**ZONGULDAK BÜLENT ECEVİT UNIVERSITY**
**GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES**

# SOLUTIONS OF SOME RATIONAL DIFFERENCE EQUATIONS VIA SPECIAL SEQUENCES

**DEPARTMENT OF MATHEMATICS**

**MASTER OF SCIENCE THESIS**

**ZEYNEP AKMEŞE**

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**ZONGULDAK BÜLENT ECEVİT UNIVERSITY**
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**Zeynep AKMEŞE**

**ADVISOR: Assoc. Prof. Dr. Melih GÖCEN**

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*"With this thesis it is declared that all the information in this thesis is obtained and presented according to academic rules and ethical principles. Also as required by the academic rules and ethical principles all works that are not result of this study are cited properly."*

Zeynep AKMEŞE

## ABSTRACT

**Master of Science Thesis**

**SOLUTIONS OF SOME RATIONAL DIFFERENCE EQUATIONS VIA SPECIAL SEQUENCES**



**Zeynep AKMEŞE**

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**Graduate School of Natural and Applied Sciences**
**Department of Mathematics**

**Thesis Advisor: Assoc. Prof. Dr. Melih GÖCEN**



**January 2024, 55 pages**



In this thesis, we investigate some rational difference equations solutions associated with special integer sequences. Moreover, we study the stability character and asymptotic behavior of the equilibrium points of these equations.

The chapters of the thesis are as follows;

In Chapter 1, we give the history of special integer sequences. A detailed literature review was conducted on the subject.

In Chapter 2, basic definitions and theorems used throughout the thesis are presented, special integer sequences were included.

Chapter 3 includes some results about following difference equations. The stability character and global asymptotic behavior of equilibrium points are investigated.

**ABSTRACT (continued)**

$$x_{n+1} = \frac{\pm 1}{x_n(x_{n-1} \pm 1)}$$

The solutions of equations are obtained via Narayana and Padovan numbers.

In Chapter 4, we investigate the following rational difference equations

$$x_{n+1} = \frac{1}{x_n(x_{n-1} - 2) - 1}$$

$$x_{n+1} = \frac{-1}{x_n(x_{n-1} + 2) - 1}$$

The exact solutions of these equations are given with Pell numbers.

**Keywords:** Difference equation, equilibrium point, asymptotic behaviour, integer sequences, Narayana Numbers, Padovan Numbers, Pell Numbers.

**Science Code:** 403.01.01

# ÖZET

**Yüksek Lisans Tezi**

**BAZI RASYONEL FARK DENKLEMLERİNİN ÖZEL SAYI DİZİLER İLE ÇÖZÜMLERİ**


**Zeynep AKMEŞE**

**Zonguldak Bülent Ecevit Üniversitesi**
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Bu tezde, özel tamsayı dizileriyle ilişkili olarak bazı rasyonel fark denklemlerinin çözümlerini inceliyoruz. Ayrıca, bu denklemlerin denge noktalarını, denge noktalarının kararlılığını ve asimptotik davranışını çalışıyoruz.

Tezin bölümleri aşağıdaki gibidir;

Bölüm 1'de, özel tamsayı dizilerinin tarihçesi araştırılıyor ve ayrıntılı bir literatür taraması yapılıyor.

Bölüm 2'de, tez boyunca kullanılan temel tanım ve teoremlere yer veriliyor, özel tamsayı dizileri ele alınıyor.

Bölüm 3 aşağıdaki fark denklemleriyle ilgili bazı sonuçları içeriyor. Denklemlerin denge noktalarının kararlılık karakteri ve global asimptotik davranışı araştırılıyor.

## ÖZET (devam ediyor)


$$x_{n+1} = \frac{\pm 1}{x_n(x_{n-1} \pm 1)}$$

Denklemlerin çözümleri Narayana ve Padovan sayıları ile elde ediliyor.

Bölüm 4'de aşağıdaki rasyonel fark denklemlerini araştırıyoruz,

$$x_{n+1} = \frac{1}{x_n(x_{n-1} - 2) - 1}$$

$$x_{n+1} = \frac{-1}{x_n(x_{n-1} + 2) - 1}$$

Bu denklemlerin tam çözümleri Pell sayılarıyla verilmektedir.

**Anahtar Kelimeler:** Fark denklemi, denge noktası, asimptotik davranış, tamsayı dizileri, Narayana Sayıları, Padovan Sayıları, Pell Sayıları.

**Bilim Kodu:** 403.01.01

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## LIST OF SYMBOLS AND ABBREVIATIONS

### SYMBOLS

| | |
|---|---|
| $N_n$ | : $n$-th Narayana number |
| $P_n$ | : $n$-th Padovan number |
| $T_n$ | : $n$-th Tribonacci number |
| $F_n$ | : $n$-th Fibonacci number |
| $J_n$ | : $n$-th Jacobsthal number |
| $L_n$ | : $n$-th Lucas number |
| $W_n$ | : $n$-th Horadam number |

# CHAPTER 1

# INTRODUCTION AND LITERATURE REVIEWS

Difference equations have contributed to solving many scientific problems in recent years. Difference equations are consist of a relation between the consecutive terms. For the solution of a difference equation, initial value conditions are needed. The initial conditions are needed to determine the exact solution. The difference equation is also a recurrence relation. Many daily life problems can be solved with the recurrence relation where the population is variable. Although many studies have been done on this subject in recent years, there are open problems that need to be worked on still.

If the change does not occur continuously, but gradually, differential equations are insufficient to solve this situation. In this case, it is more convenient to use difference equations.

The most well-known examples in this field are salmon population, compound monthly interest, etc. Difference equations are often used in some fields; probability calculations, population growth problems, statistical problems, half-life calculation, economic models, amount of radiation, field of computers, rate of spread of infectious diseases (S.I.R modeling) etc.

Since the 1950s, difference equations have become a popular field of study. Especially in the last 30 years, a large number of studies have been conducted on difference equations. Special integer sequences in mathematics are well suited for working with difference equations. Therefore, we will investigate the form of the solutions of some rational difference equations via special integer sequences in this thesis.

## 1.1 AN OVERVIEW OF THE HISTORY OF SPECIAL NUMBER SEQUENCES AND DIFFERENCE EQUATIONS

In this section, we discuss the historical development of difference equations. We will include the studies of many mathematicians and scientists on the subject.

The mathematician Fibonacci found the Fibonacci numbers to which difference equations often refer. Fibonacci's Liber Abaci(First published in 1202) was important book on mathematics in the Middle Ages, introducing Arabic numerals throughout Europe. In his book, Fibonacci introduced the modus Indorum (Indians' method), known as the Indo-Arabic numeral system. Fibonacci numbers, which can be considered as one of the first examples of difference equations, are revealed in this book.

The Fibonacci sequence is a set of integers. Each number in the series is equal to the sum of the two numbers before it. Fibonacci numbers, sometimes known as the Golden Ratio, are a popular topic in mathematics. This ratio has many examples in the natural world (sunflower, pineapple, cone, etc.). Fibonacci numbers emerged when calculating the total number of rabbits population for one year. A newborn pair of rabbits (one female, one male) is placed in a specific area, each pair of rabbits becomes adults from the second month, after which a new pair of rabbits (one female, one male) gives birth each month. Assuming that rabbits never die, how many pairs of rabbits will there be in this region at the end of a year?

Fibonacci started the number sequence as $1, 2, 3, ...$ and moved the calculation up to the 13th row, the number 233. Fibonacci skipped the "0" and started with the "1" which is included today. Terms of the Fibonacci sequence of numbers; $0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, \ldots$

This sequence of numbers was named after the French mathematician Lucas in May 1876. The generalized Fibonacci function is expressed as below.

$$F_n = \begin{cases} 0, & if \quad n = 0 \\ 1, & if \quad n = 1 \\ F_{n-1} + F_{n-2}, & if \quad n > 1 \end{cases}$$

The Fibonacci numbers have a closed-form expression, just like every sequence denoted by a linear recurrence with constant coefficients. Although Abraham de Moivre and Daniel Bernoulli knew about it first, it has come to be known as Binet's formula, after French mathematician Jacques Philippe Marie Binet.

Between 1600 and 1700, several scientists, including Bernoulli, Moivre, Pascal, and Newton, worked on difference equations. In fact, different equations emerged as recurrence relations.

Mathematician Moivre was the first to present some general concepts for recurrence relations. Moivre uncovered the information needed to solve homogeneous, linear, constant-coefficient discrete equations in closed form. The difference equations of the famous Riccati, known by the name of the mathematician Riccati, are as follows

$$y_{n+1} = \frac{a + by_n}{c + dy_n} \qquad n = 0, 1, 2, .... \tag{1.1}$$

The Eq.(1.1) is known as Riccati difference equation. Leonhard Euler, in his book Institutiones Calculi Differentialis, used mathematical symbols similar to today. The difference operator ($\Delta$), in its current form, is used in this book. Euler also systematized Moivre's methods in his work.

British Mathematician Babbage designed a kind of calculator that could be counted automatically for that period. He called it the Difference Engine so several people refer to Babbage as the "father of the computer". If the first few values of a polynomial are known, the remaining values can be calculated using consecutive differences.

In addition, Laplace also worked on various methods to solve some classes of nonlinear difference equations in closed form. Felice Casorati who studied about difference equation and revealed the Casorati matrix. The Casorati matrix is a useful method for calculating linear difference equations. Lucas worked on Fibonacci number sequences, he created the formula for the Fibonacci numbers used today. He also devised the test for Mersenne primes still used today.

While Lucas was working on Fibonacci numbers, he created the Lucas number sequence with a similar approach. The "Lucas number sequence" is a set of numbers containing $1, 3, 4, 7, 11, 18, 29, 47, ....$

The Fibonacci approach is similarly valid for the Lucas number sequence. In the 1900s, mathematicians discovered plane filling curves or fractals and began to use them with difference equations. Mathematician Peano's famous space-filling curve appeared in 1890, this was the first known instance of a fractal. Hilbert, in 1891, described similar space-filling curves. This was an early illustration of a fractal, as it is currently known.

The Weierstrass function was one of the first fractals to be studied, but the "term fractal" came into use much later. Weierstrass functions are continuous everywhere but not differentiable. Like some fractals, it is self-similar. In 1872, Weierstrass finalized the definition of continuity of functions and showed that continuity is not differentiability.

Niels Fabian Helge von Koch, a Swedish mathematician, discovered one of the oldest fractals, the Koch snowflake fractal, named after him. In the early 1900s, Carl David

T. Runge and Martin Wilhelm Kutta created this technique. The Runge-Kutta method is a widely used method for solving the initial-value problems of differential equations. Thus, numerical solution methods for differential equations started to be used by using difference equations. The Runge-Kutta method attempts to overcome the problem of Euler's method.

Since the 1950s, studies on difference equations have been systematically increasing and have come to the present day. In this section, we discussed the development of difference equations and special number sequences, which we will discuss throughout the thesis.

## 1.2 LITERATURE REVIEW

This section examines recent studies on the difference equations expressed in terms of well-known integer sequences, including the Narayana, Padovan, Pell, Padovan, Fibonacci, etc. numbers.

By conducting a detailed literature review, we focused on theses and articles where the solution elements of difference equations are given by special integer sequences. For details, see [4] and [28].

In [41], Tollu et al. considered the Riccati difference equation

$$x_{n+1} = \frac{a + bx_n}{c + dx_n}, \quad n = 0, 1, ... \tag{1.2}$$

In Riccati difference Eq.(1.2) by taking $a = c = d = 1$, $b = 0$ and $a = d = 1$, $c = -1$, $b = 0$ the equation (1.2) respectively, is transformed into the following equations

$$x_{n+1}{=}\frac{1}{1 + x_n}, \quad y_{n+1}{=}\frac{1}{-1 + y_n}, \quad n = 0, 1, ..., \tag{1.3}$$

They examined the solutions of two special types of the Riccati difference equation (1.3). Fibonacci numbers have a relation with the solutions where initial conditions are

$$x_0 \in \mathbb{R} - \left\{-\frac{F_{m+1}}{F_m}\right\}_{m=1}^{\infty} \text{ and } \ y_0 \in \mathbb{R} - \left\{\frac{F_{m+1}}{F_m}\right\}_{m=1}^{\infty}$$

respectively, and $F_m$ is the $mth$ Fibonacci number. For $n = 0, 1, 2, ...$, the solutions of equations (1.3) are as follows

$$x_n = \frac{F_n + F_{n-1}x_0}{F_{n+1} + F_n x_0} \quad for \ \ x_0 \in \mathbb{R} - \left(\left\{\frac{1}{\alpha}, \frac{1}{\beta}\right\} \cup \left\{-\frac{F_{m+1}}{F_m}\right\}_{m=1}^{\infty}\right),$$

$$y_n = \frac{F_{-n} + F_{-(n-1)}y_0}{F_{-(n+1)} + F_{-n}y_0} \text{ for } y_0 \in \mathbb{R} - \left(\{\alpha, \beta\} \cup \left\{\frac{F_{m+1}}{F_m}\right\}_{m=1}^{\infty}\right).$$

In [45], Yazlık et al. considered the following rational difference equations

$$x_{n+1} = \frac{x_{n-1\pm1}}{y_n x_{n-1}},\ y_{n+1} = \frac{y_{n-1\pm1}}{x_n y_{n-1}}, \qquad n = 0, 1, ... \tag{1.4}$$

In this study, they derived the solutions of the rational difference equation associated with Padovan numbers. The forms of solutions of system Eq.(1.4) are as follows

$$x_n = \begin{cases} \pm\dfrac{P_n x_{-1} y_0 \pm P_{n+1} x_{-1} + P_{n-1}}{P_{n-1} x_{-1} y_0 \pm P_n x_{-1} + P_{n-2}}, & \text{if } n \text{ is odd} \\ \pm\dfrac{P_n y_{-1} x_0 \pm P_{n+1} y_{-1} + P_{n-1}}{P_{n-1} y_{-1} x_0 \pm P_n y_{-1} + P_{n-2}}, & \text{if } n \text{ is even} \end{cases}$$

$$y_n = \begin{cases} \pm\dfrac{P_n y_{-1} x_0 \pm P_{n+1} y_{-1} + P_{n-1}}{P_{n-1} y_{-1} x_0 \pm P_n y_{-1} + P_{n-2}}, & \text{if } n \text{ is odd} \\ \pm\dfrac{P_n x_{-1} x_0 \pm P_{n+1} x_{-1} + P_{n-1}}{P_{n-1} x_{-1} y_0 \pm P_n x_{-1} + P_{n-2}}, & \text{if } n \text{ is even} \end{cases}$$

where $P_n$ be the $nth$ Padovan number.

