathbf{X}$ is defined such that there are three covariantly constant complex structures I, J, K in which they satisfy $I^2 = J^2 = K^2 = IJK = -1$. If we have a compact group(for our purpose, we take it as $U(k)$) which acts on a symplectic or Kähler manifold $(\mathbf{X}, \omega)$ and it preserves the symplectic and complex structure, we can define a reduced space whose symplectic two-form is inherited from $\mathbf{X}$. This can be achieved by introducing equivariant moment map $\mu : \mathbf{X} \to \mathfrak{u}_k^*$, where $\mathfrak{u}_k^*$ corresponds to dual of the the Lie algebra of $U(k)$. Then the equivariance condition on moment map is given by $d\langle \mu, \xi \rangle = \iota_{V_\xi}\omega$ where $\xi \in \mathfrak{u}_k$ and $V_\xi \in \mathrm{Vect}(\mathbf{X})$ is the Hamiltonian generating vector field. Then the quotient is determined by the condition $\mu(v \cdot x) = v^{-1}\mu(x)v$ for any $v \in U(k)$. Then the hyperKähler quotient $\mathbf{X}///U(k) = \vec{\mu}^{-1}(0)/U(k)$, where $\vec{\mu}^{-1} \in X$. Note that the reduction is also hyperKähler manifold since the $U(k)$ preserves the three complex structures as well as the symplectic structure and the moment map is also extended for hyperKähler case, i.e. $\vec{\mu} =: \mathbf{X} \to \mathfrak{u}_k^* \otimes \mathbb{R}^3$. One can also deform the $\vec{\mu}^{-1}(0)/U(k)$ in $\mu^{-1}(0)$ by $\vec{\zeta}$ in $\mathfrak{u}_k^* \otimes \mathbb{R}^3$. From the supersymmetric point of view, these modifications correspond to the Fayet-Illiopoulos terms and they have also meaning in terms of turning of Neveu-Schwarz 2-form field in brane constructions of the Higgs branch of moduli space of vacua of supersymmetric gauge theories. The deformed quotient can be understood as a moduli space of non-commutative instantons [40]. Let us return to the ADHM construction. We will look upon the gauge group $G = U(N)$ and the ADHM construction for the k-instanton solution in unitary gauge theory can be made by the following complex vector spaces:

$$V = \mathbb{C}^k\,, \qquad W = \mathbb{C}^N\,, \tag{4.31}$$

and with the following homomorphisms between the above complex vector spaces

$$\mathbf{X} = \mathrm{Hom}(V, V) \oplus \mathrm{Hom}(V, V) \oplus \mathrm{Hom}(W, V) \oplus \mathrm{Hom}(V, W)\,. \tag{4.32}$$

The corresponding linear maps $(B_{1,2}, I, J) \in \mathbf{X}$ are the ADHM data:

$$B_{1,2} \in \mathrm{Hom}(V, V) = \mathrm{End}(V)\,, \quad I \in \mathrm{Hom}(W, V)\,, \quad J \in \mathrm{Hom}(V, W)\,, \tag{4.33}$$

and they satisfy the celebrated ADHM equations

$$\mu_{\mathbb{R}} = 0, \quad \mu_{\mathbb{C}} = 0\,. \tag{4.34}$$

The moment maps that generate the Hamiltonian group action on $\mathbf{X}$ with respect to the three hyperKähler forms

$$\begin{aligned} \mu_{\mathbb{R}} &= [B_1, B_1^\dagger] + [B_2, B_2^\dagger] + II^\dagger - J^\dagger J\,, \\ \mu_{\mathbb{C}} &= [B_1, B_2] + IJ\,, \end{aligned} \tag{4.35}$$

and $\mu_i = (\mu_{\mathbb{R}}, \mu_{\mathbb{C}}) : \mathbf{X} \to \mathfrak{u}_k^\star \otimes \mathbb{R}^3$. The ADHM variables are defined as:

$$v(B_{1,2}, I, J) = (vB_{1,2}v^{-1}, vI, Jv^{-1}), \qquad v \in U(k)\,. \tag{4.36}$$

There is also framing on the instantons as we mentioned, $G = U(N)$ associated with the gauge transformation of $U(N)$ gauge theory at infinity:

$$g(B_{1,2}, I, J) = (B_{1,2}, Ig^{-1}, gJ)\,, \qquad g \in U(N)\,. \tag{4.37}$$

This rigid gauge transformation acts on the ADHM variables as a global symmetry. The ADHM variables are the coordinates for the configuration of the instantons. Therefore the ADHM instanton moduli space can be defined as a hyperKähler quotient as follows

$$\begin{aligned} \mathcal{M}_{N,k} &= \{(B_{1,2}, I, J) \mid (\mu_{\mathbb{R}}, \mu_{\mathbb{C}}) = (0, 0)\} /\!/\!/ U(k) \\ &= \mu_{\mathbb{R}}^{-1}(0) \cap \mu_{\mathbb{C}}^{-1}(0) / U(k)\,, \end{aligned} \tag{4.38}$$

where the hyperKähler condition is comprehended in terms of three conditions on the moment maps, $(\mu_{\mathbb{R}}^{-1}(0), \mathrm{Re}\,\mu_{\mathbb{C}}^{-1}(0), \mathrm{Im}\,\mu_{\mathbb{C}}^{-1}(0))$ as we have seen before. Hence the moduli space of instantons is the hyperKähler manifold.

4.4. Curing non-compactness and resolution of singularities

We have discussed that the moduli space of instantons exhibits non-smooth behavior due to the presence of small instanton singularities (UV) and the unbounded range of collective coordinates (IR). The first one comes from the fact that the instantons can be made as small as possible and the latter corresponds to the non-compactness of $\mathbb{R}^4$. The issue of UV non-compactness can be addressed through the use of Uhlenbeck compactification. This is achieved by introducing a point-like instanton to the instanton moduli space for the unitary gauge group. in the following sense

$$\widetilde{\mathcal{M}}_{N,k} = \mathcal{M}_{N,k} \cup \underbrace{\mathcal{M}_{N,k-1} \times \mathbb{R}^4}_{\substack{\text{1 zero-size} \\ \text{instanton}}} \cup \underbrace{\mathcal{M}_{N,k-2} \times Sym^2\left(\mathbb{R}^4\right)}_{\substack{\text{2 zero-size} \\ \text{instantons}}} \cup \cdots \cup \underbrace{Sym^k\left(\mathbb{R}^4\right)}_{\substack{k \text{ zero-size} \\ \text{instantons}}} \tag{4.39}$$

where $Sym^k\left(\mathbb{R}^4\right)$ denotes the k^{th} symmetric product of $\mathbb{R}^4$, associated with the point-like instanton since they are not distinguishable. The Uhlenbeck compactification allows us to add the solutions of $k-1$-instantons in which their centers are separated. However, there are still singular points in $\widetilde{\mathcal{M}}_{N,k}$. For instance, consider the first compactified part $\mathcal{M}_{N,k-1} \times \mathbb{R}^4$: The associated $(k-1)$-moduli space is smooth, while the presence of one remaining instanton gives rise to a singular point on $\mathcal{M}_{N,k-1} \times \mathbb{R}^4$. Consequently, the Uhlenbeck compactification gives the fact that instantons that are point-like are located only in a specific part of the compactified moduli space in which there are no singularities but the resulting compactified space is still problematic due to the addition of zero-size instantons.

As a further step, we need to resolve such a singularity. Recall that the hyperKähler quotient of ADHM vector space $\mathbf{X}$ at zero level set, $\mu^{-1}(0)/U(k)$ is nothing but the moduli of instantons. We can also generalize the above quotient for some non-zero parameters. In algebraic geometry, the moduli space of instantons is isomorphic to the moduli of framed torsion-free sheaves on $\mathbb{CP}^2$. The deformation occurs due to a change in the level set of $\mathbb{CP}^2$ at infinity, where $\mathbb{C}^2$ is equivalent to $\mathbb{CP}^2/l_\infty$. and the singular points can be blow-up and replaced with exceptional divisors. Ultimately, small instanton singularities are resolved. Physically, Nekrasov and Schwarz gave a physical understanding of this blowing as an non-commutative extension in terms of instantons which live on non-commutative $\mathbb{R}^4 \cong \mathbb{C}^2$ [49]. Therefore, for unitary gauge groups the deformation can be done by improving the ADHM equation (4.34) as [49]:

$$\mu_{\mathbb{R}} = \zeta \mathbf{1}_V , \quad \mu_{\mathbb{C}} = 0 . \tag{4.40}$$

The moduli space of instantons becomes

$$\mathcal{M}^{\zeta}_{N,k} = \vec{\mu}^{-1}(\vec{\zeta})/U(k) , \tag{4.41}$$

where $\vec{\mu} = (\mu_{\mathbb{R}}, \mathrm{Re}\,\mu_{\mathbb{C}}, \mathrm{Im}\,\mu_{\mathbb{C}})$ and $\vec{\zeta} = (\zeta 1_V, 0, 0)$. We will not give detailed derivation for other gauge groups in which such a procedure is not possible. The instanton partition function for non-unitary gauge groups can be rewritten in terms of character formula in a way that the resulting partition function is well-defined even if the instanton moduli space is singular, see [56]. The deformation parameters are given in terms of the two complex coordinates $[z_i, \overline{z_i}] = \zeta_i$ $(i = 1, 2)$ with $\zeta = \zeta_1 + \zeta_2$, corresponding to the coordinates of $\mathbb{R}^4 \cong \mathbb{C}^2$.

The next issue is to regularize the volume of deformed moduli space. Since the IR non-compactness gives infinity when it is integrated all over the $\mathbb{R}^4$, the resulting answer turns out to be infinity

$$\int_{\mathbb{R}^4 \cong \mathbb{C}^2} d^2 z_1 d^2 z_2 = \infty . \tag{4.42}$$

Therefore, one should introduce a regularization term for a finite answer

$$\int_{\mathbb{C}^2} d^2 z_1 d^2 z_2 \longmapsto \int_{\mathbb{C}^2} d^2 z_1 d^2 z_2 e^{-\left(\epsilon_1 |z_1|^2 + e_2 |z_2|^2\right)} \propto \frac{1}{\epsilon_1 \epsilon_2} , \tag{4.43}$$

where ϵ_1, ϵ_2 are real equivariant parameters for the Hamiltonian, $H = \epsilon_1 |z_1|^2 + \epsilon_2 |z_2|^2$ that is introduced before. This Hamiltonian generates a vector field for the $\mathbb{T}^2$ action on $\mathbf{X}$ and it has a moment map μ acting on the element $\xi = (\epsilon_1, \epsilon_2) \in \mathfrak{t}$ where $\mathfrak{t}$ is the Lie algebra of the torus action. Thus $\mathbb{T}^2$ action deletes IR non-compactness and one can use localization techniques since the above integral converges at long distances.

Before we end the discussion, it is essential to make a remark about the deformation of the hyperKähler quotient. As it was mentioned that one can compute equivariant volumes of deformed $\mathcal{M}^{\zeta}_{N,k}$ by using the closed symplectic two-form ω inherited from $\widetilde{\mathcal{M}}_{N,k}$ and the relevant symplectic form is the Kähler form. However, there can be extra contributions due to the volumes of exceptional divisors. Fortunately, these divisors have complex dimension two in our case, and the restricted Kähler form vanishes on them. This means that the exceptional divisors do not give any extra contributions to the original moduli space of commutative gauge theory.

4.4.1. Equivariant cohomology

In this discussion, we will provide a concise explanation of how equivariant volumes are integrated by utilizing the equivariant group action on a manifold. By introducing singularities and regularization of the underlying symplectic manifold, we can apply the equivariant integral formula, which localizes on fixed point loci under the equivariant group action. This is made possible thanks to the well-known Duistermaat-Heckman's formula. Duistermaat-Heckman's theorem stands as a prominent example of this localization principle concerning a compact symplectic manifold that has a $U(1)$ action linked to the moment map. It is worth accentuating that while the localization formula primarily holds for a compact manifold, it remains applicable even in cases of non-compactness and infinite-dimensional integrals, provided that the equivariant fixed point still has equivariant compactness. From the quantum field theory point of view, the equivariant localization or the Berline-Vergne localization theorem formula has found widespread use in path integrals with the identification of the BRST supercharge with the equivariant derivative. To provide a concise overview, let us delve into the equivariant group action and explore its cohomological property.

