ISTANBUL TECHNICAL UNIVERSITY ★ GRADUATE SCHOOL

CLASSICAL YANG-BAXTER EQUATION FROM DUALITY COVARIANT FORMULATION OF STRING THEORY

Ph.D. THESIS

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Department of Mathematics Engineering

Mathematical Engineering Program

JANUARY 2024

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İSTANBUL TEKNİK ÜNİVERSİTESİ ★ LİSANSÜSTÜ EĞİTİM ENSTİTÜSÜ

SİCİM KURAMININ DUALİTE KOVARYANT FORMÜLASYONUNDAN KLASİK YANG-BAXTER DENKLEMİ

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To my family,

FOREWORD

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ABBREVIATIONS

c.p.	**:** The Cyclic Permutation of the Previous Term(s)
CYBE	**:** Classical Yang-Baxter Equation
DFT	**:** Double Field Theory
GCG	**:** Generalized Complex Geometry
LHS	**:** Left Hand Side
MYBE	**:** Modified Yang-Baxter Equation
RHS	**:** Right Hand Side
YB	**:** Yang-Baxter

SYMBOLS

$\mathbf{[\,\cdot,\cdot\,]}$	**:** Lie bracket of vector fields
$\mathbf{\{\,\cdot,\cdot\,\}}$	**:** Poisson bracket of smooth functions
$\mathbf{[\,\cdot,\cdot]_{SN}}$	**:** Schouten-Nijenhuis (SN) bracket of multi-vector fields
$\mathbf{[\,\cdot,\cdot]_{\mathfrak{d}}}$	**:** Lie bracket on $g \oplus g^*$, where g and g^* are dual Lie subalgebras
$\mathbf{i_X}$	**:** Interior product of differential forms by a vector field X
$\mathbf{i}_\alpha$	**:** Interior product of multi-vector fields by a 1-form α
$\mathscr{L}_\mathbf{X}$	**:** Lie derivative of forms or multi-vector fields along a vector field X
$\mathbf{C^\infty(M)}$	**:** Smooth real-valued functions on a manifold M
$\wedge^\mathbf{k}\mathbf{A}$	**:** kth exterior power bundle of a Lie algebroid A
$\wedge^\mathbf{k}\mathbf{A}^*$	**:** kth exterior power bundle of a Lie algebroid A^*
$\Gamma(\wedge^\mathbf{k}\mathbf{A})$	**:** Smooth sections of kth exterior power bundle of a Lie algebroid A
$\Gamma(\wedge^\mathbf{k}\mathbf{A}^*)$	**:** Smooth sections of kth exterior power bundle of a Lie algebroid A^*

CLASSICAL YANG-BAXTER EQUATION FROM DUALITY COVARIANT FORMULATION OF STRING THEORY

SUMMARY

The aim of the thesis is to study the homogeneous Yang-Baxter (YB) deformation proposed in the physics literature for a generic Green-Schwarz sigma model from a geometric point of view. It has been shown that these kind of deformations are generated by a certain kind of non-constant $O(d,d)$ transformation, called β transformation, which acts as solution generating transformations in string theory. We study the construction of such an $O(d,d)$ transformation from a bi-vector field related to the Poisson structure on the manifold. It is a well-known fact that there is a Lie algebroid structure on the cotangent bundle of the manifold when there is a Poisson structure on the manifold. Moreover, this Lie algebroid structure is compatible with the standard Lie algebroid structure on the tangent bundle, so that there is a Courant algebroid structure on the direct sum of the tangent and cotangent bundle (called the generalized tangent bundle) of the manifold. We also study Courant algebroid structures in order to understand and to generalize the transformation and the YB deformation.

Given a Lie algebra with a non-degenerate inner product, if there exists an endomorphism R, which satisfies the classical Yang-Baxter equation (CYBE), then the direct sum of the Lie algebra and its dual has a natural Drinfel'd structure. Such an endomorphism can be extended to the tangent bundle of the integral Lie group by the help of the adjoint action. In this way, an automorphism called the dressed R-matrix can be constructed, which satisfies the CYBE since the adjoint action is an automorphism of the Lie bracket. It is possible to build a Poisson bi-vector field on the manifold from the dressed R-matrix. It can be shown that the Schouten-Nijenhuis bracket of the bi-vector field with itself vanishes following directly from the fact that the dressed R-matrix satisfies CYBE. The Lie algebroid structure on the cotangent bundle induced from the Poisson structure is compatible with the standard Lie algebroid structure on the tangent bundle. Then the tangent and cotangent bundles with the stated Lie algebroid structures form a Lie bialgebroid, which is an example of a triangular Lie bialgebroid. The Drinfel'd double of the resulting triangular Lie bialgebroid is a Courant algebroid with transversal Dirac structures. This geometrical structure plays a prominent role in the solution generating mechanism stated above.

The dynamical fields in the universal sector of the low energy effection action of string theory are the Riemannian metric, a 2-form field called the B-field and a scalar field called the dilaton field. The first two of these fields become the constituents of the generalized metric, which is a tensor on the generalized tangent bundle $TM \oplus T^*M$ that transforms naturally under $O(d,d)$. There is a $O(d,d)$ covariant version of string

theory, called Double Field Theory (DFT), which is written in terms of the generalized metric and the generalized dilaton field. DFT provides a suitable framework to demonstrate the fact that YB deformation preserves the solutions of string theory. From a geometric point of view, the existence of a generalized metric is equivalent to the existence of a subbundle of the generalized tangent bundle on which the inner product is positive definite.

If one starts with a generalized metric of a specific form that solves the field equations of DFT in the limit in which it reduces to the field equations of supergravity and transforms it with the $O(d,d)$ matrix generating the YB deformation, the resulting generalized metric also solves the field equations of DFT in the same limit. In the physics literature, the proof of this is based on comparing the "fluxes" before and after the transformation and showing that these fluxes do not change. From a geometrical point of view the fluxes are just the "structure functions" of the Courant algebroid structure on the generalized tangent bundle, when a specific basis is chosen for the sections of tangent and cotangent bundles. In order to understand this "flux preservation" principle from a geometrical point of view, we also study the axioms defining a Courant algebraid in local coordinates. We also work out in detail the case where the anchor of the Courant algebroid is determined by a bi-vector field associated by the YB deformation.

SİCİM KURAMININ DUALİTE KOVARYANT FORMÜLASYONUNDAN KLASİK YANG-BAXTER DENKLEMİ

ÖZET

Tezin amacı, fizik literatüründe Green-Schwarz sigma modeli için önerilen homojen Yang-Baxter deformasyonunu geometrik açıdan incelemektir. Bu tür deformasyonların, β dönüşümü adı verilen ve sicim teorisinde çözüm üreten dönüşümler olarak görev yapan belirli bir tür sabit olmayan $O(d,d)$ dönüşümü tarafından üretildiği gösterilmiştir. Manifold üzerindeki Poisson yapısına bağlı iki-vektör alanından böyle bir $O(d,d)$ dönüşümünün inşasını inceliyoruz. Manifold üzerinde bir Poisson yapısı varken manifoldun kotanjant lif demetinde bir Lie cebiroid yapısının olduğu iyi bilinen bir gerçektir. Ayrıca, bu Lie cebiroid yapısı, tanjant lif demetindeki standart Lie cebiroid yapısıyla uyumludur, böylece manifoldun tanjant ve kotanjant lif demetinin (genelleştirilmiş tanjant lif demeti olarak adlandırılır) doğrudan toplamında bir Courant cebiroid yapısı bulunur. Dönüşümü ve Yang-Baxter deformasyonunu anlamak ve genelleştirmek için Courant cebiroid yapıları üzerinde çalışıyoruz.

Dejenere olmayan bir iç çarpımı olan bir Lie cebiri verildiğinde, klasik Yang-Baxter denklemini sağlayan bir R endomorfizması varsa, Lie cebiri ile onun dual cebirinin doğrudan toplamı doğal bir Drinfel'd yapısına sahiptir. Böyle bir yapı, Manin üçlüsü adı verilen geometrik bir yapının da özel bir örneğini oluşturmaktadır. R-matrisinden ve iç çarpımdan yararlanarak Lie cebir üzerinde bir iki-vektör oluşturmak mümkündür. Bunun yöntemi, Lie cebir üzerindeki, antisimetrik R-matrisi tarafından tanımlanan dejenere olmayan, en azından Frobenius bir alt cebirde, bir iki-form alanını kullanarak tersinden bir iki-vektör alanı inşa etmektir. Böyle bir R endomorfizması, adjoint eyleminin yardımıyla integral Lie grubunun tanjant lif demetine genişletilebilir. Bu şekilde, adjoint eyleminin Lie parantezinin bir otomorfizması olması nedeniyle klasik Yang-Baxter denklemini sağlayan giydirilmiş R-matrisi (R_g) denilen bir otomorfizma inşa edilebilir. Giydirilmiş R-matrisinden manifold üzerinde bir Poisson iki-vektör alanı oluşturmak mümkündür. Bunun yöntemi, önceden inşa edilen iki-vektör alanını kullanmak ve tüm manifolda genişletmektir. Giydirilmiş R-matrisi klasik Yang-Baxter denklemini sağladığı için ondan elde edilen iki-vektör alanının da kendisiyle Schouten-Nijenhuis parantezi sıfır olur. Bu sebeple, bu bir Poisson iki-vektör alanıdır. Poisson yapısının kotanjant lif demeti üzerinde belirlediği Lie cebiroid yapısı, tanjant lif demeti üzerindeki standart Lie cebiroid yapısıyla uyumludur. Bu sayede, tanjant ve kotanjant lif demetleri belirtilen Lie cebiroid yapılarıyla birlikte bir üçgensel iki-cebiroid yapısı oluşturur. Bu Lie iki-cebiroid'inin Drinfel'd ikilisi, üzerinde transversal Dirac yapıları olan, bir Courant cebiroididir. Bu geometrik yapı yukarıda belirtilen çözüm üretme mekanizmasında önemli bir rol oynamaktadır.

Sicim teorisinin düşük enerjilerdeki efektif alan teorisinin eyleminin evrensel sektöründeki dinamik alanlar; Riemannian metriği, B alanı adı verilen iki-form alanı ve dilaton alanı adı verilen skaler bir alandır. Bu alanlardan ilk ikisi, $O(d,d)$ altında doğal olarak dönüşen $TM \oplus T^*M$ genelleştirilmiş tanjant lif demeti üzerindeki bir tensör olan genelleştirilmiş metriğin bileşenleridir. Sicim teorisinin, genelleştirilmiş metrik ve genelleştirilmiş dilaton alanı cinsinden yazılmış, Çift Alan Teorisi adı verilen bir $O(d,d)$ kovaryant bir versiyonu vardır. Yang-Baxter deformasyonunun sicim teorisinin çözümlerini koruduğunu göstermek için Çift Alan Teorisi uygun bir çerçeve sunar. Geometrik açıdan; genelleştirilmiş metriğin varlığı, iç çarpımın pozitif definit olduğu genelleştirilmiş tanjant lif demetinin bir alt lif demetinin varlığına eşdeğerdir.

Çift Alan Teorisi alan denklemlerini süper kütle çekim alan denklemlerine indirgendiği limitte çözen aşağıdaki gibi özel bir formda olan genelleştirilmiş metrik ile başladığımızı varsayalım.

$$\mathscr{H}^{MN} = (U^{-1})^{M}{}_{A}\mathscr{H}^{AB}(U^{-1})^{N}{}_{B}.$$

Yukarıda verilen denklemde U^{-1} matrisleri $O(d,d)$'nin elemanları olup, bu şekilde olan genelleştirilmiş metriğe Yang-Baxter deformasyonunu üreten $O(d,d)$ dönüşümünü uyguladığımızı varsayalım. Buradan elde edilen yeni genelleştirilmiş metrik de aşağıdaki formda olacaktır.

$$\widetilde{\mathscr{H}}^{MN} = (\widetilde{U}^{-1})^{M}{}_{A}\mathscr{H}^{AB}(\widetilde{U}^{-1})^{N}{}_{B}.$$

Yukarıda verilen denklemdeki $\widetilde{U}^{-1}$ matrisleri de $O(d,d)$'nin elemanları olup, önceki genelleştirilmiş metrik olan $\mathscr{H}$'i kullanılarak üretilen yeni genelleştirilmiş metrik $\widetilde{\mathscr{H}}$ da Çift Alan Teorisi'nin, yine aynı limitte, alan denklemlerinin çözümü olacaktır. Fizik literatüründe bunun ispatı, dönüşümden önceki ve sonraki "akıların" karşılaştırılması ve bu akıların değişmediğinin gösterilmesine dayanmaktadır. Bu sebeple tezde gösterilmiştir ki özel olarak seçilen U^{-1} ve $\widetilde{U}^{-1}$ matrislerinin akılarının hesaplandığı her iki durumda da f-akısı Lie cebirinin yapı sabitlerine eşit çıkmaktadır ve $Q-, H-, R-$akıları ise sıfır çıkmaktadır. Burada önemli rol oynayan kısım; $\widetilde{U}^{-1}$ matrisinin yapısında giydirilmiş R-matrisinden üretilen Poisson iki-vektör alanının olması ve giydirilmiş R-matrisinin de klasik Yang-Baxter denklemini sağlıyor olmasıdır (R-matrisi klasik Yang-Baxter denklemini sağladığı durumda geçerlidir). Bu sebeple R-akısı sıfırlanmıştır. Diğer taraftan, giydirilmiş R-matrisinin özelliklerini kullanarak Q-akısı sıfırlanmıştır. H-akısı ise; matrisin formu sebebiyle otomatik olarak sıfırlanmaktadır.

Sonuç olarak, dönüşümden önceki ve sonraki akıların korunması prensibi sayesinde elde edilen yeni genelleştirilmiş metrik de aynı limitte alan denklemlerinin çözümü olacaktır. Buradan Riemannian metrik ile B-alanı çözümlerinin elde edilmesi de mümkün olmaktadır.

Geometrik açıdan, akılar; tanjant ve kotanjant lif demetleri için özel bir çerçeve seçildiğinde genelleştirilmiş tanjant lif demeti üzerindeki Courant cebiroid yapısının "yapı fonksiyonlarıdır". Bu "akı korunması" ilkesini geometrik açıdan anlamak için, Courant cebiroidini tanımlayan aksiyomları lokal koordinatlarda inceledik. Ayrıca Courant cebiroidinin çapasının Yang-Baxter deformasyonuyla ilişkili bir iki-vektör alanı tarafından belirlendiği durumu da ayrıntılı olarak inceledik.

E vektör lif demetinin Courant cebiroid yapısına sahip olabilmesi için üzerindeki yapıların; iç çarpımın, parantezin ve çapanın (ρ), sağlaması gereken beş şart olduğu literatürde gösterilmiştir. (Daha sonradan gösterilmiştir ki bu şartların bazıları diğerlerinden çıkmaktadır.) Çapa denilen geometrik yapı; E'den aldığı kesitleri tanjant lif demetine taşımaktadır. İlerleyen kısımda anlatılacağı üzere; Courant cebiroid yapısındaki çapa, çalışmalarımızda önemli bir rol oynamaktadır.

$E = R \oplus R^*$ formunda yazılabilen bir vektör lif demeti olduğunu varsayalım. Çalışmalarımızda kullandığımız yapı genelleştirilmiş tanjant lif demeti, $E = TM \oplus T^*M$, olduğu için tam olarak bu formdaki bir yapıya karşılık gelmektedir. Vektör lif demetinin bu şekildeki özel formunda; iç çarpımı, çapayı, parantezi ve türev operatörünü değişmeyen bir baz sistemine göre lokal koordinatlarda yazmak mümkündür. Bu durumda, vektör lif demetinin kesitleri üzerindeki parantezden yapı fonksiyonları T_{LIJ} gelmektedir. Bu durum, Lie cebir elemanları üzerindeki parantezden yapı sabitlerinin gelmesinin genelleştirilmiş hali olarak düşünülebilir.

Courant cebiroidinin üzerindeki geometrik yapıların lokal koordinatlardaki ifadeleri kullanıldığında; vektör lif demetinin Courant cebiroid olabilmesi için sağlaması gereken beş adet aksiyom; çapanın ve yapı fonksiyonlarının sağlaması gereken aşağıda verilen üç tane lokal şarta denk gelmektedir.

$$
\begin{aligned}
&* \eta^{IJ}\rho^i_I\rho^j_J = \rho^i{}_k\rho^{jk}_* + \rho^{ik}_*\rho^j{}_k = 0,\\
&* \rho^i_I\partial_i\rho^j_J - \rho^i_J\partial_i\rho^j_I - \eta^{KL}\rho^j_K T_{LIJ} = 0,\\
&* 4\rho^i_{[L}\partial_i T_{IJK]} + 3\eta^{MN}T_{M[IJ}T_{KL]N} = 0.
\end{aligned}
$$

Yukarıda verilen şartlar, sırasıyla, güçlü kısıt, akıların tanımı ve akıların sağlaması gereken koşullardır. Bunun yanı sıra, fizik literatüründe son koşula Bianchi özdeşliği denmektedir. Tezde gösterilmiştir ki; güçlü kısıtlama altında Çift Alan Teorisi eyleminin ayar değişmezliği gibi fiziksel prensipler tarafından Çift Alan Teorisi akılarına uygulanan koşullar tam olarak bu şartlara denk gelmektedir.

Yukarıdaki çalışmalarımızın yanı sıra, Courant cebiroidin kotanjant lif demetindeki kesitlerini tanjant lif demetindeki kesitlere götüren çapanın bileşenlerinin Yang-Baxter deformasyonuyla ilişkili bir iki-vektör alanı (Poisson iki-vektör alanı) tarafından belirlendiği durum incelenmiştir. Bu standart olmayan Courant cebiroidi için lokal koordinatlarda yukarıda verilen üç koşul; akıların tanımlarını ve sağlaması gereken denklemleri vermiştir. Örneğin, R-akısı; klasik Yang-baxter denkleminin sağlanması ve çapanın bileşeninin Poisson iki-vektör alanı olması sebebiyle sıfırlanmıştır. R-akısı ile birlikte diğer akılar da $\widetilde{U}^{-1}$ matrisinin akıları ile aynı gelmektedir ki bu beklenen bir durumdur. Bunun sebebi; bu matrislerin akılarını bulmak için kullanılan $O(d,d)$ matrisinin de Poisson iki-vektör alanını içermesidir.

Son olarak, Yang-Baxter deformasyonunu Courant cebiroid lisanında, tüm çalışmalarımızı birleştirecek şekilde, inceledik. $E = TM \oplus T^*M$ üzerinde tanımlanan genelleştirilmiş metriğin yukarıda verilen özel formda nasıl seçilebileceği ve eski genelleştirilmiş metrikten yeni olanının nasıl inşa edilebileceği gösterilmiştir. Çözüm yaratan $O(d,d)$ dönüşümünün; dönüşümden önceki ve sonraki akıları koruduğu gerçeği önceden gösterilmişti. Bunun yanı sıra, bu dönüşüm bir Courant cebiroid otomorfizması olmakla birlikte parantezi korumamaktadır. Ancak her iki durumdaki

parantez yapısında akıların dışında gelen terimler grup manifoldunun koordinatlarına bağlı değildir. Dolayısıyla bu tür terimler, çözüm üretme mekanizmasında bir rol oynamamaktadır.

1. INTRODUCTION

Usual geometric structures in Riemannian geometry are defined on the tangent bundle TM of a smooth manifold M. The sections of the tangent bundle are vector fields and there is a natural Lie bracket on these sections. Riemannian geometry is the underlying geometrical structure for Einstein's general theory of relativity. String theory can be regarded as a generalization of general relativity and it has a richer symmetry structure. It turns out that to describe this richer symmetry structure, it is more useful to work on the generalized tangent bundle $TM \oplus T^*M$ where T^*M is the cotangent bundle of M. There is also a natural Lie bracket on the sections of $TM \oplus T^*M$ called the (standard) Courant bracket. The vector bundle $TM \oplus T^*M$ with the standard Courant bracket, the natural inner product between the tangent and cotangent bundle and the anchor as the projection to the tangent bundle is the prototypical Courant algebroid. The Courant algebroid structure was originally introduced by Zhang-Ju Liu, Alan Weinstein and Ping Xu in their investigation of doubles of Lie bialgebroids in 1997 in [1]. Lie algebroids are the natural generalizations of Lie algebras. Precisely, Lie algebroid is a vector bundle together with a Lie bracket on its space of sections and a vector bundle morphism from the vector bundle to the tangent bundle called the anchor. A Lie bialgebroid, which is a natural generalizations of a Lie bialgebra, is a pair of dual Lie algebroids (L, L^*) that satisfy certain compatibility conditions. We will particularly be interested in triangular Lie bialgebroids which are constructed by the help of a bundle map from L^* to L, that can also be regarded as an element of $\wedge^2 L$. The definitions of Lie algebroids, Lie bialgebroids and Courant algebroids can be found in the Section 3.

Recently, a new physical theory called double field theory (DFT) has been constructed, which takes into account some of these stringy symmetries. If the manifold on which the strings propagates has non-contractible cycles, the string can wrap itself, creating a winding state that does not exist in point particle field theories. DFT is defined on a doubled space where the usual coordinates conjugate to momentum modes are

supplemented with dual coordinates that are conjugate to winding modes. T-duality is a stringy symmetry which interchanges momentum and winding modes and DFT action is a reformulation of low energy effective action of string theory which makes this duality symmetry manifest. The gauge symmetries of DFT is governed by the C-bracket, which can be thought as a generalized version of the Courant bracket. The DFT action is consistent only when a certain constraint, called the strong constrained is imposed, which means that the physical theory depends on half of the doubled coordinates only. In this limit, the C-bracket reduces to the Courant bracket and the relevant geometric structures required to describe the dynamics of DFT fields are defined on the generalized tangent bundle $TM \oplus T^*M$. It is possible to express the action of DFT in terms of some quantities, called "fluxes" in the physics literature. From a geometrical point of view, the fluxes are just the "structure functions" of the Courant algebroid structure on the generalized tangent bundle, when a a specific basis is chosen for the sections of tangent and cotangent bundles. The conditions imposed by the axioms defining a Courant algebroid on the structure functions are equivalent to the conditions imposed on DFT fluxes by physical principles like the gauge invariance of the DFT action, under the strong constraint. This fact can be followed from the Propositions 6.2.3 and 6.2.4. One of the purposes of this thesis is to explore the fluxes in the language of Courant algebroids.

In order to understand the dynamics of the string, one should have the action written for the maps from a two-dimensional surface, which is called a two-dimensional sigma model, to 10-dimensional one which is called a target space. TsT transformations, Yang-Baxter (YB) deformations, and λ deformations are well-known examples. The development of a fundamental understanding of integrable two-dimensional field theories, their relationship to generalized worldsheet dualities, and their potential for generalizing the AdS/CFT correspondence to nonmaximally supersymmetric cases while maintaining integrability are the main driving forces behind their research. A deformation on the worldsheet of a string that is described by the sigma model. We shall concentrate on a specific category of deformations that is a member of the YB class of homogeneous deformations. Such deformations of the $AdS_5 \times S^5$ string was first studied in [2]. The relation between TsT transformations and

homogeneous YB deformations was also studied in the papers [3]- [4]. When a linear operator R, whose definitions and properties can be found in the Section 5.1 in detail, acts on the Lie algebra g of isometries of an integrable sigma model, homogeneous deformations are produced. If R is anti-symmetric and satisfies the classical Yang-Baxter equation (CYBE), integrability is preserved. The fact that YB deformation is a solution-generating transformation has been shown in various papers by embedding it in DFT. The key idea in the proof is that the DFT fluxes are kept invariant under this transformation. This is where the significance of examining the fluxes in the thesis comes from. Many anti-symmetric solutions are admissible by the CYBE, and each of these will produce a distinct deformed target-space background for the string sigma-model. When the R-matrix is unimodular, these backgrounds are supergravity solutions [5], [6], [7].

Algebraically, YB deformation is closely related with Drinfel'd doubles. Therefore, the Drinfel'd double structure plays a prominent role in generating solutions in supergravity with isometry group G with a Lie algebra g. We have employed the Drinfel'd double structure, which is in an algebra setting, in our article [8]. For the remaining part of the thesis, we have attempted to comprehend the geometric structures that underlying this technique which can be used to generate solutions. A Drinfel'd double is a particular case of Lie bialgebroid, when the base manifold is a point. Indeed, the suitable framework to describe YB deformation will be the case of a special Lie bialgebroid that is the triangular Lie bialgebroid as we have mentioned earlier. In order to define a triangular Lie bialgebroid, one needs a bi-vector field. We will show in the Subsection 5.2 how this bi-vector field is constructed, with the help of Cartan-Killing metric, in terms of the R-matrix and dressed R-matrix which satisfy the CYBE, in detail. The existence of such a triangular Lie bialgebroid allows us to determine that there is a corresponding Courant algebroid with transversal Dirac structures. We will compute the fluxes associated with the relevant matrices and discuss about how this relates to the DFT principle of flux preservation. Furthermore, we will show how this fact results in a transformation-generating solution. Details about these concepts can be found below.

A fixed, constant R-matrix associated with the background isometries satisfying the CYBE is used to generate the dressed R-matrix, which is then extended via the adjoint action to the whole group manifold. Whenever the R-matrix fulfills CYBE, so does the dressed R-matrix. Such a matrix will be advantageous in a variety of ways. One is that the dressed R-matrix and Cartan-Killing metric can be utilized to produce a bi-vector field on the manifold. As can be seen from the Proposition 5.2.6, this constructed bi-vector field is actually a Poisson bi-vector field since its Schouten-Nijenhuis (SN) bracket with itself vanishes. On the other hand, the tangent and cotangent bundles of this manifold together form a Lie bialgebroid structure, since there exists a Lie algebroid structure on the cotangent bundle attached to each Poisson manifold, as shown in Subsection 3.1.4. Lie bialgebroids are the name given to this type of Lie bialgebroid construction. Additionally, it will be seen in the Subsection 3.3.5 that there is a one-to-one correspondence between Courant algebroids with transversal Dirac structures, which are transversal subbundles of a Courant algebroid that are maximally isotropic and integrable (involutive), and (triangular) Lie bialgebroids. So, this fact leads us to conclude that the structure above gives rise to a Courant algebroid structure on the sum of these Lie algebroids constructed. We will be interested in a special type of a Courant algebroid which is decomposed as $E = TM \oplus T^*M$. For a given Riemannian metric g and a two-form B on a manifold M, positive and negative definite subbundles which are subbundles of $TM \oplus T^*M$ of a Courant algebroid can be constructed. Then, the endomomorphism $\tau : TM \oplus T^*M \to TM \oplus T^*M$ with $\tau^2 = 1$ can be built. It will be seen that the pair (g, B) defines a unique metric which will be called a generalized metric, denoted by $\mathscr{H}$, on a Courant algebroid.

New supergravity solutions relating to (integrable) deformations of string sigma models can be produced using the duality covariant formulations of string theory. Our goal in the thesis is to study the homogeneous YB deformation proposed in the physics literature for a generic Green-Schwarz sigma model in terms of a geometric point of view. We will demonstrate that these kind of deformations are generated by a certain kind of non-constant $O(d,d)$ transformation which acts as a transformation on a Courant algebroid, called β transformation (3.46) that acts as solution generating transformations in string theory. We study the construction of such an $O(d,d)$

transformation from the bi-vector field related to the Poisson structure on the manifold. This bi-vector field on the group manifold is constructed from the dressed R-matrix and the Cartan-Killing form κ, as explained previously. As we will demonstrate in the Subsection 3.3.2, a bi-vector field β can be regarded as a map from T^*M to TM. Then, it can be extended naturally to an orthogonal map on the generalized tangent bundle $TM \oplus T^*M$. The orthogonal map on the generalized tangent bundle $TM \oplus T^*M$ indeed preserves the natural metric on the generalized tangent bundle which can be found in the Subsection 3.3.2. This constructed $O(d,d)$ matrix, which represents this particular type of β transformation that generates new solutions, will be referred as YB matrix and denoted by T_{YB} (6.65). Viewing YB deformation as an $O(d,d)$ transformation in the target space is quite useful. First of all, it makes the calculations considerably simpler. Another advantage is that it gives a natural embedding of YB deformation, that is the deformation of R-matrices which satisfy CYBE, in DFT.

Precisely, let us consider a Riemannian metric g and a B-field, which are the supergravity frame solutions. The generalized metric $\mathscr{H}^{MN}$ (6.20) on a Courant algebroid $TM \oplus T^*M$ constructed from these g and B will solve the DFT field equations (6.34). As we will discuss in Subsections 6.2.2, 6.3 and 6.4, the specific form of a generalized metric will be assumed for our purposes that is (6.35):

$$\mathscr{H}^{MN}(x,Y) = (U^{-1})^M{}_A(Y)\hat{\mathscr{H}}^{AB}(x)(U^{-1})^N{}_B(Y),$$

where the twist matrix $(U^{-1})^M{}_A = L^M{}_A$ as in (6.36). Then, $\hat{\mathscr{H}}$ satisfies the deformed field equations (6.37). Finding a different twist matrix with the same $\hat{\mathscr{H}}$ that satisfies the field equations (6.34) is our aim. The method we use involves calculating the fluxes F_{ABC} associated with the twist matrices $U^{-1} = L$ given in (6.36) and the new twist matrix $\widetilde{U}^{-1} = T_{YB}$ given in (6.65). The first matrix L contains only the components of the left invariant vector fields of the group manifold whereas the second matrix T_{YB} contains the components of the constructed bi-vector field, Poisson tensor, as explained above. The calculation of the fluxes of these matrices yields the same result that is a geometric flux which is the only non-vanishing flux. Geometric flux gives nothing but the structure constants of the Lie algebra of the isometry group. This result follows from the fact that the dressed R-matrix satisfies the CYBE and also from the properties of the components of the left invariant vector fields and the dressed R-matrix.

Furthermore, the expressions of these fluxes which will be given in the Subsection 6.2.4, are identical to the conditions on fluxes that are needed to establish a Courant algebroid for special anchors which can be found in the Subsection 4.4.3. From the Theorem 6.3.1, we will conclude that this special kind of a β transformation generates a new generalized metric $\widetilde{\mathscr{H}}$ which solves the DFT equations. Subsequently, as can be observed in the Subsection 6.3, the transformed B-field and the transformed metric can be obtained from the symmetric and anti-symmetric parts of the background matrix, which are once more the supergravity frame's solutions. As a result, the principle of flux preservation implies that the constructed non-constant $O(d,d)$ transformation is a solution generating transformation. Therefore, the characteristics of the R-matrix and the supergravity deformations are directly correlated. Since it makes it possible to discover more solutions by employing suitable twist matrices that yield the same fluxes in DFT, this method can be viewed of as a solution-generating instrument in supergravity.

Additionally, in the last part of the thesis, the previously studied algebroid structures, the generalized metric on the Courant algebroid $TM \oplus T^*M$, β transformations, R-matrices, flux calculations associated with the twist matrices and the flux preservation mechanism will be merged. In other words, we will demonstrate and interpret YB deformation in the language of Courant algebroids.

The outline of the thesis can be explained as follows.

First, Lie algebra cohomology will be briefly discussed in Section 2 in order to better understand one of the criteria of Lie bialgebras that is the compatibility condition provided by cocycles. The double of Lie bialgebras (Drinfel'd doubles), Manin triples, the definition and characteristics of R-matrices and CYBE will be provided later. Since the features of R-matrices will be examined in the Section 5 in detail, in this section these will be introduced briefly.

Section 3 will introduce Lie algebroids and (triangular) Lie bialgebroids. Poisson manifolds and calculus on Lie algebroids will be presented first. It will be demonstrated that there is a Lie algebroid structure on the cotangent bundle of the provided Poisson manifold. Additionally, a Courant algebroid structure which is a

generalization of the concept of a Drinfel'd double, often known as the double of a Lie bialgebroid, will be introduced together with its properties and symmetries. It will be seen that there is a one-to-one correspondence between Courant algebroids with transversal Dirac structures and (triangular) Lie bialgebroids and hereby Dirac structures will be also introduced. Last but not least, the generalized metric, which contains two-form as well as a Riemannian metric, on Courant algebroids will be defined for further studies that are in the Subsection 6.3.

Section 4 is devoted to find the local expressions of the notions that were introduced previously for our purposes.

Section 5 covers the properties of R-matrix as endomorphisms on a Lie algebra g with a semi-simple Lie group G in detail. The definition and the features of dressed R-matrix constructed from R-matrix will be introduced in order to work on the group manifold associated with the isometries of the background by using the adjoint action. We shall be able to construct a bi-vector on the Lie algebra g using the non-degenerate Cartan-Killing form on g, which allows us to identify g and the dual vector space $g*$. We can create a Lie algebra structure in the dual vector space with the aid of this bi-vector. It will be seen that the dressed R-matrix constructs a bi-vector field β on the manifold, which will be constructed in the Subsection 5.2, and that whenever the dressed R-matrix satisfies the CYBE, SN bracket of the bi-vector field with itself vanishes, resulting in a Poisson structure. Since there exists a natural Lie algebroid structure on the cotangent bundle to attached to each Poisson manifold, these tangent and cotangent bundles construct together a Lie bialgebroid. Furthermore, there exists a Courant algebroid structure since there is a one-to-one correspondence between (triangular) Lie bialgebroids and Courant algebroids with transversal Dirac structures. Some parts of this Section are based on the paper named "Yang–Baxter deformation as an $O(d,d)$ transformation" written by my advisor Prof. Dr. Aybike Çatal-Özer and Seçil Tunalı-Çırak [8].

In the Section 6, following a brief review of DFT, it will be examined why the development of the $O(d,d)$ matrix, which contains the components of the bi-vector field β, leads to the discovery of new supergravity solutions. In order to achieve this,

we introduce firstly DFT fields and action in the Subsection 6.2. Then the fluxes associated with the general twist matrix will be also examined in order to calculate the fluxes associated with the twist matrices L and T_{YB} in the next Subsection 6.2.4. Additionally, in this part of the thesis, it will be provided that the axioms of the fluxes in DFT when the strong constraint is imposed actually correspond to the conditions imposed by the axioms defining a Courant algebroid on the structure functions. This fact can be followed from the Propositions 6.2.3 and 6.2.4. Subsection 6.3 will provide the justifications for why the new generalized metric constructed with the transformation T_{YB} satisfies the field equations of supergravity. Additionally, in the last part of the thesis, we will demonstrate and interpret YB deformation in the language of Courant algebroids.

In the last part of the thesis, Appendix will be given that covers some of the proofs of Theorems, Propositions, Lemmas, and Examples in detail.

2. LIE BIALGEBRA, DRINFEL'D DOUBLE, *R*-MATRICES AND CYBE

We will require a few definitions from the theory of Lie algebra cohomology in order to define the definition and properties of a Lie bialgebra. We will start by giving the necessary definitions first. After that some related concepts such as a Drinfel'd double that is a double of a Lie bialgebra and a Manin triple will be also given for our purposes. Last but not least, the definition and the properties of the *R*-matrices will be given briefly.

2.1 Lie Algebra Cohomology

Let g represent a Lie algebra over the field of real numbers. We say that g acts on M when M is the vector space of a representation p of g [9], [10]. Frequently, we express $(p(X))(a)$ as $X \cdot a$ when $X \in g$ and $a \in M$.

Since we will be using this representation for our studies, the adjoint representation will now be introduced.

Example 2.1.1. *[9], [10] Any Lie algebra g acts on itself by the adjoint representation, which is ad* : $X \in g \mapsto ad_X \in End\ g$, *defined by* $ad_X(Y) = [X,Y]$ *for* $Y \in g$.

For the general case; the following formula shows how g acts on any tensor product of g with itself.

Definition 2.1.1. *[9], [10] For decomposable elements,* $Y_1 \otimes \cdots \otimes Y_p$ *in* $\otimes^p g = g \otimes \cdots \otimes g$ *(p times),*

$$\begin{aligned} X \cdot (Y_1 \otimes \cdots \otimes Y_p) &= ad_X^{(p)}(Y_1 \otimes \cdots \otimes Y_p) \\ &= ad_X Y_1 \otimes Y_2 \otimes \cdots \otimes Y_p + Y_1 \otimes ad_X Y_2 \otimes Y_3 \otimes \cdots \otimes Y_p + \cdots \\ &+ Y_1 \otimes Y_2 \otimes \cdots \otimes Y_{p-1} \otimes ad_X Y_p. \end{aligned}$$

Let us give now the result of the above formula for $p = 2$ since we will usually deal with this situation in our calculations.

$$ad_X^{(2)}(Y_1 \otimes Y_2) = ad_X Y_1 \otimes Y_2 + Y_1 \otimes ad_X Y_2 = [X, Y_1] \otimes Y_2 + Y_1 \otimes [X, Y_2].$$

Since 1 and 2-cocycles appear in the definition of a Lie bialgebra and these are related to objects as cochains and coboundaries, their definitions will be given now.

Definition 2.1.2. *[9], [10] For each non-negative integer k, the vector space of skew-symmetric k-linear mappings on g with values in M, where M is the vector space of a representation of g, is called the space of k-cochains on g with values in M.*

A 1-cochain on g with values in M is a linear map from g to M, and a 0-cochain on g with values in M is an element of M [9], [10].

Definition 2.1.3. *[9], [10] The coboundary of a k-cochain u on g with values in M is the $(k+1)$-cochain, δu, with values in M defined by*

$$\begin{array}{c} \delta u\,(X_0, X_1, \ldots, X_k) = \sum_{i=0}^{k} (-1)^i X_i \cdot \left(u\left(X_0, \ldots, \hat{X}_i, \ldots, X_k\right)\right) \\ + \sum_{i<j} (-1)^{i+j} u([X_i, X_j], X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_k), \end{array}$$

for $X_0, X_1, \ldots, X_k \in g$, where $\hat{X}_i$ indicates that the element X_i is omitted.

In the calculations, the cases $k = 0$ or $k = 1$ will arise so that from the above formula, we obtain the followings for these two cases.

Definition 2.1.4. *[9], [10] The coboundaries of a k-cochain u on g with values in M are*

$$\begin{array}{c} \delta u(X) = X \cdot u, \ \textit{for} \ k = 0, \ u \in M\ , X \in g, \textit{and} \\ \delta u(X, Y) = X \cdot u(Y) - Y \cdot u(X) - u([X, Y]) \ \textit{for} \ k = 1, \ u : g \to M, \ X, Y \in g. \end{array}$$

From this definition, we have $\delta(\delta u) = 0$ for any 0-cochain u on g with values in M [9], [10].

In fact, for $X, Y \in g$ [9], [10],

$$(\delta(\delta u))(X, Y) = X \cdot (Y \cdot u) - Y \cdot (X \cdot u) - [X, Y] \cdot u,$$

and this expression vanishes identically because $X \mapsto p(X)$ is a representation of g in M, i.e., $[X, Y] \cdot u = X \cdot (Y \cdot u) - Y \cdot (X \cdot u)$.

Proposition 2.1.1. *[9], [10] The property $\delta(\delta u) = 0$ is valid for any k-cochain $u, k \geq 0$.*

Definition 2.1.5. *[9], [10] A k-cochain u is called a k-cocycle if $\delta u = 0$. A k-cochain u $(k \geq 1)$ is called a k-coboundary if there exists a $(k-1)$-cochain, v , such that $u = \delta v$.*

Remark 2.1.1. *[9], [10] The 0-cocycles of g with values in M are the invariant elements in M, i.e., the elements $u \in M$ such that $X \cdot u = 0$, for each $X \in g$.*

Now that we have enough tools, we can give the description and characteristics of a Lie bialgebra.

2.2 Lie Bialgebra

Now, let g be a Lie algebra and γ be a linear map from g to $g \otimes g$. We will denote ${}^t\gamma : g^* \otimes g^* \to g^*$ as the transpose of this map. In comparison with the beginning of the Section 2.1, we employ the representation $ad^{(2)}$ here, and the representation space is $M = g \otimes g$.

Definition 2.2.1. *[9], [10] A Lie bialgebra is a Lie algebra g with a linear map $\gamma : g \to g \otimes g$ such that*
(i) ${}^t\gamma : g^* \otimes g^* \to g^*$ defines a Lie bracket on g^*, i.e., is a skew-symmetric bilinear map on g^* satisfying the Jacobi Identity, and
(ii) γ is a 1-cocycle on g with values in $g \otimes g$, where g acts on $g \otimes g$ by the adjoint representation $ad^{(2)}$.

Lemma 2.2.1. *Condition (ii) corresponds to the 2-cochain $\delta\gamma$ to be vanished, i.e., for $X, Y \in g$,*

$$(ii') \quad ad_X^{(2)}(\gamma(Y)) - ad_Y^{(2)}(\gamma(X)) - \gamma([X,Y]) = 0. \tag{2.1}$$

The above Lemma is a straight forward result of the Definitions 2.1.4 and 2.1.5.

Proposition 2.2.1. *[9], [10] If (g, γ) is a Lie bialgebra, and μ is the Lie bracket of g, then $(g^*, {}^t\mu)$ is a Lie bialgebra, where ${}^t\gamma$ is the Lie bracket of g^* and $\mu : g \otimes g \to g$ the skew-symmetric bilinear mapping on g defining the Lie bracket of g, i.e., $[X,Y] = \mu(X \otimes Y)$.*

By definition $(g^*, {}^t\mu)$ is called the dual of Lie bialgebra (g, γ) [9], [10]. As such, every Lie bialgebra has a dual Lie bialgebra, the dual of which is the Lie bialgebra.

2.3 The Double of a Lie Bialgebra and Manin Triples

Let us define the ad-invariance of a scalar product first since it will come up in the next definition.

Definition 2.3.1. *[9], [10] A scalar product $(\cdot \mid \cdot)$ on a Lie algebra g with bracket $[\cdot,\cdot]$, is called invariant if, for any $X,Y,Z \in g$,*

$$([X,Y] \mid Z) = (X \mid [Y,Z]).$$

Proposition 2.3.1. *[9], [10] Let (g,γ) be a Lie bialgebra with dual $(g^*, {}^t\mu)$. There exists a unique Lie algebra structure on the vector space $g \oplus g^*$ such that g and g^* are Lie subalgebras and that the natural scalar product on $g \oplus g^*$ is invariant, which is defined as*

$$(X \mid Y) = 0,\ \ (\xi \mid \eta) = 0,\ \ (X \mid \xi) = \langle \xi, X \rangle,\ \ \text{for } X,Y \in g,\ \ \xi,\eta \in g^*.$$

The bracket $[\cdot,\cdot]_{\mathfrak{d}}$ with the following formulas

$$\begin{cases} [X,Y]_{\mathfrak{d}} = [X,Y] \\ [X,\xi]_{\mathfrak{d}} = -ad^*_\xi X + ad^*_X \xi \\ [\xi,\eta]_{\mathfrak{d}} = [\xi,\eta]_{g^*}, \end{cases}$$

defines a Lie algebra structure on $g \oplus g^*$. In above, $ad^*_X = -{}^t(ad_X)$ that is $X \in g \mapsto ad^*_X \in End\ g^*$ is a representation of g in g^*. Also, $[\xi,\eta]_{g^*} = {}^t\gamma(\xi \otimes \eta) = ad_\xi \eta$, where $\langle ad_\xi \eta, X \rangle = -\langle \eta, ad^*_\xi X \rangle$. That is $\xi \in g^* \mapsto ad^*_\xi \in End\ g$ is the coadjoint representation of g^*.

Definition 2.3.2. *[9], [10] When g is a Lie bialgebra, $g \oplus g^*$ equipped with the Lie bracket $[\cdot,\cdot]_{\mathfrak{d}}$ defined by the triple above is called the double (Drinfel'd double) of g, and denoted by $g \bowtie g^*$ or $\mathfrak{d}$.*

[9], [10] Note that $\mathfrak{d} = g \bowtie g^*$ is also the double of g^*. In the Lie algebra $\mathfrak{d}$, the subspaces g and g^* are complementary Lie subalgebras, and both are isotropic, i.e., the scalar product vanishes on g and on g^*.

It will be seen that for any Lie bialgebra g, $(\mathfrak{d}, g, g^*)$ is an example of a Manin triple, defined as follows:

Definition 2.3.3. *[9], [10] A Manin triple is a triple* $(\vartheta, \mathfrak{a}, \mathfrak{b})$*, where* ϑ *is a Lie algebra with an invariant, non-degenerate, symmetric bilinear form, and* $\mathfrak{a}$ *and* $\mathfrak{b}$ *are complementary isotropic Lie subalgebras.*

[9], [10] In the finite-dimensional case, we can demonstrate that, in contrast to the previously provided definition; $\mathfrak{a}$ has a Lie bialgebra structure when $(\vartheta, \mathfrak{a}, \mathfrak{b})$ is a Manin triple. Since $\mathfrak{a}$ and $\mathfrak{b}$ play symmetric roles, $\mathfrak{b}$ also has a Lie bialgebra structure. Also, the Lie bialgebra $\mathfrak{b}$ can be identified with the dual of the Lie bialgebra $\mathfrak{a}$. Let $(\cdot \mid \cdot)$ be the given scalar product on ϑ. To $m \in \mathfrak{b}$ we associate the 1-form $i(m)$ on $\mathfrak{a}$ defined by $i(m)(n) = (n \mid m)$ where $n \in \mathfrak{a}$. The linear map $m \mapsto i(m)$ from $\mathfrak{b}$ to $\mathfrak{a}^*$ is injective. In fact, if $i(m) = 0$, then $(n \mid m) = 0$ for all $n \in \mathfrak{a}$, and then for all $n \in \vartheta$ since $\mathfrak{b}$ is isotropic and $\vartheta = \mathfrak{a} \oplus \mathfrak{b}$. We obtain $m = 0$ from the non-degeneracy of the scalar product. We see then that $\mathfrak{b}$ is isomorphic to $\mathfrak{a}^*$ by dimensions. Therefore, the Lie bracket on $\mathfrak{b}$ defines a Lie bracket on $\mathfrak{a}^*$. To see that it defines a Lie bialgebra structure on $\mathfrak{a}$, we use the Jacobi Identity in ϑ, and the invariance of the scalar product. Therefore, we have

Theorem 2.3.1. *[9], [10] There is a one-to-one correspondence between finite-dimensional Lie bialgebras and finite-dimensional Manin triples.*

2.4 *R*-matrices

In this Subsection, we will examine a linear map R from g to g, where g is a Lie algebra, which defines R-bracket that will be important in our studies. By the help of this new bracket, modified and classical YB equations will be introduced and the latter will be the fundamental basis for our work in the Chapter 5. The R-matrix, an endomorphism R that satisfies the modified YB equation, defines a second Lie algebra structure as this new R-bracket on g, as will be illustrated in that Chapter. In fact, we will be working with the semi-simple Lie groups G with Lie algebras g, and by using the non-degenerate Cartan-Killing form on g, we will be able to identify g and the dual vector space g^*. Then, a bi-vector will be constructed on the Lie algebra g, which will enable us to build a Lie algebra structure in the dual vector space and this fact will

yield a Drinfel'd double structure. More features of R-matrices will be examined in the Chapter 5.

Definition 2.4.1. *[9], [10] The skew-symmetric bilinear form $\langle R,R\rangle$ on g, where R is a linear map from g to g, with values in g defined by*

$$\langle R,R\rangle(X,Y) = [RX,RY] - R([RX,Y]+[X,RY]) + [X,Y].$$

for $X,Y \in g$, and, more generally, we define $\langle R,R\rangle_k$, by

$$\langle R,R\rangle_k(X,Y) = [RX,RY] - R([RX,Y]+[X,RY]) + k^2[X,Y], \tag{2.2}$$

where k is any scalar.

Definition 2.4.2. *[9], [10] Condition*

$$\langle R,R\rangle_k = 0$$

is called modified Yang-Baxter equation (MYBE) with coefficient k^2.

Definition 2.4.3. *[9], [10] An endomorphism R of g satisfying $\langle R,R\rangle_k = 0$ for some scalar k is called a classical R-matrix, or simply an R-matrix.*

Remark 2.4.1. *For our purposes, the R-matrix with $k = 0$ will be studied and the equation* (2.2) *for $k = 0$ becomes*

$$[RX,RY] - R([RX,Y]+[X,RY]) = 0, \tag{2.3}$$

which is called classical (or homogeneous) Yang-Baxter equation (CYBE), whenever R is the R-matrix.

Proposition 2.4.1. *[9], [10] The equation* (2.3) *is a sufficient condition for the bracket $[\cdot,\cdot]_R$ to satisfy the Jacobi Identity, where*

$$[X,Y]_R = [RX,Y] + [X,RY]. \tag{2.4}$$

Proof: In order to show this fact, Jacobiator of the bracket $[\cdot,\cdot]_R$ will be examined such that

$$\begin{aligned}
J &= [X,[Y,Z]_R]_R + [Y,[Z,X]_R]_R + [Z,[X,Y]_R]_R \\
&= [RX,[Y,Z]_R] + [X,R[Y,Z]_R] + [RY,[Z,X]_R] + [Y,R[Z,X]_R] \\
&+ [RZ,[X,Y]_R] + [Z,R[X,Y]_R] \\
&= [RX,[RY,Z]] + [RX,[Y,RZ]] + [X,R[Y,Z]_R] + [RY,[RZ,X]] \\
&+ [RY,[Z,RX]] + [Y,R[Z,X]_R] \\
&+ [RZ,[RX,Y]] + [RZ,[X,RY]] + [Z,R[X,Y]_R] \\
&= [RY,[RZ,X]] + [RZ,[X,RY]] + [X,R[Y,Z]_R] + c.p. \\
&= -[RZ,[X,RY]] - [X,[RY,RZ]] + [RZ,[X,RY]] + [X,R[Y,Z]_R] + c.p. \\
&= [X,R[Y,Z]_R - [RY,RZ]] + c.p.,
\end{aligned}$$

where we have used the Jacobi Identity of the Lie bracket in the fifth equality. So, if the condition (2.3) is satisfied, then the above equation (Jacobiator of the bracket $[,]_R$) indeed vanishes.■

[9], [10] Consequently, a second Lie algebra structure $[\cdot,\cdot]_R$ on g is defined by any R-matrix on g. A Lie algebra with a R-matrix is therefore referred to as a double Lie algebra. This definition should not be confused with the definition of a Lie bialgebra given in Subsection 2.3.

Since R-matrices will be examined in Chapter 5 in detail, this brief introduction of these has been given for now.

2.4.1 Classification of R-matrices

In the thesis, we will study skew-symmetric R-matrices for our purposes. These skew-symmetric solutions for CYBE on a finite-dimensional Lie algebra g correspond exactly to quasi-Frobenius subalgebras of g, which we shall define shortly. The R-matrix acts as a non-degenerate operator on such a subalgebra.

Let us write the R-matrix as

$$R = \frac{1}{2} r^{ij} b_i \wedge b_j, \tag{2.5}$$

where, b_i and b_j are the generators of the quasi-Frobenius subalgebra.

Definition 2.4.4. *[7] Unimodularity condition for the R-matrix is*

$$R^B{}_A f^A{}_{BC} = 0. \tag{2.6}$$

Definition 2.4.5. *[7] Abelian R-matrices are the ones for which* $[b_i, b_j] = 0$, $\forall i, j$ *while non-abelian ones have* $[b_i, b_j] \neq 0$ *for some* i, j.

Proposition 2.4.2. *[7] The unimodularity condition* (2.6) *takes the following form when R is an anti-symmetric matrix and satisfies CYBE.*

$$r^{ij}[b_i, b_j] = 0.$$

The first important fact, due to Stolin [11], [12], is that there is a one-to-one correspondence between constant solutions of the CYBE for a Lie algebra g and quasi-Frobenius (or symplectic) subalgebras $h \subset g$ (see also [13]).

Definition 2.4.6. *[7] A Lie algebra is quasi-Frobenius if it has a non-degenerate* 2*-cocycle* ω*, i.e.* $\omega(X,Y) = -\omega(Y,X)$, $\quad \omega([X,Y],Z) + \omega([Z,X],Y) + \omega([Y,Z],X) = 0$, $\quad \forall X, Y, Z \in h$*. It is Frobenius if* ω *is a coboundary, i.e.* $\omega(X,Y) = f([X,Y])$ *for some linear function* f.

For the definition of a 2-cocyle, see the Definition 2.1.5.

If R is a solution to the CYBE for g, then there is a subalgebra h on which R is non-degenerate. This fact will be important in Subsection 5.2 for constructing a bi-vector field from the R-matrix. This subalgebra is necessarily quasi-Frobenius, and writing R in the form (2.5) the 2-cocycle is the inverse of the R-matrix, i.e. $\omega(b_i, b_j) = (r^{-1})_{ij}$. The converse is also true, i.e. if $h \subset g$ is quasi-Frobenius then the inverse of the 2-cocycle ω gives a solution to the CYBE. Therefore, finding solutions to the CYBE for a given g reduces to finding all quasi-Frobenius subalgebras of g.

3. LIE ALGEBROID, LIE BIALGEBROID AND COURANT ALGEBROID

In Chapter 5, dressed R-matrix, R_g, will be constructed from the R-matrix, and it will be demonstrated that the components of the dressed R-matrix produce a bi-vector field β whose SN bracket vanishes with itself whenever R_g satisfies the CYBE given in (5.9). Therefore, β defines a Poisson structure on the group manifold. Additionally, it will be demonstrated how this fact results in a Poisson structure, a natural Lie algebroid structure on the cotangent bundle of a Poisson manifold, and how they combine to form a Lie bialgebroid. Since these ideas come up in our research, our aim in this Chapter is to introduce these notions and investigate their features. In addition, a Courant algebroid, which has been constructed for finding the double of a Lie bialgebroid, will be investigated since it is a generalization of the concept of a Drinfel'd double and is one of the key structures in the thesis.

3.1 Lie Algebroid and Poisson Manifold

3.1.1 Lie algebroid

A Lie algebroid was first defined by J. Pradines in 1967 in [14] and explored in depth by K. Mackenzie in [15]. This can be thought of as a structure that is a generalized version of a Lie algebra, such that it carries the sections of a vector bundle to the tangent bundle with the help of the smooth bundle map called anchor that induces a Lie algebra homomorphism and satisfies a Leibniz rule given below.

Definition 3.1.1. *[16] A Lie algebroid is a vector bundle A on a smooth manifold M, equipped with a Lie bracket $[\cdot,\cdot]$ on $\Gamma(A)$ (smooth sections of A) and a smooth bundle map $\rho : A \to TM$, called the anchor. The anchor must induce a Lie algebra homomorphism $\rho : \Gamma(A) \to \Gamma(TM)$ (smooth sections of the tangent bundle TM), i.e.,*

$$1)\ \rho([X,Y]) = [\rho(X), \rho(Y)], \tag{3.1}$$

and the following Leibniz Rule must be satisfied

$$2)\ [X, fY] = f[X,Y] + (\rho(X)f)Y, \tag{3.2}$$

$\forall X, Y \in \Gamma(A)$ *and* $\forall f \in C^\infty(M)$ *denotes the smooth functions on the manifold* M.

Corollary 3.1.1. *First condition above is a fundamental property of Lie algebroids, and is often given as a condition as above in many sources. However, it is a consequence of the second condition together with the Jacobi Identitiy, as it has been noted by J.C. Herz, Y. Kosmann-Schwarzbach, F. Magri and J. Grabowski among others (see [17], [18], [19], [20]). (Proof can be found in Appendix in Proof A.1.)*

There is another structure called a pre-Lie algebroid which is a vector bundle with anti-symmetric and bilinear bracket instead of the Lie bracket equipped with the above conditions. Therefore, this structure eliminates the Jacobi condition that is the sections of a vector bundle do not have to satisfy the Jacobi Identity.

3.1.2 Calculus on Lie algebroids

The definitions and some properties on Lie algebroids will be provided now in order to achieve calculations on these structures. These can be thought as a generalization of the notions such as the exterior derivative, the Lie derivative, the interior product, the SN bracket and some relations between them to arbitrary Lie algebroids of the standard calculus of differential forms and multi-vector fields.

Let us consider a Lie algebroid A on base M with anchor $\rho : A \to TM$. The notation $\Gamma(\wedge^k A)$ will stand for the smooth sections of kth exterior power bundle of a Lie algebroid A whereas $\Gamma(\wedge^k A^*)$ is for for the smooth sections of kth exterior power bundle of a Lie algebroid A^*, where $k \geq 0$. For $k \geq 0$, we identify, the k-th exterior power bundle on M, $\wedge^k(A^*)$ with $\mathrm{Alt}^k(A, M \times \mathbb{R})$ where $\mathrm{Alt}^k(A, M \times \mathbb{R})$ denotes the set of k-multi-linear alternating forms on A. For more details of these notions, one may follow [15].

Definition 3.1.2. *[21] The exterior derivative, or Lie algebroid co-boundary, $d : \Gamma(\wedge^k A^*) \to \Gamma(\wedge^{k+1} A^*)$ is defined by*

$$d\phi(X_1,\ldots,X_{k+1}) = \sum_{i=1}^{k+1}(-1)^{i+1}\rho(X_i)(\phi(X_1,\ldots,\widehat{X_i},\ldots,X_{k+1})) + \sum_{i<j}(-1)^{i+j}\phi([X_i,X_j],X_1,\ldots,\widehat{X_i},\ldots,\widehat{X_j},\ldots,X_{k+1}),$$

for $\phi \in \Gamma(\wedge^k A^*), X_i \in \Gamma(A), 1 \leqslant i \leqslant k+1$ and $\widehat{X_i}, \widehat{X_j}$ are for the omitted terms. Here, the bracket $[X_i,X_j]$ is the Lie algebroid bracket on the sections of A.

The exterior derivative for $k = 0$ and $k = 1$ on Lie algebroids can be found below.

Lemma 3.1.1. *For $k = 0$ in the above definition, that is for the exterior derivative of a 0-form, which is a function $f \in C^\infty(M)$ on a Lie algebroid, we have*

$$df(X) = \rho(X)(f). \tag{3.3}$$

Lemma 3.1.2. *For $k = 1$ in the above definition, that is for the exterior derivative of a 1-form $\phi \in \Gamma(A^*)$, we obtain*

$$d\phi(X_1,X_2) = \rho(X_1)\phi(X_2) - \rho(X_2)\phi(X_1) - \phi[X_1,X_2]. \tag{3.4}$$

Definition 3.1.3. *[21] For $X \in \Gamma(A)$ and $k \geqslant 0$, the Lie derivative $L_X : \Gamma(\wedge^k A^*) \to \Gamma(\wedge^k A^*)$ is defined by*

$$\mathscr{L}_X(\phi)(Y_1,\ldots,Y_k) = \rho(X)(\phi(Y_1,\ldots,Y_k)) - \sum_{i=1}^{k}\phi(Y_1,\ldots,[X,Y_i],\ldots,Y_k), \tag{3.5}$$

for $\phi \in \Gamma(\wedge^k A^*)$ and $Y_1,\ldots,Y_k \in \Gamma(A)$.

Definition 3.1.4. *[21] The contraction, or interior multiplication, $i_X : \Gamma(\wedge^{k+1} A^*) \to \Gamma(\wedge^k A^*)$ is defined by*

$$i_X(\phi)(Y_1,\ldots,Y_k) = \phi(X,Y_1,\ldots,Y_k),$$

for $\phi \in \Gamma(\wedge^{k+1} A^*)$ and $Y_1,\ldots,Y_k \in \Gamma(A)$.

These operators satisfy the following analogues of the standard calculus of differential forms [21];

$$d(\phi \wedge \psi) = d\phi \wedge \psi + (-1)^k \phi \wedge d\psi, \tag{3.6}$$

$$d^2 = 0, \tag{3.7}$$

$$\mathscr{L}_{[X,Y]_{SN}} = \mathscr{L}_X \circ \mathscr{L}_Y - \mathscr{L}_Y \circ \mathscr{L}_X, \tag{3.8}$$

$$i_{[X,Y]_{SN}} = \mathscr{L}_X \circ i_Y - i_Y \circ \mathscr{L}_X, \tag{3.9}$$

$$\mathscr{L}_X = d \circ i_X + i_X \circ d, \tag{3.10}$$

$$\mathscr{L}_X(f\phi) = f\mathscr{L}_X(\phi) + \rho(X)(f)\phi, \tag{3.11}$$

$$\mathscr{L}_{fX}(\phi) = f\mathscr{L}_X(\phi) + df \wedge i_X(\phi), \tag{3.12}$$

where $X, Y \in \Gamma(A)$, $f \in C^\infty(M)$, $\phi \in \Gamma(\wedge^k A^*)$, $\psi \in \Gamma(\wedge^m A^*)$. Note that in (3.12), $df \in \Gamma(\wedge^1 A^*)$ refers to the Lie algebroid coboundary.

The Definition of the (generalized) SN Bracket and its Properties:

The bracket, which is called SN bracket, of the multi-vector fields on a smooth manifold will be presented now. Because of its structure, this bracket is an extension of a Lie bracket of vector fields. SN bracket can be also extended for the multi-sections of a Lie algebroid which will be used later in Section 3 and its definition will be given next.

Let $\Gamma(TM) = \bigoplus_{j=0}^{n} \wedge^j(M), (\wedge^0 = C^\infty(M), n = \dim M)$, be the contravariant exterior algebra of skew-symmetric contravariant tensor fields (multi-vectors) over M. The Lie bracket of vector fields on M may be uniquely extended to an $\mathbb{R}$-bilinear bracket on $\Gamma(TM)$, the SN bracket, in such way that $\Gamma(TM)$ becomes a graded superalgebra. The SN bracket [22], [23] is a bilinear mapping $\wedge^p(M) \times \wedge^q(M) \to \wedge^{p+q-1}(M)$.