In [17], Halim studied the explicit solution of the following difference equation system. He gave the general solutions of the system with generalized Fibonacci numbers

$$x_{n+1} = \frac{1}{1 + y_{n-2}}, \qquad y_{n+1} = \frac{1}{1 + x_{n-2}}, \qquad n = 0, 1, ... \tag{1.5}$$

where the initial values are arbitrary real numbers with the restriction that $x_{-2}$, $y_{-2}$, $x_{-1}$, $y_{-1}$, $x_0$, $y_0 \notin \left\{-\frac{F_{m+1}}{F_m};\ n = 1, 2, ...\right\}$. The solutions of Eq.(1.5) are as follows

$$x_{6n+i} = \frac{F_{2n+1} + F_{2n} y_{i-3}}{F_{2n+2} + F_{2n+1} y_{i-3}}, \quad i = 1, 2, 3,$$

$$y_{6n+i} = \frac{F_{2n+1} + F_{2n} x_{i-3}}{F_{2n+2} + F_{2n+1} x_{i-3}}, \quad i = 1, 2, 3,$$

$$x_{6n+i} = \frac{F_{2n+2} + F_{2n+1} x_{i-6}}{F_{2n+3} + F_{2n+2} x_{i-6}}, \quad i = 4, 5, 6,$$

$$y_{6n+i} = \frac{F_{2n+2} + F_{2n+1} y_{i-6}}{F_{2n+3} + F_{2n+2} y_{i-6}}, \quad i = 4, 5, 6,$$

where $F_n$ is the $nth$ Fibonacci number.

In [18], Halim and Rabago investigated the periodicity and stability of the following system of equations

$$x_{n+1} = \frac{1}{\pm1 \pm y_{n-k}}, \qquad y_{n+1} = \frac{1}{\pm1 \pm x_{n-k}}, \qquad n,\ k \in \mathbb{N}_0, \tag{1.6}$$

where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and the initial conditions $x_{-k}$, $x_{-k+1}$, ..., $x_0$, $y_{-k}$, $y_{-k+1}$, ..., $y_0$ are non zero real numbers. The solution of the system of equations (1.6) is given by associated with the Fibonacci numbers. They determined the equations system's solutions as follows,

$$\begin{aligned}
x_{2(k+1)_{n+i}} &= \frac{F_{2n+1} + F_{2n} y_{i-(k+1)}}{F_{2n+2} + F_{2n+1} y_{i-(k+1)}}, \quad i = 1, 2, ..., k+1, \\
y_{2(k+1)_{n+i}} &= \frac{F_{2n+1} + F_{2n} x_{i-(k+1)}}{F_{2n+2} + F_{2n+1} x_{i-(k+1)}}, \quad i = 1, 2, ..., k+1, \\
x_{2(k+1)_{n+i}} &= \frac{F_{2n+2} + F_{2n+1} x_{i-(2k+2)}}{F_{2n+3} + F_{2n+2} x_{i-(2k+2)}}, \quad i = k+2, ..., 2k+2, \\
y_{2(k+1)_{n+i}} &= \frac{F_{2n+2} + F_{2n+1} y_{i-(2k+2)}}{F_{2n+3} + F_{2n+2} y_{i-(2k+2)}}, \quad i = k+2, ..., 2k+2,
\end{aligned}$$

In this way, they gave the explicit form of (1.6) equations of solutions in terms of Fibonacci numbers.

In [19], the authors investigated the following rational difference equation

$$x_{n+1} = \frac{\alpha x_{n-1} + \beta}{\gamma x_n x_{n-1}}, \quad n \in \mathbb{N}_0, \tag{1.7}$$

where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\alpha$, $\beta$, $\gamma \in \mathbb{R}^+$, and the initial conditions $x_{-1}$ and $x_0$ are non zero real numbers such that generalized Padovan numbers are related to their solutions. They investigated the two-dimensional form of the equation, which was given by

$$x_{n+1} = \frac{\alpha x_{n-1} + \beta}{\gamma y_n x_{n-1}}, \quad y_{n+1} = \frac{\alpha y_{n-1} + \beta}{\gamma x_n y_{n-1}}, \quad n \in \mathbb{N}_0, \tag{1.8}$$

By defining two new coefficients for equation (1.8) $p = \frac{\alpha}{\gamma}$ and $q = \frac{\beta}{\gamma}$ they obtained the following equation

$$x_{n+1} = \frac{p x_{n-1} + q}{x_n x_{n-1}}, \quad n \in \mathbb{N}_0 \tag{1.9}$$

They assume that $S_n$, the $nth$ generalized Padovan numbers, to satisfy the recurrence equation; $S_{n+1} = pS_{n-1} + qS_{n-2}$, $n \in \mathbb{N}_0$ with initial conditions $S_{-2} = 0$, $S_{-1} = 0$, $S_0 = 1$. According to this work, the closed form solution is provided by

$$x_n = \frac{S_{n+1} x_{-1} + S_n x_0 x_{-1} + q S_{n-1}}{S_n x_{-1} + S_{n-1} x_0 x_{-1} + q S_{n-2}}, \tag{1.10}$$

where the initial conditions $x_{-1}$, $x_0 \in \mathbb{R} - F$, with $F$ is the forbidden set of equation (1.10) given by

$$F = \bigcup_{n=-1}^{\infty} \{(x_{-1}, x_0) : S_n x_{-1} + S_{n-1} x_0 x_{-1} + q S_{n-2} = 0\}.$$

If $\alpha = \beta = \gamma$, then form Eq.(1.10) they get the solution

$$x_n = \frac{P_{n+1}x_{-1} + P_n x_0 x_{-1} + qP_{n-1}}{P_n x_{-1} + P_{n-1}x_0 x_{-1} + qP_{n-2}},$$

Thus, $S_n = P_n$, $n \in \mathbb{N}$, and consequently they got the solution associated via Padovan numbers.

In [23], Khelifa et al. studied the following difference equation

$$x_{n+1} = \frac{5y_{n-k} - 5}{y_{n-k}}, \qquad y_{n+1} = \frac{5x_{n-k} - 5}{x_{n-k}}, \qquad n,\ k\ \in\ \mathbb{N}_0 \tag{1.11}$$

where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and the initial conditions $x_{-k}$, $x_{-k+1}$, $...x_0$, $y_{-k}$, $y_{-k+1}, ...y_0$ are real numbers that are not zero, and their solutions are related to Lucas numbers.

They determineted that the exact solutions of equations (1.11) are as follows,

$$x_{(k+1)2n-j} = \frac{5F_{2n} - x_{-j}L_{2n+1}}{L_{2n-1} - x_{-j}F_{2n}},$$

$$x_{(k+1)(2n+1)-j} = \frac{5L_{2n+1} - 5y_{-j}F_{2n+2}}{5F_{2n} - y_{-j}L_{2n+1}},$$

$$y_{(k+1)2n-j} = \frac{5F_{2n} - y_{-j}L_{2n+1}}{L_{2n-1} - y_{-j}F_{2n}},$$

$$x_{(k+1)(2n+1)-j} = \frac{5L_{2n+1} - 5x_{-j}F_{2n+2}}{5F_{2n} - x_{-j}L_{2n+1}},$$

where $j \in \{0, 1, ..., k\}$, $\{L_n\}_n$ the Lucas sequence, $\{F_n\}_n$ the Fibonacci sequence and the initial conditions $x_{-k}$, $x_{-k+1}$, $...x_0$, $y_{-k}$, $y_{-k+1}, ...y_1$ and $y_0 \in \mathbb{R} - G_j$, with $G_j$ is the forbidden set of system Eq.(1.11) given by

$$G_j = \bigcup_{n=-1}^{\infty} \left\{ \begin{array}{c} (x_{-k}, x_{-k+1}, ...x_0, y_{-k}, y_{-k+1}, ...y_0) \\ L_{2n-1} - x_{-j}F_{2n} = 0,\ 5F_{2n} - y_{-j}L_{2n+1} = 0, \quad j = 0, 1, ..., k \end{array} \right\}.$$

Then, in [32] Okumuş investigated the form of solutions, stability character and asymptotic behavior of the following equations

$$x_{n+1} = \frac{1}{x_n(x_{n-1} - 1) - 1},\ n = 0, 1, ..., \tag{1.12}$$

$$x_{n+1} = \frac{1}{x_n(x_{n-1} + 1) + 1},\ n = 0, 1, ..., \tag{1.13}$$

$$x_{n+1} = \frac{-1}{x_n(x_{n-1} - 1) + 1},\ n = 0, 1, ..., \tag{1.14}$$

$$x_{n+1} = \frac{-1}{x_n(x_{n-1} + 1) - 1},\ n = 0, 1, ..., \tag{1.15}$$

The generalized Tribonacci solution of Eq.(1.12) is given as follows. The form of solutions $\{x_n\}_{n=-1}^{\infty}$ is given by

$$x_n=\frac{T_{-n}x_{-1}x_0+(T_{-(n+1)}+T_{-(n+2)})x_0+T_{-(n+1)}}{T_{-(n+1)}x_{-1}x_0+(T_{-n}-T_{-(n+1)})x_0+T_{-(n+2)}}$$

where $T_n$ is the $nth$ Tribonacci number and initial conditions are $\mathrm{x}_{-1}$, $x_0\in\mathbb{R}-F_1$ with $F_1$ is the forbidden set of given by

$$F_1=\bigcup_{n=-1}^{\infty}\left\{(x_{-1},x_0):T_{-(n+1)}x_{-1}x_0+(T_{-n}-T_{-(n+1)})x_0+T_{-(n+2)}=0\right\}.$$

In this study, the solutions of equations (1.13)-(1.14) and (1.15) which are associated with generalized Tribonacci numbers are obtained.

In [10], Elsayed et al. studied the behavior of the difference equation

$$x_{n+1}=ax_n+\frac{bx_nx_{n-1}}{cx_{n-1}+dx_{n-2}},\qquad n=0,1,...,\tag{1.16}$$

where $a,b,c$, and $d$ are positive constants and the initial conditions are arbitrary positive real numbers, $x_{-2}$, $x_{-1}$ and $x_0$.

They gave some special cases of Eq.(1.16). We will consider one of them, as follows

$$x_{n+1}=x_n+\frac{x_nx_{n-1}}{x_{n-1}+x_{n-2}},\tag{1.17}$$

The Eq.(1.17) is a special circumstance of the Eq.(1.16). For $n=0,1,2,...$, the solution of equations (1.17) are given as

$$x_{2n}=h\prod_{i=0}^{n-1}(\frac{f_{2i+3}h+f_{2i+2}k}{f_{2i+2}h+f_{2i+1}k})(\frac{f_{2i+3}k+f_{2i+2}r}{f_{2i+2}k+f_{2i+1}r})$$

$$x_{2n+1}=h\prod_{i=0}^{n}(\frac{f_{2i+1}h+f_{2i}k}{f_{2i}h+f_{2i-1}k})(\frac{f_{2i+3}k+f_{2i+2}r}{f_{2i+2}k+f_{2i+1}r})$$

where $x_{-2}=r$, $x_{-1}=k$, $x_0=h$, $\{f_m\}_{m=-1}^{\infty}=\{0,\ 0,1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,...\}$.