Consider $\mathbf{X}$, a manifold endowed with a free group operation G. In such a scenario, the G-equivariant cohomologies of $\mathbf{X}$ manifest isomorphism with the de Rham cohomologies of $\mathbf{X}/G$

$$H^\star_G(\mathbf{X}) \cong H^\star(\mathbf{X}/G)\,. \tag{4.44}$$

In the event that the group action G on manifold $\mathbf{X}$ is not free, the resulting quotient $\mathbf{X}/G$ ceases to be a conventional manifold. Consequently, in such cases, the G-equivariant cohomology can be defined with the help of the universal bundle EG, which undergoes free action from the group G:

$$H^\star_G(\mathbf{X}) = H^\star(\mathbf{X}\times_G EG) = H^\star((\mathbf{X}\times EG)/G)\,. \tag{4.45}$$

Significantly, when considering the scenario where $\mathbf{X}$ reduces to a single point, we have $H^\star_G(\text{pt}) = H^\star(BG)$, where $BG = EG/G$ is so-called the classifying space [44]. Let us return the case of the group action is free. The construction proceeds as follows. Let $\mathfrak{g} = \text{Lie}(G)$, and define the G-equivariant differential forms as $\Omega^\star_G(\mathbf{X})$,and we can write the G-invariant complex:

$$\Omega^\star_G(\mathbf{X}) = (\Omega^\star(\mathbf{X}) \otimes \mathbb{C}[\mathfrak{g}])^G \, . \tag{4.46}$$

The induced action on $\alpha(\mathbf{X}) \in \Omega^\star_G(\mathbf{X})$, where $\mathbf{X} \in \mathfrak{g}$, can be obtained via the pullback operation:

$$g^\star \alpha(\mathbf{X}) = \alpha(g^{-1}\mathbf{X}g) \qquad \text{for} \qquad \forall g \in G \, . \tag{4.47}$$

Consider the vector field $V(\mathbf{X})$ which belongs to the Lie algebra on $\mathbf{X}$. Then, the Lie derivative is defined as $\mathcal{L}_V : \Omega^\star(\mathbf{X}) \to \Omega^\star(\mathbf{X}) : \mathcal{L}_V = d\iota_V + \iota_V d$, where $\iota_V : \Omega^\star(\mathbf{X}) \to \Omega^{\star-1}(\mathbf{X})$ stands for the contraction. Additionally, the equivariant exterior derivative $D : \Omega^\star_G(\mathbf{X}) \to \Omega^{\star+1}_G(\mathbf{X})$ is defined incorporating the interior multiplication ι_V:

$$D = d + \iota_V \, , \tag{4.48}$$

with nilpotency i.e., $D^2 = \mathcal{L}_V$. The Lie derivative on the equivariant forms $\alpha(\mathbf{X}) \in \Omega^\star_G(\mathbf{X})$ gives the fact that $\mathcal{L}_V \alpha(\mathbf{X}) = 0$. the equivariant cohomology is defined based on the nilpotent equivariant derivative:

$$H^\star_G(\mathbf{X}) = \frac{\text{Ker}\, D}{\text{Im}\, D} \, . \tag{4.49}$$

For our purpose, the symplectic manifold is the moduli space of instantons and the group action is given by $U(k)$ as we shall see.

4.5. Equivariant localization on instanton moduli space

According to the Equation (4.27) the Nekrasov partition function $Z(a, \epsilon_1, \epsilon_2)$ is the partition function of the $\mathcal{Q}_E$ twisted $\mathcal{N} = 2$ supersymmetric gauge theory that is evaluated in a non-trivial background a corresponding to the vacuum state $|0\rangle_a$. Notice that before the spontaneous symmetry breaking of the gauge group of supersymmetric gauge theory, there is no vacuum expectation value for the adjoint scalar in the vector multiplet. In other words, all observables are evaluated from the expectation value of the pure vacuum, $\langle 0|1|0\rangle$. However, in perturbation theory, we should choose a perturbed vacuum state $|0\rangle_a$ in the presence of the non-trivial classical value for the adjoint scalar and therefore we need to evaluate all observables from $|0\rangle_a$, as well as the partition function itself, after the Higgsing. Furthermore, one can also deform the trivial observable by any observable of form $1 + \mathcal{Q}_E\mathcal{O}$ by using the topological BRST invariance of the $\mathcal{Q}_E$ twisted theory. Then it is possible to write the equivalent version of the Nekrasov partition function by making such an identification. In order to do that, let us consider twisted $\mathcal{N} = 2$ supersymmetric gauge theory with the gauge groups G on $\mathbb{R}^4 \cong \mathbb{C}^2$ with the following observable

$$\mathcal{O}(\phi, \psi, \overline{\psi}, A) = \left\langle \exp\left\{ \frac{1}{(2\pi i)^2} \int_{\mathbb{R}^4 \cong \mathbb{C}^2} \left[\omega \wedge \mathrm{Tr}(\phi F + \frac{1}{2}\psi \wedge \psi) - \frac{1}{2} H \,\mathrm{Tr}(F \wedge F) \right] \right\} \right\rangle_a . \tag{4.50}$$

Additional terms come from highest-ghost-number operators of the twisted Yang-Mills theory through quadratic Casimir invariants in addition to the first quadratic term that is proportional to $\mathrm{Tr}(\phi^2)$ [46],

$$\frac{1}{4\pi^2}\,\mathrm{Tr}(\phi F + \frac{1}{2}\psi \wedge \psi)\,, \tag{4.51}$$

where ϕ, ψ and A denote the content of vector multiplet and ω is the complex structure of $\mathbb{R}^4$ with the Hamiltonian H. With the help of $\mathcal{Q}_E$ twisting or Ω-deformation, we can use the equivariant localization for the original partition function as the twisted theory is topological and the partition function does not alter under the change of coupling constants. Then the partition function can be computed in the UV limit, where the classical solutions dominate the vacuum. To put it in another way, the partition function is reduced to an integral over the moduli space of instantons. First of all, we write the partition function of undeformed theory by summing all over instanton sectors

$$Z(a, \epsilon_1, \epsilon_2 \mathfrak{q}) = \sum_{k=0}^{\infty} \mathfrak{q}^k Z_k(a, \epsilon_1, \epsilon_2)\,, \tag{4.52}$$

where $Z_k(a, \epsilon_1, \epsilon_2) = \oint_{\mathcal{M}^{\zeta}_{N,k}} 1$ and 1 stands for the localization of the pushforward of a single point on $G \times \mathbb{T}^2$-equivariant cohomology of moduli space of instantons. The torus action for the equivariant parameters of the Ω-background $\mathbb{R}^4_{\epsilon_1\epsilon_2}$ is denoted by $\mathbb{T}^2 = U(1) \times U(1)$. We can replace 1 by a cohomologically equal form to the $\oint 1$

$$\oint_{\mathcal{M}^{\zeta}_{N,k}} 1 \longrightarrow \int_{\mathcal{M}^{\zeta}_{N,k}} \mathcal{O}\,. \tag{4.53}$$

Then the partition function becomes

$$Z(a,\epsilon_1,\epsilon_2\mathfrak{q}) = \sum_{k=0}^{\infty} \mathfrak{q}^k \int_{\mathcal{M}^{\zeta}_{N,k}} \mathcal{O}\,. \tag{4.54}$$

Let us consider the only instanton sectors in the observables $\mathcal{O}_{\epsilon_1\epsilon_2}$ in which $\phi = a$ and $\psi = 0$,

$$\mathcal{O}(a,0,0,A) = \left\langle \exp\left\{ \frac{1}{(2\pi i)^2} \int_{\mathbb{R}^4\cong\mathbb{C}^2} \Big[\omega\wedge \mathrm{Tr}(\phi F)\Big] - 2kH\,\mathrm{dvol}_{\mathbb{R}^4} \right\}\right\rangle_a , \tag{4.55}$$

where we have used the fact that $k = \frac{1}{8\pi^2}\int_{\mathbb{R}^4}\mathrm{Tr}(F\wedge F)$ and $\mathrm{dvol}_{\mathbb{R}^4}$ is the volume of the Euclidean spacetime. The partition function in the UV limit where the instantons dominate is given by

$$Z(a,\epsilon_1,\epsilon_2\mathfrak{q}) = \sum_{k=0}^{\infty} \mathfrak{q}^k \int_{\mathcal{M}^{\zeta}_{N,k}} \left\langle \exp\left\{ \frac{1}{(2\pi i)^2} \int_{\mathbb{R}^4\cong\mathbb{C}^2} \Big[\omega\wedge \mathrm{Tr}(\phi F)\Big] - 2kH\,\mathrm{dvol}_{\mathbb{R}^4} \right\}\right\rangle_a . \tag{4.56}$$

We can also write the Nekrasov partition function in terms of moment maps

$$Z(a,\epsilon_1,\epsilon_2\mathfrak{q}) = \sum_{k=0}^{\infty} \mathfrak{q}^k \int_{\mathcal{M}^{\zeta}_{N,k}} \exp\left\{\omega + \mu_G(a) - \mu_{\mathbb{T}^2}(\epsilon_1,\epsilon_2)\right\}, \tag{4.57}$$

where $\mu_G(a)$ and $\mu_{\mathbb{T}^2}(\epsilon_1,\epsilon_2)$ are the moment maps corresponding to the global gauge rotation of the instantons at infinity, i.e $A = g^{-1}dg$, and the maximal torus action for the $SO(4)$ rotations of $\mathbb{R}^4$ introduced before through the Hamiltonian action on the $\mathcal{M}^{\zeta}_{N,k}$. Note that we rewrite the field strength F in terms of Cartan subalgebra of G in the instanton background, the gauge connection is trivial at the infinity and the adjoint complex scalar is given by the a. To illustrate this fact, consider the gauge group G is given by $SU(N)$. In the background of a nontrivial vacuum expectation value of ϕ, which breaks the gauge symmetry from $SU(N)$ to its maximal torus can be written as $U(1)^{N-1}$. Then the instanton configuration A_μ can be chosen to be in the Cartan subalgebra of the $\mathfrak{su}_N = Lie(SU(N))$ and the gauge potential A_μ can be diagonalized and written in terms of $N-1$ diagonal elements, $a_1, a_2, \ldots, a_{N-1}$, where a_i corresponds to the i^{th} generator of the Cartan subalgebra. In this background, the field strength tensor can be expressed as $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$. When A_μ is diagonal, one can write $A_\mu = \mathrm{diag}(a_1, a_2, \ldots, a_{N-1})$, where a_i corresponds to the i^{th} generator of the Cartan subalgebra. Then

$$F_{ij} = \partial_i A_j - \partial_j A_i + [A_i, A_j] = \partial_i a_j - \partial_j a_i + [a_i, a_j] = i(a_i - a_j)\,, \tag{4.58}$$

where $i, j = 1, 2, \ldots, N-1$. The off-diagonal components vanish because A_μ is diagonal. Therefore, the non-zero components of $F_{\mu\nu}$ are given by $F_{ij} = -F_{ji} = i(a_i - a_j)$, where

$i, j = 1, 2, \ldots, N-1$. The trace of $\phi F_{\mu\nu}$ in this background can then be written as

$$\mathrm{Tr}(\phi F_{\mu\nu}) = \sum_{i,j=1}^{N-1} \phi_{ij} F_{ij} = i \sum_{i,j=1}^{N-1} \phi_{ij}(a_i - a_j)\,. \tag{4.59}$$

Thus the moment map associated with the rigid gauge transformation at infinity depends only on the vacuum expectation value a.

The Equation (4.57) is the final form that we should evaluate but in order to do that we should use equivariant localization with respect to these moment maps by using the Duistermaat-Heckman formula as follows. Let $(\mathbf{X}^{2n}, \omega)$ be a symplectic manifold with a Hamiltonian action of a torus $\mathbb{T}^r, \mu : \mathbf{X} \to \mathfrak{t}^*$ with the corresponding moment map $\xi \in \mathfrak{t} = \mathrm{Lie}(\mathbb{T})$ the action generator, $V_\xi \in \mathrm{Vect}(\mathbf{X})$ the vector field on $\mathbf{X}$ corresponding to the $\mathbb{T}$ action generated by Hamiltonian vector field ξ and $f \in \mathbf{X}$ a fixed point of the action, and $w_i[\xi](f)$ are the corresponding weights of the $\mathbb{T}$ action on tangent space, i.e. $T_f\mathbf{X}$. Then the Duistermaat-Heckman formula states:

$$\int_{\mathbf{X}} \frac{\omega^n}{n!} e^{-\langle \mu, \xi \rangle} = \sum_{f : V_\xi(f) = 0} \frac{e^{-\langle \mu(f), \xi \rangle}}{\prod_{i=1}^n w_i[\xi](f)}\,, \tag{4.60}$$

where $\underbrace{\omega \wedge \cdots \wedge \omega}_{\text{n-times}} = \omega^n$ and $w_i[\xi](f)$ are the weights of the torus action on the tangent space at the fixed point $T_f\mathbf{X}$. For our purpose, $\mathbf{X} = \mathcal{M}^{\zeta}_{N,k}$ and $r = N + 2$ so that the full torus action is given by $U(1)^{N+2}$ with the associated parameters $\xi = (a, \epsilon_1, \epsilon_2)$. It can be seen that the ω-dependent part comes from the expansion of the $\int e^\omega$ since the only highest form in the series expansion makes sense due to integration over $2n$-dimensional space. As an example of the illustration of how the remaining moment maps contribute to the partition function, consider the only spacetime torus action with corresponding parameters given by $\xi = (\epsilon_1, \epsilon_2)$. The only fixed point f is the origin of $\mathbb{R}^4$, where $\langle \mu(f), \epsilon_i \rangle = 0$ whereby the Hamiltonian is zero, i.e., $z_1 = z_2 = 0$. Therefore, the localization with respect to the $\mathbb{T}^2$ action forbids the contributions of instantons that run off to infinity.