Definition 3.1.5. *[24] Let $X_1, \ldots, X_p, Y_1, \ldots, Y_q$ be vector fields over M. Then*

$$[X_1 \wedge \ldots \wedge X_p, Y_1 \wedge \ldots \wedge Y_q]_{SN} = \sum (-1)^{i+j} X_1 \wedge \ldots \widehat{X}_j \ldots \wedge X_p \wedge [X_j, Y_i] \wedge Y_1 \wedge \ldots \widehat{Y}_i \ldots \wedge Y_q, \tag{3.13}$$

where $[\cdot,\cdot]_{SN}$ is the SN bracket and $\widehat{X}$ stands for the omission of X.

The above definition can be generalized to the context of Lie algebroids as follows which has actually the same form of (3.13).

Definition 3.1.6. *[16] (Generalized) SN bracket,*

$$[\cdot,\cdot]_{SN} : \Gamma(\wedge^k A) \times \Gamma(\wedge^m A) \to \Gamma(\wedge^{k+m-1} A),$$

on the sections $X_1 \wedge \ldots \wedge X_p \in \Gamma(\wedge^p A)$, $Y_1 \wedge \ldots \wedge Y_q \in \Gamma(\wedge^q A)$ *of a Lie algebroid A is defined as follows*

$$[X_1 \wedge \ldots \wedge X_p, Y_1 \wedge \ldots \wedge Y_q]_{SN} = \Sigma_{i,j}(-1)^{i+j}[X_i, Y_j] \wedge X_1 \wedge \ldots \wedge \widehat{X}_i \wedge \ldots \wedge X_p \wedge Y_1 \wedge \ldots \wedge \widehat{Y}_j \wedge \ldots \wedge Y_q. \tag{3.14}$$

Some of the features of the SN bracket will be stated now since these arise in our studies.

Corollary 3.1.2. *For* $X \in \Gamma(A)$ *and* $f \in C^\infty(M)$*, the SN bracket reduces to the followings [15]:*

$$\mathscr{L}_X(f) = [X, f] = -[f, X] = \rho(X)f. \tag{3.15}$$

Also, we have [25]

$$\mathscr{L}_X(Y) = [X, Y]_{SN}, \tag{3.16}$$

where $X \in \Gamma(A)$ *and* $Y \in \Gamma(\wedge^q A)$*. This property is also true when* $X \in \Gamma(\wedge^p A)$*.*

Corollary 3.1.3. *The SN bracket satisfies [25]*

$$[fX, gY]_{SN} = fg[X, Y]_{SN} + gX \wedge [f, Y]_{SN} + (-1)^{degree\ of\ X} f[g, X]_{SN} \wedge Y, \tag{3.17}$$

for all $X \in \Gamma(\wedge^p A)$ *and* $Y \in \Gamma(\wedge^q A)$*. Here,* $[f, Y]_{SN} = -\rho(Y)f$ *and* $[g, X]_{SN} = -\rho(X)g$ *from* (3.15)*. (Proof can be found in Appendix in Proof A.2.)*

Corollary 3.1.4. *The SN bracket yields for* $Y \in \Gamma(\wedge^q A)$ *[26]*

$$[f, Y_1 \wedge \cdots \wedge Y_n]_{SN} = \sum_{i=1}^{n} (-1)^i Y_i(f) Y_1 \wedge \cdots \wedge \hat{Y}_i \wedge \cdots \wedge Y_n. \tag{3.18}$$

For $Y \in \Gamma(\wedge^2 A)$, the above equation yields the following.

$$[f, Y_1 \wedge Y_2]_{\mathrm{SN}} = -Y_1(f)Y_2 + Y_2(f)Y_1,$$

or by using (3.15)

$$[f, Y_1 \wedge Y_2]_{\mathrm{SN}} = -\rho(Y_1)(f)Y_2 + \rho(Y_2)(f)Y_1. \tag{3.19}$$

Examples of Lie Algebroids

Examples 3.1.1. 1) *Every Lie algebra is a Lie algebroid over the one point manifold such that the anchor is* $\rho = 0$ *and* $A = g$ *is a vector space with a bracket that has to fulfill Jacobi Identity.*

2) *With the identity map acting as its anchor, the tangent bundle is a Lie algebroid in and of itself. A Lie algebra can be thought of as a generalization of the tangent bundle.*

3) *Every bundle of Lie algebras over a smooth manifold defines a Lie algebroid where the Lie bracket is defined pointwise and the anchor map is equal to zero.*

4) *Given a Poisson manifold* (M, Π)*, which will be defined in the next Subsection, there exists a natural Lie algebroid structure on its cotangent bundle together with an anchor induced by the Poisson bi-vector and the induced Lie bracket on the sections of this cotangent bundle.*

Remark 3.1.1. *The last example above will be examined in the Subsection 3.1.4 in detail since this result is important for our studies. Also, the Poisson manifolds not only set an example for Lie algebroids but also form a Lie bialgebroid structure on* $T^*M \oplus TM$ *(see the Proposition 3.1.4) whose definition will be given in the Subsection 3.2. Last but not least, we will see that the SN bracket of a Poisson bi-vector* Π *with itself vanishes and this fact will be also important for us in the future.*

Considering all these features, Poisson manifolds have an importance in our studies so that the next Subsection will be devoted to this notion and to some properties of this structure. We will start with the definition of a Poisson structure.

3.1.3 Poisson structure

Let M be a smooth manifold of finite dimension m. We denote by $C^\infty(M)$ the algebra of smooth real-valued functions on M. The trivial Lie algebroid structure $A = TM$ will be used in this case.

Definition 3.1.7. *[27] A Poisson structure on M is an $\mathbb{R}$-bilinear map*

$$\{\cdot,\cdot\}_M : C^\infty(M) \times C^\infty(M) \to C^\infty(M),$$

called the Poisson bracket (we will drop the M below the bracket for short), which satisfies the following conditions:

$$1)\,\{f,g\} = -\{g,f\},$$

$$2)\quad \{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\} = 0,$$

$$3)\,\{fg,h\} = f\{g,h\} + \{f,h\}g,$$

for all $f,g,h \in C^\infty(M)$. The pair $(M,\{\cdot,\cdot\})$ is called a Poisson manifold.

$\mathbb{R}$-bilinearity, the skew-symmetry property (1) and the Jacobi Identity (2) together mean that $\{,\}$ is a Lie bracket on $C^\infty(M)$, making the space of smooth functions into a Lie algebra. Property (3), the Leibniz identity, means that, for all $f \in C^\infty(M)$, the operator $\{f,\cdot\}$ is a derivation of the algebra of smooth functions in $C^\infty(M)$.

Example 3.1.1. *One can define a trivial Poisson structure on any manifold by setting all functions f and g in C^∞ to $\{f,g\} = 0$.*

Example 3.1.2. *On $\mathbb{R}^{2n}$ with linear coordinates $\left(q^1,\ldots,q^n,p_1,\ldots,p_n\right)$ the formula:*

$$\{f,g\} := \sum_{i=1}^{n}\left(\frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i} - \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i}\right).$$

defines a Poisson bracket, called the canonical Poisson bracket on $\mathbb{R}^{2n}$.

Definition 3.1.8. *[28] A 2-vector field (bi-vector field)* Π*, such that the bracket* $\{f,g\} := \Pi(f,g) = \langle \Pi, df \wedge dg \rangle$ *is a Poisson bracket, is called a Poisson tensor, or also a Poisson structure. The corresponding Poisson bracket is often denoted by* $[\cdot,\cdot]_\pi$*. If the Poisson tensor* Π *is a* C^k*-smooth 2-vector field, then we say that we have a* C^k*-smooth Poisson structure, and so on.*

Proposition 3.1.1. *In a local system of coordinates* $(x_1,\ldots,x_n)$ *we have*

$$\Pi = \sum_{i<j} \Pi_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j} = \frac{1}{2} \sum_{i,j} \Pi_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j},$$

where $\Pi_{ij} = \langle \Pi, dx_i \wedge dx_j \rangle = \{x_i, x_j\}$*, and*

$$\{f,g\} = \left\langle \sum_{i<j} \{x_i,x_j\} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}, \sum_{i,j} \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j} dx_i \wedge dx_j \right\rangle = \sum_{i,j} \Pi_{ij} \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j}.$$

Lemma 3.1.3. *[28] For any* C^1*-smooth* 2*-vector field* Π*, one can associate to it a 3-vector field* Λ *defined by*

$$\Lambda(f,g,h) = \{\{f,g\},h\} + \{\{g,h\},f\} + \{\{h,f\},g\},$$

where $\{k,l\}$ *denotes* $\langle \Pi, dk \wedge dl \rangle$ *(i.e., the bracket of* Π *).*

Proposition 3.1.2. *Direct calculations in local coordinates show that*

$$\Lambda(f,g,h) = \sum_{ijk} \left(\sum_s \frac{\partial \Pi_{ij}}{\partial x_s} \Pi_{sk} + c.p. \right) \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j} \frac{\partial h}{\partial x_k},$$

where c.p. stands for the cyclic permutations of Π_{ij} *depending on* i,j,k*. (Proof can be found in* (A.1)*.)*

Alternatively,

$$\Lambda = \sum_{i<j<k} \left(\sum_{\mathrm{s}} \frac{\partial \Pi_{ij}}{\partial x_{\mathrm{s}}} \Pi_{sk} + c.p. \right) \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j} \wedge \frac{\partial}{\partial x_k}.$$

It is evident that the requirement that $\Lambda = 0$ is equivalent to the Jacobi Identity for Π. So, we obtain:

Proposition 3.1.3. *[28] A 2-vector field* $\Pi = \sum_{i<j} \Pi_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}$ *expressed in terms of a given system of coordinates* $(x_1,\ldots,x_n)$ *is a Poisson tensor if and only if it satisfies the following system of equations:*

$$\boxed{\frac{\partial \Pi_{ij}}{\partial x_s} \Pi_{sk} + c.p. = 0} \quad (\forall i,j,k). \tag{3.20}$$

(Proof can be found in Appendix in Proof A.3.)

The preceding proposition has the noticeable conclusion that the components of the bi-vector field must fulfill a requirement for a 2-vector field to be a Poisson structure.

The important result will be given now that the Jacobi Identity (2) for the Poisson bracket given in 3.1.7, which corresponds to (3.20) as shown above, is equivalent to the vanishing of the SN bracket of the bi-vector field with itself that is $[\Pi,\Pi]_{\mathrm{SN}} = 0$.

Lemma 3.1.4. *Let β be a bi-vector field on a manifold M. Then the SN bracket of β with itself is given by [29]*

$$[\beta,\beta]_{SN} = [\beta^{il}(\partial_l\beta^{jk}) + \beta^{jl}(\partial_l\beta^{ki}) + 2\beta^{kl}(\partial_l\beta^{ij})]\partial_i \wedge \partial_j \wedge \partial_k,$$

by writing $\beta = \beta^{ij}\partial_i \wedge \partial_j$. Here, the SN bracket is as given in (3.13).

The equivalent condition for $[\beta,\beta]_{SN} = 0$ is as follows.

$$\boxed{\beta^{il}(\partial_l\beta^{jk}) + \beta^{jl}(\partial_l\beta^{ki}) + \beta^{kl}(\partial_l\beta^{ij}) = 0}. \tag{3.21}$$

Then β defines a Poisson structure on M with the following identity since it has the same form with (3.20). *(Proof can be found in Appendix in Proof A.4.)*

For a Poisson tensor, we will use Π in further studies.

Example 3.1.3. *[28] Let V be a finite-dimensional vector space over $\mathbb{R}$. When two linear functions have a Poisson bracket that is also a linear function, we have a linear Poisson structure on V. The components of the corresponding Poisson tensor are equivalently linear functions in local coordinates. The operation $(f,g) \mapsto \{f,g\}$ in this instance, restricted to linear functions, results in an operation $[\cdot,\cdot] : V^* \times V^* \longrightarrow V^*$, a Lie algebra structure on V^*, where V^* is the dual linear space of V.*

On the other hand, a linear Poisson structure on V is determined by any Lie algebra structure on V^*. Consider a finite-dimensional Lie algebra, denoted as $(g,[\cdot,\cdot])$. The element of g corresponding to each linear function $f : g^* \longrightarrow \mathbb{R}$ is denoted by $\tilde{f}$. For

each α in g^*, we put $\{f,g\}(\alpha) = (\alpha, [\tilde{f}, \tilde{g}])$ if f and g are two linear functions on g^*. $\{x_i, x_j\} = \sum {c_{ij}}^k x_k$ is obtained if we select a set of basis vectors as $e_1, \ldots, e_n$ of g, with $[e_i, e_j] = \sum {c_{ij}}^k e_k$. Here, x_l is the function that satisfies $\tilde{x}_l = e_l$. From the Jacobi Identity for $[\cdot,\cdot]$, we can take $(x_1, \ldots, x_n)$ as a linear system of coordinates on g^*. This means that the functions $\Pi_{ij} = \{x_i, x_j\}$ verify the Equation (3.20).

A Poisson structure on g^* is thus obtained. The following formula can be used to define this Poisson structure intrinsically:

$$\{f,g\}(\alpha) = \langle \alpha, [df(\alpha), dg(\alpha)] \rangle,$$

where $df(\alpha)$ and $dg(\alpha)$ are thought as elements of g from the identification $(g^*)^* = g$. Finite-dimensional linear Poisson structures and finite-dimensional Lie algebras are therefore naturally bijective. Even Lie algebras can be studied by considering them to be linear Poisson structures (see, e.g., [30]).

As we have promised in the Remark 3.1.1, the following Subsection is devoted to the fact that there exists a natural Lie algebroid structure on the cotangent bundle of a given Poisson manifold. This structure will be used in the Proposition 3.1.4 in order to show that there is a Lie bialgebroid structure on the direct sum of the tangent and cotangent bundles with the induced Poisson structure and also in the Lemma 5.2.1 where we will have a bi-vector field constructed by using an R-matrix that satisfies the SN bracket to be vanished and becomes a Poisson tensor.

3.1.4 Lie algebroid structure on the cotangent bundle of a Poisson manifold

Example 3.1.4. *([28], Example* 8.1.12*) As an important example (which was discovered by Dazord and Sondaz [31]) for a Lie algebroid, it can be given that attached to each Poisson manifold (M, Π), there is a natural Lie algebroid structure $\left(T^*M, [\cdot,\cdot]_*, \Lambda^\sharp\right)$, on the cotangent bundle of M, where $\Lambda^\sharp : T^*M \to TM$ is the anchor map of Π, $\langle \Lambda^\sharp \alpha, \beta \rangle = \Pi(\alpha, \beta), \forall \alpha, \beta \in \Gamma(A^*)$, and whose Lie bracket is defined by the*

Koszul bracket:

$$[\alpha,\beta]_* = \mathscr{L}_{\Lambda^\sharp\alpha}\beta - \mathscr{L}_{\Lambda^\sharp\beta}\alpha - d(\Pi(\alpha,\beta)) \tag{3.22}$$
$$= i_{\Lambda^\sharp\alpha}d\beta - i_{\Lambda^\sharp\beta}d\alpha + d(\Pi(\alpha,\beta)).$$

$\left(T^*M,[\cdot,\cdot]_*,\Lambda^\sharp\right)$*, is called the cotangent algebroid of* (M,Π).
(Proof can be found in Appendix in Proof A.5.)

This bracket on 1-forms of a Poisson manifold was first discovered by Magri, Franco and Morosi [32].

Proposition 3.1.4. *[21], [33] A Poisson manifold* (M,Π) *gives naturally rise to a Lie bialgebroid, whose definition will be given in the next Subsection, on* TM *with the commutator bracket of tangent vector fields and* T^*M *with the Lie bracket induced by the Poisson structure. (Proof can be found in Appendix in Proof A.6.)*

It has been seen that for a given Poisson manifold $(M,\{\cdot,\cdot\})$, which is determined by a Poisson bi-vector Π, the cotangent bundle of M carries a natural Lie algebroid structure. That is on differential 1-forms, which are sections of T^*M, the bracket (3.22) can be defined. On the other hand, for a given Lie algebroid $\left(T^*M,[\cdot,\cdot]_*,\Lambda^\sharp\right)$ on the cotangent bundle T^*M, there exists a Poisson bracket $\{\cdot,\cdot\}$ on M such that $\Pi=\Lambda^\sharp$: $T^*M\to TM$ if and only if some conditions are satisfied. These are the followings: $\Lambda^\sharp$ must be skew-symmetric and if $\alpha,\beta\in\Gamma(A^*)$ are closed then $[\alpha,\beta]_*$ must be closed. These facts can be followed from [34] in detail.

3.2 Lie Bialgebroid

Lie bialgebroids were introduced and studied by Kirill C. H. Mackenzie and Ping Xu in [21] in 1994. These structures are the straightforward extension of the concept of Lie bialgebras. While two compatible Lie algebras (g,g^*) (where g^* is a dual vector space of g) constructs a Lie bialgebra, Lie bialgebroids are constructed by two Lie algebroids (A,A^*) (where A^* is the dual vector bundle of A) which satisfy again a compatibility condition given below.

Remark 3.2.1. *Just as the Lie algebroid (bracket) structure on A induces an exterior derivative (d) on* A^* *given in Subsection 3.1.2, the bracket introduced below (*$[\cdot,\cdot]_*$*)*

on A^* induces also an exterior derivative on A and it will be denoted by call it d_*. The compatibility condition will be the condition that d_* must be the derivation of the bracket on the sections of A which is given below in (3.24) *and this condition will be used in our calculations later.*

Definition 3.2.1. *Let A be a Lie algebroid and suppose its dual bundle A^* also has the structure of a Lie algebroid. Then (A,A^*) is a Lie bialgebroid if the Lie algebroid derivative $d:\Gamma(A^*)\to\Gamma(\wedge^2 A^*)$ is a derivation of the SN bracket on $\Gamma(A^*)$, in the sense that [16]*

$$d[\xi,\eta]_* = [d\xi,\eta]_* + [\xi,d\eta]_*, \tag{3.23}$$

or d_* must be the derivation of the bracket on the sections of A, that is [21]

$$\begin{aligned} d_*[X,Y]_{SN} &= [X,d_*Y]_{SN} + [d_*X,Y]_{SN} \\ &= \mathscr{L}_X d_*Y - \mathscr{L}_Y d_*X, \end{aligned} \tag{3.24}$$

where $d_*:\Gamma(A)\to\Gamma(\wedge^2 A)$ is the induced differential operator.

3.2.1 Exact and triangular Lie bialgebroids

Any given Lie algebroid A can be associated with a dual vector bundle A^*. We will be interested in whether this dual vector bundle is also equipped with a Lie algebroid structure. As will be demonstrated, if there exists a bundle map $\Lambda^\sharp : A^* \to A$ with $\Lambda \in \Gamma(\wedge^2 A)$, then the condition (3.26) to be satisfied to have a Lie algebroid structure on the dual vector bundle A^*. Then, Lie algebroids A and A^* together construct a Lie bialgebroid (A,A^*) since the compatibility condition between A and A^*, given in (3.24), automatically holds. This Lie bialgebroid is called an exact Lie bialgebroid, but we shall work on a special case of this condition and it defines a triangular Lie bialgebroid.

All these notions have been studied in [35] and they will be verified in detail now since these works will help in the future calculations.

Remark 3.2.2. *In our calculations below, we will see how the Lie bialgebroid structure is constructed under a condition using the bundle map $\Lambda^\sharp$. Lie bialgebroids constructed*

in this way are called exact or triangular Lie bialgebroids whose definitions can be seen in the Definition 3.2.3. We will be interested in the triangular Lie bialgebroids, as mentioned above, later in Subsection 4.3.

Let A be a Lie algebroid with anchor ρ, and Λ a section of $\Gamma(\wedge^2 A)$ (sometimes Λ will be called a bi-section). Let us denote the bundle map $\Lambda^\sharp : A^* \to A$ defined by $\Lambda^\sharp(\xi)(\eta) = \Lambda(\xi,\eta)$ for all $\xi,\eta \in \Gamma(A^*)$. By the help of this map, let us introduce a new bracket on $\Gamma(A^*)$ by [35]

$$\boxed{[\xi,\eta]_* = \mathscr{L}_{\Lambda^\sharp\xi}\eta - \mathscr{L}_{\Lambda^\sharp\eta}\xi - d[\Lambda(\xi,\eta)]}, \tag{3.25}$$

where $[\cdot,\cdot]_* : \Gamma(\wedge^2 A^*) \to \Gamma(A^*)$.

Remark 3.2.3. *Note that the construction of a Lie bialgebroid will be done by using the bundle map $\Lambda^\sharp : A^* \to A$ defined by $\Lambda^\sharp(\xi)(\eta) = \Lambda(\xi,\eta)$ for all $\xi,\eta \in \Gamma(A^*)$ and introducing a new bracket on the smooth sections of the dual vector bundle A^*, given in* (3.25) *together with the induced differential operator $d_* : \Gamma(A) \to \Gamma(\wedge^2 A)$ which will be given by $d_* X = [\Lambda, X]_{SN}$ (see the Proposition 3.2.1). This construction can be thought as a generalization of the idea given in Example 3.1.4. In that example, we had an anchor instead of $\Lambda^\sharp$ such that $\Lambda^\sharp : T^*M \to TM$, and we had a Poisson bi-vector Π instead of Λ, and new bracket on the smooth sections of the cotangent bundle T^*M given in* (3.22) *which has the same form as above.*

Definition 3.2.2. *Let ρ_* be the composition $\rho \circ \Lambda^\sharp : A^* \to TM$, where TM is the tangent bundle on the manifold M.*

Theorem 3.2.1. *([35], Theorem* 2.1*) A^* with the bracket and anchor ρ_* defined above becomes a Lie algebroid if and only if*

$$\mathscr{L}_X[\Lambda,\Lambda]_{SN} = 0, \ \textit{for all} \ X \in \Gamma(A), \tag{3.26}$$

where $[\Lambda,\Lambda]^\sharp$ is the map $A^* \wedge A^* \to A$ defined by

$$\langle [\Lambda,\Lambda]^\sharp(\xi_1,\xi_2),\xi_3\rangle = [\Lambda,\Lambda]_{SN}(\xi_1,\xi_2,\xi_3), \tag{3.27}$$

and $[\Lambda,\Lambda]_{SN} \in \Gamma(\wedge^3 A)$ *is the SN bracket of* Λ *with itself. (Proof of this Theorem will be given after the Remark and the following Lemmas.)*

Remark 3.2.4. *In order to have a Lie algebroid structure on the dual vector bundle, one more condition has been put forward in the first place in the above Theorem in [35] such that*

$$\rho \circ [\Lambda,\Lambda]^\sharp = 0. \tag{3.28}$$

However, they discovered in [36] later that the condition (3.26) *implies the above condition.*

In order to prove the Theorem 3.2.1, the following Lemmas will be used so that these will be given now. Proofs of each can be found in Appendix.

Lemma 3.2.1. *([35], Lemma* 2.2*) For any* ξ_1 *and* $\xi_2 \in \Gamma(A^*)$*,*

$$\Lambda^\sharp[\xi_1,\xi_2] - [\Lambda^\sharp\xi_1, \Lambda^\sharp\xi_2] = \frac{1}{2}[\Lambda,\Lambda]^\sharp(\xi_1,\xi_2). \tag{3.29}$$

(Proof can be found in Appendix in Proof A.7.)

Lemma 3.2.2. *([35], Lemma* 2.3*) For any* $X \in \Gamma(A)$ *and* $\xi_i \in \Gamma(A^*)$*,* $i = 1,2,3$*, we have*

$$\begin{aligned}(\mathscr{L}_X[\Lambda,\Lambda]_{SN})(\xi_1,\xi_2,\xi_3) = \Big\{\langle \mathscr{L}_X([\Lambda,\Lambda]^\sharp(\xi_1,\xi_2)),\xi_3\rangle + c.p.\Big\} \\ - 2\mathscr{L}_X([\Lambda,\Lambda]_{SN}(\xi_1,\xi_2,\xi_3)).\end{aligned}$$

(Proof can be found in Appendix in Proof A.8.)

Lemma 3.2.3. *([35], Lemma* 2.4*)*

$$J(\xi_1,\xi_2,\xi_3) = \left\{\frac{1}{2}\mathscr{L}_{[\Lambda,\Lambda]^\sharp(\xi_1,\xi_2)}\xi_3 + c.p.\right\} - d([\Lambda,\Lambda]_{SN}(\xi_1,\xi_2,\xi_3)), \tag{3.30}$$

where the Jacobiator is given by

$$J(\xi_1,\xi_2,\xi_3) = [[\xi_1,\xi_2],\xi_3] + c.p., \tag{3.31}$$

for any $\xi_1,\xi_2,\xi_3 \in \Gamma(A^*)$*. (Proof can be found in Appendix in Proof A.9.)*

Proof of the Theorem 3.2.1: In order to prove the the condition (3.26) in this Theorem, Lemma 3.2.3 will be used such that

$$\begin{aligned}
\langle J(\xi_1,\xi_2,\xi_3),X\rangle &= \left\{\frac{1}{2}\langle \mathscr{L}_{[\Lambda,\Lambda]^\sharp(\xi_1,\xi_2)}\xi_3,X\rangle + c.p\right\} - \mathscr{L}_X([\Lambda,\Lambda]_{\mathrm{SN}}(\xi_1,\xi_2,\xi_3)) \\
&= \left\{\frac{1}{2}\mathscr{L}_{[\Lambda,\Lambda]^\sharp(\xi_1,\xi_2)}\langle\xi_3,X\rangle - \frac{1}{2}\langle\xi_3,\mathscr{L}_{[\Lambda,\Lambda]^\sharp(\xi_1,\xi_2)}X\rangle + c.p\right\} \\
&- \mathscr{L}_X([\Lambda,\Lambda]_{\mathrm{SN}}(\xi_1,\xi_2,\xi_3)) \\
&= \left\{-\frac{1}{2}\langle\xi_3,\mathscr{L}_{[\Lambda,\Lambda]^\sharp(\xi_1,\xi_2)}X\rangle + c.p\right\} - \mathscr{L}_X([\Lambda,\Lambda]_{\mathrm{SN}}(\xi_1,\xi_2,\xi_3)) \\
&= \left\{\frac{1}{2}\langle\mathscr{L}_X([\Lambda,\Lambda]^\sharp(\xi_1,\xi_2)),\xi_3\rangle + c.p\right\} - \mathscr{L}_X([\Lambda,\Lambda]_{\mathrm{SN}}(\xi_1,\xi_2,\xi_3)) \\
&= \frac{1}{2}(\mathscr{L}_X([\Lambda,\Lambda]_{\mathrm{SN}}))(\xi_1,\xi_2,\xi_3),
\end{aligned}$$

where we have also used $\langle df,X\rangle = \mathscr{L}_X f$ in the first line and the Leibniz Rule of $\mathscr{L}_X$ in the second line, and $\mathscr{L}_{[\Lambda,\Lambda]^\sharp(\xi_1,\xi_2)}\langle\xi_3,X\rangle = \rho([\Lambda,\Lambda]^\sharp(\xi_1,\xi_2))\langle\xi_3,X\rangle = 0$ from (3.28) in the third line, and the last line follows from Lemma 3.2.2. Therefore, $\langle J(\xi_1,\xi_2,\xi_3),X\rangle = 0$, which is non-degenerate, if and only if $\mathscr{L}_X([\Lambda,\Lambda]_{\mathrm{SN}}) = 0$ for all $X \in \Gamma(A)$ which is the condition (3.26) and this ends the proof of Theorem 3.2.1. ■

Proposition 3.2.1. *[35] If the condition in Theorem 3.2.1 holds so that A^* becomes a Lie algebroid. The induced differential operator $d_* : \Gamma(A) \to \Gamma(\wedge^2 A)$ is given by $d_*X = [\Lambda,X]_{SN}$ for any $X \in \Gamma(A)$ and Λ is a section of $\Gamma(\wedge^2 A)$. Then the compatibility condition between A and A^*, which is* (3.24) *is satisfied automatically, namely, $d_*[X,Y] = \mathscr{L}_X d_*Y - \mathscr{L}_Y d_*X$. This leads a Lie bialgebroid structure (A,A^*).*

Proof: We should firstly prove the fact that $d_*X = [\Lambda,X]_{\mathrm{SN}}$ which as follows:

For any $\phi, \psi \in \Gamma(A^*)$, we have 1001[21]

$$\begin{aligned}
(d_*X)(\phi,\psi) &= \rho_*(\phi)(X.\psi) - \rho_*(\psi)(X.\phi) - X[\phi,\psi] \\
&= \rho(\Lambda^\sharp\phi)(X.\psi) - \rho(\Lambda^\sharp\psi)(X.\phi) - X(\mathscr{L}_{\Lambda^\sharp\phi}\psi - \mathscr{L}_{\Lambda^\sharp\psi}\phi - d(\Lambda(\phi,\psi))) \\
&= \mathscr{L}_{\Lambda^\sharp\phi}(X.\psi) - \mathscr{L}_{\Lambda^\sharp\psi}(X.\phi) - X.\mathscr{L}_{\Lambda^\sharp\phi}\psi + X.\mathscr{L}_{\Lambda^\sharp\psi}\phi + X.d(\Lambda(\phi,\psi)) \\
&= (\mathscr{L}_{\Lambda^\sharp\phi}X).\psi - (\mathscr{L}_{\Lambda^\sharp\psi}X).\phi + \rho(X)(\Lambda(\phi,\psi)) \\
&= -\mathscr{L}_X(\Lambda^\sharp\phi).\psi + \mathscr{L}_X(\Lambda^\sharp\psi).\phi + \mathscr{L}_X(\Lambda(\phi,\psi)) \\
&= -\langle\mathscr{L}_X(\Lambda^\sharp\phi),\psi\rangle + \langle\mathscr{L}_X(\Lambda^\sharp\psi),\phi\rangle + \mathscr{L}_X(\Lambda(\phi,\psi)) \\
&= -\mathscr{L}_X\langle\Lambda^\sharp\phi,\psi\rangle + \langle\Lambda^\sharp\phi,\mathscr{L}_X\psi\rangle + \mathscr{L}_X\langle\Lambda^\sharp\psi,\phi\rangle - \langle\Lambda^\sharp\psi,\mathscr{L}_X\phi\rangle + \mathscr{L}_X(\Lambda(\phi,\psi)) \\
&= \Lambda(\phi,\mathscr{L}_X\psi) + \mathscr{L}_X(\Lambda(\psi,\phi)) - \Lambda(\psi,\mathscr{L}_X\phi) \\
&= \Lambda(\mathscr{L}_X(\phi,\psi)) + \mathscr{L}_X(\Lambda(\psi,\phi)) \\
&= (\mathscr{L}_X\Lambda)(\psi,\phi) \\
&= [\Lambda,X]_{\mathrm{SN}}(\phi,\psi), \qquad (3.32)
\end{aligned}$$

where the first equality follows from the definition in 1001[37]. So, the above result gives that $(d_*X)(\phi,\psi) = [\Lambda,X]_{\mathrm{SN}}(\phi,\psi)$ for all $\phi,\psi \in \Gamma(A^*)$. Then it yields $d_*X = [\Lambda,X]_{\mathrm{SN}}$. Therefore,

$$\begin{aligned}
d_*[X,Y] &= [\Lambda,[X,Y]]_{\mathrm{SN}} \\
&= [\Lambda,\mathscr{L}_XY]_{\mathrm{SN}} \\
&= \mathscr{L}_X[\Lambda,Y]_{\mathrm{SN}} - [\mathscr{L}_X\Lambda,Y]_{\mathrm{SN}} \\
&= \mathscr{L}_Xd_*Y - \mathscr{L}_Yd_*X.\blacksquare \qquad (3.33)
\end{aligned}$$

Therefore, these lead (A,A^*) to be a Lie bialgebroid if the requirement of Theorem 3.2.1 is satisfied. Then,

Definition 3.2.3. *[35] If the condition in Theorem 3.2.1 holds, then (A,A^*) is called an exact Lie bialgebroid and Λ is called an r-matrix. If the condition is changed with the requirement $[\Lambda,\Lambda]_{SN} = 0$, then it is called a triangular Lie bialgebroid.*

3.3 Courant Algebroid

Courant algebroid was originally introduced by Zhang-Ju Liu, Alan Weinstein and Ping Xu in their investigation of doubles of Lie bialgebroids in 1997 in [1]. Liu, Weinstein and Xu named it after Courant, who had implicitly devised earlier in 1990, in [38], the standard prototype of Courant algebroid through his discovery of a skew symmetric bracket on $TM \oplus T^*M$, called Courant bracket today, which fails to satisfy the Jacobi Identity. Both this standard example and the double of a Lie bialgebra are special instances of Courant algebroids which will be shown later.

Definition 3.3.1. *A Courant algebroid is a vector bundle E equipped with a non-degenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$ as well as a skew-symmetric bracket $[\cdot, \cdot]$ on $\Gamma(E)$, and with a smooth bundle map $\rho : E \to T$ called the anchor. This induces a natural differential operator $D : C^\infty(M) \to \Gamma(E)$ via the definition $\langle Df, A \rangle = \frac{1}{2}\rho(A)f$ for all $f \in C^\infty(M)$ and $A \in \Gamma(E)$. These structures must be compatible in the following sense: [16]*

$$
\begin{aligned}
&C1)\ \rho([A,B]) = [\rho(A), \rho(B)], \quad \forall A, B \in \Gamma(E), \\
&C2)\ Jac(A,B,C) = D(N_{ij}(A,B,C)), \quad \forall A, B, C \in \Gamma(E), \\
&C3)\ [A, fB] = f[A,B] + (\rho(A)f)B - \langle A, B \rangle Df, \quad \forall A, B \in \Gamma(E), f \in C^\infty(M), \\
&C4)\ \rho \circ D = 0,\ i.e.,\ \langle Df, Dg \rangle = 0, \quad \forall f, g \in C^\infty(M), \\
&C5)\ \rho(A)\langle B, C \rangle = \langle [A,B] + D\langle A, B \rangle, C \rangle + \langle B, [A,C] + D\langle A, C \rangle \rangle, \quad \forall A, B, C \in \Gamma(E),
\end{aligned}
\tag{3.34}
$$

where the Jacobiator $Jac(A,B,C) = [[A,B],C] + c.p.$, and the Nijenhuis operator defined by

$$N_{ij}(A,B,C) = \frac{1}{3}(\langle [A,B], C \rangle + \langle [B,C], A \rangle + \langle [C,A], B \rangle).$$

The last condition (C5) above can be thought as a generalization of the metric invariance in a Lie algebra.

Theorem 3.3.1. *[39] Let $D : C^\infty(M) \to \Gamma(E)$ be the map given by the composition of the dual map ρ_* of ρ and the usual differential d, $D = \frac{1}{2}\beta^{-1}\rho_* d$, β being the isomorphism between E and E^* given by the bilinear form $(\beta A)(\xi) = \langle A, \xi \rangle$ and it*

is equivalent to $\langle Df, A\rangle = \frac{1}{2}\rho(A)f, \forall A \in \Gamma E, \forall f \in C^\infty(M)$. *Then, the conditions* [C3] *and* [C4] *and the property of D, i.e.,* $D = \frac{1}{2}\beta^{-1}\rho_* d$, *follow from [C1], [C5] and the Leibniz Rule for D that is* $D(fg) = fD(g) + gD(f)$ *for all* $f, g \in C^\infty(M)$. *(Proof can be followed from [39] in detail.)*

Example 3.3.1. *(Standard Courant Algebroid:* $E = TM \oplus T^*M$*) [38] If we take the usual Lie bracket on the sections of* TM *that is* $[\cdot,\cdot]_s$ *with the anchor* $\rho_s : TM \oplus T^*M \to TM$ *such that* $\rho_s(X+\xi) = X$, *then it constructs a Lie algebroid structure itself. On the other hand, taking zero bracket on the sections of* T^*M *together with the anchor* $\rho_* = 0$, *the cotangent bundle has also a Lie algebroid structure. These together constructs a Lie bialgebroid structure since the compatibility condition in* (3.23) *or* (3.24) *holds automatically. Therefore, it can be constructed a Courant algebroid structure on* $TM \oplus T^*M$ *which will be given in the Theorem 3.3.4. The bracket on the sections of* $TM \oplus T^*M$ *is called the standard Courant bracket and is given by*

$$[X+\xi, Y+\eta]_s = [X,Y] + \mathscr{L}_X\eta - \mathscr{L}_Y\xi + \frac{1}{2}d(i_Y\xi - i_X\eta), \tag{3.35}$$

and the standard inner product is

$$\langle X+\xi, Y+\eta\rangle_s = \frac{1}{2}(\xi(Y) + \eta(X)). \tag{3.36}$$

This special case of a Courant algebroid will be investigated in the Example 4.4.1 in detail.

Note that in above, we have the followings: $[\xi,\eta]_* = 0$, $\mathscr{L}_\xi Y = \rho_*(\xi)Y = 0$, *and* $d_*(i_\xi Y) = 0$.

Alternative Definition of a Courant Algebroid in terms of a Dorfman Bracket

It has been proven in [40] and [37] that the definition of a Courant algebroid can be given in terms of a Dorfman bracket, which is not necessarily skew-symmetric operation, and which is a binary operation defined by

$$A \circ B = [A,B] + D\langle A,B\rangle, \tag{3.37}$$

and two definitions are equivalent. Without giving the proof of this fact now, the expressions in order to have a Courant algebroid structure will be examined in local

coordinates in Section 4.4, which are (4.37), and it will be verified that these yield the same axioms as expected.

Notice that according to this definition, the Courant bracket is the anti-symmetrization of the Dorfman bracket such that

$$\begin{aligned}\frac{1}{2}(A\circ B - B\circ A) &= \frac{1}{2}([A,B]+D\langle A,B\rangle - [B,A]+D\langle B,A\rangle)\\ &= [A,B],\end{aligned}$$

since $[\cdot,\cdot]$ is a skew-symmetric bracket and $\langle\cdot,\cdot\rangle$ is a symmetric bilinear form.

Definition 3.3.2. *[37] A Courant algebroid is a vector bundle $E \to M$ together with a non-degenerate symmetric bilinear form $\langle\cdot,\cdot\rangle_E$ on the bundle, a bilinear operation "$\circ$" on $\Gamma(E)$, and a bundle map $\rho : E \to TM$ satisfying the following properties:*

1. $A\circ(B\circ C) = (A\circ B)\circ C + B\circ(A\circ C),\ \forall A,B,C\in\Gamma(E)$;
2. $\rho(A\circ B) = [\rho(A),\rho(B)],\ \forall A,B\in\Gamma(E)$;
3. $A\circ(fB) = f(A\circ B) + (\rho(A)f)B,\ \forall A,B\in\Gamma(E), f\in C^\infty(M)$;
4. $A\circ A = D\langle A,A\rangle,\ \forall A\in\Gamma(E)$;
5. $\rho(A)\langle B,C\rangle = \langle A\circ B,C\rangle + \langle B,A\circ C\rangle,\ \forall A,B,C\in\Gamma(E)$,

where $D : C^\infty(M)\to\Gamma(E)$ via the definition $\langle Df,A\rangle = \frac{1}{2}\rho(A)f$ for all $f\in C^\infty(M)$ and $A\in\Gamma(E)$ as in the usual definition of a Courant algebroid.

Remark 3.3.1. *In the notation of [37], the coefficient of the equality* (4.) *has been taken as $\frac{1}{2}$. Instead of that, we will take as* 1. *Also, it will be taken as $\frac{1}{2}$ in the definition of D instead of* 1 *in order to be consistent with the usual definition.*

Theorem 3.3.2. *[1] Let (L,L^*) be a pair of Lie algebroids in duality, with anchors a, a_* and exterior derivatives d, d_*, respectively. Then, the followings can be defined on $E = L\oplus L^*$:*

$$\langle X+\xi, Y+\eta\rangle_{L\oplus L^*} = \frac{1}{2}(i_Y\xi + i_X\eta) = \frac{1}{2}(\xi(Y)+\eta(X)), \tag{3.38}$$

$$[(X+\xi)\circ(Y+\eta)]_{L\oplus L^*} = ([X,Y]+\mathscr{L}_\xi Y - i_\eta d_* X) + ([\xi,\eta]+\mathscr{L}_X\eta - i_Y d\xi), \tag{3.39}$$

$$\rho_{new}(X+\xi) = \rho_L(X) + \rho_*(\xi), \tag{3.40}$$

$$D = d + d_*. \tag{3.41}$$

If (L, L^*) is a Lie bialgebroid, then $(L \oplus L^*, \langle \cdot, \cdot \rangle_{L \oplus L^*}, \circ, \rho_{new})$ is a Courant algebroid.

The above proposition was given in terms of the Dorfman bracket and the similar one will be given also in Theorem 3.3.4 in terms of the Courant bracket. Additionally, the above result will be proven by using the Courant bracket in Subsection 3.3.5 in local coordinates.

Example 3.3.2. *The Dorfman bracket defined in* (3.39) *reduces to the following bracket on $E = TM \oplus T^*M$, which can be called standard Dorfman bracket, equipped with TM has an anchor a and the Lie bracket $[\cdot, \cdot]$ and with T^*M has an anchor $a_* = 0$, and the zero bracket.*

$$[(X+\xi) \circ (Y+\eta)]_s = [X,Y] + \mathscr{L}_X \eta - i_Y d\xi. \tag{3.42}$$

In above, we have the followings: $[\xi, \eta]_* = 0$, $\mathscr{L}_\xi Y = a_*(\xi)Y = 0$, and $i_\eta d_* X = 0$.

We will be working on Courant algebroids of the special type called the exact Courant algebroids. P. Ševera in [40] showed that every exact Courant algebroid is isomorphic to an H-twisted version of $TM \oplus T^*M$ for some closed three form H on the manifold M. This fact and the symmetries of exact Courant algebroids, and the Courant algebroids with transversal Dirac structures will be important for our studies. It will be seen that there is a one-to-one relation between Lie bialgebroids and Courant algebroids with transversal Dirac structures. So, these objects will be introduced in the following Subsections.

3.3.1 Exact Courant algebroid

If $E \to M$ is a Courant algebroid with anchor map ρ then $\rho \circ \rho^t = 0$ from the conditions $(C1)$ and $(C5)$ in (3.34), that is

$$0 \to T^*M \xrightarrow{\rho^t} E \underset{\sigma}{\overset{\rho}{\rightleftarrows}} TM \to 0, \tag{3.43}$$

is a chain complex.

Definition 3.3.3. *[41] A Courant algebroid $E \to M$ is exact if* (3.43) *is an exact sequence.*

The simplest example of an exact Courant algebroid can be given as the standard Courant algebroid, that is every exact Courant algebroid is locally standard which can be seen below.

Our goal is to verify that exact Courant algebroids over M are classified by closed three-forms on the manifold which is known as Ševera's classification [40]. If E is an exact Courant algebroid then there is a Lagrangian subbundle $L \subset E$ (i.e. $L^{\perp} = L$) such that $\rho|_L : L \to TM$ is an isomorphism. If L is a subbundle in this way, then [41]

$$H(X,Y)(Z) := \langle [\sigma(X), \sigma(Y)], \sigma(Z) \rangle,$$

where $X, Y, Z \in TM$, defines a closed 3-form H on the manifold. Let us take $H(X,Y) = \sigma^*[\sigma X, \sigma Y]$ where $\sigma^* : E \to T^*M$ and show that H is C^{∞}-linear and skew in X, Y such that

$$H(X, fY) = f\sigma^*[\sigma X, \sigma Y] + \sigma^*(X(f)\sigma Y) = f\sigma^*[\sigma X, \sigma Y],$$

and

$H(fX,Y) = \sigma^*[f\sigma X, \sigma Y] = fH(X,Y) - \sigma^*((Yf)\sigma X) + 2\langle \sigma X, \sigma Y \rangle df = fH(X,Y).$

Furthermore,

$$[\sigma X, \sigma Y] + [\sigma Y, \sigma X] = \rho^t d\langle \sigma X, \sigma Y \rangle.$$

Secondly, $H(X,Y)(Z)$ is totally symmetic in X, Y, Z as below:

$$H(X,Y)(Z) = \langle [\sigma X, \sigma Y], \sigma Z \rangle = X\langle \sigma Y, \sigma Z \rangle - \langle \sigma Y, [\sigma X, \sigma Z] \rangle.$$

So, we have $[\sigma X, \sigma Y] = [X,Y] - i_Y i_X H$ for a closed 3-form H on the manifold.

If we use $\sigma \oplus \rho^t$ in order to identify E with $TM \oplus T^*M$ then its anchor ρ and pairing $\langle \cdot, \cdot \rangle$ are as in the standard Courant algebroid and the bracket which is called H-twisted standard Courant bracket, is

$$[X + \xi, Y + \eta] = [X,Y] + \mathscr{L}_X \eta - \mathscr{L}_Y \xi + \frac{1}{2} d(i_Y \xi - i_X \eta) + i_Y i_X H, \tag{3.44}$$

which was actually the bracket (3.35) without the last term.

On the other hand, for any closed three-form H the above bracket makes $TM \oplus T^*M$ to an exact Courant algebroid. If σ is changed by a two-form τ on the manifold, then H will be replaced by $H + d\tau$. Therefore,

Theorem 3.3.3. *[40] [41] Exact Courant algebroids over M are classified by closed three-forms on the manifold with the bracket given by* (3.44).

The above result will be investigated later in terms of Courant algebroids for special anchors, namely in Proposition 4.4.3, in local coordinates and the same result will be found in there, as expected.

It will be seen in the next Subsection that the orthogonal automorphisms of exact Courant algebroids are only diffeomorphisms and closed three-forms.

3.3.2 Symmetries of an exact Courant algebroid

It is known that the symmetries of the Lie bracket are only the diffeormorphisms and its proof can be found in [16] in detail. At this point, a natural question can be asked as to whether the Courant bracket possesses extra symmetries. However, our interest will be on the symmetries of an exact Courant algebroid.

We will firstly examine the symmetries of the standard Courant algebroid which is $E = TM \oplus T^*M$ over a manifold M. From the definition of its bracket given in (3.35), one can find out that it is invariant under diffeomorphisms. Also, it has an extra symmetry called B-field transformation defined below.

Definition 3.3.4. *[16] Let B be a smooth two-form such that $B : TM \to T^*M$ by interior product $X \mapsto i_X B$. Then the invertible bundle map given by exponentiating B as*

$$e^B = \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix}, \tag{3.45}$$

that is $e^B : X + \xi \mapsto X + \xi + i_X B$, is orthogonal since $B^* = -B$ implies that $(e^B)^* e^B = I$.

Let us show that it is an orthogonal transformation, namely it preserves the inner product, such that

$$\langle e^B(X+\xi), e^B(Y+\eta)\rangle = \langle X+\xi+i_XB, Y+\eta+i_YB\rangle = i_Y\xi + i_X\eta = \langle X+\xi, Y+\eta\rangle,$$

since $i_X(i_YB) = -i_Y(i_XB)$ for all $B : TM \to T^*M$.

Proposition 3.3.1. *([16], Proposition* 3.23*) The map e^B is an automorphism of the standard Courant bracket* (3.35) *if and only if B is closed, i.e. $dB = 0$.*

Proof: Let $X+\xi$, $Y+\eta \in C^\infty(TM \oplus T^*M)$ and let B be a smooth two-form. Then

$$\begin{aligned}
[e^B(X+\xi), e^B(Y+\eta)]_{T\oplus T^*} &= [X+\xi+i_XB, Y+\eta+i_YB]_{T\oplus T^*} \\
&= [X+\xi, Y+\eta]_{T\oplus T^*} + [X, i_YB]_{T\oplus T^*} + [i_XB, Y]_{T\oplus T^*} \\
&= [X+\xi, Y+\eta]_{T\oplus T^*} + \mathscr{L}_X(i_YB) - \frac{1}{2}d(i_X(i_YB)) \\
&\quad - i_Yd(i_XB) - d(i_Y(i_XB)) + \frac{1}{2}d(i_Y(i_XB)) \\
&= [X+\xi, Y+\eta]_{T\oplus T^*} + \mathscr{L}_X(i_YB) - i_Yd(i_XB) \\
&= [X+\xi, Y+\eta]_{T\oplus T^*} + [\mathscr{L}_X, i_Y]B + i_Yi_X(dB) \\
&= [X+\xi, Y+\eta]_{T\oplus T^*} + i_{[X,Y]}B + i_Yi_X(dB) \\
&= e^B([X+\xi, Y+\eta]_{T\oplus T^*}) + i_Yi_X(dB),
\end{aligned}$$

where we have used the definition of the standard Courant bracket given in (3.35), $i_X(i_YB) = -i_Y(i_XB)$ for all B, and the Cartan formula with other properties of the Lie derivative and interior product. This proves that e^B is an automorphism of the standard Courant bracket if and only if $i_Yi_X(dB) = 0$ for all X, Y, which holds precisely when $dB = 0$. ■

It is straightforward to note that, from the above calculation, e^B is also an automorphism of the H-twisted standard Courant bracket (3.44) if $dB = 0$ again, which can also be seen in [16] in Proposition 3.42. Another result can be followed from [16] that this bracket defines a Courant algebroid structure on $TM \oplus T^*M$ (using the same inner product and anchor) if and only if $dH = 0$.

One can wonder now that whether B-field transformations and diffeomorphisms are the only orthogonal automorphisms of the standard Courant bracket or not.

Proposition 3.3.2. *([16], Proposition* 3.24*) Let (f,F) be an orthogonal automorphism of the direct sum $T \oplus T^*$ for a smooth manifold M. Suppose also that F preserves the standard Courant bracket. Hence, F must be the composition of a diffeomorphism of M and a B-field transformation. (Proof can be followed from [16].)*

Another orthogonal transformation called β transformation which is not an automorphism of the standard Courant bracket will be introduced now for further studies.

Definition 3.3.5. *Let $\beta : T^*M \to TM$, which is skew-symmetric and then it can be assumed as a bi-vector field. Hence, the orthogonal transformation on $T \oplus T^*$ given as follows [16]*

$$e^{\beta} = \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}, \tag{3.46}$$

which is sending $X + \xi \mapsto X + \xi + i_{\xi}\beta$. This transformation will be called a β-transform.

Let us show that it is an orthogonal transformation, namely it preserves the inner product, such that

$$\langle e^{\beta}(X+\xi), e^{\beta}(Y+\eta)\rangle = \langle X+\xi+i_{\xi}\beta, Y+\eta+i_{\eta}\beta\rangle = i_Y\xi + i_X\eta = \langle X+\xi, Y+\eta\rangle. \tag{3.47}$$

Although β transformations are not automorphisms of a standard Courant bracket, they will be important in our studies later in such a way that these will be orthogonal automorphisms of Courant algebroids as one can see in Subsection 6.2.3. Also, it will be seen that β takes T^* to an integrable Dirac structure which will be defined in the next part (see the Example 3.3.4).

3.3.3 Isomorphism of Courant Algebroids

Definition 3.3.6. *[42] Let $(E,\rho,\langle\cdot,\cdot\rangle_E,[\cdot,\cdot]_E)$ and $(\widetilde{E},\rho_{\widetilde{E}},\langle\cdot,\cdot\rangle_{\widetilde{E}},[\cdot,\cdot]_{\widetilde{E}})$ be two Courant algebroids over the same base manifold M. Let $F \in \mathrm{Hom}(E,\widetilde{E})$ (which*

denotes the set of vector bundle morphisms from E to $\widetilde{E}$ over an identity map on the base space) be a morphism of the underlying vector bundles. We say that F is a morphism of Courant algebroids, if it preserves all involved structures, that is

$$\rho = \rho_{\widetilde{E}} \circ F, \ \ \langle U,V\rangle_E = \langle F(U),F(V)\rangle_{\widetilde{E}}, \ \ F([U,V]_E) = [F(U),F(V)]_{\widetilde{E}},$$

for all $U,V \in \Gamma(E)$. If F is invertible, it is called an isomorphism of Courant algebroids. As usual in linear algebra, the inverse F^{-1} of a Courant algebroid morphism is automatically a Courant algebroid morphism. When $E = \widetilde{E}$, the Courant algebroid isomorphism becomes Courant algebroid automorphism.

After defining Dirac structures, our next aim is to present the fact that there exists a one-to-one correspondence between Lie bialgebroids and Courant algebroids with transversal Dirac structures.

3.3.4 Dirac structures of a Courant algebroid

Definition 3.3.7. *[1] Let E be a Courant algebroid. A subbundle L of E is called isotropic if it is isotropic under the symmetric bilinear form $\langle\cdot,\cdot\rangle$. It is called integrable if $\Gamma(L)$ is closed under the bracket $[\cdot,\cdot]$. A Dirac structure, or Dirac subbundle, is a subbundle L which is maximally isotropic and integrable (involutive).*

Proposition 3.3.3. *[1] Suppose that L is an integrable isotropic subbundle of a Courant algebroid $(E,[,],\langle,\rangle,\rho)$. Then $(L,[,],\rho_L)$ is a Lie algebroid.*

Example 3.3.3. *[16] Since the tangent bundle TM is maximally isotropic and involutive (the standard Courant bracket* (3.35) *vanishes on TM), it establishes a Dirac structure. Any closed 2-form on the manifold M can be applied to this fundamental Dirac structure to produce another involutive maximal isotropic structure. In fact, the maximal subspace that is isotropic*

$$e^{\omega}(TM) = \{X + i_X\omega : X \in TM\},$$

is involutive if and only if $d\omega = 0$ (see the Proposition 3.3.1). A Dirac structure can therefore be used to characterize the geometry defined by a closed 2-form.

Likewise,

Example 3.3.4. *[16] The cotangent bundle T^*M defines a Dirac structure since it is maximally isotropic and involutive (the standard Courant bracket* (3.35) *vanishes on T^*M). Using this fundamental Dirac structure, we can apply a bi-vector field β on the manifold and obtain the following.*

$$L_\beta = e^\beta(T^*M) = \{\xi + i_\xi\beta : \xi \in T^*M\}. \tag{3.48}$$

It suffices to check the involutivity of sections of the form $\xi + i_\xi\beta$ where $\xi = df$ for $f \in C^\infty(M)$ due to the fact that $\mathrm{N_{ij}}|_{L_\beta}$ is tensorial. The bi-vector field β determines a bracket on functions as follows.

$$\{f,g\} = \beta(df,dg).$$

Then, we have

$$\mathrm{N_{ij}}(df + i_{df}\beta, dg + i_{dg}\beta, dh + i_{dh}\beta) = \{\{f,g\},h\} + \{\{g,h\},f\} + \{\{h,f\},g\},$$

demonstrating that $\mathrm{N_{ij}}|_{L_\beta} = 0$ if and only if $[\beta,\beta]_{SN} = 0$, that is, the bracket $\{\cdot,\cdot\}$ satisfies the Jacobi identity, i.e., β is a Poisson structure. Here, we observe that although β is not a symmetry of the Courant bracket, it does lead to an integrable Dirac structure for T^*M.

3.3.5 One-to-one correspondence between Lie bialgebroids and Courant algebroids with transversal Dirac structures

The Courant algebroids can be built by Lie bialgebroids through a generalization of the Drinfel'd double construction which has been mentioned in the Theorem 3.3.2 in terms of a Dorfman bracket and more details can be found in [1]. Additionally, an inner product and bracket on the sections of $L \oplus L^*$ can be defined for a given Lie bialgebroid (L, L^*), converting this bundle into a Courant algebroid and both L, L^* into Dirac structures in this Courant algebroid. It has been also demonstrated that $L' = L^*$ can be found using the inner product given any two transversal Dirac structures L, L' in a Courant algebroid, and (L, L^*) is a Lie bialgebroid. With regard to this one-to-one correspondence, the following theorems apply.

In the following Theorem, we shall regard X,Y as sections of a bundle L and ξ,η as sections of the dual bundle L^*, where (L,L^*) forms a Lie bialgebroid, and the following bracket (3.50) on the sections of $L\oplus L^*$ becomes the key element in making $L\oplus L^*$ into a Courant algebroid.

Theorem 3.3.4. *([16], Theorem* 3.35*) Let (L,L^*) be a Lie bialgebroid. The following inner product $\langle\cdot,\cdot\rangle$ can be defined on the bundle $L\oplus L^*$:*

$$\langle X+\alpha, Y+\beta\rangle_{L\oplus L^*} = \frac{1}{2}(\alpha(Y)+\beta(X)). \tag{3.49}$$

We also define the following skew-symmetric bracket operation on $\Gamma(L\oplus L^*)$:

$$\begin{aligned}[X+\xi, Y+\eta]_{L\oplus L^*} = [X,Y]_L + \mathscr{L}_X\eta - \mathscr{L}_Y\xi + \frac{1}{2}d(i_Y\xi - i_X\eta)\\ + [\xi,\eta]_{L^*} + \mathscr{L}_\xi Y - \mathscr{L}_\eta X - \frac{1}{2}d_*(i_\xi Y - i_\eta X).\end{aligned} \tag{3.50}$$

If ρ, ρ_* are the anchors for L,L^*, we define the bundle map

$$\rho_{new} = \rho + \rho_* \ : \ L\oplus L^* \to T. \tag{3.51}$$

With these definitions, the structure $(L\oplus L^*, [\cdot,\cdot]_{L\oplus L^*}, \langle\cdot,\cdot\rangle_{L\oplus L^*}, \rho_{new})$ is a Courant algebroid. The operator D: $C^\infty(M)\to\Gamma(L\oplus L^*)$ clearly becomes

$$D = d + d_*, \tag{3.52}$$

where $d : C^\infty(M)\to\Gamma(L^*)$ and $d_* : C^\infty(M)\to\Gamma(L)$ are the usual differential operators associated with Lie algebroids.

Remark 3.3.2. *Notice that the (general) Courant bracket in* (3.50) *is more symmetric than the standard Courant bracket in* (3.35)*, as it treats the sections of L and L^* (which are the one-forms and vector fields, respectively, in a standard Courant algebroid) completely in a symmetric way.*

Theorem 3.3.5. *([16], Theorem* 3.36*) Let $(E,[\cdot,\cdot]_E,\langle\cdot,\cdot\rangle_E,\rho_E)$ be a Courant algebroid, and let $L,L' < E$ be Dirac sub-bundles transverse to each other, i.e.,*

$E = L \oplus L'$. Then $L' = L^*$ using the inner product, and (L, L') is a Lie bialgebroid. Applying the construction in the previous theorem to (L, L'), we recover the original Courant algebroid structure on E.

Comprehensive proof of this one-to-one correspondence between Lie bialgebroids and Courant algebroids with transversal Dirac structures can be followed from [1]. We shall prove the Theorem 3.3.4 for triangular Lie bialgebroids in local coordinates in the Subsection 4.4.2.

[16] As with $TM \oplus T^*M$ in the Examples given in 3.3.3 and 3.3.4, given a splitting of a Courant algebroid which has such Dirac subbundles L, L^* such that $E = L \oplus L^*$ into a Lie bialgebroid, we could try to construct new Dirac structures as graphs of elements in $\wedge^2 L$ or $\wedge^2 L^*$. Let $\varepsilon \in \wedge^2 L^*$, and without losing generality, consider the maximal isotropic subbundle as

$$L_\varepsilon = \{A + i_A \varepsilon : A \in L\},$$

The last main result of [1] is the condition for the involutivity of the almost Dirac structure L_ε, which is a subbundle that satisfies only maximal isotropic condition:

Theorem 3.3.6. *[1] The almost Dirac structure L_ε, for $\varepsilon \in \wedge^2 L^*$, is integrable if and only if ε satisfies the generalized Maurer-Cartan equation*

$$d_* \varepsilon + \frac{1}{2}[\varepsilon, \varepsilon] = 0.$$

Here $d_* : \Gamma(\wedge^k L^*) \to \Gamma(\wedge^{k+1} L^*)$ and $[\cdot, \cdot]$ is the Lie algebroid bracket on L^*.

If one has a Poisson bi-vector β that is $[\beta, \beta]_{\mathrm{SN}} = 0$, then the above equation vanishes identically that means the almost Dirac structure L_ε, for $\varepsilon \in \wedge^2 L^*$, is integrable. So it yields a Dirac structure.

The definition of the generalized metric on an exact Courant algebroid will be introduced now since this notion will be important in further studies in the Chapter 6.

3.3.6 Definition of a generalized metric on a Courant algebroid $TM \oplus T^*M$

For a given Riemannian metric g and a two-form B on a manifold M, positive and negative definite subbundles S_+ and S_- which are subsets of $TM \oplus T^*M$ of a Courant algebroid E can be constructed, respectively. After the construction of the subbundles S_+ and S_-, the endomorphism $\tau : TM \oplus T^*M \to TM \oplus T^*M$ with $\tau^2 = 1$ can be defined and it will be seen that the pair (g,B) defines a unique metric which will be called a generalized metric denoted by $\mathscr{H}$ on a Courant algebroid.