In [22], Karaoğlu dealt with the next difference equations

$$x_{n+1}=\frac{2}{x_n(x_{n-1}-1)-1},\qquad n=0,1,...,\tag{1.18}$$

$$x_{n+1}=\frac{-2}{x_n(x_{n-1}+1)-1},\qquad n=0,1,...,\tag{1.19}$$

The solutions of Eq.(1.18) and Eq.(1.19) are associated with third order Jacobstal numbers. She presented that the closed form solutions of Eq.(1.18) is given by

$$x_n=\begin{cases}\dfrac{J_nx_{-1}x_0-(J_{(n-1)}+J_{(n+1)})x_0+J_n}{J_{(n-1)}x_{-1}x_0+J_{(n-1)}x_0-(J_n+J_{(n+1)})}, & n=0\pmod 3\\ \dfrac{J_{n-1}x_{-1}x_0+J_{(n-1)}x_0-(J_n+J_{(n+1)})}{-(J_n+J_{(n-1)})x_{-1}x_0+J_{(n+1)}x_0+J_{(n+1)}}, & n=1\pmod 3\\ \dfrac{-(J_n+J_{(n-1)})x_{-1}x_0+J_{(n+1)})x_0+J_{(n+1)}}{J_nx_{-1}x_0-(J_{(n-1)}+J_{(n+1)})x_0+J_n}, & n=2\pmod 3\end{cases}$$

where $J_n$ is the $nth$ Jacobsthal number and the initial conditions $x_{-1},\ x_0 \in \mathbb{R} - F_1$, with $F_1$ is the forbidden set of Eq.(1.18) given by

$$F_1 = \bigcup_{n=-1}^{\infty} \left\{(x_{-1}, x_0) : A_n^1 = 0 \ or \ B_n^1 = 0 \ or \ C_n^1 = 0\right\}.$$

and

$$A_n^1 = J_{(n-1)}x_{-1}x_0 + J_{(n-1)}x_0 - (J_n + J_{(n+1)}).$$

$$B_n^1 = -(J_n + J_{(n-1)})x_{-1}x_0 + J_{(n+1)}x_0 + J_{(n+1)}.$$

$$C_n^1 = J_n x_{-1}x_0 - (J_{(n-1)} + J_{(n+1)})x_0 + J_n.$$

Also for the Eq.(1.19), she examined that the explicit solutions of Eq.(1.19) is given by

$$x_n = \begin{cases} \dfrac{J_n x_{-1}x_0 + (J_{(n-1)} + J_{(n+1)})x_0 + J_n}{(-J_{(n-1)}x_{-1}x_0 + J_{(n-1)}x_0 + (J_n + J_{(n+1)}))}, & n = 0 \pmod 3 \\ \dfrac{J_{n-1}x_{-1}x_0 - J_{(n-1)}x_0 - (J_n + J_{(n+1)})}{(J_n + J_{(n-1)})x_{-1}x_0 + J_{(n+1)}x_0 - J_{(n+1)})}, & n = 1 \pmod 3 \\ \dfrac{-(J_n + J_{(n-1)})x_{-1}x_0 - J_{(n+1)})x_0 + J_{(n+1)}}{(-J_n x_{-1}x_0 - (J_{(n-1)} + J_{(n+1)})x_0 - J_n)}, & n = 2 \pmod 3 \end{cases}$$

where $J_n$ is the $nth$ Jacobsthal number and the initial conditions $x_{-1},\ x_0 \in \mathbb{R} - F_2$, with $F_2$ is the forbidden set of equation(1.19) given by

$$F_2 = \bigcup_{n=-1}^{\infty} \left\{(x_{-1}, x_0) : A_n^2 = 0 \ or \ B_n^2 = 0 \ or \ C_n^2 = 0\right\}.$$

where

$$A_n^2 = (-J_{(n-1)}x_{-1}x_0 + J_{(n-1)}x_0 + (J_n + J_{(n+1)}))$$

$$B_n^2 = (J_n + J_{(n-1)})x_{-1}x_0 + J_{(n+1)}x_0 - J_{(n+1)})$$

$$C_n^2 = (-J_n x_{-1}x_0 - (J_{(n-1)} + J_{(n+1)})x_0 - J_{(n)}).$$

In [20], Halim et al. considered the following the higher order rational difference equation system

$$x_{n+1} = \frac{1 + 2y_{n-k}}{3 + y_{n-k}}, \quad y_{n+1} = \frac{1 + 2z_{n-k}}{3 + z_{n-k}}, \quad z_{n+1} = \frac{1 + 2x_{n-k}}{3 + x_{n-k}} \tag{1.20}$$

where $n,\ k \in \mathbb{N}_0$, the initial values $x_{-k},\ x_{-k+1}, ..., x_0,\ y_{-k},\ y_{-k+1}, ..., y_0,\ z_{-k},\ z_{-k+1}, ...z_1$ and zero are arbitrary real numbers do not equal $-3$. Using the Fibonacci and Lucas numbers, they got at a closed form solution to the system.

Then, in [3], Hüseyin et al. investigated the form of the following rational difference equation system

$$x_n = \frac{z_{n-1}z_{n-3}}{x_{n-2+}2z_{n-3}}, \quad y_n = \frac{x_{n-1}x_{n-3}}{-y_{n-2+}6x_{n-3}}, \quad z_n = \frac{y_{n-1}y_{n-3}}{z_{n-2+}14y_{n-3}}, \quad n \in \mathbb{N}_0 \tag{1.21}$$

where initial values $x_{-3}$, $x_{-2}$, $x_{-1}$, $y_{-3}$, $y_{-2}$, $y_{-1}$, $z_{-3}$, $z_{-2}$, $z_{-1}$ are real numbers that are nonzero. The solutions are related to pell numbers.

In [14], Göcen presented the following difference equations

$$x_{n+1} = \frac{\alpha}{x_n x_{n-1} - 1}, \quad n = 0, 1, ..., \tag{1.22}$$

He obtained the exact solutions of the Eq.(1.22) for the case $\alpha = 2$ and $\alpha = -2$ via adjusted Jacobsthal-Padovan numbers.

In [44], Yalçınkaya et. al studied the positive solutions of the fuzzy difference equation

$$z_{n+1} = A + \frac{B}{z_{n-m_1}} + \frac{C}{z_{n-m_2}}, \quad n \in \mathbb{N}_0$$

where $(z_n)$ is a sequence of positive fuzzy numbers. $A$, $B$, $C$ and the initial values $z_j$, $j = 0, 1, 2, ....s$ are positive fuzzy numbers and $m_1$, $m_2$ are nonnegative integers with $s = max\{m_1, m_2\}$.

They investigated the behavior of the solution of the following system

$$x_{n+1} = a_1 + \frac{b_1}{y_{n-m_1}} + \frac{c_1}{y_{n-m_2}} \tag{1.23}$$

$$y_{n+1} = a_2 + \frac{b_2}{x_{n-m_1}} + \frac{c_2}{x_{n-m_2}}$$

where the parameters $a_1, b_1, c_1, a_2, b_2, c_2$, and the initial values $x_{-j}$, $y_{-j}$, ( $j = 0, 1, ..., s$) are positive real numbers and $m_1$, $m_2$ are nonnegative integers with

s =max$\{m_1, m_2\}$.

Let $\{(x_n,\ y_n)\}_{n=-s}^{\infty}$ is a positive solution of system Eq.(1.23) .Then they got from Eq.(1.23) that

$$x_n \geq a_1 > 0 \text{ and } y_n \geq a_2 > 0 \text{ for all } n \in \mathbb{N}_0 \tag{1.24}$$

Thus, the sequences are bounded from below and very far from zero. They got the following system from the system Eq.(1.23) and Eq.(1.24)

$$x_{n+1} = a_1 + \frac{b_1}{y_{n-m_1}} + \frac{c_1}{y_{n-m_2}} \leq a_1 + \frac{b_1}{a_2} + \frac{c_1}{a_2} < \infty$$

$$y_{n+1} = a_2 + \frac{b_2}{x_{n-m_1}} + \frac{c_2}{x_{n-m_2}} \leq a_2 + \frac{b_2}{a_1} + \frac{c_2}{a_1} < \infty$$

It is clear that above, the sequences are far from infinity and bounded from the top thus the proof is completed.

In [1], the authors represent the system of second-order rational difference equations in terms of Lucas and Fibonacci sequences

$$\begin{aligned} x_{n+1} &= \frac{L_{m+2} + L_{m+1}y_{n-1}}{L_{m+3} + L_{m+2}y_{n-1}} \qquad & y_{n+1} &= \frac{L_{m+2} + L_{m+1}z_{n-1}}{L_{m+3} + L_{m+2}z_{n-1}} \\ z_{n+1} &= \frac{L_{m+2} + L_{m+1}w_{n-1}}{L_{m+3} + L_{m+2}w_{n-1}} \qquad & w_{n+1} &= \frac{L_{m+2} + L_{m+1}x_{n-1}}{L_{m+3} + L_{m+2}x_{n-1}} \end{aligned}$$

where $n \in \mathbb{N}_0$, $\{L_m\}_{m\,=\,-\infty}^{+\infty}$ is Lucas sequence and the initial contitions $x_{-1}, x_0, y_{-1}, y_0, z_{-1}, z_0, w_{-1}, w_0$ are arbitrary real numbers such that $v_{-i} \neq \dfrac{L_{m+3}}{L_{m+2}}$, where $v_{-i} = x_{-i}, y_{-i}, z_{-i}, w_{-i}$, $i = 0, 1$ and $m \in \mathbb{Z}$.

# CHAPTER 2

## BASIC DEFINITIONS AND THEOREMS

Basic definitions and theorems about difference equations that are necessary for the thesis are included in this section. For details, see [4], [28].
Then, special integer sequences that form the basis of this thesis are discussed.

**Definition 2.1** *Let $f : I^{k+1} \to I$ be a continuously differentiable function and let $I$ be some interval of real numbers. A difference equation of order $(k+1)$ is an equation of the following form*

$$x_{n+1} = f(x_n, x_{n-1}, \ldots, x_{n-k}), \ n = 0, 1, \ldots, \tag{2.1}$$

*where $f$ is a function that maps some set $I^{k+1}$ into $I$. A solution of Equation (2.1) is a sequence $\{x_n\}_{n=-k}^{\infty}$ that satisfies Eq.(2.1) for all $n \geq -k$.*

**Definition 2.2** *A solution of Eq. (2.1) that is constant for all $n \geq -k$ is called an equilibrium solution of Eq.(2.1), if*

$x_n = \overline{x}$, *for all* $n \geq -k$

*is an equilibrium solution of Eq.(2.1), then $\overline{x}$ is called an equilibrium of Eq.(2.1).*

**Definition 2.3** *(Stability) Let $\overline{x}$ an equilibrium point of Equation (2.1).*

**a)** *An equilibrium point $\overline{x}$ of Eq.(2.1) is called locally stable if, for every $\epsilon > 0$; there exists $\delta > 0$ such that if $\{x_n\}_{n=-k}^{\infty}$ is a solution of Eq.(2.1) with*

$$|x_{-k} - \overline{x}| + |x_{1-k} - \overline{x}| + \ldots + |x_0 - \overline{x}| < \delta$$

*then*

$$|x_n - \overline{x}| < \epsilon, \quad \text{for all} \quad n \geq -k$$

**b)** *An equilibrium point $\overline{x}$ of Eq.(2.1) is called locally asymptotically stable if, $\overline{x}$ is locally stable, and if in addition there exists $\gamma > 0$ such that if $\{x_n\}_{n=-k}^{\infty}$ is a solution of Eq.(2.1) with*

$$|x_{-k} - \overline{x}| + |x_{1-k} - \overline{x}| + ... + |x_0 - \overline{x}| < \gamma,$$

*then we have*

$$\lim_{n\to\infty} x_n = \overline{x}.$$

**c)** *An equilibrium point $\overline{x}$ of Eq. (2.1) is called a global attractor if, for every solution $\{x_n\}_{n=-k}^{\infty}$ of Eq.(2.1), we have*

$$\lim_{n\to\infty} x_n = \overline{x}.$$

**d)** *An equilibrium point $\overline{x}$ of Eq. (2.1) is called globally asymptotically stable if it is locally stable, and $\overline{x}$ is also a global attractor of Eq.(2.1).*

**e)** *An equilibrium point $\overline{x}$ of Eq. (2.1) is called unstable if it is not locally stable.*

**Definition 2.4** *(Linearized Stability Analysis) Assume that there is an open neighborhood around an equilibrium point $\overline{x}$ where the function $F$ is continuously differentiable. Let*

$$q_i = \frac{\partial F}{\partial u_i}(\overline{x}, \overline{x}, ...\overline{x}), \qquad for \quad i = 0, 1, ...k$$

*denote the partial derivative of $F(u_0,\ u_1, ..., u_k)$ with respect to $u_i$ evaluated at the equilibrium point $\overline{x}$ of Equation (2.1). Then the equation*

$$y_{n+1} = q_0 y_n + q_1 y_{n-1} + .... + q_k y_{n-k}, \qquad n = 0, 1, ... \tag{2.2}$$

*is called the linearized equation of Eq.(2.1) about the equilibrium point $\overline{x}$, and the equation*

$$\lambda^{k+1} - q_0 \lambda^k - ..... - q_{k-1}\lambda - q_k = 0 \tag{2.3}$$

*is called the characteristic equation of Equation of (2.2) about $\overline{x}$.*

**Theorem 2.1** *(The Linearized Stability Theorem) Suppose that the function $f$ is a continuously differentiable function defined on some open neighborhood of an equilibrium point $\overline{x}$. Then the following statements are true:*

**a)** *The equilibrium point $\overline{x}$ of characteristic Eq. (2.1) is locally asymptotically stable when all of the roots of Eq. (2.3) have an absolute value less than one.*

**b)** *The equilibrium point $\overline{x}$ of Eq.(2.1) is unstable if at least one root of Eq.(2.3) has an absolute value larger than one.*

**c)** *The equilibrium point $\overline{x}$ of Eq.(2.1) is called hyperbolic if no root of characteristic Eq. (2.3) has absolute value equal to one. If there exists a root of characteristic Eq. (2.3) with absolute value equal to one, then the equilibrium $\overline{x}$ is called nonhyperbolic.*

**d)** *An equilibrium point $\overline{x}$ of Eq. (2.1) is called a repeller if all roots of Eq. (2.3) have absolute value greater than one.*

**e)** *An equilibrium point $\overline{x}$ of Eq. (2.1) is called a saddle if one of the roots of characteristic Eq.(2.3) is greater than one and another is less than one in absolute value.*

The three theorems that follow give sufficient and necessary condition for every root of a real polynomial of degree two, three, or four to have modulus less than one.
For each equation of order three that we examine in this thesis, we must use one of these three theorems to determine the locally asymptotically stable of the equilibrium points.