Let us take a closer look at the equivariant action on the ADHM moduli space and its associated fixed points. In ADHM construction, we observed that the ADHM coordinates are subject to group actions by $G^\vee = U(k)$ and $G = U(N)$, as described in Equation (4.36) and Equation (4.37). Here, the symbol $\vee$ indicates the dual of the gauge group in the context of the ADHM construction. In addition, there is a Lorentz rotation of $\mathbb{C}^2$ that operates on the quadruples of ADHM construction

$$(e^{\epsilon_1}, e^{\epsilon_2}) \cdot (B_1, B_2, I, J) = (e^{-\epsilon_1} B_1, e^{-\epsilon_2} B_2, I, e^{-(\epsilon_1+\epsilon_2)} J)\,, \tag{4.61}$$

where $(e^{\epsilon_1}, e^{\epsilon_2}) \in \mathbb{T}^2 := U(1) \times U(1) \subset SO(4)$. To use the localization formula, it is necessary to define the fixed point in the equivariant action. We will thoroughly investigate the fixed point within the ADHM moduli space, specifically by analyzing the group actions of $U(N)$ and $U(k)$ given by

$$g = e^{a}, \qquad v = e^{\phi}\,. \tag{4.62}$$

Those group actions are parametrized by the Lie algebras of $U(N)$ and $U(k)$. i.e. $a \in \mathfrak{u}_N = Lie(U(N))$ and $\phi \in \mathfrak{u}_k = Lie(U(k))$. We can linearize the ADHM equations with the infinitesimal version of the above transformations to find the fixed point equations:

$$\begin{aligned} \phi B_1 - B_1\phi &= \epsilon_1 B_1\,, \\ \phi B_2 - B_2\phi &= \epsilon_2 B_2\,, \\ \phi I - Ia &= 0\,, \\ aJ - J\phi &= (\epsilon_1+\epsilon_2) J\,. \end{aligned} \tag{4.63}$$

We have used the fact that the complex part of the moment map is zero, $\mu_{\mathbb{C}} = 0$ as we evaluate it at the fixed point. Note that the elements of the $U(N)$ group is diagonalized with the help of the Lie algebra parameters $a \in \mathfrak{u}_N$ and they are determined by the Cartan subalgebra. This can be seen in the following the last two fixed point equations transform under $U(N)$ as, $a \mapsto hah^{-1}$, where $h \in U(N)$, we have $\phi I - I(hah^{-1}) = 0 \longrightarrow \phi\widetilde{I} - \widetilde{I}a = 0$, in which $\widetilde{I} = Ih$. Same argumentation applies to J, $+hah^{-1}J - J\phi - (\epsilon_1+\epsilon_2)J = 0$, $a\widetilde{J} - \widetilde{J}\phi - (\epsilon_1+\epsilon_2)\widetilde{J} = 0$ with $\widetilde{J} = h^{-1}J$. Accordingly, diagonalization arises in terms of parameters, $a \in \mathfrak{u}_N$

$$a = \bigoplus_{I=1}^{N} a_1\,. \tag{4.64}$$

Similarly, the half part of the quadruple (I, J) can be also decomposed with respect to the torus action of $U(N)$ or $SU(N)$, $I = I_1 \oplus \cdots \oplus I_N$ and $J = J_1 \oplus \cdots \oplus J_N$. Then the eigenvalue equations together with the eigenvectors of ϕ I_l and $J_l's$ can be obtained by using the fixed point equations under the full torus action $\mathbb{T}^{N+2} = U(1)^{N+2}$

$$\phi I_l = a_l I_l, \quad J_l\phi = \big(a_l - (\epsilon_1+\epsilon_2)\big) J_l\,. \tag{4.65}$$

Hence we can observe that the different eigenvalues do not match for non-zero Ω-background parameters ϵ_1, ϵ_2 and, $a_l \neq a_{l'} + \epsilon_1 + \epsilon_2$ where l, l' runs from 1 to N. This leads to the conclusion that the eigenvectors which have different eigenvalues are orthogonal: $J_l I_{l'} = 0$.

This orthogonality condition implies that B_1 and B_2 commutes at the fixed point $[B_1, B_2] = 0$ by the use of $\mu_{\mathbb{C}} = [B_1, B_2] + IJ = 0$ and $IJ = 0$. Expanding the ADHM variables in terms of eigenvectors of the gauge group and spacetime torus action, by multiplying the above equations with the eigenvectors $(I_l, J_l)_{l=1,\dots,N}$, the fixed point equations can be expressed as follows:

$$\begin{aligned} \phi B_1 I_l &= B_1 \phi I_l + \epsilon_1 B_1 I_l = (a_l + \epsilon_1) B_1 I_l \,, \\ \phi B_2 I_l &= B_2 \phi I_l + \epsilon_2 B_1 I_l = (a_l + \epsilon_2) B_2 I_l \,, \\ J_l B_1 \phi &= J_l \phi B_1 - \epsilon_1 J_\alpha B_1 = J_l B_1 (a_l - \epsilon_1 - \epsilon_2 - \epsilon_1) \,, \\ J_l B_2 \phi &= J_l \phi B_2 - \epsilon_2 J_\alpha B_2 = J_l B_1 (a_l - \epsilon_1 - \epsilon_2 - \epsilon_2) \,. \end{aligned} \tag{4.66}$$

The most general version of the above equations can be written recursively [50]

$$\begin{aligned} \phi \left(B_1^{i-1} B_2^{j-1} I_l \right) &= \left(a_l + (i-1)\epsilon_1 + (j-1)\epsilon_2 \right) \left(B_1^{i-1} B_2^{j-1} I_l \right) , \\ \left(J_l B_1^{i-1} B_2^{j-1} \right) \phi &= \left(a_l - i\epsilon_1 - j\epsilon_2 \right) \left(J_l B_1^{i-1} B_2^{j-1} \right) , \end{aligned} \tag{4.67}$$

for $(i, j) \in (1, \dots, \infty)$. Note that formally, there are an infinite number of linearly independent eigenvectors. However, k of them are independent because the rank of ϕ is equal to k. The above construction can also be rephrased in terms of partitions in the following sense [51,52]. Let $s = (i, j)$ be a partition, then the fixed point is determined by the N-tuple partition $\vec{Y} = (Y_1, \cdots, Y_l)$ with $|\vec{Y}| = \sum_{l=1}^{N} |Y_l| = k$, and each Y_l is characterized by $Y_l = (\lambda_{L,1} \geq \lambda_{L,2} \geq \cdots \geq 0)$. The partition $s = (i, j)$ corresponds to a box $\Box \in Y_l$ whose location is identified with i'th row and j'th column of l'th Young diagram. Each box $s = (i, j) \in Y_l$ can be represented with a monomial $z_1^{i-1} z_2^{j-1}$ with $i \in (1, \dots, \infty)$, $j \in (1, \dots, Y_{l,i})$. Each box $\Box \in Y_l$ has an associated eigenvalue of ϕ, which is denoted by ϕ_l and can be found using Equation (4.67) [51]

$$\vec{Y} \longrightarrow \phi_I(a_l, s) = a_l + (i-1)\epsilon_1 + (j-1)\epsilon_2 \,. \tag{4.68}$$

4.5.1. Integrating out ADHM variables

In this part, we will integrate the ADHM coordinates using the path integral of the twisted $\mathcal{N}=2$ gauge theory with a gauge group G. To begin with, the ADHM moduli space $\mathcal{M}_{N,k}$ is defined as $\mathbf{X}=\mathrm{Hom}(V,V)\oplus\mathrm{Hom}(W,W)\oplus\mathrm{Hom}(W,V)\oplus\mathrm{Hom}(V,W)$, where the coordinates are $(B_{1,2},I,J)$. In this moduli space, the group $G^{\vee}=U(k)$ acts on the ADHM quadruple. Furthermore, the deformed ADHM moduli space is obtained by taking the quotient of the level set $\vec{s}^{-1}(0)\subset\mathbf{X}$ as $\mathcal{M}^{\zeta}_{N,k}=\vec{s}^{-1}(0)/G^{\vee}$ in which the section is defined as $\vec{s}=\vec{\mu}-\vec{\zeta}$ with $\vec{\zeta}=(\zeta,0,0)$. The ADHM equations can be incorporated into equivariant integration using the Mathai-Quillen formalism [53, 54]. We have two coordinate types: bosonic and fermionic. The former represents the ADHM construction coordinates, while the latter is obtained by applying the equivariant derivative (scalar supercharge) $D:=\mathcal{Q}_E$, associated with the anti-commuting one-forms $(\Psi_{B_{1,2}},\Psi_I,\Psi_J)$. Following the Mathai-Quillen formalism, it is necessary to introduce anti-ghost multiplets $\big(\vec{\chi}:=(\chi_{\mathbf{R}},\chi_{\mathbf{C}}),\vec{H}:=(H_{\mathbf{R}},H_{\mathbf{C}})\big)$ and $(\overline{\phi},\eta)$, arising from the gauge-fixing condition. Note that these anti-ghost multiplets are fiber bundles of $\mathbf{X}$ with a section s, and they belong to $\mathfrak{u}_k=\mathrm{Lie}\,G^{\vee}$. Ensuring compatibility between topological BRST transformation and equivariant action is crucial. Hence, all transformations must be suitably defined [51].

$$
\begin{aligned}
\mathcal{Q}_E B_{1,2} &= \Psi_{B_{1,2}}, & \mathcal{Q}_E\Psi_{B_{1,2}} &= [\phi,B_{1,2}]-\epsilon_{1,2}B_{1,2}\,,\\
\mathcal{Q}_E I &= \Psi_I, & \mathcal{Q}_E\Psi_I &= \phi I - Ia\,,\\
\mathcal{Q}_E J &= \Psi_J, & \mathcal{Q}_E\Psi_J &= -J\phi + aJ-(\epsilon_1+\epsilon_2)J\,,\\
\mathcal{Q}_E\chi_{\mathbb{R}} &= H_{\mathbb{R}}, & \mathcal{Q}_E H_{\mathbb{R}} &= [\phi,\chi_{\mathbb{R}}]\,,\\
\mathcal{Q}_E\chi_{\mathbb{C}} &= H_{\mathbb{C}}, & \mathcal{Q}_E H_{\mathbb{C}} &= [\phi,\chi_{\mathbb{C}}]+(\epsilon_1+\epsilon_2)\chi_{\mathbb{C}}\,,\\
\mathcal{Q}_E\overline{\phi} &= \eta, & \mathcal{Q}_E\eta &= [\phi,\overline{\phi}]\,,\\
\mathcal{Q}_E B^{\dagger}_{1,2} &= \overline{\Psi}_{B_{1,2}}, & \mathcal{Q}_E\overline{\Psi}_{B_{1,2}} &= -[\phi,B^{\dagger}_{1,2}]+\epsilon_{1,2}B^{\dagger}_{1,2}\,,\\
\mathcal{Q}_E I^{\dagger} &= \overline{\Psi}_I, & \mathcal{Q}_E\overline{\Psi}_I &= -I^{\dagger}\phi + aI^{\dagger}\,,\\
\mathcal{Q}_E J^{\dagger} &= \overline{\Psi}_J, & \mathcal{Q}_E\overline{\Psi}_J &= \phi J^{\dagger} - J^{\dagger}a+(\epsilon_1+\epsilon_2)J^{\dagger}\,.
\end{aligned}
\tag{4.69}
$$

It can be understood that the fixed point Equations (4.69) are translated into the condition $\mathcal{Q}_E\Psi_{B_{1,2},I,J}=\mathcal{Q}_E\overline{\Psi}_{B_{1,2},I,J}=0$. Therefore, the fixed points exhibit the nilpotency of the equivariant derivative on $\mathcal{M}^{\zeta}_{N,k}$. To carry out an integral over the moduli space of instantons, it is necessary to map the equivariant cohomology on $\mathbf{X}$ to the ordinary cohomology on $\mathcal{M}^{\zeta}_{N,k} = \vec{s}^{-1}(0)/G^{\vee}$ by using the inclusion map $i : \vec{s}^{-1}(0) \hookrightarrow \mathbf{X}$. This is accomplished by defining the map $I \circ i^{\star}$ in which the pullback is given by $i^{\star} : H^{\star}_{G^{\vee}}(\mathbf{X}) \hookrightarrow H^{\star}_{G^{\vee}}(\vec{s}^{-1}(0))$ and the isomorphism $I : H^{\star}_{G^{\vee}}(\vec{s}^{-1}(0)) \cong H^{\star}(\vec{s}^{-1}(0)/G^{\vee})$ where we assume that the $G^{\vee}$-action is free on $\vec{s}^{-1}(0)$. The cohomology class in $\vec{s}^{-1}(0)/G^{\vee}$ is determined as $\widetilde{\alpha} = I \circ i^{\star}\alpha(\phi)$, where $\alpha(\phi)$ is equivariant differential form, i.e., $D\alpha(\phi) = 0$, the element of $\Omega^{\star}_{G^{\vee}}(\mathbf{X})$. Hence, the integral of this equivariant form, $\int_{\mathbf{X}} \alpha(\phi)$ is nothing bu the the pushforward map $\Omega^{\star}_{G^{\vee}}(\mathbf{X}) \to \Omega^{\star}_{G^{\vee}}(\text{pt})$, which descends to $H^{\star}_{G^{\vee}}(\mathbf{X}) \to H^{\star}_{G^{\vee}}(\text{pt})$.