Let us define $S_+ \subset E = TM \oplus T^*M$ be the positive definite subbundle (rank n) such that

$$S_+ = \{X + g^\sharp X : X \in TM\Big\} \subset TM \oplus T^*M,$$

where Riemannian map $g^\sharp$ is defined as [43]

$$g^\sharp : TM \to T^*M,$$
$$X \mapsto \xi,$$

where $\xi(Y) = (g^\sharp X)(Y) = g(X,Y)$ which is the positive definite inner product $\langle X+\alpha, Y+\beta\rangle = \frac{1}{2}(\alpha(Y)+\beta(X))$, where $X,Y \in TM$ and $\alpha,\beta \in T^*M$, restricted to S_+, say $\langle\cdot,\cdot\rangle|_{S_+}$. The map $g^\sharp$ shall hereafter be referred to as g in order to avoid confusion.

Also, S_- be the negative definite subbundle $S_- \subset TM \oplus T^*M$ (rank n) of a Courant algebroid $TM \oplus T^*M$ with respect to the nondegenerate inner product (3.36) and S_- is to be the orthogonal complement, namely $S_- = S_+^\perp$ such that

$$S_- = \{X - gX : X \in TM\} \subset TM \oplus T^*M.$$

Here $-g(X,Y)$ can be defined as the negative definite inner product on $TM \oplus T^*M$ restricted to S_-, say $\langle\cdot,\cdot\rangle|_{S_-}$.

Therefore, we obtain $TM \oplus T^*M = S_+ \oplus S_-$.

Proposition 3.3.4. *[16] $S_\pm$ are the ± 1-eigenbundles of the endomorphism*

$$\upsilon = \begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix},$$

with $\upsilon^2 = 1$ acting on $TM \oplus T^*M$.

Assuming that g and g^{-1} are maps in the above Proposition such that $g : TM \to T^*M$ and $g^{-1} : T^*M \to TM$, respectively.

Proof: Let $X + \xi \in S_+$, then the following shows that it is $+1$-eigenbundle of G.

$$\begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix} \begin{pmatrix} X \\ \xi \end{pmatrix} = 1 \cdot \begin{pmatrix} X \\ \xi \end{pmatrix},$$

then $g^{-1}\xi = X$ and $gX = \xi$.

Similarly, let $X + \xi \in S_-$, then we can show it is -1-eigenbundle of υ:

$$\begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix} \begin{pmatrix} X \\ \xi \end{pmatrix} = (-1) \cdot \begin{pmatrix} X \\ \xi \end{pmatrix},$$

then $g^{-1}\xi = -X$ and $gX = -\xi$. ■

By applying the orthogonal transformation e^B, which was given in (3.45), to υ, we obtain the following [16].

$$\tau = e^B \upsilon e^{-B} = \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix} \begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -B & 1 \end{pmatrix} = \begin{pmatrix} -g^{-1}B & g^{-1} \\ g - Bg^{-1}B & Bg^{-1} \end{pmatrix}, \tag{3.53}$$

where τ is an endomorphism such as

$$\tau : TM \oplus T^*M \to TM \oplus T^*M,$$

with $\tau^2 = 1$. In above, a two-form B is assumed to be a map such that $B : TM \to T^*M$. Therefore, we have $B \pm g : TM \to T^*M$ which will be used below.

Similar to the case of υ, we can choose a positive definite subbundle $\tilde{S}_+ \subset TM \oplus T^*M$ and the orthogonal complement $\tilde{S}_-$ as the negative definite subbundle to have $\tilde{S}_+ \oplus \tilde{S}_- \subset TM \oplus T^*M$. These subbundles can be defined as follows.

$$\tilde{S}_+ = \{X + (B+g)X : X \in TM\} \subset TM \oplus T^*M, \tag{3.54}$$

and

$$\tilde{S}_- = \{X + (B-g)X : X \in TM\} \subset TM \oplus T^*M. \tag{3.55}$$

Proposition 3.3.5. *[44]* $\tilde{S}_{\pm}$ *defined in* (3.54) *and* (3.55)*, respectively, are the* ± 1*-eigenbundles of the endomorphism* τ *with* $\tau^2 = 1$ *acting on* $TM \oplus T^*M$*.*

Proof: We will show firstly that $\tilde{S}_+$ is $+1$-eigenbundle of τ and has elements of the form $(X,(g+B)X)$. Let $X+g \in \tilde{S}_+$, then

$$\begin{pmatrix} -g^{-1}B & g^{-1} \\ g - Bg^{-1}B & Bg^{-1} \end{pmatrix} \begin{pmatrix} X \\ \xi \end{pmatrix} = 1. \begin{pmatrix} X \\ \xi \end{pmatrix},$$

and so that $\xi = (B+g)X$.

Similarly, it will be shown that $\tilde{S}_-$ is -1-eigenbundle of τ and has elements of the form $(X,(B-g)X)$. Let $X+g \in \tilde{S}_-$, then

$$\begin{pmatrix} -g^{-1}B & g^{-1} \\ g - Bg^{-1}B & Bg^{-1} \end{pmatrix} \begin{pmatrix} X \\ \xi \end{pmatrix} = (-1) \cdot \begin{pmatrix} X \\ \xi \end{pmatrix},$$

and then $\xi = (B-g)X$, which completes the proof. ■

Proposition 3.3.6. *The subbundles* $\tilde{S}_+$ *and* $\tilde{S}_-$ *defined in* (3.54) *and* (3.55)*, respectively, are orthogonal subbundles.*

Proof:

$$\begin{aligned} \langle X+(B+g)X, Y+(B-g)Y \rangle &= \langle X,(B-g)Y \rangle + \langle (B+g)X, Y \rangle \\ &= \langle X, BY \rangle - \langle X, gY \rangle + \langle BX, Y \rangle + \langle gX, Y \rangle \\ &= 0. \blacksquare \end{aligned}$$

Definition 3.3.8. *[16] Let the splitting* $TM \oplus T^*M = \tilde{S}_+ \oplus \tilde{S}_-$ *exists. A generalized metric* $\mathscr{H}$ *is defined as a positive definite metric on* $TM \oplus T^*M$ *and*

$$\mathscr{H} = \langle \cdot, \cdot \rangle|_{\tilde{S}_+} - \langle \cdot, \cdot \rangle|_{\tilde{S}_-}. \tag{3.56}$$

The above definition of a generalized metric on $TM \oplus T^*M$ makes it possible to take inner product of the elements projected onto $\tilde{S}_+$ and $\tilde{S}_-$, leading a positive definite inner product. This process is distinct from using the standard inner product in a Courant algebroid.

An equivalent definitions of a generalized metric is as follows. The equivalence of the above and the below definitions can be seen in the Proposition 6.2.2.

Definition 3.3.9. *Let E be a vector bundle with a fiber-wise metric $\langle\cdot,\cdot\rangle$. A generalized Riemannian metric is an automorphism $\tau \in \mathrm{Aut}(E)$, such that $\tau^2 = 1$, and the formula [42]*

$$\mathscr{H}(A,B) := \langle A, \tau(B)\rangle, \tag{3.57}$$

for all $A,B \in \Gamma(E)$, defines a positive definite fiber-wise metric $\mathscr{H}$ on E.

Definition 3.3.10. *Let $\mathscr{H}$ be a positive definite fiber-wise metric on E. We say that $\mathscr{H}$ is a generalized metric if the induced isomorphism $\mathscr{H} \in \mathrm{Hom}(E,E^*)$ defines a map orthogonal with respect to the fiber-wise metric $g = \langle\cdot,\cdot\rangle$ on E and the fiber-wise metric g^{-1} on E^*: [42]*

$$g(A,B) = g^{-1}(\mathscr{H}(A),\mathscr{H}(B)), \tag{3.58}$$

for all $A,B \in \Gamma(E)$.

Let $O(E) \subseteq \mathrm{Aut}(E)$ denote the group of orthogonal automorphisms with respect to $\langle\cdot,\cdot\rangle$, that is every $\mathscr{O} \in O(E)$ satisfies $\langle\mathscr{O}(A),\mathscr{O}(B)\rangle = \langle A,B\rangle$ for all $A,B \in \Gamma(E)$. There is a natural action of $O(E)$ on the space of generalized metrics, that is if we set [42]

$$\tilde{\mathscr{H}}(A,B) = \mathscr{H}(\mathscr{O}(A),\mathscr{O}(B)), \tag{3.59}$$

for all $A,B \in \Gamma(E)$. Other quantities transform accordingly, namely $\tilde{\tau} = \mathscr{O}^{-1}\tau\mathscr{O}$ and $\tilde{S}_\pm = \mathscr{O}^{-1}(S_\pm)$. Then $\tilde{\mathscr{H}}, \tilde{\tau}$ and $\tilde{S}_+$ again define a generalized metric.

Proposition 3.3.7. *[42] Let $E = TM \oplus T^*M$ and $\langle\cdot,\cdot\rangle$ be the fiber-wise metric defined by* (3.49)*. Let $\mathscr{H}$ be a generalized metric on E. Then $\mathscr{H}$ defines a unique pair (g,B), where $g > 0$ is a Riemannian metric on M and B is a two-form on the manifold. Conversely, any pair (g,B) defines a unique generalized metric.*

4. LOCAL EXPRESSIONS AND PROPERTIES OF DRINFEL'D DOUBLES AND ALGEBROIDS

This chapter is devoted to finding and interpreting the expressions in local coordinates of the concepts given in the previous Chapters since the resulting local expressions of these notions will be useful for our studies in the next Chapters.

4.1 Properties of the Drinfel'd Double in Local Coordinates

Proposition 4.1.1. *The Drinfel'd double of a Lie algebra g defined in the Definition 2.3.2 can be written in local coordinates as follows. Let T_a, T^b be fixed bases for the two compatible Lie algebras g, g^*, respectively, then*

$$[T_a, T_b] = f_{ab}{}^c T_c, \quad [T_a, T^b] = Q_a{}^{bc} T_c - f_{ac}{}^b T^c, \quad [T^a, T^b] = Q_c{}^{ab} T^c,$$

where $f_{ab}{}^c$ are the structure constants of the Lie algebra g, and $Q_a{}^{bc}$ are the structure constants of the dual Lie algebra g^*. The proof of the fact that the structure constants of the second bracket result in terms of $f_{ab}{}^c$ and $Q_c{}^{ab}$ can be followed from the proof of the Example 4.4.5.

The Jacobi identity for the Lie algebra on $g \oplus g^*$ means that

$$f_{ab}^j f_{jd}^i + f_{bd}{}^j f_{ja}{}^i + f_{da}{}^j f_{jb}{}^i = 0,$$

$$f_{dj}{}^i Q_a{}^{jb} + f_{ja}{}^b Q_d{}^{ij} + f_{ad}{}^j Q_j{}^{ib} + f_{ja}^i Q_d^{jb} + f_{dj}^b Q_a^{ij} = 0,$$

and

$$Q_j{}^{id} Q_b{}^{ja} + Q_j{}^{da} Q_b{}^{ji} + Q_j{}^{ai} Q_b{}^{jd} = 0,$$

where the first equality is the Jacobi Identity for g, the second one is for the compatibility condition between g and g^*, and the last one is for the Jacobi Identity for g^*. (The proof can be found in the proof of the Example 4.4.5.)

4.2 Properties of the Lie Algebroid in Local Coordinates

The definition of the Lie algebroid has been given in the Subsection 3.1.1. Here, the local expressions will be examined and the properties given in the Subsection 3.1.2 will be used throughout the following parts.

Let M be a differentiable, n-dimensional manifold and (TM, π_M, M) its tangent bundle. Recall that a Lie algebroid over the manifold M, is the triple $(A, [\cdot,\cdot], \rho)$ where $\pi : A \to M$ is a vector bundle of rank m over M, whose $C^\infty(M)$-module of sections $\Gamma(A)$ is equipped with a Lie algebra structure $[\cdot,\cdot]$ and the anchor $\rho : A \to TM$ is a vector bundle map, which satisfies the conditions given in the Definition 3.1.1.

If we take the local coordinates (X^i) on an open $U \subset M$, a local basis $\{S_a\}$ of sections of A. The local functions $\rho_a^i(X)$, and $L_{ab}{}^c(X)$ on M defined by [45]

$$\rho(S_a) = \rho_a^i \frac{\partial}{\partial X^i}, \tag{4.1}$$

and

$$[S_a, S_b] = L_{ab}{}^c S_c, \tag{4.2}$$

where $L_{ab}{}^c$ are called the structure functions of the bracket of the elements of $\Gamma(A)$, and $i = (1, \ldots n)$ and $a, b, c = (1, \ldots m)$.

In the following Proposition, the Definition 3.1.1 will be written in local coordinates, by using the above Definition, such that the first condition will be obtained from (3.1) and the latter will be obtained from the Jacobi identity.

Proposition 4.2.1. *The local functions defined above are called structure functions and they satisfy the structure equations given below.*
From the first property given in (3.1)*, we have*

$$\boxed{\rho_a^j \frac{\partial \rho_b^i}{\partial X^j} - \rho_b^j \frac{\partial \rho_a^i}{\partial X^j} = \rho_c^i L_{ab}{}^c}, \tag{4.3}$$

and from the Jacobi identity of the Lie bracket on $\Gamma(A)$*, we obtain*

$$\boxed{\rho_a^m \frac{\partial L_{bc}{}^d}{\partial X^m} + \rho_b^m \frac{\partial L_{ca}{}^d}{\partial X^m} + \rho_c^m \frac{\partial L_{ab}{}^d}{\partial X^m} + L_{ae}{}^d L_{bc}{}^e + L_{be}{}^d L_{ca}{}^e + L_{ce}{}^d L_{ab}{}^e = 0}. \tag{4.4}$$

Their proofs can be found in Appendix in A.10.

Remark 4.2.1. *In the rest of the thesis, we will sometimes express* (4.4) *in the following way:*

$$\boxed{\left(\rho_a^i \frac{\partial L_{bc}{}^d}{\partial X^i} + L_{ae}{}^d L_{bc}{}^e\right) + c.p. = 0}. \tag{4.5}$$

The structure functions of the bracket of sections of a Lie algebroid defined above can be thought as a generalization of the structure constants of the Lie bracket of vector fields of a Lie algebra given in Proposition 4.1.1.

Calculus on Lie Algebroids in Local Coordinates

Our goal is to express the notions given in the Subsections 3.1.2 and 3.2.1 in local coordinates, since these will be used in further studies.

Let A be a Lie algebroid on a smooth manifold M with the dual vector bundle A^*. Let $X = \psi^a S_a$, and $\theta = \theta_a S^a$ be the smooth sections of A and A^*. Let $[S_a, S_b] = L_{ab}^c S_c$ as defined above in (4.2). There is a natural inner product between A and A^* and we choose the basis such that

$$\langle S_a, S^b \rangle = \delta_a{}^b, \tag{4.6}$$

is satisfied.

Then, we assume that there exists a Lie bracket on the dual vector bundle A^*, whose sections satisfy the above equality, in the following way:

$$[S^a, S^b] = \widetilde{L}_c^{ab} S^c, \tag{4.7}$$

where $\widetilde{L}^{ab}{}_c$ are the structure functions of the bracket of the elements of $\Gamma(A^*)$.

Additionally, $\rho : A \to TM$ is an anchor with local functions ρ_a^i on M whereas $\rho_* : A^* \to TM$ is a dual anchor with local functions ρ_*^{ia} on M which are

$$\rho_*(S^a) = \rho_*^{ia} \frac{\partial}{\partial X^i}, \tag{4.8}$$

and $d : \Gamma(A^*) \to \Gamma(^2A^*)$ whereas $d_* : \Gamma(A) \to \Gamma(^2A)$, and $f \in C^\infty(M)$.

Remark 4.2.2. *Our focus in the following Subsection will be on formulating the requirements for a dual Lie algebroid structure, for a given Lie algebroid A, in local coordinates such that* (A,A^*) *becomes a triangular Lie bialgebroid as in the Subsection 3.2.1. Here, the* $\Lambda^\sharp$ *which is defined in the Definition 3.2.2 as* $\rho\circ\Lambda^\sharp=\rho_*$*, will be utilized where* $\Lambda^\sharp:A^*\to A$*. The following results in local coordinates from this formula.*

$$\begin{aligned}
&(\rho\circ\Lambda^\sharp)S^a=\rho_*(S^a)\\
\Rightarrow&\rho\circ(\Lambda^{ab}S_b)=\rho_*^{ia}\frac{\partial}{\partial X^i}\\
\Rightarrow&\Lambda^{ab}\rho_b^i\frac{\partial}{\partial X^i}=\rho_*^{ia}\frac{\partial}{\partial X^i}\\
\Rightarrow&\Lambda^{ab}\rho_b^i=\rho_*^{ia}.
\end{aligned}\tag{4.9}$$

Using all the given above and the rules already given, we find the following equalities in local coordinates.

From (3.3):

$$df(S_a)=\rho(S_a)(f)=\rho_a^i\frac{\partial f}{\partial X^i},$$

and so that

$$df=\frac{\partial f}{\partial X^i}\rho_a^iS^a,\tag{4.10}$$

and similarly

$$d_*f=\frac{\partial f}{\partial X^i}\rho_*^{ia}S_a.\tag{4.11}$$

From (3.4) and (4.6):

$$\begin{aligned}
dS^a(S_b,S_c)&=\rho(S_b)S^a(S_c)-\rho(S_c)S^a(S_b)-S^a[S_b,S_c]\\
&=-S^aL_{bc}{}^eS_e,\\
&=-\frac{1}{2}L_{bc}{}^a,
\end{aligned}$$

and hence

$$dS^a=-\frac{1}{2}L_{bc}{}^aS^b\wedge S^c,\tag{4.12}$$

and similarly

$$d_*S_a=-\frac{1}{2}\widetilde{L}^{bc}{}_aS_b\wedge S_c.\tag{4.13}$$

Therefore,

$$\begin{aligned} d\theta &= d(\theta_a S^a) = (d\theta_a) \wedge S^a + \theta_a \wedge (dS^a) \\ &= \frac{\partial \theta_a}{\partial X^i} \rho_b^i S^b \wedge S^a + \theta_a \wedge \left(-\frac{1}{2} L_{bc}{}^a S^b \wedge S^c \right) \\ &= \frac{\partial \theta_a}{\partial X^i} \rho_b^i S^b \wedge S^a - \frac{1}{2} L_{bc}{}^a \theta_a S^b \wedge S^c \\ &= \left(\frac{\partial \theta_a}{\partial X^i} \rho_b^i - \frac{1}{2} \theta_c L_{ba}{}^c \right) S^b \wedge S^a, \end{aligned} \tag{4.14}$$

where $d : \Gamma(A^*) \to \Gamma(^2A^*)$, and

$$\begin{aligned} d_* X &= d_*(\psi^a S_a) = (d_* \psi^a) \wedge S_a + \psi^a \wedge (d_* S_a) \\ &= \frac{\partial \psi^a}{\partial X^i} \rho_*^{ib} S_b \wedge S_a + \psi^a \wedge \left(-\frac{1}{2} \widetilde{L}^{bc}{}_a S_b \wedge S_c \right) \\ &= \left(\frac{\partial \psi^a}{\partial X^i} \rho_*^{ib} - \frac{1}{2} \psi^c \widetilde{L}^{ba}{}_c \right) S_b \wedge S_a, \end{aligned} \tag{4.15}$$

where $d_* : \Gamma(A) \to \Gamma(^2A)$.

Also, we will use (3.12), (4.6), Cartan's formula, (4.14), (4.10), the equality $i_X(\beta \wedge \gamma) = (i_X \beta) \wedge \gamma + (-1)^p \beta \wedge (i_X \gamma)$ for $X \in \Gamma(A)$, $\beta \in (^pA^*)$, and $\gamma \in (^qA^*)$ for the following first equality. Then, together with some of the previous properties, (4.15) and (4.11) will be used in order to obtain the second equality below.

$$\begin{aligned} \mathscr{L}_X(\theta) &= \mathscr{L}_{\psi^a S_a}(\theta_b S^b) \\ &= \left(\psi^a \frac{\partial \theta_b}{\partial X^i} \rho_a^i + \theta^a \frac{\partial \psi^a}{\partial X^i} \rho_b^i + \theta_d \psi^a L_{ba}{}^d \right) S^b, \end{aligned} \tag{4.16}$$

and

$$\begin{aligned} \mathscr{L}_\theta(X) &= \mathscr{L}_{\theta_a S^a}(\psi^b S_b) \\ &= \left(\theta_a \frac{\partial \psi^b}{\partial X^i} \rho_*^{ia} + \psi^a \frac{\partial \theta_a}{\partial X^i} \rho_*^{ib} + \theta_a \psi^d \widetilde{L}^{ba}{}_d \right) S_b. \end{aligned} \tag{4.17}$$

The special cases are

$$\mathscr{L}_{S_a} S^b = -L_{ad}{}^b S^d, \tag{4.18}$$

and

$$\mathscr{L}_{S^b} S_a = -\widetilde{L}^{bd}{}_a S_d. \tag{4.19}$$

4.3 Properties of Triangular Lie Bialgebroid in Local Coordinates

It has been seen in the Subsection 3.2.1 that if the condition given in Theorem 3.2.1 is satisfied for a given Lie algebroid A with a new bracket (3.25) on the sections of the dual vector bundle A^* derived from the bi-section Λ, then there exists also a Lie algebroid structure on the dual vector bundle. Additionally, it has been verified that they define together a Lie bialgebroid since induced differential operator is $d_* X = [\Lambda, X]_{\mathrm{SN}}$ and automatically satisfies the compatibility condition $d_*[X,Y] = \mathscr{L}_X d_* Y - \mathscr{L}_Y d_* X$ which was given in (3.24) (see the Proposition 3.2.1).

In this Subsection, our goal is to determine the local expressions of these conditions that must be satisfied in terms of the new bracket and the dual vector bundle to be a Lie algebroid since these will be helpful for our studies.

As it was stated in the Remark 3.2.2, a special case of Lie bialgebroids will be studied here which are triangular Lie bialgebroids that are constructed by $\Lambda \in \Gamma(^2 A)$.

Proposition 4.3.1. *The new bracket given in* (3.25)*, which is*

$$[\xi, \eta]_* = L_{\Lambda^\sharp \xi} \eta - L_{\Lambda^\sharp \eta} \xi - d[\Lambda(\xi, \eta)],$$

reads in local coordinates

$$\boxed{\widetilde{L}^{ab}{}_d = L_{id}{}^a \Lambda^{bi} - L_{id}{}^b \Lambda^{ai} + \frac{\partial \Lambda^{ab}}{\partial X^i} \rho^i_d}. \tag{4.20}$$

Proof: Let $\xi = S^a$ and $\eta = S^b$, then we have

$$\begin{aligned}
\widetilde{L}^{ab}{}_{d}S^d &= [S^a, S^b]_* \\
&= \mathscr{L}_{\Lambda^\sharp S^a}S^b - \mathscr{L}_{\Lambda^\sharp S^b}S^a - d[\Lambda(S^a, S^b)] \\
&= \mathscr{L}_{\Lambda^{ac}S_c}S^b - \mathscr{L}_{\Lambda^{bc}S_c}S^a - d\langle \Lambda^\sharp S^a, S^b\rangle \\
&= \Lambda^{ac}\mathscr{L}_{S_c}S^b + (d\Lambda^{ac})\wedge(i_{S_c}S^b) - \Lambda^{bc}\mathscr{L}_{S_c}S^a - (d\Lambda^{bc})\wedge(i_{S_c}S^a) - d\langle \Lambda^{ac}S_c, S^b\rangle \\
&= -\Lambda^{ac}L_{cd}{}^{b}S^d + \frac{\partial\Lambda^{ac}}{\partial X^i}\rho^i_d S^d \wedge \delta^b{}_c + \Lambda^{bc}L_{cd}{}^{a}S^d - \frac{\partial\Lambda^{bc}}{\partial X^i}\rho^i_d S^d \wedge \delta^a{}_c - d(\Lambda^{ac}\delta^b{}_c) \\
&= -\Lambda^{ac}L_{cd}{}^{b}S^d + \frac{\partial\Lambda^{ab}}{\partial X^i}\rho^i_d S^d + \Lambda^{bc}L_{cd}{}^{a}S^d - \frac{\partial\Lambda^{ba}}{\partial X^i}\rho^i_d S^d - \frac{\partial\Lambda^{ab}}{\partial X^i}\rho^i_c S^c \\
&= \left(-\Lambda^{ac}L_{cd}{}^{b} + \Lambda^{bc}L_{cd}{}^{a} + \frac{\partial\Lambda^{ab}}{\partial X^i}\rho^i_d\right)S^d,
\end{aligned}$$

where we have used the fact $\Lambda^\sharp S^a = \Lambda^{ac}S_c$ in the third line and (3.12) in the fourth line, and this result gives

$$\widetilde{L}^{ab}{}_{d} = L_{cd}{}^{a}\Lambda^{bc} - L_{cd}{}^{b}\Lambda^{ac} + \frac{\partial\Lambda^{ab}}{\partial X^i}\rho^i_d. \blacksquare$$

Therefore, we have derived a formula for the structure functions of the dual Lie algebroid in terms of the structure functions of the associated Lie algebroid and the components of the bi-section Λ.

Lemma 4.3.1. *When the the components of Λ are constants and anchors are trivial ones (zero), then the equation* (4.20) *reduces to the equation below:*

$$\boxed{\widetilde{L}^{ab}{}_{d} = L_{cd}{}^{a}\Lambda^{bc} - L_{cd}{}^{b}\Lambda^{ac}}, \tag{4.21}$$

Here, $L_{id}{}^{a}$ and $\widetilde{L}^{ab}{}_{d}$ are structure constants.

In order to check that there is also a Lie algebroid structure on the dual vector bundle A^*, one must verify the conditions given in the Proposition 4.2.1 hold in terms of an anchor $\rho_* : A^* \to TM$ and structure functions of the new bracket, $\widetilde{L}^{ab}{}_{d}$, of A^*.

Proposition 4.3.2. *The first condition in terms of the Lie algebroid A^* of the Proposition 4.2.1, namely,*

$$\boxed{\rho_*^{ia}\frac{\partial\rho_*^{jb}}{\partial X^i} - \rho_*^{ib}\frac{\partial\rho_*^{ja}}{\partial X^i} = \widetilde{L}^{ab}{}_{i}\rho_*^{ji}}, \tag{4.22}$$

is satisfied if and only if the following equation holds:

$$\boxed{\Lambda^{ak}\Lambda^{bi}L_{ki}{}^{e}+\Lambda^{bk}\Lambda^{ei}L_{ki}{}^{a}+\Lambda^{ek}\Lambda^{ai}L_{ki}{}^{b}+\Lambda^{ak}\rho_{k}^{i}\frac{\partial\Lambda^{be}}{\partial X^{i}}+\Lambda^{bk}\rho_{k}^{i}\frac{\partial\Lambda^{ea}}{\partial X^{i}}+\Lambda^{ek}\rho_{k}^{i}\frac{\partial\Lambda^{ab}}{\partial X^{i}}=0}. \tag{4.23}$$

The equation $[\Lambda,\Lambda]_{SN}=0$*, is also equivalent to the condition above in local coordinates. (Proof can be found in Appendix in Proof A.11.)*

Note that in the above condition in local expression includes the derivatives of the bi-sections together with the anchors and bi-sections.

Lemma 4.3.2. *The equation* $[\Lambda,\Lambda]_{SN}=0$ *which is equivalent to the condition* (4.23) *above reduces to the following when the components of* Λ *are constants.*

$$\Lambda^{ai}\Lambda^{bk}L_{ik}{}^{e}+\Lambda^{bk}\Lambda^{ei}L_{ki}{}^{a}+\Lambda^{ei}\Lambda^{ak}L_{ik}{}^{b}=0. \tag{4.24}$$

Here, $L_{ik}{}^{e}$ *are structure constants.*

Proposition 4.3.3. *The second condition in terms of the Lie algebroid* A^{*} *of the Proposition 4.2.1, namely,*

$$\boxed{\rho_{*}^{ia}\frac{\partial\widetilde{L}^{bc}{}_{e}}{\partial X^{i}}+\rho_{*}^{ib}\frac{\partial\widetilde{L}^{ca}{}_{e}}{\partial X^{i}}+\rho_{*}^{ic}\frac{\partial\widetilde{L}^{ab}{}_{e}}{\partial X^{i}}+\widetilde{L}^{ai}{}_{e}\widetilde{L}^{bc}{}_{i}+\widetilde{L}^{bi}{}_{e}\widetilde{L}^{ca}{}_{i}+\widetilde{L}^{ci}{}_{e}\widetilde{L}^{ab}{}_{i}=0}, \tag{4.25}$$

is satisfied if and only if $\mathscr{L}_{X}[\Lambda,\Lambda]_{SN}=0$*.*

However, we will assume that $[\Lambda,\Lambda]_{\mathrm{SN}}=0$ (to be a triangular Lie bialgebroid) is satisfied for the above condition which is equivalent to (4.23). (Proof can be found in Appendix in Proof A.12.)

The above fact is also an equivalent local condition for the new bracket $[\cdot,\cdot]_{*}$ to satisfy the Jacobi identity.

It has been verified in the Proposition 3.2.1 that the compatibility condition between A and A^{*} is satisfied since the induced differential operator is $d_{*}X=[\Lambda,X]_{\mathrm{SN}}$ and then they together define a Lie bialgebroid structure. Our goal is to analyze the local expression for the compatibility condition between A and A^{*} in terms of their structure functions which will be important in the following parts of the thesis.

Proposition 4.3.4. *For $X = S_\delta, Y = S_\omega \in \Gamma(A)$, the compatibility condition* (3.24)*, which is $d_*[X,Y] = \mathscr{L}_X d_* Y - \mathscr{L}_Y d_* X$, reads in local coordinates as follows.*

$$\begin{aligned}\rho_*^{ib}\frac{\partial L_{cd}{}^{e}}{\partial X^i} - \rho_*^{ie}\frac{\partial L_{cd}{}^{b}}{\partial X^i} + \rho_d^i\frac{\partial \widetilde{L}_c{}^{eb}}{\partial X^i} - \rho_c^i\frac{\partial \widetilde{L}_d{}^{eb}}{\partial X^i} + L_{cd}{}^{i}\widetilde{L}_i{}^{eb} \\ - L_{ci}{}^{b}\widetilde{L}_d{}^{ei} + L_{ci}{}^{e}\widetilde{L}_d{}^{bi} - L_{di}{}^{b}\widetilde{L}_c{}^{ie} + L_{di}{}^{e}\widetilde{L}_c{}^{ib} = 0. \end{aligned} \tag{4.26}$$

(Proof can be found in Appendix in Proof A.13.)

If $L_{cd}{}^{e}$ and $\widetilde{L}_c{}^{eb}$ are taken to be structure constants on a Lie algebra in above equation, then it yields the compatibility condition in Drinfel'd double which was given in the Proposition 4.1.1 as a second condition. Also, both of these structure constants must satisfy the Jacobi identities which are the same as the first and the third conditions in that Proposition. Therefore, a triangular Lie bialgebroid constructed above yields a Drinfel'd double structure when the structure functions are taken to be structure constants.

Remark 4.3.1. *As a special case, if we take $A = TM$ and when the local basis of sections are chosen to be coordinate basis then the structure functions $L_{cd}{}^{e}$ vanish since $S_a = \partial_a$. Also, Λ^{ac} will be the components of the bi-vector field and the anchors ρ_c^i will be the identity map, then the equation* (4.23) *reduces to*

$$\Lambda^{ai}\frac{\partial \Lambda^{be}}{\partial X^i} + \Lambda^{bi}\frac{\partial \Lambda^{ea}}{\partial X^i} + \Lambda^{ei}\frac{\partial \Lambda^{ab}}{\partial X^i} = 0,$$

which yields that Λ^{ai} is a Poisson tensor from the Lemma 3.1.4 in (3.21)*.*

4.4 Properties of the Courant Algebroid in Local Coordinates

The definition of the Courant algebroid has been given in the Definition 3.3.1. In this part of the thesis, our goal is to examine the local expressions correspond to the conditions of a Courant algebroid given in that definition. Let $E \to M$ be a Courant algebroid. If the dimension of the base manifold M is d, we will assume that the vector bundle will have rank $2d$ and the inner product will be of signature (d,d). In this case, it will be enough to regard that the vector bundle E is of this form: $E = R \oplus R^*$. The Courant algebroids constructed in this form are called split Courant algebroids [46]. So, we will assume that R and R^* do not have to be involutive subbundles of E.

For the sections of E, let us introduce a local basis for the sections $\Gamma(R \oplus R^*)$ as S^I where $I = (1,\ldots,2d)$ such that $S^I = (S_i, S^i)$ where S_i and S^i are local basis for $\Gamma(R)$ and $\Gamma(R^*)$, respectively with $i = (1,\ldots,d)$, such that [47]

$$\langle S^I, S^J \rangle = \frac{1}{2}\eta^{IJ}, \tag{4.27}$$

where

$$\eta = \begin{pmatrix} 0 & 1^d \\ 1^d & 0 \end{pmatrix} \tag{4.28}$$

is $O(d,d)$ metric and I,J are two-dimensional indices, and the Courant bracket is given by

$$[S^I, S^J] = \eta^{IK}\eta^{JL}T_{KLM}S^M, \tag{4.29}$$

where T_{KLM} is a generalized three-form.

Remark 4.4.1. *The generalized three-form T_{KLM} are the structure functions of the introduced bracket and it is anti-symmetric in all its indices and multilinear (trilinear).*

Proof:

We use the last condition in the Definition 3.3.1 of a Courant algebroid (C5) in order to show this fact.

$$\begin{aligned} 0 &= \langle [S^I, S^J], S^K \rangle + \langle S^J, [S^I, S^K] \rangle \\ &= \langle \eta^{IN}\eta^{JL}T_{NLM}S^M, S^K \rangle + \langle S^J, \eta^{IN}\eta^{KL}T_{NLM}S^M \rangle \\ &= \frac{1}{2}\eta^{IN}\eta^{JL}T_{NLM}\eta^{MK} + \frac{1}{2}\eta^{IN}\eta^{KL}T_{NLM}\eta^{JM} \\ &= \frac{1}{2}\eta^{IN}\eta^{JL}\eta^{MK}(T_{NLM} + T_{NML}) \\ &\Longrightarrow T_{NLM} = -T_{NML}, \end{aligned} \tag{4.30}$$

which means that it is skew-symmetric in last two indices.

Directly from the definition, we also obtain the following:

$$\begin{aligned}
&[S^J, S^I] = \eta^{JK}\eta^{IL}T_{KLM}S^M = -[S^I, S^J] = -\eta^{IK}\eta^{JL}T_{KLM}S^M \\
&\implies \eta^{JL}\eta^{IK}T_{LKM}S^M = -\eta^{IK}\eta^{JL}T_{KLM}S^M \\
&\implies \eta^{IK}\eta^{JL}(T_{LKM} + T_{KLM})S^M = 0 \\
&\implies T_{LKM} = -T_{KLM},
\end{aligned}$$

so it is anti-symmetric in first two indices and trilinearity follows from again the above property (4.30) together with the anti-symmetry of T_{KLM} shown above, as follows. For a smooth function f:

$$\begin{aligned}
&\langle [S^I, S^J], fS^K \rangle = -\langle S^J, [S^I, fS^K] \rangle \\
&\implies \langle f[S^I, S^J], S^K \rangle = -\langle [S^I, fS^K], S^J \rangle,
\end{aligned}$$

and by expanding the LHS, we obtain

$$\begin{aligned}
\langle f[S^I, S^J], S^K \rangle &= \frac{1}{2} f\eta^{IN}\eta^{JL}T_{NLM}\eta^{KM} \\
&= -\frac{1}{2} f\eta^{IN}\eta^{KL}T_{NLM}\eta^{JM} \\
&= -\langle f[S^I, S^K], S^J \rangle,
\end{aligned}$$

which completes the proof. ■

Some Local Expressions of the Notions in a Courant Algebroid

Since $\rho : E = R \oplus R^* \to TM$ is an anchor map, we have

$$\rho : S^I = (S^i, S_i) \to X^I = (X^i, X_i), \tag{4.31}$$

where $X^I \in \Gamma(TM)$. That is [47]

$$\rho(S^i) = X^i = X^{ki}\partial_k, \ \ where \ \ X^{ki} := \rho_*^{ki},$$

$$\rho(S_i) = X_i = X^k{}_i\partial_k, \ \ where \ \ X^k{}_i := \rho^k{}_i,$$

where we use the Einstein summation notation and it will be used in the next calculations as well. Then,

$$\rho(S^i)f = \rho_*^{ki}\partial_k f, \tag{4.32}$$

$$\rho(S_i)f = \rho^k{}_i \partial_k f, \tag{4.33}$$

for all $f \in C^\infty(M)$, or these equalities can be written in a compact way as follows.

$$\rho(S^I)f = \eta^{IJ}\rho^k_J \partial_k f, \tag{4.34}$$

where the components of the η matrix decompose as $\eta^i_j = \delta^i_j$, $\eta^j_i = \delta^j_i$, and the other terms vanish.

Also, we have the differential operator $D : C^\infty(M) \to \Gamma(E)$ and this yields in local coordinates as follows.

$$\begin{aligned}\langle Df, A\rangle &= \langle (D_I f S^I), A_J S^J\rangle \\ &= (D_I f) A_J \langle S^I, S^J\rangle \\ &= \frac{1}{2}(D_I f) A_J \eta^{IJ},\end{aligned}$$

where we have expanded $Df = (D_I f)S^I$ and $A = A_J S^J \in \Gamma(E)$. On the other hand,

$$\begin{aligned}\frac{1}{2}\rho(A)f &= \frac{1}{2}\rho(A_I S^I)f \\ &= \frac{1}{2}A_I \rho(S^I)f \\ &= \frac{1}{2}A_I \eta^{IJ}\rho^k_J \partial_k f,\end{aligned}$$

where we have used ρ is C^∞-linear in the second line and (4.34) in the last line. Since $\langle Df, A\rangle = \frac{1}{2}\rho(A)f$ as defined in the Definition 3.3.1, we obtain

$$D_I f = \rho^k_I \partial_k f, \tag{4.35}$$

which gives, from $Df = (D_I f)S^I$,

$$Df = \rho^k_I(\partial_k f)S^I = \rho^k{}_i(\partial_k f)S^i + \rho^{ki}_*(\partial_k f)S_i. \tag{4.36}$$

The above equation stands for the local expression of $Df = df + d_* f$ given in (3.52), where $d : C^\infty(M) \to \Gamma(R^*)$ and $d_* : C^\infty(M) \to \Gamma(R)$.

Theorem 4.4.1. *[47], [48] The axioms of a Courant algebroid given in the Definition 3.3.1 can be written as the following three independent properties in local coordinates*

using the local expressions given above and the Leibniz rule (C3) in the Definition.

$$
\begin{aligned}
&* \eta^{IJ}\rho_I^i\rho_J^j = \rho^i{}_k\rho_*^{jk} + \rho_*^{ik}\rho^j{}_k = 0,\\
&* \rho_I^i\partial_i\rho_J^j - \rho_J^i\partial_i\rho_I^j - \eta^{KL}\rho_K^j T_{LIJ} = 0,\\
&* 4\rho_{[L}^i\partial_i T_{IJK]} + 3\eta^{MN}T_{M[IJ}T_{KL]N} = 0.
\end{aligned}
\tag{4.37}
$$

(Proof can be found in Appendix in Proof A.14.)

Note that the anchors ρ_I^i in the second and the third conditions will be expanded as in the first equation which will be seen in future calculations in detail.

In the above Theorem (see the Proof), we have seen that five abstract axioms given in the Definition of a Courant algebroid correspond to only three local conditions. This fact comes from the followings: The Jacobiator identity, which is given as $C2)$ $Jac(A,B,C) = D(N_{ij}(A,B,C))$ in (3.34), yields to all these three local conditions with the help of the Leibniz Rule $C3)$ $[A,fB] = f[A,B] + (\rho(A)f)B - \langle A,B\rangle Df$. Besides, the other axioms $C4)$ $\rho\circ D = 0$, *i.e.*, $\langle Df,Dg\rangle = 0$ and $C1)$ $\rho([A,B]) = [\rho(A),\rho(B)]$ coincides with the above first and second local conditions, respectively. The left property which is $C5)$ $\rho(A)\langle B,C\rangle = \langle [A,B] + D\langle A,B\rangle,C\rangle + \langle B,[A,C] + D\langle A,C\rangle\rangle$ is satisfied automatically.

Theorem 4.4.2. *The properties of the Courant algebroid in terms of a Dorfman bracket given in* (3.37) *correspond to the three independent axioms in local coordinates same with* (4.37)*, as expected. (Proof can be found in Appendix in Proof A.15.)*

Remark 4.4.2. *One of the aim of the thesis is to demonstrate a close relationship between the DFT fluxes and the structure functions T_{KLM} in the bracket* (4.29) *in the local basis of a Courant algebroid which will be provided in Subsection 6.2.3. Namely, the conditions imposed by the axioms defining a Courant algebroid on the structure functions are equivalent to the conditions imposed on DFT fluxes by physical principles when the strong constraint, which will be introduced in* (6.7) *and* (6.8)*, is imposed. This fact can be seen in the Propositions 6.2.3 and 6.2.4. These structure functions T_{KLM} are called* ***fluxes*** *in physics literature and it can be identified with the four types of fluxes such as $f_{ij}{}^k$ (f-flux), H_{ijk} (H-flux), $Q^{ij}{}_k$ (Q-flux) and R^{ijk} (R-flux) (since T_{KLM}*

is totally anti-symmetric (see the Remark 4.4.1), only these type of fluxes arise) [49]. First two are called geometric and the rest are called non-geometric fluxes [50].

Remark 4.4.3. *We will investigate the axioms in* (4.37) *for the anchor* $\rho^i{}_J = (\rho^i{}_j, \rho^{ij}_*)$*, one by one since these results will be used in our studies.*

The first axiom can be written as follows.

$$\begin{aligned}
&\eta^{IJ}\rho^i_I\rho^j_J = 0\\
&\implies \eta^a_{\ b}\rho^i_a\rho^{jb}_* + \eta_a^{\ b}\rho^{ia}_*\rho^j_b = 0\\
&\implies \rho^i_a\rho^{ja}_* + \rho^{ia}_*\rho^j_a = 0,
\end{aligned} \tag{4.38}$$

where we have used (4.28) *in the last line.*

In the following two axioms, we will use the fact that T_{KLM} *is totally anti-symmetric from the Remark 4.4.1.*

The second axiom become the following.

$$\begin{aligned}
&\rho^i_I\partial_i\rho^j_J - \rho^i_J\partial_i\rho^j_I - \eta^{KL}\rho^j_K T_{LIJ} = 0\\
&\implies \rho^i_a\partial_i\rho^j_b - \rho^i_b\partial_i\rho^j_a - \eta^c_{\ d}\rho^j_c T^d_{\ ab} - \eta_c^{\ d}\rho^{jc}_* T_{dab}\\
&+ \rho^{ia}_*\partial_i\rho^{jb}_* - \rho^{ib}_*\partial_i\rho^{ja}_* - \eta^c_{\ d}\rho^j_c T^{dab} - \eta_c^{\ d}\rho^{jc}_* T_d^{\ ab}\\
&+ \rho^{ia}_*\partial_i\rho^j_b - \rho^i_b\partial_i\rho^{ja}_* - \eta^c_{\ d}\rho^j_c T^{da}_{\ \ b} - \eta_c^{\ d}\rho^{jc}_* T_d^{\ a}{}_b\\
&+ \rho^i_a\partial_i\rho^{jb}_* - \rho^{ib}_*\partial_i\rho^j_a - \eta^c_{\ d}\rho^j_c T^d_{\ a}{}^b - \eta_c^{\ d}\rho^{jc}_* T_{da}{}^b = 0\\
&\implies \rho^i_a\partial_i\rho^j_b - \rho^i_b\partial_i\rho^j_a - \rho^j_d f_{ab}{}^d - \rho^{jd}_* H_{dab}\\
&+ \rho^{ia}_*\partial_i\rho^{jb}_* - \rho^{ib}_*\partial_i\rho^{ja}_* - \rho^j_d R^{dab} - \rho^{jd}_* Q_d{}^{ab}\\
&+ \rho^{ia}_*\partial_i\rho^j_b - \rho^i_b\partial_i\rho^{ja}_* - \rho^j_d Q_b{}^{da} + \rho^{jd}_* f_{db}{}^a\\
&+ \rho^i_a\partial_i\rho^{jb}_* - \rho^{ib}_*\partial_i\rho^j_a + \rho^j_d Q_a{}^{db} - \rho^{jd}_* f_{da}{}^b = 0.
\end{aligned} \tag{4.39}$$

Lastly, the third condition can be expressed as follows.

$$
\begin{aligned}
&4\rho^i_{[L}\partial_i T_{IJK]} + 3\eta^{MN} T_{M[IJ} T_{KL]N} = 0 \\
&\implies \rho^i_L \frac{\partial T_{IJK}}{\partial X^i} - \rho^i_I \frac{\partial T_{JKL}}{\partial X^i} + \rho^i_J \frac{\partial T_{KLI}}{\partial X^i} - \rho^i_K \frac{\partial T_{LIJ}}{\partial X^i} \\
&+ \eta^{MN}(T_{MIJ}T_{KLN} + T_{MIK}T_{LJN} + T_{MIL}T_{JKN}) = 0 \\
&\implies \rho^i_d \partial_i H_{abc} - \rho^i_a \partial_i H_{bcd} + \rho^i_b \partial_i H_{cda} - \rho^i_c \partial_i H_{dab} + \rho^{id}_* \partial_i R^{abc} - \rho^{ia}_* \partial_i R^{bcd} + \rho^{ib}_* \partial_i R^{cda} \\
&- \rho^{ic}_* \partial_i R^{dab} + \rho^i_d \partial_i R^{abc} - \rho^{ia}_* \partial_i Q_d{}^{bc} - \rho^{ib}_* \partial_i Q_d{}^{ca} - \rho^{ic}_* \partial_i Q_d{}^{ab} + \rho^{id}_* \partial_i Q_a{}^{bc} - \rho^i_a \partial_i R^{bcd} \\
&+ \rho^{ib}_* \partial_i Q_a{}^{cd} + \rho^{ic}_* \partial_i Q_a{}^{db} - \rho^{id}_* \partial_i Q_b{}^{ac} - \rho^{ia}_* \partial_i Q_b{}^{cd} + \rho^i_b \partial_i R^{cda} - \rho^{ic}_* \partial_i Q_b{}^{da} + \rho^{id}_* \partial_i Q_c{}^{ab} \\
&+ \rho^{ia}_* \partial_i Q_c{}^{bd} + \rho^{ib}_* \partial_i Q_c{}^{da} - \rho^i_c \partial_i R^{dab} + \rho^i_d \partial_i Q_a{}^{bc} - \rho^i_a \partial_i Q_d{}^{bc} + \rho^{ib}_* \partial_i f_{da}{}^c - \rho^{ic}_* \partial_i f_{da}{}^b \\
&- \rho^i_d \partial_i Q_b{}^{ac} + \rho^{ia}_* \partial_i f_{bd}{}^c - \rho^i_b \partial_i Q_d{}^{ca} + \rho^{ic}_* \partial_i f_{db}{}^a + \rho^i_d \partial_i Q_c{}^{ab} - \rho^{ia}_* \partial_i f_{cd}{}^b + \rho^{ib}_* \partial_i f_{cd}{}^a \\
&- \rho^i_c \partial_i Q_d{}^{ab} + \rho^{id}_* \partial_i f_{ab}{}^c - \rho^i_a \partial_i Q_b{}^{cd} + \rho^i_b \partial_i Q_a{}^{cd} - \rho^{ic}_* \partial_i f_{ab}{}^d + \rho^{id}_* \partial_i f_{bc}{}^a - \rho^{ia}_* \partial_i f_{bc}{}^d \\
&+ \rho^i_b \partial_i Q_c{}^{da} - \rho^i_c \partial_i Q_b{}^{da} - \rho^{id}_* \partial_i f_{ac}{}^b + \rho^i_a \partial_i Q_c{}^{bd} - \rho^{ib}_* \partial_i f_{ca}{}^d + \rho^i_c \partial_i Q_a{}^{db} + \rho^i_d \partial_i f_{ab}{}^c \\
&+ \rho^i_a \partial_i f_{bd}{}^c + \rho^i_b \partial_i f_{da}{}^c - \rho^{ic}_* \partial_i H_{dab} + \rho^{id}_* \partial_i H_{abc} - \rho^i_a \partial_i f_{bc}{}^d - \rho^i_b \partial_i f_{ca}{}^d - \rho^i_c \partial_i f_{ab}{}^d \\
&+ \rho^i_d \partial_i f_{bc}{}^a - \rho^{ia}_* \partial_i H_{bcd} + \rho^i_b \partial_i f_{cd}{}^a + \rho^i_c \partial_i f_{db}{}^a - \rho^i_d \partial_i f_{ac}{}^b - \rho^i_a \partial_i f_{cd}{}^b + \rho^{ib}_* \partial_i H_{cda} \\
&- \rho^i_c \partial_i f_{da}{}^b + H_{mab} f_{cd}{}^m + H_{mac} f_{db}{}^m + H_{mad} f_{bc}{}^m + f_{ab}{}^m H_{cdm} + f_{ac}{}^m H_{dbm} + f_{ad}{}^m H_{bcm} \\
&+ Q_m{}^{ab} R^{cdm} + Q_m{}^{ac} R^{dbm} + Q_m{}^{ad} R^{bcm} + R^{mab} Q_m{}^{cd} + R^{mac} Q_m{}^{db} + R^{mad} Q_m{}^{bc} \\
&- Q_m{}^{ab} Q_d{}^{cm} + Q_m{}^{ac} Q_d{}^{bm} - f_{md}{}^a R^{bcm} + R^{mab} f_{dm}{}^c - R^{mac} f_{dm}{}^b + Q_d{}^{ma} Q_m{}^{bc} \\
&+ f_{ma}{}^b R^{cdm} + f_{ma}{}^c R^{dbm} + f_{ma}{}^d R^{bcm} - Q_a{}^{mb} Q_m{}^{cd} - Q_a{}^{mc} Q_m{}^{db} - Q_a{}^{md} Q_m{}^{bc} \\
&- f_{mb}{}^a R^{cdm} - Q_m{}^{ac} Q_b{}^{dm} + Q_m{}^{ad} Q_b{}^{cm} + Q_b{}^{ma} Q_m{}^{cd} + R^{mac} f_{bm}{}^d - R^{mad} f_{bm}{}^c \\
&+ Q_m{}^{ab} Q_c{}^{dm} - f_{mc}{}^a R^{dbm} - Q_m{}^{ad} Q_c{}^{bm} - R^{mab} f_{cm}{}^d + Q_c{}^{ma} Q_m{}^{db} + R^{mad} f_{cm}{}^b \\
&- f_{ma}{}^b Q_d{}^{cm} + f_{ma}{}^c Q_d{}^{bm} + H_{mad} R^{bcm} - Q_a{}^{mb} f_{dm}{}^c + Q_a{}^{mc} f_{dm}{}^b + f_{ad}{}^m Q_m{}^{bc} \\
&+ f_{mb}{}^a Q_d{}^{cm} + Q_m{}^{ac} f_{db}{}^m - f_{md}{}^a Q_b{}^{cm} + Q_b{}^{ma} f_{dm}{}^c + R^{mac} H_{dbm} - Q_d{}^{ma} f_{bm}{}^c \\
&+ Q_m{}^{ab} f_{cd}{}^m - f_{mc}{}^a Q_d{}^{bm} + f_{md}{}^a Q_c{}^{bm} + R^{mab} H_{cdm} - Q_c{}^{ma} f_{dm}{}^b + Q_d{}^{ma} f_{cm}{}^b \\
&+ H_{mab} R^{cdm} - f_{ma}{}^c Q_b{}^{dm} + f_{ma}{}^d Q_b{}^{cm} + f_{ab}{}^m Q_m{}^{cd} - Q_a{}^{mc} f_{bm}{}^d + Q_a{}^{md} f_{bm}{}^c \\
&- f_{mb}{}^a Q_c{}^{dm} + f_{mc}{}^a Q_b{}^{dm} + Q_m{}^{ad} f_{bc}{}^m - Q_b{}^{ma} f_{cm}{}^d + Q_c{}^{ma} f_{bm}{}^d + R^{mad} H_{bcm} \\
&+ f_{ma}{}^b Q_c{}^{dm} + H_{mac} R^{dbm} - f_{ma}{}^d Q_c{}^{bm} + Q_a{}^{mb} f_{cm}{}^d + f_{ac}{}^m Q_m{}^{db} - Q_a{}^{md} f_{cm}{}^b \\
&- H_{mab} Q_d{}^{cm} + f_{ma}{}^c f_{db}{}^m + H_{mad} Q_b{}^{cm} + f_{ab}{}^m f_{dm}{}^c - Q_a{}^{mc} H_{dbm} - f_{ad}{}^m f_{bm}{}^c \\
&+ H_{mab} Q_c{}^{dm} - H_{mac} Q_b{}^{dm} + f_{ma}{}^d f_{bc}{}^m - f_{ab}{}^m f_{cm}{}^d + f_{ac}{}^m f_{bm}{}^d - Q_a{}^{md} H_{bcm} \\
&- f_{mb}{}^a f_{cd}{}^m - f_{mc}{}^a f_{db}{}^m - f_{md}{}^a f_{bc}{}^m + Q_b{}^{ma} H_{cdm} + Q_c{}^{ma} H_{dbm} + Q_d{}^{ma} H_{bcm} \\
&+ f_{ma}{}^b f_{cd}{}^m + H_{mac} Q_d{}^{bm} - H_{mad} Q_c{}^{bm} - Q_a{}^{mb} H_{cdm} - f_{ac}{}^m f_{dm}{}^b + f_{ad}{}^m f_{cm}{}^b .
\end{aligned}
\tag{4.40}
$$

We will list the terms one by one according to the position of the indices in the last equation above in order to find the conditions on the fluxes.

Examining the terms whose all indices are lowered yields the following condition.

$$\rho_a^i \partial_i H_{bcd} - \rho_b^i \partial_i H_{cda} + \rho_c^i \partial_i H_{dab} - \rho_d^i \partial_i H_{abc} - H_{mab} f_{cd}{}^m + H_{mbc} f_{da}{}^m - H_{mcd} f_{ab}{}^m + H_{mda} f_{bc}{}^m + H_{mac} f_{db}{}^m + H_{mdb} f_{ac}{}^m = 0. \tag{4.41}$$

The terms whose only one index is raised, which is d, give rise to the following condition. The other terms in a,b,c yield the same condition.

$$\rho_a^i \partial_i f_{bc}{}^d + \rho_b^i \partial_i f_{ca}{}^d + \rho_c^i \partial_i f_{ab}{}^d - \rho_*^{id} \partial_i H_{abc} + f_{ab}{}^m f_{cm}{}^d + f_{bc}{}^m f_{am}{}^d + f_{ca}{}^m f_{bm}{}^d - Q_a{}^{dm} H_{bcm} - Q_b{}^{dm} H_{cam} - Q_c{}^{dm} H_{abm} = 0. \tag{4.42}$$

Now the terms whose two indices are raised, which are c,d, gives the following condition. When the other indices raised, these yield the same condition.

$$\rho_a^i \partial_i Q_b{}^{cd} - \rho_b^i \partial_i Q_a{}^{cd} + \rho_*^{ic} \partial_i f_{ab}{}^d - \rho_*^{id} \partial_i f_{ab}{}^c - f_{ab}{}^m Q_m{}^{cd} - H_{abm} R^{mcd} + Q_a{}^{mc} f_{bm}{}^d - Q_b{}^{mc} f_{am}{}^d - Q_a{}^{md} f_{bm}{}^c + Q_b{}^{md} f_{am}{}^c = 0. \tag{4.43}$$

The terms whose three indices are raised, which are a,b,c, yield the following condition. The terms in other raised indices yield the same condition.

$$\rho_*^{ia} \partial_i Q_d{}^{bc} + \rho_*^{ib} \partial_i Q_d{}^{ca} + \rho_*^{ic} \partial_i Q_d{}^{ab} - \rho_d^i \partial_i R^{abc} + Q_m{}^{ab} Q_d{}^{cm} + Q_m{}^{bc} Q_d{}^{am} + Q_m{}^{ca} Q_d{}^{bm} - f_{dm}{}^a R^{bcm} - f_{dm}{}^b R^{cam} - f_{dm}{}^c R^{abm} = 0. \tag{4.44}$$

Lastly, examining the terms whose all indices are raised yields the following condition.

$$\rho_*^{ia} \partial_i R^{bcd} - \rho_*^{ib} \partial_i R^{cda} + \rho_*^{ic} \partial_i R^{dab} - \rho_*^{id} \partial_i R^{abc} - R^{mab} Q_m{}^{cd} + R^{mbc} Q_m{}^{da} - R^{mcd} Q_m{}^{ab} + R^{mda} Q_m{}^{bc} + R^{dbm} Q_m{}^{ac} + R^{mac} Q_m{}^{db} = 0. \tag{4.45}$$

These conditions on fluxes are actually the Bianchi identities of DFT fluxes as will be discussed in the Proposition 6.2.3.

Remark 4.4.4. *If the vector bundle E in a Courant algebroid has the subbundles R and R^* which are taken to be also involutive as well as maximally isotropic and transversal, these become transversal Dirac subbundles of the vector bundle E and it has been seen in the Theorem 3.3.5 that these together construct a Lie bialgebroid. If R is taken to be an involutive subbundle of E, then from* (4.29) *we obtain the following for S_i which is a local basis for $\Gamma(R)$.*

$$[S_i, S_j] = \eta_i{}^l \eta_j{}^m T_{lm}{}^k S_k = T_{ij}{}^k S_k = f_{ij}{}^k S_k.$$

Also, $H_{ijk} = 0$ as required by the assumption of involutivity of R.

Similarly, if R^* is taken to be an involutive subbundle of E, (4.29) *yields the following for S^i which is a local basis for $\Gamma(R^*)$.*

$$[S^i, S^j] = \eta^i{}_l \eta^j{}_m T^{lm}{}_k S^k = T^{ij}{}_k S^k = Q^{ij}{}_k S^k = Q_k{}^{ij} S^k,$$

since T_{KLM} is anti-symmetric in all indices which can be seen in the Remark 4.4.1. Also, $R^{ijk} = 0$ as required by the assumption of involutivity of R^*.

For the mixed terms, we have

$$[S_i, S^j] = \eta_i{}^l \eta^j{}_m T_l{}^m{}_k S^k + \eta_i{}^l \eta^j{}_m T_l{}^{mk} S_k = -f'_{ik}{}^j S^k + Q'_i{}^{jk} S_k,$$

and similarly

$$[S^j, S_i] = f'_{ik}{}^j S^k - Q'_i{}^{jk} S_k = -[S_i, S^j].$$

By using (C5) in the Definition 3.3.1, we will show that the metric compatibility condition imposes that $f'_{ik}{}^j = f_{ik}{}^j$ and $Q'_i{}^{jk} = Q_i{}^{jk}$ as follows.

$$\begin{aligned}
&0 = \langle [S_i, S^j], S_k \rangle + \langle S^j, [S_i, S_k] \rangle \\
&\implies 0 = \langle -f'_{im}{}^j S^m + Q'_i{}^{jm} S_m, S_k \rangle + \langle S^j, f_{ik}{}^m S_m \rangle \\
&\implies 0 = -f'_{im}{}^j \eta^m{}_k + Q'_i{}^{jm} \eta_{mk} + f_{ik}{}^m \eta^j{}_m \\
&\implies f'_{ik}{}^j = f_{ik}{}^j,
\end{aligned}$$

and

$$
\begin{aligned}
&0 = \langle [S^i, S^j], S_k \rangle + \langle S^j, [S^i, S_k] \rangle \\
&\Longrightarrow 0 = \langle Q_m{}^{ij} S^m, S_k \rangle + \langle S^j, f'_{km}{}^i S^m - Q'_k{}^{im} S_m \rangle \\
&\Longrightarrow 0 = Q_m{}^{ij} \eta^m{}_k + f'_{km}{}^i \eta^{jm} - Q'_k{}^{im} \eta^j{}_m \\
&\Longrightarrow Q'_k{}^{ij} = Q_k{}^{ij}.
\end{aligned}
$$

Note that in the Subsection 4.3, which have explored the properties of triangular Lie bialgebroids in local coordinates, we have encountered only f- and Q-fluxes.

In the following Subsection, the sections of a Courant algebroid will be considered generally which means the fluxes will be considered as $H \neq 0$ and $R \neq 0$.

4.4.1 Courant bracket in local coordinates

Our aim now is to express the (general) Courant bracket in terms of local coordinates and to see also the relation of this expression with the standard Courant bracket in local coordinates.

Proposition 4.4.1. *The Courant bracket in local coordinates reads [51]*

$$
[A,B] = \left[\rho^l_J \left(A^J \partial_l B_K - B^J \partial_l A_K \right) - \frac{1}{2} \rho^l_K \left(A^J \partial_l B_J - B^J \partial_l A_J \right) \right] S^K + A^L B^M T_{LMK} S^K, \tag{4.46}
$$

where $A = A_I S^I$ and $B = B_J S^J$ are the elements of $\Gamma(E)$.

Proof:

The following property and its proof will be given firstly.

Lemma 4.4.1.

$$
[fA, gB] = fg[A,B] + f(\rho(A)g)B - g(\rho(B)f)A + \langle A,B \rangle (gDf - fDg), \tag{4.47}
$$

where A, B are only the local bases of $\Gamma(E)$ here.