**Theorem 2.2** *Suppose that $a_1$ and $a_0$ are real numbers. Then there is a necessary and sufficient condition for all roots of the equation*

$\lambda^2 - a_1\lambda - a_0 = 0.$

*to lie inside the unit disk is*

$|a_1| < 1 + a_0 < 2.$

**Theorem 2.3** *Assume that $a_2$, $a_1$, and $a_0$ are real numbers. Then a necessary and sufficient condition for all roots of the equation*

$\lambda^3 + a_2\lambda^2 - a_1\lambda - a_0 = 0.$

*to lie inside the unit disk is*

$|a_2 + a_0| < 1 + a_1, \quad |a_2 - 3a_0| < 3 - a_1, \quad \textit{and} \quad a_0^2 + a_1 - a_0a_2 < 1.$

**Theorem 2.4** *Suppose that $a_3$, $a_2$, $a_1$, and $a_0$ are real numbers. Then a sufficient and necessary condition for each of the equations roots*

$$\lambda^4 + a_3\lambda^3 + a_2\lambda^2 - a_1\lambda - a_0 = 0.$$

*to lie inside the unit disk is*

$$|a_1 + a_3| < 1 + a_0 + a_2, \quad |a_1 - a_3| < 2(1 - a_0), \quad a_2 - 3a_0 < 3,$$

*and*

$$a_0 + a_2 + a_0^2 + a_1^2 + a_0^2 a_2 + a_0 a_3^2 < 1 + 2a_0 a_2 + a_1 a_3 + a_0 a_1 a_3 + a_0^3.$$

*This result is sufficient for all roots of an equation of any order to lie inside the unit disk.*

**Theorem 2.5** ***(Clark Theorem)*** *Assume that $q_0, q_1, ...q_k$ are real numbers such that*

$$|q_0| + |q_1| + ... + |q_k| < 1$$

*Then all roots of Eq.(2.3) lie inside the unit disk.*

## 2.1 INTEGER SEQUENCES

In recent years, studies on obtaining solutions to difference equations with special integer sequences (Fibonacci, Lucas, Pell, Jacobsthal, Padovan, Perrin, Narayana, etc.) have attracted the attention of many researchers. Applications of special integer sequences include mathematics, biology, economics, computer science, etc. In the literature section, we examined previous studies on difference equations whose solutions are provided by integer sequences.

The aim of this thesis is to find the solutions of difference equations with special integer sequences. Now we will explain special integer sequences in detail.

### 2.1.1 Fibonacci Numbers

The Fibonacci numbers are the sequence of numbers $\{F_n\}_{n=1}^{\infty}$defined by the linear recurrence equation

$$F_n = F_{n-1} + F_{n-2}, \; n \geq 2 \tag{2.4}$$

with initial conditions $F_0 = 0,\ F_1 = 1,\ F_2 = 1$. The sequence begins $0, 1, 1, 2, 3, 5, 8, ....$

**Table 2.1** A few values of the Fibonacci Numbers.

| $F_0$ | $F_1$ | $F_2$ | $F_3$ | $F_4$ | $F_5$ | $F_6$ | $F_7$ | $F_8$ | $F_9$ | $F_{10}$ | $F_{11}$ | $F_{12}$ | $F_{13}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 1 | 2 | 3 | 5 | 8 | 13 | 21 | 34 | 55 | 87 | 144 | 233 |

It is possible to extend the value of $F_n$ to negative integers in accordance with

$$F_{-n} = (-1)^{n+1} F_n \tag{2.5}$$

A homogeneous linear recurrence has the form

$$x_n = a_0 x_{n-1} + a_1 x_{n-2} + a_2 x_{n-3} + ..... + a_{k-1} x_{n-k}, \quad for \ \ n \geq k \tag{2.6}$$

where $a_0,\ a_1, ...a_k$ are constants. The values $x_0,\ x_{n-1}, ...$ are initial conditions. The Eq.(2.4) is a linear recurrence relation

$$x_n = a_0 x_{n-1} + a_1 x_{n-2},\ n \geq 2 \tag{2.7}$$

Consequently, the closed form for $F_n$ is provided by

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \tag{2.8}$$

where $\alpha$ and $\beta$ are roots of $x^2 = a_0 x + a_1$. Here, $a_0 = a_1 = 1$, so the characteristic equation is $x^2 - x - 1 = 0$. The roots are

$$\alpha = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad \beta = \frac{1-\sqrt{5}}{2}$$

The division of two consecutive Fibonacci numbers gives the golden ratio. This ratio is especially useful in the fields of engineering, architecture, and physics. Also, the following limit is available

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \alpha$$

where $F_n$ is nth Fibonacci number. The closed form is therefore given by

$$F_n = \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n \sqrt{5}} \tag{2.9}$$

The Eq.(2.9) is kwown as Binet's formula. A few examples of fibonacci sequences identities are listed below

$$\begin{aligned} F_{2n} &= F_{n+1}^2 - F_{n-1}^2 \\ &= F_n(F_{n+1} + F_{n-1}) \\ &= F_n(2F_{n+1} - F_n) \\ F_{3n} &= F_{n+1}^3 + F_n^3 - F_{n-1}^3 \\ F_{n+1}^2 &= 4F_nF_{n-1} + F_{n-2}^2 \end{aligned} \tag{2.10}$$

Catalan's identity

$$F_n^2 - F_{n+r}F_{n-r} = (-1)^{n-r}F_r^2 \tag{2.11}$$

d'Ocagne's identity

$$F_mF_{n+1} - F_nF_{m+1} = (-1)^nF_{m-n} \tag{2.12}$$

and the Gelin-Cesàro identity

$$F_n^4 - F_{n-2}F_{n-1}F_{n+1}F_{n+2} = 1 \tag{2.13}$$

Taking $r = 1$ in (2.11) gives Cassini's identity

$$F_{n-1}F_{n+1} - F_n^2 = (-1)^n. \tag{2.14}$$

The (2.14) identity is also known as Simson's formula because it was discovered by Simson. De Moivre-type identity

$$\left(\frac{1 \pm \sqrt{5}}{2}\right)^n = \frac{L_n \mp \sqrt{5}F_n}{2}$$

where $L_n$ denotes the n-th lucas number and $F_n$ denotes the n-th Fibonacci numbers. The addition formula

$$F_{m+n} = \frac{1}{2}(F_mL_n + L_mF_n)$$

where $L_n$ is a Lucas number, the subtraction formula

$$F_{m-n} = \frac{1}{2}(-1)^n(F_mL_n - L_mF_n).$$

In terms of the Lucas number $L_n$, we have

$$\begin{aligned} F_{2n} &= F_nL_n \\ F_{2n}(L_{2n}^2 - 1) &= F_{6n} \\ F_{m+p} + (-1)^{p+1}F_{m-p} &= F_pL_m \end{aligned}$$

$$\sum_{k=a+1}^{a+4n} F_k = F_{a+4n+2} - F_{a+2} = F_{2n}L_{a+2n+2}.$$

### 2.1.2 Padovan Numbers

The integer sequence $P_n$ that is defined by initial values is known as the Padovan sequence. The Padovan sequence initial values are $P_0 = 1,\ P_1 = P_2 = 1$. The Padovan numbers recurrence relation is following

$$P_n = P_{n-2} + P_{n-3} \quad \text{for} \quad n \geq 3 \tag{2.15}$$

The Padovan numbers are $1, 1, 1, 2, 2, 3, 4, 5, 7, 9, 12, 16, 21, 28, 37, 49, 65, 86, 114, \ldots$.
The characteristic equation of (2.15) is

$$x^3 - x - 1 = 0 \tag{2.16}$$

Here $\alpha$, $\beta$, $\gamma$ are the roots of the cubic equation

$$\begin{aligned}
\alpha &= \left(\frac{1}{2} + \sqrt{\frac{23}{108}}\right)^{1/3} + \left(\frac{1}{2} - \sqrt{\frac{23}{108}}\right)^{1/3} \\
\beta &= \omega\left(\frac{1}{2} + \sqrt{\frac{23}{108}}\right)^{1/3} + \omega^2\left(\frac{1}{2} - \sqrt{\frac{23}{108}}\right)^{1/3} \\
\gamma &= \omega^2\left(\frac{1}{2} + \sqrt{\frac{23}{108}}\right)^{1/3} + \omega\left(\frac{1}{2} - \sqrt{\frac{23}{108}}\right)^{1/3}.
\end{aligned}$$

where $\alpha = \left(\frac{1}{2} + \sqrt{\frac{23}{108}}\right)^{1/3} + \left(\frac{1}{2} - \sqrt{\frac{23}{108}}\right)^{1/3} \approx 1.32471795724$ is called plastic number (or plastic ratio or plastic constant). There is also the following limit

$$\lim_{n\to\infty} \frac{P_{n+1}}{P_n} = \alpha \tag{2.17}$$

where $P_n$ is $nth$ Padovan number.

Padovan numbers were discovered by Italian architect Richard Padovan, who was born in 1935. He studied architecture in London (1952–1957). Richard Padovan, in his 1994 essay "Dom", attributed its discovery to the Dutch architect Hans van der Lahn.

In June of 1996, Ian Stewart wrote about this sequence in his Scientific American column "The Mathematical Reproduction". Thus, Padovan numbers were defined. The golden ratio is a mathematical constant used in architecture and geometry to describe recurring situations in symmetrical structures. Just as the Fibonacci number sequence is related

to the golden ratio, Padovan numbers are related to plastic numbers. Two consecutive Padovan numbers ratio converges to the plastic number (radiant number).
In 1924, Gérard Cordonnier, a student of architecture from France, created the plastic number. This sequence of numbers is also known as the Cordonnier sequence.
Extensions to negative values can be made on any sequence defined by a recurrence relation. The $\{P_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$P_{-n} = -P_{-(n-1)} + P_{-(n-3)}.$$

**Table 2.2** A few values of the Negative Padovan Numbers.

| $P_{-11}$ | $P_{-10}$ | $P_{-9}$ | $P_{-8}$ | $P_{-7}$ | $P_{-6}$ | $P_{-5}$ | $P_{-4}$ | $P_{-3}$ | $P_{-2}$ | $P_{-1}$ | $P_0$ | $P_1$ | $P_2$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $-2$ | 2 | $-1$ | 0 | 1 | $-1$ | 1 | 0 | 0 | 1 | 0 | 1 | 1 | 1 |

The generalized Padovan sequence is defined by

$$S_n = pS_{n-2} + qS_{n-3}, \quad n \ \in \ \mathbb{N} \tag{2.18}$$

$S_{-2} = 0$, $S_{-1} = 0$ and $S_0 = 0$ are initial conditions, where $p$ and $q$ are arbitrary real numbers. The characteristic equation of (2.18) is $x^3 - px - q = 0$ such that the roots

$$\begin{aligned} \phi &= \frac{R^2 + 12p}{6R} \\ \varphi &= -\frac{R^2 + 12p}{12R} + i\frac{\sqrt{3}}{2}(\frac{R}{6} - \frac{2p}{R}) \\ \psi &= -\frac{R^2 + 12p}{12R} - i\frac{\sqrt{3}}{2}(\frac{R}{6} - \frac{2p}{R}) \end{aligned}$$

where $R = 108q + 12\sqrt{-12p^3 + 81q^2}$. The following limit exists

$$\lim_{n\to\infty} \frac{S_{n+1}}{S_n} = \phi$$

where $S_n$ is $nth$ generelized Padovan number. Padovan polynomials are a generalization of Padovan sequence numbers. These polynomials are defined by

$$P_n(x) = \begin{cases} 1, & if \quad n = 1 \\ 0, & if \quad n = 1 \\ x, & if \quad n = 3 \\ xP_{n-2}(x) + P_{n-3}(x) & if \quad n \geq 4 \end{cases}$$

The first few Padovan polynomials are

$P_1(x) = 1$, $P_2(x) = 0$, $P_3(x) = x$, $P_4(x) = 1$, $P_5(x) = x^2$, $P_6(x) = 2x$, $P_7(x) = x^3 + 1$. By evaluating the polynomials, the Padovan numbers can be obtained the $P_{n-3}(x)$ at $x = 1$. Evaluating $P_{n-3}(x)$ at $x = 2$ gives the nth Fibonacci number plus $(-1)^n$.
The ordinary generating function for the sequence is

$$\sum_{n=1}^{\infty} P_n(x)t^n = \frac{t}{1 - xt^2 - t^3} .$$

The Padovan numbers are generated by a matrix Q. The matrix is in the form of 3 x 3. To obtain the $nth$ term of this sequence, look at the nth power of the matrix without using the recurrence relation. The relationship represented by the matrix was put forward by Sokhuma (2013) and Seenukul (2015), see [42]. For n $\geq 1$ we have that the generating matrix Q of Padovan sequence having initial values $P_0 = P_1 = 0$, $P_2 = 1$, is given by

$$Q = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \text{ and } Q^n = \begin{bmatrix} P_{n-1} & P_{n+1} & P_n \\ P_n & P_{n+2} & P_{n+1} \\ P_{n+1} & P_{n+3} & P_{n+2} \end{bmatrix} .$$

So we have

$$Q^0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; Q^1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} .$$

The Padovan numbers and their properties have been studied by some other authors, see [6], [7], [8], [9], [12].