After all this mathematical machinery, we can evaluate the path integral over the regularized equivariant volume of the moduli space $\mathcal{M}^{\zeta}_{N,k}$. Remember that we can compute the path integral in the weak coupling limit in which the relevant contributions come from the instanton configurations. The instanton sector of the path integral can be written as [51, 55]

$$\begin{aligned} Z_k(a, \epsilon_1, \epsilon_2) &:= \oint_{\mathcal{M}^{\zeta}_{N,k}} 1 \\ &= \int \mathcal{D}\phi\mathcal{D}\overline{\phi}\mathcal{D}\vec{H}\mathcal{D}\vec{\chi}\mathcal{D}\eta\mathcal{D}\Psi\mathcal{D}B\mathcal{D}I\mathcal{D}J e^{\mathcal{Q}_E \operatorname{tr}\left(\vec{\chi}\cdot\vec{s}(B,I,J) + t_H \vec{\chi}\cdot\vec{H} + \frac{1}{t_V}\Psi\cdot V\left(\overline{\phi}\right) + \frac{1}{t_\eta}\eta[\phi,\overline{\phi}]\right)} , \end{aligned} \tag{4.70}$$

where $\vec{\chi} \cdot \vec{s} = \chi_{\mathbf{R}}\mu_{\mathbf{R}} + \chi_{\mathbf{C}}\mu_{\mathbf{C}}$ and $\vec{\chi} \cdot \vec{H} = \chi_{\mathbf{R}}H_{\mathbf{R}} + \chi_{\mathbf{C}}H_{\mathbf{C}}$. The remaining term is determined by the $V(\overline{\phi})$, which is the dual group flow vector field which is generated by the dual group action $G^{\vee}$:

$$\begin{aligned} \Psi \cdot V(\overline{\phi}) = {} & \Psi_{B_1}[\overline{\phi}, B_1] + \Psi_{B_2}[\overline{\phi}, B_2] + \Psi_I\overline{\phi}I - J\overline{\phi}\Psi_J - \overline{\Psi}_{B_1}[\overline{\phi}, B_1^{\dagger}] \\ & - \overline{\Psi}_{B_2}[\overline{\phi}, B_2^{\dagger}] - I^{\dagger}\overline{\phi}\overline{\Psi}_I + \overline{\Psi}_J\overline{\phi}J^{\dagger} . \end{aligned} \tag{4.71}$$

We can see that the action is written as the $\mathcal{Q}_E$-exact form, and as we shall see in a moment, it is independent of the coupling constants that we introduced. As a further step, we can add the corresponding quadratic or kinetic terms for $(B_{1,2}, I, J)$ without altering the action since it is $\mathcal{Q}_E$-exact [55]

$$\frac{t_{\text{quad}}}{2}\,\mathcal{Q}_E\,\text{tr}\left[B_{1,2}^{\dagger}\Psi_{B_{1,2}}-\overline{\Psi}_{B_{1,2}}B_{1,2}+I^{\dagger}\Psi_I-\overline{\Psi}_I I+J^{\dagger}\Psi_J-\overline{\Psi}_J J\right]$$
$$=t_{\text{quad}}\,\text{tr}\left[B_{1,2}^{\dagger}\left([\phi,B_{1,2}]-\epsilon_{1,2}B_{1,2}\right)+I^{\dagger}\left(\phi I-Ia\right)+J^{\dagger}\left(-J\phi+aJ-(\epsilon_1+\epsilon_2)J\right)\right]$$
$$+\,t_{\text{quad}}\,\text{tr}\left[\overline{\Psi}_{B_{1,2}}\Psi_{B_{1,2}}+\overline{\Psi}_I\Psi_I+\overline{\Psi}_J\Psi_J\right]\,. \tag{4.72}$$

The BRST transformations were used to derive the second line of the Equation (4.72). As mentioned earlier, we can evaluate the path integral within the semiclassical regime, i.e. $t_{\text{quad}}\to\infty$. Consequently, this results in the inclusion of the aforementioned term, along with $\vec{\chi}\cdot\vec{H}$, as crucial components in the path integral. To accomplish this, it becomes necessary to diagonalize (ϕ,a) according to Equation (4.73) and Equation (4.64). Upon expressing ϕ using that basis, the expansion becomes:

$$\phi=\text{diag}(\phi_1,\ldots,\phi_k)\,. \tag{4.73}$$

Note that there is an additional subtlety, which is to fix the path integral measure in terms of a constant factor. This stems from the translation invariance of $\mathfrak{g}^{\vee}$. We can approach this as follows: The group $G^{\vee}$ is naturally isomorphic to the tangent space to the identity element of $G^{\vee}$, which is the Lie algebra of $G^{\vee}$, denoted as $\mathfrak{g}^{\vee}$. The corresponding invariant volume element of the group, known as the Haar measure on $G^{\vee}$, defines a measure on $G^{\vee}$. Now, we can select an arbitrary Haar measure on $G^{\vee}$ as $\phi_1,\ldots,\phi_k$ in a manner that this measure $d\phi_1 d\phi_2\ldots d\phi_k$ on the group is equal to the chosen Haar measure at the identity of $G^{\vee}$. The subsequent measure is then as follows, the measure

$$\frac{d\phi_1\ldots d\phi_k}{\text{vol}\,G^{\vee}}\,, \tag{4.74}$$

is a natural measure on $G^{\vee}$, where $\text{vol}\,G^{\vee}$ denotes the total volume of Haar measure. Note that Equation (4.74) is independent of the chosen Haar measure on $G^{\vee}$. Hence, we can describe the equivariant integral as an integral over $\mathbf{X}\times\mathfrak{g}^{\vee}$ with the above setting. The corresponding Haar measure is obtained via the Vandermonde determinant:

$$\frac{d\phi}{\text{vol}\,U(k)}=\frac{1}{k!}\frac{d\phi_I}{(2\pi i)}\prod_{I\neq J}^{k}(\phi_I-\phi_J)\longrightarrow\mathcal{D}\phi=\frac{1}{k!}\prod_{I=1}^{k}\frac{d\phi_I}{(2\pi i)}\prod_{I\neq J}^{k}(\phi_I-\phi_J)^2\,, \tag{4.75}$$

where we utilized the fact that $\mathcal{D}M=\prod_{i=1}^{k}d\lambda_i\Delta(\lambda)^2$ with $\Delta(\lambda)=\prod_{i<j}(\lambda_i-\lambda_j)$ and M represents the $k\times k$ matrix defined as $M=U\Lambda U^{\dagger}$ where $\Lambda=\text{diag}(\lambda_1,\ldots,\lambda_k)$ and U corresponds to the unitary matrix. Consequently, we can calculate the bosonic component of the quadratic $\mathcal{Q}_E$ term

$$t_{\rm quad}\,{\rm tr}\left[B_{1,2}^{\dagger}\left([\phi,B_{1,2}]-\epsilon_{1,2}B_{1,2}\right)\right]=t_{\rm quad}\sum_{\substack{1\le I,J\le k\\ m=1,2}}(\phi_{IJ}-\epsilon_m)|B_{m,IJ}|^2\,,$$
$$t_{\rm quad}\left[I^{\dagger}\left(\phi I-Ia\right)\right]=t_{\rm quad}\sum_{\substack{I=1,\dots,k\\ l=1,\dots,N}}(\phi_I-a_l)|I_{Il}|^2\,,$$
$$t_{\rm quad}\,{\rm tr}\left[J^{\dagger}\left(-J\phi+aJ-(\epsilon_1+\epsilon_2)J\right)\right]=t_{\rm quad}\sum_{\substack{I=1,\dots,k\\ l=1,\dots,N}}(-\phi_I+a_l-(\epsilon_1+\epsilon_2))|J_{lI}|^2\,,\tag{4.76}$$

with $\phi_{IJ}=\phi_I-\phi_J$. Therefore, the total of the three terms is equal to

$$\sum_{\substack{1\le I,J\le k\\ m=1,2}}(\phi_{IJ}-\epsilon_m)|B_{m,IJ}|^2+\sum_{\substack{I=1,\dots,k\\ l=1,\dots,N}}(\phi_I-a_l)|I_{Il}|^2+\sum_{\substack{I=1,\dots,k\\ l=1,\dots,N}}(-\phi_I+a_l-(\epsilon_1+\epsilon_2))|J_{lI}|^2\,.\tag{4.77}$$

All three terms are quadratic and we can perform the path integral of the corresponding degrees of freedom by noticing that all of them are bosonic. For instance, the variable $B_{1,2}$ can be integrated out $\int\prod_{m=1,2}dB_{IJm}\exp\left(t^{k^2}\sum_{\substack{1\le I,J\le k\\ m=1,2}}(\phi_{IJ}-\epsilon_m)|B_{m,IJ}|^2\right)\cong t^{-k^2}\prod_{\substack{1\le I,J\le k\\ m=1,2}}(\phi_{IJ}-\epsilon_m)^{-1}$. If we do the same computations, we find that

$$t_{\rm quad}^{-k^2-2Nk}\prod_{\substack{1\le I,J\le k\\ m=1,2}}(\phi_{IJ}-\epsilon_m)^{-1}\prod_{\substack{I=1,\dots,k\\ l=1,\dots,N}}(\phi_I-a_l)^{-1}(-\phi_I+a_l-(\epsilon_1+\epsilon_2))^{-1}\,.\tag{4.78}$$

The fermionic counterparts of the $B_{1,2}, I, J$ can also be achieved by performing fermionic Gaussian integral and the final result depends on $t_{\rm quad}^{k^2+2Nk}$ rather than the $(\Psi_{B_{1,2}},\Psi_I,\Psi_J)$. Fermionic integration is fixed by $\int d\overline{\Psi}d\Psi\exp(\overline{\Psi}\Psi)=1$, and an appropriate constant factor comes out as the inverse of the bosonic one. Therefore, the final answer has no dependence on the coupling constants as we anticipated before. The other relevant term can be dealt with along the same lines:

$$t_H\mathcal{Q}_E\,{\rm tr}\,\vec{\chi}\cdot\vec{H}=t_H\,{\rm tr}\left[H_{\mathbb{R}}H_{\mathbb{R}}+H_{\mathbb{C}}^{\dagger}H_{\mathbb{C}}+\chi_{\mathbb{R}}[\phi,\chi_{\mathbb{R}}]+\chi_{\mathbb{C}}^{\dagger}\left([\phi,\chi_{\mathbb{C}}]-(\epsilon_1+\epsilon_2)\chi_{\mathbb{C}}\right)\right]\,.\tag{4.79}$$

The quadratic terms for the $\vec{H}$ and $\chi_{\mathbb{C}}$ can be made similarly

$$t_H\,{\rm tr}\,\chi_{\mathbb{C}}^{\dagger}\left([\phi,\chi_{\mathbb{C}}]-(\epsilon_1+\epsilon_2)\chi_{\mathbb{C}}\right)=t_H\sum_{1\le I,J\le k}(\phi_{IJ}-(\epsilon_1+\epsilon_2))|\chi_{\mathbb{C},IJ}|^2\,,\tag{4.80}$$

and if we carry out the Gaussian integral, we find that the result

$$t_H^{k^2}\prod_{I,J}^{k}(\phi_{IJ}-(\epsilon_1+\epsilon_2))=t_H^{k^2}\left(-(\epsilon_1+\epsilon_2)\right)^k\prod_{I\neq J}^{k}(\phi_{IJ}-(\epsilon_1+\epsilon_2))\,.\tag{4.81}$$

Note that the $\chi_{\mathbb{C}}$ term is fermionic, contrary to the $H_{\mathbb{C}}$ term. The Gaussian integral of $\chi_{\mathbb{C}}$ yields the factor $t_H^{k^2}$. Consequently, this is compensated by the Gaussian integral of $H_{\mathbb{C}}$, resulting in no dependence on the coupling constant. The $\chi_{\mathbb{R}}$ term provides:

$$t_H \operatorname{tr} \chi_{\mathbb{R}}^{\dagger}([\phi, \chi_{\mathbb{R}}]) = t_H \sum_{1 \le I,J \le k} (\phi_{IJ}) |\chi_{\mathbb{R},IJ}|^2 \longrightarrow t_H^{k^2} \prod_{I \ne J}^{k} \phi_{IJ} \,. \tag{4.82}$$

The Gaussian integral of $H_{\mathbb{R}}$ gives the inverse power of the coupling constant, so the result is independent of t_H again. The full path integral of the ADHM variables can be expressed as a multivariable contour integral by combining equations Equation (4.75), Equation (4.78), and Equation (4.81) together with the Vandermonde determinants. The equivariant integral over the instanton moduli space, given by Equation (4.70), is then localized on this contour integral through equivariant localization [51, 53], [54]

$$Z_k(a, \epsilon_1, \epsilon_2) = \frac{1}{k!} \frac{(\epsilon_1 + \epsilon_2)^k}{(2\pi i \epsilon_1 \epsilon_2)^k} \oint \prod_{I=1}^{k} \frac{d\phi_I}{P(\phi_I) P(\phi_I + \epsilon_1 + \epsilon_2)} \prod_{I \ne J}^{k} \frac{\phi_{IJ}^2 (\phi_{IJ}^2 - (\epsilon_1 + \epsilon_2)^2)}{(\phi_{IJ}^2 - \epsilon_1^2)(\phi_{IJ}^2 - \epsilon_2^2)} \,, \tag{4.83}$$

where $P(x) = \prod_{l=1}^{N} (x - a_l)$ and note that we absorb the two -1 that come from the determinant associated with J and $\chi_{\mathbb{C}}$. The integral (4.83) can be interpreted as the contour integral, taking into account the poles located at the denominator. This means that these poles are identified with the eigenvalue of the infinitesimal equivariant full torus action $\mathbb{T}_G \times \mathbb{T}_{G^\vee} \times \mathbb{T}^2$ on the ADHM variables as

$$\begin{aligned}
&(\phi_I - \phi_J - \epsilon_m) && \text{for } B_{m,IJ}, (m = 1, 2) \\
&(\phi_I - a_l) && \text{for } I_{Il}, \\
&(a_l - \phi_I - (\epsilon_1 + \epsilon_2)) && \text{for } J_{lI} \\
&(\phi_I - \phi_J - (\epsilon_1 + \epsilon_2)) && \text{for } \chi_{\mathbb{C},IJ} \,.
\end{aligned} \tag{4.84}$$

The denominators correspond to the weights of the equivariant actions at the fixed point. This can be seen by looking at the torus action on the χ-term since it is the counterpart of the moment map. Then we have that

$$\mu_{\mathbb{C},IJ} \longmapsto [\phi, \mu_{\mathbb{C}}] + (\epsilon_1 + \epsilon_2) \mu_{\mathbb{C}} \overset{(4.73)}{\Longrightarrow} (\phi_I - \phi_J - (\epsilon_1 + \epsilon_2)) \, \mu_{\mathbb{C},IJ} \,. \tag{4.85}$$

The equivariant localization can be simplified by initially localizing about the groups $U(k) \times G \times \mathbb{T}^2$ that act on the vector space of ADHM matrices. Afterward, the $U(k)$ part of the localization multiplet can be integrated to include the quotient. We should also remark on the contour integration. To one must shift the poles located at $\phi_{IJ} = \pm\epsilon_{1,2}$ by either altering the contour or adjusting $\epsilon_{1,2} \to \epsilon_{1,2} + i\delta$, where $\delta \ll 1$. Likewise, to avoid the zeroes of $P(\phi_I)$ in the denominator, we should set $a_l \to a_l + I\delta'$. Hence, the origin of these poles can be described through the application of the Duistermaat-Heckman formula and everything is consistent. When matter fields are present in the moduli space in terms of fundamental or antifundamental representation of the gauge group, the path integral localizes on the Weyl zero mode locus. This solution can be described by additional fermionic variables, namely $(\lambda_{lf})^{f=1,\ldots,N^f}_{l=1,\ldots,k}$ and $(\widetilde{\lambda}_{fl})^{f=1,\ldots,N^{af}}_{l=1,\ldots,k}$, as discussed in [55]. We can utilize the path integral formalism by defining multiplets, together with equivariant torus actions $\mathbb{T}_M = U(1)^{N^{\mathrm{f}}} \subset U(N^{\mathrm{f}})$ and $\mathbb{T}_{\widetilde{M}} = U(1)^{N^{\mathrm{af}}} \subset U(N^{\mathrm{af}})$ stands for the Cartan subalgebra of the flavor symmetry groups and the corresponding parameters are the masses, $m = (m_1, \cdots, m_{N_f})$ and $\widetilde{m} = (\widetilde{m}_1, \cdots, \widetilde{m}_{N_{af}})$. See more detailed expression and derivation [51, 52], [55].