Proof of the Lemma 4.4.1:

By using the condition (C3) in the definition of a Courant algebroid given in 3.3.1, we obtain

$$
\begin{aligned}
[fA,gB] &= g[fA,B] + (\rho(fA)g)B - \langle fA,B\rangle Dg \\
&= -g[B,fA] + (f\rho(A)g)B - f\langle A,B\rangle Dg \\
&= -g\left\{f[B,A] + (\rho(B)f)A - \langle B,A\rangle Df\right\} + (f\rho(A)g)B - f\langle A,B\rangle Dg \\
&= fg[A,B] - g(\rho(B)f)A + g\langle A,B\rangle Df + (f\rho(A)g)B - f\langle A,B\rangle Dg \\
&= fg[A,B] + (f\rho(A)g)B - g(\rho(B)f)A + \langle A,B\rangle(gDf - fDg),
\end{aligned}
$$

where we have used that Courant bracket is skew-symmetric in the second equality. □

Now, the expansion of the Courant bracket in local coordinates using (4.47) gives the following.

$$
\begin{aligned}
[A_I S^I, B_J S^J] &= A_I B_J [S^I, S^J] + A_I(\rho(S^I)B_J)S^J - B_J(\rho(S^J)A_I)S^I \\
&\quad + \langle S^I, S^J\rangle [B_J D(A_I) - A_I D(B_J)] \\
&= A_I B_J \eta^{IK}\eta^{JL} T_{KLM} S^M + A_I \eta^{IK}\rho^i_K(\partial_i B_J)S^J - B_J\eta^{JK}\rho^i_K(\partial_i A_I)S^I \\
&\quad + \frac{1}{2}\eta^{IJ}B_J\rho^i_K(\partial_i A_I)S^K - \frac{1}{2}\eta^{IJ}A_I\rho^i_K(\partial_i B_J)S^K \\
&= \left[\rho^l_J\left(A^J\partial_l B_K - B^J\partial_l A_K\right) - \frac{1}{2}\rho^l_K\left(A^J\partial_l B_J - B^J\partial_l A_J\right)\right]S^K + A^L B^M T_{LMK} S^K,
\end{aligned}
$$

where we have used (4.29), (4.34), (4.27) and (4.36) in the second equality. ■

The (general) Courant bracket in local coordinates given in (4.46) can be expanded as follows for the decomposed anchor $\rho^i{}_J = (\rho^i_j, \rho^{ij}_*)$ and $S^K = (S_k, S^k)$.

$$
\begin{aligned}
[A,B] &= \left[\rho^l_J\left(A^J\partial_l B_K - B^J\partial_l A_K\right) - \frac{1}{2}\rho^l_K\left(A^J\partial_l B_J - B^J\partial_l A_J\right)\right]S^K + A^L B^M T_{LMK} S^K \\
&= \rho^l_j A^j(\partial_l B_k)S^k + \rho^l_j A^j(\partial_l B^k)S_k + \rho^{lj}_* A_j(\partial_l B_k)S^k + \rho^{lj}_* A_j(\partial_l B^k)S_k - \rho^l_j B^j(\partial_l A_k)S^k \\
&\quad - \rho^l_j B^j(\partial_l A^k)S_k - \rho^{lj}_* B_j(\partial_l A_k)S^k - \rho^{lj}_* B_j(\partial_l A^k)S_k - \frac{1}{2}\rho^l_k A^j(\partial_l B_j)S^k \\
&\quad - \frac{1}{2}\rho^l_k A_j(\partial_l B^j)S^k - \frac{1}{2}\rho^{lk}_* A^j(\partial_l B_j)S_k - \frac{1}{2}\rho^{lk}_* A_j(\partial_l B^j)S_k + \frac{1}{2}\rho^l_k B^j(\partial_l A_j)S^k \\
&\quad + \frac{1}{2}\rho^l_k B_j(\partial_l A^j)S^k + \frac{1}{2}\rho^{lk}_* B^j(\partial_l A_j)S_k + \frac{1}{2}\rho^{lk}_* B_j(\partial_l A^j)S_k + A^L B^M T_{LMK} S^K,
\end{aligned}
\tag{4.48}
$$

where the last term is left the same.

The (general) Courant bracket given in (4.46) can be also written as follows for the anchors $\rho^i{}_J = (\rho^i_j, \rho^{ij}_*)$ and for $A = A_V + A_F \in \Gamma(E)$ with $A_V \in \Gamma(TM)$ and $A_F \in \Gamma(T^*M)$ and $B = B_V + B_F \in \Gamma(E)$ with $B_V \in \Gamma(TM)$ and $B_F \in \Gamma(T^*M)$ [51].

$$\begin{aligned}[A,B] &= [A_V, B_V] + \mathscr{L}_{A_V} B_F - \mathscr{L}_{B_V} A_F - \frac{1}{2} d\left(i_{A_V} B_F - i_{B_V} A_F\right) \\ &\quad + [A_F, B_F] + \mathscr{L}_{A_F} B_V - \mathscr{L}_{B_F} A_V + \frac{1}{2} d_*\left(i_{A_V} B_F - i_{B_V} A_F\right) + T(A,B), \end{aligned} \tag{4.49}$$

which has been introduced in (3.50) in terms of the direct sum of Lie algebroids (L, L^*) (which was actually a Lie bialgebroid structure). In above, $T(A,B) := A_i B_j T^{ijk} S_k + A^i B^j T_{ijk} S^k$, and

$$[A_V, B_V] = (\rho^l_j A^j \partial_l B^k - \rho^l_j B^j \partial_l A^k + A^i B^j T_{ij}{}^k) S_k, \tag{4.50}$$

$$\mathscr{L}_{A_V} B_F = (\rho^l_j A^j \partial_l B_k + \rho^l_k B_j \partial_l A^j + A^j B_i T_{kj}{}^i) S^k, \tag{4.51}$$

$$d(i_{A_V} B_F) = (\rho^l_k A^j \partial_l B_j + \rho^l_k B_j \partial_l A^j) S^k, \tag{4.52}$$

$$[A_F, B_F] = (\rho^{lj}_* A_l \partial_j B_k - \rho^{lj}_* B_l \partial_j A_k + A_i B_j T_k{}^{ji}) S^k, \tag{4.53}$$

$$\mathscr{L}_{A_F} B_V = (\rho^{lj}_* A_j \partial_l B^k + \rho^{lk}_* B^j \partial_l A_j + A_j B^i T_i{}^{kj}) S_k, \tag{4.54}$$

$$d_*(i_{A_V} B_F) = (\rho^{lk}_* B_j \partial_l A^j + \rho^{lk}_* A^j \partial_l B_j) S_k. \tag{4.55}$$

These operators have been given in the Subsection 4.2 in calculus on Lie algebroids. Also, the bracket on 1-forms in (4.53) has been introduced in (3.22) which is the Koszul bracket. Note that the derivatives of the Poisson bi-vector tensor are hidden in the Q-fluxes. This bracket can be followed from the Proposition 4.4.2 in detail.

Note that by substituting these defined operators in (4.49), we obtain the Courant bracket's expanded version given in (4.48) if and only if $T(A,B) := A_i B_j T^{ijk} S_k + A^i B^j T_{ijk} S^k$.

Also note that (4.50) can be found from (4.47) for $A = A^i S_i$ and $B = B^j S_j$ in that equation since $[S_i, S_j] = 0$ and $\langle S_i, S_j \rangle = 0$. Similarly, (4.53) can be found also from (4.47) for $A = A_i S^i$ and $B = B_j S^j$ in that equation since $[S^i, S^j] = 0$ and $\langle S^i, S^j \rangle = 0$. Additionally, (4.51) and (4.54) are the same as (4.16) for $T_{kj}{}^i = L_{kj}{}^i$ and (4.17) for $T^{kj}{}_i = \widetilde{L}^{kj}{}_i$, respectively, as expected. The other operators will behave in the same way

with the previous ones which have not been written in local coordinates.
The following examples of a Courant algebroid will be important for further studies.

Example 4.4.1. *The special case of a Courant algebroid can be given by the standard Courant algebroid which is firstly demonstrated in the Example 3.3.1. Standard Courant algebroid consists of the anchor $\rho : E \to TM$ which is the projection to the tangent bundle, the metric which is induced by the natural pairing between TM and T^*M, and the map $D : C^\infty(M) \to \Gamma(E)$ that is given by $Df = df$ where d is the exterior differential operator on the tangent bundle of the manifold. In that case, we have $\rho^i{}_J = (\delta^i_j, 0)$ and by substituting this together with $S^K = (\partial_k, dx^k)$ in* (4.50) - (4.55)*, the bracket* (4.49) *reads [51]*

$$\begin{aligned}[A,B]_s =&(A^l\partial_l B^k - B^l\partial_l A^k)\partial_k + \Big(A^l\partial_l B_k - B^l\partial_l A_k - \frac{1}{2}A^l\partial_k B_l + \frac{1}{2}B^l\partial_k A_l \\ &-\frac{1}{2}A_l\partial_k B^l + \frac{1}{2}B_l\partial_k A^l + A^l B^m H_{lmk}\Big)\, dx^k, \end{aligned} \tag{4.56}$$

which is called H-twisted Courant bracket and introduced in (3.44).

Note that only H-flux arises in the above equality. The reason of this fact will be given in the Proposition 4.4.3.

Example 4.4.2. *If we take the Lie bracket on the sections of TM as zero bracket and zero anchor (trivial ones), then it constructs a Lie algebroid structure. On the other hand, if we take the bracket on the sections of T^*M as the Koszul bracket together with the non-standard anchor on T^*M as the Poisson bi-vector field, then this constructs a Lie algebroid structure from the Example 3.1.4. These structures satisfy the compatibility condition given in* (3.23) *or* (3.24) *automatically. Therefore, there exists a Lie bialgebroid structure. On the direct sum of these structures, it can be constructed a Courant algebroid from the Theorem 3.3.4.*

In order to find the bracket on $TM \oplus T^*M$, we will use the operators (4.50) - (4.55) *for the anchor $\rho^i{}_J = (0, \Pi^{ij})$ and $S^K = (\partial_k, dx^k)$. Then* (4.49) *turns into the following.*

$$\begin{aligned}[A,B]_k = &\left(\Pi^{lj} A_j \partial_l B^k + \frac{1}{2}\Pi^{lk} B^j \partial_l A_j - \Pi^{lj} B_j \partial_l A^k - \frac{1}{2}\Pi^{lk} A^j \partial_l B_j + \frac{1}{2}\Pi^{lk} B_j \partial_l A^j \right. \\ &\left. -\frac{1}{2}\Pi^{lk} A_j \partial_l B^j \right) \partial_k + \Pi^{lj}(A_l \partial_j B_k - B_l \partial_j A_k) dx^k + A_i B_j Q_k{}^{ji} dx^k \\ &+ A_j B^i Q_i{}^{kj} \partial_k - B_j A^i Q_i{}^{kj} \partial_k + A_i B_j R^{ijk} \partial_k .\end{aligned}$$

In above example, the fluxes H and f are taken to be zero which will be proved in the Lemma 4.4.2. Also, the Q- flux can be written as $Q_i{}^{jk} = \partial_i \Pi^{jk}$ from this Lemma.

Proposition 4.4.2. *The bracket on* 1*-forms in* (4.53) *gives the following for $\rho^i{}_J = (0, \Pi^{ij})$ and $S^K = (\partial_k, dx^k)$.*

$$[A_F, B_F]_k = \Pi^{ij}(A_i \partial_j B_m - B_i \partial_j A_m) dx^m + A_j B_i (\partial_m \Pi^{ji}) dx^m, \tag{4.57}$$

which is the same expression with the Koszul bracket given in (3.22)*.*

Proof: The Koszul bracket (3.22) can be expanded in local bases as follows.

$$\begin{aligned}&[A_i dx^i, B_j dx^j]_* = B_j [A_i dx^i, dx^j]_* + \mathscr{L}_{\Lambda^\sharp(A_i dx^i)}(B_j dx^j) \\ &= -B_j \left(A_i [dx^j, dx^i]_* + \mathscr{L}_{\Lambda^\sharp(dx^j)}(A_i dx^i) \right) + B_j \mathscr{L}_{\Lambda^\sharp(A_i dx^i)}(dx^j) + \rho(\Lambda^\sharp(A_i dx^i))(B_j) dx^j \\ &= -B_j \left(A_i (\mathscr{L}_{\Lambda^\sharp(dx^j)}(dx^i) - \mathscr{L}_{\Lambda^\sharp(dx^i)}(dx^j) - d(\Pi(dx^j, dx^i)) + \mathscr{L}_{\Lambda^\sharp(dx^j)}(A_i dx^i) \right) \\ &+ B_j A_i \mathscr{L}_{\Lambda^\sharp(dx^i)}(dx^j) + A_i \Pi^{ik}(\partial_k B_j) dx^j \\ &= A_i B_j \partial_k \Pi^{ji} dx^k - B_j \Pi^{jk}(\partial_k A_i) dx^i + A_i \Pi^{ik}(\partial_k B_j) dx^j ,\end{aligned}$$

where we have used the Leibniz rule for the Koszul bracket in the first line, Leibniz rule again and (3.11) in the second line, (3.22) and (3.12) in the third line, and (3.10), (3.7) in the last one. This result ends the proof. ■

Example 4.4.3. *If we take the usual Lie bracket on the sections of TM and the identity anchor on TM, it constructs a Lie algebroid structure as we discussed before. There also exists a Lie algebroid structure on T^*M equipped with the Koszul bracket on the sections of T^*M and the non-standard anchor on T^*M as the Poisson bi-vector*

field which was also discussed before. These together constructs a triangular Lie bialgebroid structure which is demonstrated generally in the Subsection 3.2.1 in detail. Therefore, a Courant algebroid structure can be constructed on the direct sum of these structures from the Theorem 3.3.4.

In order to determine the bracket on $TM \oplus T^*M$, we will use the operators (4.50) - (4.55) *for the anchor $\rho^i{}_J = (\delta^i{}_j, \Pi^{ij})$ and $S^K = (\partial_k, dx^k)$. Then* (4.49) *becomes*

$$
\begin{aligned}
[A,B]_{TM\oplus T^*M} &= (A^l\partial_l B^k - B^l\partial_l A^k + \Pi^{lj}A_j\partial_l B^k + \frac{1}{2}\Pi^{lk}B^j\partial_l A_j - \Pi^{lj}B_j\partial_l A^k \\
&- \frac{1}{2}\Pi^{lk}A^j\partial_l B_j + \frac{1}{2}\Pi^{lk}B_j\partial_l A^j - \frac{1}{2}\Pi^{lk}A_j\partial_l B^j)\partial_k + (A^l\partial_l B_k + \frac{1}{2}B_j\partial_k A^j \\
&- B^l\partial_l A_k - \frac{1}{2}A_j\partial_k B^j - \frac{1}{2}A^j\partial_k B_j + \frac{1}{2}B^j\partial_k A_j + \Pi^{lj}A_l\partial_j B_k - \Pi^{lj}B_l\partial_j A_k)dx^k \\
&+ A_iB_jQ_k{}^{ij}dx^k + A_jB^iQ_i{}^{kj}\partial_k - B_jA^iQ_i{}^{kj}\partial_k,
\end{aligned}
$$

where the fluxes H, f and R are taken as zero from the Lemma 4.4.3. Also, the Q-flux can be written as $Q_i{}^{jk} = \partial_i\Pi^{jk}$ from this Lemma. Note that the derivatives of the Poisson bi-vector field are only contained in the fluxes, specifically in Q-fluxes.

Above example demonstrates the Courant algebroid bracket on the direct sum of the tangent and cotangent bundles. Although the tangent bundle is taken to be as usual with the identity anchor and the standard Lie bracket, the cotangent bundle is not taken to be trivial. Instead, the bracket on the sections of cotangent bundle is a Koszul bracket in this case. So these together yield the above non-standard Courant bracket on $TM \oplus T^*M$. This case will play a prominent role in further studies, in the Subsection6.4, in such a way that this Courant algebroid structure is the algebroid structure underlying the solution generating mechanism.

We will now investigate Courant algebroids in local coordinates under some particular conditions.

Example 4.4.4. *A Courant algebroid over $M = \{pt\}$, a point, is just a quadratic Lie algebra, i.e. a Lie algebra g with an invariant non-degenerate symmetric bilinear form.*

Proof: Since the base manifold is a point, the anchor is the trivial one that is $\rho = 0$ and that gives $Df = 0$. Also, one has the structure constants, instead of structure functions, of the Lie algebra. Therefore, the conditions of being a Courant algebroid, which are given in the Definition 3.3.1 turn into the followings.

(C1) and (C4) is satisfied identically. Also, (C2) become the following for the elements S_i of the manifold.

$$\begin{aligned} Jac(S_i,S_j,S_k) &= [[S_i,S_j],S_k] + [[S_j,S_k],S_i] + [[S_k,S_i],S_j] = 0 \\ &\Longrightarrow (f_{ij}{}^m f_{mk}{}^l + f_{jk}{}^m f_{mi}{}^l + f_{ki}{}^m f_{mj}{}^l)S_l = 0, \end{aligned}$$

which gives

$$f_{ij}{}^m f_{mk}{}^l + f_{jk}{}^m f_{mi}{}^l + f_{ki}{}^m f_{mj}{}^l = 0, \tag{4.58}$$

for every $S_l \in M$, which is one of the conditions for L to be a Lie algebra. Besides, (C3) becomes $[S_i, fS_j] = f[S_i,S_j]$. Lastly, (C5) becomes

$$0 = \langle [S_i,S_j],S_k\rangle + \langle S_j,[S_i,S_k]\rangle,$$

which is a condition for being an invariant inner product.

As a result, a Courant algebroid over a point is just a quadratic Lie algebra. ■

Example 4.4.5. *Let $E = L \oplus L^*$ as in the Theorem 3.3.5. If the base manifold of E is a point, then the conditions on structure constants (see the Remark 4.4.4) become exactly the same conditions for a Manin triple structure (see the Proposition 4.1.1).*

Proof: Let us assume that L, L^* are transversal Dirac subbundles in a Courant algebroid. Then one could use the definitions in local coordinates given in Subsection 4.4. Since base manifold is a point, we will have again structure constants instead of structure functions. One needs to work on the Courant bracket before looking at the requirements for being a Courant algebroid as stated in the Definition 3.3.1.

The brackets on the sections of L, L^* for the elements S_i, S^i, respectively, will be the same as discussed in the Remark 4.4.4 such that

$$[S_i, S_j] = f_{ij}{}^{k} S_k,$$
$$[S^i, S^j] = Q_k{}^{ij} S^k,$$
$$[S_i, S^j] = -f_{ik}{}^{j} S^k + Q_i{}^{jk} S_k = -[S^j, S_i].$$

Here, $f_{ij}{}^{k}$ and $Q_k{}^{ij}$ are now the structure constants of the Lie algebra and its dual Lie algebra, respectively, since we take the base manifold as a point.

Now is the time to look at each condition of a Courant algebroid given in the Definition 3.3.1 in local coordinates individually. Firstly, (C1) and (C4) is satisfied identically. Also, (C2) yields the same result found in (4.58) which is one of the conditions for L to be a Lie algebra.

Similary,

$$Jac(S^i, S^j, S^k) = (Q_m{}^{ij} Q_l{}^{mk} + Q_m{}^{jk} Q_l{}^{mi} + Q_m{}^{ki} Q_l{}^{mj}) S^l = 0,$$

which gives

$$Q_m{}^{ij} Q_l{}^{mk} + Q_m{}^{jk} Q_l{}^{mi} + Q_m{}^{ki} Q_l{}^{mj} = 0,$$

which is one of the conditions for L^* to be a Lie algebra.

From the mixed sections, we obtain

$$\begin{aligned}
&Jac(S^i, S_j, S_k) = [[S^i, S_j], S_k] + [[S_j, S_k], S^i] + [[S_k, S^i], S_j] = 0 \\
&\implies [f_{jm}{}^{i} S^m - Q_j{}^{im} S_m, S_k] + [f_{jk}{}^{m} S_m, S^i] + [-f_{km}{}^{i} S^m + Q_k{}^{im} S_m, S_j] = 0 \\
&\implies -f_{jm}{}^{i} [S^m, S_k] + Q_j{}^{im} [S_m, S_k] - f_{jk}{}^{m} [S_m, S^i] + f_{km}{}^{i} [S^m, S_j] - Q_k{}^{im} [S_m, S_j] = 0 \\
&\implies -f_{jm}{}^{i} (f_{kl}{}^{m} S^l - Q_k{}^{ml} S_l) + Q_j{}^{im} f_{mk}{}^{l} S_l - f_{jk}{}^{m} (-f_{ml}{}^{i} S^l + Q_m{}^{il} S_l) \\
&+ f_{km}{}^{i} (f_{jl}{}^{m} S^l - Q_j{}^{ml} S_l) - Q_k{}^{im} f_{mj}{}^{l} S_l = 0 \\
&\implies (-f^i_{jm} f^m_{kl} + f_{jk}{}^{m} f_{ml}{}^{i} + f_{km}{}^{i} f_{jl}{}^{m}) S^l \\
&+ (f_{jm}{}^{i} Q_k{}^{ml} + Q_j{}^{im} f_{mk}{}^{l} - f_{jk}{}^{m} Q_m{}^{il} - f^i_{km} Q^{ml}_j - Q^{im}_k f^l_{mj}) S_l = 0
\end{aligned} \tag{4.59}$$

then we get (4.58) which is the Jacobi Identity for L.

Also, we obtain

$$f_{jm}{}^{i}Q_{k}^{ml}+Q_{j}^{im}f_{mk}{}^{l}-f_{jk}{}^{m}Q_{m}^{il}-f_{km}^{i}Q_{j}^{ml}-Q_{k}^{im}f_{mj}^{l}=0,$$

which is the compatibility condition between L and L^* given in Proposition 4.1.1 and this yields a Lie bialgebra structure (L,L^*). Note that we have used the condition (C3), which becomes $[A,fB]=[fA,B]=f[A,B]$, in above calculations. Also note that the other combinations of the sections will give similar kind of results. (In some of them, we have Jacobi Identity for L^* together with the compatibility condition again.)

As a result, when the base manifold is taken a point and if there exist transversal Dirac structures in a Courant algebroid, then there is a Drinfel'd double structure and so a Manin triple structure. ■

4.4.2 Triangular Lie bialgebroid yields a Courant algebroid with transversal Dirac structures in local coordinates

The correspondence between (general) Lie bialgebroids and Courant algebroids with transversal Dirac structures has been given in the Theorems 3.3.4 and 3.3.5. Our aim in this part of the thesis is to verify that for a given triangular Lie bialgebroid, which is firstly introduced in Subsection 3.2.1 and whose local expressions have been given in Subsection 4.3, the axioms to be a Courant algebroid with transversal Dirac structures are satisfied in local coordinates.
The Subsection 4.2 that covers Lie algebroid's local expressions will be used in this part.

Theorem 4.4.3. *Let us assume that there is a triangular Lie bialgebroid structure* $(A\oplus A^*,[\cdot,\cdot]_{A\oplus A^*},\langle\cdot,\cdot\rangle_{A\oplus A^*},\rho_{A\oplus A^*})$*, which is constructed by using the bundle map* $\Lambda^\sharp:A^*\to A$ *defined by* $\Lambda^\sharp(\xi)(\eta)=\Lambda(\xi,\eta)$ *for all* $\xi,\eta\in\Gamma(A^*)$ *satisfying some conditions given in the Subsection 3.2.1. Also, let us have a new bracket on the smooth sections of the dual vector bundle* A^* *given in* (3.25) *together with the induced differential operator* $d_*:\Gamma(A)\to\Gamma(^2A)$ *that is* $d_*X=[\Lambda,X]_{SN}$*. Then, this structure yields a Courant algebroid with transversal Dirac structures. In other words, the three*

local properties given in (4.37) *are satisfied in terms of this structure of a triangular Lie bialgebroid.*

Proof: Let $E = A \oplus A^*$ where (A, A^*) is a triangular Lie bialgebroid. Then we assume the followings: A is a Lie algebroid that satisfies (4.1), (4.2), (4.3) and (4.5) or (4.4) and A^* is a Lie algebroid that satisfies (4.8), (4.7), (4.22) and (4.25). Besides, there is a natural inner product between these Lie algebroids given in (4.6) and they fulfill (4.16) and (4.17). The compatibility condition between A and A^* given in (4.26) is also satisfied. By using the bundle map, which satisfies (4.9), we assume also that the introduced new bracket on the smooth sections of the dual vector bundle A^* satisfies (4.20). We define the following skew-symmetric bracket operation on $\Gamma(A \oplus A^*)$ as given in (3.50):

$$\begin{aligned}[S_a + S^a, S_b + S^b]_{A\oplus A^*} &= [S_a, S_b]_A + \mathscr{L}_{S_a} S^b - \mathscr{L}_{S_b} S^a + \frac{1}{2} d(i_{S_b} S^a - i_{S_a} S^b) \\ &\quad + [S^a, S^b]_{A^*} + \mathscr{L}_{S^a} S_b - \mathscr{L}_{S^b} S_a - \frac{1}{2} d_*(i_{S^a} S_b - i_{S^b} S_a),\end{aligned}$$

where $S_a, S_b \in \Gamma(A)$ and $S^a, S^b \in \Gamma(A^*)$. Therefore,

$$[S_a, S_b]_{A\oplus A^*} = [S_a, S_b]_A = L_{ab}{}^c S_c, \tag{4.60}$$

and

$$[S^a, S^b]_{A\oplus A^*} = [S^a, S^b]_{A^*} = \widetilde{L}^{ab}{}_c S^c. \tag{4.61}$$

For the mixed terms, we have

$$\begin{aligned}[S_a, S^b]_{A\oplus A^*} &= \mathscr{L}_{S_a} S^b - \mathscr{L}_{S^b} S_a \\ &= -L_{ac}{}^b S^c + \widetilde{L}^{bc}{}_a S_c,\end{aligned} \tag{4.62}$$

and

$$\begin{aligned}[S^a, S_b]_{A\oplus A^*} &= -\mathscr{L}_{S_b} S^a + \mathscr{L}_{S^a} S_b \\ &= L_{bc}{}^a S^c - \widetilde{L}^{ac}{}_b S_c.\end{aligned} \tag{4.63}$$

Since T_{KLM} are the structure functions of the Courant bracket (4.29), we need to comprehend the relation between these and the structure functions L and $\widetilde{L}$ of Lie algebroids A and A^*, respectively. Hereby,

$$[S_a, S_b] = T_{abc} S^c + T_{ab}{}^c S_c,$$

which gives from (4.60): $T_{ab}{}^{c} = L_{ab}{}^{c}$ and $T_{abc} = 0$, and

$$[S^a, S^b] = T^{abc}S_c + T^{ab}{}_c S^c,$$

which gives from (4.61): $T^{ab}{}_c = \widetilde{L}^{ab}{}_c$ and $T^{abc} = 0$, and

$$[S_a, S^b] = T_a{}^b{}_c S^c + T_a{}^{bc} S_c,$$

which gives from (4.62): $T_a{}^b{}_c = -L_{ac}{}^b S^c$ and $T_a{}^{bc} = \widetilde{L}^{bc}{}_a$, and

$$[S^a, S_b] = T^a{}_{bc} S^c + T^a{}_b{}^c S_c,$$

which gives from (4.63): $T^a{}_{bc} = L_{bc}{}^a$ and $T^a{}_b{}^c = -\widetilde{L}^{ac}{}_b$. In above calculations, we have used the facts $\eta^{ab} = \eta_{ab} = 0$ and these will be used in the next calculations together with the above results.

We are ready now to prove that the axioms to be a Courant algebroid given in (4.37) are satisfied in terms of this triangular Lie bialgebroid structure one by one. Note that since the structure functions satisfy $T_{abc} = T^{abc} = 0$, which means that the fluxes H and R vanish, then we have transversal Dirac structures from the Remark 4.4.4.

1) We would like to show now the first axiom given in (4.37): $\eta^{IJ}\rho_I^i\rho_J^j = 0$ is satisfied by the constructed triangular Lie bialgebroid. For this purpose, we will use (4.38) and obtain the following.

$$\begin{aligned}\eta^{IJ}\rho_I^i\rho_J^j &= \eta_b^a\rho_a^i\rho_*^{jb} + \eta_a^b\rho_*^{ia}\rho_b^j \\ &= \rho_a^i\Lambda^{ac}\rho_c^j + \rho_c^i\Lambda^{ac}\rho_a^j \\ &= 0,\end{aligned}$$

where we have used the facts: (4.9) and Λ^{ac} is skew-symmetric. Hence, it has been proven that the first condition for being a Courant algebroid is satisfied in terms of the constructed triangular Lie bialgebroid (A, A^*).

2) Our next goal is to prove that the second axiom in (4.37), that is

$$\rho_I^i\partial_i\rho_J^j - \rho_J^i\partial_i\rho_I^j - \eta^{KL}\rho_K^j T_{LIJ} = 0,$$

holds for this triangular Lie bialgebroid. From (4.39), we have

$$
\begin{aligned}
&\rho^i_I\partial_i\rho^j_J-\rho^i_J\partial_i\rho^j_I-\eta^{KL}\rho^j_K T_{LIJ}\\
&=\rho^i_a\partial_i\rho^j_b-\rho^i_b\partial_i\rho^j_a-\rho^j_d L_{ab}{}^d+\rho^{ia}_*\partial_i\rho^{jb}_*-\rho^{ib}_*\partial_i\rho^{ja}_*-\rho^{jd}_*\widetilde{L}^{ab}{}_d\\
&+\Lambda^{ac}\rho^i_c\partial_i\rho^j_b-\rho^i_b(\partial_i\Lambda^{ac})\rho^j_c-\rho^i_b(\partial_i\rho^j_c)\Lambda^{ac}-\rho^j_d\widetilde{L}^{da}{}_b+\Lambda^{dc}\rho^j_c L_{db}{}^a\\
&+\rho^i_a(\partial_i\Lambda^{bc})\rho^j_c+\rho^i_a(\partial_i\rho^j_c)\Lambda^{bc}-\Lambda^{bc}\rho^i_c\partial_i\rho^j_a+\rho^j_d\widetilde{L}^{db}{}_a-\Lambda^{dc}\rho^j_c L_{da}{}^b\\
&=\Lambda^{ac}\rho^i_c\partial_i\rho^j_b-\rho^i_b(\partial_i\Lambda^{ac})\rho^j_c-\rho^i_b(\partial_i\rho^j_c)\Lambda^{ac}-\rho^j_d\widetilde{L}^{da}{}_b+\Lambda^{dc}\rho^j_c L_{db}{}^a\\
&+\rho^i_a(\partial_i\Lambda^{bc})\rho^j_c+\rho^i_a(\partial_i\rho^j_c)\Lambda^{bc}-\Lambda^{bc}\rho^i_c\partial_i\rho^j_a+\rho^j_d\widetilde{L}^{db}{}_a-\Lambda^{dc}\rho^j_c L_{da}{}^b\\
&=\Lambda^{ac}\rho^i_c\partial_i\rho^j_b-\rho^i_b(\partial_i\Lambda^{ac})\rho^j_c-\rho^i_b(\partial_i\rho^j_c)\Lambda^{ac}-\rho^j_d L_{cb}{}^d\Lambda^{ac}+\rho^j_d L_{cb}{}^a\Lambda^{dc}\\
&-\rho^j_d(\partial_i\Lambda^{da})\rho^i_b+\Lambda^{dc}\rho^j_c L_{db}{}^a+\rho^i_a(\partial_i\Lambda^{bc})\rho^j_c+\rho^i_a(\partial_i\rho^j_c)\Lambda^{bc}-\Lambda^{bc}\rho^i_c\partial_i\rho^j_a\\
&+\rho^j_d L_{ca}{}^d\Lambda^{bc}-\rho^j_d L_{ca}{}^b\Lambda^{dc}+\rho^j_d(\partial_i\Lambda^{db})\rho^i_a-\Lambda^{dc}\rho^j_c L_{da}{}^b\\
&=-\rho^i_b(\partial_i\Lambda^{ac})\rho^j_c+\rho^j_d L_{cb}{}^a\Lambda^{dc}-\rho^j_d(\partial_i\Lambda^{da})\rho^i_b+\Lambda^{dc}\rho^j_c L_{db}{}^a+\rho^i_a(\partial_i\Lambda^{bc})\rho^j_c\\
&-\rho^j_d L_{ca}{}^b\Lambda^{dc}+\rho^j_d(\partial_i\Lambda^{db})\rho^i_a-\Lambda^{dc}\rho^j_c L_{da}{}^b\\
&=0,
\end{aligned}
$$

where we have used (4.9) in the second equality, (4.3) and (4.22) in the third equality, due to the facts that A and A^* are Lie algebroids. Also, (4.20) has been used in the fourth equality and (4.3) has been used in the fifth equality. Lastly, it has been used that Λ^{ca} is skew-symmetric in the last equality and the terms cancelled each other identically.

Therefore, it has been proven that the second condition for being a Courant algebroid is satisfied in terms of the constructed Lie bialgebroid (A,A^*).

3) Finally, the last equality (4.37) should be satisfied by the constructed Lie bialgebroid which is $4\rho^i_{[L}\partial_i T_{IJK]}+3\eta^{MN}T_{M[IJ}T_{KL]N}=0$. When only K is raised, the LHS of this equation yields to the following from (4.42).

$$
\begin{aligned}
&\rho^i_a\frac{\partial L_{bc}{}^d}{\partial X^i}+\rho^i_b\frac{\partial L_{ca}{}^d}{\partial X^i}+\rho^i_c\frac{\partial L_{ab}{}^d}{\partial X^i}+L_{eb}{}^dL_{ac}{}^e+L_{bc}{}^eL_{ae}{}^d-L_{ba}{}^eL_{ce}{}^d\\
&=0,
\end{aligned}
$$

where (4.4) has been applied since A is a Lie algebroid.

Raising L and I at the same time, we have from the equation (4.43):

$$\begin{aligned}
&\eta^{MN}(T_M{}^b{}_c T_d{}^a{}_N + T_M{}^b{}_d T^a_{cN} + T_M{}^{ba} T_{cdN}) \\
&= \rho_*^{ia} \frac{\partial L_{cd}{}^b}{\partial X^i} - \rho_*^{ib} \frac{\partial L_{cd}{}^a}{\partial X^i} + \rho_c^i \frac{\partial \widetilde{L}^{ab}{}_d}{\partial X^i} - \rho_d^i \frac{\partial \widetilde{L}^{ab}{}_c}{\partial X^i} \\
&- L_{ec}{}^b \widetilde{L}^{ae}{}_d + L_{ed}{}^b \widetilde{L}^{ae}{}_c + \widetilde{L}^{ba}{}_e L_{cd}{}^e - \widetilde{L}^{eb}{}_c L_{de}{}^a + \widetilde{L}^{eb}{}_d L_{ce}{}^a \\
&= 0,
\end{aligned}$$

where we have used (4.26) in the last equality since the compatibility condition is satisfied between A and A^*.

By raising LIJ, one could obtain the following equation from (4.44).

$$\begin{aligned}
&\rho_*^{ia} \frac{\partial \widetilde{L}^{bc}{}_d}{\partial X^i} + \rho_*^{ib} \frac{\partial \widetilde{L}^{ca}{}_d}{\partial X^i} + \rho_*^{ic} \frac{\partial \widetilde{L}^{ab}{}_d}{\partial X^i} + \widetilde{L}^{bc}{}_e \widetilde{L}^{ae}{}_d - \widetilde{L}^{ba}{}_e \widetilde{L}^{ce}{}_d + \widetilde{L}^{eb}{}_d \widetilde{L}^{ac}{}_e \\
&= \rho_*^{ia} \frac{\partial \widetilde{L}^{bc}{}_d}{\partial X^i} + \rho_*^{ib} \frac{\partial \widetilde{L}^{ca}{}_d}{\partial X^i} + \rho_*^{ic} \frac{\partial \widetilde{L}^{ab}{}_d}{\partial X^i} + \widetilde{L}^{ae}{}_d \widetilde{L}^{bc}{}_e + \widetilde{L}^{be}{}_d \widetilde{L}^{ca}{}_e + \widetilde{L}^{ce}{}_d \widetilde{L}^{ab}{}_e \\
&= 0,
\end{aligned}$$

where we have used the fact (4.25) in the last line since A^* is a Lie algebroid.

When the all indices are raised or lowered as in (4.45) and (4.41), respectively, we have identically zero since $T^{bcd} = T_{bcd} = 0$.

Consequently, the last condition in (4.37) of the Courant algebroid in local coordinates is satisfied by the constructed triangular Lie bialgebroid (A, A^*) and it completes the proof of the fact that the structure $(A \oplus A^*, [\cdot,\cdot]_{A\oplus A^*}, \langle\cdot,\cdot\rangle, \rho)$, where $\rho^i{}_J = (\rho_a^i, \rho_*^{ia})$, is a Courant algebroid with transversal Dirac structures. ■

4.4.3 Courant algebroids for some special anchors in local coordinates

According to the local conditions of a Courant algebroid that are stated in (4.37), the axioms that must be satisfied by a Courant algebroid for some special anchors will be

examined in this Subsection and can be followed in Propositions below. The setting will be taken as the same with the Subsection 4.4 and so that our consideration will be on the vector bundle E in such a way that $E = R \oplus R^*$ along with the anchors in the form of $\rho^i{}_J = (\rho^i_a, \rho^{ia}_*)$. The last special anchor, that is the most general one among them, is important for our upcoming goals.

The second condition in (4.37) can be thought as the definition of fluxes in terms of anchors and other fluxes (sometimes they vanish as in the first example) whereas the third condition known as Bianchi identity in physics literature, is the requirement that fluxes must satisfy. These notions will be examined in Subsection 6.2.3 in detail.

Proposition 4.4.3. *If the anchor is taken as $\rho^i_J = (\delta^i_j, 0)$ such that $\rho^i_j = \delta^i_j$ and $\rho^{ij}_* = 0$ in a Courant algebroid (which gives a standard Courant algebroid, see the Example 4.4.1), the three equations in* (4.37) *take the following forms [51].*

The first condition (4.38) *is satisfied identically.*
The second condition (4.39)*: $f_{ab}{}^j + R^{jab} + Q_b{}^{ja} - Q_a{}^{jb} = 0$ which implies $R = f = Q = 0$.*
The third condition (4.40) *for $f = R = Q = 0$ gives (which can be also seen directly from* (4.41)*):*

$$\partial_a H_{bcd} - \partial_b H_{cda} + \partial_c H_{dab} - \partial_d H_{abc} = 0, \tag{4.64}$$

or $dH = 0$.

Note that the second condition yields vanishing fluxes except H-flux in above theorem. Also, the third condition yields the Bianchi identities that H-flux must obey leading to the H-twisted standard Courant algebroid.

The bi-vector field β will be introduced in the Propositions and Lemmas that follow, where β is some $(2,0)$-tensor as $\beta \in \Gamma(TM \otimes TM)$ with associated bundle map $\beta^\sharp : T^*M \to TM$, induced by the canonical dual pairing between the tangent and cotangent bundles, in a Courant algebroid which consistently produces a non-standard Courant algebroid.

Proposition 4.4.4. *If the anchor is taken as* $\rho^i_J = (0, \beta^{ij})$ *such that* $\rho^i_j = 0$ *and* $\rho^{ij}_* = \beta^{ij}$*, then the three equations in* (4.37) *become the followings [51].*

The first condition (4.38) *is again satisfied identically.*
From the second condition (4.39)*, we have*

$$\beta^{jd} H_{dab} = 0,$$
$$\beta^{jd} Q_d{}^{ab} = \beta^{ia}(\partial_i \beta^{jb}) - \beta^{ib}(\partial_i \beta^{ja}), \tag{4.65}$$

and

$$\beta^{jd} f_{da}{}^{b} = 0.$$

These results lead to the conditions $H = f = 0$ *and* $\beta^{jd} Q_d{}^{ab} = \beta^{ia}(\partial_i \beta^{jb}) - \beta^{ib}(\partial_i \beta^{ja})$.

The second requirement above does not restrict R-flux as it does not occur and the other fluxes do not have to be zero. Besides, R-flux does not even appear in the second condition however it arises in the third condition which stands for the Bianchi identities. These identities can be easily seen from (4.41)-(4.45) for $H = f = 0$ for $\rho^i_J = (0, \beta^{ij})$.

Lemma 4.4.2. *When* $\beta = \Pi$ *is a non-degenerate Poisson tensor, that is the SN bracket with itself vanishes,* $[\Pi, \Pi]_{SN} = 0$ *given in* (3.21) *in local coordinates, then a Courant algebroid for* $\rho^i_J = (0, \Pi^{ij})$ *is obtained as* $H_{ijk} = f_{ij}{}^{k} = 0$ *and,* $Q_i{}^{jk} = \partial_i \Pi^{jk}$ *from* (4.65)*. The third condition stands for an additional requirement for* $\beta = \Pi$*.*

Proposition 4.4.5. *If the anchor taken to be* $\rho^i_J = (\delta^i_j, \beta^{ij})$ *such that* $\rho^i_j = \delta^i_j$ *and* $\rho^{ij}_* = \beta^{ij}$*, then it gives a non-standard Courant algebroid again and the three equations in* (4.37) *yield the following [51].*

The first condition (4.38) *is satisfied when* $\beta^{ij} = -\beta^{ji}$*, which means* β *is skew-symmetric.*

From the second condition (4.39)*, we obtain*

$$
\begin{aligned}
&f_{ab}{}^{j} + \beta^{jd} H_{dab} = 0, \\
&R^{jab} + \beta^{jd} Q_d{}^{ab} = \beta^{ia}(\partial_i \beta^{jb}) - \beta^{ib}(\partial_i \beta^{ja}), \\
&Q_a{}^{jb} = \beta^{jd} f_{da}{}^{b} - \partial_a \beta^{jb}.
\end{aligned}
$$

The skew-symmetry of the Q-flux may be observed by writing the second condition as follows, which can be achieved by applying the first equality mentioned above.

$$Q_a{}^{jb} = \beta^{ji}\beta^{kb} H_{kia} - \partial_a \beta^{jb}.$$

In above Theorem, we can see from the results of the second condition, once more the fluxes do not have to be zero. Besides, all of the fluxes appears now for the second condition to be vanished. It is evident that the third condition, which represents the Bianchi identities, gives rise to further conditions which can be found from (4.41)-(4.45) for $\rho^i_j = (\delta^i_j, \beta^{ij})$.

Lemma 4.4.3. *When $\beta = \Pi$ is a non-degenerate Poisson tensor and $H = 0$ (that yields $f = 0$), then a Courant algebroid for $\rho^i_J = (\delta^i_j, \Pi^{ij})$ is obtained as*

$$\Pi^{ji} = -\Pi^{ij},$$

since Poisson tensor is skew-symmetric, it is expected. Also,

$$Q_a{}^{jb} = \partial_a \Pi^{bj}.$$

The above result together with the fact that Π is a non-degenerate Poisson tensor and so it satisfies $[\Pi, \Pi]_{SN} = 0$ given in (3.21)*, give the following.*

$$R^{jab} = 0.$$

Also, the third condition again stands for an additional requirement for $\beta = \Pi$.

The following case will be especially important for us as explained below and note that it is kind of a generalization of the previous case with taking δ^i_j as the anchor ρ^i_j. As a result, it produces a different, non-standard Courant algebroid that is also more inclusive.

Proposition 4.4.6. *If the anchor is taken to be* $\rho^i_J = (a^i_j, a^i_k\beta^{kj})$ *where* $\rho^i_j = a^i_j$ *and* $\rho^{ij}_* = a^i_k\beta^{kj}$*, i.e., the most general case in comparison to the earlier cases, then the three equations in* (4.37) *take the following forms.*

The first condition (4.38) *is satisfied again when* $\beta^{ij} = -\beta^{ji}$.
The second condition (4.39) *yields*

$$a^i_c\partial_i a^j_{\ b} - a^i_{\ b}\partial_i a^j_c - a^j_{\ i}f_{cb}^{\ \ i} - a^j_i\beta^{ik}H_{kcb} = 0,$$

$$a^i_k\beta^{kc}\partial_i a^j_{\ b} - a^i_{\ b}\beta^{kc}\partial_i a^j_k - a^i_{\ b}a^j_k\partial_i\beta^{kc} - a^j_{\ i}Q_b^{\ \ ic} + a^j_k\beta^{ki}f_{ib}^{\ \ c} = 0,$$

$$a^i_k\beta^{kc}\beta^{lb}\partial_i a^j_l - a^i_k\beta^{kb}\beta^{lc}\partial_i a^j_l + a^i_k a^j_l\beta^{kc}\partial_i\beta^{lb} - a^i_k a^j_l\beta^{kb}\partial_i\beta^{lc} - a^j_i R^{icb} - a^j_k\beta^{ki}Q_i^{\ \ cb} = 0.$$

These conditions on structure functions in a Courant algebroid will be essential when we compare with constraints in DFT fluxes in the Subsection 6.2.3, and these can be expressed as follows.

$$\begin{aligned} f_{cb}^{\ \ m} &= a^m_j a^i_c\partial_i a^j_{\ b} - a^m_j a^i_{\ b}\partial_i a^j_c - \beta^{mi}H_{icb}, \\ &= C_{cb}^{\ \ m} - \beta^{mi}H_{icb}, \end{aligned} \tag{4.66}$$

where we have defined $a^m_j a^i_c\partial_i a^j_{\ b} - a^m_j a^i_{\ b}\partial_i a^j_c = C_{cb}^{\ \ m}$. Also,

$$Q_b^{\ \ mc} = \beta^{ic}C_{ib}^{\ \ m} + \beta^{mi}C_{ib}^{\ \ c} - a^i_{\ b}\partial_i\beta^{mc} + \beta^{jc}\beta^{mi}H_{ijb}, \tag{4.67}$$

$$\begin{aligned} R^{mcb} &= \beta^{ic}\beta^{jb}C_{ij}^{\ \ m} - \beta^{mi}\beta^{jb}C_{ji}^{\ \ c} - \beta^{mi}\beta^{cj}C_{ji}^{\ \ b} + a^i_j\beta^{jc}\partial_i\beta^{mb} \\ &\quad - a^i_j\beta^{jb}\partial_i\beta^{mc} + a^i_j\beta^{mj}\partial_i\beta^{cb} + 2\beta^{jc}\beta^{kb}\beta^{mi}H_{ijk}. \end{aligned} \tag{4.68}$$

In above Theorem, we can see from the results of the second condition, once more the fluxes do not have to be zero. Besides, all of the fluxes appears now for the second condition to be vanished. The other conditions arise from the third condition which stands for the Bianchi identities which can be checked from (4.41)-(4.45) for $\rho^i_J = (a^i_j, a^i_k\beta^{kj})$.

Remark 4.4.5. *If* $H = 0$ *will be imposed in the above axioms* (4.66)*,* (4.67)*, and* (4.68)*, respectively, we obtain the followings.*

$$f_{cb}^{\ \ m} = C_{cb}^{\ \ m}, \tag{4.69}$$

$$Q_b{}^{mc} = \beta^{ic} C_{ib}{}^{m} + \beta^{mi} C_{ib}{}^{c} - a^i{}_b \partial_i \beta^{mc}, \tag{4.70}$$

$$R^{mcb} = \beta^{ic}\beta^{jb} C_{ij}{}^{m} - \beta^{im}\beta^{jb} C_{ij}{}^{c} + \beta^{im}\beta^{jc} C_{ij}{}^{b} + a^i_j \beta^{jc} \partial_i \beta^{mb} - a^i_j \beta^{jb} \partial_i \beta^{mc} + a^i_j \beta^{mj} \partial_i \beta^{cb}. \tag{4.71}$$

Lemma 4.4.4. *When $H = 0$ and $\beta = \Pi$ is a non-degenerate Poisson tensor, the anchor becomes $\rho^i_j = (a^i_j, a^i_k \Pi^{kj})$. This case will produce again $R = 0$ as in the Lemma 4.4.3 and this result gives $[\Pi, \Pi]_{SN} = 0$, as expected.*

The following section covers the properties of R-matrix on a Lie algebra g of a semi-simple Lie group G in detail. Some parts of this discussion have already been introduced in the Section 2. The definition and the features of dressed R-matrix R_g constructed from R-matrix will be introduced in order to work on the group manifold associated with the isometries of the background by using the adjoint action. Firstly, it will be seen that for a given R-matrix which is an endomorphism on a Lie algebra g, one can construct a bi-vector associated with it by the help of the non-degenerate Cartan-Killing form on g. This enables us to identify g and the dual vector space g^* and one can create a Lie algebra structure in the dual vector space with the aid of this bi-vector. Then one can construct a bi-vector field which is defined on the whole manifold. This bi-vector field constructed from the dressed R-matrix defines a Poisson structure whenever the dressed R-matrix satisfies the CYBE. This result leads to the fact that there is actually a Courant algebroid structure.

5. DRESSED R-MATRIX AND THE CONSTRUCTION OF BI-VECTOR FIELD

In this part of the thesis, we will assume that we have the R-matrix on the group manifold together with the non-degenerate and Ad-invariant Cartan-Killing metric. By using these notions, it will be demonstrated that we can construct a bi-vector field, which is the components of the dressed R-matrix that is built from the R-matrix. Here, we will use the endomorphism R on a Lie algebra g that was first introduced in Subsection 2.4. When the dressed R-matrix satisfies the CYBE, then it will be seen that the SN bracket of this constructed bi-vector field with itself vanishes. This result will give rise to the fact that this bi-vector field defines a Poisson structure on the group manifold. We have seen in Subsection 3.1.4 that there is a natural Lie algebroid structure on the cotangent bundle of a Poisson manifold, and the tangent and cotangent bundles together construct a Lie bialgebroid. It has been seen in Examples 3.3.3 and 3.3.4 that the tangent and cotangent bundles are transversal Dirac structures, respectively. Additionally, there exists a natural Courant algebroid structure with these transversal Dirac structures since there is a one-to-one correspondence between these structures, as stated in Subsection 3.3.5. Therefore, we will be able to verify that the above picture actually represents a Courant algebroid structure.

As a preparation for these computations, we need to present more properties of the R-matrices and the related algebraic structures in this Chapter. These studies are based on the paper named "Yang–Baxter deformation as an $O(d,d)$ transformation" written by my advisor Prof. Dr. Aybike Çatal-Özer and Seçil Tunalı-Çırak [8].

Our index conventions will be as follows and will be used in the following Subsections.
A : Doubled coordinates such that $^{A} = (_{\mu}, ^{\mu})$,

B : Doubled coordinates; ${}^{B} = ({}_{\alpha}, {}^{\alpha})$, where

$$\begin{array}{llll} \mu = (i,m), & \mu = 1,\cdots,10; & i = 1,\cdots,d, & d = \dim G, \\ \alpha = (I,a), & \alpha = 1,\cdots,10; & I = 1,\cdots,d, & \end{array}$$

where we are using μ indices for the coordinate basis and α indices for the arbitrary basis or for the choice of the left-invariant (or right-invariant) one-forms as basis in our situation.

5.1 Properties of the R-matrix and the CYBE

The definition and the properties of the R-matrices have been given in the Subsections 2.4 and 2.4.1 together with the MYBE and the CYBE. In this part of the thesis, these notions will be examined in detail. In this Subsection, additional R-matrix features will be offered and examined with regard to a fixed basis. These characteristics will be essential for use in current studies as well as comparisons with past research. Our approach will depend on the classical skew-symmetric R-matrices which are endomorphisms on g such that they satisfy CYBE.

Let g be the Lie algebra of a semi-simple Lie group G. Let R be an endomorphism on g.

Definition 5.1.1. *[9], [10] The CYBE for the operator R is the following:*

$$\boxed{[RX,RY] - R([RX,Y] + [X,RY]) = 0}, \quad \forall X,Y \in g,$$

which has been stated in Remark 2.4.1.

If the R endomorphism satisfies the CYBE, then it is an R-matrix which can be seen in the Definition 2.4.3. The components of the corresponding R-matrix with respect to a fixed basis T_I of g will be denoted by $R^I{}_J$. Then, R-matrix with respect to that basis is given by [9], [10]

$$RX = (RX)^I T_I = R^I{}_J X^J T_I, \tag{5.1}$$

where $X \in g$.

Then, the equation (2.3) can be expressed in local coordinates as follows.

$$R^I{}_J R^K{}_L C_{IK}{}^M - R^I{}_J R^M{}_K C_{IL}{}^K - R^I{}_L R^M{}_K C_{JI}{}^K = 0, \tag{5.2}$$

where $C_{IJ}{}^K$ are the structure constants of the Lie algebra g with respect to the basis T_I.

Proof: Let $X = T_I$ and $Y = T_J$ for fixed bases $T_I, T_J \in g$ in the equation (2.3), then

$$\begin{aligned}
&[RT_I, RT_J] - R([RT_I, T_J] + [T_I, RT_J]) = 0 \\
&\Rightarrow [R^K_I T_K, R^L_J T_L] - R([R^K_I T_K, T_J] + [T_I, R^K_J T_K]) = 0 \\
&\Rightarrow R^K_I R^L_J C_{KL}{}^M T_M - R(R^K_I C_{KJ}{}^M T_M + R^K_J C_{IK}{}^M T_M) = 0 \\
&\Rightarrow R^K_I R^L_J C_{KL}{}^M T_M - R^K_I C_{KJ}{}^M R^N_M T_N - R^K_J C_{IK}{}^M R^N_M T_N = 0 \\
&\Rightarrow (R^K_I R^L_J C_{KL}{}^M - R^K_I R^M_L C_{KJ}{}^L - R^K_J R^M_L C_{IK}{}^L) T_M = 0,
\end{aligned}$$

where we have used that $R(T_I) = R^K{}_I T_K$ in the second and in the fourth lines, and the fact that R operator is linear has been used in the third line. The above equation gives the desired result for a fixed basis T_M in a Lie algebra. ■

Definition 5.1.2. *[9], [10] Let R be a linear map from g to g. R-bracket of $X, Y \in g$ is defined by*

$$\boxed{[X,Y]_R = [RX,Y] + [X,RY]},$$

which has been firstly introduced in the Proposition 2.4.1.

[9], [10] The bracket $[\cdot,\cdot]_R$ has been previously shown to satisfy the Jacobi Identity when (2.3) is fulfilled. As a result, any R-matrix on g defines a second Lie algebra structure $[\cdot,\cdot]_R$ on g from the Proposition 2.4.1. Therefore, using the new bracket $[\cdot,\cdot]_R$ above and the same underlying vector space as that of g, a new Lie algebra g_R is constructed.

Proposition 5.1.1. *[8] The structure constants of a Lie algebra g, which are C_{KJ}^L, can be used to compute the structure constants $\tilde{C}_{IJ}{}^L$ of g_R with regard to the basis T_I as follows.*

$$\boxed{\tilde{C}_{IJ}{}^L = R^K_I C_{KJ}{}^L - R^K_J C_{KI}{}^L = -\tilde{C}_{JI}{}^L}. \tag{5.3}$$

Proof: The equation (2.4) for $X = T_I$ and $Y = T_J$, where $T_I, T_J \in g$ are fixed bases, turns into the following:

$$\begin{aligned}
&[T_I, T_J]_R = [RT_I, T_J] + [T_I, RT_J] \\
&\Longrightarrow \tilde{C}_{IJ}{}^{L} T_L = [R^L_I T_L, T_J] + [T_I, R^L_J T_L] \\
&\Longrightarrow \tilde{C}_{IJ}{}^{L} T_L = (R^L_I C_{LJ}{}^{M} + R^L_J C_{IL}{}^{M}) T_M \\
&\Longrightarrow \tilde{C}_{IJ}{}^{L} = R^K_I C_{KJ}{}^{L} + R^K_J C_{IK}{}^{L},
\end{aligned}$$

where we have used that $R(T_I) = R^L{}_I T_L$ in the second line, and the fact that R operator is linear has been used in the third line. So, the proof has been completed for a fixed basis T_L. ■

We focus on Lie algebras that admit non-degenerate and ad-invariant inner products. This category of Lie algebras includes particular cases such as semi-simple Lie algebras with ad-invariant and non-degenerate Cartan-Killing forms. The Lie algebras we study do not necessarily have to be semi-simple. Any Lie algebra with a non-degenerate inner product is also suitable for our formulation.

Construction of a Bi-vector from *R*-matrix:

Using the non-degenerate Cartan-Killing form on g, which enables us to identify g and the dual vector space g^*, our aim is to build a bi-vector on the Lie algebra g. It has been generally discussed in Subsections 3.1.4 and 3.2.1 that with the help of a bi-vector, one can build a Lie algebra structure in the dual vector space. A non-degenerate 2-form on g will initially be introduced in order to construct such a bi-vector.

Let ω be a non-degenerate 2-form on the Lie algebra defined by

$$\kappa(RX, Y) = \omega(X, Y), \quad X, Y \in g, \tag{5.4}$$

where κ is the non-degenerate Cartan-Killing form on g defined as $B(X,Y) = tr(ad_X \circ ad_Y)$. When the above equation is written in a fixed bases $X = T_I$ and $Y = T_J$, where

$T_I, T_J \in g$, we obtain

$$\omega_{IJ} = R^L{}_I \kappa_{LJ}. \tag{5.5}$$

Since the skew-symmetry condition of R implies that $\omega(X,Y) = \langle RX, Y\rangle = -\langle X, RY\rangle = -\langle RY, X\rangle = -(Y,X)$, then ω mentioned above is in fact a 2-form on the Lie algebra. Furthermore, it is non-degenerate, at least on the Frobenius subalgebra, a subalgebra of g (see the Definition 2.4.6 and the information below it). On this subalgebra, there exists an inverse of this two-form and so that one can define a bi-vector $\beta_0 \in \wedge^2 g$ on the Lie algebra g such that

$$\beta_0(\kappa X, \kappa Y) = \omega(X,Y), \tag{5.6}$$

where $\kappa : g \to g^*$. In order to express this bi-vector on g in local coordinates, we will firstly investigate the identification between g and g^* by using the non-degenerate Cartan-Killing form κ on g.

The inverse metric κ^{-1} will induce a map from g^* to g and the components of κ^{-1} will be shown by κ^{IJ}. Let $T_J \in g$ and $\tilde{T}^I \in g^*$ such that $\tilde{T}^I(T_J) = \delta^I_J$. Since $\kappa(T_I)(T_J) = \kappa_{IJ}$, we obtain $\kappa(T_I) = \kappa_{IL}\tilde{T}^L$. Similarly, $\kappa^{-1}(\tilde{T}^I)(\tilde{T}^J) = \kappa^{IJ}$.

We define an endomorphism $\widetilde{R}$ on g^* such that $\widetilde{R}\kappa = \kappa R$ which can be written in local coordinates as

$$\begin{aligned}
&\widetilde{R}\kappa(T_I) = \kappa R(T_I) \\
&\Rightarrow \widetilde{R}(\kappa_{IL}\tilde{T}^L) = \kappa(R^L_I T_L) \\
&\Rightarrow \kappa_{IL}\widetilde{R}(\tilde{T}^L) = \kappa_{LJ} R^L_I \tilde{T}^J,
\end{aligned}$$

then

$$\widetilde{R}(\tilde{T}^L) = R^M_I \kappa^{IL} \kappa_{MJ} \tilde{T}^J. \tag{5.7}$$

Therefore, the bi-vector on g can be written in local coordinates as follows.

$$\begin{aligned}(\beta_0)^{LN} &= \beta_0(\tilde{T}^L, \tilde{T}^N) = \kappa^{-1}(\widetilde{R}(\tilde{T}^L), \tilde{T}^N) \\ &= \kappa^{-1}(R^M_{\ I}\kappa^{IL}\kappa_{MJ}\tilde{T}^J, \tilde{T}^N) \\ &= R^M_{\ I}\kappa^{IL}\kappa_{MJ}\kappa^{JN} \\ &= R^N_{\ I}\kappa^{IL} := R^{LN}. \end{aligned} \tag{5.8}$$

Here, we have used $\kappa^{JM}\kappa_{NM} = \delta^J_{\ N}$ and we have chosen bases such that the components of κ are constants. Now it can be seen that (5.6) holds with the above definitions.

If R is taken to be skew-symmetric with respect to κ, then $R^{IJ} = -R^{JI}$.

Proof: When the operator R is skew-symmetric, we have $\langle RX, Y\rangle = \langle X, R^*Y\rangle = -\langle X, RY\rangle$, where R^* stands for the transpose of the matrix R, since its entries are real numbers only. Since, it is true also for the dual elements with $\widetilde{R}$, let us expand this equality with respect to fixed bases $\tilde{T}^I$ and $\tilde{T}^J$ of g^* as follows.

$$\begin{aligned}&\langle \widetilde{R}\tilde{T}^I, \tilde{T}^J\rangle = -\langle \tilde{T}^I, \widetilde{R}\tilde{T}^J\rangle \\ &\Rightarrow \langle R^M_{\ L}\kappa^{LI}\kappa_{MN}\tilde{T}^N, \tilde{T}^J\rangle = -\langle \tilde{T}^I, R^M_{\ L}\kappa^{LJ}\kappa_{MN}\tilde{T}^M\rangle \\ &\Rightarrow R^M_{\ L}\kappa^{LI}\kappa_{MN}\kappa^{NJ} = -R^M_{\ L}\kappa^{LJ}\kappa_{MN}\kappa^{IN} \\ &\Rightarrow R^J_{\ L}\kappa^{LI} = -R^I_{\ L}\kappa^{LJ} \\ &\Rightarrow R^{IJ} = -R^{JI}, \end{aligned}$$

where we have used (5.7) in the second line and (5.8) in the last line. ■

Proposition 5.1.2. *[8] With respect to the fixed basis T_I, the CYBE reads in local coordinates the following equation by using* (5.2)*:*

$$\boxed{R^{LI}R^{MJ}C_{LM}^{\ \ K} + R^{LJ}R^{MK}C_{LM}^{\ \ I} + R^{LK}R^{MI}C_{LM}^{\ \ J} = 0} \ (\textit{CYBE in local}), \tag{5.9}$$

where $C_{IJ}^{\ \ K}$ are the structure constants of the Lie algebra g with respect to the basis T_I. Also, all indices are raised and lowered by the Killing form κ.