### 2.1.3 Horadam Numbers

The Horadam sequence is a direct generalization of the Fibonacci numbers in the complex plane. It depend on a family of four complex parameters; its contain two recurrence coefficients and two initial conditions. It gives rise to some well-known sequences (Fibonacci, Pell, Lucas, Pell-Lucas, Jacobsthal, Jacobsthal-Lucas, Fermat, Fermat Lucas, for exampleA Horadam sequence $\{W_n\}_{n=0}^{\infty} = \{W_n(a, b; p, q)\}_{n=0}^{\infty}$ is defined by the recurrence

$$W_{n+2} - p.W_{n+1} + q.W_{n+1} = 0, \quad W_0 = a, \; W_1 = b, \quad n \geqslant 2 \tag{2.19}$$

where the parameters a, b, p, q are complex numbers. The recursion formula (2.19) has been considered as describing a generalized Fibonacci-Lucas sequence recurrence.

The sequence $\{U_n(p,\ q)\}$ is defined as $U_n = W_n(1,p;\ p,q)\}$ and the sequence $\{V_n(p,$ $q)\}$ defined as $V_n = W_n(1,p;\ p,q)\}$. The sequences $\{U_n\}$ and $\{V_n\}$, respectively, are generalized Fibonacci and Lucas sequences. Another generalization of the Fibonacci and Lucas sequence, named as the bi-periodic Fibonacci sequence $\{q_n\}$ is defined by

$$q_n = \begin{cases} aq_{n-1} + q_{n-2}, & if \;\; n \; is \; even \; n \geqslant 2 \\ bq_{n-1} + q_{n-2}, & if \;\; n \; is \; odd \; n \geqslant 2 \end{cases}$$

with initial values $q_0 = 0$, $q_1 = 1$ and $a$, $b$ are nonzero numbers and the bi-periodic Lucas sequence $\{p_n\}$ is defined by

$$p_n = \begin{cases} bp_{n-1} + p_{n-2}, & if \;\; n \; is \; even \; n \geqslant 2 \\ ap_{n-1} + p_{n-2}, & if \;\; n \; is \; odd \; n \geqslant 2 \end{cases}$$

with the initial conditions $p_0 = 2$, $p_1 = a$. If we take $a = b = 1$ in $\{q_n\}$, we get the classical Fibonacci sequence and if we take $a = b = 1$ in $\{p_n\}$, we get the classical Lucas sequence. The Horadam sequences have been characterized in (2.19), and arise when zeros of the characteristic equation

$$x^2 - px + q = 0.$$

There are two distinct characteristic roots

$$\begin{aligned} \alpha(p,q) &= \frac{(p + \sqrt{p^2 - 4q})}{2} \qquad (p, q \neq 0) \\ \beta(p,q) &= \frac{(p - \sqrt{p^2 - 4q})}{2} \qquad (p, q \neq 0) \end{aligned}$$

Also, there exists the following limit,

$$\lim_{n\to\infty} \frac{W_{n+1}}{W_n} = \lambda$$

where $W_n$ is nth Horadam number.

### 2.1.4 Tribonacci Numbers

The Tribonacci sequence is defined as

$$T_{n+1} = T_n + T_{n-1} + T_{n-2}, \quad n > 1. \tag{2.20}$$

Tribonacci numbers have three predetermined terms at the beginning of the sequence, and each subsequent term is the sum of the three terms that came before it. The first

tribonacci numbers are 0, 0, 1, 1, 2, 4, 7, 13, 24, 44, 81, 149, 274, 504, 927, 1705, 3136, 5768, 10609, 19513, 35890, 66012,...
The Tribonacci numbers are a generalization of the Fibonacci numbers defined initial conditions with $T_1 = 1$, $T_2 = 1$, $T_3 = 2$ and the recurrence equation

$$T_n = T_{n-1} + T_{n-2} + T_{n-3}, \quad n \geq 4$$

The Tribonacci sequence can be extend to negative subscript as follow

$$T_{-n} = T_{-n+3} - T_{-n+2} - T_{-n+1}.$$

The characteristic equation is

$$x^3 - x^2 - x - 1 = 0$$

Here $\alpha$, $\beta$, $\gamma$ are the roots of the cubic equation

$$\begin{aligned}
\alpha &= (1 + \sqrt[3]{19 + 3\sqrt{33}} + \sqrt[3]{19 - 3\sqrt{33}})/3 \\
\beta &= (1 + \omega\sqrt[3]{19 + 3\sqrt{33}} + \omega^2\sqrt[3]{19 - 3\sqrt{33}})/3 \\
\gamma &= (1 + \omega^2\sqrt[3]{19 + 3\sqrt{33}} + \omega\sqrt[3]{19 - 3\sqrt{33}})/3
\end{aligned}$$

where $\omega = (1 + i\sqrt{3})/2 = \exp(2\pi i/3)$ is the primitive cube root of unity. The Binet formula of Tribonacci sequence is given by

$$\begin{aligned}
T_n &= \frac{\alpha^{n+1}}{(\alpha - \beta)(\alpha - \gamma)} + \frac{\beta^{n+1}}{(\beta - \alpha)(\beta - \gamma)} + \frac{\gamma^{n+1}}{(\gamma - \alpha)(\gamma - \beta)} \\
&= \frac{\alpha^n}{-\alpha^2 + 4\alpha - 1} + \frac{\beta^n}{-\beta^2 + 4\beta - 1} + \frac{\gamma^n}{-\gamma^2 + 4\gamma - 1}
\end{aligned}$$

where $(\alpha, \beta, \gamma)$ are the three roots of

$$p(x) = x^3 - x^2 - x - 1.$$

This can be written in slightly more concise form as

$$T_n = r_1\alpha^n + r_2\beta^n + r_3\gamma^n.$$

The generating function can also be used to calculate the Tribonacci numbers

$$\begin{aligned}
\sum_{n=0}^{\infty} T_n x^n &= \frac{x}{1 - x - x^2 - x^3} \\
&= 1 + x + 2x^2 + 4x^3 + 7x^4 + 13x^5 + 24x^6 + 44x^7 + 81x^8 + 149x^9 + ...
\end{aligned}$$

The proof is obtained from following equality

$$(1-x-x^2-x^3)(\sum_{n=0}^{\infty} T_n x^n) = T_0+(T_1-T_0)x+(T_2-T_1-T_0)x^2+\sum_{n=3}^{\infty}(T_n-T_{n-1}-T_{n-2}-T_{n-3})x^n.$$

Another explicit formula for $T_n$ is also given by

$$\left[ 3\frac{\left\{\frac{1}{3}(19+3\sqrt{33})^{1/3}+\frac{1}{3}(19-3\sqrt{33})^{1/3}+\frac{1}{3}\right\}^n (586+102\sqrt{33})^{1/3}}{(586+102\sqrt{33})^{2/3}+4-(586+102\sqrt{33})^{1/3}} \right]$$

where $r_n$ is the $nth$ root of polynomial. Additionally, there is the following restriction limit

$$\lim_{n\to\infty}\frac{T_{n+1}}{T_n} = \alpha^r$$

where $T_n$ is the nth Tribonacci number and $r \epsilon \mathbb{Z}$.

#### 2.1.5 Lucas Numbers

The Fibonacci sequence and the Lucas sequence both have a recursive relation in which each term is the sum of the two terms before it. T he first two Lucas numbers are $L_0 = 2$ and $L_1 = 1$. The Lucas numbers are defined as follows

$$L_n = \begin{cases} 2, & if \quad n = 0 \\ 1, & if \quad n = 1 \\ L_{n-1} + L_{n-2}, & if \quad n > 1 \end{cases}$$

where n is a natural number. The first few Lucas numbers are $2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, 322, 521, 843, 1364, 2207, 3571, 5778, 9349, ....$

Also, negative recurrence relation can be expressed as

$$L_{n-2} = L_n - L_{n-1}.$$

In this sequence, the formula for terms with negative indices is

$$L_{-n} = (-1)^n.L_n$$

The Fibonacci numbers and the Lucas numbers are related in numerous ways. Some identities regarding these are as follows

$$\begin{aligned} L_n &= F_{n-1} + F_{n+1} \\ F_{2n} &= L_n F_n \\ F_{n+k} + (-1)^k F_{n-k} &= L_k F_n \\ L_{2n} &= 5F_n^2 + 2.(-1)^n = L_n^2 - 2.(-1)^n \\ \lim_{n\to\infty} \frac{L_n}{F_n} &= \sqrt{5} \\ \left| L_n - \sqrt{5} F_n \right| &= \frac{2}{\varphi^n}, \qquad \varphi : golden\ ratio \\ F_n &= \frac{L_{n-1} + L_{n+1}}{5} \\ 5F_n + L_n &= 2L_{n+1}. \end{aligned}$$

The Lucas sequence closed form is given below

$$L_n = \varphi^n + (1-\varphi)^n = \varphi^n + (-\varphi)^{-n} = ((1+\sqrt{5}/2))^n + ((1-\sqrt{5}/2))^n.$$

Most fibonacci identities are similar to Lucas numbers. For example, the Cassini identity becomes

$$\begin{aligned} L_n^2 - L_{n-1}L_{n+1} &= 5.(-1)^n \\ \sum_{k=0}^{n} L_k &= L_{n+2} - 1 \\ \sum_{k=0}^{n} L_k^2 &= L_n L_{n+1} + 2 \\ 2L_{n-1}^2 + L_n^2 &= L_{2n+1} + 5F_{n-2}^2 \end{aligned}$$

where

$$F_n = \frac{L_{n-1} + L_{n+1}}{5}.$$

### 2.1.6 Pell Numbers

The following recurrence relation defines the Pell numbers

$$P_n = \begin{cases} 0, & if \quad n = 0 \\ 1, & if \quad n = 1 \\ 2P_{n-1} + P_{n-2}, & otherwise \end{cases}$$

The Pell numbers are beginning with 0 and 1, and each subsequent Pell number is the sum of the two Pell numbers that came before it. The first few terms in the sequence are $0, 1, 2, 5, 12, 29, 70, 169, 408, 985, 2378, 5741, 13860, ....$

The Pell numbers closed form can be express below

$P_n = \dfrac{(1+\sqrt{2})^n - (1-\sqrt{2})^n}{2\sqrt{2}}.$

Both Pell numbers and Fibonacci numbers increase exponentially in proportion to the powers of the silver ratio $(1+\sqrt{2})$. Pell numbers are useful for finding square triangular numbers, solving combinatorial problems, and calculate approximating the square root of two. The Pell sequence has been studied by many authors and is given extensive attention in the literature, see for example; [2], [5], [11], [11], [15], [21], [24], [27], [30], [43]. For higher order Pell sequences, see [38], [35], [36], [37], [25].

A third order Pell sequence $\{P_n\}_{n\geq 0}$ is defined by recurrence relation as follows

$$P_n = 2P_{n-1} + P_{n-2} + P_{n-3} \tag{2.21}$$

with initial conditions are $P_0 = 0,\ P_1 = 1,\ P_2 = 2.$

Also, Pell sequence can be extended the negative subscripts as

$$P_{-n} = -P_{-(n-1)} - 2P_{-(n-2)} + P_{-(n-3)} \tag{2.22}$$

The cubic equation for Eq.(4.3) is

$$x^3 - 2x^2 - x - 1 = 0. \tag{2.23}$$

Assume that, the cubic equation roots are $\alpha,\ \beta$ and $\gamma$ are as follows

$$\begin{aligned}
\alpha &= \frac{2}{3} + (\frac{61}{54} + \sqrt{\frac{29}{36}})^{1/3} + (\frac{61}{54} - \sqrt{\frac{29}{36}})^{1/3} \qquad (2.24)\\
\beta &= \frac{2}{3} + \omega(\frac{61}{54} + \sqrt{\frac{29}{36}})^{1/3} + \omega^2(\frac{61}{54} - \sqrt{\frac{29}{36}})^{1/3}\\
\gamma &= \frac{2}{3} + \omega^2(\frac{61}{54} + \sqrt{\frac{29}{36}})^{1/3} + \omega(\frac{61}{54} - \sqrt{\frac{29}{36}})^{1/3}
\end{aligned}$$

where

$$\omega = \frac{-1 + i\sqrt{3}}{2} = \exp(2\pi i/3)$$

is a primitive cube root of unity. It seems that the only real root is $\alpha$. Note that

$$\alpha + \beta + \gamma = 2,$$

$\alpha\beta + \alpha\gamma + \beta\gamma \;=\; -1,$

and

$\alpha\beta\gamma = 1.$

We present the first few values of the third-order Pell numbers with positive and negative subscripts in table.

### 2.1.7 Pell–Lucas Numbers

The recurrence relation defines the companion Pell numbers, also known as Pell–Lucas numbers. The recurrence relation is defined by

$$Q_n = \begin{cases} Q_0 = 2, & if \quad n = 0 \\ Q_1 = 2, & if \quad n = 1 \\ Q_n = 2Q_{n-1} + Q_{n-2}, & \text{otherwise} \end{cases}.$$

The first two terms in this series are also the number 2, each next term is created by multiplying the previous Pell-Lucas number by two and adding the next term. The Pell-Lucas numbers can be expressed by the closed form formula

$$Q_n = (1 + \sqrt{2})^n + (1 - \sqrt{2})^n.$$

### 2.1.8 Narayana Numbers

In the 14th century, Indian mathematician Narayana developed the Narayana numbers while researching the problem of a herd of cows and calves. Narayana's cows problem; a cow gives birth to one calf each year, and starting in its fourth year, each calf gives birth to one calf at the start of each year. How many calf there are in total after 20 years? As a recurrence relation, the Narayana problem can be defined as follows

$$N_{n+3} = N_{n+2} + N_n, \quad n \geq 0$$

where n is the year and the initial conditions $N_0 = 0,\ N_1 = 1,\ N_2 = 1$. The sequence $\{N_n\}_{n\geq 0}$ can be defined to include negative subscripts

$$N_{-n} = -N_{-(n-2)} + N_{-(n-3)} \tag{2.25}$$

Narayana's cows problem sequence is a sequence of positive integers which are defined by $1, 1, 1, 2, 3, 4, 6, 9, 13, 19, 28, 41, 60, 88, 129, 189, 277, 406, 595, 872, 1278, ...$, see [39].