Note that the entire discussion is based on the gauge group $U(N)$ so far and we can express the instanton part of the partition function in its most general form using a geometric perspective as

$$Z_k\left(a, m, \widetilde{m}, \epsilon_1, \epsilon_2\right) = \oint_{\mathcal{M}_k^G} \mathrm{Eu}_{G\times\mathbb{T}^2\times U(N_f)\times U(N_{af})}(V \otimes M \otimes \widetilde{M})\,. \tag{4.86}$$

Here the torus action of the symmetry groups is represented by $G\times\mathbb{T}^2\times U(N_f)\times U(N_{af})$, which is a combination of the flavor group, gauge group, rigid gauge transformations, and the Ω-deformation. The bundles over the instanton moduli space have fibers given by the vector spaces $(V, M, \widetilde{M})$, and their equivariant Euler class is denoted by $\mathrm{Eu}_{G\times\mathbb{T}^2\times U(N_f)\times U(N_{af})}$. We can also add the adjoint matter as well as the quiver generalizations of the instanton part of the Nekrasov partition function, we will discuss this when we write the instanton partition function in terms of partitions.

4.6. Extracting The Prepotential

We can make use of the prepotential by establishing a connection between the partition function and the prepotential. This can be achieved through the following approach:

$$\begin{aligned} Z\left(a, \epsilon_1, \epsilon_2\Lambda\right) = \langle 1\rangle_a = \int \mathcal{D}X e^{-S_{\mathrm{micro}(X)}} &= \int_{|k|<\Lambda} \mathcal{D}\widetilde{X} e^{-S_{\mathrm{eff}}(\widetilde{X})} \\ &= \exp\left\{\frac{1}{4\pi}\,\mathrm{Im}\,\frac{1}{2\pi i}\int d^4x d^4\theta \mathcal{F}(-2\sqrt{2}a, \Lambda(x,\theta))\right\} \\ &= \exp\frac{1}{\epsilon_1\epsilon_2}\mathcal{F}\left(a, \epsilon_1, \epsilon_2\Lambda\right)\,. \end{aligned} \tag{4.87}$$

This correspondence can be demonstrated in the following manner. Firstly, consider the case in which there is no matter. In the first line of Equation (4.87), we integrated out the massive degrees of freedom in a Wilsonian sense and localized the integral to the zero modes, which are parametrized by the vacuum expectation value of the adjoint scalar. The resulting function becomes dependent on the cutoff scale Λ. That is because the coupling is superspace dependent and related, causing the cut-off scale to also be superspace dependent, i.e., $\mathcal{H}(x,\theta) = H(x) + \frac{1}{2}\omega_{\mu\nu}\theta^\mu\theta^\nu$. Then we can replace the prepotential in terms of superspace dependent prepotential, $\mathcal{F}(a, \Lambda e^{-\mathcal{H}})$ and we can expand as

$$\mathcal{F}(a, \Lambda e^{-\mathcal{H}}) = \mathcal{F}(a, \Lambda e^{-H}) + \omega\ \Lambda\frac{\partial}{\partial\Lambda}\mathcal{F}(a, \Lambda e^{-H}) + \frac{1}{2}\omega^2\Lambda^2\frac{\partial^2}{\partial\Lambda^2}\mathcal{F}(a, \Lambda e^{-H})\,. \tag{4.88}$$

Upon rescaling the metric of $\mathbb{R}^4$ by a factor of coupling constant t and taking the limit as $t \to \infty$, the main contribution comes from the third term of Equation (4.27) [51]. This gives together with the fact that the derivatives of H with respect to z_1 and z_2, are proportional to $\epsilon_{1,2}$

$$\begin{aligned} Z(a,\epsilon_1,\epsilon_2) &= \exp\left\{\frac{1}{2(2\pi i)^2}\int_{\mathbb{R}^4}\omega\wedge\omega\frac{\partial^2\mathcal{F}(a,\Lambda e^{-H})}{\partial\log\Lambda^2}\right\} + O(\epsilon_1,\epsilon_2) \\ &\approx \exp\frac{\mathcal{F}_{\text{inst}}(a,\Lambda) + O(\epsilon_1,\epsilon_2)}{\epsilon_1\epsilon_2}\,, \end{aligned} \tag{4.89}$$

where the instanton part of the prepotential of the low-energy theory is given by

$$\mathcal{F}_{\text{inst}}(a,\Lambda) = \int_0^\infty \frac{\partial^2\mathcal{F}(a,\Lambda e^{-H})}{\partial H^2}\,H dH\,. \tag{4.90}$$

We can evaluate the Seiberg-Witten prepotential by turning off the Ω-background and obtaining the quantities in $\mathbb{R}^4$. Hence, we have derived a formula that can be used to evaluate the prepotential along with the fundamental matter

$$\lim_{\epsilon_1,\epsilon_2\to 0}\epsilon_1\epsilon_2\log Z\left(a,m,\epsilon_1,\epsilon_2\Lambda\right) = \mathcal{F}_{\text{inst}}\left(a,m,\epsilon_1,\epsilon_2\Lambda\right)\,. \tag{4.91}$$

Accordingly, instanton part of the prepotential of the $\mathcal{N} = 2$ supersymmetric gauge theory with fundamental hypermultiplets are given by the logarithm of the Nekrasov partition function at $\epsilon_1 = \epsilon_2 = 0$, which recovers the Seiberg-Witten prepotential.

4.7. Nekrasov Factors

In the previous discussion, we uncovered that the equivariant torus actions at the fixed points of the instanton moduli space can be expressed in terms of partitions.

This section will delve into this topic with a more comprehensive approach. Let us consider a Young tableau $Y_l = (\lambda_{l,1} \geq \lambda_{l,2} \geq \cdots \geq 0)$ denotes the height of the i-th column. If i is greater than the width of the tableau, then we set $Y_l = 0$. Let $Y_l^\dagger = (\lambda_{l,1}^\dagger \geq \lambda_{l,2}^\dagger \geq \cdots \geq 0)$ be the transpose of Y. For a box s located at the coordinate (i, j), we define its arm-length $a_Y(s)$ and leg-length $l_Y(s)$ with respect to the tableau Y

$$a_Y(s) = \lambda_i - j, \qquad \ell_Y(s) = \lambda_j^\dagger - i, \tag{4.92}$$

where $a_Y(s)$ stands for the distance from the partition to the right end of the Y and $\ell_Y(s)$ gives the distance from s to the bottom of the Young diagram. The Nekrasov factor is defined as [57]

$$E_{Y_1 Y_2}(a, s) = a - \epsilon_1 \ell_{Y_2}(s) + \epsilon_2 (a_{Y_1}(s) + 1)\,. \tag{4.93}$$

where we have denoted $\vec{a}$ as $(a_1, \ldots, a_N)$. Then, the $SU(N)$ vector multiplet from the fixed points f labeled by an N-tuple of Young diagrams $\vec{Y}$ can be written as [57]

$$Z_{\text{vect}}(\vec{a}, \vec{Y}) = \prod_{l,n=1}^{N} \frac{1}{\prod_{s \in Y_l} E_{Y_l Y_n}(a_l - a_n, s) \prod_{t \in Y_l}(\epsilon_1 + \epsilon_2 - E_{Y_n Y_l}(a_l - a_n, t))}\,, \tag{4.94}$$

and the following formula gives the matter part in fundamental representation with N_f flavors

$$Z_{\text{fund}}(\vec{a}, \vec{Y}, m) = \prod_{l=1}^{N_f} \prod_{s \in Y_l} (\phi(a_l, s) - m + \epsilon_1 + \epsilon_2)\,. \tag{4.95}$$

The anti-fundamental hypermultiplet can be determined from the fundamental one as

$$Z_{\text{antifund}}(\vec{a}, \vec{Y}, m) = Z_{\text{fund}}(\vec{a}, \vec{Y}, \epsilon_1 + \epsilon_2 - m)\,, \tag{4.96}$$

where $\phi(a_l, s)$ is defined at Equation (4.68). In the context of a gauge group that comprises two distinct groups, namely $SU(N_1)$ and $SU(N_2)$, a field that is charged under both groups sits in the bifundamental representation. The contribution of the bifundamental part can be expressed as follows [57], [65–67]

$$\begin{aligned} Z_{\text{bifund}}(\vec{a}, \vec{Y}\vec{b}, \vec{W}m) = &\prod_{l=1}^{N_1} \prod_{n=1}^{N_2} \prod_{s \in Y_l} (E_{Y_l W_n}(a_l - b_n, s) - m_{\text{bf}}) \\ &\times \prod_{t \in W_n} (\epsilon_1 + \epsilon_2 - E_{W_n Y_l}(b_n - a_l, t) - m_{\text{bf}})\,, \end{aligned} \tag{4.97}$$

where we have denoted Coulomb branch parameters for $SU(N_1)$ and $SU(N_2)$ as $\vec{a}$ and $\vec{b}$, respectively. The adjoint hypermultiplet can be determined by

$$Z_{\text{adj}}(\vec{a}, \vec{Y}, 0) = 1/Z_{\text{vector}}(\vec{a}, \vec{Y})\,. \tag{4.98}$$

Additionally, $Z_{\rm adj}(\vec{a},\vec{Y},0)$ is obtained by setting $\vec{a}=\vec{b}$ in the bifundamental part

$$Z_{\rm adj}(\vec{a},\vec{Y},m)=Z_{\rm bifund}(\vec{a},\vec{Y},\vec{a},\vec{Y},m)\,. \tag{4.99}$$

We can glue all of those different contributions as [51,66]

$$\begin{aligned} Z=&\sum_{\vec{Y}^{(1)}}\cdots\sum_{\vec{Y}^{(M-1)}}\left(\mathfrak{q}^{(1)}\right)^{|\vec{Y}^{(1)}|}\cdots\left(\mathfrak{q}^{(M-1)}\right)^{|\vec{Y}^{(M-1)}|}\\ &\times\prod_{\alpha=1}^{M-1}Z_{\rm vect}(\vec{a}^{(\alpha)},\vec{Y}^{(\alpha)})\prod_{\alpha=1}^{M-2}Z_{\rm bifund}(\vec{a}^{(\alpha)},\vec{Y}^{(\alpha)},\vec{a}^{(\alpha+1)},\vec{Y}^{(\alpha+1)},m_{\rm bf}^{(\alpha,\alpha+1)})\\ &\times\prod_{\delta=1}^{N_f}Z_{\rm fund}(\vec{a}^{(M-1)},\vec{Y}^{(M-1)},m^{\rm f}_{N+\delta})\prod_{\gamma=1}^{N_{af}}Z_{\rm antifund}(\vec{a}^{(1)},\vec{Y}^{(1)},m^{\rm af}_{\gamma})\,, \end{aligned} \tag{4.100}$$

and the Coulomb branch moduli parameters and Young diagrams for the gauge group factors can be found by

$$\vec{a}^{(\alpha)}=(a_1^{(\alpha)},\cdots,a_N^{(\alpha)})\qquad\text{and}\qquad\vec{Y}^{(\alpha)}=(Y_1^{(\alpha)},\cdots,Y_N^{(\alpha)})\,. \tag{4.101}$$

The index $\alpha=1,\ldots,M$ is employed to enumerate nodes of the hypermultiplets along the quiver gauge group with the associated instanton parameters $\mathfrak{q}$.