Proof: By multiplying (5.2) by $\kappa^{JP}\kappa^{LQ}$, we obtain

$$\begin{aligned}
&R^{PI}R^{QK}C_{IK}{}^{M} - R^{PI}R^{M}{}_{K}C_{I}{}^{QK} - R^{QI}R^{M}{}_{K}C^{P}{}_{I}{}^{K} = 0 \\
&\Rightarrow R^{PI}R^{QK}C_{IK}{}^{M} - R^{PI}R^{KM}C_{I}{}^{Q}{}_{K} - R^{QI}R^{KM}C^{P}{}_{IK} = 0 \\
&\Rightarrow R^{IP}R^{KQ}C_{IK}{}^{M} + R^{IM}R^{KP}C_{IK}{}^{Q} + R^{IQ}R^{KM}C_{IK}{}^{P} = 0,
\end{aligned}$$

where we have used the fact that $R^{M}{}_{K}C_{I}{}^{QK} = R^{KM}C_{I}{}^{Q}{}_{K}$ together with that $C_{IJK} = C_{IJ}{}^{L}\kappa_{LK}$ are totally anti-symmetric in all of its indices (see the Lemma 1.0.1 with its Proof in Appendix), and R is skew-symmetric with respect to κ, and this completes the proof. ■

We have used the endomorphism R in order to construct a new Lie algebra structure on g in the preceding part and after that the bi-vector has been built by using κ in (5.8). The fact that the bi-vector satisfies the CYBE (5.9) gives that it is the same expression with the (4.24). Therefore, (4.23) can be thought as a generalized CYBE in local coordinates on the triangular Lie bialgebroids. We have seen in the Section 4.3 that for a given bi-section in triangular Lie bialgebroids, a dual Lie algebroid structure on the dual vector bundle can be constructed. Therefore, we can use this fact in order to construct a Lie algebra structure on the dual vector space by using the bi-vector.

Moreover, the structure constants of this dual Lie algebra $\tilde{C}^{IJ}{}_{L}$ can be written in terms of the structure constants of the associated Lie algebra and the components of the R-matrix as follows from (4.21). So, (4.20) stands for the general expression on the triangular Lie bialgebroids.

Lemma 5.1.1. *[8] The structure constants $\tilde{C}^{IJ}{}_{L}$ of the dual Lie algebra g_R^* of the Lie algebra g_R can be calculated as follows.*

$$\boxed{\tilde{C}^{IJ}{}_{L} = C_{KL}{}^{J}R^{IK} - C_{KL}{}^{I}R^{JK}}. \tag{5.10}$$

In the following part of the thesis, dressed R-matrix will be introduced in order to move this picture to the whole group manifold associated with the isometries of the background by using the adjoint action. With the help of this matrix and the

Cartan-Killing form κ, a bi-vector field will be constructed on the group manifold. As we have discussed in the Subsection 3.3.2, a bi-vector field β can be regarded as a map from T^*M to TM. Then, it can be extended naturally to an orthogonal map on the generalized tangent bundle $TM \oplus T^*M$. The orthogonal map on the generalized tangent bundle $TM \oplus T^*M$ indeed preserves the natural metric on the generalized tangent bundle as stated in the Subsection 3.3.2. The components of the bi-vector field will be used in order to construct such an $O(d,d)$ matrix. It will be seen that this bi-vector field actually defines a Poisson structure since the dressed R-matrix satisfies the CYBE when the R-matrix does. This constructed $O(d,d)$ matrix which represents this particular type of β transformation will be referred as YB matrix and denoted by T_{YB} (6.65).

It will be demonstrated after that this transformation is actually a solution-generating transformation of a string theory. This fact will be seen from the preservation of the fluxes associated with the related matrices: L and T_{YB} which can be followed from the Subsection 6.2.4. The generalized metric in terms of the twist matrix L satisfies the field equations of 10-dimensional supergravity which can be found in the Subsection 6.2.2. Together with the flux preservation this fact yields that the $O(d,d)$ transformation, constructed from the bi-vector field and left invariant vector fields, acts as a solution generating transformation in string theory which can be seen in the Subsection 6.3.

5.2 The Dressed R-matrix

Definition 5.2.1. *The dressed R-matrix is defined as [52]*

$$R_g = Ad_{g^{-1}}\, R\, Ad_g, \tag{5.11}$$

where g is the Lie algebra of the isometry group G.

Proposition 5.2.1. *([8], and references there) The dressed R-matrix also satisfies the CYBE (whenever R does).*

Proof: Our goal is to prove that R_g defined above satisfies the CYBE given in (2.3) whenever R-matrix does. The LHS of that equation become the following expression

for all $X,Y \in \mathfrak{g}$ where $\mathfrak{g} = T_eG$ is a Lie algebra associated with the semi-simple Lie group G.

$$
\begin{aligned}
&[R_gX,R_gY] - R_g([R_gX,Y]+[X,R_gY]) \\
&= [Ad_{g^{-1}}\, R\, Ad_gX, Ad_{g^{-1}}\, R\, Ad_gY] - (Ad_{g^{-1}}\, R\, Ad_g)([Ad_{g^{-1}}\, R\, Ad_gX,Y]+[X,Ad_{g^{-1}}\, R\, Ad_gY]) \\
&= Ad_{g^{-1}}([R\, Ad_gX, R\, Ad_gY] - \, R\, Ad_g([Ad_{g^{-1}}\, R\, Ad_gX, Ad_{g^{-1}}Ad_gY] \\
&+ [Ad_{g^{-1}}Ad_gX, Ad_{g^{-1}}\, R\, Ad_gY])) \\
&= Ad_{g^{-1}}\left([R\,\tilde{X}, R\,\tilde{Y}] - \, R\,([R\,\tilde{X},\tilde{Y}]+[\tilde{X}, R\,\tilde{Y}])\right),
\end{aligned}
$$

where $\tilde{X} = Ad_gX$ and $\tilde{Y} = Ad_gY$ are elements of g and we have used the facts that $[Ad_{g^{-1}}X, Ad_{g^{-1}}Y] = Ad_{g^{-1}}[X,Y]$ since Ad_g is a Lie algebra homomorphism in the third and the fourth lines and $(Ad_{g^{-1}})^{-1} = Ad_g$ in the third line. The above expression vanishes due to the fact that R-matrix satisfies CYBE that is (2.3). ■

Before revealing the useful equality of the dressed R-operator, which is (5.20), let us first present some essential definitions.

The right invariant vector fields k_I:

$$k_I = k_I{}^{\mu}\partial_{\mu}, \tag{5.12}$$

where $\partial_{\mu} = \frac{\partial}{\partial X^{\mu}}$ where $X^{\mu} = (X_1, \cdots, X_d)$ are a set of local coordinates on G and d is the dimension of G, which satisfies

$$[k_I, k_J] = -C_{IJ}{}^{K} k_K.$$

These are Killing vector fields for the left invariant metric on g.

The left invariant vector fields l_I:

$$l_I = l_I{}^{\mu}\partial_{\mu}, \tag{5.13}$$

and we have

$$[l_I, l_J] = C_{IJ}{}^{K} l_K.$$

The left invariant one-form l^I:

$$l^I = l^I{}_\mu d\theta^\mu, \tag{5.14}$$

generated by k_I and $\theta^\mu = (\theta_1, \cdots, \theta_d)$ are a set of local coordinates on the manifold M. The Maurer-Cartan one-form σ, which is the unique g-valued one-form on G, is defined by

$$\sigma = \sigma^I T_I = l^I{}_\mu d\theta^\mu T_I, \tag{5.15}$$

where T_I are elements of $g \in T_eG$. It also satisfies the following property:

$$d\sigma^I = \frac{1}{2} C_{JK}{}^I \sigma^J \wedge \sigma^K. \tag{5.16}$$

The right invariant one-form k^I:

$$k^I = k^I{}_\mu d\theta^\mu. \tag{5.17}$$

The components of the above notions satisfy the followings.

$$l^I{}_\mu l_I{}^\eta = \delta_\mu{}^\eta, \tag{5.18}$$

$$k^I{}_\mu k_I{}^\eta = \delta_\mu{}^\eta. \tag{5.19}$$

Proposition 5.2.2. *([8], and references there) The dressed R-operator satisfies*

$$R_g^{IJ} = K^I{}_N R^{NL} K_L{}^J, \tag{5.20}$$

where

$$K^I{}_N = k^I{}_\mu l_N{}^\mu,$$

where $k^I{}_\mu, l_N{}^\mu$ are the components of the right invariant one-form and the left invariant vector field given in (5.17) *and* (5.13)*, respectively, and*

$$K_L{}^J = k_L{}^\mu l^J{}_\mu, \tag{5.21}$$

where $k_L{}^\mu, l^J{}_\mu$ are the components of the right invariant vector fields and the left invariant one-forms given in (5.12) *and* (5.14)*, respectively.*

In order to prove the Proposition above, the following Proposition with its proof will be given first.

Proposition 5.2.3. *[6]*

$$(Ad_{g^{-1}})^J_I = K^J_I, \tag{5.22}$$

which follows directly from the well-known relation

$$(Ad_{g^{-1}})\sigma = r_g{}^*\sigma, \tag{5.23}$$

where the RHS corresponds to the pull-back of the Maurer-Cartan form under the right translation r_g.

In the following calculations, $(\cdot)_*$ will denote the push-forward.

Proof: The proof of the equality (5.22) is follow from the evaluation of (5.23) at T_eG as follows.

$$(r_g{}^*\sigma)_e(T_J) = (Ad_{g^{-1}})\sigma_e(T_J) = (Ad_{g^{-1}})(T_J) = (Ad_{g^{-1}})^I{}_J(T_I),$$

and the LHS gives

$$\begin{aligned}(r_g{}^*\sigma)_e(T_J) &= \sigma_{g(e)}(r_{g_*}T_J) = \sigma_g(r_{g_*}T_J)\\ &= (l^I{}_\mu d\theta^\mu T_I)(k^\mu_J\partial_\mu)\\ &= l^I{}_\mu k^\mu_J T_I.\end{aligned}$$

Then we obtain $(Ad_{g^{-1}})^I{}_J = l^I{}_\mu k^\mu_J$ which completes the proof. ■.

Now, we are ready to prove the Proposition 5.2.2.

Proof of the Proposition 5.2.2: The dressed R-operator in (5.11) can be written as

$$(R_g)^I{}_L = K^I_J R^J_K \tilde{K}^K_L, \tag{5.24}$$

where $\tilde{K}^I_J = (Ad_g)^I_J$.

Raising the indices of the LHS of (5.24) with the Cartan-Killing metric κ^{LM}, we have

$$\begin{aligned}(R_g)^{IM} &= (R_g)^I{}_L\kappa^{LM} = K^I_J R^J{}_K \tilde{K}^K_L \kappa^{LM}\\ &= K^I_J R^J{}_K (\tilde{K}^{-1})^M_L \kappa^{LK}\\ &= K^I_J R^{JL} K^M_L,\end{aligned} \tag{5.25}$$

where we have used $\tilde{K} = K^{-1}$ in the last line, which follows from $Ad_{g^{-1}} = Ad_g^{-1}$. Also, we have used that the Cartan-Killing metric is Ad-invariant in the second line, that is, Ad_g is orthogonal with respect to κ, or in other words

$$\tilde{K}_L^K \kappa^{LM} \tilde{K}_M^R = \kappa^{RK},$$

and this implies that

$$\tilde{K}_L^K \kappa^{LM} = (\tilde{K}^{-1})_L^M \kappa^{LK}. \blacksquare$$

The functions K_J^I are the foundation of the dressed R-matrix's coordinate dependency. In terms of the structure constants, we can formulate expressions of the type $\partial_L K_J^I$ as follows since we will face in our subsequent research.

Proposition 5.2.4. *[8]*

$$l^i{}_L \partial_i K_I^J = \partial_L K_J^I = -C_{PL}{}^I K_J^P \ \ or \ \ \partial_L K_J^I = C_{LP}{}^I K_J^P. \tag{5.26}$$

Proof:

In order to prove the above result, first note that the functions K_I^J are equal to $i_{k_I}\sigma^J$.

$$i_{k_I}\sigma^J = \langle \sigma^J, k_I \rangle = \langle l^J{}_i d\theta^i, k_I{}^\mu \partial_\mu \rangle = k_I{}^i l^J{}_i = K_I^J, \tag{5.27}$$

where we have used (5.12) and (5.21). Since k_I are the right-invariant vector fields in (5.12) and σ^I are the left-invariant one-forms, we have the following fact by using (5.16):

$$\mathscr{L}_{k_I}\sigma^J = d i_{k_I}\sigma^J + i_{k_I} d\sigma^J = d i_{k_I}\sigma^J + i_{k_I}\left(\frac{1}{2} C_{MN}{}^J \sigma^M \wedge \sigma^N\right) = 0,$$

then by using (5.27):

$$\begin{aligned} & dK_I^J + \frac{1}{2} C_{MN}{}^J (i_{k_I}\sigma^M) \wedge \sigma^N - \frac{1}{2} C_{MN}{}^J \sigma^M \wedge (i_{k_I}\sigma^N) = 0 \\ & \Rightarrow dK_I^J + C_{MN}{}^J K_I^M \sigma^N = 0 \\ & \Rightarrow \partial_i K_I^J + C_{MN}{}^J K_I^M l^N_i = 0. \end{aligned}$$

Multiplication of both sides of the above equation by $l^i{}_L$ gives

$$l^i{}_L \partial_i K_I^J = \partial_L K_J^I = -C_{ML}{}^J K_I^M,$$

where we have used (5.18). ■

Another result we will use in the Subsection 6.2.4 is the following one.

Proposition 5.2.5. *[8]*

$$C_{IJ}{}^{K} = -l^{i}{}_{I}(\partial_i l^{j}{}_{J})l^{K}_{j} + l^{i}{}_{J}(\partial_i l^{j}{}_{I})l^{K}_{j}. \tag{5.28}$$

Proof: We will use the following equality, which has been also used in the above proof, and then the proof is straightforward.

$$\begin{aligned} & d\sigma^{I} = \frac{1}{2}C_{JK}{}^{I}\sigma^{J}\wedge\sigma^{K} \\ & \Rightarrow d(l^{I}{}_{i}d\theta^{i}) = \frac{1}{2}C_{JK}{}^{I}(l^{J}{}_{i}d\theta^{i}\wedge l^{K}_{j}d\theta^{j}) \\ & \Rightarrow \frac{1}{2}(\partial_j l^{I}{}_{i} - \partial_i l^{I}{}_{j})d\theta^{j}\wedge d\theta^{i} = \frac{1}{2}C_{JK}{}^{I}l^{J}{}_{j}l^{K}_{i}d\theta^{j}\wedge d\theta^{i}, \end{aligned}$$

so that

$$C_{IJ}{}^{K} = -l^{i}{}_{I}(\partial_i l^{j}{}_{J})l^{K}_{j} + l^{i}{}_{J}(\partial_i l^{j}{}_{I})l^{K}_{j}, \tag{5.29}$$

where we have used $(\partial_j l^{I}{}_{i})l^{i}{}_{K} = -(\partial_j l^{i}{}_{K})l^{I}{}_{i}$■.

Construction of a Bi-vector Field from the Dressed *R*-matrix:

Our goal is to extend the bi-vector constructed in (5.8) to the whole group manifold and to have a bi-vector field by using the dressed R-matrix R_g and Cartan-Killing form κ. Let us define

$$\beta = (r_g)_*\beta_0.$$

Then expanding in local basis and using the properties of the left and right translations, pull-back and the adjoint action, we obtain the following.

$$\begin{aligned}\beta(l_g^I, l_g^J) &= (r_g)_*\beta_0(l_g^I, l_g^J) = \beta_0\left((r_g)^* l_g^I, (r_g)^* l_g^J\right)\\ &= \beta_0\left((r_g)^*(l_{g^{-1}})^*\widetilde{T}^I, (r_g)^*(l_{g^{-1}})^*\widetilde{T}^J\right)\\ &= \beta_0\left((Ad_{g^{-1}})^*\widetilde{T}^I, (Ad_{g^{-1}})^*\widetilde{T}^J\right)\\ &= (Ad_{g^{-1}})_*\beta_0(\widetilde{T}^I, \widetilde{T}^J)\\ &= (Ad_{g^{-1}})^I{}_K (Ad_{g^{-1}})^J{}_L \beta_0^{KL}\\ &= K_M^I K_L^J R^{ML} := \beta^{IJ}, \end{aligned} \tag{5.30}$$

which is actually the dressed R-matrix in local coordinates given in (5.25).

Therefore, we have

$$\beta = K_M^I K_L^J R^{ML} l_I \wedge l_J. \tag{5.31}$$

As for the components of the bi-vector field β in the coordinate basis we have

$$\begin{aligned}\beta &= k_M^i l_i^I k_L^j l_j^J R^{ML} l_I \wedge l_J\\ &= k_M^i k_L^j R^{ML} \partial_i \wedge \partial_j\\ &= \beta^{ij}\partial_i \wedge \partial_j. \end{aligned} \tag{5.32}$$

So, we have seen that the components of the bi-vector field in the coordinate basis given by

$$\beta^{ij} = k_M^i k_L^j R^{ML}. \tag{5.33}$$

Proposition 5.2.6. *[8] The bi-vector field β given in* (5.31) *satisfies $[\beta,\beta]_{SN} = 0$ whenever the R-matrix satisfies the CYBE given in* (5.9)*. Therefore, β defines a Poisson structure on the group manifold from the Lemma 3.1.4 equipped with* (3.21)*.*

Proof: Our aim is to prove that $[\beta,\beta]_{\mathrm{SN}} = 0$ is satisfied by the bi-vector field (5.31). The equation $[\beta,\beta]_{\mathrm{SN}} = 0$ corresponds to the following expression.

$$\beta^{ID}\beta^{JB}C_{IJ}^{\ M} - \beta^{IM}\beta^{JB}C_{IJ}^{\ D} + \beta^{IM}\beta^{JD}C_{IJ}^{\ B} + l_J^i\beta^{JD}\partial_i\beta^{MB} - l_J^i\beta^{JB}\partial_i\beta^{MD} + l_J^i\beta^{MJ}\partial_i\beta^{DB} = 0.$$

The LHS of the above equation yields

$$
\begin{aligned}
&\beta^{DI}\beta^{LJ}C_{DL}{}^{M} - \beta^{DM}\beta^{LJ}C_{DL}{}^{I} + \beta^{DM}\beta^{LI}C_{DL}{}^{J} \\
&+ l_L^i\beta^{LI}\partial_i\beta^{MJ} - l_L^i\beta^{LJ}\partial_i\beta^{MI} + l_L^i\beta^{ML}\partial_i\beta^{IJ} \\
&= \beta^{DI}\beta^{LJ}C_{DL}{}^{M} + \beta^{DJ}\beta^{LM}C_{DL}{}^{I} + \beta^{DM}\beta^{LI}C_{DL}{}^{J} \\
&+ l_L^i K_N^L K_T^I R^{NT}\partial_i(K_P^M K_S^J R^{PS}) - l_L^i K_N^L K_T^J R^{NT}\partial_i(K_P^M K_S^I R^{PS}) + l_L^i K_N^M K_T^L R^{NT}\partial_i(K_P^I K_S^J R^{PS}) \\
&= \beta^{DI}\beta^{LJ}C_{DL}{}^{M} + \beta^{DJ}\beta^{LM}C_{DL}{}^{I} + \beta^{DM}\beta^{LI}C_{DL}{}^{J} \\
&+ l_L^i K_N^L K_T^I R^{NT} K_S^J R^{PS}\partial_i K_P^M + l_L^i K_N^L K_T^I R^{NT} K_P^M R^{PS}\partial_i K_S^J - l_L^i K_N^L K_T^J R^{NT} K_S^I R^{PS}\partial_i K_P^M \\
&- l_L^i K_N^L K_T^J R^{NT} K_P^M R^{PS}\partial_i K_S^I + l_L^i K_N^M K_T^L R^{NT} K_S^J R^{PS}\partial_i K_P^I + l_L^i K_N^M K_T^L R^{NT} K_P^I R^{PS}\partial_i K_S^J \\
&= \beta^{DI}\beta^{LJ}C_{DL}{}^{M} + \beta^{DJ}\beta^{LM}C_{DL}{}^{I} + \beta^{DM}\beta^{LI}C_{DL}{}^{J} \\
&- K_N^L K_T^I R^{NT} K_S^J R^{PS} C_{DL}{}^{M} K_P^D - K_N^L K_T^I R^{NT} K_P^M R^{PS} C_{DL}{}^{J} K_S^D + K_N^L K_T^J R^{NT} K_S^I R^{PS} C_{DL}{}^{M} K_P^D \\
&+ K_N^L K_T^J R^{NT} K_P^M R^{PS} C_{DL}{}^{I} K_S^D - K_N^M K_T^L R^{NT} K_S^J R^{PS} C_{DL}{}^{I} K_P^D - K_N^M K_T^L R^{NT} K_P^I R^{PS} C_{DL}{}^{J} K_S^D \\
&= 3(\beta^{DI}\beta^{LJ}C_{DL}{}^{M} + \beta^{DJ}\beta^{LM}C_{DL}{}^{I} + \beta^{DM}\beta^{LI}C_{DL}{}^{J}) = 0,
\end{aligned}
$$

where we have used (5.26) in the fourth equality and the dressed R-matrix satisfies the CYBE (5.9), see the Proposition 5.2.1. This completes the proof with $R = [\beta, \beta]_{\mathrm{SN}} = 0$.
■

Lemma 5.2.1. *From the Proposition 3.1.4, there exists a Lie algebroid structure on the cotangent bundle of this Poisson manifold which together yield a Lie bialgebroid structure on TM and T^*M where M is the group manifold. (TM is itself a Lie algebroid with the Lie bracket, see the second one in Examples 3.1.1.) Furthermore, there exists a Courant algebroid structure since there is a one-to-one correspondence between (triangular) Lie bialgebroids and Courant algebroids with transversal Dirac structures, as stated in Subsection 3.3.5.*

6. DUALITY COVARIANT FORMULATION OF STRING THEORY

String theory is a promising candidate to unify quantum mechanics and Einstein's general theory of relativity (GR). Riemannian geometry is the right mathematical framework to describe GR, and the structure of the theory is largely dictated by diffeomorphism invariance. String theory is endowed by additional symmetries besides diffeomorphisms. The theory called double field theory (DFT) has been constructed by Hull and Zwiebach that takes into account some of these stringy symmetries.

[8] DFT is a field theory defined on a doubled space, which implements the $O(d,d)$ T-duality symmetry of string theory as a manifest symmetry. In addition to the standard space-time coordinates, the doubled space also includes dual coordinates, which are associated with the winding excitations of closed strings on backgrounds with non-trivial cycles. The space-time and dual coordinates $X^M = (x^m, \widetilde{x}_m)$ transform as a vector under the T-duality group $O(d,d)$. In DFT, the semi-Riemannian metric and the B-field are encoded in the generalized metric $\mathscr{H}$ defined in (3.3.9). A generalized tensor field is defined according to how they transform under generalized diffeomorphisms of DFT. Infinitesimal transformations under generalized diffeomorphisms are generated by the generalized Lie derivative, which defines the D-bracket that generalizes the Lie bracket. Anti-symmetrization of the D-bracket gives the C-bracket. For further details see [53] and [54]. In investigation of the geometric features of the DFT, the generalized versions of the Courant algebroids that are DFT algebroids have been studied in literature. However, our consideration will be on the Courant algebroids since the strong constraint should be imposed in DFT which is explained below.

[8] In its current formulation, DFT is a consistent field theory only when a certain constraint, called the strong constraint is satisfied. When the strong constraint is satisfied generalized tensor fields become sections of the direct sum of the tangent and the cotangent bundle, generalized diffeomorphisms reduce to the semi-direct product of space-time diffeomorphisms and B-field gauge transformations, and the C-bracket becomes the standard Courant bracket which is defined in (3.35). Also, when the strong constraint is imposed, conditions on the DFT fluxes arising from the consistency requirement of the physical theory are the same as the conditions on the structure functions of the associated Courant algebroid determined by the axioms, as discussed in the Propositions 6.2.3 and 6.2.4.

[8] A trivial solution of the strong constraint occurs when all the fields and gauge parameters in the theory are independent of the winding type coordinates. Such fields are said to belong to the supergravity frame and it can be shown that the DFT action reduces to the standard supergravity action in the supergravity frame. This makes DFT a good framework to study solutions of supergravity.

The associated matrix with the specific non-constant $O(d,d)$ transformation will be called YB matrix: $\mathcal{T}_{YB}$ (6.65) which is constructed from the bi-vector field β, demonstrated in the Subsection 5.2, together with the isometry components will act as a solution-generating transformation in DFT. This fact will be seen from the calculation of the fluxes associated with the matrices L and $\mathcal{T}_{YB}$ which gives exactly the same results in terms of the conditions on fluxes from the properties of β which can be followed from the Subsection 6.2.4. This fact stands for the preservation of the fluxes and transformation associated with the YB matrix then acts as a solution generating transformation in string theory. The Subsection 6.3 describes how the new string theory solutions result from this $O(d,d)$ transformation which is described briefly as follows. It has been seen earlier that this bi-vector field is actually a Poisson tensor and it defines not only a triangular Lie bialgebroid structure but also a Courant algebroid structure with transversal Dirac structures. We have demonstrated in the Subsection 3.3.6 that for a given Riemannian metric and two-form, one can construct

a generalized metric defined in (3.57) on an exact Courant algebroid. By using β-transformation generated by the dressed R-matrix, a new generalized metric can be constructed, which contains the new Riemannian metric and the two-form field that are again solutions for the supergravity equations. Since the fluxes are preserved, these geometric structures are still solutions. Namely, fluxes associated with the YB matrix and the conditions must be satisfied by the structure functions in a Courant algebroid $TM \oplus T^*M$ with a Poisson bi-vector are exactly the same.

We will start by introducing some features of the DFT briefly.

6.1 A Brief Review of Double Field Theory

In this part of the thesis; after presenting basic geometric ingredients in DFT such as doubled coordinates and the constraint, we will focus on some of the important mathematical structures arising in this theory. More precisely, the notions such as C-bracket and the relation with the standard Courant bracket, D-bracket and the generalized Lie derivative, will be examined. These mathematical structures are important for determining the symmetries and the gauge algebra of DFT and these will be used in further Subsections.

Let us give firstly basic geometric ingredients of DFT of NS-NS sector. The terminology comes from physics and we will be using the same terminology without giving any details about the physical meaning. The basic symmetry group that defines the DFT of the NS-NS sector is $O(d,d)$ and the DFT of the NS-NS sector will be studied throughout the thesis.

Doubled Coordinates

As it was stated in above, the most important idea in DFT is to double coordinates and the idea is as follows. Closed string theory with the target space geometries contains two different sets of momenta. On the one hand, there are the standard momenta

canonically conjugate to the center of mass coordinates of the string and on the other hand, there is the possibility of a string winding around compact directions, resulting in winding degrees of freedom. Therefore, these together set up both types of the coordinates: the standard momentum (x) as well as the resulting coordinates conjugate to winding which give a second set of coordinates referred to as winding coordinates ($\widetilde{x}$). The physical fields depend on both sets of coordinates.

Remark 6.1.1. *[55] In DFT, it is the coordinates that are doubled and 'double fields' refer to fields which depend on both sets of coordinates. Therefore, we also refer to double functions, double vector fields, double forms, etc.*

Under $O(d,d)$ transformations, x and $\widetilde{x}$ transform as vectors such that [56]

$$\begin{pmatrix} \widetilde{x} \\ x \end{pmatrix} \to h \begin{pmatrix} \widetilde{x} \\ x \end{pmatrix}, \tag{6.1}$$

where

$$h = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{6.2}$$

Here, h is $2d \times 2d$ invertible matrix belonging to the $O(d,d)$ group.

Previously, the notions that we have studied have no dependence to the dual coordinates $\widetilde{x}$ and hence these are indeed the generalizations of the previous ideas.

Definition of C- and D-brackets and Their Properties

The bracket which determines the gauge algebra of DFT is the C-bracket. It will be seen that C-bracket reduces to the standard Courant bracket when the strong constraint (the trivial solution of it), which is (6.8), imposed.

Definition 6.1.1. *The C-bracket of the doubled sections is defined by [57]*

$$[\Sigma_1, \Sigma_2]_C^M = \Sigma_{12}^M = \left(\Sigma_1^N \partial_N \Sigma_2^M - \frac{1}{2} \Sigma_1^N \partial^M \Sigma_{2N} \right) - c.p., \tag{6.3}$$

where Σ is the pair consisting of

$$\Sigma^M = \begin{pmatrix} \Sigma^m \\ \Sigma_m \end{pmatrix}, \tag{6.4}$$

and ∂^M (generalized derivative) is the pair consisting of

$$\partial^M = \begin{pmatrix} \partial_i \\ \widetilde{\partial}^i \end{pmatrix}, \tag{6.5}$$

both depending on doubled coordinates $(x, \widetilde{x})$.

Constraint

Definition 6.1.2. *In DFT, all fields and gauge parameters should satisfy the conditions called weak constraint and strong constraint [56]:*

$$\partial_M \partial^M A = 0, \tag{6.6}$$

and

$$\partial_M A \partial^M B = 0, \tag{6.7}$$

respectively.

Lemma 6.1.1. *The trivial solution of these constraints is*

$$\widetilde{\partial}^i(\ldots) = 0. \tag{6.8}$$

Proposition 6.1.1. *When the strong constraint (the trivial solution of it), which is* (6.8)*, is imposed, the C-bracket reduces to the standard Courant bracket given in* (3.35)*.*

Proof: Let us start with the equation (6.3) which has an upper index as

$$\begin{aligned} ([\Sigma_1, \Sigma_2]_C)^i = {} & \xi_1^j \partial_j \xi_2^i + \widetilde{\xi}_{1j} \widetilde{\partial}^j \xi_2^i - \xi_2^j \partial_j \xi_1^i - \widetilde{\xi}_{2j} \widetilde{\partial}^j \xi_1^i \\ & - \frac{1}{2} \xi_1^j \widetilde{\partial}^i \widetilde{\xi}_{2j} - \frac{1}{2} \widetilde{\xi}_{1j} \widetilde{\partial}^i \xi_2^j + \frac{1}{2} \xi_2^j \widetilde{\partial}^i \widetilde{\xi}_{1j} + \frac{1}{2} \widetilde{\xi}_{2j} \widetilde{\partial}^i \xi_1^j, \end{aligned} \tag{6.9}$$

where $\Sigma = \xi + \widetilde{\xi}$.

Applying the strong constraint (the trivial solution of it) (6.8) ($\widetilde{\partial}^i(\ldots) = 0$) in above equation yields

$$([\Sigma_1,\Sigma_2]_C)^i = \xi_1^j\partial_j\xi_2^i - \xi_2^j\partial_j\xi_1^i = \left(\mathscr{L}_{\xi_1}\xi_2\right)^i = ([\xi_1,\xi_2])^i, \tag{6.10}$$

which is actually a Lie bracket.

If we consider the lower index, we get

$$\begin{aligned}([\Sigma_1,\Sigma_2]_C)_i = {}& \xi_1^j\partial_j\widetilde{\xi}_{2i} + \widetilde{\xi}_{1j}\widetilde{\partial}^j\widetilde{\xi}_{2i} - \xi_2^j\partial_j\widetilde{\xi}_{1i} - \widetilde{\xi}_{2j}\widetilde{\partial}^j\widetilde{\xi}_{1i} \\ & - \frac{1}{2}\xi_1^j\partial_i\widetilde{\xi}_{2j} - \frac{1}{2}\widetilde{\xi}_{1j}\partial_i\xi_2^j + \frac{1}{2}\xi_2^j\partial_i\widetilde{\xi}_{1j} + \frac{1}{2}\widetilde{\xi}_{2j}\partial_i\xi_1^j.\end{aligned} \tag{6.11}$$

After applying the strong constraint (the trivial solution of it) given in (6.8), we obtain

$$\begin{aligned}([\Sigma_1,\Sigma_2]_C)_i &= \xi_1^j\partial_j\widetilde{\xi}_{2i} - \frac{1}{2}(\xi_1^j\partial_i\widetilde{\xi}_{2j} - \widetilde{\xi}_{2j}\partial_i\xi_1^j) - (1\leftrightarrow 2) \\ &= \xi_1^j\partial_j\widetilde{\xi}_{2i} - \frac{1}{2}\{\partial_i(\xi_1^j\widetilde{\xi}_{2j}) - 2\widetilde{\xi}_{2j}\partial_i\xi_1^j\} - (1\leftrightarrow 2) \\ &= \xi_1^j\partial_j\widetilde{\xi}_{2i} + \widetilde{\xi}_{2j}\partial_i\xi_1^j - \frac{1}{2}\partial_i(\xi_1^j\widetilde{\xi}_{2j}) - (1\leftrightarrow 2) \\ &= \left(\mathscr{L}_{\xi_1}\widetilde{\xi}_2 - \frac{1}{2}d(i_{\xi_1}\widetilde{\xi}_2)\right)_i - (1\leftrightarrow 2).\end{aligned} \tag{6.12}$$

Adding (6.10) and (6.12) up gives

$$[\Sigma_1,\Sigma_2]_C = [\xi_1,\xi_2] + \left\{\mathscr{L}_{\xi_1}\widetilde{\xi}_2 - \frac{1}{2}d(i_{\xi_1}\widetilde{\xi}_2) - (1\leftrightarrow 2)\right\}. \tag{6.13}$$

For $\Sigma_1 = A+\alpha$ and $\Sigma_2 = B+\beta$ elements of $T\oplus T^*$,

$$\begin{aligned}[A+\alpha,B+\beta]_C &= [A,B] + \mathscr{L}_A\beta - \frac{1}{2}d(i_A\beta) - \mathscr{L}_B\alpha + \frac{1}{2}d(i_B\alpha) \\ &= [A,B] + \mathscr{L}_A\beta - \mathscr{L}_B\alpha - \frac{1}{2}d(i_A\beta - i_B\alpha) \\ &= [A+\alpha,B+\beta]_{T\oplus T*},\end{aligned} \tag{6.14}$$

which is indeed the standard Courant bracket given in (3.35). ■

D-bracket is defined from the C-bracket such that its skew-symmetrization gives the C-bracket:

$$[\Sigma_1,\Sigma_2]_C = \frac{1}{2}([\Sigma_1,\Sigma_2]_D - [\Sigma_2,\Sigma_1]_D), \tag{6.15}$$

and the generalized Lie derivative of generalized vectors can be defined from the D-bracket, since it reduces to a Dorfman bracket under strong constraint which satisfies the Jacobi identity, that is

$$[\Sigma_1, \Sigma_2]_D = \mathscr{L}_{\Sigma_1}\Sigma_2. \tag{6.16}$$

Proposition 6.1.2. *D-bracket defined in* (6.15) *reduces to the standard Dorfman bracket given in* (3.42) *by using* (6.18).

Definition 6.1.3. *Local coordinates of the action of the generalized Lie derivative* $\mathscr{L}_\Sigma$ *is as follows [55].*

$$\mathscr{L}_\Sigma \phi = \Sigma^M \partial_M \phi, \tag{6.17}$$

$$(\mathscr{L}_\Sigma V)_M = \Sigma^K \partial_K V_M + (\partial_M \Sigma^K - \partial^K \Sigma_M) V_K, \tag{6.18}$$

respectively. In above, ϕ *is a doubled function on the base manifold and* V *is a doubled vector field.*

6.2 DFT Fields and Action

6.2.1 DFT fields

The fundamental fields in the low energy effective action of the universal sector of string theory are the semi-Riemannian metric g and the 2-form field B. The main element of the DFT utilized to write the action, introduced in the Definitions 3.3.8 and 3.3.9, is the generalized metric which is called the DFT field. This metric consists of the Riemannian metric g and the two-form B of the manifold required to represent the string theory. For a chosen basis of sections for TM and T^*M, the generalized metric is expressed in the literature by the following form.

$$\mathscr{H} = \begin{pmatrix} \mathscr{H}_{ij} & \mathscr{H}_i{}^j \\ \mathscr{H}^i{}_j & \mathscr{H}^{ij} \end{pmatrix} = \begin{pmatrix} g_{ij} - B_{in}g^{nk}B_{kj} & B_{ik}g^{kj} \\ -g^{ik}B_{kj} & g^{ij} \end{pmatrix} \in O(d,d), \tag{6.19}$$

or it can be written independently of local coordinates as follows.

$$\mathscr{H} = \begin{pmatrix} g - Bg^{-1}B & Bg^{-1} \\ -g^{-1}B & g^{-1} \end{pmatrix} \in O(d,d). \tag{6.20}$$

Here, we assume the Riemannian metric g, the inverse of g and B-field as maps in such a way that $g : TM \to T^*M$, $g^{-1} : T^*M \to TM$ and $B : TM \to T^*M$, respectively, as in the Subsection 3.3.6.

Proposition 6.2.1. *The local form of the generalized metric given in* (6.19) *is actually the same with the generalized metric on a Courant algebroid given in* (3.56)*.*

Proof: Our first aim is to find the local expressions of the generalized metric $\mathscr{H}$ defined in (3.56). Given any $A = X + \eta \in TM \oplus T^*M$, where $X \in TM$ and $\eta \in T^*M$, there exist elements $A_+ \in \tilde{S}_+$ and $A_- \in \tilde{S}_-$ such that

$$A = A_+ + A_-.$$

Therefore, we have seen that $TM \oplus T^*M = \tilde{S}_+ \oplus \tilde{S}_-$.

Let us call $A_+ = P_+A$ and $A_- = P_-A$. Let

$$\begin{aligned} 2A_+ &= X + Y + (B+g)(X+Y), \\ 2A_- &= X - Y + (B-g)(X-Y), \end{aligned}$$

then

$$\begin{aligned} A_+ + A_- &= X + B(X) + g(Y) \\ &= X + \eta \\ &= A, \end{aligned}$$

where we have chosen Y as an element of TM such that $g(Y) = \eta - B(X)$ which gives us $Y = g^{-1}(\eta) - g^{-1}B(X) \in TM$. Note that $B(X) + g(Y)$ is indeed an element of T^*M.

If we take $A = e_i = \partial_i$ which is the local basis of TM, then $X = e_i$ and $\eta = 0$, and so that $Y = -g^{-1}B(e_i)$. Therefore,

$$e_{i+} := P_+(e_i) = \frac{1}{2}[e_i - g^{-1}B(e_i) + g(e_i) - Bg^{-1}B(e_i)], \tag{6.21}$$

$$e_{i-} := P_-(e_i) = \frac{1}{2}[e_i + g^{-1}B(e_i) - g(e_i) + Bg^{-1}B(e_i)]. \tag{6.22}$$

Here, note that $e_i, g^{-1}B(e_i) \in TM$ while $g(e_i), Bg^{-1}B(e_i) \in T^*M$.

Now, let us take $A = e^i = dx^i$ which is the local basis of T^*M, then $X = 0$ and $\eta = e^i$, and so that $Y = g^{-1}(e^i)$. Hence,

$$e^i_+ = P_+(e^i) = \frac{1}{2}[g^{-1}(e^i) + e^i + Bg^{-1}(e^i)], \tag{6.23}$$

$$e^i_- = P_-(e^i) = \frac{1}{2}[-g^{-1}(e^i) + e^i - Bg^{-1}(e^i)]. \tag{6.24}$$

Here, note that $g^{-1}(e^i) \in TM$ while $e^i, Bg^{-1}(e^i) \in T^*M$.

The local expressions of the terms in the above equations and the related ones will be used so that these will be given now.

$$\begin{aligned}
g(e_i) &= g_{in}e^n, \ \textit{where} \ g_{in} = g(e_i, e_n), \\
g^{-1}(e^i) &= g^{in}e_n, \\
B(e_i) &= B_{in}e^n, \ \textit{where} \ B_{in} = B(e_i, e_n), \\
g^{-1}B(e_i) &= B_{ik}g^{kn}e_n, \\
Bg^{-1}(e^i) &= g^{in}B_{nk}e^k, \\
g(e_j, g^{-1}B(e_i)) &= g(e_j, B_{ik}g^{kn}e_n) = B_{ik}g^{kn}g_{jn} = B_{ij}, \\
Bg^{-1}B(e_i) &= B_{nm}B_{ik}g^{kn}e^m, \\
i_{e_i}[Bg^{-1}B(e_j)] &= B_{ni}B_{jk}g^{kn} = i_{e_j}[Bg^{-1}B(e_i)], \\
i_{[g^{-1}B(e_i)]}[Bg^{-1}B(e_j)] &= B_{im}g^{mp}B_{np}B_{jk}g^{kn} = -i_{[g^{-1}B(e_j)]}[Bg^{-1}B(e_i)], \\
g(e_j, g^{-1}B(e_i)) &= B_{ij} = -g(e_i, g^{-1}B(e_j)).
\end{aligned}$$

We are ready to calculate now $\mathscr{H}(e_i, e_j) = \mathscr{H}_{ij}$, $\mathscr{H}(e^i, e^j) = \mathscr{H}^{ij}$, $\mathscr{H}(e_i, e^j) = \mathscr{H}_i{}^j$ and $\mathscr{H}(e^i, e_j) = \mathscr{H}^i{}_j$, respectively, by using the Definition (3.56), (6.21), (6.22), (6.23), (6.24) and the above rules.

$$\begin{aligned}
\mathscr{H}_{ij} = \mathscr{H}(e_i, e_j) &= \langle P_+(e_i), P_+(e_j)\rangle - \langle P_-(e_i), P_-(e_j)\rangle \\
&= \frac{1}{4}(\langle e_i - g^{-1}B(e_i) + g(e_i) - Bg^{-1}B(e_i), e_j - g^{-1}B(e_j) + g(e_j) - Bg^{-1}B(e_j)\rangle \\
&- \langle e_i + g^{-1}B(e_i) - g(e_i) + Bg^{-1}B(e_i), e_j + g^{-1}B(e_j) - g(e_j) + Bg^{-1}B(e_j)\rangle) \\
&= g_{ij} + B_{in}B_{jk}g^{kn},
\end{aligned} \tag{6.25}$$

$$\begin{aligned}\mathscr{H}^{ij} &= \mathscr{H}(e^i,e^j) = \langle P_+(e^i),P_+(e^j)\rangle - \langle P_-(e^i),P_-(e^j)\rangle \\ &= \frac{1}{4}(\langle g^{-1}(e^i)+e^i+Bg^{-1}(e^i),g^{-1}(e^j)+e^j+Bg^{-1}(e^j)\rangle \\ &- \langle -g^{-1}(e^i)+e^i-Bg^{-1}(e^i),-g^{-1}(e^j)+e^j-Bg^{-1}(e^j)\rangle) \\ &= g^{ij}, \end{aligned} \tag{6.26}$$

$$\begin{aligned}\mathscr{H}_i{}^j &= \mathscr{H}(e_i,e^j) = \langle P_+(e_i),P_+(e^j)\rangle - \langle P_-(e_i),P_-(e^j)\rangle \\ &= \frac{1}{4}(\langle e_i - g^{-1}B(e_i)+g(e_i)-Bg^{-1}B(e_i),g^{-1}(e^j)+e^j+Bg^{-1}(e^j)\rangle \\ &- \langle e_i + g^{-1}B(e_i)-g(e_i)+Bg^{-1}B(e_i),-g^{-1}(e^j)+e^j-Bg^{-1}(e^j)\rangle) \\ &= -B_{ik}g^{kj}, \end{aligned} \tag{6.27}$$

and lastly,

$$\mathscr{H}^i{}_j = \mathscr{H}(e^i,e_j) = -B_{jk}g^{ki} = g^{ik}B_{kj}. \tag{6.28}$$

By using the results (6.25), (6.26), (6.27), and (6.28), we obtain the local form of the generalized metric in the basis ∂_i and dx^i as follows.

$$\mathscr{H} = \begin{pmatrix} \mathscr{H}_{ij} & \mathscr{H}_i{}^j \\ \mathscr{H}^i{}_j & \mathscr{H}^{ij} \end{pmatrix} = \begin{pmatrix} g_{ij}-B_{in}g^{kn}B_{kj} & -B_{ik}g^{kj} \\ g^{ik}B_{kj} & g^{ij} \end{pmatrix}, \tag{6.29}$$

which completes the proof. ■

The equivalency of the following definitions has been provided in Subsection 3.3.6 without any supporting evidence. For our purposes, it is now time to prove in local coordinates.

Proposition 6.2.2. *The Definitions 3.3.8 and 3.3.9 are equivalent definitions.*

Proof: It has been given in the Proposition 3.3.5 that the subbundles $\tilde{S}_\pm$ defined in (3.54) and (3.55), respectively, are the ± 1-eigenbundles of the endomorphism τ, which is as defined in (3.53), with $\tau^2 = 1$ acting on $TM \oplus T^*M$. By using these facts, we have

$$\tau(P_+(e_i)) = P_+(e_i),$$

and

$$\tau(P_-(e_i)) = -P_-(e_i).$$

Therefore,

$$\begin{aligned}\langle e_i, \tau(e_j)\rangle &= \langle P_+(e_i)+P_-(e_i), \tau(P_+(e_j)+P_-(e_j))\rangle \\ &= \langle P_+(e_i)+P_-(e_i), P_+(e_j)-P_-(e_j)\rangle \\ &= \langle P_+(e_i), P_+(e_j)\rangle - \langle P_-(e_i), P_-(e_j)\rangle \\ &= \mathscr{H}(e_i, e_j), \end{aligned} \tag{6.30}$$

where we have used the fact that the subbundles $\tilde{S}_+$ and $\tilde{S}_-$ are orthogonal (see the Proposition 3.3.6) in the third line and this completes the proof. ■

Also, it can be easily shown that $\tau^p_j = \eta^{pi}\mathscr{H}_{ij}$ by using the equality (6.30).

6.2.2 Action

The DFT action is the following [58] (and see references in there):

$$\int \mathscr{L}_{NS-NS} dx d\tilde{x}, \tag{6.31}$$

where the Lagrangian of the NS-NS sector is defined by

$$\mathscr{L}_{NS-NS} = e^{-2d}\mathscr{R}(\mathscr{H}, d). \tag{6.32}$$

Here, the term $\mathscr{R}(\mathscr{H}, d)$ is the generalized Ricci scalar and its explicit form is as follows:

$$\begin{aligned}\mathscr{R}(\mathscr{H}, d) =& 4\mathscr{H}^{MN}\partial_M\partial_N d - \partial_M\partial_N\mathscr{H}^{MN} - 4\mathscr{H}^{MN}\partial_M d\partial_N d + 4\partial_M\mathscr{H}^{MN}\partial_N d \\ &+ \frac{1}{8}\mathscr{H}^{MN}\partial_M\mathscr{H}^{KL}\partial_N\mathscr{H}_{KL} - \frac{1}{2}\mathscr{H}^{MN}\partial_M\mathscr{H}^{KL}\partial_K\mathscr{H}_{NL}. \end{aligned} \tag{6.33}$$

The field equations of 10-dimensional supergravity obtained by varying the DFT action with respect to the generalized dilaton field d is [58]

$$\mathscr{R} = 0, \tag{6.34}$$

where $\mathscr{R}$ is as in (6.33). We will be working with generalized metrics which satisfy the field equations above. If the generalized metric depends on the space-time coordinates only so that we are in the supergravity frame, the associated Riemannian metric g,

the B-field, and the dilaton field d, which together construct (6.19), will solve the supergravity equations.

Our consideration will be on the solutions such that $\mathscr{H}$ can be written in the following separated form [58]

$$\mathscr{H}^{MN}(x,Y) = (U^{-1})^{M}{}_{A}(Y)\hat{\mathscr{H}}^{AB}(x)(U^{-1})^{N}{}_{B}(Y), \tag{6.35}$$

where the matrix $U^{-1}(Y)$ is usually called a twist matrix that is an element of $O(d,d)$ and (x,Y) are DFT coordinates which are a set of coordinates that to not mixed up. x and Y are referred to as internal and external coordinates, respectively. In our case, the group manifold's coordinates are Y coordinates, and the other coordinates are x coordinates. Additionally, $\hat{\mathscr{H}}^{AB}(x)$ will generally be assumed to be constant. The explanation for this choice will be provided in Subsection 6.4.

An example for this situation is when g and B-field in the generalized metric $\mathscr{H}$ admits isometries. Here, the twist matrix, which is an $O(d,d)$ matrix as in (6.2), takes the form

$$U^{-1} = L = \begin{pmatrix} l^{t} & 0 \\ 0 & l^{-1} \end{pmatrix}, \tag{6.36}$$

which will be used again in the Subsection 6.2.4. In the above equation l's are the components of the left invariant one-forms, see the Subsection 5.2 for further details. If the action of the isometry group has no fixed points and is transitive, then [58]

$$\begin{aligned} ds^2 &= g_{ij}d\theta^i d\theta^j \\ &= g_{\alpha\beta}\sigma^\alpha\sigma^\beta, \end{aligned}$$

where θ^i, $i = 1,\cdots,d$, are the coordinates for the isometry group G and σ^α, $\alpha = 1,\cdots,dimG$, are the left invariant one-forms $\sigma^\alpha = l^\alpha_i d\theta^i$ on G defined by the Maurer-Cartan form. Here, g_{ij} are independent of the coordinates θ and are constants. All the θ dependence is inside of σ^α and σ^β.

Another example is when G is a Poisson Lie group. In this case, it can be shown that the generalized tangent bundle $TG\oplus T^*G$ can be trivialized to the direct product bundle

$G \times \mathfrak{d}$ where $\mathfrak{d}$ is the natural Drinfel'd double structure on $g \oplus g^*$. Then, if $\mathscr{H}^{AB}$ is the tensor corresponding to the generalized metric in $G \times \mathfrak{d}$ (which is just $G \times h$, where h is a maximally isotropic subalgebra), its image under the isomorphism to $TG \oplus T^*G$ will be just of the form (6.35) where

$$U^{-1} = \begin{pmatrix} l^t & 0 \\ l^{-1}\Pi & l^{-1} \end{pmatrix},$$

where Π is the Poisson bi-vector field corresponding to the Poisson Lie group structure on G.

If the generalized metric in the form of (6.35) satisfies the field equations given in (6.34), then $\hat{\mathscr{H}}$ satisfies [58]

$$\mathscr{R}_{\text{def}} = 0, \tag{6.37}$$

where the deformed Ricci scalar is given by

$$\mathscr{R} \to \mathscr{R}_{\text{def}} = \mathscr{R} + \mathscr{R}_F, \tag{6.38}$$

with

$$\begin{aligned} \mathscr{R}_F = &-\frac{1}{2}F^A_{BC}\hat{\mathscr{H}}^{BD}\hat{\mathscr{H}}^{CE}\partial'_D\hat{\mathscr{H}}_{AE} - \frac{1}{12}F^A_{BC}F^D{}_{EF}\hat{\mathscr{H}}_{AD}\hat{\mathscr{H}}^{BE}\hat{\mathscr{H}}^{CF} \\ &-\frac{1}{4}F^A_{BC}F^B_{AD}\hat{\mathscr{H}}^{CD} - 2\eta_A\partial'_B\hat{\mathscr{H}}^{AB} + 4\eta_A\hat{\mathscr{H}}^{AB}\partial'_B d - \eta_A\eta_B\hat{\mathscr{H}}^{AB}, \end{aligned} \tag{6.39}$$

where $\partial'_D = (U^{-1})^M{}_D\partial_M$.

Recall that the theory is in the supergravity frame when the strong constraint given in (6.8) is satisfied because for this solution of the constraint, (6.32) reduces to the standard NS-NS action for the massless fields of string theory [58]. On the other hand, the theory described by the action (6.38) is called gauged double field theory (GDFT) and it reduces to the action of gauged supergravity in the supergravity frame. The GDFT action is a deformation of the DFT action determined by the fluxes F_{ABC}. Recall that the notion of flux was introduced in Remark 4.4.2 as structure functions of a Courant algebroid in local coordinates. These two seemingly unrelated notions of flux are in fact the same, as will be demonstrated in the next subsection in the

Propositions 6.2.3 and 6.2.4.

It will be investigated in the following Subsection that the calculation of the fluxes associated with the matrix L, which is given in (6.36), yields only non-vanishing flux as f-flux which is just the structure constants of the Lie algebra of the isometry group G. Also, the fluxes associated with the T_{YB} matrix will yield the same results. This result means that the YB transformation is a flux preservation transformation and it leads a new generalized metric $\widetilde{H}$ constructs a solution of the DFT. Therefore, the transformation associated with the YB matrix acts as a solution generating transformation in DFT. These results will be discussed in the Subsection 6.3 in detail.

We would like to first present the DFT fluxes in order to obtain all of these conclusions.

6.2.3 Fluxes in DFT

The geometric flux and non-geometric flux can be unified into a single $O(d,d)$ tensor known as the covariant flux.

Definition 6.2.1. *The covariant flux is defined in terms of the C-bracket of the generalized components of an $O(d,d)$ transformation on $TM \oplus T^*M$ as follows [50].*

$$F_{ABC} = [E_A, E_B]_C^M E_{CM}, \tag{6.40}$$

where the C-bracket is defined as in (6.3)*, and where twist matrix $E = U^{-1}$ is given by*

$$E = \begin{pmatrix} a^t & a^t\, B \\ a\,\beta & a + a\,\beta\, B \end{pmatrix},$$

where a is a diffeomorphism on TM which also induces a diffeomorphism on T^*M, B is a smooth two-form such that $B : TM \to T^*M$ and β is an anti-symmetric bi-vector field such that $\beta : T^*M \to TM$, which have been introduced in the Subsection 3.3.2. The twist matrix can be written in terms of a chosen coordinate basis as follows.

$$E_M^L = \begin{pmatrix} E_l{}^i & E_{li} \\ E^{li} & E^l_i \end{pmatrix} = \begin{pmatrix} a_l{}^i & a_l{}^k B_{ki} \\ a^l{}_k \beta^{ki} & a^l{}_i + a^l{}_k \beta^{kp} B_{pi} \end{pmatrix}. \tag{6.41}$$

Remark 6.2.1. *In the physics literature, $E_M^L = \langle E(\partial_M), \partial^L \rangle$ is called the generalized vielbein. This is because when E acts on the coordinates basis $\partial^M = (\partial_i, dx^i)$, it yields*

a new non-coordinate basis for the sections of the generalized tangent bundle as given below.

If B-field is taken to be zero, then

$$\begin{pmatrix} a_l{}^i & 0 \\ a^l{}_k\beta^{ki} & a^l{}_i \end{pmatrix}\begin{pmatrix} \partial_i \\ dx^i \end{pmatrix} = \begin{pmatrix} a_l{}^i\partial_i \\ a^l{}_k\beta^{ki}\partial_i + a^l{}_i dx^i \end{pmatrix}, \tag{6.42}$$

where the new non-coordinate basis (e_l, e^l) *is*

$$e_l = a_l{}^i\partial_i, \tag{6.43}$$

and

$$e^l = a^l{}_k\beta^{ki}\partial_i + a^l{}_i dx^i, \tag{6.44}$$

which will be used in the Subsection 6.4.

By using (6.3), the equation (6.40) becomes

$$\begin{aligned} F_{ABC} =& E_A^N\partial_N E_B{}^M E_{CM} - \frac{1}{2}E_{AN}\partial^M E_B^N E_{CM} - E_B^N\partial_N E_A{}^M E_{CM} \\ &+ \frac{1}{2}E_{BN}\partial^M E_A^N E_{CM}. \end{aligned} \tag{6.45}$$

Defining Ω_{ABC} such that [50]

$$\Omega_{ABC} = E_A{}^N(\partial_N E_B{}^M)E_{CM}, \tag{6.46}$$

where E_A^N is an $O(d,d)$ matrix, the covariant flux becomes

$$F_{ABC} = \Omega_{ABC} - \frac{1}{2}\Omega_{CBA} - \Omega_{BAC} + \frac{1}{2}\Omega_{CAB}.$$

We have $E_{BM} = \eta_{MN}E_B^N$, where η_{MN} is the matrix given in (4.28). Since η_{MN} is an $O(d,d)$ invariant metric and $E_A{}^M$ are $O(d,d)$ matrices, we have

$$\eta_{AB} = E_A{}^M E_{BM} = E_A{}^M\eta_{MN}E_B^N,$$

and its derivative $\partial_N\eta_{AB} = 0$. Then, Ω_{ABC} is anti-symmetric in the last two indices such that

$$\begin{aligned} \Omega_{ABC} = E_A{}^N(\partial_N E_B{}^M)E_{CM} &= -E_A{}^N(\partial_N E_{CM})E_B{}^M \\ &= -E_A{}^N(\partial_N(\eta_{MK}E_C^K))E_B{}^M \\ &= -E_A{}^N\partial_N E_C{}^K E_{BK} \\ &= -\Omega_{ACB}. \end{aligned} \tag{6.47}$$

Hence, the covariant flux can be rewritten as [50]

$$\begin{aligned} F_{ABC} &= \Omega_{ABC} - \frac{1}{2}\Omega_{CBA} - \Omega_{BAC} + \frac{1}{2}\Omega_{CAB}, \\ &= \Omega_{ABC} + \Omega_{BCA} + \Omega_{CAB}, \\ &= 3\Omega_{[ABC]}, \end{aligned} \tag{6.48}$$

which is totally anti-symmetric and these elements are corresponding to the fluxes $H_{abc}, f^{a}{}_{bc}, Q_{a}{}^{bc}$, and R^{abc} as we mentioned before in the Remark 4.4.2 and these notations will be used in the following results.

Note that by using the properties above, (6.46) can be written as follows and this equality will be used in the next Subsection.

$$\Omega_{ABC} = -E^{N}_{A}\partial_{N}E^{M}_{B}(E^{-1})^{D}_{M}\eta_{CD}. \tag{6.49}$$

It can be demonstrated that the fact (6.48) gives rise to the fluxes associated with generalized twist matrix (6.41) should meet the following conditions [50], [59].

$$\begin{aligned} &H_{abc} = 3\left(\nabla_{[a}B_{bc]} - B_{d[a}\tilde{\nabla}^{d}B_{bc]}\right), \\ &f_{ab}{}^{c} = 2\Gamma_{[ab]}{}^{c} + \tilde{\nabla}^{c}B_{ab} + 2\Gamma^{mc}{}_{[a}B_{b]m} + \beta^{cm}H_{mab}, \\ &Q_{c}{}^{ab} = 2\Gamma^{[ab]}{}_{c} + \partial_{c}\beta^{ab} + B_{cm}\tilde{\partial}^{m}\beta^{ab} + 2f_{mc}{}^{[a}\beta^{b]m} - H_{mnc}\beta^{ma}\beta^{nb}, \\ &R^{abc} = 3\left(\beta^{[\underline{a}m}\nabla_{m}\beta^{\underline{bc}]} + \tilde{\nabla}^{[a}\beta^{bc]} + B_{mn}\tilde{\nabla}^{n}\beta^{[ab}\beta^{c]m} + \beta^{[\underline{a}m}\beta^{\underline{b}n}\tilde{\nabla}^{\underline{c}]}B_{mn}\right) + \beta^{am}\beta^{bn}\beta^{cl}H_{mnl}, \end{aligned} \tag{6.50}$$

which stand for the definitions of the fluxes associated with that matrix and in above equations we have

$$\begin{aligned} B_{cb} = a_{c}{}^{i}a_{b}{}^{j}B_{ij}, \quad & \beta^{cb} = a^{c}{}_{i}a^{b}{}_{j}\beta^{ij}, \\ \Gamma_{cb}{}^{d} = a_{c}{}^{i}\partial_{i}a_{b}{}^{j}a^{d}_{j}, \quad & \Gamma^{cb}{}_{d} = a^{c}{}_{i}\tilde{\partial}^{i}a^{b}{}_{j}a^{j}_{d}, \\ \nabla_{a}B_{bc} = \partial_{a}B_{bc} - \Gamma_{ab}{}^{d}B_{dc} - \Gamma_{ac}{}^{d}B_{bd}, \quad & \tilde{\nabla}^{a}B_{bc} = \tilde{\partial}^{a}B_{bc} + \Gamma^{ad}{}_{b}B_{dc} + \Gamma^{ad}{}_{c}B_{bd}, \\ \nabla_{a}\beta^{bc} = \partial_{a}\beta^{bc} + \Gamma_{ad}{}^{b}\beta^{dc} + \Gamma_{ad}{}^{c}\beta^{bd}, \quad & \tilde{\nabla}^{a}\beta^{bc} = \tilde{\partial}^{a}\beta^{bc} - \Gamma^{ab}{}_{d}\beta^{dc} - \Gamma^{ac}{}_{d}\beta^{bd}, \end{aligned}$$

where $\partial_{i} = a_{i}{}^{j}\partial_{j}$ and $\tilde{\partial}^{i} = a^{i}{}_{j}\tilde{\partial}^{j}$. Also, the coefficients as 3, 2, etc., comes from the antisymmetrization.