**Table 2.3** A few values of the Narayana Numbers.

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $N_n$ | 0 | 1 | 1 | 1 | 2 | 3 | 4 | 6 | 9 | 13 | 19 | 28 | 41 | 60 |
| $N_{-n}$ | | 0 | 1 | 0 | $-1$ | 1 | 1 | $-2$ | 0 | 3 | $-2$ | $-3$ | 5 | 1 |

Narayana numbers can be expressed using Binet's formulas for all integers $n$, see [40]. The cubic equation

$$x^3 - x^2 - 1 = 0.$$

Here, $\alpha$, $\beta$ and $\gamma$ are the roots of the cubic equation

$$\alpha = \frac{1}{3} + (\frac{29}{54} + \sqrt{\frac{31}{108}})^{1/3} + (\frac{29}{54} - \sqrt{\frac{31}{108}})^{1/3}$$

$$\beta = \frac{1}{3} + \omega(\frac{29}{54} + \sqrt{\frac{31}{108}})^{1/3} + \omega^2(\frac{29}{54} - \sqrt{\frac{31}{108}})^{1/3}$$

$$\gamma = \frac{1}{3} + \omega^2(\frac{29}{54} + \sqrt{\frac{31}{108}})^{1/3} + \omega(\frac{29}{54} - \sqrt{\frac{31}{108}})^{1/3}$$

where

$$\omega = \frac{-1 + i\sqrt{3}}{2} = \exp(2\pi i/3)$$

is a primitive cube root of unity. Note that

$$\alpha + \beta + \gamma = 2$$

$$\alpha\beta + \alpha\gamma + \beta\gamma = 0, \quad \alpha\beta\gamma = 1.$$

## CHAPTER 3

## THE EXACT SOLUTIONS OF SOME DIFFERENCE EQUATIONS VIA NARAYANA AND PADOVAN NUMBERS

In the literature section of Chapter 1, we dealt with the difference equations whose solutions are obtained by special integer sequences. In Chapter 3 and Chapter 4 we obtained solutions of some difference equations with special integer sequences such as Padovan, Narayana and Pell numbers.

In Chapter 3, we investigated the solutions to the following four rational difference equations and their solutions which is related to the Narayana and Padovan numbers. We studied with the stability character of these four equations, their equilibrium points, and the asymptotic behavior of the equilibrium points.

$$x_{n+1} = \frac{1}{x_n(x_{n-1} - 1)}, \quad n = 0, 1, ..., \tag{3.1}$$

$$x_{n+1} = \frac{-1}{x_n(x_{n-1} - 1)}, \quad n = 0, 1, ..., \tag{3.2}$$

$$x_{n+1} = \frac{1}{x_n(x_{n-1} + 1)}, \quad n = 0, 1, ..., \tag{3.3}$$

$$x_{n+1} = \frac{-1}{x_n(x_{n-1} + 1)}, \quad n = 0, 1, ..., \tag{3.4}$$

We found an interesting relation between the above difference equations solutions with Narayana and Padovan numbers.

### 3.1 MAIN RESULTS

In this section, we present our findings about these equations. We obtained the general solutions of the difference equations in explicit form. We also analyzed asymptotic behavior of the solutions.

### 3.1.1 The Difference Equation (3.1)

In this part, we deal with the Eq.(3.1), that is

$$x_{n+1} = \frac{1}{x_n(x_{n-1}-1)}, \quad n = 0, 1, ...,$$

and we analyze the asymptotic behavior of solutions Eq.(3.1).

**Theorem 3.1** *Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of Eq.(3.1). Then, for $n = 0, 1, 2, ...$, the form of solutions $\{x_n\}_{n=-1}^{\infty}$ is given by*

$$x_n = \frac{N_{-n}x_{-1}x_0 + N_{-(n+2)}x_0 + N_{-(n+1)}}{N_{-(n+1)}x_{-1}x_0 + N_{-(n+3)}x_0 + N_{-(n+2)}}, \tag{3.5}$$

*where $N_n$ is the nth third order Narayana numbers and the initial conditions $x_{-1}$, $x_0 \in \mathbb{R} - F_1$, with $F_1$ is the forbidden set of Eq.(3.1) given by*

$$F_1 = \bigcup_{n=-1}^{\infty} \left\{ (x_{-1}, x_0) : N_{-(n+1)}x_{-1}x_0 + N_{-(n+3)}x_0 + N_{-(n+2)} = 0 \right\}.$$

**Proof. (First proof)** This theorem will be proved by induction on k. For $k = 0$, from Eq.(3.1),

$$x_1 = \frac{1}{x_0(x_{-1}-1)} = \frac{1}{x_0x_{-1}-x_0} = \frac{N_{-1}x_{-1}x_0 + N_{-3}x_0 + N_{-2}}{N_{-2}x_{-1}x_0 + N_{-4}x_0 + N_{-3}}$$

Let us assume that the equality is true for $n = k$,

$$x_k = \frac{N_{-k}x_{-1}x_0 + N_{-(k+2)}x_0 + N_{-(k+1)}}{N_{-(k+1)}x_{-1}x_0 + N_{-(k+3)}x_0 + N_{-(k+2)}}$$

$$x_{k-1} = \frac{N_{-(k-1)}x_{-1}x_0 + N_{-(k+1)}x_0 + N_{-k}}{N_{-k}x_{-1}x_0 + N_{-(k+2)}x_0 + N_{-(k+1)}}.$$

Thus, we have to prove that it is true for $k+1$. Considering that (3.1) and (3.5), we get $x_{k+1}$ as follows,

$$\begin{aligned} x_{k+1} &= \frac{1}{x_k(x_{k-1}-1)} \\ &= \frac{1}{\frac{N_{-(k+2)}x_0 + N_{-k}x_{-1}x_0 + N_{-(k+1)}}{N_{-(k+1)}x_{-1}x_0 + N_{-(k+3)}x_0 + N_{-(k+2)}}\left(\frac{N_{-(k+1)}x_0 + N_{-(k-1)}x_{-1}x_0 + N_{-k}}{N_{-k}x_{-1}x_0 + N_{-(k+2)}x_0 + N_{-(k+1)}} - 1\right)} \\ &= \frac{N_{-(k+3)}x_0 + N_{-(k+2)} + N_{-(k+1)}x_{-1}x_0}{(N_{-k} - N_{-k-1}) + (N_{1-k} - N_{-k})x_{-1}x_0 + (N_{-k-1} - N_{-k-2})x_0} \\ &= \frac{N_{-(k+3)}x_0 + N_{-(k+2)} + N_{-(k+1)}x_{-1}x_0}{N_{-(k+3)} + N_{-(k+2)}x_{-1}x_0 + N_{-(k+4)}x_0} \end{aligned}$$

which completes the proof and the induction.

**Proof.** **(Second proof)** We will show the proof of this situation in a different way, so let's take $n = 0, 1, 2, ....$

$$
\begin{aligned}
n &= 0 \quad \Longrightarrow \quad x_1 = \frac{1}{x_0 x_{-1} - x_0} \\
n &= 1 \quad \Longrightarrow \quad x_2 = \frac{x_0 x_{-1} - x_0}{x_0 - 1}, \\
n &= 2 \quad \Longrightarrow \quad x_3 = \frac{x_0 - 1}{x_0 - x_0 x_{-1} + 1}, \\
n &= 3 \quad \Longrightarrow \quad x_4 = \frac{-x_0 x_{-1} + x_0 + 1}{x_0 x_{-1} - 2x_0 + 1}, \\
n &= 4 \quad \Longrightarrow \quad x_5 = \frac{x_0 x_{-1} - 2x_0 + 1}{x_0 x_{-1} - 2}, \\
n &= 5 \quad \Longrightarrow \quad x_6 = \frac{x_0 x_{-1} - 2}{3x_0 - 2x_0 x_{-1}}, \\
n &= 6 \quad \Longrightarrow \quad x_7 = \frac{2x_0 x_{-1} - 3x_0}{2x_0 - 3}, \\
n &= 7 \quad \Longrightarrow \quad x_8 = \frac{2x_0 - 3}{3x_0 - 3x_0 x_{-1} + 2}, \\
n &= 8 \quad \Longrightarrow \quad x_9 = \frac{-3x_0 x_{-1} + 3x_0 + 2}{2x_0 x_{-1} - 5x_0 + 3}, \\
n &= 9 \quad \Longrightarrow \quad x_{10} = \frac{-2x_0 x_{-1} + 5x_0 - 3}{-3x_0 x_{-1} + x_0 + 5}, \\
& \qquad \vdots
\end{aligned}
$$

According to the third order recurrence relation (2.25), the negative narayana numbers are obtained. Thus, the theorem is proved.

**Theorem 3.2** *The Eq.(3.1) has unique positive equilibrium point $\overline{x} = \alpha$ and $\alpha$ is repeller.*

**Proof.** The equilibrium point of Eq.(3.1) is the real root of the equation

$$\overline{x} = \frac{1}{\overline{x}\,(\overline{x} - 1)}.$$

After simplification, we obtain the following cubic equation

$$\overline{x}^3 - \overline{x}^2 - 1 = 0. \tag{3.6}$$

The Eq.(3.6) has three roots and one of them is real. Therefore, the unique real equilibrium point of Eq.(3.6) is $\overline{x} = \alpha$.

Let $I$ be an interval of real numbers and

$f : I^2 \to I$

be a continuous function with the definition of

$$f(x,y)=\frac{1}{x(y-1)}$$

Consequently, it follows that

$$\begin{aligned}\frac{\partial f(x,y)}{\partial x} &= \frac{-(y-1)}{(x(y-1))^2},\\ \frac{\partial f(x,y)}{\partial y} &= \frac{-x}{(x(y-1))^2}.\end{aligned}$$

Then, we get

$$\begin{aligned}\frac{\partial f(\overline{x},\overline{x})}{\partial x} &= \frac{-(\overline{x}-1)}{\left(\overline{x}^2-\overline{x}\right)^2}\\ &= \frac{-(\overline{x}-1)}{\left(\frac{1}{\overline{x}}\right)^2}\\ &= \frac{-(\frac{1}{\overline{x}^2})}{\left(\frac{1}{\overline{x}}\right)^2}\\ &= -1\\ \frac{\partial f(\overline{x},\overline{x})}{\partial y} &= \frac{-\overline{x}}{\left(\overline{x}^2-\overline{x}\right)^2}\\ &= \frac{-\overline{x}}{\left(\frac{1}{\overline{x}}\right)^2}\\ &= -\alpha^3\end{aligned}$$

and the equation of linearization of Eq.(3.1) about $\overline{x}=\alpha$ is

$$y_{n+1}=-y_n+\left(-\alpha^3\right)y_{n-1}$$

its equal to

$$y_{n+1}+y_n+\alpha^3 y_{n-1}=0.$$

As a result, the associated characteristic equation is

$$\lambda^2+\lambda+\alpha^3=0.$$

Then, it is clearly seen that, the roots of characteristic equations are

$$\lambda_1=\frac{1}{2}\sqrt{-4\alpha^3+1}-\frac{1}{2}$$

$$\lambda_2=-\frac{1}{2}\sqrt{-4\alpha^3+1}-\frac{1}{2}.$$

So, the solution is obtained numerically as

$|\lambda_1| = |\lambda_2| \approx 1.772 > 1.$

According to the Theorem (2.1), the equilibrium point $\overline{x} = \alpha$ is repeller point. The proof of the theorem is completed.

### 3.1.2 The Difference Equation (3.2)

In this section, we consider that the Eq.(3.2)

$$x_{n+1} = \frac{-1}{x_n(x_{n-1} - 1)}, \quad n = 0, 1, ...,$$

and we study the dynamical behavior of solutions of Eq.(3.2).