4.8. The Five-Dimensional Lift

In this part, we will give a genuine correspondence between the Gopakumar-Vafa formalism and the five-dimensional lift of the Nekrasov partition function. This relation can be understood in terms of various approaches. To begin with, the instanton part of the prepotential of $\mathcal{N}=2$ supersymmetric gauge theory can be expressed as the one-loop corrections in the 5d $\mathcal{N}=1$ theory when the circle shrinks to the zero size. The five-dimensional theory lives on the $\mathbb{R}^4\times\mathbf{S}^1$ background [58,59]. To better understand this fact, it is necessary to review some basic facts about five-dimensional $\mathcal{N}=1$ supersymmetry. Let us recall the spinor representation of $SO(4,1)$. The supercharges of 5d $\mathcal{N}=1$ supersymmetry are transformed in the pseudo-Majorana representation and after Kaluza-Klein reduction around the circle, the supercharges decompose to the $\mathcal{N}=2$ supersymmetry in four dimensions. This can be understood from the fact that the supercharge is decomposed under the little group $SO(4)=SU(2)_L\times SU(2)_R$ resulting the following supercharge

$$Q_\alpha=\begin{pmatrix} Q^1_\alpha \\ \epsilon_{\dot{\alpha}\dot{\beta}}(Q^2_\beta)^\dagger \end{pmatrix}\,. \tag{4.102}$$

The anticommutation relations can be written bu using the above decompostion

$$\begin{aligned} \{Q_a^i, Q_b^j\} &= \epsilon_{ab}\epsilon^{ij}\,(Z + iP_5)\ , \\ \{Q_a^i, Q_b^{j\dagger}\} &= \gamma^\mu_{ab} P_\mu\ . \end{aligned} \tag{4.103}$$

Z is a hermitian central charge in four dimensions and the four-dimensional $\mathcal{N} = 2$ supersymmetry Equation (3.30) can be read from by noticing that the remaining part of the four-dimensional central charge comes from the momentum along the circle direction $\mathbf{S}^1$ up to some coefficients. As in the case of $\mathcal{N} = 2$ supersymmetry, 5d $\mathcal{N} = 1$ has vector multiplets and hypermultiplets. However, the vector multiplets of $\mathcal{N} = 1$ have some difference compared to the four-dimensional vector multiplet in terms of their spins. The scalar fields in five dimensions are real contrary to the four dimensions. Under Kaluza-Klein reduction, four-dimensional complex scalar field $\varphi \in \mathbf{t}$ can be obtained by the combination of real adjoint scalar in five dimensions φ with the Wilson line along the circle direction. This brings about that we have $a_{4d} = ge^{\beta\varphi}$ in which the Wilson line is given by the holonomy $g = \int_{\mathbf{S}^1} A_{5d} \in \mathbb{T}$ of the gauge field around the $\mathbf{S}^1$ at infinity of $\mathbb{R}^4$. Furthermore, we are interested in the twisted version of the $\mathcal{N} = 2$ supersymmetry in four dimensions and dimensional reduction must be compatible with that. The twisting gives rise to the fact the $\mathbb{R}^4$ is rotated in a nontrivial way if one goes around the circle, $\mathbb{R}^4 \to \mathbb{R}^4_{\epsilon_1\epsilon_2}$ or more precisely the spacetime is fibered over the circle. This implies that the loop wrapping the $\mathbf{S}^1$ localizes at the origin of spacetime. Hence, regularization for the equivariant volume integral naturally emerges, with the localization of instantons being equivalent to the loop corrections affecting the prepotential of the 5d. The partition function of the five-dimensional (Ω-background) $\mathcal{N} = 1$ supersymmetric gauge theory can be computed by the supersymmetric quantum mechanics which lives on the instanton moduli space. The degrees of freedom can be implemented from the quantization of the moduli space of collective coordinates. Then, the partition function is given by the following Witten index [24, 25], [51]

$$Z(a, m, \epsilon_1, \epsilon_2\beta) = \mathrm{Tr}_{\mathcal{H}}(-1)^{2(j_L+j_R)} \exp\left[-\epsilon_1(J_1^3 + J_{\mathcal{R}}^3) - \epsilon_2(J_2^3 + J_{\mathcal{R}}^3) - a\cdot Q + m\cdot F - \beta H\right] \tag{4.104}$$

J_1, J_2 are the two Cartans of $SO(4)$ which rotate the spatial $\mathbb{R}^4$ and they are related by the left and right part of the $SO(4)$, i.e., J_L, R by $J_R = (J_1 + J_2)/2$ $J_L = (J_1 - J_2)/2$.

The generator for the Cartan subgroup of the five-dimensional $SU(2)_{\mathcal{R}}$ R-symmetry is given by $J_{\mathcal{R}}$. The H is the Hamiltonian of the supersymmetric quantum mechanics, the parameter β is linked with the size of the circle, and the trace is taken with respect to the Hilbert space of the supersymmetric quantum states. Q and F are the gauge and the flavor charges associated with a and m are the four-dimensional chemical potentials for the gauge and the flavor symmetries, respectively and the sums are represented as $a \cdot Q = \sum_{l=1}^{r} a_l Q_l$ and $m \cdot F = \sum_{s=1}^{N_f} m_s F_s$. Note that we evaluate the gauge field at the infinity in such a way that a belongs to the complexified Cartan torus, $\mathbb{T}_{\mathbb{C}}$, including the holonomy factor that we stated before. The construction mentioned above can be explained through the effective gauge coupling constant from a physical perspective. The five-dimensional coupling constant is running due to one-loop contributions which come from BPS states as well as their bound states of them. Consequently, five-dimensional W-bosons correspond to four-dimensional instanton solutions and the effective coupling constant depends on the various charges, masses, and spin degeneracies of the BPS particles which are running on the loops [58, 59]. The relation between the four-dimensional partition function and the above Witten index can be determined by taking the limit $\beta \to 0$, which means the size of the circle of the Ω background goes to zero. Additionally, after compactification on $\mathbf{S}^1$, the resulting preserved supersymmetry is given by the twisted supercharge $\mathcal{Q}_E$. The four-dimensional limit can be taken while $\epsilon_1, \epsilon_2, a$ and the masses are kept finite. This results from the fact that the five-dimensional partition function is reduced to a four-dimensional one with a factor $\propto \beta^{-(2h^\vee(G)-\kappa(R))k}$ [57]. Where $2h^\vee(G)k$ stands for the complex dimension of the moduli space of instantons with a given gauge group by G and $\kappa(R)k$ is the complex dimension of the hypermultiplets with a given representation R of the gauge group. The $\kappa(R)$ is the so-called quadratic Casimir and this factor affects the four-dimensional instanton parameters as

$$-\frac{8\pi^2\beta}{g_{5d}^2} = (2h^\vee(G) - k(R))\log(-i\beta) + \log \mathfrak{q}\,, \tag{4.105}$$

where the four-dimensional instanton fugacity is fixed. On the other hand, dimensional reduction along the circle gives rise to the connection between the four-dimensional coupling g_{4d} and the five-dimensional coupling g_{5d}

$$\frac{8\pi^2}{g_{\mathrm{4d}}^2} = \frac{8\pi^2\beta}{g_{\mathrm{5d}}^2}\,. \tag{4.106}$$

We can see that upon variation of β, the above relation gives that the logarithmic one-loop running of the four-dimensional coupling constant can be understood in terms of UV cut-off scale β^{-1} since the one-loop beta function is determined the dynamically generated scale Λ through $\mathfrak{q} = \Lambda^{2h^\vee(G)-k(R)}$.

Let us return to the geometric side of this story from string theory construction. As we have mentioned, the twist for the five-dimensional supersymmetric gauge theory results in that the noncompact four-dimensional part of the flat five-dimensional space-time turns out to be $\mathbb{R}^4_{\epsilon_1\epsilon_2}$ or the total space $X_{\epsilon_1\epsilon_2}$ becomes the $\mathbb{R}^4$ is fibration over the base space $\mathbf{S}^1$. According to the Geometric Engineering, the five-dimensional gauge theory can be obtained by the M-theory compactification on non-compact Calabi-Yau threefold which is given by the Asymptotically Locally Euclidean space(ALE). This can be rephrased in terms of the local neighborhood of the $K3$ fibered over $\mathbb{P}^1$, non-compact Calabi-Yau threefold [60]. Recall that Type II-A string theory is the perturbative limit of the M-theory in a way that the M-theory circle $\mathbf{S}^1_M$, dictates the string coupling. If the M-theory circle goes to zero size, we recover the Type II-A string theory. Then if the M-theory circle is chosen as the base circle of $X_{\epsilon_1\epsilon_2}$, the four-dimensional limit can be studied in terms of string theory. This configuration is similar to the Gopakumar-Vafa background, then the graviphoton field strength which comes from compactification of the ten-dimensional metric along the non-compact Calabi-Yau threefold is identified with

$$F = \epsilon_1 dx^1 \wedge dx^2 + \epsilon_2 dx^3 \wedge dx^4 \,. \tag{4.107}$$

This is the same field strength after the complexification of $\mathbb{R}^4$ that we discussed in the refinement of the topological string free energy of the Gopakumar-Vafa formalism, surely this is not a coincidence. Since the five-dimensional particles running on the loops wrapping the base circle become the four-dimensional particles coupled to the graviphoton field strength. Hence this story is the same as the Gopakumar-Vafa case. This leads to the conclusion that the two pictures are equivalent and we can write the relation between the partition function and the free energy schematically as

$$\log Z(a, m, \epsilon_1, \epsilon_2 \beta) := \frac{1}{\epsilon_1\epsilon_2} F(Q_a, Q_m, q, t) \,, \tag{4.108}$$

where q and t are defined as $q = e^{\epsilon_1}$ and $t = e^{\epsilon_2}$, and Q_a and Q_m are Kähler parameters for the associated 2-cycles wrapped by branes. The general map between the geometric engineering parameters and field theory ones is more subtle, but we made this simplification for clarity. The right-hand side of Equation (4.108) can be computed via the state-of-the-art topological vertex [63, 64]. On the other hand, one can compute the partition function without relying on any geometric construction, and it is possible to compute the refined Gopakumar-Vafa invariants from the Nekrasov partition function of the five-dimensional supersymmetric gauge theory on the Ω-background directly. Nevertheless, the mapping between them is generally not trivial.

We can also extend this remarkable result to the worldsheet instantons as well as the higher genus corrections by means of the genus expansion of topological string theory. Let us ignore the effect of the Ω-background by setting $\epsilon_1 = -\epsilon_2 = \hbar$, then we can expand the logarithm of the partition function as

$$\log Z(a, \hbar, -\hbar\beta) = -\frac{1}{\hbar^2} F(a, \hbar, -\hbar) = \sum_{g=0}^{\infty} \hbar^{2g-2} F_g(a) \,. \tag{4.109}$$

The parameter $\hbar$ can be thought of as string coupling constant λ. Therefore, the prepotential of the gauge theory engineered by underlying local toric Calabi-Yau geometry can be achieved by the genus zero worldsheet instanton contribution. Furthermore, the gravitational couplings $R_+^2 F_+^{2g-2}$ can also be found from the expansion Equation (4.109) in which F_+ and R_+ is self-dual part of the graviphoton field strength and Riemann curvature tensor, respectively. Let us discuss the more general case for $\epsilon_1 + \epsilon_2 \neq 0$. In that case the Nekrasov partition function also has gravitational couplings through the coefficients of $R^2 F^{2g-2}$ terms in the effective action. The gravitational couplings can be computed by the expansion of the Nekrasov's partition function around small ϵ_1, ϵ_2 parameters [61, 62]

$$\log Z(a, \epsilon_1, \epsilon_2\beta) = \sum_{g,n=0}^{\infty} (\epsilon_1 + \epsilon_2)^{2n} (\epsilon_1\epsilon_2)^{g-1} F_{n,g}(a) \,. \tag{4.110}$$

Note that this expansion is not unique [61, 62]. So far, we consider the relation between the Nekrasov partition function and the topological string theory free energy. However, we did not discuss how the five-dimensional Nekrasov partition function can be computed explicitly. Let us take the five-dimensional $SU(N)$ gauge theory with eight supercharges. The partition function of this theory can be constructed by the five-dimensional lift of the four-dimensional Nekrasov factor that we introduced. The five-dimensional Nekrasov factor is given by:

$$N_{Y_1Y_2}(a_1-a_2,s,t)=\prod_{s\in Y_l}\left(1-e^{\ell_{Y_2}(s)\epsilon_1-(a_{Y_1}(s)+1)\epsilon_2+a_1-a_2}\right)\times\prod_{t\in Y_n}\left(1-e^{-(\ell_{Y_1}(t)+1)\epsilon_1+a_{Y_2}(t)\epsilon_2+a_1-a_2}\right)\,. \tag{4.111}$$

Then, the five-dimensional $SU(N)$ vector multiplet with eight supercharges can be written as

$$Z_{\text{vect}}(\vec{a},\vec{Y})=\prod_{l,n=1}^{N}\frac{1}{N_{Y_lY_n}(a_l-a_n,s,t)}\,. \tag{4.112}$$

Additionally, the five-dimensional bifundamental part can be found from

$$Z_{\text{bifund}}(\vec{a},\vec{Y}\vec{b},\vec{W}m)=\prod_{l=1}^{N_1}\prod_{n=1}^{N_2}N_{Y_lW_n}(a_l-b_n-m_{\text{bf}},s,t)\,. \tag{4.113}$$

The other contributions, such as fundamental, antifundamental or adjoint hypermultiplet ones can be determined as in the four-dimensional story without any further modification, then we can write the general partition function which has all such contributions

$$\begin{aligned}Z=&\sum_{\vec{Y}^{(1)}}\cdots\sum_{\vec{Y}^{(M-1)}}\left(e^{-\frac{\epsilon_1+\epsilon_2}{2}}(\mathfrak{q}^{(1)})^2\right)^{N^{(1)}|\vec{Y}^{(1)}|}\cdots\left(e^{-\frac{\epsilon_1+\epsilon_2}{2}}(\mathfrak{q}^{(M-1)})^2\right)^{N^{(M-1)}|\vec{Y}^{(M-1)}|}\\&\times\prod_{\alpha=1}^{M-1}Z_{\text{vect}}(\vec{a}^{(\alpha)},\vec{Y}^{(\alpha)})\prod_{\alpha=1}^{M-2}Z_{\text{bifund}}(\vec{a}^{(\alpha)},\vec{Y}^{(\alpha)},\vec{a}^{(\alpha+1)},\vec{Y}^{(\alpha+1)},m_{\text{bf}}^{(\alpha,\alpha+1)})\\&\times\prod_{\delta=1}^{N_f}Z_{\text{fund}}(\vec{a}^{(M-1)},\vec{Y}^{(M-1)},m^{\text{f}}_{N+\delta})\prod_{\gamma=1}^{N_{af}}Z_{\text{antifund}}(\vec{a}^{(1)},\vec{Y}^{(1)},m^{\text{af}}_{\gamma})\,.\end{aligned} \tag{4.114}$$

Note that each instanton parameter is multiplied by the additional factor which depends on the Ω-background parameters.