When the dual coordinates have been taken to be zero, in other words under the strong constraint, the definition of the fluxes in (6.50) takes the following forms.

$$\begin{aligned} H_{abc} &= 3\nabla_{[a}B_{bc]}, \\ f_{ab}{}^{c} &= 2\Gamma_{[ab]}{}^{c} + \beta^{cm}H_{mab}, \\ Q_{c}{}^{ab} &= \partial_c\beta^{ab} + 2f_{mc}{}^{[a}\beta^{b]m} - H_{mnc}\beta^{ma}\beta^{nb}, \\ R^{abc} &= 3\beta^{[\underline{a}m}\nabla_m\beta^{\underline{bc}]} + \beta^{am}\beta^{bn}\beta^{cl}H_{mnl}, \end{aligned} \tag{6.51}$$

Proposition 6.2.3. *These definition of the fluxes in DFT under the strong constraint are exactly the same with the equalities of the structure functions that have been found from the second axiom in* (4.37) *to be a Courant algebroid in local coordinates for the anchor* $\rho^i_J = (\rho^i_j, \rho^{ij}_*) = (a^i_j, a^i_k\beta^{kj})$*. Precisely, the above definitions of fluxes correspond to* (4.66)*,* (4.67) *and* (4.68)*, respectively.*

Also, the definition of the fluxes in the DFT in (6.51) *become the followings for the assumption* $B_{bc} = 0$*.*

$$\begin{aligned} H_{abc} &= 0, \\ f_{bc}{}^{d} &= 2(a^i_{[b}\partial_i a^j_{c]}a^d_j), \\ Q_{c}{}^{ab} &= \partial_c\beta^{ab} + 2f_{cm}{}^{[a}\beta^{mb]}, \\ R^{abc} &= 3(\beta^{[am}\partial_m\beta^{bc]} + f^{[a}_{mn}\beta^{bm}\beta^{c]n}). \end{aligned}$$

These are also the same equalities as we found in Courant algebroid's axioms in local coordinates for the anchor $\rho^i_J = (a^i_j, a^i_k\beta^{kj})$ *which have been found in* (4.69)*,* (4.70)*, and* (4.71) *where H-flux is taken to be zero.*

[59] Also, the following conditions must be satisfied in terms of fluxes associated with that matrix and these conditions are the Bianchi identities as we mentioned earlier.

$$\begin{aligned} &\mathscr{D}_{[a}H_{bcd]} - \frac{3}{2}H_{e[ab}f_{cd]}{}^{e} = Z_{abcd} \\ &3\mathscr{D}_{[a}f_{bc]}{}^{d} - \mathscr{D}^{d}H_{abc} + 3f_{[ab}{}^{e}f_{c]e}{}^{d} - 3Q_{[a}{}^{de}H_{bc]e} = Z_{abc}{}^{d} \\ &2\mathscr{D}_{[a}Q_{b]}{}^{cd} + 2\mathscr{D}^{[c}f_{ab}{}^{d]} - f_{ab}{}^{e}Q_{e}{}^{cd} - H_{abe}R^{ecd} + 4Q_{[a}{}^{e[c}f_{b]e}{}^{d]} = Z_{ab}{}^{cd} \\ &3\mathscr{D}^{[a}Q_{d}{}^{bc]} - \mathscr{D}_{d}R^{abc} + 3Q_{e}{}^{[ab}Q_{d}{}^{c]e} - 3f_{de}{}^{[a}R^{bc]e} = Z^{abc}{}_{d} \\ &\mathscr{D}^{[a}R^{bcd]} - \frac{3}{2}R^{e[ab}Q_{e}{}^{cd]} = Z^{abcd}, \end{aligned} \tag{6.52}$$

where $\mathscr{D}_A = \rho_A{}^M \partial_M$ and Z_{ABCD} are terms which become zero when the strong constraint is imposed. Also, the coefficients as $\frac{3}{2}$, 3, etc., comes from the antisymmetrization. Although the precise form of these terms does not interest us, people who are interested can find further information together with the index notations at [59].

When the dual coordinates have been taken to be zero, the above identities take the following forms.

$$\begin{aligned}
&\mathscr{D}_{[a}H_{bcd]} - \frac{3}{2}H_{e[ab}f_{cd]}{}^e = 0,\\
&3\mathscr{D}_{[a}f_{bc]}{}^d - \mathscr{D}^d H_{abc} + 3f_{[ab}{}^e f_{c]e}{}^d - 3Q_{[a}{}^{de}H_{bc]e} = 0,\\
&2\mathscr{D}_{[a}Q_{b]}{}^{cd} + 2\mathscr{D}^{[c}f_{ab}{}^{d]} - f_{ab}{}^e Q_e{}^{cd} - H_{abe}R^{ecd} + 4Q_{[a}{}^{e[c}f_{b]e}{}^{d]} = 0,\\
&3\mathscr{D}^{[a}Q_d{}^{bc]} - \mathscr{D}_d R^{abc} + 3Q_e{}^{[ab}Q_d{}^{c]e} - 3f_{de}{}^{[a}R^{bc]e} = 0,\\
&\mathscr{D}^{[a}R^{bcd]} - \frac{3}{2}R^{e[ab}Q_e{}^{cd]} = 0,
\end{aligned}$$

where $\mathscr{D}_A = \rho_A{}^i \partial_i$.

Proposition 6.2.4. *These Bianchi identities in DFT under the strong constraint are exactly the same with the conditions on the structure functions that have been found from the third axiom in* (4.37) *to be a Courant algebroid in local coordinates for the anchor* $\rho_J^i = (\rho_j^i, \rho_*^{ij})$. *Precisely, the above identities correspond to* (4.41), (4.42), (4.43), (4.44) *and* (4.45), *respectively, when* $\rho_J^i = (\rho_j^i, \rho_*^{ij}) = (a_j^i, a_k^i \beta^{kj})$.

As a result, when the strong constraint is imposed in the definition and in the Bianchi identities of the fluxes that fluxes in the DFT must satisfy are the same as the conditions that structure functions must follow to be a Courant algebroid structure for a general anchor. This demonstrates that the fluxes determining the deformation of DFT to GDFT are just the structure functions of the Courant algebroid on which the theory is defined, with respect to the coordinate basis for the generalized tangent bundle.

It is now time to explore the fluxes associated with the related matrices and to determine that these fluxes are the same which fact will be used in the Section 6.3.

6.2.4 YB Matrix as a Twist Matrix

In this section we will calculate the fluxes associated with the twist matrix L given in (6.36) and the matrix T_{YB}. It will be demonstrated that these fluxes produce identical results, indicating that the flux preservation principle is hold for the fluxes associated with these matrices. The index convention can be found in the beginning of the Subsection 5. In the following computations, we take the coordinates on which the twist matrices depend on the standard coordinates and not the dual ones so that $\widetilde{\partial}^i = 0$ will be taken.

Theorem 6.2.1. *([58], and references there) Fluxes associated with the twist matrix $U^{-1} = L$ in the form of* (6.2)*, which is given in* (6.36)*, which can be written in local coordinates*

$$(U^{-1})^A{}_B = L^A{}_B = \begin{pmatrix} L_i{}^a & L_{ia} \\ L^{ia} & L^i{}_a \end{pmatrix} = \begin{pmatrix} l_i{}^a & 0 \\ 0 & l^i{}_a \end{pmatrix}, \tag{6.53}$$

are

$$\boxed{f_{ab}{}^c = C_{ab}{}^c},$$

and

$$\boxed{R^{abc} = H_{abc} = Q_a{}^{bc} = 0}.$$

Proof: The inverse matrix of L can be found as

$$(L^A{}_B)^{-1} = \begin{pmatrix} L_a{}^i & L_{ai} \\ L^{ai} & L^a{}_i \end{pmatrix} = \begin{pmatrix} l_a{}^i & 0 \\ 0 & l^a{}_i \end{pmatrix}.$$

The formula (6.49) for the fluxes becomes

$$\begin{aligned} \Omega_{ABC} &= -E^M{}_A(\partial_M E^F{}_B)(E^{-1})^D{}_F \eta_{CD} \\ &= -E^\alpha{}_A(\partial_\alpha E^F{}_B)(E^{-1})^D{}_F \eta_{CD} \\ &= -E^I{}_A(\partial_I E^J{}_B)(E^{-1})^D{}_J \eta_{CD}. \end{aligned} \tag{6.54}$$

Here, we have utilized the facts that the twist matrix L does not depend on the winding type coordinates in the second line, and that it only depends on the coordinates along the isometry directions in the third line, where the indices I that label the isometry directions are located.

When we expand the indices I,J,D in (6.54), this reduces to the following result and it will produces some formulas which will be used in the calculation of the fluxes.

$$\begin{aligned}\Omega_{ABC} = &-E^i{}_A(\partial_i E^j{}_B)(E^{-1})^d{}_j\eta_{Cd} - E^i{}_A(\partial_i E^j{}_B)(E^{-1})_{dj}\eta_C{}^d \\ &- E^i{}_A(\partial_i E_{jB})(E^{-1})^{dj}\eta_{Cd} - E^i{}_A(\partial_i E_{jB})(E^{-1})_d{}^j\eta_C{}^d,\end{aligned}$$

from which we obtain the followings.

$$\Omega_{abc} = -E^i{}_a(\partial_i E^j{}_b)(E^{-1})_{cj} - E^i{}_a(\partial_i E_{jb})(E^{-1})_c{}^j, \tag{6.55}$$

$$\Omega_{ab}{}^c = -E^i{}_a(\partial_i E^j{}_b)(E^{-1})^c{}_j - E^i{}_a(\partial_i E_{jb})(E^{-1})^{cj}, \tag{6.56}$$

$$\Omega_a{}^{bc} = -E^i{}_a(\partial_i E^{jb})(E^{-1})^c{}_j - E^i{}_a(\partial_i E_j{}^b)(E^{-1})^{cj}, \tag{6.57}$$

$$\Omega^{ab}{}_c = -E^{ia}(\partial_i E^{jb})(E^{-1})_{cj} - E^{ia}(\partial_i E_j{}^b)(E^{-1})_c{}^j, \tag{6.58}$$

$$\Omega^a{}_{bc} = -E^{ia}(\partial_i E^j{}_b)(E^{-1})_{cj} - E^{ia}(\partial_i E_{jb})(E^{-1})_c{}^j, \tag{6.59}$$

$$\Omega^{abc} = -E^{ia}(\partial_i E^{jb})(E^{-1})^c{}_j - E^{ia}(\partial_i E_j{}^b)(E^{-1})^{cj}, \tag{6.60}$$

due to the components of the η matrix given in (4.28). Note that $\Omega_a{}^b{}_c = -\Omega_{ac}{}^b$ and similarly $\Omega^a{}_b{}^c = -\Omega^{ac}{}_b$ since it is anti-symmetric in last two indices as shown in (6.47).

Let us calculate the fluxes associated with the L matrix now. We will have $E = U^{-1} = L$ in (6.54) in this case and η matrix is of the form (4.28).

Firstly, the f-flux is obtained from (6.48), (6.56) and (6.59) as follows.

$$\begin{aligned}f_{ab}{}^c &= \Omega_{ab}{}^c - \Omega_{ba}{}^c + \Omega^c{}_{ab} \\ &= -l^i{}_a(\partial_i l^j{}_b)l^c{}_j + l^i{}_b(\partial_i l^j{}_a)l^c{}_j \\ &= C_{ab}{}^c,\end{aligned} \tag{6.61}$$

where we have used (5.29) in the last line.

In order to calculate R-flux associated with the L matrix, we will use again (6.48) and also (6.60), then

$$\begin{aligned}R^{abc} &= \Omega^{abc} + \Omega^{bca} + \Omega^{cab} \\ &= 0.\end{aligned} \tag{6.62}$$

Similarly, the H-flux associated with the L matrix will be calculated as follows from (6.48) and (6.55).

$$\begin{aligned} H_{abc} &= \Omega_{abc} + \Omega_{bca} + \Omega_{cab} \\ &= 0. \end{aligned} \tag{6.63}$$

Lastly, the Q-flux associated with the L matrix is obtained by using (6.48), (6.57), and (6.58) as follows.

$$\begin{aligned} Q_a^{bc} &= \Omega_a^{bc} + \Omega^{bc}{}_a - \Omega^{cb}{}_a \\ &= 0.\blacksquare \end{aligned} \tag{6.64}$$

Note that these results of the fluxes are exactly the same with the expressions of fluxes for $\beta = 0$ in the Remark 4.4.5, which have been found from the conditions on fluxes to be a Courant algebroid for a special anchor.

Now the fluxes associated with the matrix T_{YB}, which will be called as the YB matrix, will be calculated and it will be shown that these fluxes completely match the results found above. This fact is crucial for our studies.

Theorem 6.2.2. *[8] The twist matrix is the YB matrix in this case which is denoted by* $\widetilde{U}^{-1} = T_{YB}$ *and is of the form* (6.2)*:*

$$\widetilde{U}^{-1} = T_{YB} = (e^{\beta})^T L = \begin{pmatrix} 1 & 0 \\ \beta & 1 \end{pmatrix} \begin{pmatrix} l^t & 0 \\ 0 & l^{-1} \end{pmatrix} = \begin{pmatrix} l^t & 0 \\ l^t\beta & l^{-1} \end{pmatrix}, \tag{6.65}$$

where β *stands for the constructed bi-vector field from the R-matrix which is* (5.31) *with the components* (5.30) *and L matrix is* (6.53)*, or it has the following form in local coordinates,*

$$(\widetilde{U}^{-1})^A{}_B = (T_{YB})^A{}_B = \begin{pmatrix} \widetilde{U}_i{}^a & \widetilde{U}_{ia} \\ \widetilde{U}^{ia} & \widetilde{U}^i{}_a \end{pmatrix} = \begin{pmatrix} l_i{}^a & 0 \\ l^i{}_j\beta^{ja} & l^i{}_a \end{pmatrix}.$$

Fluxes associated with the YB matrix are

$$\boxed{f_{ab}^c = C_{ab}^c},$$

and

$$\boxed{R^{abc} = H_{abc} = Q_a^{bc} = 0},$$

where R-flux vanishes whenever R-matrix (and hence by the dressed R-matrix) satisfies the CYBE that is given in (5.9)*.*

We will have $E = \widetilde{U}^{-1} = T_{YB}$ in this case in (6.54) and η matrix is of the form (4.28) for the calculation of the fluxes.

Please take note that the indices above were used in Subsection 5.2 for the group manifold. To avoid misunderstanding with doubled indices, we shall utilize the same in this section and the sections that are connected to it.

Proof: The inverse matrix of $\widetilde{U}^{-1}$ can be found as

$$\widetilde{U}^{A}_{\ B} = \begin{pmatrix} \widetilde{U}_a^{\ i} & \widetilde{U}_{ai} \\ \widetilde{U}^{ai} & \widetilde{U}_i^{\ a} \end{pmatrix} = \begin{pmatrix} l_a^{\ i} & 0 \\ -\beta^{aj} l_j^{\ i} & l_i^{\ a} \end{pmatrix}.$$

Let us calculate the fluxes associated with this matrix now. Firstly, the f-flux is obtained from (6.48), (6.56), (6.59) and (6.61) as follows.

$$\begin{aligned} f_{ab}^{\ \ c} &= -l^i_{\ a}(\partial_i l'^j_{\ b}) l^c_{\ j} + l^i_{\ b}(\partial_i l'^j_{\ a}) l^c_{\ j} \\ &= C_{ab}^{\ \ c}, \end{aligned}$$

where we have used (5.29) in the last line. Note that this result is the same with (4.69), as expected.

In order to calculate R-flux associated with the $\widetilde{U}^{-1}$ matrix, we will use again (6.48), (6.60) and also (6.62), then

$$\begin{aligned} R^{abc} &= -l^i_{\ k}\beta^{ka}(\partial_i l^j_{\ l})\beta^{lb} l^c_{\ j} + l^i_{\ k}\beta^{ka}(\partial_i l_j^{\ b})\beta^{cl} l_l^{\ j} - l^i_{\ k}\beta^{kb}(\partial_i l^j_{\ l})\beta^{lc} l^a_{\ j} + l^i_{\ k}\beta^{kb}(\partial_i l_j^{\ c})\beta^{al} l_l^{\ j} \\ &\quad - l^i_{\ k}\beta^{kc}(\partial_i l^j_{\ l})\beta^{la} l^b_{\ j} + l^i_{\ k}\beta^{kc}(\partial_i l_j^{\ a})\beta^{bl} l_l^{\ j} - l^i_{\ k}\beta^{ka}(\partial_i \beta^{cb}) - l^i_{\ k}\beta^{kb}(\partial_i \beta^{ac}) - l^i_{\ k}\beta^{kc}(\partial_i \beta^{ba}) \\ &= \beta^{al}\beta^{kb} C_{kl}^{\ \ c} - \beta^{kc}\beta^{la} C_{lk}^{\ \ b} + \beta^{bl}\beta^{kc} C_{kl}^{\ \ a} - l^i_{\ k}\beta^{ka}(\partial_i \beta^{cb}) - l^i_{\ k}\beta^{kb}(\partial_i \beta^{ac}) - l^i_{\ k}\beta^{kc}(\partial_i \beta^{ba}) \\ &= 0, \end{aligned} \tag{6.66}$$

from the fact that $[\beta,\beta]_{\mathrm{SN}} = 0$ which can be followed from the equation (4.71) being zero that is β is a Poisson bi-vector field as demonstrated in the Proposition 5.2.6 whenever the R-matrix (and hence the bi-vector field β) satisfies the CYBE 5.9. Note

that the equation (6.66) is the same with (4.71) for $a^i_j = l^i_j$, as expected.

The H-flux associated with the $\widetilde{U}^{-1}$ matrix will be calculated as follows from (6.48), (6.55) and (6.63).

$$H_{abc} = 0.$$

Lastly, the Q-flux associated with the $\widetilde{U}^{-1}$ matrix is obtained by using (6.48), (6.57), (6.58), and (6.64) as follows.

$$\begin{aligned}
Q_a^{bc} &= -l^i{}_a(\partial_i l^j{}_k)\beta^{kb}l^c{}_j - l^i{}_k(\partial_i l^c{}_j)\beta^{kb}l_a{}^j + l^i{}_a(\partial_i l^b{}_j)\beta^{ck}l_k{}^j + l^i{}_k(\partial_i l^b{}_j)\beta^{kc}l_a{}^j - l^i{}_a(\partial_i\beta^{cb}) \\
&= -\beta^{kb}C_{ka}{}^c - \beta^{ck}C_{ka}{}^b - l^i{}_a(\partial_i\beta^{cb}) \qquad (6.67) \\
&= -\beta^{kb}C_{ka}{}^c - \beta^{ck}C_{ka}{}^b - l^i{}_a(\partial_i(K^c{}_p R^{pn} K^b{}_n)) \\
&= -\beta^{kb}C_{ka}{}^c - \beta^{ck}C_{ka}{}^b - l^i{}_a R^{pn} K^b{}_n(\partial_i K^c{}_p) - l^i{}_a R^{pn} K^c{}_p(\partial_i K^b{}_n) \\
&= -\beta^{kb}C_{ka}{}^c - \beta^{ck}C_{ka}{}^b + C_{pa}{}^c K^p{}_n R^{nt} K^b{}_t + C_{pa}{}^b K^p{}_t R^{nt} K^c{}_t \\
&= -\beta^{kb}C_{ka}{}^c - \beta^{ck}C_{ka}{}^b + \beta^{pb}C_{pa}{}^c + \beta^{cp}C_{pa}{}^b \\
&= 0,
\end{aligned}$$

where we have used (5.30) in the third line and (5.26) in the fifth line. ■
Note that the equation (6.67) is the same result with (4.70) for $a^i_j = l^i_j$, as expected.

Consequently, the fluxes associated with the matrices $U^{-1} = L$ and $\widetilde{U}^{-1} = T_{YB}$ are identical. Given that the fluxes are preserved in such a way that the L matrix only generates non-vanishing flux as f-flux, which is nothing but the structure constants. This ensures that the transformation generated by T_{YB} is a solution generating transformation, as will be discussed in the following subsection in detail.

6.3 New Solutions of String Theory from $O(d,d)$ Transformation

In this section, we will start with a known solution of the universal sector of supergravity containing the Riemannian metric g and the B-field. We assume that the isometry group is G and the B-field is also invariant under G. The relevant Courant algebroid is then $TG \oplus T^*G$. We will assume that there is an R-matrix on the Lie algebra of the isometry group G, which satisfies the CYBE. In this case, the bi-vector

field on the group manifold can be constructed from the dressed R-matrix as we have discussed in the Subsection 5.2. Once we have a bi-vector field, we can construct the T_{YB} matrix given in (6.65). We will obtain a new generalized metric when we act this $O(d,d)$ transformation to the old generalized metric. This yields a new Riemannian metric $\widetilde{g}$ and $\widetilde{B}$ field which are again supergravity solutions. Let us discuss why this is the case in detail.

Let us assume that we have a generalized metric $\mathscr{H}$ which satisfies (6.34). Therefore, it is a solution of the DFT. Also, suppose that g and B-field in the generalized metric $\mathscr{H}$ admits isometries and the isometry algebra is determined by the Lie group G. So $\mathscr{H}$ can be written of the following form as discussed in (6.35).

$$\mathscr{H}^{MN}(x,Y) = (U^{-1})^{M}{}_{A}(Y)\hat{\mathscr{H}}^{AB}(x)(U^{-1})^{N}{}_{B}(Y),$$

and the RHS satisfies (6.34) since $\mathscr{H}^{MN}(x,Y)$ does. The deformed form of $\mathscr{R}$, which is (6.38), then vanishes, that is $\mathscr{R}_{\text{def}} = \mathscr{R} + \mathscr{R}_F = 0$ where $\mathscr{R}$ is written in terms of $\hat{\mathscr{H}}$ only and $\mathscr{R}_F$ contains $\hat{\mathscr{H}}$ together with the fluxes associated with the twist matrix U^{-1} and d.

The solution generating mechanism relies on the following question: Is it possible to discover a new generalized metric in the form of (6.35) that contains another twist matrix with the same $\hat{\mathscr{H}}$ which once again satisfies the field equations (6.34)?

For this purpose, let $\widetilde{H}$ be the new generalized metric of the following form which has the same form (6.35). (Construction of such a new generalized metric by utilizing the old one will be provided in the Subsection 6.4.)

$$\widetilde{H}^{MN}(x,\widetilde{Y}) = (\widetilde{U}^{-1})^{M}{}_{A}(\widetilde{Y})\hat{\mathscr{H}}^{AB}(x)(\widetilde{U}^{-1})^{N}{}_{B}(\widetilde{Y}). \tag{6.68}$$

In order to conclude that this new generalized metric is a solution of a string in DFT, one needs to check that the equation (6.34) is satisfied or not. Its deformed form (6.38) will contain $\hat{\mathscr{H}}$ as well as the fluxes associated with the $\widetilde{U}^{-1}$. If the fluxes associated with the $\widetilde{U}^{-1}$ are the same with the fluxes associated with the matrix U^{-1}, then it is

also a solution since it solves the equation (6.34). We have the following Theorem with its proof for this fact.

Theorem 6.3.1. *Let*

$$\mathscr{H}(x,Y) = U^{-1}(Y)\hat{\mathscr{H}}(x)(U^t)^{-1}(Y),$$

and

$$\widetilde{\mathscr{H}}(x,\widetilde{Y}) = \widetilde{U}^{-1}(\widetilde{Y})\hat{\mathscr{H}}(x)(\widetilde{U}^t)^{-1}(\widetilde{Y}),$$

which have been introduced before. Assume that $U^{-1}(Y)$ and $\widetilde{U}^{-1}(\widetilde{Y})$ generate the same constant fluxes. Then, $\mathscr{H}(x,Y)$ satisfies the DFT equations if and only if $\widetilde{\mathscr{H}}(x,\widetilde{Y})$ satisfies the DFT equations.

Proof: It is known that the generalized metric $\mathscr{H}$ of the form (6.35) satisfies (6.34) that is

$$\mathscr{R}[\mathscr{H}] = 0.$$

Then, $\hat{\mathscr{H}}$ satisfies (6.37) which is

$$\mathscr{R}_{\text{def}}^{U^{-1}}[\hat{\mathscr{H}}] = 0.$$

Our goal is to conclude that the new generalized metric satisfies the field equations that is $\mathscr{R}[\widetilde{\mathscr{H}}] = 0$. We have

$$\mathscr{R}_{\text{def}}^{U^{-1}}[\hat{\mathscr{H}}] = \mathscr{R}_{\text{def}}^{\widetilde{U}^{-1}}[\hat{\mathscr{H}}],$$

since the fluxes associated with the twist matrices U^{-1} and $\widetilde{U}^{-1}$ are assumed to be the same. Another reason for the above equation to be satisfied is the following fact and this completes the proof.
U and $\widetilde{U}$ appear in $(U^{-1})^M{}_D\partial_M = \partial'_D$ and $(\widetilde{U}^{-1})^M{}_D\partial_M = \widetilde{\partial}'_D$, respectively. These are acting on $\hat{\mathscr{H}}$ and d in (6.39) and $(U^{-1})^M{}_D = \delta^M{}_D$ when acting on fields depending only on x. In other words, $\partial' = \widetilde{\partial}' = \partial$ when acting on $\mathscr{H}(x)$ and $d(x)$ because these fields depend only on x. ■

Note that

$$\widetilde{\mathscr{H}}(x,\widetilde{Y}) = \psi(\widetilde{Y})\mathscr{H}(x,\widetilde{Y})\psi^t(\widetilde{Y}),$$

where we define

$$\psi(\widetilde{Y}) = \widetilde{U}^{-1}(\widetilde{Y})U(\widetilde{Y}).$$

Also note that according to this definition,

$$\widetilde{U}^{-1}(\widetilde{Y}) = \psi(\widetilde{Y})U^{-1}(\widetilde{Y}).$$

We will now start by choosing another twist matrix as $\widetilde{U}^{-1} = T_{YB}$, which contains the elements of the left-invariant vector fields and the components of the constructed bi-vector field from the dressed R-matrix which can be seen in (6.65). The twist matrix $U^{-1} = L$, provided in (6.36), yields only non-vanishing flux as a geometric flux, which is equivalent to the structure constants of the isometry group's Lie algebra. New twist matrix gives exactly the same result, which has been shown in Subsection 6.2.4. So the flux preservation principle associated with these twist matrices holds. Therefore, the condition in the Theorem 6.3.1 holds which leads that the generated generalized metric $\widetilde{\mathscr{H}}$ from new twist matrix $\widetilde{U}^{-1} = T_{YB}$ satisfies the DFT equations that is $\mathscr{R}[\widetilde{\mathscr{H}}] = 0$. This new generalized metric yields a new Riemannian metric $\widetilde{g}$ and $\widetilde{B}$ field which are again supergravity solutions as discussed below. So it will be inferred that the transformation generated by T_{YB} is a solution generating transformation.

Let us define $E = g + B$ which is the background matrix [60]. If we write T, which will be taken as the transformation associated with the YB matrix (6.65), as in the equation (6.2) [56]:

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in O(d,d),$$

then, it is known that the action of $O(d,d)$ on the background matrix is by fractional linear transformations [56]:

$$E \to \widetilde{E} = T.E = (aE+b)(cE+d)^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} E. \tag{6.69}$$

By using $T\eta T^t = \eta$ and $T^t\eta T = \eta$, where η is an also $O(d,d)$ matrix given in (4.28), we obtain

$$a^t c + c^t a = 0,\ \ b^t d + d^t b = 0,\ \ a^t d + c^t b = I.$$

T can be embedded in $O(D,D,\mathbb{R})$ as follows [60].

$$\hat{T} = \begin{pmatrix} \hat{a} & \hat{b} \\ \hat{c} & \hat{d} \end{pmatrix},$$

where $\hat{a},\hat{b},\hat{c},\hat{d}$ are $D\times D$ matrices defined below

$$\hat{a}=\begin{pmatrix} a & 0\\ 0 & I\end{pmatrix},\ \hat{b}=\begin{pmatrix} b & 0\\ 0 & 0\end{pmatrix},\ \hat{c}=\begin{pmatrix} c & 0\\ 0 & 0\end{pmatrix},\ \hat{d}=\begin{pmatrix} d & 0\\ 0 & I\end{pmatrix}. \tag{6.70}$$

The action of the $O(D,D)$ matrix $\hat{T}$ on the background matrix E is defined as below [60]:

$$\widetilde{E}\left(\widetilde{g},\widetilde{B}\right)=\hat{T}\cdot E(g,B)\equiv(\hat{a}E+\hat{b})(\hat{c}E+\hat{d})^{-1},$$

which is the new background matrix.

$\widetilde{E}$ includes the transformed metric and the transformed B-field which are symmetric and antisymmetric parts of $\widetilde{E}$, respectively, given in the Definition 3.3.9. These are

$$\widetilde{g}=\frac{\widetilde{E}+\widetilde{E}^t}{2},$$

and

$$\widetilde{B}=\frac{\widetilde{E}-\widetilde{E}^t}{2},$$

respectively.

The equation (6.69) is equivalent to

$$\widetilde{E}=T\widetilde{E}T^t.$$

So, it has been shown that $\widetilde{g}$ and $\widetilde{B}$ are also satisfy the field equations of 10-dimensional supergravity by using the flux preservation principle.

6.4 YB Deformation in the Language of Courant Algebroids

We will demonstrate and interpret YB deformation, introduced in the Subsections 6.2 and 6.3, in the language of Courant algebroids. All the geometric structures that have been previously explored will be integrated in this way.

Let us assume that we have a bi-invariant metric G on the group manifold. Let $e_a(g)=l_{g^*}(T_a)$ where $T_a\in\mathfrak{g}$ and $l_{g^*}:T_eM\to T_gM$. Let $e_a=l^i{}_a\partial_i$ with $i=1,\cdots,d$ where d is the dimension of the group manifold. Therefore, we have

$$G(e_a,e_b)=G(l_{g^*}(T_a),l_{g^*}(T_b))=G(T_a,T_b)=G_{ab}=constant.$$

It is a well known fact that any diffeomorphism $\Phi : M \to M$ defines an automorphism [61]

$$\mathbb{T}\Phi : TM \oplus T^*M \to TM \oplus T^*M,$$

by taking the sum of the tangent and cotangent maps. Here, $\mathbb{T}M$ is for the generalized tangent bundle: $TM \oplus T^*M$. This automorphism preserves the Courant bracket.

Let us define now the following transformation:

$$\begin{aligned} Tl_g : \ \mathfrak{g} \oplus \mathfrak{g}^* \ &\to T_gM \oplus T_g^*M \\ (T_a, \widetilde{T}^a) &\mapsto (l_{g^*}(T_a), l^*_{g^{-1}}(\widetilde{T}^a)), \end{aligned}$$

where $l_g^* : T_g^*M \to T_e^*M$ and $l^*_{g^{-1}} : T_e^*M \to T_g^*M$.

Consider now the Courant algebroid $E = TG \oplus T^*G$ where G is a semi-simple Lie group G with the Lie algebra g and let R be a non-degenerate R-matrix on g as introduced in the Subsection 5.1. Let g, B defined on G from which we can construct the generalized metric $\mathscr{H}$ (6.20).

Matrix of the transformation Tl_g given above will be the twist matrix L in our case. Then the generalized metric $\mathscr{H}$ becomes to the following if it is bi-invariant under the action of Tl_g.

$$\begin{aligned} \mathscr{H}^{AB} &= \mathscr{H}(e^A, e^B) \\ &= \mathscr{H}(Tl_g(T^A), Tl_g(T^B)) \\ &= \mathscr{H}(T^A, T^B) \\ &= \hat{\mathscr{H}}^{AB} = constant \\ \Longrightarrow &\mathscr{H}(e^A, e^B) = \hat{\mathscr{H}}^{AB} = constant, \end{aligned} \tag{6.71}$$

where $e^A = (U^{-1})^A{}_M \partial^M$, $\partial^M = (\partial_I, \partial^I)$ together with the twist matrix $(U^{-1})^A{}_M = L^A{}_M$ given in (6.36), (6.53), and $T^A = (T_a, \widetilde{T}^a)$.

Remark 6.4.1. *The above ansatz means that $\mathscr{H}^{MN} = \mathscr{H}(\partial^M, \partial^N)$ is of the form* (6.35) *with $e^M = (U^{-1})^M{}_A \partial^A$. This fact can be seen immediately as follows.*

$$\begin{aligned}
&\mathscr{H}(e^A, e^B) = \mathscr{H}((U^{-1})^A{}_M \partial^M, (U^{-1})^B{}_N \partial^N) \\
&= (U^{-1})^A{}_M \mathscr{H}(\partial^M, \partial^N)((U^t)^{-1})^B{}_N \\
&\Longrightarrow \mathscr{H}^{MN} = (U^{-1})^M{}_A \mathscr{H}(e^A, e^B)(U^{-1})^N{}_B \\
&\Longrightarrow \mathscr{H}^{MN} = (U^{-1})^M{}_A \hat{\mathscr{H}}^{AB}(U^{-1})^N{}_B,
\end{aligned}$$

where we have used (6.71) *in the last line.*

We define the new generalized metric on the generalized tangent bundle $E = TG \oplus T^*G$ as

$$\widetilde{\mathscr{H}}(FU, FV) = \mathscr{H}(U, V), \tag{6.72}$$

where $F = e^\beta$ and β is the bi-vector field constructed in the Subsection 5.2, and U, V are the sections on E. Note that β-transform given in (3.46) acts on the sections of the generalized tangent bundle and the generalized metric is a tensor again defined on $T \oplus T^*$. This transform will act on the generalized metric and it transforms the Riemannian metric and B-field encoded in the generalized metric.

Our first goal is to demonstrate that the generalized metric $\widetilde{\mathscr{H}}$ defined in (6.72) is indeed a generalized metric on the generalized tangent bundle E. Since $\mathscr{H}$ is a generalized metric, we can write it as $\mathscr{H}(U, V) = \langle U, \tau(V) \rangle$ for the smooth sections U, V on E from the Definition (3.57). Here, $\tau \in \mathrm{Aut}(E)$, such that $\tau^2 = 1$ which is defined in (3.53). We define the following operator on $\mathrm{Aut}(E)$.

$$\widetilde{\tau} := F \circ \tau \circ F^{-1} : E \to E, \tag{6.73}$$

which immediately satisfies $\widetilde{\tau}^2 = 1$. Our claim is to prove that there exists such an automorphism $\widetilde{\tau}$ which satisfies

$$\widetilde{\mathscr{H}}(U, V) = \langle U, \widetilde{\tau}(V) \rangle,$$

together with $\tau^2 = 1$. In order to show this fact, let us write (6.73) in the RHS of the above equation:

$$\begin{aligned}\langle U, \widetilde{\tau}(V)\rangle &= \langle U, (F\circ\tau\circ F^{-1})(V)\rangle \\ &= \langle F^{-1}(U), \tau\circ F^{-1}(V)\rangle \\ &= \mathscr{H}(F^{-1}(U), F^{-1}(V)) \\ &= \widetilde{\mathscr{H}}(U,V),\end{aligned}$$

where we have used the fact that $F = e^{\beta}$ is an orthogonal transformation, which can be seen from (3.47), in the second equality, $\mathscr{H}(U,V) = \langle U, \tau(V)\rangle$ in the third one, and (6.72) in the last one.

Also, let us assume that the subbundles $\tilde{S}_{\pm}$ defined in (3.54) and (3.55), respectively, are the ± 1-eigenbundles of the endomorphism τ. Then, from the proof of the Proposition 6.2.2, we have

$$\tau(P_+(e_i)) = P_+(e_i),$$

and

$$\tau(P_-(e_i)) = -P_-(e_i).$$

Therefore, we can find the followings.

$$\widetilde{\tau}(F(P_+(e_i))) = (F\circ\tau\circ F^{-1})(F(P_+(e_i))) = F(\tau(P_+(e_i))) = F(P_+(e_i)),$$

and

$$\widetilde{\tau}(F(P_-(e_i))) = (F\circ\tau\circ F^{-1})(F(P_-(e_i))) = F(\tau(P_-(e_i))) = -F(P_-(e_i)),$$

which yields that the subbundles $F(S'_+)$ and $F(S'_-)$ are the ± 1-eigenbundles, respectively, of the endomorphism $\widetilde{\tau}$ on E.

The result (6.72) leads us to the following fact.

$$\widetilde{\mathscr{H}}(F(e^A), F(e^B)) = \mathscr{H}(e^A, e^B) = \hat{\mathscr{H}}^{AB}.$$

So if $\widetilde{e}^A = F(e^A)$ and $\widetilde{e}^B = F(e^B)$ be fixed bases of $TG\oplus T^*G$, then

$$\widetilde{\mathscr{H}}(\widetilde{e}^A, \widetilde{e}^B) = \hat{\mathscr{H}}^{AB}.$$

In other words, for the twist matrices $(\widetilde{U}^{-1})^A{}_M$ of the form (6.65), we have

$$\begin{aligned}
&\widetilde{\mathscr{H}}((\widetilde{U}^{-1})^A{}_M\partial^M,(\widetilde{U}^{-1})^B{}_N\partial^N) = \hat{\mathscr{H}}^{AB}\\
&\Longrightarrow (\widetilde{U}^{-1})^A{}_M\widetilde{\mathscr{H}}(\partial^M,\partial^N)((\widetilde{U}^t)^{-1})^B{}_N = \hat{\mathscr{H}}^{AB}\\
&\Longrightarrow \widetilde{\mathscr{H}}^{MN} = (\widetilde{U}^{-1})^M{}_A\hat{\mathscr{H}}^{AB}(\widetilde{U}^{-1})^N{}_B \qquad (6.74)\\
&\Longrightarrow \widetilde{\mathscr{H}}^{MN} = (e^{\beta}L)^M{}_A\hat{\mathscr{H}}^{AB}(e^{\beta}L)^N{}_B = (T_{YB})^M{}_A\hat{\mathscr{H}}^{AB}(T_{YB})^N{}_B.
\end{aligned}$$

Note that (6.74) is in the form of (6.68).

Now the results from physics say that $\widetilde{\mathscr{H}}$ will also be a solution if

$$[\tilde{e}^A,\tilde{e}^B] = F^{AB}{}_C\tilde{e}^C,$$

if

$$[e^A,e^B] = F^{AB}{}_Ce^C,$$

where the bracket is the standard Courant bracket on $TG\oplus T^*G$. This follows directly from the fact that the DFT fluxes are computed from the C-bracket as was given in (6.40) and the C-bracket reduces to the standard Courant bracket in the supergravity frame which was given in the Proposition 6.1.1. In other words we need:

$$F[F^{-1}(\tilde{e}^A),F^{-1}(\tilde{e}^B)] = [\tilde{e}^A,\tilde{e}^B]. \qquad (6.75)$$

If F were a Courant algebroid automorphism (see the Definition 3.3.6) preserving the standard Courant bracket, (6.75) would be automatic. However, we know that the only such bracket preserving automorphisms are the ones induced from diffeomorphisms and B-transformations with closed 2-form B (see the Proposition 3.3.2). F is a β-transformation and as such, it does not preserve the bracket. In order to understand how F changes the bracket let us refer to our discussion in the Subsection 4.4.1 and the equations (4.50)-(4.55).

For the standard Courant bracket the anchor is the projection: $\rho(X+\eta) = X$, where $X\in TM$ and $\eta\in T^*M$ (see Example 3.3.1). Recall that in the notation of Subsection 4.4 we have from (4.32) and (4.33):

$$\rho(e^A) = \rho^{Ak}\partial_k.$$

Note that we use here e^A instead of the notation S^A in there. Then for the frame $e^A = L^A{}_K \partial^K$ which corresponds to (6.42) for $\beta = 0$, we have the followings from (6.43) and (6.44):

$$\rho(e_i) = \rho(l_i{}^k \partial_k) = l_i{}^k \partial_k,$$

$$\rho(e^i) = \rho(l^i{}_k dx^k) = 0.$$

Then $\rho_i{}^k = l_i{}^k$ and $\rho_*^{ik} = 0$ and we are in the case of the Proposition 4.4.6 with $\beta = 0$ and $a_i{}^j = l_i{}^j$. On the other hand, for the frame $\widetilde{e}^A = F(e^A) = ((e^\beta)^T L)^A{}_K \partial^K$, where $(e^\beta)^T L = T_{YB}$ as in (6.65), which corresponds to (6.42) for non-vanishing β, we have the followings from (6.43) and (6.44):

$$\rho(\widetilde{e}_i) = l_i{}^k \partial_k = \tilde{\rho}_i{}^k \partial_k,$$

$$\rho(\widetilde{e}^i) = l_j{}^i \beta^{jk} \partial_k = \tilde{\rho}_*^{ik} \partial_k,$$

where β is the bi-vector field we constructed from the dressed R-matrix in the Subsection 5.2. So, we are again in the case of the Proposition 4.4.6, this time with non-vanishing β. Although the fluxes computed in the two case are the same due to the properties of the bi-vector field, the corresponding brackets are different as can be checked directly from (4.50)-(4.55). However, the different parts are the terms involving derivatives of the form $l_j{}^i \beta^{jk} \partial_k$. As we discussed in the proof of the Theorem 6.3.1, the fields in the physical theory ($\hat{\mathscr{H}}$) do not depend on the coordinates of the group manifold, so such terms do not play a role in the gauged double field theory action.

So we see here that the main role of the YB deformation relies on two features: Existence of e^A, g and B such that $\mathscr{H}$ can be written as in (6.71), and existence of an orthogonal map F preserving the flux part of the Courant bracket. We hope that describing the YB deformation in this geometrical setup makes it possible to generalize it to other solution generating transformations.

7. CONCLUSION

In this thesis, we studied the homogeneous YB deformation proposed in [6] for generic GS sigma models and showed that the deformation in the target space is generated by a non-constant $O(d,d)$ transformation. The associated matrix, which we call the YB matrix, is determined by an R-matrix satisfying the CYBE. The coordinate dependence comes from extending this fixed R-matrix to a dressed R-matrix on the whole group manifold (of the isometry group) by the adjoint action. Since the adjoint action is an automorphism of the Lie bracket, the resulting dressed R-matrix also satisfies the CYBE, whenever the initial R-matrix does. This shows that the bi-Killing ansazt employed in the physics literature [62], [63], [64], arises naturally from this construction with the adjoint action.

In addition to making computations easier, our geometrical approach provides us the option of generalizing the solution generating process. We have viewed the YB deformation as the generalized diffeomorphism on the generalized tangent bundle of the isometry group of the solution. Namely, the $O(d,d)$ transformation acts as a transformation on a Courant algebroid and is determined by the bi-vector field related to the Poisson structure on the base manifold. We have thoroughly examined the process of the construction of this bi-vector field. It is a well-known fact that there is a Lie algebroid structure on the cotangent bundle of the manifold when there is a Poisson structure on the manifold. Additionally, this Lie algebroid structure is compatible with the standard Lie algebroid structure on the tangent bundle, so that there is a Courant algebroid structure on the direct sum of the generalized tangent bundle of the manifold. Because of all these reasons, we have also studied Courant algebroid structures in detail, in order to understand and to generalize the transformation and the YB deformation.

If a generalized metric of a certain form solves the DFT field equations in the limit where it reduces to the supergravity field equations, and is transformed with the

$O(d,d)$ matrix producing the YB deformation, then the resulting generalized metric also solves the DFT field equations in the same limit. This fact is demonstrated in the literature on physics by demonstrating the preservation of the "fluxes" associated with the generalized metric before and after the transformation. When a particular basis is used for the sections of tangent and cotangent bundles, the fluxes are simply the "structure functions" of the Courant algebroid structure on the generalized tangent bundle. To gain a geometric understanding of this "flux preservation" principle, we have also examined the axioms that define a Courant algebraid in local coordinates. We have also thoroughly examined the situation in which a bi-vector field linked to the YB deformation determines the anchor of the Courant algebroid. The details of these concepts are as follows.

We have started by establishing an endomorphism R on a Lie algebra with a non-degenerate Cartan-Killing metric which satisfies CYBE, and we then demonstrated that the sum of the Lie algebra and its dual has a natural Drinfel'd structure. In order to have a β-transformation, one needs to have a bi-vector field and it is constructed from the components of the dressed R-matrix in our case. The dressed R-matrix is constructed by extending R-matrix to the whole group manifold of the isometry group G with the adjoint action. This constructed dressed R-matrix also satisfies CYBE, whenever the initial R-matrix does, since the adjoint action is an automorphism of the Lie bracket. It has been shown that the bi-vector field that built from the dressed R-matrix has a vanishing SN bracket with itself since the R-matrix satisfies CYBE and hence determines a Poisson structure. This yields a triangular Lie bialgebroid structure since there exists a Lie algebroid structure on the cotangent bundle of a Poisson manifold. Additionally, there exists a Courant algebroid structure on the direct sum of these Lie algebroids due to the fact that there is a one-to-one correspondence between (triangular) Lie bialgebroids and Courant algebroids with transversal Dirac structures. Since these geometric structures, namely Lie algebroids, Lie bialgebroids, Poisson structures, Courant algebroids, and others, have arisen in our studies, we have examined their definitions, features, local expressions as well as the connections between them in detail.

A Riemannian metric g and a B-field on the manifold M, which are taken to be solutions of the supergravity frame, yield a generalized metric on the generalized tangent bundle $TM \oplus T^*M$. Applying the constructed β-transformation to the generalized metric gives a new generalized metric parameterized by the transformed Riemannian metric and the transformed B-field. We have demonstrated that this transformation can be understood as a different splitting of the generalized tangent bundle to Dirac subbundles. After the transformation, the tangent bundle remains intact, whereas the cotangent bundle is transformed to its graph under the β-transformation. The resulting subbundle is still a Dirac subbundle since the β-transformation preserves the inner product, leaving it maximally isotropic, and it is involutive since the R-flux vanishes, giving rise to the vanishing of the SN bracket. Therefore, one still has the structure of a Courant algebroid on the direct sum of these Dirac subbundles.

It has been seen that previously discussed geometric structures as algebroid structures, the generalized metric on the Courant algebroid $TM \oplus T^*M$, β transformations, R-matrices and flux calculations associated with the twist matrices can be merged in the YB deformation in the language of Courant algebroids. Precisely, for a given Riemannian metric and B-field taken as the solutions in supergravity, the generalized metric on a Courant algebroid constructed from these satisfies the DFT field equations. The generalized metric can be defined in the form of (6.71) by using isometries of the group manifold. By utilizing this form of the generalized metric and the bi-vector field β constructed from the R-matrix, one can define a new generalize metric which does again solve the field equations of the supergravity frame. In order to obtain this result, the law of the flux preservation must be satisfied in terms of the associated twist matrices in the forms (6.53) and (6.65). Also, the expressions of the fluxes associated with these twist matrices are actually the same with the conditions on the structure functions of the Courant algebroid. Also, this preservation corresponds to the β transformation to be an orthogonal transformation of the Courant bracket. It has been also demonstrated that the axioms of the fluxes in DFT when the strong constraint is imposed actually correspond to the conditions imposed by the axioms defining a Courant algebroid on the structure functions.

With this geometrical method, we expect to advance the knowledge on the interesting relationship between the CYBE and supergravity. We hope that understanding the underlying geometrical structure might yield new solution generating transformations for supergravity.

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APPENDICES

APPENDIX A : Proofs

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Proof A.1. *By the Jacobi Identity and the Leibniz Rule, we obtain*

$$\begin{aligned}
0 =& [[X,Y],fZ]+[[Y,fZ],X]+[[fZ,X],Y] \\
=& f[[X,Y],Z]+(\rho[X,Y](f))Z \\
&+f[[Y,Z],X]-(\rho X(f))[Y,Z]+(\rho Y(f))[Z,X]-(\rho X(\rho Y(f)))Z \\
&+f[[Z,X],Y]-(\rho Y(f))[Z,X]-(\rho X(f))[Z,Y]+(\rho Y(\rho X(f)))Z \\
=& ((\rho[X,Y]-[\rho X,\rho Y])(f))Z.
\end{aligned}$$

Since this is true for any $X,Y,Z\in\Gamma(A)$ and function f, one concludes that $\rho[X,Y]=[\rho X,\rho Y]$. ■

Proof A.2. *For $X\in\Gamma(\wedge^2 A)$ and $Y\in\Gamma(\wedge^2 A)$, the LHS of the equation* (3.17) *gives*

$$\begin{aligned}
&[f(X_1\wedge X_2),g(Y_1\wedge Y_2)]_{SN}=[fX_1,gY_1]\wedge X_2\wedge Y_2-[fX_1,Y_2]\wedge X_2\wedge(gY_1) \\
&-[X_2,gY_1]\wedge(fX_1)\wedge Y_2+[X_2,Y_2]\wedge(fX_1)\wedge(gY_1) \\
&=fg[X_1,Y_1]\wedge X_2\wedge Y_2-g(\rho(Y_1)f)X_1\wedge X_2\wedge Y_2+f(\rho(X_1)g)Y_1\wedge X_2\wedge Y_2 \\
&+fg[Y_2,X_1]\wedge X_2\wedge Y_1+g(\rho(Y_2)f)X_1\wedge X_2\wedge Y_1-fg[X_2,Y_1]\wedge X_1\wedge Y_2 \\
&-f(\rho(X_2)g)Y_1\wedge X_1\wedge Y_2+fg[X_2,Y_2]\wedge X_1\wedge Y_1 \\
&=fg[X_1\wedge X_2,Y_1\wedge Y_2]_{SN}+gX_1\wedge X_2\wedge[f,Y_1\wedge Y_2]_{SN}+f[g,X_1\wedge X_2]_{SN}\wedge Y_1\wedge Y_2,
\end{aligned}$$

where we have used (3.14) *in the first equality,* (3.2) *in the second equality, and* (3.19) *in the last equality. The above result is the same with the RHS and it completes the proof since it can be generalized to $X\in\Gamma(\wedge^p A)$ and $Y\in\Gamma(\wedge^q A)$.* ■

Proof A.3. *Let a bi-vector field Π is a Poisson tensor. So, it is skew-symmetric and satisfies Leibniz Rule. In order to prove* (3.20)*, Jacobiator should vanish and it will be expanded in local coordinates such as*

$$\begin{aligned}
&J=\{f,\{g,h\}\}+\{g,\{h,f\}\}+\{h,\{f,g\}\}=0 \\
&\Longrightarrow\{f,\Pi^{ij}(\partial_i g)(\partial_j h)\}+\{g,\Pi^{ij}(\partial_i h)(\partial_j f)\}+\{h,\Pi^{ij}(\partial_i f)(\partial_j g)\}=0 \\
&\Longrightarrow\Pi^{ij}\{f,(\partial_i g)(\partial_j h)\}+\{f,\Pi^{ij}\}(\partial_i g)(\partial_j h)+\Pi^{ij}\{g,(\partial_i h)(\partial_j f)\} \\
&+\{g,\Pi^{ij}\}(\partial_i h)(\partial_j f)+\Pi^{ij}\{h,(\partial_i f)(\partial_j g)\}+\{h,\Pi^{ij}\}(\partial_i f)(\partial_j g)=0 \\
&\Longrightarrow\Pi^{ij}(\partial_i g)\{f,(\partial_j h)\}+\Pi^{ij}\{f,(\partial_i g)\}(\partial_j h)+\{f,\Pi^{ij}\}(\partial_i g)(\partial_j h) \\
&+\Pi^{ij}(\partial_i h)\{g,(\partial_j f)\}+\Pi^{ij}\{g,(\partial_i h)\}(\partial_j f)+\{g,\Pi^{ij}\}(\partial_i h)(\partial_j f) \\
&+\Pi^{ij}(\partial_i f)\{h,(\partial_j g)\}+\Pi^{ij}\{h,(\partial_i f)\}(\partial_j g)+\{h,\Pi^{ij}\}(\partial_i f)(\partial_j g)=0 \\
&\Longrightarrow\Pi^{ij}(\partial_i g)\Pi^{kl}(\partial_k f)(\partial_l\partial_j h)+\Pi^{ij}\Pi^{kl}(\partial_k f)(\partial_l\partial_i g)(\partial_j h)+\Pi^{kl}(\partial_k f)(\partial_l\Pi^{ij})(\partial_i g)(\partial_j h) \\
&+\Pi^{ij}(\partial_i h)\Pi^{kl}(\partial_k g)(\partial_l\partial_j f)+\Pi^{ij}\Pi^{kl}(\partial_k g)(\partial_l\partial_i h)(\partial_j f)+\Pi^{kl}(\partial_k g)(\partial_l\Pi^{ij})(\partial_i h)(\partial_j f) \\
&+\Pi^{ij}(\partial_i f)\Pi^{kl}(\partial_k h)(\partial_l\partial_j g)+\Pi^{ij}\Pi^{kl}(\partial_k h)(\partial_l\partial_i f)(\partial_j g)+\Pi^{kl}(\partial_k h)(\partial_l\Pi^{ij})(\partial_i f)(\partial_j g)=0,
\end{aligned}$$

where we have used the Leibniz Rule (third property) of a Poisson bracket twice. The following terms in the above result vanish from the anti-symmetry property of Π^{ij}*:*

$$\textcircled{1}+\textcircled{5}=\Pi^{ij}(\partial_i g)\Pi^{kl}(\partial_k f)(\partial_l\partial_j h)+\Pi^{lk}\Pi^{ij}(\partial_i g)(\partial_j\partial_l h)(\partial_k f)=0,$$

$$\textcircled{2}+\textcircled{7}=\Pi^{ij}\Pi^{kl}(\partial_k f)(\partial_l\partial_i g)(\partial_j h)+\Pi^{kl}(\partial_k f)\Pi^{ji}(\partial_j h)(\partial_i\partial_l g)=0,$$

and

$$\textcircled{4}+\textcircled{8}=\Pi^{ij}(\partial_i h)\Pi^{kl}(\partial_k g)(\partial_l\partial_j f)+\Pi^{lk}\Pi^{ij}(\partial_i h)(\partial_j\partial_l f)(\partial_k g)=0.$$

Therefore, we obtain

$$[\Pi^{kl}(\partial_l\Pi^{ij})+\Pi^{il}(\partial_l\Pi^{jk})+\Pi^{jl}(\partial_l\Pi^{ki})](\partial_k f)(\partial_i g)(\partial_j h)=0. \qquad (A.1)$$

For $(\partial_k f)(\partial_i g)(\partial_j h)\neq 0$*, it yields*

$$\begin{aligned}&\Pi^{kl}(\partial_l\Pi^{ij})+\Pi^{il}(\partial_l\Pi^{jk})+\Pi^{jl}(\partial_l\Pi^{ki})=0\\ \Longrightarrow\ &\Pi^{lk}(\partial_l\Pi^{ij})+\Pi^{li}(\partial_l\Pi^{jk})+\Pi^{lj}(\partial_l\Pi^{ki})=0\\ \Longrightarrow\ &\Pi^{li}(\partial_l\Pi^{jk})+\Pi^{lj}(\partial_l\Pi^{ki})+\Pi^{lk}(\partial_l\Pi^{ij})=0,\end{aligned} \qquad (A.2)$$

or

$$\sum_{l=1}^{m}\left(\Pi^{li}\frac{\partial\Pi^{jk}}{\partial x_l}+\Pi^{lj}\frac{\partial\Pi^{ki}}{\partial x_l}+\Pi^{lk}\frac{\partial\Pi^{ij}}{\partial x_l}\right)=\sum_{l}\frac{\partial\Pi^{ij}}{\partial x_l}\Pi^{lk}+c.p.=0. \qquad (A.3)$$

The another side of the proof is trivial since the above condition gives rise to the Jacobiator to be vanished and 2-vector field Π *becomes a Poisson tensor.* ■

Proof A.4. *Let* $\beta=\beta^{ij}\partial_i\wedge\partial_j$*, then*

$$\begin{aligned}[\beta,\beta]_{SN}&=[\beta^{ij}\partial_i\wedge\partial_j,\beta^{kl}\partial_k\wedge\partial_l]_{SN}\\ &=\beta^{ij}\beta^{kl}[\partial_i\wedge\partial_j,\partial_k\wedge\partial_l]_{SN}+\beta^{kl}\partial_i\wedge\partial_j\wedge[\beta^{ij},\partial_k\wedge\partial_l]_{SN}+\beta^{ij}[\beta^{kl},\partial_i\wedge\partial_j]_{SN}\wedge\partial_k\wedge\partial_l\\ &=\beta^{ij}\beta^{kl}([\partial_i,\partial_k]\wedge\partial_j\wedge\partial_l-[\partial_i,\partial_l]\wedge\partial_j\wedge\partial_k-[\partial_j,\partial_k]\wedge\partial_i\wedge\partial_l+[\partial_j,\partial_l]\wedge\partial_i\wedge\partial_k)\\ &+\beta^{kl}\partial_i\wedge\partial_j\wedge[-(\partial_k\beta^{ij})\partial_l+(\partial_l\beta^{ij})\partial_k]+\beta^{ij}[-(\partial_i\beta^{kl})\partial_j+(\partial_j\beta^{kl})\partial_i]\wedge\partial_k\wedge\partial_l\\ &=-\beta^{kl}(\partial_k\beta^{ij})\partial_i\wedge\partial_j\wedge\partial_l+\beta^{kl}(\partial_l\beta^{ij})\partial_i\wedge\partial_j\wedge\partial_k-\beta^{ij}(\partial_i\beta^{kl})\partial_j\wedge\partial_k\wedge\partial_l\\ &+\beta^{ij}(\partial_j\beta^{kl})\partial_i\wedge\partial_k\wedge\partial_l\\ &=-\beta^{lk}(\partial_l\beta^{ij})\partial_i\wedge\partial_j\wedge\partial_k+\beta^{kl}(\partial_l\beta^{ij})\partial_i\wedge\partial_j\wedge\partial_k-\beta^{lj}(\partial_l\beta^{ki})\partial_j\wedge\partial_k\wedge\partial_i\\ &+\beta^{il}(\partial_l\beta^{kj})\partial_i\wedge\partial_k\wedge\partial_j\\ &=[-\beta^{lk}(\partial_l\beta^{ij})+\beta^{kl}(\partial_l\beta^{ij})-\beta^{lj}(\partial_l\beta^{ki})-\beta^{il}(\partial_l\beta^{kj})]\partial_i\wedge\partial_j\wedge\partial_k\\ &=[\beta^{il}(\partial_l\beta^{jk})+\beta^{jl}(\partial_l\beta^{ki})+2\beta^{kl}(\partial_l\beta^{ij})]\partial_i\wedge\partial_j\wedge\partial_k,\end{aligned}$$

where (3.17) *and* (3.18) *have been used in the second and third equalities, respectively. Also, the fact* $[\partial_i,\partial_k]=0$ *has been used in the fourth equality.*

For the SN bracket of a bi-vector field with itself above to be vanished, the expansion in terms of cyclic terms must be vanished as follows.

$$
\begin{aligned}
&[\beta,\beta]_{SN} = 0 \\
&\Longrightarrow \beta^{il}(\partial_l\beta^{jk}) + \beta^{jl}(\partial_l\beta^{ki}) + 2\beta^{kl}(\partial_l\beta^{ij}) + \beta^{jl}(\partial_l\beta^{ki}) + \beta^{kl}(\partial_l\beta^{ij}) + 2\beta^{il}(\partial_l\beta^{jk}) \\
&+ \beta^{kl}(\partial_l\beta^{ij}) + \beta^{il}(\partial_l\beta^{jk}) + 2\beta^{jl}(\partial_l\beta^{ki}) = 0 \\
&\Longrightarrow 4[\beta^{il}(\partial_l\beta^{jk}) + \beta^{jl}(\partial_l\beta^{ki}) + \beta^{kl}(\partial_l\beta^{ij})] = 0 \\
&\Longrightarrow \beta^{il}(\partial_l\beta^{jk}) + \beta^{jl}(\partial_l\beta^{ki}) + \beta^{kl}(\partial_l\beta^{ij}) = 0,
\end{aligned}
$$

which is exactly the same equation with (A.2) *for* β *instead of* Π*. So, the SN bracket of a bi-vector field with itself in local coordinates coincides with the Jacobi Identity for a Poisson bracket where the coefficients are bi-vector fields, in local coordinates.* ■

Proof A.5.