**Theorem 3.3** *Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of Eq.(3.2). Then, for $n = 0, 1, 2, ...$, the form of solutions $\{x_n\}_{n=-1}^{\infty}$ is given by*

$$x_n = \frac{-P_{(n-3)}x_{-1}x_0 + P_{(n-1)}x_0 - P_{(n-2)}}{P_{(n-2)}x_{-1}x_0 - P_n x_0 + P_{(n-1)}}, \tag{3.7}$$

*where $P_n$ is the nth Padovan number and initial conditions $x_{-1}$, $x_0 \in \mathbb{R} - F_2$, with $F_2$ is the forbidden set of Eq.(3.2) given by*

$$F_2 = \bigcup_{n=-1}^{\infty} \left\{(x_{-1}, x_0) : P_{(n-2)}x_{-1}x_0 - P_n x_0 + P_{(n-1)} = 0\right\}.$$

**Proof.** By using induction on k, we will prove this theorem. For $k = 0$, from Eq.(3.2),

$$x_1 = \frac{-1}{x_0(x_{-1} - 1)} = \frac{-1}{x_0 x_{-1} - x_0} = \frac{-P_{-3}x_{-1}x_0 + P_{-1}x_0 - P_{-2}}{P_{-2}x_{-1}x_0 - P_0 x_0 + P_{-1}}$$

Suppose that the equality holds for $n = k$, that is

$$x_k = \frac{-P_{(k-3)}x_{-1}x_0 + P_{(k-1)}x_0 - P_{(k-2)}}{P_{(k-2)}x_{-1}x_0 - P_k x_0 + P_{(k-1)}},$$

$$x_{k-1} = \frac{-P_{(k-4)}x_{-1}x_0 + P_{(k-2)}x_0 - P_{(k-3)}}{P_{(k-3)}x_{-1}x_0 - P_{k-1}x_0 + P_{(k-2)}}.$$

Therefore, we need to show that it holds true for $k+1$. When we take into account (3.2) and (3.7), we get $x_{k+1}$ as follows,

$$\begin{aligned}
x_{k+1} &= \frac{-1}{x_k(x_{k-1}-1)} \\
&= \frac{-1}{\frac{-P_{(k-3)}x_{-1}x_0+P_{(k-1)}x_0-P_{(k-2)}}{P_{(k-2)}x_{-1}x_0-P_kx_0+P_{(k-1)}}\left(\frac{-P_{(k-4)}x_{-1}x_0+P_{(k-2)}x_0-P_{(k-3)}}{P_{(k-3)}x_{-1}x_0-P_{k-1}x_0+P_{(k-2)}}-1\right)} \\
&= \frac{-1}{\frac{-P_{(k-3)}x_{-1}x_0+P_{(k-1)}x_0-P_{(k-2)}}{P_{(k-2)}x_{-1}x_0-P_kx_0+P_{(k-1)}}\left(\frac{-P_{(k-1)}x_{-1}x_0+P_{(k+1)}x_0-P_k}{P_{(k-3)}x_{-1}x_0-P_{k-1}x_0+P_{(k-2)}}\right)} \\
&= \frac{1}{\left(\frac{-P_{(k-1)}x_{-1}x_0+P_{(k+1)}x_0-P_k}{P_{(k-2)}x_{-1}x_0-P_kx_0+P_{(k-1)}}\right)} \\
&= \frac{P_{(k-2)}x_{-1}x_0-P_kx_0+P_{(k-1)}}{-P_{(k-1)}x_{-1}x_0+P_{(k+1)}x_0-P_k}
\end{aligned}$$

which concludes the proof and the induction.

**Theorem 3.4** *The Eq.(3.2) has unique positive equilibrium point $\overline{x}=a$ and $a$ is locally asymptotically stable.*

**Proof.** The Equilibrium point of Eq.(3.2) is satisfy the equation

$$\overline{x}=\frac{-1}{\overline{x}\left(\overline{x}-1\right)}$$

After simplification, it is simple to obtain cubic equation of Eq.(3.2)

$$\overline{x}^3-\overline{x}^2+1=0. \tag{3.8}$$

The unique real equilibrium point of Eq.(3.8) is $\overline{x}=a$.

Let $I$ be an interval of real numbers and $f: I^2 \to I$ be a continuous function with the definition of asymptotically stable.

$$f\left(x,y\right)=\frac{-1}{x\left(y-1\right)}$$

Therefore, it follows that

$$\begin{aligned}
\frac{\partial f\left(x,y\right)}{\partial x} &= \frac{\left(y-1\right)}{\left(x\left(y-1\right)\right)^2}, \\
\frac{\partial f\left(x,y\right)}{\partial y} &= \frac{x}{\left(x\left(y-1\right)\right)^2}.
\end{aligned}$$

If we edit the equations, we get

$$\begin{aligned}
\frac{\partial f\left(\overline{x},\overline{x}\right)}{\partial x} &= \frac{\left(\overline{x}-1\right)}{\left(\overline{x}^2-\overline{x}\right)^2} \\
&= \frac{\left(\overline{x}-1\right)}{\left(\overline{x}.\left(\overline{x}-1\right)\right)^2} \\
&= \frac{1}{\overline{x}^2.\left(\overline{x}-1\right)} \\
&= \frac{+1}{-1} \\
&= -1 \\
\frac{\partial f\left(\overline{x},\overline{x}\right)}{\partial y} &= \frac{\overline{x}}{\left(\overline{x}^2-\overline{x}\right)^2} \\
&= \frac{1}{\overline{x}.\left(\overline{x}-1\right)^2} \\
&= \frac{1}{\overline{x}.\left(\frac{1}{\overline{x}}\right)^4} \\
&= a^3
\end{aligned}$$

and the equation of linearization of Eq.(3.2) about $\overline{x}=a$ is

$$y_{n+1} = -y_n + \left(a^3\right) y_{n-1}$$

or equivalently

$$y_{n+1} + y_n - a^3 y_{n-1} = 0.$$

Therefore, the corresponding characteristic equation is

$$\lambda^2 + \lambda - a^3 = 0. \tag{3.9}$$

Then, it is clearly seen that

$$\lambda_1 = \frac{1}{2}\sqrt{4a^3+1} - \frac{1}{2}$$

$$\lambda_1 = -\frac{1}{2}\sqrt{4a^3+1} - \frac{1}{2}$$

Solution is and numerically

$$|\lambda_1| = |\lambda_2| \approx 0.65586 < 1$$

So, from Theorem (2.1), the equilibrium points $\overline{x}=a$ is locally asymptotically stable and this completes the proof.

**Theorem 3.5** *The equilibrium point of Eq.(3.2) is globally asymptotically stable.*

**Proof.** Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of Eq.(3.2), we need to prove that the equilibrium point $a$ is global attractor. According to Definition (2.3), we will prove that

$$\lim_{n\to\infty} x_n = a.$$

From Eq. (3.7), (2.16) and (2.17), we get following limit

$$\begin{aligned}
\lim_{n\to\infty} x_n &= \lim_{n\to\infty} \frac{P_{n-3}(-x_{-1}x_0 + \frac{P_{n-1}}{P_{n-3}}x_0 - \frac{P_{n-2}}{P_{n-3}})}{P_{n-2}(x_{-1}x_0 - \frac{P_n}{P_{n-2}}x_0 + \frac{P_{n-1}}{P_{n-2}})} \\
&= \left(\frac{-x_{-1}x_0 + \frac{P_{n-1}}{P_{n-3}}x_0 - \frac{P_{n-2}}{P_{n-3}}}{x_{-1}x_0 - \frac{P_n}{P_{n-2}}x_0 + \frac{P_{n-1}}{P_{n-2}}}\right) \lim_{n\to\infty} \frac{P_{n-3}}{P_{n-2}} \\
&= \left(\frac{-x_{-1}x_0 + \alpha^2 x_0 - \alpha}{x_{-1}x_0 - \alpha^2 x_0 + \alpha}\right) \lim_{n\to\infty} \frac{P_{n-3}}{P_{n-2}} \\
&= -\lim_{n\to\infty} \frac{P_{n-3}}{P_{n-2}} \\
&= -\frac{1}{\alpha} \\
&= a
\end{aligned}$$

The proof is complete.

### 3.1.3 The Difference Equation(3.3)

In this section, we consider the Eq.(3.3)

$$x_{n+1} = \frac{1}{x_n(x_{n-1}+1)}, \quad n = 0, 1, ...,$$

we study the dynamical behavior of solutions of Eq.(3.3).

**Theorem 3.6** *Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of Eq.(3.3). Then, for $n = 0, 1, 2, ...,$the form of solutions $\{x_n\}_{n=-1}^{\infty}$ is given by*

$$x_n = \frac{P_{(n-3)}x_{-1}x_0 + P_{(n-1)}x_0 + P_{(n-2)}}{P_{(n-2)}x_{-1}x_0 + P_n x_0 + P_{(n-1)}}, \tag{3.10}$$

*where $P_n$ is the nth Padovan number and initial conditions $x_{-1}$, $x_0 \in \mathbb{R} - F_3$, with $F_3$ is the forbidden set of Eq.(3.3) given by*

$$F_3 = \bigcup_{n=-1}^{\infty} \left\{(x_{-1}, x_0) : P_{(n-2)}x_{-1}x_0 + P_n x_0 + P_{(n-1)} = 0\right\}.$$

**Proof.** By induction on k, we will demonstrate this theorem. For $k = 0$, from Eq.(3.3).

$$x_1 = \frac{1}{x_0(x_{-1}+1)} = \frac{1}{x_0x_{-1}+x_0} = \frac{P_{-3}x_{-1}x_0 + P_{-1}x_0 + P_{-2}}{P_{-2}x_{-1}x_0 + P_0x_0 + P_{-1}}$$

Assume that the equality holds for $n = k$, that is

$$x_k = \frac{P_{(k-3)}x_0x_{-1} + P_{(k-1)}x_0 + P_{(k-2)}}{P_{(k-2)}x_0x_{-1} + P_kx_0 + P_{(k-1)}}.$$

$$x_{k-1} = \frac{P_{(k-4)}x_0x_{-1} + P_{(k-2)}x_0 + P_{(k-3)}}{P_{(k-3)}x_0x_{-1} + P_{k-1}x_0 + P_{(k-2)}}.$$

Thus, we have to prove that it is true for $k+1$. Taking into account (3.3) and (3.10), we have $x_{k+1}$

$$\begin{aligned}
x_{k+1} &= \frac{1}{x_k(x_{k-1}+1)} \\
&= \frac{1}{\left(\frac{P_{(k-3)}x_0x_{-1} + P_{(k-1)}x_0 + P_{(k-2)}}{P_{(k-2)}x_0x_{-1} + P_kx_0 + P_{(k-1)}}\right)\left(\frac{P_{(k-4)}x_0x_{-1} + P_{(k-2)}x_0 + P_{(k-3)}}{P_{(k-3)}x_0x_{-1} + P_{k-1}x_0 + P_{(k-2)}} + 1\right)} \\
&= \frac{1}{\left(\frac{P_{(k-3)}x_0x_{-1} + P_{(k-1)}x_0 + P_{(k-2)}}{P_{(k-2)}x_0x_{-1} + P_kx_0 + P_{(k-1)}}\right)\left(\frac{P_{(k-1)}x_0x_{-1} + P_{(k+1)}x_0 + P_k}{P_{(k-3)}x_0x_{-1} + P_{k-1}x_0 + P_{(k-2)}}\right)} \\
&= \frac{P_{(k-2)}x_0x_{-1} + P_kx_0 + P_{(k-1)}}{P_{(k-1)}x_0x_{-1} + P_{(k+1)}x_0 + P_k}
\end{aligned}$$

which ends the proof and the induction.

**Theorem 3.7** *The Eq.(3.3) has unique positive equilibrium point $\overline{x} = b$ and $b$ is locally asymptotically stable.*

**Proof.** The Equilibrium point of Eq.(3.3) is the real root of the equation

$$\overline{x} = \frac{1}{\overline{x}\,(\overline{x}+1)}$$

After simplification, one can easily obtain the cubic equation of Eq.(3.3)

$$\overline{x}^3 + \overline{x}^2 - 1 = 0. \tag{3.11}$$

The unique real equilibrium point of Eq.(3.11) is $\overline{x} = b$.

Let $I$ be an interval of real numbers and

$f : I^2 \to I$

defined by

$$f(x,y)=\frac{1}{x(y+1)}$$

Therefore, it follows that

$$\begin{aligned}\frac{\partial f(x,y)}{\partial x} &= \frac{-(y+1)}{(x(y+1))^2},\\ \frac{\partial f(x,y)}{\partial y} &= \frac{-x}{(x(y+1))^2}.\end{aligned}$$

Then

$$\begin{aligned}\frac{\partial f(\overline{x},\overline{x})}{\partial x} &= \frac{-(\overline{x}+1)}{(\overline{x}^2+\overline{x})^2}\\ &= \frac{-1}{\overline{x}^2.(\overline{x}+1)}\\ &= \frac{-1}{\overline{x}^3+\overline{x}^2}\\ &= \frac{-1}{+1}\\ &= -1\\ \frac{\partial f(\overline{x},\overline{x})}{\partial y} &= \frac{-\overline{x}}{\overline{x}^2(\overline{x}+1)^2}\\ &= \frac{-1}{\overline{x}.(\overline{x}+1)^2}\\ &= \frac{-1}{\overline{x}.\left(\frac{1}{\overline{x}}\right)^4}\\ &= -b^3\end{aligned}$$

and the linearized equation of Eq.(3.3) about $\overline{x}=b$ is

$$y_{n+1}=-y_n+\left(-b^3\right)y_{n-1}$$

or equivalently

$$y_{n+1}+y_n+b^3y_{n-1}=0.$$

Therefore, the characteristic equation that corresponds to it is

$$\lambda^2+\lambda+b^3=0.$$

Then, it is clearly seen that

$$t\lambda_1=\frac{1}{2}\sqrt{-4b^3+1}-\frac{1}{2}$$

$\lambda_2 = -\frac{1}{2}\sqrt{-4b^3+1} - \frac{1}{2}$

Solution is and numerically

$|\lambda_1| = |\lambda_2| \approx 0.65586 < 1$

Thus, from Theorem (2.1) the equilibrium point $\overline{x} = b$ is locally asymptotically stable and this completes the proof.

**Theorem 3.8** *The equilibrium point of Eq.(3.3) is globally asymptotically stable.*

**Proof.** Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of Eq.(3.3), we need to prove that the equilibrium point $b$ is global attractor. According to Definition (2.3), we will prove that

$$\lim_{n\to\infty} x_n = b.$$

From Eq. (3.10), (2.16) and (2.17), it follows that

$$\begin{aligned}
\lim_{n\to\infty} x_n &= \lim_{n\to\infty} \frac{P_{(n-3)}x_{-1}x_0 + P_{(n-1)}x_0 + P_{(n-2)}}{P_{(n-2)}x_{-1}x_0 + P_n x_0 + P_{(n-1)}} \\
&= \lim_{n\to\infty} \frac{P_{n-3}(x_{-1}x_0 + \frac{P_{n-1}}{P_{n-3}}x_0 + \frac{P_{n-2}}{P_{n-3}})}{P_{n-2}(x_{-1}x_0 + \frac{P_n}{P_{n-2}}x_0 + \frac{P_{n-1}}{P_{n-2}})} \\
&= \frac{(x_{-1}x_0 + \alpha^2 x_0 + \alpha)}{(x_{-1}x_0 + \alpha^2 x_0 + \alpha)} \lim_{n\to\infty} \frac{P_{n-3}}{P_{n-2}} \\
&= \lim_{n\to\infty} \frac{P_{n-3}}{P_{n-2}} \\
&= \frac{1}{\alpha} \\
&= b
\end{aligned}$$

The proof is complete.