4.9. Refined G_2 Instanton Sum

In this section, we shall give a brief review of [70], in particular the computation of the refined G_2 instanton partition function from "ADHM-like" formalism. Classical gauge theories admit ADHM description and one can understand the dynamics of moduli space of instantons via ADHM [47]. By using ADHM construction, It is possible to write down closed expressions for partition functions of $ABCD$-type gauge theories in terms of contour integrals, and residues of these integrals are nothing but Jeffrey-Kirwan residues [56, 69]. On the other hand, these contour integrals can be written as a residue sum only for A-type gauge theories in which fixed points of the equivariant action match with ordered sets of Young diagrams, and the instanton sum can be made by the combinatorics of Young diagrams. Apart from that, there is little known about exceptional gauge theories, however, as discussed in [70], one can get a closed formula for $SO(7)$ and G_2 gauge theories from colored Young diagrams of $SU(4)$ and $SU(3)$ respectively. The ADHM construction for exceptional instantons is the following firstly, consider d dimensional gauge theory with a gauge group G_r whose rank is given by r and H_r is classical gauge group as a subgroup of G_r. The relevant phase for this construction is the Coulomb phase rather than the symmetric phase of the exceptional gauge theory. In the Coulomb phase, the moduli space of instantons of G_r are lifted to the saddle points and these saddle points are instanton moduli space of classical gauge group H_r. These saddle points are identified by massive parameters since we are in the Coulomb phase. When G_r is an exceptional gauge theory, there is no standard ADHM construction. However, when the G_r and H_r have the same rank, in some cases, it is tractable to separate the whole moduli into the instanton moduli space of H_r which admits the standard ADHM formalism and the moduli which comes from the extra fields. To sum up, there are two moduli, the first one comes from the vector multiplets of H_r, and the second one comes from the extra light degrees of freedom in UV. The latter one is crucial to capture the full dynamics of the moduli space of instantons of G_r. In group theoretical language, this means that adjoint representation vector multiplet in G_r decomposes as

$$\mathbf{adj}(G) \to \mathbf{adj}(H) \bigoplus_i \mathbf{R}_i(H)\,, \tag{4.115}$$

where $\mathbf{adj}(H)$ and $\mathbf{R}_i(H)$ stand for vector multiplet of classical gauge group and extra fields respectively. When $G_r = G_2$, then $H_r = SU(3)$ we have

$$SU(3) \subset G_2 : \mathbf{14} \to \mathbf{8} \oplus \mathbf{3} \oplus \overline{\mathbf{3}}\,, \tag{4.116}$$

where the first part of the decomposition corresponds to $SU(3)$ and the remainder is the extra moduli. In standard ADHM formalism, d dimensional gauge theory can be represented by $d-4$ dimensional gauge theory. In our case, $d = 5$ and with Ω-background deformations, $5d$ $\mathcal{N} = 1$ G_2 gauge theory is reduced to supersymmetric quantum mechanics on the moduli space of k $SU(3)$ instantons with extra vector multiplets. The field content for standard ADHM given by $SU(3)$ [70]

$$\begin{aligned} A_\mu, \lambda_0, \lambda &: \mathcal{N}=(0,4) \quad U(k) \text{ vector multiplet}\,, \\ q_l, \widetilde{q}^l &: (\mathbf{k}, \overline{\mathbf{3}}) + (\overline{\mathbf{k}}, \mathbf{3}) \quad (l=1,2,3)\,, \\ q_{\text{adj}}, \widetilde{q}_{\text{adj}} &: (\mathbf{adj}, \mathbf{1})\,. \end{aligned} \tag{4.117}$$

The 2-dimensional gauge field is denoted by $A_\mu = (A_0, A_1)$, q_l and $\widetilde{q}^l$ are hypermultiplets with bosonic fields which sit in $(\mathbf{k}, \overline{\mathbf{3}}) + (\overline{\mathbf{k}}, \mathbf{3})$. The hypermultiplets with bosonic fields that sit in $(\mathbf{adj}, \mathbf{1})$ labeled by $q_{\text{adj}}, \widetilde{q}_{\text{adj}}$. The extra bosonic field content can be expressed as [70]

$$\begin{aligned} \phi_i, \phi_4 &: \text{chiral bosonic fields in } (\overline{\mathbf{k}}, \overline{\mathbf{3}})_{J=\frac{1}{2}} + (\overline{\mathbf{k}}, \mathbf{1})_{J=\frac{1}{2}}\,, \\ b, \widetilde{b} &: \text{chiral bosonic fields in } (\overline{\mathbf{anti}_2}, \mathbf{1})_{J=\frac{1}{2}}\,, \\ \hat{\lambda} &: \text{Fermi fields in } (\mathbf{sym}_2, \mathbf{1})_{J=0}\,, \\ \check{\lambda} &: \text{Fermi fields in } (\mathbf{sym}_2, \mathbf{1})_{J=-1}\,. \end{aligned} \tag{4.118}$$

J is defined as the generators of the symmetry group $SU(2)_L \times SU(2)_R$

$$J = \frac{J_R + J_{\mathcal{R}}}{2}\,. \tag{4.119}$$

J_R and $J_{\mathcal{R}}$ are the Cartan's of $SU(2)_R$ and $SU(2)_{\mathcal{R}}$, where $SU(2)_{\mathcal{R}}$ is the R-symnmetry of the five dimensions. The matter can also be added to the above field content by introducing extra Fermi multiplets. However, since we are interested only pure G_2 gauge theory, thus we ignore these fields. There are also left-right symmetry $SO(4) = SU(2)_R \times SU(2)_L$, of the gauge group in 5d Ω-background which is $\mathbb{C}^2 \times \mathbf{S}^1$. The $SO(4)$ group acts on the spatial part of Ω-background, $\mathbb{C}^2 \cong \mathbb{R}^4$. The partition function for the above field contour, in other words, the refined pure G_2 instanton index with an instanton number k can be written as the Witten index. This index can be expressed as contour integral [70]

$$\begin{aligned} Z_k^{G_2} = \frac{1}{k!} \oint \prod_{I=1}^{k} \frac{d\phi_I}{2\pi i} \cdot \frac{\prod_{I\neq J} 2\sinh\frac{\phi_{IJ}}{2} \cdot \prod_{I,J} 2\sinh\frac{2\epsilon_+ - \phi_{IJ}}{2}}{\prod_{I=1}^{k}\prod_{l=1}^{3} 2\sinh\frac{\epsilon_+ \pm(\phi_I - a_l)}{2} \cdot \prod_{I,J} 2\sinh\frac{\epsilon_{1,2}+\phi_{IJ}}{2}} \\ \times \frac{\prod_{I\leq J}\left(2\sinh\frac{\phi_I+\phi_J}{2} \cdot 2\sinh\frac{\phi_I+\phi_J-2\epsilon_+}{2}\right)}{\prod_I\left(\prod_{l=1}^{3} 2\sinh\frac{\epsilon_+ - \phi_I - a_l}{2} \cdot 2\sinh\frac{\epsilon_+ - \phi_I}{2}\right) \cdot \prod_{I<J} 2\sinh\frac{\epsilon_{1,2}-\phi_I-\phi_J}{2}}\,, \end{aligned} \tag{4.120}$$

where $\phi_{IJ} = \phi_I - \phi_J$ and $\epsilon_\pm \equiv (\epsilon_1 \pm \epsilon_2)/2$. $\pm$ and $\epsilon_{1,2}$ means that all terms are multiplied. The chemical potentials of the $SU(3)$ gauge theory are given by the parameters a_l with the constraint $\sum_{l=1}^{3} a_l = 0$.

The integrand on the first line represents the standard $SU(3)$ ADHM fields $(q, \widetilde{q}, q_{\text{adj}}, \widetilde{q}_{\text{adj}})$ and $U(k)$ vector multiplet fermions. The second line indicates the extra vector multiples. The poles with non-zero residue contributions to the contour integral, Jeffrey-Kirwan residues, come from only the first line of Equation (4.120) and the relevant pole locations for ϕ_I are given by colored $SU(3)$ Young diagrams. There are three distinct Young diagrams $Y = (Y_1, Y_2, Y_3)$ and every Y_l contains the number of boxes running from 1 to instanton number k. Let $s = (i, j)$ be a partition and the partition $s = (i, j)$ represents a box $\Box \in Y_l$ whose location is identified with i'th row and j'th column of l'th Young diagram. The correspondent pole to $Y(s)$ is specified by

$$\phi(s) = a_l - \epsilon_+ - (i-1)\epsilon_2 - (j-1)\epsilon_1 \quad , \quad s = (i,j) \in Y_l \quad (l = 1,2,3)\,. \tag{4.121}$$

One can get the residue sum by using the pole specification for Equation (4.121),

$$Z_k^{G_2} = \\ (-1)^k \sum_{\vec{Y}|\vec{Y}|=k} \prod_{l=1}^{3} \prod_{s\in Y_l} \frac{2\sinh\phi(s)\cdot 2\sinh(\epsilon_+ - \phi(s))\cdot 2\sinh\frac{\epsilon_+-\phi(s)}{2}}{\prod_{j=1}^{3}\left(2\sinh\frac{E_{ln}(s)}{2}\cdot 2\sinh\frac{E_{ln}(s)-2\epsilon_+}{2}\cdot 2\sinh\frac{\epsilon_+-\phi(s)-a_n}{2}\right)} \tag{4.121}$$
$$\times \prod_{l\leq n}^{3} \prod_{s_{l,n}\in Y_{l,n} s_l<s_n} \frac{2\sinh\frac{\phi(s_l)+\phi(s_n)}{2}\cdot 2\sinh\frac{\phi(s_l)+\phi(s_n)-2\epsilon_+}{2}}{2\sinh\frac{\epsilon_{1,2}-\phi(s_l)-\phi(s_n)}{2}}\,.$$

$E_{ln}(s)$ is defined with Equation (4.93)

$$E_{ln}(s) = a_l - a_n - \epsilon_1 \ell_{Y_n}(s) + \epsilon_2(a_{Y_l}(s)+1)\,, \tag{4.123}$$

where $s_l < s_n$ means that

$$s_l < s_n \quad \text{if} \quad \begin{cases} l < n, \\ l = n, \quad i_l < j_n, \\ l = n, \quad i_n = i_n, \quad j_l < j_n\,. \end{cases} \tag{4.124}$$

The Coulomb branch parameters are given by $e^{-v_i} = A_i$ and we switch to Kähler parameters via [71]

$$\begin{aligned} A_1 &= Q_2\,, \\ A_2 &= Q_1 Q_2\,, \\ A_3 &= Q_1^{-1} Q_2^{-2}\,. \end{aligned} \tag{4.125}$$

We compute the refined G_2 instanton partition function that is Equation (4.121) up to 3 instantons and presents 1 instanton and 2 instanton results for simplicity. The partition function for the $k=1$ instanton and $k=2$ instanton sectors are given by

$$
\begin{aligned}
Z_1^{G_2} &= \frac{1}{(1-q)(1-t)} \\
&\times \left(\frac{q^2 Q_1^3 Q_2^6 t^2 \,(q Q_1 Q_2^2 + t)}{(Q_1 Q_2^3 - 1)\,(Q_1^2 Q_2^3 - 1)\,(q Q_2 - t)\,(q Q_1 Q_2 - t)\,(q Q_1 Q_2^3 - t)\,(q Q_1^2 Q_2^3 - t)} \right. \\
&- \frac{q^2 Q_1^3 Q_2^4 t^2 \,(q + Q_2 t)}{(Q_1 - 1)\,(Q_1 Q_2^3 - 1)\,(q Q_1 - t)\,(q Q_1 Q_2 - t)\,(q - Q_1 Q_2^2 t)\,(q - Q_1 Q_2^3 t)} \\
&+ \left. \frac{q^2 Q_1^3 Q_2^4 t^2 \,(q + Q_1 Q_2 t)}{(Q_1 - 1)\,(Q_1^2 Q_2^3 - 1)\,(q - Q_1 t)\,(q Q_2 - t)\,(q - Q_1 Q_2^2 t)\,(q - Q_1^2 Q_2^3 t)} \right),
\end{aligned}
\tag{4.126}
$$

$$
\begin{aligned}
Z_2^{G_2} &= \frac{q^4 Q_1^6 Q_2^8 t^4}{(q-1)^2 (t-1)^2} \left(\frac{q^3 (t-1)}{(q+1)(Q_1-1)(Q_1 Q_2^3 - 1)(q Q_1 - 1)(q - Q_1 Q_2^3)} \right. \\
&\times \frac{(q^2 + Q_2 t)}{(qt-1)(q Q_1 - t)(q^2 Q_1 - t)(q - Q_2 t)(q Q_1 Q_2 - t)} \\
&\times \frac{(q^3 - Q_2^2 t^2)}{(q^2 Q_1 Q_2 - t)(q - Q_1 Q_2^2 t)(q^2 - Q_1 Q_2^2 t)(q - Q_1 Q_2^3 t)(q^2 - Q_1 Q_2^3 t)} \\
&- \frac{(q-1) t^3}{(Q_1 - 1)(Q_1 Q_2^3 - 1)(t+1)(qt-1)(t - Q_1)} \\
&\times \frac{(q + Q_2 t^2)}{(Q_1 Q_2^3 t - 1)(t - q Q_1)(t^2 - q Q_1)(Q_2 t - q)(t - q Q_1 Q_2)} \\
&\times \frac{(q^2 - Q_2^2 t^3)}{(t^2 - q Q_1 Q_2)(Q_1 Q_2^2 t - q)(Q_1 Q_2^2 t^2 - q)(Q_1 Q_2^3 t - q)(Q_1 Q_2^3 t^2 - q)} \\
&- \frac{Q_2^2 (q + Q_2 t)}{(Q_1 - 1)(Q_1^2 Q_2^3 - 1)(q - Q_1 Q_2^3)(q Q_1 Q_2^3 - 1)(t - Q_1 Q_2^3)} \\
&\times \frac{(q^2 Q_1 Q_2 - t^2)}{(Q_1 Q_2^3 t - 1)(q Q_1 - t)(q Q_2 - t)(q Q_1 Q_2 - t)(q Q_1 Q_2 - t^2)} \\
&\times \left. \frac{(q Q_1 Q_2^2 + t)}{(q^2 Q_1 Q_2 - t)(q - Q_1 Q_2^2 t)(q Q_1^2 Q_2^3 - t)} + \cdots \right),
\end{aligned}
\tag{4.127}
$$

where we set $q = e^{-\epsilon_1}$ and $t = e^{\epsilon_2}$.