$$
\begin{aligned}
[\alpha,\beta]_* &= \mathscr{L}_{\rho\alpha}\beta - \mathscr{L}_{\rho\beta}\alpha - d(\Pi(\alpha,\beta)) \\
&= i_{\rho\alpha}d\beta + di_{\rho\alpha}\beta - i_{\rho\beta}d\alpha - di_{\rho\beta}\alpha - d(\Pi(\alpha,\beta)) \\
&= i_{\rho\alpha}d\beta + d(\langle\rho\alpha,\beta\rangle) - i_{\rho\beta}d\alpha - d(\langle\rho\beta,\alpha\rangle) - d(\Pi(\alpha,\beta)) \\
&= i_{\rho\alpha}d\beta + d(\Pi(\alpha,\beta)) - i_{\rho\beta}d\alpha - d(\Pi(\beta,\alpha)) - d(\Pi(\alpha,\beta)) \qquad \text{(A.4)} \\
&= i_{\rho\alpha}d\beta - i_{\rho\beta}d\alpha + d(\Pi(\alpha,\beta)),
\end{aligned}
$$

since Π *is skew-symmetric.*

The Leibniz Rule is immediately satisfied by this bracket [28]:

$$
\begin{aligned}
[\alpha,f\beta]_* &= i_{\rho\alpha}d(f\beta) - i_{\rho(f\beta)}d\alpha + d(\Pi(\alpha,f\beta)) \\
&= i_{\rho\alpha}(fd\beta + df\wedge\beta) - fi_{\rho\beta}d\alpha + d(f\Pi(\alpha,\beta)) \\
&= fi_{\rho\alpha}d\beta + i_{\rho\alpha}(df\wedge\beta) - fi_{\rho\beta}d\alpha + fd(\Pi(\alpha,\beta)) + \Pi(\alpha,\beta)df \\
&= fi_{\rho\alpha}d\beta + i_{\rho\alpha}(df\wedge\beta) - fi_{\rho\beta}d\alpha + f([\alpha,\beta]_* - i_{\rho\alpha}d\beta + i_{\rho\beta}d\alpha) + \Pi(\alpha,\beta)df \\
&= i_{\rho\alpha}(df\wedge\beta) + f[\alpha,\beta]_* + \Pi(\alpha,\beta)df \\
&= (i_{\rho\alpha}df)\wedge\beta - df(i_{\rho\alpha}\beta) + f[\alpha,\beta] + (i_{\rho\alpha}\beta)df \\
&= (i_{\rho\alpha}df)\wedge\beta + f[\alpha,\beta]_* \\
&= f[\alpha,\beta]_* + ((\rho\alpha)(f))\beta.
\end{aligned}
$$

[28] Let's confirm that the Jacobi Identity is satisfied by this bracket. By definition, $[df,dg] = d\{f,g\}$ *and so on if* $\alpha = df$*,* $\beta = dg$*,* $\gamma = dh$ *are exact 1-forms. Consequently, the Jacobi Identity for the triple* (df,dg,dh) *follows from the Jacobi Identity for the triple* (f,g,h) *with regard to the Poisson bracket. The Leibniz Rule can be applied to reduce to the case of exact 1-forms from more general 1-forms.* ■

Proof A.6. *We have to show that the compatibility condition* $d[\alpha,\beta]_\pi = [d\alpha,\beta]_\pi + [\alpha,d\beta]_\pi$*, which is* (3.23)*, holds for all* $\alpha,\beta\in\Omega^1(M)$*.*

The formula is trivial for exact forms. For the general case we have to check that both sides gain the same terms when multiplying a form with a smooth function.

$$\begin{aligned}
d\,[df, g_1 dg_2]_\pi &= d(g_1[df, dg_2]_\pi + \rho(df)(g_1)dg_2) \\
&= dg_1 \wedge [df, dg_2]_\pi + d\,(\rho(df)(g_1)dg_2) \\
&= dg_1 \wedge [df, dg_2]_\pi + d(\langle \pi, df \wedge dg_1 \rangle dg_2) \\
&= dg_1 \wedge [df, dg_2]_\pi + d\langle \pi, df \wedge dg_1 \rangle \wedge dg_2 \\
&= dg_1 \wedge [df, dg_2]_\pi + d\{f, g_1\} \wedge dg_2 \\
&= dg_1 \wedge [df, dg_2]_\pi + [df, dg_1]_\pi \wedge dg_2 \\
&= [df, d\,(g_1 dg_2)]_\pi + \left[d^2 f, g_1 dg_2\right]_\pi .
\end{aligned}$$

The same formula holds for the first argument of the bracket due to skew-symmetry.■

Proof A.7. *For any $\xi_1, \xi_2, \xi_3 \in \Gamma(A^*)$, we have (see the equation* (2.31) *in [65]):*

$$[\Lambda, \Lambda]_{SN}(\xi_1, \xi_2, \xi_3) = 2\langle \Lambda^\sharp L_{\Lambda^\sharp \xi_1} \xi_2, \xi_3 \rangle + c.p. \qquad (A.5)$$

By using (3.27)*, the above equation can be written as*

$$\left\langle \frac{1}{2}[\Lambda, \Lambda]^\sharp(\xi_1, \xi_2), \xi_3 \right\rangle = \langle \Lambda^\sharp L_{\Lambda^\sharp \xi_1} \xi_2, \xi_3 \rangle + c.p. \qquad (A.6)$$

The LHS of this equation should be the following with respect to the Lemma 3.2.1 and the bracket in (3.25)*:*

$$\langle \Lambda^\sharp L_{\Lambda^\sharp \xi_1} \xi_2, \xi_3 \rangle - \langle \Lambda^\sharp L_{\Lambda^\sharp \xi_2} \xi_1, \xi_3 \rangle - \langle \Lambda^\sharp d[\Lambda(\xi_1, \xi_2)], \xi_3 \rangle - \langle L_{\Lambda^\sharp \xi_1} \Lambda^\sharp \xi_2, \xi_3 \rangle, \qquad (A.7)$$

here we used $[\Lambda^\sharp \xi_1, \Lambda^\sharp \xi_2] = L_{\Lambda^\sharp \xi_1} \Lambda^\sharp \xi_2$ since it is a Lie bracket.
We now expand the RHS of (A.6) *and obtain the above expression. The RHS is then*

$$\begin{aligned}
&\langle \Lambda^\sharp L_{\Lambda^\sharp \xi_1} \xi_2, \xi_3 \rangle + \langle \Lambda^\sharp L_{\Lambda^\sharp \xi_2} \xi_3, \xi_1 \rangle + \langle \Lambda^\sharp L_{\Lambda^\sharp \xi_3} \xi_1, \xi_2 \rangle \\
&\quad = \langle \Lambda^\sharp L_{\Lambda^\sharp \xi_1} \xi_2, \xi_3 \rangle - \langle L_{\Lambda^\sharp \xi_2} \xi_3, \Lambda^\sharp \xi_1 \rangle - \langle L_{\Lambda^\sharp \xi_3} \xi_1, \Lambda^\sharp \xi_2 \rangle - \langle \Lambda^\sharp L_{\Lambda^\sharp \xi_2} \xi_1, \xi_3 \rangle - \langle L_{\Lambda^\sharp \xi_2} \xi_1, \Lambda^\sharp \xi_3 \rangle \\
&\quad = \langle \Lambda^\sharp L_{\Lambda^\sharp \xi_1} \xi_2, \xi_3 \rangle - \langle \Lambda^\sharp L_{\Lambda^\sharp \xi_2} \xi_1, \xi_3 \rangle - L_{\Lambda^\sharp \xi_3} \langle \xi_1, \Lambda^\sharp \xi_2 \rangle - \langle L_{\Lambda^\sharp \xi_1} \Lambda^\sharp \xi_2, \xi_3 \rangle \\
&\quad = \langle \Lambda^\sharp L_{\Lambda^\sharp \xi_1} \xi_2, \xi_3 \rangle - \langle \Lambda^\sharp L_{\Lambda^\sharp \xi_2} \xi_1, \xi_3 \rangle - \langle d(\langle \xi_1, \Lambda^\sharp \xi_2 \rangle), \Lambda^\sharp \xi_3 \rangle - \langle L_{\Lambda^\sharp \xi_1} \Lambda^\sharp \xi_2, \xi_3 \rangle \\
&\quad = \langle \Lambda^\sharp L_{\Lambda^\sharp \xi_1} \xi_2, \xi_3 \rangle - \langle \Lambda^\sharp L_{\Lambda^\sharp \xi_2} \xi_1, \xi_3 \rangle - \langle \Lambda^\sharp d[\Lambda(\xi_1, \xi_2)], \xi_3 \rangle - \langle L_{\Lambda^\sharp \xi_1} \Lambda^\sharp \xi_2, \xi_3 \rangle,
\end{aligned}$$

where we have added and subtracted $\langle \Lambda^\sharp L_{\Lambda^\sharp \xi_2} \xi_1, \xi_3 \rangle$ in the second line and used Leibniz Rule to obtain the third line, and we have used the fact $L_X f = a(X) f = \langle df, X \rangle$ for the fourth line. Since the LHS and the RHS equals, it ends the proof of the Lemma.■

Proof A.8.

$$\begin{aligned}
&L_X([\Lambda, \Lambda]_{SN}(\xi_1, \xi_2, \xi_3)) = (L_X[\Lambda, \Lambda]_{SN})(\xi_1, \xi_2, \xi_3) + \{[\Lambda, \Lambda]_{SN}(\xi_1, \xi_2, L_X \xi_3) + c.p.\} \\
&= (L_X[\Lambda, \Lambda]_{SN})(\xi_1, \xi_2, \xi_3) + \left\{ \langle [\Lambda, \Lambda]^\sharp(\xi_1, \xi_2), L_X \xi_3 \rangle + c.p. \right\} \\
&= (L_X[\Lambda, \Lambda]_{SN})(\xi_1, \xi_2, \xi_3) + 3L_X(\langle [\Lambda, \Lambda]^\sharp(\xi_1, \xi_2), \xi_3 \rangle) - \left\{ \langle L_X([\Lambda, \Lambda]^\sharp(\xi_1, \xi_2)), \xi_3 \rangle + c.p. \right\} \\
&= (L_X[\Lambda, \Lambda]_{SN})(\xi_1, \xi_2, \xi_3) + 3L_X([\Lambda, \Lambda]_{SN}(\xi_1, \xi_2, \xi_3)) - \left\{ \langle L_X([\Lambda, \Lambda]^\sharp(\xi_1, \xi_2)), \xi_3 \rangle + c.p. \right\},
\end{aligned}$$

where we used the Leibniz Rule of L_X in the first and the third line. Therefore, we obtain

$$(L_X[\Lambda,\Lambda]_{SN})(\xi_1,\xi_2,\xi_3)=\Big\{\langle L_X([\Lambda,\Lambda]^\sharp(\xi_1,\xi_2)),\xi_3\rangle+c.p.\Big\}-2L_X([\Lambda,\Lambda]_{SN}(\xi_1,\xi_2,\xi_3)).\blacksquare \tag{A.8}$$

Proof A.9. *We start from the definition of the Jacobiator;*

$$\begin{aligned}
J(\xi_1,\xi_2,\xi_3) &= [[\xi_1,\xi_2],\xi_3]+c.p.\\
&=\Big\{L_{\Lambda^\sharp[\xi_1,\xi_2]}\xi_3-L_{\Lambda^\sharp\xi_3}[\xi_1,\xi_2]-d[\Lambda([\xi_1,\xi_2],\xi_3)]\Big\}+c.p.\\
&=\Big\{L_{\Lambda^\sharp[\xi_1,\xi_2]}\xi_3-L_{\Lambda^\sharp\xi_3}[\xi_1,\xi_2]-d\langle\Lambda^\sharp[\xi_1,\xi_2],\xi_3\rangle\Big\}+c.p.\\
&=\{\frac{1}{2}L_{[\Lambda,\Lambda]^\sharp(\xi_1,\xi_2)}\xi_3+L_{[\Lambda^\sharp\xi_1,\Lambda^\sharp\xi_2]}\xi_3-L_{\Lambda^\sharp\xi_3}[\xi_1,\xi_2]\\
&-\frac{1}{2}d([\Lambda,\Lambda]_{SN}(\xi_1,\xi_2,\xi_3))-d[\langle[\Lambda^\sharp\xi_1,\Lambda^\sharp\xi_2],\xi_3\rangle]\}+c.p.\\
&=\Big\{\frac{1}{2}L_{[\Lambda,\Lambda]^\sharp(\xi_1,\xi_2)}\xi_3-\frac{1}{2}d([\Lambda,\Lambda]_{SN}(\xi_1,\xi_2,\xi_3))\Big\}+c.p.\\
&+\Big\{L_{[\Lambda^\sharp\xi_1,\Lambda^\sharp\xi_2]}\xi_3-L_{\Lambda^\sharp\xi_3}[\xi_1,\xi_2]-d\langle[\Lambda^\sharp\xi_1,\Lambda^\sharp\xi_2],\xi_3\rangle\Big\}+c.p.\\
&=I_1+I_2,
\end{aligned}$$

where we have used (3.25) *for the second line and Lemma 3.2.1 for the fourth line. Let us try to simplify I_2 now.*

$$\begin{aligned}
&\Big\{L_{[\Lambda^\sharp\xi_1,\Lambda^\sharp\xi_2]}\xi_3-L_{\Lambda^\sharp\xi_3}[\xi_1,\xi_2]-d\langle[\Lambda^\sharp\xi_1,\Lambda^\sharp\xi_2],\xi_3\rangle\Big\}+c.p.\\
&=\{L_{[\Lambda^\sharp\xi_1,\Lambda^\sharp\xi_2]}\xi_3-L_{\Lambda^\sharp\xi_3}(L_{\Lambda^\sharp\xi_1}\xi_2)+L_{\Lambda^\sharp\xi_3}(L_{\Lambda^\sharp\xi_2}\xi_1)-d(L_{\Lambda^\sharp\xi_3}\langle\xi_1,\Lambda^\sharp\xi_2\rangle)\\
&-d\langle[\Lambda^\sharp\xi_1,\Lambda^\sharp\xi_2],\xi_3\rangle\}+c.p.,
\end{aligned}$$

where we have used (3.25)*. By expanding cyclic permutations for the first three terms in the above equation, we get*

$$\begin{aligned}
&\Big\{L_{[\Lambda^\sharp\xi_1,\Lambda^\sharp\xi_2]}\xi_3-L_{\Lambda^\sharp\xi_3}(L_{\Lambda^\sharp\xi_1}\xi_2)+L_{\Lambda^\sharp\xi_3}(L_{\Lambda^\sharp\xi_2}\xi_1)\Big\}+c.p.\\
&=L_{[\Lambda^\sharp\xi_1,\Lambda^\sharp\xi_2]}\xi_3+L_{[\Lambda^\sharp\xi_2,\Lambda^\sharp\xi_3]}\xi_1+L_{[\Lambda^\sharp\xi_3,\Lambda^\sharp\xi_1]}\xi_2\\
&-L_{\Lambda^\sharp\xi_3}(L_{\Lambda^\sharp\xi_1}\xi_2)-L_{\Lambda^\sharp\xi_1}(L_{\Lambda^\sharp\xi_2}\xi_3)-L_{\Lambda^\sharp\xi_2}(L_{\Lambda^\sharp\xi_3}\xi_1)\\
&+L_{\Lambda^\sharp\xi_3}(L_{\Lambda^\sharp\xi_2}\xi_1)+L_{\Lambda^\sharp\xi_2}(L_{\Lambda^\sharp\xi_1}\xi_3)+L_{\Lambda^\sharp\xi_1}(L_{\Lambda^\sharp\xi_3}\xi_2).
\end{aligned}$$

However, the above equation becomes zero from $L_{[X,Y]}=L_X\circ L_Y-L_Y\circ L_X$. Then,

$$\begin{aligned}
I_2 &= -d\Big\{L_{\Lambda^\sharp\xi_3}\langle\xi_1,\Lambda^\sharp\xi_2\rangle+\langle[\Lambda^\sharp\xi_1,\Lambda^\sharp\xi_2],\xi_3\rangle\Big\}+c.p.\\
&=-d\Big\{L_{\Lambda^\sharp\xi_3}\langle\xi_1,\Lambda^\sharp\xi_2\rangle+L_{\Lambda^\sharp\xi_1}\langle\xi_2,\Lambda^\sharp\xi_3\rangle+L_{\Lambda^\sharp\xi_2}\langle\xi_3,\Lambda^\sharp\xi_1\rangle\\
&+\langle[\Lambda^\sharp\xi_1,\Lambda^\sharp\xi_2],\xi_3\rangle+\langle[\Lambda^\sharp\xi_2,\Lambda^\sharp\xi_3],\xi_1\rangle+\langle[\Lambda^\sharp\xi_3,\Lambda^\sharp\xi_1],\xi_2\rangle\Big\}\\
&=-d\{\langle L_{\Lambda^\sharp\xi_3}\xi_1,\Lambda^\sharp\xi_2\rangle+\langle\xi_1,L_{\Lambda^\sharp\xi_3}\Lambda^\sharp\xi_2\rangle+\langle L_{\Lambda^\sharp\xi_1}\xi_2,\Lambda^\sharp\xi_3\rangle+\langle\xi_2,L_{\Lambda^\sharp\xi_1}\Lambda^\sharp\xi_3\rangle\\
&+\langle L_{\Lambda^\sharp\xi_2}\xi_3,\Lambda^\sharp\xi_1\rangle+\langle\xi_3,L_{\Lambda^\sharp\xi_2}\Lambda^\sharp\xi_1\rangle+\langle L_{\Lambda^\sharp\xi_1}\Lambda^\sharp\xi_2,\xi_3\rangle\\
&+\langle L_{\Lambda^\sharp\xi_2}\Lambda^\sharp\xi_3,\xi_1\rangle+\langle L_{\Lambda^\sharp\xi_3}\Lambda^\sharp\xi_1,\xi_2\rangle\},
\end{aligned}$$

where we have used the Leibniz Rule of L_X in the last equality. If we use the property $L_XY = -L_YX$ for the last three terms in the above equation and after simplifications, we obtain

$$\begin{aligned}
I_2 &= -d\left\{\langle L_{\Lambda^\sharp\xi_3}\xi_1, \Lambda^\sharp\xi_2\rangle + \langle L_{\Lambda^\sharp\xi_1}\xi_2, \Lambda^\sharp\xi_3\rangle + \langle L_{\Lambda^\sharp\xi_2}\xi_3, \Lambda^\sharp\xi_1\rangle\right\} \\
&= -d\langle L_{\Lambda^\sharp\xi_1}\xi_2, \Lambda^\sharp\xi_3\rangle + c.p. \\
&= d\langle \Lambda^\sharp L_{\Lambda^\sharp\xi_1}\xi_2, \xi_3\rangle + c.p. \\
&= \frac{1}{2}d[\Lambda,\Lambda]_{SN}(\xi_1,\xi_2,\xi_3) \\
&= \frac{1}{2}d\langle [\Lambda,\Lambda]^\sharp(\xi_1,\xi_2),\xi_3\rangle,
\end{aligned}$$

where the fourth equality follows from (A.5). *So, the Jacobiator becomes*

$$\begin{aligned}
J(\xi_1,\xi_2,\xi_3) &= \left\{\frac{1}{2}L_{[\Lambda,\Lambda]^\sharp(\xi_1,\xi_2)}\xi_3 + c.p\right\} + \left\{-\frac{1}{2}d([\Lambda,\Lambda]_{SN}(\xi_1,\xi_2,\xi_3)) + c.p.\right\} \\
&\quad + \frac{1}{2}d\langle [\Lambda,\Lambda]^\sharp(\xi_1,\xi_2),\xi_3\rangle \\
&= \left\{\frac{1}{2}L_{[\Lambda,\Lambda]^\sharp(\xi_1,\xi_2)}\xi_3 + c.p\right\} - \frac{1}{2}d\langle [\Lambda,\Lambda]^\sharp(\xi_2,\xi_3),\xi_1\rangle - \frac{1}{2}d\langle [\Lambda,\Lambda]^\sharp(\xi_3,\xi_1),\xi_2\rangle \\
&= \left\{\frac{1}{2}L_{[\Lambda,\Lambda]^\sharp(\xi_1,\xi_2)}\xi_3 + c.p\right\} - \frac{1}{2}d([\Lambda,\Lambda]_{SN}(\xi_2,\xi_3,\xi_1)) - \frac{1}{2}d([\Lambda,\Lambda]_{SN}(\xi_3,\xi_1,\xi_2)) \\
&= \left\{\frac{1}{2}L_{[\Lambda,\Lambda]^\sharp(\xi_1,\xi_2)}\xi_3 + c.p\right\} - d([\Lambda,\Lambda]_{SN}(\xi_1,\xi_2,\xi_3)).\blacksquare
\end{aligned}$$

Proof A.10. *Since ρ is a Lie algebra homomorphism, this gives us the first property in local coordinates such as [48]*

$$\begin{aligned}
&[\rho(S_a),\rho(S_b)] = \rho([S_a,S_b]) \\
\Rightarrow &\left[\rho_a^i\frac{\partial}{\partial X^i},\rho_b^j\frac{\partial}{\partial X^j}\right] = \rho(L_{ab}^c S_c) \\
\Rightarrow &\left(\rho_a^i\frac{\partial\rho_b^j}{\partial X^i} - \rho_b^i\frac{\partial\rho_a^j}{\partial X^i}\right)\frac{\partial}{\partial X^j} = L_{ab}^c\rho_c^k\frac{\partial}{\partial X^k} \\
\Rightarrow &\rho_a^m\frac{\partial\rho_b^i}{\partial X^m} - \rho_b^m\frac{\partial\rho_a^i}{\partial X^m} = L_{ab}^c\rho_c^i.\square
\end{aligned}$$

The second property follows from the Jacobi Identity: $[[S_a,S_b],S_c]+[[S_b,S_c],S_a]+[[S_c,S_a],S_b]=0$ *because of A being a Lie algebroid. In local coordinates, it yields*

$$
\begin{aligned}
&[L^d_{ab}S_d,S_c]+[L^d_{bc}S_d,S_a]+[L^d_{ca}S_d,S_b]=0\\
&\Rightarrow [S_c,L^d_{ab}S_d]+[S_a,L^d_{bc}S_d]+[S_b,L^d_{ca}S_d]=0\\
&\Rightarrow L^d_{ab}[S_c,S_d]+(\rho(S_c)L^d_{ab})S_d+L^d_{bc}[S_a,S_d]+(\rho(S_a)L^d_{bc})S_d\\
&+L^d_{ca}[S_b,S_d]+(\rho(S_b)L^d_{ca})S_d=0\\
&\Rightarrow L^d_{ab}L^m_{cd}S_m+\rho^m_c\frac{\partial L^d_{ab}}{\partial X^m}S_d+L^d_{bc}L^m_{ad}S_m+\rho^m_a\frac{\partial L^d_{bc}}{\partial X^m}S_d+L^d_{ca}L^m_{bd}S_m+\rho^m_b\frac{\partial L^d_{ca}}{\partial X^m}S_d=0\\
&\Rightarrow \left(\rho^m_a\frac{\partial L^d_{bc}}{\partial X^m}+\rho^m_b\frac{\partial L^d_{ca}}{\partial X^m}+\rho^m_c\frac{\partial L^d_{ab}}{\partial X^m}+L^d_{ae}L^e_{bc}+L^d_{be}L^e_{ca}+L^d_{ce}L^e_{ab}\right)S_d=0\\
&\Rightarrow \rho^m_a\frac{\partial L^d_{bc}}{\partial X^m}+\rho^m_b\frac{\partial L^d_{ca}}{\partial X^m}+\rho^m_c\frac{\partial L^d_{ab}}{\partial X^m}+L^d_{ae}L^e_{bc}+L^d_{be}L^e_{ca}+L^d_{ce}L^e_{ab}=0,
\end{aligned}
$$

where we have used the property (3.2) *of a Lie algebroid in the third line. Or, it can be written as*

$$\left(\rho^i_\alpha\frac{\partial L^\delta_{\beta\gamma}}{\partial X^i}+L^\delta_{\alpha\eta}L^\eta_{\beta\gamma}\right)+c.p.=0, \tag{A.9}$$

and this completes the proof of (4.5). ■

Proof A.11. *Let us write the condition:*

$$
\begin{aligned}
&\rho^{ia}_*\frac{\partial\rho^{jb}_*}{\partial X^i}-\rho^{ib}_*\frac{\partial\rho^{ja}_*}{\partial X^i}=\widetilde{L}^{ab}{}_c\rho^{jc}_*\\
&\Rightarrow \Lambda^{ac}\rho^i_c\frac{\partial(\Lambda^{bd}\rho^j_d)}{\partial X^i}-\Lambda^{bc}\rho^i_c\frac{\partial(\Lambda^{ad}\rho^j_d)}{\partial X^i}=\left(\Lambda^{bd}L_{dc}{}^a-\Lambda^{ad}L_{dc}{}^b+\frac{\partial\Lambda^{ab}}{\partial X^i}\rho^i_c\right)\Lambda^{ce}\rho^j_e\\
&\Rightarrow \left(\Lambda^{ac}\rho^i_c\rho^j_d\frac{\partial\Lambda^{bd}}{\partial X^i}+\Lambda^{ac}\rho^i_c\Lambda^{bd}\frac{\partial\rho^j_d}{\partial X^i}-\Lambda^{bc}\rho^i_c\rho^j_d\frac{\partial\Lambda^{ad}}{\partial X^i}-\Lambda^{bc}\rho^i_c\Lambda^{ad}\frac{\partial\rho^j_d}{\partial X^i}-\Lambda^{bd}L_{dc}{}^a\Lambda^{ce}\rho^j_e\right.\\
&\left.+\Lambda^{ad}L_{dc}{}^b\Lambda^{ce}\rho^j_e-\frac{\partial\Lambda^{ab}}{\partial X^i}\rho^i_c\Lambda^{ce}\rho^j_e\right)=0\\
&\Rightarrow \left(\Lambda^{ac}\Lambda^{bd}\rho^j_eL_{cd}{}^e+\Lambda^{ac}\rho^i_c\rho^j_d\frac{\partial\Lambda^{bd}}{\partial X^i}-\Lambda^{bc}\rho^i_c\rho^j_d\frac{\partial\Lambda^{ad}}{\partial X^i}-\Lambda^{bd}L_{dc}{}^a\Lambda^{ce}\rho^j_e+\Lambda^{ad}L_{dc}{}^b\Lambda^{ce}\rho^j_e\right.\\
&\left.-\frac{\partial\Lambda^{ab}}{\partial X^i}\rho^i_c\Lambda^{ce}\rho^j_e\right)=0\\
&\Rightarrow \rho^j_e\left(\Lambda^{ac}\Lambda^{bd}L_{cd}{}^e-\Lambda^{bd}\Lambda^{ce}L_{dc}{}^a+\Lambda^{ad}\Lambda^{ce}L_{dc}{}^b+\Lambda^{ac}\rho^i_c\frac{\partial\Lambda^{be}}{\partial X^i}-\Lambda^{bc}\rho^i_c\frac{\partial\Lambda^{ae}}{\partial X^i}-\Lambda^{ce}\rho^i_c\frac{\partial\Lambda^{ab}}{\partial X^i}\right)
\end{aligned}
\tag{A.10}
$$

$$=0,$$

where we have used the identity $\rho^{ia}_*=\Lambda^{ab}\rho^i_b$ *(from* (4.9)*) and* (4.20) *in the second line, and* (4.3)*, because of A being a Lie algebroid, in the fifth line.*

If the condition:

$$\Lambda^{ac}\Lambda^{bd}L_{cd}{}^{e}+\Lambda^{bd}\Lambda^{ec}L_{dc}{}^{a}+\Lambda^{ec}\Lambda^{ad}L_{cd}{}^{b}+\Lambda^{ac}\rho_c^i\frac{\partial\Lambda^{be}}{\partial X^i}+\Lambda^{bc}\rho_c^i\frac{\partial\Lambda^{ea}}{\partial X^i}+\Lambda^{ec}\rho_c^i\frac{\partial\Lambda^{ab}}{\partial X^i}=0.$$

is satisfied for $\rho_e^j\neq 0$*, then the result* (A.10) *vanishes.* □

Our next goal is to prove that $[\Lambda,\Lambda]_{SN}=0$ *is equivalent to the condition* (4.23).
Let $\Lambda=\frac{1}{2}\Lambda^{\alpha\beta}S_\alpha\wedge S_\beta$*, then*

$$\begin{aligned}
[\Lambda,\Lambda]_{SN} &= \left[\frac{1}{2}\Lambda^{\alpha\beta}S_\alpha\wedge S_\beta,\frac{1}{2}\Lambda^{\gamma\delta}S_\gamma\wedge S_\delta\right]_{SN}\\
&=\frac{1}{4}\Lambda^{\alpha\beta}\Lambda^{\gamma\delta}[S_\alpha\wedge S_\beta,S_\gamma\wedge S_\delta]_{SN}+\frac{1}{2}\Lambda^{\gamma\delta}S_\alpha\wedge S_\beta\wedge\left[\frac{1}{2}\Lambda^{\alpha\beta},S_\gamma\wedge S_\delta\right]_{SN}\\
&+\frac{1}{2}\Lambda^{\alpha\beta}\left[\frac{1}{2}\Lambda^{\gamma\delta},S_\alpha\wedge S_\beta\right]_{SN}\wedge S_\gamma\wedge S_\delta\\
&=\frac{1}{4}\Lambda^{\alpha\beta}\Lambda^{\gamma\delta}([S_\alpha,S_\gamma]\wedge S_\beta\wedge S_\delta-[S_\alpha,S_\delta]\wedge S_\beta\wedge S_\gamma-[S_\beta,S_\gamma]\wedge S_\alpha\wedge S_\delta\\
&+[S_\beta,S_\delta]\wedge S_\alpha\wedge S_\gamma)\\
&+\frac{1}{2}\Lambda^{\gamma\delta}S_\alpha\wedge S_\beta\wedge\left\{-\rho(S_\gamma)\left(\frac{1}{2}\Lambda^{\alpha\beta}\right)S_\delta+\rho(S_\delta)\left(\frac{1}{2}\Lambda^{\alpha\beta}\right)S_\gamma\right\}\\
&+\frac{1}{2}\Lambda^{\alpha\beta}\left\{-\rho(S_\alpha)\left(\frac{1}{2}\Lambda^{\gamma\delta}\right)S_\beta+\rho(S_\beta)\left(\frac{1}{2}\Lambda^{\gamma\delta}\right)S_\alpha\right\}\wedge S_\gamma\wedge S_\delta\\
&=\frac{1}{4}\Lambda^{\alpha\beta}\Lambda^{\gamma\delta}(L^\varepsilon_{\alpha\gamma}S_\varepsilon\wedge S_\beta\wedge S_\delta-L^\varepsilon_{\alpha\delta}S_\varepsilon\wedge S_\beta\wedge S_\gamma-L^\varepsilon_{\beta\gamma}S_\varepsilon\wedge S_\alpha\wedge S_\delta+L^\varepsilon_{\beta\delta}S_\varepsilon\wedge S_\alpha\wedge S_\gamma)\\
&+\frac{1}{2}\Lambda^{\gamma\delta}S_\alpha\wedge S_\beta\wedge\left\{-\rho^i_\gamma\frac{\partial\left(\frac{1}{2}\Lambda^{\alpha\beta}\right)}{\partial X^i}S_\delta+\rho^i_\delta\frac{\partial\left(\frac{1}{2}\Lambda^{\alpha\beta}\right)}{\partial X^i}S_\gamma\right\}\\
&+\frac{1}{2}\Lambda^{\alpha\beta}\left\{-\rho^i_\alpha\frac{\partial\left(\frac{1}{2}\Lambda^{\gamma\delta}\right)}{\partial X^i}S_\beta+\rho^i_\beta\frac{\partial\left(\frac{1}{2}\Lambda^{\gamma\delta}\right)}{\partial X^i}S_\alpha\right\}\wedge S_\gamma\wedge S_\delta\\
&=\frac{1}{4}\Lambda^{\alpha\beta}\Lambda^{\gamma\delta}L^\varepsilon_{\alpha\gamma}S_\varepsilon\wedge S_\beta\wedge S_\delta-\frac{1}{4}\Lambda^{\alpha\beta}\Lambda^{\gamma\delta}L^\varepsilon_{\alpha\delta}S_\varepsilon\wedge S_\beta\wedge S_\gamma\\
&-\frac{1}{4}\Lambda^{\alpha\beta}\Lambda^{\gamma\delta}L^\varepsilon_{\beta\gamma}S_\varepsilon\wedge S_\alpha\wedge S_\delta+\frac{1}{4}\Lambda^{\alpha\beta}\Lambda^{\gamma\delta}L^\varepsilon_{\beta\delta}S_\varepsilon\wedge S_\alpha\wedge S_\gamma\\
&-\frac{1}{4}\Lambda^{\gamma\delta}\rho^i_\gamma\frac{\partial\Lambda^{\alpha\beta}}{\partial X^i}S_\alpha\wedge S_\beta\wedge S_\delta+\frac{1}{4}\Lambda^{\gamma\delta}\rho^i_\delta\frac{\partial\Lambda^{\alpha\beta}}{\partial X^i}S_\alpha\wedge S_\beta\wedge S_\gamma\\
&-\frac{1}{4}\Lambda^{\alpha\beta}\rho^i_\alpha\frac{\partial\Lambda^{\gamma\delta}}{\partial X^i}S_\beta\wedge S_\gamma\wedge S_\delta+\frac{1}{4}\Lambda^{\alpha\beta}\rho^i_\beta\frac{\partial\Lambda^{\gamma\delta}}{\partial X^i}S_\alpha\wedge S_\gamma\wedge S_\delta,
\end{aligned}$$

where we have used (3.17) *in the second line, the definition of the SN bracket* (3.14) *in the third line and* (3.19) *in the fourth and fifth lines,* (4.2) *in the sixth line and* (4.1) *in the seventh and eighth lines. We will now write all the terms in form of* $S_\varepsilon\wedge S_\alpha\wedge S_\delta$ *in*

the above equality. Let us look at one by one;

$$\textcircled{1}(\alpha \leftrightarrow \beta) = \frac{1}{4}\Lambda^{\beta\alpha}\Lambda^{\gamma\delta}L^{\varepsilon}_{\beta\gamma}S_{\varepsilon}\wedge S_{\alpha}\wedge S_{\delta}$$

$$\textcircled{2}(\alpha \leftrightarrow \beta, \gamma \leftrightarrow \delta) = -\frac{1}{4}\Lambda^{\beta\alpha}\Lambda^{\delta\gamma}L^{\varepsilon}_{\beta\gamma}S_{\varepsilon}\wedge S_{\alpha}\wedge S_{\delta} = \frac{1}{4}\Lambda^{\beta\alpha}\Lambda^{\gamma\delta}L^{\varepsilon}_{\beta\gamma}S_{\varepsilon}\wedge S_{\alpha}\wedge S_{\delta}$$

$$\textcircled{3} = -\frac{1}{4}\Lambda^{\alpha\beta}\Lambda^{\gamma\delta}L^{\varepsilon}_{\beta\gamma}S_{\varepsilon}\wedge S_{\alpha}\wedge S_{\delta} = \frac{1}{4}\Lambda^{\beta\alpha}\Lambda^{\gamma\delta}L^{\varepsilon}_{\beta\gamma}S_{\varepsilon}\wedge S_{\alpha}\wedge S_{\delta}$$

$$\textcircled{4}(\gamma \leftrightarrow \delta) = \frac{1}{4}\Lambda^{\alpha\beta}\Lambda^{\delta\gamma}L^{\varepsilon}_{\beta\gamma}S_{\varepsilon}\wedge S_{\alpha}\wedge S_{\delta} = \frac{1}{4}\Lambda^{\beta\alpha}\Lambda^{\gamma\delta}L^{\varepsilon}_{\beta\gamma}S_{\varepsilon}\wedge S_{\alpha}\wedge S_{\delta},$$

where we have used the property $\Lambda^{\alpha\beta} = -\Lambda^{\beta\alpha}$ *because* $\Lambda \in \Gamma(\wedge^2 A)$*. So, the addition of these first four terms gives that*

$$\textcircled{1}+\textcircled{2}+\textcircled{3}+\textcircled{4} = \Lambda^{\beta\alpha}\Lambda^{\gamma\delta}L^{\varepsilon}_{\beta\gamma}S_{\varepsilon}\wedge S_{\alpha}\wedge S_{\delta}. \tag{A.11}$$

We will focus on the last four terms now;

$$\textcircled{5}(\beta \to \varepsilon) = -\frac{1}{4}\Lambda^{\gamma\delta}\rho^{i}_{\gamma}\frac{\partial\Lambda^{\alpha\varepsilon}}{\partial X^{i}}S_{\alpha}\wedge S_{\varepsilon}\wedge S_{\delta} = \frac{1}{4}\Lambda^{\gamma\delta}\rho^{i}_{\gamma}\frac{\partial\Lambda^{\alpha\varepsilon}}{\partial X^{i}}S_{\varepsilon}\wedge S_{\alpha}\wedge S_{\delta}$$

$$\textcircled{6}(\beta \to \varepsilon, \gamma \leftrightarrow \delta) = \frac{1}{4}\Lambda^{\delta\gamma}\rho^{i}_{\gamma}\frac{\partial\Lambda^{\alpha\varepsilon}}{\partial X^{i}}S_{\alpha}\wedge S_{\varepsilon}\wedge S_{\delta} = \frac{1}{4}\Lambda^{\gamma\delta}\rho^{i}_{\gamma}\frac{\partial\Lambda^{\alpha\varepsilon}}{\partial X^{i}}S_{\varepsilon}\wedge S_{\alpha}\wedge S_{\delta}$$

$$\textcircled{7}(\alpha \leftrightarrow \gamma, \beta \to \delta, \delta \to \varepsilon) = -\frac{1}{4}\Lambda^{\gamma\delta}\rho^{i}_{\gamma}\frac{\partial\Lambda^{\alpha\varepsilon}}{\partial X^{i}}S_{\delta}\wedge S_{\alpha}\wedge S_{\varepsilon} = \frac{1}{4}\Lambda^{\gamma\delta}\rho^{i}_{\gamma}\frac{\partial\Lambda^{\alpha\varepsilon}}{\partial X^{i}}S_{\varepsilon}\wedge S_{\alpha}\wedge S_{\delta}$$

$$\textcircled{8}(\gamma \to \varepsilon, \alpha \leftrightarrow \delta, \beta \to \gamma) = \frac{1}{4}\Lambda^{\delta\gamma}\rho^{i}_{\gamma}\frac{\partial\Lambda^{\varepsilon\alpha}}{\partial X^{i}}S_{\delta}\wedge S_{\varepsilon}\wedge S_{\alpha} = \frac{1}{4}\Lambda^{\gamma\delta}\rho^{i}_{\gamma}\frac{\partial\Lambda^{\alpha\varepsilon}}{\partial X^{i}}S_{\varepsilon}\wedge S_{\alpha}\wedge S_{\delta}.$$

Therefore, the addition of these terms yields

$$\textcircled{5}+\textcircled{6}+\textcircled{7}+\textcircled{8} = \Lambda^{\gamma\delta}\rho^{i}_{\gamma}\frac{\partial\Lambda^{\alpha\varepsilon}}{\partial X^{i}}S_{\varepsilon}\wedge S_{\alpha}\wedge S_{\delta}. \tag{A.12}$$

After all, we find that

$$[\Lambda,\Lambda]_{SN} = \left(\Lambda^{\beta\alpha}\Lambda^{\gamma\delta}L^{\varepsilon}_{\beta\gamma} + \Lambda^{\gamma\delta}\rho^{i}_{\gamma}\frac{\partial\Lambda^{\alpha\varepsilon}}{\partial X^{i}}\right)S_{\varepsilon}\wedge S_{\alpha}\wedge S_{\delta}, \tag{A.13}$$

or

$$[\Lambda,\Lambda]_{SN} = \left(\Lambda^{\alpha\beta}\rho^{i}_{\beta}\frac{\partial\Lambda^{\varepsilon\delta}}{\partial X^{i}} + \Lambda^{\alpha\beta}\Lambda^{\gamma\delta}L^{\varepsilon}_{\beta\gamma}\right)S_{\varepsilon}\wedge S_{\delta}\wedge S_{\alpha}, \tag{A.14}$$

which is an element of $\Gamma(\wedge^3 A)$*, as expected. Whether the above equation vanishes or not, one needs to write down its expansion which is as below.*

$$\begin{aligned}
&\Lambda^{\alpha\beta}\Lambda^{\gamma\delta}L^{\varepsilon}_{\beta\gamma}+\Lambda^{\delta\beta}\Lambda^{\gamma\varepsilon}L^{\alpha}_{\beta\gamma}+\Lambda^{\varepsilon\beta}\Lambda^{\gamma\alpha}L^{\delta}_{\beta\gamma}+\Lambda^{\alpha\beta}\rho^{i}_{\beta}\frac{\partial\Lambda^{\varepsilon\delta}}{\partial X^{i}}+\Lambda^{\varepsilon\beta}\rho^{i}_{\beta}\frac{\partial\Lambda^{\delta\alpha}}{\partial X^{i}}+\Lambda^{\delta\beta}\rho^{i}_{\beta}\frac{\partial\Lambda^{\alpha\varepsilon}}{\partial X^{i}}\\
&=-\Lambda^{\alpha\beta}\Lambda^{\delta\gamma}L^{\varepsilon}_{\beta\gamma}-\Lambda^{\delta\beta}\Lambda^{\varepsilon\gamma}L^{\alpha}_{\beta\gamma}-\Lambda^{\varepsilon\beta}\Lambda^{\alpha\gamma}L^{\delta}_{\beta\gamma}-\Lambda^{\alpha\beta}\rho^{i}_{\beta}\frac{\partial\Lambda^{\delta\varepsilon}}{\partial X^{i}}\\
&-\Lambda^{\delta\beta}\rho^{i}_{\beta}\frac{\partial\Lambda^{\varepsilon\alpha}}{\partial X^{i}}-\Lambda^{\varepsilon\beta}\rho^{i}_{\beta}\frac{\partial\Lambda^{\alpha\delta}}{\partial X^{i}}\\
&=-\Big(\Lambda^{\alpha\beta}\Lambda^{\delta\gamma}L^{\varepsilon}_{\beta\gamma}+\Lambda^{\delta\beta}\Lambda^{\varepsilon\gamma}L^{\alpha}_{\beta\gamma}+\Lambda^{\varepsilon\beta}\Lambda^{\alpha\gamma}L^{\delta}_{\beta\gamma}+\Lambda^{\alpha\beta}\rho^{i}_{\beta}\frac{\partial\Lambda^{\delta\varepsilon}}{\partial X^{i}}\\
&+\Lambda^{\delta\beta}\rho^{i}_{\beta}\frac{\partial\Lambda^{\varepsilon\alpha}}{\partial X^{i}}+\Lambda^{\varepsilon\beta}\rho^{i}_{\beta}\frac{\partial\Lambda^{\alpha\delta}}{\partial X^{i}}\Big),
\end{aligned} \tag{A.15}$$

which yields the local expression of $[\Lambda,\Lambda]_{SN}$ *and if the condition* (4.23) *is correct then it indeed vanishes.* ■

Proof A.12. *Let us expand the LHS of the condition:*

$$
\begin{aligned}
&\widetilde{L}^{ab}{}_{d}\widetilde{L}^{dc}{}_{e}+\widetilde{L}^{bc}{}_{d}\widetilde{L}^{da}{}_{e}+\widetilde{L}^{ca}{}_{d}\widetilde{L}^{db}{}_{e}-\rho_*^{ic}\frac{\partial\widetilde{L}^{ab}{}_{e}}{\partial X^i}-\rho_*^{ia}\frac{\partial\widetilde{L}^{bc}{}_{e}}{\partial X^i}-\rho_*^{ib}\frac{\partial\widetilde{L}^{ca}{}_{e}}{\partial X^i}\\
&=\left(\Lambda^{bm}L_{md}{}^{a}-\Lambda^{am}L_{md}{}^{b}+\frac{\partial\Lambda^{ab}}{\partial X^i}\rho_d^i\right)\left(\Lambda^{cf}L_{fe}{}^{d}-\Lambda^{df}L_{fe}{}^{c}+\frac{\partial\Lambda^{dc}}{\partial X^j}\rho_e^j\right)\\
&+\left(\Lambda^{cm}L_{md}{}^{b}-\Lambda^{bm}L_{md}{}^{c}+\frac{\partial\Lambda^{bc}}{\partial X^i}\rho_d^i\right)\left(\Lambda^{af}L_{fe}{}^{d}-\Lambda^{df}L_{fe}{}^{a}+\frac{\partial\Lambda^{da}}{\partial X^j}\rho_e^j\right)\\
&+\left(\Lambda^{am}L_{md}{}^{c}-\Lambda^{cm}L_{md}{}^{a}+\frac{\partial\Lambda^{ca}}{\partial X^i}\rho_d^i\right)\left(\Lambda^{bf}L_{fe}{}^{d}-\Lambda^{df}L_{fe}{}^{b}+\frac{\partial\Lambda^{db}}{\partial X^j}\rho_e^j\right)\\
&-\Lambda^{cf}\rho_f^i\frac{\partial}{\partial X^i}\left(\Lambda^{bm}L_{me}{}^{a}-\Lambda^{am}L_{me}{}^{b}+\frac{\partial\Lambda^{ab}}{\partial X^j}\rho_e^j\right)-\Lambda^{af}\rho_f^i\frac{\partial}{\partial X^i}\left(\Lambda^{cm}L_{me}{}^{b}-\Lambda^{bm}L_{me}{}^{c}+\frac{\partial\Lambda^{bc}}{\partial X^j}\rho_e^j\right)\\
&-\Lambda^{bf}\rho_f^i\frac{\partial}{\partial X^i}\left(\Lambda^{am}L_{me}{}^{c}-\Lambda^{cm}L_{me}{}^{a}+\frac{\partial\Lambda^{ca}}{\partial X^j}\rho_e^j\right)\\
&=\Lambda^{bm}L_{md}{}^{a}\Lambda^{cf}L_{fe}{}^{d}-\Lambda^{bm}L_{md}{}^{a}\Lambda^{df}L_{fe}{}^{c}+\Lambda^{bm}L_{md}{}^{a}\frac{\partial\Lambda^{dc}}{\partial X^j}\rho_e^j-\Lambda^{am}L_{md}{}^{b}\Lambda^{cf}L_{fe}{}^{d}\\
&+\Lambda^{am}L_{md}{}^{b}\Lambda^{df}L_{fe}{}^{c}-\Lambda^{am}L_{md}{}^{b}\frac{\partial\Lambda^{dc}}{\partial X^j}\rho_e^j+\frac{\partial\Lambda^{ab}}{\partial X^i}\rho_d^i\Lambda^{cf}L_{fe}{}^{d}-\frac{\partial\Lambda^{ab}}{\partial X^i}\rho_d^i\Lambda^{df}L_{fe}{}^{c}\\
&+\frac{\partial\Lambda^{ab}}{\partial X^i}\rho_d^i\frac{\partial\Lambda^{dc}}{\partial X^j}\rho_e^j+\Lambda^{cm}L_{md}{}^{b}\Lambda^{af}L_{fe}{}^{d}-\Lambda^{cm}L_{md}{}^{b}\Lambda^{df}L_{fe}{}^{a}+\Lambda^{cm}L_{md}{}^{b}\frac{\partial\Lambda^{da}}{\partial X^j}\rho_e^j\\
&-\Lambda^{bm}L_{md}{}^{c}\Lambda^{af}L_{fe}{}^{d}+\Lambda^{bm}L_{md}{}^{c}\Lambda^{df}L_{fe}{}^{a}-\Lambda^{bm}L_{md}{}^{c}\frac{\partial\Lambda^{da}}{\partial X^j}\rho_e^j+\frac{\partial\Lambda^{bc}}{\partial X^i}\rho_d^i\Lambda^{af}L_{fe}{}^{d}\\
&-\frac{\partial\Lambda^{bc}}{\partial X^i}\rho_d^i\Lambda^{df}L_{fe}{}^{a}+\frac{\partial\Lambda^{bc}}{\partial X^i}\rho_d^i\frac{\partial\Lambda^{da}}{\partial X^j}\rho_e^j+\Lambda^{am}L_{md}{}^{c}\Lambda^{bf}L_{fe}{}^{d}-\Lambda^{am}L_{md}{}^{c}\Lambda^{df}L_{fe}{}^{b}\\
&+\Lambda^{am}L_{md}{}^{c}\frac{\partial\Lambda^{db}}{\partial X^j}\rho_e^j-\Lambda^{cm}L_{md}{}^{a}\Lambda^{bf}L_{fe}{}^{d}+\Lambda^{cm}L_{md}{}^{a}\Lambda^{df}L_{fe}{}^{b}-\Lambda^{cm}L_{md}{}^{a}\frac{\partial\Lambda^{db}}{\partial X^j}\rho_e^j\\
&+\frac{\partial\Lambda^{ca}}{\partial X^i}\rho_d^i\Lambda^{bf}L_{fe}{}^{d}-\frac{\partial\Lambda^{ca}}{\partial X^i}\rho_d^i\Lambda^{df}L_{fe}{}^{b}+\frac{\partial\Lambda^{ca}}{\partial X^i}\rho_d^i\frac{\partial\Lambda^{db}}{\partial X^j}\rho_e^j-\Lambda^{cf}\rho_f^i\frac{\partial\Lambda^{bm}}{\partial X^i}L_{me}{}^{a}\\
&-\Lambda^{cf}\rho_f^i\Lambda^{bm}\frac{\partial L_{me}{}^{a}}{\partial X^i}+\Lambda^{cf}\rho_f^i\frac{\partial\Lambda^{am}}{\partial X^i}L_{me}{}^{b}+\Lambda^{cf}\rho_f^i\Lambda^{am}\frac{\partial L_{me}{}^{b}}{\partial X^i}-\Lambda^{cf}\rho_f^i\frac{\partial^2\Lambda^{ab}}{\partial X^i\partial X^j}\rho_e^j\\
&-\Lambda^{cf}\rho_f^i\frac{\partial\Lambda^{ab}}{\partial X^j}\frac{\partial\rho_e^j}{\partial X^i}-\Lambda^{af}\rho_f^i\frac{\partial\Lambda^{cm}}{\partial X^i}L_{me}{}^{b}-\Lambda^{af}\rho_f^i\Lambda^{cm}\frac{\partial L_{me}{}^{b}}{\partial X^i}+\Lambda^{af}\rho_f^i\frac{\partial\Lambda^{bm}}{\partial X^i}L_{me}{}^{c}\\
&+\Lambda^{af}\rho_f^i\Lambda^{bm}\frac{\partial L_{me}{}^{c}}{\partial X^i}-\Lambda^{af}\rho_f^i\frac{\partial^2\Lambda^{bc}}{\partial X^i\partial X^j}\rho_e^j-\Lambda^{af}\rho_f^i\frac{\partial\Lambda^{bc}}{\partial X^j}\frac{\partial\rho_e^j}{\partial X^i}-\Lambda^{bf}\rho_f^i\frac{\partial\Lambda^{am}}{\partial X^i}L_{me}{}^{c}\\
&-\Lambda^{bf}\rho_f^i\Lambda^{am}\frac{\partial L_{me}{}^{c}}{\partial X^i}+\Lambda^{bf}\rho_f^i\frac{\partial\Lambda^{cm}}{\partial X^i}L_{me}{}^{a}+\Lambda^{bf}\rho_f^i\Lambda^{cm}\frac{\partial L_{me}{}^{a}}{\partial X^i}-\Lambda^{bf}\rho_f^i\frac{\partial^2\Lambda^{ca}}{\partial X^i\partial X^j}\rho_e^j\\
&-\Lambda^{bf}\rho_f^i\frac{\partial\Lambda^{ca}}{\partial X^j}\frac{\partial\rho_e^j}{\partial X^i}.
\end{aligned}
\tag{A.16}
$$

By using (4.4) *(since A is a Lie algebroid) for the above result, we obtain the followings*

$$\begin{aligned}
(1)+(22)+(29)+(43) &= -\Lambda^{bm}\Lambda^{cf}\left(L_{md}{}^{a}L_{ef}{}^{d}+L_{fd}{}^{a}L_{me}{}^{d}+\rho_{f}^{i}\frac{\partial L_{me}{}^{a}}{\partial X^{i}}+\rho_{m}^{i}\frac{\partial L_{ef}{}^{a}}{\partial X^{i}}\right)\\
&= \Lambda^{bm}\Lambda^{cf}\left(\rho_{e}^{i}\frac{\partial L_{fm}{}^{a}}{\partial X^{i}}+L_{ed}{}^{a}L_{fm}{}^{d}\right),
\end{aligned}$$

$$\begin{aligned}
(31)+(35)+(4)+(10) &= \Lambda^{am}\Lambda^{cf}\left(\rho_{f}^{i}\frac{\partial L_{me}{}^{b}}{\partial X^{i}}+\rho_{m}^{i}\frac{\partial L_{ef}{}^{b}}{\partial X^{i}}+L_{md}{}^{b}L_{ef}{}^{d}+L_{fd}{}^{b}L_{me}{}^{d}\right)\\
&= -\Lambda^{am}\Lambda^{cf}\left(\rho_{e}^{i}\frac{\partial L_{fm}{}^{b}}{\partial X^{i}}+L_{ed}{}^{b}L_{fm}{}^{d}\right),
\end{aligned}$$

$$\begin{aligned}
(37)+(41)+(13)+(19) &= -\Lambda^{am}\Lambda^{bf}\left(\rho_{f}^{i}\frac{\partial L_{me}{}^{c}}{\partial X^{i}}+\rho_{m}^{i}\frac{\partial L_{ef}{}^{c}}{\partial X^{i}}+L_{fd}{}^{c}L_{me}{}^{d}+L_{md}{}^{c}L_{ef}{}^{d}\right)\\
&= \Lambda^{am}\Lambda^{bf}\left(\rho_{e}^{i}\frac{\partial L_{fm}{}^{c}}{\partial X^{i}}+L_{ed}{}^{c}L_{fm}{}^{d}\right).
\end{aligned}$$

Therefore, the result (A.16) *becomes*

$$\begin{aligned}
&= \Lambda^{bm}\Lambda^{cf}\rho_{e}^{i}\frac{\partial L_{fm}{}^{a}}{\partial X^{i}}+\Lambda^{bm}\Lambda^{cf}L_{ed}{}^{a}L_{fm}{}^{d}-\Lambda^{am}\Lambda^{cf}\rho_{e}^{i}\frac{\partial L_{fm}{}^{b}}{\partial X^{i}}-\Lambda^{am}\Lambda^{cf}L_{ed}{}^{b}L_{fm}{}^{d}\\
&+\Lambda^{am}\Lambda^{bf}\rho_{e}^{i}\frac{\partial L_{fm}{}^{c}}{\partial X^{i}}+\Lambda^{am}\Lambda^{bf}L_{ed}{}^{c}L_{fm}{}^{d}-\Lambda^{bm}L_{md}{}^{a}\Lambda^{df}L_{fe}{}^{c}+\Lambda^{bm}L_{md}{}^{a}\frac{\partial\Lambda^{dc}}{\partial X^{j}}\rho_{e}^{j}\\
&+\Lambda^{am}L_{md}{}^{b}\Lambda^{df}L_{fe}{}^{c}-\Lambda^{am}L_{md}{}^{b}\frac{\partial\Lambda^{dc}}{\partial X^{j}}\rho_{e}^{j}+\frac{\partial\Lambda^{ab}}{\partial X^{i}}\rho_{d}^{i}\Lambda^{cf}L_{fe}{}^{d}-\frac{\partial\Lambda^{ab}}{\partial X^{i}}\rho_{d}^{i}\Lambda^{df}L_{fe}{}^{c}\\
&+\frac{\partial\Lambda^{ab}}{\partial X^{i}}\rho_{d}^{i}\frac{\partial\Lambda^{dc}}{\partial X^{j}}\rho_{e}^{j}-\Lambda^{cm}L_{md}{}^{b}\Lambda^{df}L_{fe}{}^{a}+\Lambda^{cm}L_{md}{}^{b}\frac{\partial\Lambda^{da}}{\partial X^{j}}\rho_{e}^{j}+\Lambda^{bm}L_{md}{}^{c}\Lambda^{df}L_{fe}{}^{a}\\
&-\Lambda^{bm}L_{md}{}^{c}\frac{\partial\Lambda^{da}}{\partial X^{j}}\rho_{e}^{j}+\frac{\partial\Lambda^{bc}}{\partial X^{i}}\rho_{d}^{i}\Lambda^{af}L_{fe}{}^{d}-\frac{\partial\Lambda^{bc}}{\partial X^{i}}\rho_{d}^{i}\Lambda^{df}L_{fe}{}^{a}+\frac{\partial\Lambda^{bc}}{\partial X^{i}}\rho_{d}^{i}\frac{\partial\Lambda^{da}}{\partial X^{j}}\rho_{e}^{j}\\
&-\Lambda^{am}L_{md}{}^{c}\Lambda^{df}L_{fe}{}^{b}+\Lambda^{am}L_{md}{}^{c}\frac{\partial\Lambda^{db}}{\partial X^{j}}\rho_{e}^{j}+\Lambda^{cm}L_{md}{}^{a}\Lambda^{df}L_{fe}{}^{b}-\Lambda^{cm}L_{md}{}^{a}\frac{\partial\Lambda^{db}}{\partial X^{j}}\rho_{e}^{j}\\
&+\frac{\partial\Lambda^{ca}}{\partial X^{i}}\rho_{d}^{i}\Lambda^{bf}L_{fe}{}^{d}-\frac{\partial\Lambda^{ca}}{\partial X^{i}}\rho_{d}^{i}\Lambda^{df}L_{fe}{}^{b}+\frac{\partial\Lambda^{ca}}{\partial X^{i}}\rho_{d}^{i}\frac{\partial\Lambda^{db}}{\partial X^{j}}\rho_{e}^{j}-\Lambda^{cf}\rho_{f}^{i}\frac{\partial\Lambda^{bm}}{\partial X^{i}}L_{me}{}^{a}\\
&+\Lambda^{cf}\rho_{f}^{i}\frac{\partial\Lambda^{am}}{\partial X^{i}}L_{me}{}^{b}-\Lambda^{cf}\rho_{f}^{i}\frac{\partial^{2}\Lambda^{ab}}{\partial X^{i}\partial X^{j}}\rho_{e}^{j}-\Lambda^{cf}\rho_{f}^{i}\frac{\partial\Lambda^{ab}}{\partial X^{j}}\frac{\partial\rho_{e}^{j}}{\partial X^{i}}-\Lambda^{af}\rho_{f}^{i}\frac{\partial\Lambda^{cm}}{\partial X^{i}}L_{me}{}^{b}\\
&+\Lambda^{af}\rho_{f}^{i}\frac{\partial\Lambda^{bm}}{\partial X^{i}}L_{me}{}^{c}-\Lambda^{af}\rho_{f}^{i}\frac{\partial^{2}\Lambda^{bc}}{\partial X^{i}\partial X^{j}}\rho_{e}^{j}-\Lambda^{af}\rho_{f}^{i}\frac{\partial\Lambda^{bc}}{\partial X^{j}}\frac{\partial\rho_{e}^{j}}{\partial X^{i}}-\Lambda^{bf}\rho_{f}^{i}\frac{\partial\Lambda^{am}}{\partial X^{i}}L_{me}{}^{c}\\
&+\Lambda^{bf}\rho_{f}^{i}\frac{\partial\Lambda^{cm}}{\partial X^{i}}L_{me}{}^{a}-\Lambda^{bf}\rho_{f}^{i}\frac{\partial^{2}\Lambda^{ca}}{\partial X^{i}\partial X^{j}}\rho_{e}^{j}-\Lambda^{bf}\rho_{f}^{i}\frac{\partial\Lambda^{ca}}{\partial X^{j}}\frac{\partial\rho_{e}^{j}}{\partial X^{i}},
\end{aligned}$$

and if the condition (4.23) *is correct then it indeed vanishes.* ■

Proof A.13. *From $d_*[X,Y]-\mathscr{L}_X d_* Y+\mathscr{L}_Y d_* X=0$, we obtain for local bases;*

$$
\begin{aligned}
&d_*[S_\delta,S_\omega]-\mathscr{L}_{S_\delta}d_*(S_\omega)+\mathscr{L}_{S_\omega}d_*(S_\delta)=0\\
&\Longrightarrow d_*(L_{\delta\omega}{}^{\alpha}S_\alpha)-\mathscr{L}_{S_\delta}\left(-\frac{1}{2}\widetilde{L}^{\alpha\beta}{}_{\omega}S_\alpha\wedge S_\beta\right)+\mathscr{L}_{S_\omega}\left(-\frac{1}{2}\widetilde{L}^{\alpha\beta}{}_{\delta}S_\alpha\wedge S_\beta\right)=0\\
&\Longrightarrow\left(\frac{\partial L_{\delta\omega}{}^{\alpha}}{\partial X^\gamma}\rho_*^{\gamma\beta}-\frac{1}{2}L_{\delta\omega}{}^{\eta}\widetilde{L}^{\beta\alpha}{}_{\eta}\right)S_\beta\wedge S_\alpha+\frac{1}{2}\widetilde{L}^{\alpha\beta}{}_{\omega}\mathscr{L}_{S_\delta}(S_\alpha\wedge S_\beta)\\
&-\rho(S_\delta)\left(-\frac{1}{2}\widetilde{L}^{\alpha\beta}{}_{\omega}\right)S_\alpha\wedge S_\beta-\frac{1}{2}\widetilde{L}^{\alpha\beta}{}_{\delta}\mathscr{L}_{S_\omega}(S_\alpha\wedge S_\beta)+\rho(S_\omega)\left(-\frac{1}{2}\widetilde{L}^{\alpha\beta}{}_{\delta}\right)S_\alpha\wedge S_\beta=0\\
&\Longrightarrow\left(\frac{\partial L_{\delta\omega}{}^{\alpha}}{\partial X^\gamma}\rho_*^{\gamma\beta}-\frac{1}{2}L_{\delta\omega}{}^{\eta}\widetilde{L}^{\beta\alpha}{}_{\eta}\right)S_\beta\wedge S_\alpha+\frac{1}{2}\widetilde{L}^{\alpha\beta}{}_{\omega}[S_\delta,S_\alpha\wedge S_\beta]_{SN}+\frac{1}{2}\rho_\delta^i\frac{\partial\widetilde{L}^{\alpha\beta}}{\partial X^i}{}_{\omega}S_\alpha\wedge S_\beta\\
&-\frac{1}{2}\widetilde{L}^{\alpha\beta}{}_{\delta}[S_\omega,S_\alpha\wedge S_\beta]_{SN}-\frac{1}{2}\rho_\omega^i\frac{\partial\widetilde{L}^{\alpha\beta}}{\partial X^i}{}_{\delta}S_\alpha\wedge S_\beta=0\\
&\Longrightarrow\left(\frac{\partial L_{\delta\omega}{}^{\alpha}}{\partial X^\gamma}\rho_*^{\gamma\beta}-\frac{1}{2}L_{\delta\omega}{}^{\eta}\widetilde{L}^{\beta\alpha}{}_{\eta}\right)S_\beta\wedge S_\alpha+\frac{1}{2}\widetilde{L}^{\alpha\beta}{}_{\omega}L_{\delta\alpha}{}^{\theta}S_\theta\wedge S_\beta-\frac{1}{2}\widetilde{L}^{\alpha\beta}{}_{\omega}L_{\delta\beta}{}^{\theta}S_\theta\wedge S_\alpha\\
&+\frac{1}{2}\rho_\delta^i\frac{\partial\widetilde{L}^{\alpha\beta}}{\partial X^i}{}_{\omega}S_\alpha\wedge S_\beta-\frac{1}{2}\widetilde{L}^{\alpha\beta}{}_{\delta}L_{\omega\alpha}{}^{\theta}S_\theta\wedge S_\beta+\frac{1}{2}\widetilde{L}^{\alpha\beta}{}_{\delta}L_{\omega\beta}{}^{\theta}S_\theta\wedge S_\alpha-\frac{1}{2}\rho_\omega^i\frac{\partial\widetilde{L}^{\alpha\beta}}{\partial X^i}{}_{\delta}S_\alpha\wedge S_\beta=0\\
&\Longrightarrow\frac{\partial L_{\delta\omega}{}^{\alpha}}{\partial X^\gamma}\rho_*^{\gamma\beta}S_\beta\wedge S_\alpha-\frac{1}{2}L_{\delta\omega}{}^{\eta}\widetilde{L}^{\beta\alpha}{}_{\eta}S_\beta\wedge S_\alpha-\widetilde{L}^{\alpha\beta}{}_{\omega}L_{\delta\beta}{}^{\theta}S_\theta\wedge S_\alpha+\frac{1}{2}\rho_\delta^i\frac{\partial\widetilde{L}^{\alpha\beta}}{\partial X^i}{}_{\omega}S_\alpha\wedge S_\beta\\
&-\widetilde{L}^{\alpha\beta}{}_{\delta}L_{\omega\alpha}{}^{\theta}S_\theta\wedge S_\beta-\frac{1}{2}\rho_\omega^i\frac{\partial\widetilde{L}^{\alpha\beta}}{\partial X^i}{}_{\delta}S_\alpha\wedge S_\beta=0\\
&\Longrightarrow(\frac{\partial L_{\delta\omega}{}^{\beta}}{\partial X^\gamma}\rho_*^{\gamma\alpha}-\frac{1}{2}L_{\delta\omega}{}^{\eta}\widetilde{L}^{\alpha\beta}{}_{\eta}-\widetilde{L}^{\beta\gamma}{}_{\omega}L_{\delta\gamma}{}^{\alpha}+\frac{1}{2}\rho_\delta^i\frac{\partial\widetilde{L}^{\alpha\beta}}{\partial X^i}{}_{\omega}\\
&-\widetilde{L}^{\gamma\beta}{}_{\delta}L_{\omega\gamma}{}^{\alpha}-\frac{1}{2}\rho_\omega^i\frac{\partial\widetilde{L}^{\alpha\beta}}{\partial X^i}{}_{\delta})S_\alpha\wedge S_\beta\\
&=0.
\end{aligned}
$$