5. CONCLUSION

In conclusion, this thesis delves into the intriguing areas of research that connect geometry, topology, and quantum field theory, namely topological string theory, Seiberg-Witten theory, and the Nekrasov partition function. In the first part, the focus is on topological string theory. The concept of two-dimensional $\mathcal{N} = (2, 2)$ supersymmetry is introduced, and the A-model and B-model are explored through the process of topological twist. The target space interpretation and Gopakumar-Vafa invariants, providing insights into Calabi-Yau threefold, are also discussed. The second chapter explores Seiberg-Witten theory, which provides a robust framework for comprehending supersymmetric gauge theories. It delves into topics such as the moduli space of vacua, breaking of R-symmetry, and electric-magnetic duality, culminating in the exact results obtained through the Seiberg-Witten solution. Furthermore, the importance of monopoles and dyons in characterizing gauge theory behavior is emphasized. Finally, the third part focuses on the Nekrasov partition function and its role in studying instantons in supersymmetric Yang-Mills theory. The ADHM construction of instantons, non-compactness, and resolution of singularities are explored, leading to the obtaining of the prepotential and the study of four and five-dimensional Nekrasov factors.

REFERENCES

1. Witten, E., “Topological Sigma Models”, *Communications in Mathematical Physics*, Vol. 118, pp. 411-449, 1988.

2. Witten, E., “Mirror manifolds and topological field theory”, *American Physical Studies/International Press Studies in Advanced Mathematics*, Vol. 9, pp. 121-160 1991.

3. Witten, E., “Topological Quantum Field Theory”, *Communications in Mathematical Physics*, Vol. 117, pp. 353-386, 1988.

4. Pioline, B., “Lectures on black holes, topological strings and quantum attractors”, arXiv:hep-th/0607227, 2006.

5. Neitzke, A. and C. Vafa, “Topological strings and their physical applications”, arXiv:hep-th/0410178, 2004.

6. Alim, M., M. Hecht, P. Mayr and A. Mertens, “Mirror Symmetry for Toric Branes on Compact Hypersurfaces”, arXiv:0901.2937, 2009.

7. Vonk, M., “A Mini-course on topological strings”, arXiv:hep-th/0504147, 2005.

8. Marino, M., *“Chern-Simons theory, matrix models, and topological strings”*, Oxford University Press, Oxford, 2005.

9. Hori, H., S. Katz, A. Klemm, R. Pandharipande, R. Thomas, C. Vafa, R. Vakil and E. Zaslow, *“Mirror symmetry”*, American Mathematical Society, 2003.

10. Antoniadis, I., E. Gava, K. S. Narain and T. R. Taylor, “Topological amplitudes in string theory”, arXiv:hep-th/9307158, 1994.

11. Bershadsky, M., S. Cecotti, H. Ooguri, and C. Vafa, "Kodaira-Spencer theory of gravity and exact results for quantum string amplitudes", arXiv:hep-th/9309140, 1994.

12. Polchinski, J.,"Dirichlet-Branes and Ramond-Ramond Charges", *Physical Review Letters*, Vol. 75, pp. 4724-4727, 1995.

13. Witten, E., "String theory dynamics in various dimensions", *Nuclear Physics B*, Vol. 443, pp. 85–126, 1995.

14. Strominger, A. and C. Vafa, *"Microscopic origin of the Bekenstein-Hawking entropy"*, Physics Letters B, Vol. 379, pp. 99-104, 1996.

15. Schomerus, V., *"A Primer on String Theory"*, Cambridge University Press, Cambridge, 2017.

16. Blumenhagen, R., E. Plauschinn, *" Introduction to Conformal Field Theory With Applications to String Theory"*, Springer, Heidelberg, 2009.

17. Blumenhagen, R., E. Plauschinn, *" Introduction to Conformal Field Theory With Applications to String Theory"*, Springer, Heidelberg, 2009.

18. Green, M. B., J. H. Schwarz, and E. Witten, *"Superstring theory"*, Cambridge University Press, Cambridge, 1987.

19. Polchinski, J., *"String theory"*, Cambridge University Press, Cambridge, 1998.

20. Becker, K., M. Becker and J. H. Schwarz, *"String theory and M-theory"*, Cambridge University Press, Cambridge, 2007.

21. Ibanez, L. E and A. Uranga, *"String theory and Particle Physics"*, Cambridge University Press, Cambridge, 2012.

22. Johnson, C., *"D-branes"*, Cambridge University Press, Cambridge, 2003.

23. Blumenhagen, R., D.Lüst, S.Theisen, *"Basic Concepts of String Theory"* , Springer, Heidelberg, 2013.

24. Gopakumar, R. and C. Vafa, "M-Theory and Topological Strings–I", arXiv:hep-th/9809187, 1998.

25. Gopakumar, R. and C. Vafa, "M-Theory and Topological Strings–II", arXiv:hepth/9812127, 1998.

26. Vafa, C., "A Stringy test of the fate of the conifold", *Physics B*, Vol. 447, pp. 252-260, 1995.

27. Faber, C., and R. Pandharipande, "Hodge integrals and Gromov-Witten theory", *Inventiones mathematical*, Vol. 139, No.1, pp. 173-199, 1998.

28. Hollowood, T. J., A. Iqbal and C. Vafa,"Matrix Models, Geometric Engineering and Elliptic Genera", arXiv:hep-th/0310272, 2008.

29. Müller-Kirsten, H. and A. Wiedemann, *"Introduction to Supersymmetry"*, World Scientific lecture notes in physics. World Scientific, 2010.

30. Wess, J. and J. Bagger, *"Supersymmetry and supergravity"*. Princeton University Press, Princeton, New Jersey, 1992.

31. Bilal, A., "Introduction to supersymmetry", arXiv:hep-th/0101055, 2001.

32. Alvarez-Gaume, L. and S. F. Hassan, "Introduction to S duality in N=2 supersymmetric gauge theories: A Pedagogical review of the work of Seiberg and Witten", *Fortschritte der Physik*, Vol. 45, pp. 159-236, 1997.

33. Seiberg, N., "Supersymmetry and Nonperturbative beta Functions", *Physics Letters B*, Vol. 206, pp. 75-80, 1988.

34. Seiberg, N. and E. Witten, “Electric - magnetic duality, monopole condensation, and confinement in $\mathcal{N} = 2$ supersymmetric Yang-Mills theory”, *Physics Letters B*, Vol. 426, pp. 19-52, 1994.

35. Seiberg, N. and E. Witten, “Monopoles, duality and chiral symmetry breaking in $\mathcal{N} = 2$ supersymmetric QCD”, *Physics Letters B*, Vol. 431, pp. 484-550, 1994.

36. Martone, M., “The constraining power of Coulomb Branch Geometry:lectures on Seiberg-Witten theory”, arXiv:2006.14038, 2020.

37. Lerche, W., “Introduction to Seiberg-Witten theory and its stringy origin,” *Nuclear Physics B*, Vol. 55, pp. 83-117, 1997.

38. Donaldson, S. K., “Instanton and Geometric invariant Theory”, *Communications in Mathematical Physics*, Vol. 93, pp. 453-460, 1984.

39. Witten, E., “Monopoles and four manifolds”, arXiv:hep-th/9411102, 1994.

40. Tong, D., “TASI lectures on solitons: Instantons, monopoles, vortices and kinks”, arXiv:hep-th/0509216, 2005.

41. Shadchin, S., “On certain aspects of string theory/gauge theory correspondence”, arXiv:hep-th/0502180, 2005.

42. Nekrasov, N.A., “Noncommutative instantons revisited”, arXiv:hep-th/0010017, 2003.

43. Vandoren, S. and P. van Nieuwenhuizen, “Lectures on instantons”, arXiv:0802.1862, 2008.

44. Cordes, S., G. W. Moore and S. Ramgoolam, “Lectures on 2-d Yang-Mills theory, equivariant cohomology and topological field theories”, *Nuclear Physics B*, Vol. 41, pp. 184-244, 1995.

45. Moore, G. W., N. Nekrasov and S. Shatashvili, "D particle bound states and generalized instantons", *Communications in Mathematical Physics*, Vol. 209, pp. 77-95, 2000.

46. Labastida, J. M. F. and C. Lozano, "Lectures in topological quantum field theory", arXiv:hep-th/9709192, 1998.

47. Atiyah, M. F., N. J. Hitchin, V. G. Drinfeld and Y. I. Manin, "Construction of Instantons", *Physics Letters A*, Vol. 65, pp. 185–187, 1978.

48. Baulieu, L. and I. M. Singer, "Topological Yang-Mills Symmetry", *Nuclear Physics B*, Vol. 5, pp. 12-19, 1988.

49. Nekrasov, N. and A. S. Schwarz, "Instantons on noncommutative $\mathbb{R}^4$ and (2,0) superconformal six-dimensional theory", *Communications in Mathematical Physics*, Vol. 198, pp. 689-703, 1998.

50. Tachikawa, Y., *"Seiberg-Witten Theory and Instanton Counting"*, MSc Thesis, Tokyo University, 2003.

51. Nekrasov, N. A., "Seiberg-Witten prepotential from instanton counting", *Advances in Theoretical and Mathematical Physics*, Vol. 7, No.5, pp. 831-864, 2003.

52. Nekrasov, N. A. and A. Okounkov, "Seiberg-Witten theory and random partitions", *Progress in Mathematics*, Vol. 244, pp. 525-596, 2006.

53. Moore, G. W. ,N. Nekrasov and S. Shatashvili, "Integrating over Higgs branches", *Communications in Mathematical Physics*, Vol. 209, pp. 97-121, 2000.

54. Moore, G. W., N. Nekrasov and S. Shatashvili, "D particle bound states and generalized instantons", *Communications in Mathematical Physics*, Vol. 209, pp. 77-95, 2000.

55. Kimura, T., “Instanton Counting, Quantum Geometry, and Algebra”, arXiv:2012.11711, 2021.

56. Nekrasov, N. A. and S. Shadchin, “ABCD of instantons”, *Communications in Mathematical Physics*, Vol. 252, pp. 359–391, 2004.

57. Tachikawa, Y., “A review on instanton counting and W-algebras”, arXiv:1412.7121, 2014.

58. Nekrasov, N. A., “Five dimensional gauge theories and relativistic integrable systems”, *Nuclear Physics B*, Vol. 531, pp. 323-344, 1998.

59. Lawrence, A.E. and N. Nekrasov, “Instanton sums and five-dimensional gauge theories”, *Nuclear Physics B*, Vol. 513, pp. 239-265, 1998.

60. Katz, S.H., A. Klemm and C. Vafa, “Geometric engineering of quantum field theories”, *Nuclear Physics B*, Vol. 497, pp. 173-195, 1997.

61. Huang, M. x. and A. Klemm, “Direct integration for general Ω backgrounds”, *Advances in Theoretical and Mathematical Physics*, Vol. 16, No.3, pp. 805-849, 2012.

62. Prudenziati, A., “Double genus expansion for general Ω background”, arXiv:1204.2322, 2016.

63. Iqbal, A., C. Kozcaz and C. Vafa, “The Refined topological vertex”, *Journal of High Energy Physics*, Vol. 10, No. 147, 2009.

64. Awata, H. and H. Kanno, “Refined BPS state counting from Nekrasov’s formula and Macdonald functions”, *International Journal of Modern Physics A*, Vol. 24, pp. 2253-2306, 2009.

65. Alday, L.F., D. Gaiotto and Y. Tachikawa, “Liouville Correlation Functions from Four-dimensional Gauge Theories”, *Letters in Mathematical Physics*, Vol. 91, pp. 167-197, 2010.

66. Fucito, F., J.F. Morales, and R. Poghossian, “Instantons on Quivers and Orientifolds”, *Journal of High Energy Physics*, Vol. 10, No. 37, 2004.

67. Shadchin, S., “Cubic Curves from Instanton Counting”, *Journal of High Energy Physics*, Vol. 03, No. 46, 2006.

68. Hollands, L., C. A. Keller and J. Song, “From SO/Sp instantons to W-algebra blocks”, *Journal of High Energy Physics*, Vol. 03, No. 53, 2011.

69. Kim, H.C., S.S. Kim and K. Lee, “5-dim Superconformal Index with Enhanced E_n Global Symmetry”, *Journal of High Energy Physics*, Vol. 10, No 142, 2012.

70. Kim, H.C., J. Kim, S. Kim, K.-H. Lee and J. Park, “6d strings and exceptional instantons”, *Physical Review D*, Vol. 103, 025012, 2018.

71. Nawata, S. and R. D. Zhu, “Instanton counting and O-vertex”, *Journal of High Energy Physics*, Vol. 09, No. 190, 2021.