From the anti-symmetrization, we obtain

$$
\begin{aligned}
&\frac{\partial L_{\delta\omega}{}^{\beta}}{\partial X^\gamma}\rho_*^{\gamma\alpha}-\frac{\partial L_{\delta\omega}{}^{\alpha}}{\partial X^\gamma}\rho_*^{\gamma\beta}-\frac{1}{2}L_{\delta\omega}{}^{\eta}\widetilde{L}^{\alpha\beta}{}_{\eta}+\frac{1}{2}L_{\delta\omega}{}^{\eta}\widetilde{L}^{\beta\alpha}{}_{\eta}-\widetilde{L}^{\beta\gamma}{}_{\omega}L_{\delta\gamma}{}^{\alpha}+\widetilde{L}^{\alpha\gamma}{}_{\omega}L_{\delta\gamma}{}^{\beta}\\
&+\frac{1}{2}\rho_\delta^i\frac{\partial\widetilde{L}^{\alpha\beta}}{\partial X^i}{}_{\omega}-\frac{1}{2}\rho_\delta^i\frac{\partial\widetilde{L}^{\beta\alpha}}{\partial X^i}{}_{\omega}-\widetilde{L}^{\gamma\beta}{}_{\delta}L_{\omega\gamma}{}^{\alpha}+\widetilde{L}^{\gamma\alpha}{}_{\delta}L_{\omega\gamma}{}^{\beta}-\frac{1}{2}\rho_\omega^i\frac{\partial\widetilde{L}^{\alpha\beta}}{\partial X^i}{}_{\delta}+\frac{1}{2}\rho_\omega^i\frac{\partial\widetilde{L}^{\beta\alpha}}{\partial X^i}{}_{\delta}=0\\
&\Longrightarrow\frac{\partial L_{\delta\omega}{}^{\beta}}{\partial X^\gamma}\rho_*^{\gamma\alpha}-\frac{\partial L_{\delta\omega}{}^{\alpha}}{\partial X^\gamma}\rho_*^{\gamma\beta}+L_{\delta\omega}{}^{\eta}\widetilde{L}^{\beta\alpha}{}_{\eta}-\widetilde{L}^{\beta\gamma}{}_{\omega}L_{\delta\gamma}{}^{\alpha}+\widetilde{L}^{\alpha\gamma}{}_{\omega}L_{\delta\gamma}{}^{\beta}-\rho_\delta^i\frac{\partial\widetilde{L}^{\beta\alpha}}{\partial X^i}{}_{\omega}\\
&-\widetilde{L}^{\gamma\beta}{}_{\delta}L_{\omega\gamma}{}^{\alpha}+\widetilde{L}^{\gamma\alpha}{}_{\delta}L_{\omega\gamma}{}^{\beta}+\rho_\omega^i\frac{\partial\widetilde{L}^{\beta\alpha}}{\partial X^i}{}_{\delta}=0.
\end{aligned}
$$

Whence, the compatibility condition in local coordinates yields

$$
\begin{aligned}
&\rho_*^{ib}\frac{\partial L_{cd}{}^{e}}{\partial X^i}-\rho_*^{ie}\frac{\partial L_{cd}{}^{b}}{\partial X^i}+\rho_d^i\frac{\partial\widetilde{L}_c{}^{eb}}{\partial X^i}-\rho_c^i\frac{\partial\widetilde{L}_d{}^{eb}}{\partial X^i}\\
&\qquad+L_{cd}{}^{\eta}\widetilde{L}_{\eta}{}^{eb}-L_{ca}{}^{b}\widetilde{L}_d{}^{ea}+L_{ca}{}^{e}\widetilde{L}_d{}^{ba}-L_{da}{}^{b}\widetilde{L}_c{}^{ae}+L_{da}{}^{e}\widetilde{L}_c{}^{ab}=0.\blacksquare
\end{aligned}
$$

Proof A.14. *Our goal is to obtain the following facts, respectively.*

(1) *All these conditions will arise from the Jacobiator identity which is given as C2)* $Jac(A,B,C) = D(N_{ij}(A,B,C))$ *in* (3.34) *with the help of the Leibniz Rule C3)* $[A, fB] = f[A,B] + (\rho(A)f)B - \langle A,B\rangle Df$.
(2) *The other properties such as C4)* $\rho \circ D = 0$, *i.e.,* $\langle Df, Dg\rangle = 0$ *and C1)* $\rho([A,B]) = [\rho(A), \rho(B)]$ *will coincide with the above first and second conditions, respectively.*
(3) *The left property which is C5)* $\rho(A)\langle B,C\rangle = \langle [A,B] + D\langle A,B\rangle, C\rangle + \langle B, [A,C] + D\langle A,C\rangle\rangle$ *will be satisfied automatically.*

For the proof of (1)***:***
The first term of the LHS of the condition

$$Jac(A,B,C) = D(N_{ij}(A,B,C)),$$

which is

$$[[A,B],C] + [[B,C],A] + [[C,A],B],$$

can be expanded in a local basis as follows.

$$
\begin{aligned}
&[[A_I e^I, B_J e^J], C_N e^N] \\
&= [A_I B_J [e^I, e^J] + A_I(\rho(e^I)B_J)e^J - B_J(\rho(e^J)A_I)e^I + \langle e^I, e^J\rangle (B_J D(A_I) - A_I D(B_J)), C_N e^N] \\
&= [\,(A_I B_J \eta^{IK}\eta^{JL} T_{KLM} + A_I \eta^{IK}\rho^i_K(\partial_i B_M) - B_J \eta^{JK}\rho^i_K(\partial_i A_M) + \frac{1}{2}\eta^{IJ}B_J\rho^i_M(\partial_i A_I) \\
&- \frac{1}{2}\eta^{IJ}A_I\rho^i_M(\partial_i B_J))e^M, C_N e^N] \\
&= A_I B_J \eta^{IK}\eta^{JL} T_{KLM} C_N \eta^{MP}\eta^{NS} T_{PST} e^T + A_I \eta^{IK}\rho^i_K(\partial_i B_M) C_N \eta^{MP}\eta^{NS} T_{PST} e^T \\
&- B_J \eta^{JK}\rho^i_K(\partial_i A_M) C_N \eta^{MP}\eta^{NS} T_{PST} e^T + \frac{1}{2}\eta^{IJ}B_J\rho^i_M(\partial_i A_I) C_N \eta^{MP}\eta^{NS} T_{PST} e^T \\
&- \frac{1}{2}A_I C_N \eta^{IJ}\eta^{MP}\eta^{NS} T_{PST}\rho^i_M(\partial_i B_J) e^T + A_I B_J \eta^{IK}\eta^{JL}\eta^{MP} T_{KLM}\rho^l_P(\partial_l C_N) e^N \\
&+ A_I \eta^{IK}\eta^{MP}\rho^i_K\rho^l_P(\partial_i B_M)(\partial_l C_N) e^N - B_J \eta^{JK}\eta^{MP}\rho^i_K\rho^l_P(\partial_i A_M)(\partial_l C_N) e^N \\
&+ \frac{1}{2}B_J \eta^{IJ}\eta^{MP}\rho^i_M\rho^l_P(\partial_i A_I)(\partial_l C_N) e^N - \frac{1}{2}A_I \eta^{IJ}\eta^{MP}\rho^i_M\rho^l_P(\partial_i B_J)(\partial_l C_N) e^N \\
&- B_J C_N \eta^{IK}\eta^{JL}\eta^{NP} T_{KLM}\rho^l_P(\partial_l A_I) e^M - A_I C_N \eta^{IK}\eta^{JL}\eta^{NP} T_{KLM}\rho^l_P(\partial_l B_J) e^M \\
&- A_I B_J C_N \eta^{IK}\eta^{JL}\eta^{NP}\rho^l_P(\partial_l T_{KLM}) e^M - C_N \eta^{IK}\eta^{NP}\rho^i_K\rho^l_P(\partial_l A_I)(\partial_i B_M) e^M \\
&- A_I C_N \eta^{IK}\eta^{NP}\rho^l_P(\partial_l \rho^i_K)(\partial_i B_M) e^M - A_I C_N \eta^{IK}\eta^{NP}\rho^i_K\rho^l_P(\partial_l \partial_i B_M) e^M \\
&+ C_N \eta^{JK}\eta^{NP}\rho^i_K\rho^l_P(\partial_l B_J)(\partial_i A_M) e^M + B_J C_N \eta^{JK}\eta^{NP}\rho^l_P(\partial_l \rho^i_K)(\partial_i A_M) e^M \\
&+ B_J C_N \eta^{JK}\eta^{NP}\rho^i_K\rho^l_P(\partial_l \partial_i A_M) e^M - \frac{1}{2}C_N \eta^{IJ}\eta^{NP}\rho^i_M\rho^l_P(\partial_l B_J)(\partial_i A_I) e^M \\
&- \frac{1}{2}B_J C_N \eta^{IJ}\eta^{NP}\rho^l_P(\partial_l \rho^i_M)(\partial_i A_I) e^M - \frac{1}{2}B_J C_N \eta^{IJ}\eta^{NP}\rho^i_M\rho^l_P(\partial_l \partial_i A_I) e^M \\
&+ \frac{1}{2}C_N \eta^{IJ}\eta^{NP}\rho^i_M\rho^l_P(\partial_l A_I)(\partial_i B_J) e^M + \frac{1}{2}A_I C_N \eta^{IJ}\eta^{NP}\rho^l_P(\partial_l \rho^i_M)(\partial_i B_J) e^M \\
&+ \frac{1}{2}A_I C_N \eta^{IJ}\eta^{NP}\rho^i_M\rho^l_P(\partial_l \partial_i B_J) e^M + \frac{1}{2}B_J C_N \eta^{IK}\eta^{JL}\eta^{MN} T_{KLM}\rho^l_P(\partial_l A_I) e^P \\
&+ \frac{1}{2}A_I C_N \eta^{IK}\eta^{JL}\eta^{MN} T_{KLM}\rho^l_P(\partial_l B_J) e^P + \frac{1}{2}A_I B_J C_N \eta^{IK}\eta^{JL}\eta^{MN}\rho^l_P(\partial_l T_{KLM}) e^P \\
&+ \frac{1}{2}C_N \eta^{IK}\eta^{MN}\rho^l_P\rho^i_K(\partial_l A_I)(\partial_i B_M) e^P + \frac{1}{2}A_I C_N \eta^{IK}\eta^{MN}\rho^l_P(\partial_l \rho^i_K)(\partial_i B_M) e^P \\
&+ \frac{1}{2}A_I C_N \eta^{IK}\eta^{MN}\rho^l_P\rho^i_K(\partial_l \partial_i B_M) e^P - \frac{1}{2}C_N \eta^{JK}\eta^{MN}\rho^l_P\rho^i_K(\partial_l B_J)(\partial_i A_M) e^P \\
&- \frac{1}{2}B_J C_N \eta^{JK}\eta^{MN}\rho^l_P(\partial_l \rho^i_K)(\partial_i A_M) e^P - \frac{1}{2}B_J C_N \eta^{JK}\eta^{MN}\rho^l_P\rho^i_K(\partial_l \partial_i A_M) e^P \\
&+ \frac{1}{4}C_N \eta^{IJ}\eta^{MN}\rho^l_P\rho^i_M(\partial_l B_J)(\partial_i A_I) e^P + \frac{1}{4}B_J C_N \eta^{IJ}\eta^{MN}\rho^l_P(\partial_l \rho^i_M)(\partial_i A_I) e^P \\
&+ \frac{1}{4}B_J C_N \eta^{IJ}\eta^{MN}\rho^l_P\rho^i_M(\partial_l \partial_i A_I) e^P - \frac{1}{4}C_N \eta^{IJ}\eta^{MN}\rho^l_P\rho^i_M(\partial_l A_I)(\partial_i B_J) e^P \\
&- \frac{1}{4}A_I C_N \eta^{IJ}\eta^{MN}\rho^l_P(\partial_l \rho^i_M)(\partial_i B_J) e^P - \frac{1}{4}A_I C_N \eta^{IJ}\eta^{MN}\rho^l_P\rho^i_M(\partial_l \partial_i B_J) e^P \\
&- \frac{1}{2}A_I B_J \eta^{IK}\eta^{JL}\eta^{MN} T_{KLM}\rho^l_P(\partial_l C_N) e^P - \frac{1}{2}A_I \eta^{IK}\eta^{MN}\rho^l_P\rho^i_K(\partial_i B_M)(\partial_l C_N) e^P \\
&+ \frac{1}{2}B_J \eta^{JK}\eta^{MN}\rho^i_K\rho^l_P(\partial_l C_N)(\partial_i A_M) e^P - \frac{1}{4}B_J \eta^{IJ}\eta^{MN}\rho^i_M\rho^l_P(\partial_i A_I)(\partial_l C_N) e^P \\
&+ \frac{1}{4}A_I \eta^{IJ}\eta^{MN}\rho^l_P\rho^i_M(\partial_i B_J)(\partial_l C_N) e^P, \qquad \text{(A.17)}
\end{aligned}
$$

where we have used (4.47) *in the first and the third equalities, and the formulas* (4.29)*,* (4.34)*,* (4.27) *and* (4.36) *were used in the second and the last equalities. Expansion of the second and the third (cyclic) terms at the LHS can be found similarly.*

On the other hand, the first term at the RHS of the condition: $Jac(A,B,C) = \frac{1}{3}D(\langle[A,B],C\rangle + \langle[B,C],A\rangle + \langle[C,A],B\rangle)$ *can be expanded in such a way that*

$$
\begin{aligned}
&\frac{1}{3}D\langle[A_Ie^I,B_Je^J],C_Ne^N\rangle \\
&= \frac{1}{3}D\Big\langle\Big[A_IB_J\eta^{IK}\eta^{JL}T_{KLM} + A_I\eta^{IK}\rho^i_K(\partial_iB_M) - B_J\eta^{JK}\rho^i_K(\partial_iA_M) + \frac{1}{2}\eta^{IJ}B_J\rho^i_M(\partial_iA_I) \\
&- \frac{1}{2}\eta^{IJ}A_I\rho^i_M(\partial_iB_J)\Big]e^M, C_Ne^N\Big\rangle \\
&= \frac{1}{3}D\Big[A_IB_JC_N\eta^{IK}\eta^{JL}T_{KLM} + A_IC_N\eta^{IK}\rho^i_K(\partial_iB_M) - B_JC_N\eta^{JK}\rho^i_K(\partial_iA_M) \\
&+ \frac{1}{2}\eta^{IJ}B_JC_N\rho^i_M(\partial_iA_I) - \frac{1}{2}\eta^{IJ}A_IC_N\rho^i_M(\partial_iB_J)\Big]\langle e^M,e^N\rangle \\
&= \frac{1}{6}\eta^{MN}\eta^{IK}\eta^{JL}D(A_IB_JC_NT_{KLM}) + \frac{1}{6}\eta^{MN}\eta^{IK}D[A_IC_N\rho^i_K(\partial_iB_M)] \\
&- \frac{1}{6}\eta^{MN}\eta^{JK}D[B_JC_N\rho^i_K(\partial_iA_M)] + \frac{1}{12}\eta^{IJ}\eta^{MN}D[B_JC_N\rho^i_M(\partial_iA_I)] \\
&- \frac{1}{12}\eta^{IJ}\eta^{MN}D[A_IC_N\rho^i_M(\partial_iB_J)] \\
&= \frac{1}{6}B_JC_NT_{KLM}\eta^{MN}\eta^{IK}\eta^{JL}\rho^l_P(\partial_lA_I)e^P + \frac{1}{6}A_IC_NT_{KLM}\eta^{MN}\eta^{IK}\eta^{JL}\rho^l_P(\partial_lB_J)e^P \\
&+ \frac{1}{6}A_IB_JT_{KLM}\eta^{MN}\eta^{IK}\eta^{JL}\rho^l_P(\partial_lC_N)e^P + \frac{1}{6}A_IB_JC_N\eta^{MN}\eta^{IK}\eta^{JL}\rho^l_P(\partial_lT_{KLM})e^P \\
&+ \frac{1}{6}C_N\rho^i_K(\partial_iB_M)\eta^{MN}\eta^{IK}\rho^l_P(\partial_lA_I)e^P + \frac{1}{6}A_I\rho^i_K(\partial_iB_M)\eta^{MN}\eta^{IK}\rho^l_P(\partial_lC_N)e^P \\
&+ \frac{1}{6}A_IC_N(\partial_iB_M)\eta^{MN}\eta^{IK}\rho^l_P(\partial_l\rho^i_K)e^P + \frac{1}{6}A_IC_N\rho^i_K\eta^{MN}\eta^{IK}\rho^l_P(\partial_l\partial_iB_M)e^P \\
&- \frac{1}{6}C_N\rho^i_K(\partial_iA_M)\eta^{MN}\eta^{JK}\rho^l_P(\partial_lB_J)e^P - \frac{1}{6}B_J\rho^i_K(\partial_iA_M)\eta^{MN}\eta^{JK}\rho^l_P(\partial_lC_N)e^P \\
&- \frac{1}{6}B_JC_N(\partial_iA_M)\eta^{MN}\eta^{JK}\rho^l_P(\partial_l\rho^i_K)e^P - \frac{1}{6}B_JC_N\rho^i_K\eta^{MN}\eta^{JK}\rho^l_P(\partial_l\partial_iA_M)e^P \\
&+ \frac{1}{12}C_N\rho^i_M(\partial_iA_I)\eta^{IJ}\eta^{MN}\rho^l_P(\partial_lB_J)e^P + \frac{1}{12}B_J\rho^i_M(\partial_iA_I)\eta^{IJ}\eta^{MN}\rho^l_P(\partial_lC_N)e^P \\
&+ \frac{1}{12}B_JC_N(\partial_iA_I)\eta^{IJ}\eta^{MN}\rho^l_P(\partial_l\rho^i_M)e^P + \frac{1}{12}B_JC_N\rho^i_M\eta^{IJ}\eta^{MN}\rho^l_P(\partial_l\partial_iA_I)e^P \\
&- \frac{1}{12}C_N\rho^i_M(\partial_iB_J)\eta^{IJ}\eta^{MN}\rho^l_P(\partial_lA_I)e^P - \frac{1}{12}A_I\rho^i_M(\partial_iB_J)\eta^{IJ}\eta^{MN}\rho^l_P(\partial_lC_N)e^P \\
&- \frac{1}{12}A_IC_N(\partial_iB_J)\eta^{IJ}\eta^{MN}\rho^l_P(\partial_l\rho^i_M)e^P - \frac{1}{12}A_IC_N\rho^i_M\eta^{IJ}\eta^{MN}\rho^l_P(\partial_l\partial_iB_J)e^P, \qquad \text{(A.18)}
\end{aligned}
$$

where the formulas (4.47)*,* (4.27) *and* (4.36) *were used in the first, in the third and in the fourth equalities, respectively. Expansion of the second and the third (cyclic) terms at the RHS can be found similarly.*

Making equal LHS and RHS and after some challenging simplifications (the facts that η is symmetric and the three-form is anti-symmetric in all indices, and component functions (A_I, B_J, C_N) are smooth, were used generally), the remaining terms can be found as

$$
\begin{aligned}
0 = {} & A_I B_J C_N \eta^{IK}\eta^{JL}\eta^{MP}\eta^{NS}T_{KLM}T_{PST}e^T + \frac{1}{2}B_J C_N \eta^{IJ}\eta^{MP}\eta^{NS}T_{PST}\rho^i_M(\partial_i A_I)e^T \\
& - \frac{1}{2}A_I C_N \eta^{IJ}\eta^{MP}\eta^{NS}T_{PST}\rho^i_M(\partial_i B_J)e^T + A_I B_J \eta^{IK}\eta^{JL}\eta^{MP}T_{KLM}\rho^l_P(\partial_l C_N)e^N \\
& + \frac{1}{2}B_J \eta^{IJ}\eta^{MP}\rho^i_M\rho^l_P(\partial_i A_I)(\partial_l C_N)e^N - \frac{1}{2}A_I \eta^{IJ}\eta^{MP}\rho^i_M\rho^l_P(\partial_i B_J)(\partial_l C_N)e^N \\
& - A_I B_J C_N \eta^{IK}\eta^{JL}\eta^{NP}\rho^l_P(\partial_l T_{KLM})e^M - A_I C_N \eta^{IK}\eta^{NP}\rho^l_P(\partial_l \rho^i_K)(\partial_i B_M)e^M \\
& + B_J C_N \eta^{JK}\eta^{NP}\rho^l_P(\partial_l \rho^i_K)(\partial_i A_M)e^M - \frac{1}{2}B_J C_N \eta^{IJ}\eta^{NP}\rho^l_P(\partial_l \rho^i_M)(\partial_i A_I)e^M \\
& + \frac{1}{2}A_I C_N \eta^{IJ}\eta^{NP}\rho^l_P(\partial_l \rho^i_M)(\partial_i B_J)e^M - \frac{1}{2}B_J C_N \eta^{JK}\eta^{MN}\rho^l_P(\partial_l \rho^i_K)(\partial_i A_M)e^P \\
& + \frac{1}{2}B_J C_N \eta^{IJ}\eta^{MN}\rho^l_P(\partial_l \rho^i_M)(\partial_i A_I)e^P + A_I B_J C_N \eta^{JL}\eta^{NM}T_{LMK}\eta^{KP}\eta^{IV}T_{PVR}e^R \\
& + \frac{1}{2}A_I \eta^{JN}C_N\rho^i_K(\partial_i B_J)\eta^{KP}\eta^{IV}T_{PVR}e^R - \frac{1}{2}A_I \eta^{JN}B_J\rho^i_K(\partial_i C_N)\eta^{KP}\eta^{IV}T_{PVR}e^R \\
& + B_J C_N \eta^{JL}\eta^{NM}T_{LMK}\eta^{KS}\rho^l_{SN}(\partial_l A_I)e^I + \frac{1}{2}\eta^{JN}C_N\rho^i_K(\partial_i B_J)\eta^{KS}\rho^l_{SN}(\partial_l A_I)e^I \\
& - \frac{1}{2}\eta^{JN}B_J\rho^i_K(\partial_i C_N)\eta^{KS}\rho^l_{SN}(\partial_l A_I)e^I - A_I \eta^{IS}\rho^l_{SN}(\partial_l T_{LMK})B_J C_N \eta^{JL}\eta^{NM}e^K \\
& - A_I \eta^{IS}\rho^l_{SN}(\partial_l \rho^i_M)B_J \eta^{JM}(\partial_i C_K)e^K + A_I \eta^{IS}\rho^l_{SN}(\partial_l \rho^i_M)C_N \eta^{NM}(\partial_i B_K)e^K \\
& - \frac{1}{2}A_I \eta^{IS}\rho^l_{SN}(\partial_l \rho^i_K)\eta^{JN}C_N(\partial_i B_J)e^K + \frac{1}{2}A_I \eta^{IS}\rho^l_{SN}(\partial_l \rho^i_K)\eta^{JN}B_J(\partial_i C_N)e^K \\
& + \eta^{KI}A_I\rho^l_{SN}(\partial_l T_{LMK})B_J C_N \eta^{JL}\eta^{NM}e^S + \frac{1}{2}\eta^{KI}A_I\rho^l_{SN}(\partial_l \rho^i_M)B_J\eta^{JM}(\partial_i C_K)e^S \\
& - \frac{1}{2}\eta^{KI}A_I\rho^l_{SN}(\partial_l \rho^i_M)C_N\eta^{NM}(\partial_i B_K)e^S + \frac{1}{2}\eta^{KI}A_I\rho^l_{SN}(\partial_l \rho^i_K)\eta^{JN}C_N(\partial_i B_J)e^S \\
& - \frac{1}{2}\eta^{KI}A_I\rho^l_{SN}(\partial_l \rho^i_K)\eta^{JN}B_J(\partial_i C_N)e^S + B_J C_N A_I \eta^{NL}\eta^{IM}T_{LMK}\eta^{KP}\eta^{JV}T_{PVR}e^R \\
& + \frac{1}{2}B_J\eta^{NI}A_I\rho^i_K(\partial_i C_N)\eta^{KP}\eta^{JV}T_{PVR}e^R - \frac{1}{2}B_J\eta^{NI}C_N\rho^i_K(\partial_i A_I)\eta^{KP}\eta^{JV}T_{PVR}e^R \\
& + C_N A_I \eta^{NL}\eta^{IM}T_{LMK}\eta^{KS}\rho^l_{SN}(\partial_l B_J)e^J + \frac{1}{2}\eta^{NI}A_I\rho^i_K(\partial_i C_N)\eta^{KS}\rho^l_{SN}(\partial_l B_J)e^J \\
& - \frac{1}{2}\eta^{NI}C_N\rho^i_K(\partial_i A_I)\eta^{KS}\rho^l_{SN}(\partial_l B_J)e^J - B_J\eta^{JS}\rho^l_{SN}(\partial_l T_{LMK})C_N A_I\eta^{NL}\eta^{IM}e^K \\
& - B_J\eta^{JS}\rho^l_{SN}(\partial_l \rho^i_M)C_N\eta^{NM}(\partial_i A_K)e^K + B_J\eta^{JS}\rho^l_{SN}(\partial_l \rho^i_M)A_I\eta^{IM}(\partial_i C_K)e^K \\
& - \frac{1}{2}B_J\eta^{JS}\rho^l_{SN}(\partial_l \rho^i_K)\eta^{NI}A_I(\partial_i C_N)e^K + \frac{1}{2}B_J\eta^{JS}\rho^l_{SN}(\partial_l \rho^i_K)\eta^{NI}C_N(\partial_i A_I)e^K. \quad \text{(A.19)}
\end{aligned}
$$

If the following conditions are satisfied, then the above equation indeed vanishes, namely when

$$\eta^{IJ}\rho^i_I\rho^j_J = 0$$

is satisfied, then

$$\textcircled{5} + \textcircled{6} + \textcircled{19} + \textcircled{20} + \textcircled{36} + \textcircled{37} = 0,$$

and if

$$\rho_I^i \partial_i \rho_J^j - \rho_J^i \partial_i \rho_I^j - \eta^{KL} \rho_K^j T_{LIJ} = 0$$

is satisfied, then

$$\begin{aligned}
(9)+(17)+(37)&=0\\
(8)+(22)+(33)&=0\\
(4)+(21)+(38)&=0\\
(3)+(11)+(27)&=0\\
(2)+(10)+(13)&=0\\
(16)+(24)+(29)&=0\\
(15)+(23)+(28)&=0\\
(12)+(32)+(40)&=0\\
(26)+(31)+(39)&=0,
\end{aligned}$$

and when

$$\begin{aligned}
4\rho^i_{[L}\partial_i T_{IJK]} + 3\eta^{MN} T_{M[IJ} T_{KL]N} = 0 \Leftrightarrow \rho_L^i \frac{\partial T_{IJK}}{\partial X^i} - \rho_I^i \frac{\partial T_{JKL}}{\partial X^i} + \rho_J^i \frac{\partial T_{KLI}}{\partial X^i} - \rho_K^i \frac{\partial T_{LIJ}}{\partial X^i}\\
+ \eta^{MN}(T_{MIJ}T_{KLN} + T_{MIK}T_{LJN} + T_{MIL}T_{JKN}) = 0
\end{aligned}$$

is satisfied, then

$$(1)+(7)+(14)+(20)+(25)+(30)+(36)=0.$$

From which it could be concluded that the five properties of being a Courant algebroid indeed reduces to the three independent conditions in local coordinates which are given in (4.37). □

For the proof of (2)***:***
Another property of Courant algebroid is $\langle Df, Dg\rangle = 0$*. One can write this property in local coordinates such that*

$$\begin{aligned}
\langle Df, Dg\rangle &= \langle \rho_I^i(\partial_i f)e^I, \rho_J^j(\partial_j g)e^J\rangle\\
&= \frac{1}{2}\eta^{IJ}\rho_I^i(\partial_i f)\rho_J^j(\partial_j g)\\
&= 0,
\end{aligned} \tag{A.20}$$

for all smooth functions f *and* g*.*

If the first condition in (4.37)*:*

$$\eta^{IJ}\rho_I^i\rho_J^j = 0, \tag{A.21}$$

is satisfied then the equation (A.20) *indeed vanishes.*

The violation of this property can be thought as a strong constraint because then it yields $\eta^{IJ}(\partial_I f)(\partial_J g) = 0$.

The first property of Courant algebroid is $\rho([A,B]) = [\rho(A),\rho(B)]$ *in* (3.34). *The LHS in local coordinates can be found as*

$$
\begin{aligned}
&\rho([A,B])f = \rho([A_I e^I, B_J e^J])f \\
&= \rho(A_I B_J [e^I,e^J] + A_I(\rho(e^I)B_J)e^J - B_J(\rho(e^J)A_I)e^I + \langle e^I,e^J\rangle(B_J D(A_I) - A_I D(B_J))f \\
&= \rho(A_I B_J \eta^{IK}\eta^{JS}T_{KSM} + A_I\eta^{IK}\rho^i_K(\partial_i B_M) - B_J\eta^{JK}\rho^i_K(\partial_i A_M) + \frac{1}{2}\eta^{IJ}B_J\rho^i_M(\partial_i A_I) \\
&- \frac{1}{2}\eta^{IJ}A_I\rho^i_M(\partial_i B_J))e^M f \\
&= (A_I B_J \eta^{IK}\eta^{JS}T_{KSM} + A_I\eta^{IK}\rho^i_K(\partial_i B_M) - B_J\eta^{JK}\rho^i_K(\partial_i A_M) + \frac{1}{2}\eta^{IJ}B_J\rho^i_M(\partial_i A_I) \\
&- \frac{1}{2}\eta^{IJ}A_I\rho^i_M(\partial_i B_J))\rho(e^M)f \\
&= (A_I B_J \eta^{IK}\eta^{JS}T_{KSM} + A_I\eta^{IK}\rho^i_K(\partial_i B_M) - B_J\eta^{JK}\rho^i_K(\partial_i A_M) + \frac{1}{2}\eta^{IJ}B_J\rho^i_M(\partial_i A_I) \\
&- \frac{1}{2}\eta^{IJ}A_I\rho^i_M(\partial_i B_J))\eta^{ML}\rho^n_L\partial_n f \\
&= (A_I B_J \eta^{IN}\eta^{JK}T_{NKM}\eta^{ML}\rho^n_L + A_I\eta^{IK}\rho^i_K(\partial_i B_M)\eta^{ML}\rho^n_L - B_J\eta^{JK}\rho^i_K(\partial_i A_M)\eta^{ML}\rho^n_L)\partial_n f,
\end{aligned}
\tag{A.22}
$$

where we have used $[fA,gB] = fg[A,B] + f(\rho(A)g)B - g(\rho(B)f)A + \langle A,B\rangle(gDf - fDg)$ *(see Lemma 4.4.1) in the second equality and the last two terms are cancelled from* (A.21) *in the fifth equality. Also, the RHS in local coordinates can be written as*

$$
\begin{aligned}
[\rho(A),\rho(B)]f &= [\rho(A_I e^I),\rho(B_J e^J)]f \\
&= [A_I\rho(e^I), B_J\rho(e^J)]f \\
&= [A_I\eta^{IJ}\rho^i_J\partial_i, B_J\eta^{JK}\rho^n_K\partial_n]f \\
&= \left\{A_I\eta^{IJ}\rho^i_J\partial_i(B_N\eta^{NK}\rho^n_K) - B_J\eta^{JK}\rho^i_K\partial_i(A_I\eta^{IN}\rho^n_N)\right\}\partial_n f \\
&= \left\{A_I\eta^{IJ}\rho^i_J(\partial_i B_N)\eta^{NK}\rho^n_K + A_I\eta^{IJ}\rho^i_J B_N\eta^{NK}(\partial_i\rho^n_K)\right. \\
&- \left. B_J\eta^{JK}\rho^i_K(\partial_i A_I)\eta^{IN}\rho^n_N - B_J\eta^{JK}\rho^i_K A_I\eta^{IN}(\partial_i\rho^n_N)\right\}\partial_n f \\
&= \left\{A_I\eta^{IK}\rho^i_K(\partial_i B_M)\eta^{ML}\rho^n_L + A_I\eta^{IN}\rho^i_N B_J\eta^{JK}(\partial_i\rho^n_K)\right. \\
&- \left. B_J\eta^{JK}\rho^i_K(\partial_i A_M)\eta^{ML}\rho^n_L - B_J\eta^{JK}\rho^i_K A_I\eta^{IN}(\partial_i\rho^n_N)\right\}\partial_n f \\
&= \left\{A_I\eta^{IK}\rho^i_K(\partial_i B_M)\eta^{ML}\rho^n_L + A_I B_J\eta^{IN}\eta^{JK}(\rho^i_N\partial_i\rho^n_K - \rho^i_K\partial_i\rho^n_N)\right. \\
&- \left. B_J\eta^{JK}\rho^i_K(\partial_i A_M)\eta^{ML}\rho^n_L\right\}\partial_n f,
\end{aligned}
\tag{A.23}
$$

where we have used the fact $[X^i\partial_i, Y^n\partial_n]f = (X^i\partial_i Y^n - Y^i\partial_i X^n)\partial_n f$ *in the fourth equality because* $[\rho(A),\rho(B)]$ *is the usual Lie bracket.*
Since the left and right hand sides should be the same for all smooth functions f, *we*

obtain the following

$$\begin{aligned}
&A_I B_J \eta^{IN}\eta^{JK}T_{NKM}\eta^{ML}\rho^n_L + A_I\eta^{IK}\rho^i_K(\partial_i B_M)\eta^{ML}\rho^n_L - B_J\eta^{JK}\rho^i_K(\partial_i A_M)\eta^{ML}\rho^n_L \\
&= A_I\eta^{IK}\rho^i_K(\partial_i B_M)\eta^{ML}\rho^n_L + A_I B_J\eta^{IN}\eta^{JK}(\rho^i_N\partial_i\rho^n_K - \rho^i_K\partial_i\rho^n_N) \\
&- B_J\eta^{JK}\rho^i_K(\partial_i A_M)\eta^{ML}\rho^n_L,
\end{aligned}$$

which means

$$T_{NKM}\eta^{ML}\rho^n_L = \rho^i_N\partial_i\rho^n_K - \rho^i_K\partial_i\rho^n_N, \tag{A.24}$$

or

$$\rho^i_I\partial_i\rho^j_J - \rho^i_J\partial_i\rho^j_I + \eta^{KL}\rho^j_K T_{JIL} = 0. \tag{A.25}$$

The above result can be also written as

$$\rho^i_I\partial_i\rho^j_J - \rho^i_J\partial_i\rho^j_I - \eta^{KL}\rho^j_K T_{LIJ} = 0, \tag{A.26}$$

by the antisymmetry property of the three form: $T_{JIL} = -T_{LIJ}$ which can be seen from (4.29)*. Hence, this property coincides with the second property in* (4.37)*.* □

For the proof of (3)***:***
The last property $\rho(A)\langle B,C\rangle = \langle [A,B] + D\langle A,B\rangle, C\rangle + \langle B, [A,C] + D\langle A,C\rangle\rangle$ in (3.34) *is satisfied automatically. In order to prove that we will expand the LHS and the RHS one by one. The LHS yields the following in local coordinates;*

$$\begin{aligned}
\rho(A)\langle B,C\rangle &= \rho(A_I e^I)\langle B_J e^J, C_K e^K\rangle \\
&= \frac{1}{2}\eta^{JK}A_I\eta^{IM}\rho^i_M\partial_i(B_J C_K) \\
&= \frac{1}{2}C_K\eta^{JK}A_I\eta^{IM}\rho^i_M(\partial_i B_J) + \frac{1}{2}B_J\eta^{JK}A_I\eta^{IM}\rho^i_M(\partial_i C_K),
\end{aligned}$$

where we have used (4.27) *in the second equality.*

The RHS gives

$$\begin{aligned}
&\langle [A,B]+D\langle A,B\rangle,C\rangle+\langle B,[A,C]+D\langle A,C\rangle\rangle=\langle [A_Ie^I,B_Je^J]+D\langle A_Ie^I,B_Je^J\rangle,C_Ke^K\rangle\\
&+\langle B_Je^J,[A_Ie^I,C_Ke^K]+D\langle A_Ie^I,C_Ke^K\rangle\rangle\\
&=\langle A_IB_J\eta^{IN}\eta^{JL}T_{NLM}e^M+A_I\eta^{IM}\rho^i_M(\partial_iB_J)e^J-B_J\eta^{JM}\rho^i_M(\partial_iA_I)e^I+\frac{1}{2}\eta^{IJ}(B_J\rho^i_M(\partial_iA_I)e^M\\
&-A_I\rho^i_M(\partial_iB_J)e^M)+\frac{1}{2}\eta^{IJ}D(A_IB_J),C_Ke^K\rangle+\langle B_Je^J,A_IC_K\eta^{IL}\eta^{KM}T_{LMN}e^N\\
&+A_I\eta^{IN}\rho^i_N(\partial_iC_K)e^K-C_K\eta^{KM}\rho^i_M(\partial_iA_I)e^I+\frac{1}{2}\eta^{IK}(C_K\rho^i_M(\partial_iA_I)e^M-A_I\rho^i_M(\partial_iC_K)e^M)\\
&+\frac{1}{2}\eta^{IK}D(A_IC_K)\rangle\\
&=\frac{1}{2}A_IB_J\eta^{IN}\eta^{JL}T_{NLM}C_K\eta^{MK}+\frac{1}{2}A_I\eta^{IM}\rho^i_M(\partial_iB_J)C_K\eta^{JK}-\frac{1}{2}B_J\eta^{JM}\rho^i_M(\partial_iA_I)C_K\eta^{IK}\\
&+\frac{1}{4}\eta^{IJ}B_J\rho^i_M(\partial_iA_I)C_K\eta^{MK}-\frac{1}{4}\eta^{IJ}A_I\rho^i_M(\partial_iB_J)C_K\eta^{MK}+\frac{1}{4}\eta^{IJ}\rho^i_M(\partial_iA_I)B_JC_K\eta^{MK}\\
&+\frac{1}{4}\eta^{IJ}\rho^i_M(\partial_iB_J)A_IC_K\eta^{MK}+\frac{1}{2}B_JA_IC_K\eta^{IL}\eta^{KM}T_{LMN}\eta^{JN}+\frac{1}{2}B_JA_I\eta^{IN}\rho^i_N(\partial_iC_K)\eta^{JK}\\
&-\frac{1}{2}B_JC_K\eta^{KM}\rho^i_M(\partial_iA_I)\eta^{JI}+\frac{1}{4}B_J\eta^{IK}C_K\rho^i_M(\partial_iA_I)\eta^{JM}-\frac{1}{4}B_J\eta^{IK}A_I\rho^i_M(\partial_iC_K)\eta^{JM}\\
&+\frac{1}{4}B_J\eta^{IK}\rho^i_M(\partial_iA_I)C_K\eta^{JM}+\frac{1}{4}B_J\eta^{IK}\rho^i_M(\partial_iC_K)A_I\eta^{JM}\\
&=\frac{1}{2}A_I\eta^{IM}\rho^i_M(\partial_iB_J)C_K\eta^{JK}+\frac{1}{2}B_JA_I\eta^{IN}\rho^i_N(\partial_iC_K)\eta^{JK},
\end{aligned}$$

where (4.47) *and* (4.27) *were used in the second equality, and* (4.27) *and* (4.36) *were used in the third equality. Therefore, it can be seen that the LHS and RHS equals and this completes the proof.*□

Consequently, the five requirements of a Courant algebroid indeed reduce to the three independent conditions in local coordinates. ■

Proof A.15. *Our goal is to show this fact from the conditions one by one. We will use the same notions in local coordinates as in the usual case.*

♦ *Firstly, the LHS of the property* (2.)*, which is* $\rho\left(A\circ B\right)=\left[\rho\left(A\right),\rho\left(B\right)\right]$*, in local coordinates becomes*

$$
\begin{aligned}
&\rho\left(A\circ B\right)f=\rho\left(\left[A,B\right]+D\langle A,B\rangle\right)f\\
&=\rho([A,B])f+\rho(D\langle A,B\rangle)f\\
&=(A_IB_J\eta^{IN}\eta^{JK}T_{NKM}\eta^{ML}\rho^n_L+A_I\eta^{IK}\rho^i_K(\partial_iB_M)\eta^{ML}\rho^n_L-B_J\eta^{JK}\rho^i_K(\partial_iA_M)\eta^{ML}\rho^n_L)\partial_nf\\
&+\rho\left(D\left(\frac{1}{2}A_IB_J\eta^{IJ}\right)\right)f\\
&=(A_IB_J\eta^{IN}\eta^{JK}T_{NKM}\eta^{ML}\rho^n_L+A_I\eta^{IK}\rho^i_K(\partial_iB_M)\eta^{ML}\rho^n_L-B_J\eta^{JK}\rho^i_K(\partial_iA_M)\eta^{ML}\rho^n_L)\partial_nf\\
&+\frac{1}{2}\eta^{IJ}\rho(D(A_I)B_J)f+\frac{1}{2}\eta^{IJ}\rho(D(B_J)A_I)f\\
&=(A_IB_J\eta^{IN}\eta^{JK}T_{NKM}\eta^{ML}\rho^n_L+A_I\eta^{IK}\rho^i_K(\partial_iB_M)\eta^{ML}\rho^n_L-B_J\eta^{JK}\rho^i_K(\partial_iA_M)\eta^{ML}\rho^n_L)\partial_nf\\
&+\frac{1}{2}\eta^{IJ}\rho(\rho^i_K(\partial_iA_I)B_Je^K)f+\frac{1}{2}\eta^{IJ}\rho(\rho^i_K(\partial_iB_J)A_Ie^K)f\\
&=(A_IB_J\eta^{IN}\eta^{JK}T_{NKM}\eta^{ML}\rho^n_L+A_I\eta^{IK}\rho^i_K(\partial_iB_M)\eta^{ML}\rho^n_L-B_J\eta^{JK}\rho^i_K(\partial_iA_M)\eta^{ML}\rho^n_L)\partial_nf\\
&+\frac{1}{2}\eta^{IJ}\rho^i_K(\partial_iA_I)B_J\rho(e^K)f+\frac{1}{2}\eta^{IJ}\rho^i_K(\partial_iB_J)A_I\rho(e^K)f\\
&=(A_IB_J\eta^{IN}\eta^{JK}T_{NKM}\eta^{ML}\rho^n_L+A_I\eta^{IK}\rho^i_K(\partial_iB_M)\eta^{ML}\rho^n_L-B_J\eta^{JK}\rho^i_K(\partial_iA_M)\eta^{ML}\rho^n_L)\partial_nf\\
&+\frac{1}{2}\eta^{IJ}\eta^{KL}\rho^i_K\rho^j_LB_J(\partial_iA_I)(\partial_jf)+\frac{1}{2}\eta^{IJ}\eta^{KL}\rho^i_K\rho^j_LA_I(\partial_iB_J)(\partial_jf)\\
&=(A_IB_J\eta^{IN}\eta^{JK}T_{NKM}\eta^{ML}\rho^n_L+A_I\eta^{IK}\rho^i_K(\partial_iB_M)\eta^{ML}\rho^n_L-B_J\eta^{JK}\rho^i_K(\partial_iA_M)\eta^{ML}\rho^n_L)\partial_nf\\
&+\frac{1}{2}\eta^{IJ}\eta^{KL}\rho^i_K\rho^j_L(\partial_jf)(B_J(\partial_iA_I)+A_I(\partial_iB_J)),
\end{aligned}
$$

where we have used the definition of the Dorfman bracket (3.37) *in the first equality and the result of the first term in the third equality in local coordinates has been written directly from* (A.22).

The RHS of the property (2.) *can be written directly from* (A.23)*;*

$$
\begin{aligned}
\left[\rho\left(A\right),\rho\left(B\right)\right]f=\big\{&A_I\eta^{IK}\rho^i_K(\partial_iB_M)\eta^{ML}\rho^n_L+A_IB_J\eta^{IN}\eta^{JK}(\rho^i_N\partial_i\rho^n_K-\rho^i_K\partial_i\rho^n_N)\\
&-B_J\eta^{JK}\rho^i_K(\partial_iA_M)\eta^{ML}\rho^n_L\big\}\,\partial_nf.
\end{aligned}
$$

Since the LHS and the RHS should be the same, one can obtain

$$
\begin{aligned}
&(A_IB_J\eta^{IN}\eta^{JK}T_{NKM}\eta^{ML}\rho^n_L+A_I\eta^{IK}\rho^i_K(\partial_iB_M)\eta^{ML}\rho^n_L-B_J\eta^{JK}\rho^i_K(\partial_iA_M)\eta^{ML}\rho^n_L)\partial_nf\\
&\quad+\frac{1}{2}\eta^{IJ}\eta^{KL}\rho^i_K\rho^j_L(\partial_jf)(B_J(\partial_iA_I)+A_I(\partial_iB_J))\\
&\quad=(A_I\eta^{IK}\rho^i_K(\partial_iB_M)\eta^{ML}\rho^n_L+A_IB_J\eta^{IN}\eta^{JK}(\rho^i_N\partial_i\rho^n_K-\rho^i_K\partial_i\rho^n_N)\\
&\quad-B_J\eta^{JK}\rho^i_K(\partial_iA_M)\eta^{ML}\rho^n_L)\partial_nf.
\end{aligned}
$$

If the first two conditions of (4.37) *are satisfied then the above result is indeed satisfied such that the term* $\frac{1}{2}\eta^{IJ}\eta^{KL}\rho^i_K\rho^j_L(\partial_jf)(B_J(\partial_iA_I)+A_I(\partial_iB_J))$ *vanishes due to the second axiom in* (4.37) *and the other terms will be equal due to the first axiom in* (4.37).

♦ *Secondly, it will be shown that* (3.) *property, which is* $A\circ(fB)=f(A\circ B)+(\rho(A)f)B$*, is identically satisfied when the local expressions are used. The LHS and the RHS will be expanded at the same time as there will be direct simplifications.*

$$
\begin{aligned}
&A\circ(fB)=f(A\circ B)+(\rho(A)f)B\\
&\Rightarrow [A,fB]+D\langle A,fB\rangle = f[A,B]+fD\langle A,B\rangle+(\rho(A)f)B\\
&\Rightarrow f[A,B]+(\rho(A)f)B-\langle A,B\rangle Df+D\langle A,fB\rangle = f[A,B]+fD\langle A,B\rangle+(\rho(A)f)B\\
&\Rightarrow -\langle A,B\rangle Df+D\langle A,fB\rangle = fD\langle A,B\rangle.
\end{aligned}
$$

When the local expressions are used, the above equality yields

$$
\begin{aligned}
&\Rightarrow -\frac{1}{2}\eta^{IJ}A_IB_J\rho^i_K(\partial_i f)e^K+\frac{1}{2}\eta^{IJ}D(A_IfB_J)=\frac{1}{2}\eta^{IJ}fD(A_IB_J)\\
&\Rightarrow -\frac{1}{2}\eta^{IJ}A_IB_J\rho^i_K(\partial_i f)e^K+\frac{1}{2}\eta^{IJ}fB_JD(A_I)+\frac{1}{2}\eta^{IJ}A_IB_JD(f)+\frac{1}{2}\eta^{IJ}A_IfD(B_J)\\
&=\frac{1}{2}\eta^{IJ}fB_JD(A_I)+\frac{1}{2}\eta^{IJ}fA_ID(B_J),
\end{aligned}
$$

which is identically equal since $D(f)=\rho^i_K(\partial_i f)e^K$*.*

♦ *From the property* (4.)*, one could obtain the rule about the Dorfman bracket in terms of the three-form as in the Lie bracket. It will be considered generally such that*

$$
\begin{aligned}
&A\circ B=[A,B]+D\langle A,B\rangle\\
&\Rightarrow (A_Ie^I)\circ(B_Je^J)=\{A_IB_J\eta^{IK}\eta^{JL}T_{KLM}+A_I\eta^{IK}\rho^i_K(\partial_iB_M)-B_J\eta^{JK}\rho^i_K(\partial_iA_M)\\
&+\frac{1}{2}\eta^{IJ}B_J\rho^i_M(\partial_iA_I)-\frac{1}{2}\eta^{IJ}A_I\rho^i_M(\partial_iB_J)\}e^M+\frac{1}{2}\eta^{IJ}D(A_IB_J)\\
&\Rightarrow (A_Ie^I)\circ(B_Je^J)=(A_IB_J\eta^{IK}\eta^{JL}T_{KLM}+A_I\eta^{IK}\rho^i_K(\partial_iB_M)-B_J\eta^{JK}\rho^i_K(\partial_iA_M)\\
&+\frac{1}{2}\eta^{IJ}B_J\rho^i_M(\partial_iA_I)-\frac{1}{2}\eta^{IJ}A_I\rho^i_M(\partial_iB_J))e^M+\frac{1}{2}\eta^{IJ}B_J\rho^i_M(\partial_iA_I)e^M+\frac{1}{2}\eta^{IJ}A_I\rho^i_M(\partial_iB_J)e^M\\
&\Rightarrow (A_Ie^I)\circ(B_Je^J)=(A_IB_J\eta^{IK}\eta^{JL}T_{KLM}+A_I\eta^{IK}\rho^i_K(\partial_iB_M)\\
&-B_J\eta^{JK}\rho^i_K(\partial_iA_M)+\eta^{IJ}B_J\rho^i_M(\partial_iA_I))e^M.
\end{aligned}
$$

In above, we have substituted directly the equality $[A,B]$ *in local coordinates which was found in the beginning of the result* (A.17)*.*

♦ *Finally, it will be shown that the last property* (5.)*, which is* $\rho(A)\langle B,C\rangle=\langle A\circ B,C\rangle+\langle B,A\circ C\rangle$*, will be automatically satisfied when the local expressions are used. The LHS of this property yields in local coordinates the following:*

$$
\begin{aligned}
\rho(A)\langle B,C\rangle&=\frac{1}{2}\rho(A_Ie^I)(B_JC_K\eta^{JK})\\
&=\frac{1}{2}\eta^{JK}A_I\rho(e^I)(B_JC_K)\\
&=\frac{1}{2}\eta^{JK}A_I\eta^{IL}\rho^i_L\partial_i(B_JC_K)\\
&=\frac{1}{2}\eta^{JK}A_I\eta^{IL}\rho^i_LC_K(\partial_iB_J)+\frac{1}{2}\eta^{JK}A_I\eta^{IL}\rho^i_LB_J(\partial_iC_K),
\end{aligned}
$$

and the RHS of this property can be found as

$$\begin{aligned}
&\langle A\circ B,C\rangle+\langle B,A\circ C\rangle=\langle[A,B]+D\langle A,B\rangle,C\rangle+\langle B,[A,C]+D\langle A,C\rangle\rangle\\
&=\left\langle\left(A_IB_J\eta^{IN}\eta^{JL}T_{NLM}+A_I\eta^{IN}\rho^i_N(\partial_iB_M)-B_J\eta^{JN}\rho^i_N(\partial_iA_M)+\eta^{IJ}B_J\rho^i_M(\partial_iA_I)\right)e^M,C_Ke^K\right\rangle\\
&+\left\langle B_Je^J,\left(A_IC_K\eta^{IL}\eta^{KM}T_{LMN}+A_I\eta^{IL}\rho^i_L(\partial_iC_N)-C_K\eta^{KL}\rho^i_L(\partial_iA_N)+\eta^{IK}C_K\rho^i_N(\partial_iA_I)\right)e^N\right\rangle\\
&=\frac{1}{2}\eta^{MK}C_KA_IB_J\eta^{IN}\eta^{JL}T_{NLM}+\frac{1}{2}\eta^{MK}C_KA_I\eta^{IN}\rho^i_N(\partial_iB_M)-\frac{1}{2}\eta^{MK}C_KB_J\eta^{JN}\rho^i_N(\partial_iA_M)\\
&+\frac{1}{2}\eta^{MK}C_K\eta^{IJ}B_J\rho^i_M(\partial_iA_I)+\frac{1}{2}\eta^{JN}B_JA_IC_K\eta^{IL}\eta^{KM}T_{LMN}+\frac{1}{2}\eta^{JN}B_JA_I\eta^{IL}\rho^i_L(\partial_iC_N)\\
&-\frac{1}{2}\eta^{JN}B_JC_K\eta^{KL}\rho^i_L(\partial_iA_N)+\frac{1}{2}\eta^{JN}B_J\eta^{IK}C_K\rho^i_N(\partial_iA_I)\\
&=\frac{1}{2}\eta^{JK}A_I\eta^{IL}\rho^i_LC_K(\partial_iB_J)+\frac{1}{2}\eta^{JK}A_I\eta^{IL}\rho^i_LB_J(\partial_iC_K),
\end{aligned}$$

where the local expression of $[A,B]+D\langle A,B\rangle$ *and* $[A,C]+D\langle A,C\rangle$ *in second equality have been substituted directly from the previous calculation's* $(4.)$ *result.*

The equality of the LHS and the RHS of this property ends the proof. (The proof of the property $(1.)$ *is left since the similar calculations will be done as in the above proofs.)* ■

Proof A.16. *Our goal is to prove the following Lemma.*

Lemma 1.0.1. C_{IJK} *are totally anti-symmetric in all of its indices.*

We need the following Lemma first with its proof.

Lemma: For all $A,X\in g$,

$$(ad(X_I))_{KJ}=C_{IJ}{}^K,$$

where $ad(A)X=[A,X]$.

Proof:

$$\begin{aligned}
&ad(X_I)X_J=\sum_K(ad(X_I))_{KJ}X_K\\
&\Longrightarrow[X_I,X_J]=\sum_K(ad(X_I))_{KJ}X_K\\
&\Longrightarrow\sum_KC_{IJ}{}^KX_K=\sum_K(ad(X_I))_{KJ}X_K\\
&\Longrightarrow C_{IJ}{}^K=(ad(X_I))_{KJ}.\square
\end{aligned}$$

By using the Lemma above and the Jacobi Identity that the structure constants of the Lie algebra obey, one obtains

$$\begin{aligned}
[ad(X_K), ad(X_I)]_{MJ} &= \sum_L ((ad(X_K))_{ML}(ad(X_I))_{LJ} - (ad(X_I))_{ML}(ad(X_K))_{LJ}) \\
&= \sum_L \left(C_{KL}{}^M C_{IJ}{}^L - C_{IL}{}^M C_{KJ}{}^L\right) \\
&= \sum_L \left(C_{KL}{}^M C_{IJ}{}^L + C_{IL}{}^M C_{JK}{}^L\right) \\
&= \sum_L \left(-C_{JL}{}^M C_{KI}{}^L\right),
\end{aligned}$$

thus

$$\begin{aligned}
&[ad(X_K), ad(X_L)]_{IJ} = \sum_M C_{MJ}{}^I C_{KL}{}^M \\
&\Longrightarrow [ad(X_K), ad(X_L)]_{IJ} = \sum_M C_{KL}{}^M ad(X_M)_{IJ}.
\end{aligned}$$

With the help of this adjoint representation, the invariant bilinear form (the non-degenerate Cartan-Killing form) can be defined on any Lie algebra and this form is to be

$$\begin{aligned}
B(X_L, X_K) = \kappa_{LK} &= tr(ad(X_L) \circ ad(X_K)) \\
&= tr \sum_M (ad(X_L))_{NM}(ad(X_K))_{MI} \\
&= tr \sum_M C_{LM}{}^N C_{KI}{}^M \\
&= \sum_{M,N} C_{LM}{}^N C_{KN}{}^M.
\end{aligned}$$

Therefore, we have $\kappa_{LK} = C_{LM}{}^N C_{KN}{}^M$. *By using this equality,* C_{IJK} *can be written as*

$$\begin{aligned}
C_{IJK} &= C_{IJ}{}^L \kappa_{LK} \\
&= C_{IJ}{}^L C_{LM}{}^N C_{KN}{}^M \\
&= -C_{ML}{}^N C_{IJ}{}^L C_{KN}{}^M \\
&= (C_{IL}{}^N C_{JM}{}^L + C_{JL}{}^N C_{MI}{}^L) C_{KN}{}^M \\
&= C_{IL}{}^N C_{JM}{}^L C_{KN}{}^M + C_{LJ}{}^N C_{MI}{}^L C_{NK}{}^M,
\end{aligned}$$

where we have used the fact that the structure constants of the Lie algebra do satisfy the Jacobi Identity in the fourth line. By using the above result, it can be seen from the following calculation that C_{IJK} *is anti-symmetric;*

$$\begin{aligned}
C_{IKJ} &= C_{IL}{}^N C_{KM}{}^L C_{JN}{}^M + C_{LK}{}^N C_{MI}{}^L C_{NJ}{}^M \\
&= C_{IM}{}^L C_{KN}{}^M C_{JL}{}^N + C_{NK}{}^M C_{LI}{}^N C_{MJ}{}^L \\
&= -C_{MI}{}^L C_{NK}{}^M C_{LJ}{}^N - C_{KN}{}^M C_{IL}{}^N C_{JM}{}^L \\
&= -C_{IJK}.
\end{aligned}$$

Proofs for other indices can be done similarly.■

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