### 3.1.4 The Difference Equation (3.4)

In this section, we assume that the Eq.(3.4)

$$x_{n+1} = \frac{-1}{x_n(x_{n-1}+1)}, \quad n = 0, 1, ...,$$

and we investigate the dynamical behavior of solutions of Eq.(3.4)

**Theorem 3.9** *Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of of Eq.(3.4). Then, for $n = 0, 1, 2, ...,$ the form of solutions $\{x_n\}_{n=-1}^{\infty}$ is given by*

$$x_n = \frac{N_{-(n+2)}x_0 - N_{-n}x_{-1}x_0 - N_{-(n+1)}}{-N_{-(n+3)}x_0 + N_{-(n+1)}x_{-1}x_0 + N_{(n+2)}}, \tag{3.12}$$

*where $P_n$ is the nth Narayana number and initial conditions $x_{-1}$, $x_0 \in \mathbb{R} - F_4$, with $F_4$ is the forbidden set of Eq.(3.4) given by*

$$F_4 = \bigcup_{n=-1}^{\infty} \left\{ (x_{-1}, x_0) : -N_{-(n+3)}x_0 + N_{-(n+1)}x_{-1}x_0 + N_{(n+2)} = 0 \right\}.$$

**Proof.** By induction on $k$, we will prove this theorem. For $k$=0, from Eq.(3.4).

$$x_1 = \frac{-1}{x_0(x_{-1}+1)} = \frac{-1}{x_0x_{-1}+x_0} = \frac{-N_{-1}x_{-1}x_0 + N_{-3}x_0 - N_{-2}}{N_{-2}x_{-1}x_0 - N_{-4}x_0 + N_{-3}}$$

Assume that the equality holds for $n = k$, that is

$$x_k = \frac{-N_{-k}x_{-1}x_0 + N_{-(k+2)}x_0 - N_{-(k+1)}}{N_{-(k+1)}x_{-1}x_0 - N_{-(k+3)}x_0 + N_{-(k+2)}}. \tag{3.13}$$

$$x_{k-1} = \frac{-N_{-(k-1)}x_{-1}x_0 + N_{-(k+1)}x_0 - N_{-k}}{N_{-k}x_{-1}x_0 - N_{-(k+2)}x_0 + N_{-(k+1)}}.$$

Thus, we have to prove that it is true for $k+1$. Taking into account (3.4) and (3.12), we have $x_{k+1}$

$$\begin{aligned} x_{k+1} &= \frac{-1}{x_k(x_{k-1}+1)} \\ &= \frac{-1}{\left(\frac{N_{-(k+2)}x_0 - N_{-k}x_{-1}x_0 - N_{-(k+1)}}{-N_{-(k+3)}x_0 + N_{-(k+1)}x_{-1}x_0 + N_{-(k+2)}}\right)\left(\frac{N_{-(k+1)}x_0 - N_{-(k-1)}x_{-1}x_0 - N_{-k}}{-N_{-(k+2)}x_0 + N_{-k}x_{-1}x_0 + N_{-(k+1)}} + 1\right)} \\ &= \frac{-1}{\left(\frac{N_{-(k+2)}x_0 - N_{-k}x_{-1}x_0 - N_{-(k+1)}}{-N_{-(k+3)}x_0 + N_{-(k+1)}x_{-1}x_0 + N_{-(k+2)}}\right)\left(\frac{-N_{-(k+2)}x_{-1}x_0 + N_{-(k+4)}x_0 - N_{-(k+3)}}{-N_{-(k+2)}x_0 + N_{-k}x_{-1}x_0 + N_{-(k+1)}}\right)} \\ &= \frac{N_{-(k+1)}x_{-1}x_0 - N_{-(k+3)}x_0 + N_{-(k+2)}}{-N_{-(k+2)}x_{-1}x_0 + N_{-(k+4)}x_0 - N_{-(k+3)}} \end{aligned}$$

which concludes the proof and the induction.

**Theorem 3.10** *The Eq.(3.4) has unique positive equilibrium point $\overline{x} = c$ and $c$ is repeller..*

**Proof.** The Equilibrium point of Eq.(3.4) is the real root of the equation

$$\overline{x} = \frac{-1}{\overline{x}\left(\overline{x}+1\right)}$$

After simplification, one can get the unique real equilibrium points of Eq.(3.4). The cubic equation

$$\overline{x}^3 + \overline{x}^2 + 1 = 0. \tag{3.14}$$

The unique real equilibrium point of Eq.(3.14) is $\overline{x} = c$.

Let $I$ be an interval of real numbers and

$$f : I^2 \to I$$

be a continuous function with the definition of

$$f(x, y) = \frac{-1}{x(y+1)}$$

Therefore, it follows that

$$\begin{aligned}
\frac{\partial f(x,y)}{\partial x} &= \frac{y+1}{(x(y+1))^2} \\
\frac{\partial f(x,y)}{\partial y} &= \frac{x}{(x(y+1))^2}.
\end{aligned}$$

Then, we get

$$\begin{aligned}
\frac{\partial f(\overline{x},\overline{x})}{\partial x} &= \frac{\overline{x}+1}{(\overline{x}(\overline{x}+1))^2} \\
&= \frac{1}{\overline{x}^2(\overline{x}+1)} \\
&= \frac{1}{\overline{x}^3+\overline{x}^2} \\
&= -1 \\
\frac{\partial f(\overline{x},\overline{x})}{\partial y} &= \frac{\overline{x}}{\overline{x}^2(\overline{x}+1)^2} \\
&= \frac{1}{\overline{x}(\overline{x}+1)^2} \\
&= \frac{1}{\overline{x}\left(\frac{1}{\overline{x}}\right)^4} \\
&= c^3
\end{aligned}$$

and the linearized equation of Eq.(3.4) about $\overline{x} = c$ is

$$y_{n+1} = -y_n + c^3 y_{n-1}$$

or equivalently

$$y_{n+1} + y_n - c^3 y_{n-1} = 0.$$

Therefore, the corresponding characteristic equation is

$$\lambda^2 + \lambda - c^3 = 0.$$

Then, it is clearly seen that the roots of equation are

$$\lambda_1 = \frac{1}{2}\sqrt{4c^3+1} - \frac{1}{2}$$

$$\lambda_2 = -\frac{1}{2}\sqrt{4c^3+1} - \frac{1}{2}$$

The solution is numerically as

$$|\lambda_1| = |\lambda_2| \approx 1.7742 > 1$$

So, from Theorem (2.1), the equilibrium point $\overline{x} = c$ is repeller point and the proof is complete.

# CHAPTER 4

## ON THE SOLUTIONS OF DIFFERENCE EQUATIONS ASSOCIATED WITH THIRD-ORDER PELL NUMBERS

In this chapter, we obtained the solutions of following rational difference equations via Pell numbers.

$$x_{n+1} = \frac{1}{x_n\left(x_{n-1}-2\right)-1}, \tag{4.1}$$

$$x_{n+1} = \frac{-1}{x_n\left(x_{n-1}+2\right)-1}, \tag{4.2}$$

We find a relation between with the exact solution of Eq.(4.1) and Eq.(4.2) the third order Pell sequence. The Pell sequence $\{P_n\}$ that is defined by

$$P_{n+2} = 2P_{n+1} + P_n, \quad n \geq 0$$

in which $P_0 = 0$ and $P_1 = 1$. The Pell sequence terms of second order are $0, 1, 2, 5, 12,$ $29, 70, 169, 408, 985, 2378, 5741, 13860, 33461, \cdots$.
A third order Pell sequence $\{P_n\}_{n\,\geq\,0}$ is defined by the third order recurrence relation

$$P_n = 2P_{n-1} + P_{n-2} + P_{n-3} \tag{4.3}$$

with initial conditions are $P_0 = 0,\ P_1 = 1,\ P_2 = 2.$
Also, Pell sequence can be extended the negative subscripts as

$$P_{-n} = -P_{-(n-1)} - 2P_{-(n-2)} + P_{-(n-3)} \tag{4.4}$$

We present the first few values of the third-order Pell numbers with positive and negative subscripts in table.

**Table 4.1** The first few values of the third-order Pell Numbers.

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $P_n$ | 0 | 1 | 2 | 5 | 13 | 33 | 84 | 214 | 545 | 1388 | 3535 | 9003 |
| $P_{-n}$ | 0 | 0 | 1 | $-1$ | $-1$ | 4 | $-3$ | $-6$ | 16 | $-7$ | $-31$ | 61 |

## 4.1 MAIN RESULTS

The studies dealt with the relation between closed form of the solutions of nonlinear difference equations and linear sequences such as Fibonacci, Lucas, Pell, Jacobsthal, Padovan, Perrin numbers have received great attention by researchers, for example, see [33], [16], [17], [18], [19], [34], [41], [45].
In this section, we studied the general solution in explicit form and analyzed asymptotic behavior of Eq.(4.1), Eq.(4.2). Furthermore, we find a interesting relation between the exact solution of these Equations and the third order Pell sequence.

### 4.1.1 The Difference Equation (4.1)

In this subsection, we consider the Eq.(4.1), that is

$$x_{n+1} = \frac{1}{x_n\left(x_{n-1}-2\right)-1}, \quad n = 0, 1, ...,$$

and analyze the dynamical behavior of solutions Eq.(4.1).

**Theorem 4.1** *Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of the Eq.(4.1). Then, for $n = 0, 1, 2, ...$, the form of solutions $\{x_n\}_{n=-1}^{\infty}$ is given by*

$$x_n = \frac{-P_{-n}x_{-1}x_0 - (P_{-(n+1)} + P_{-(n+2)})x_0 - P_{-(n+1)}}{-P_{-(n+1)}x_{-1}x_0 - (P_{-(n+2)} + P_{-(n+3)})x_0 - P_{-(n+2)}}, \tag{4.5}$$

*where $P_n$ is the nth third-order Pell number and the initial conditions $x_{-1}$, $x_0 \in \mathbb{R} - F_1$, with $F_1$ is the forbidden set of Eq.(4.1) given by*

$$F_1 = \bigcup_{n=-1}^{\infty} \left\{(x_{-1}, x_0) : -P_{-(n+1)}x_{-1}x_0 - (P_{-(n+2)} + P_{-(n+3)})x_0 - P_{-(n+2)} = 0\right\}.$$

**Proof. (First proof)** We will prove this theorem using induction on $k$, we will demonstrate this theorem. In the case where $k = 0$, from Eq.(4.1),

$$x_1 = -\frac{1}{2x_0 - x_0x_{-1} + 1} = \frac{-P_{-1}x_{-1}x_0 - (P_{-2} + P_{-3})x_0 - P_{-2}}{-P_{-2}x_{-1}x_0 - (P_{-3} + P_{-4})x_0 - P_{-3}},$$

Assume that the equality holds for $n = k$, that is

$$x_k = \frac{-P_{-k}x_{-1}x_0 - (P_{-(k+1)} + P_{-(k+2)})x_0 - P_{-(k+1)}}{-P_{-(k+1)}x_{-1}x_0 - (P_{-(k+2)} + P_{-(k+3)})x_0 - P_{-(k+2)}},$$

$$x_{k-1} = \frac{-P_{-(k-1)}x_{-1}x_0 - (P_{-k} + P_{-(k+1)})x_0 - P_{-k}}{-P_{-k}x_{-1}x_0 - (P_{-(k+1)} + P_{-(k+2)})x_0 - P_{-(k+1)}}.$$

Therefore, we must show that it holds true for $k+1$. Considering equations (4.1) and (4.5), we get $x_{k+1}$ as

$$\begin{aligned}
x_{k+1} &= \frac{1}{x_k\left(x_{k-1}-2\right)-1} \\
&= \frac{1}{\frac{-(P_{-(k+1)}+P_{-(k+2)})x_0-P_{-k}x_{-1}x_0-P_{-(k+1)}}{-P_{-(k+1)}x_{-1}x_0-(P_{-(k+2)}+P_{-(k+3)})x_0-P_{-(k+2)}}\left(\frac{-(P_{-k}+P_{-(k+1)})x_0-P_{-(k-1)}x_{-1}x_0-P_{-k}}{-P_{-k}x_{-1}x_0-(P_{-(k+1)}+P_{-(k+2)})x_0-P_{-(k+1)}}-2\right)-1} \\
&= -\frac{x_0P_{-k-2}+x_0P_{-k-3}+P_{-k-2}+x_0P_{-k-1}x_{-1}}{x_0P_{-k-1}-P_{-k}-x_0P_{-k}+3x_0P_{-k-2}+x_0P_{-k-3}+2P_{-k-1}+P_{-k-2}+2x_0P_{-k}x_{-1}+x_0P_{-k-1}x_{-1}-x_0P_{1-k}x_{-1}} \\
&= \frac{(P_{-(k+2)}+P_{-(k+3)})x_0+P_{-(k+1)}x_{-1}x_0+P_{-(k+2)}}{(P_{-(k+3)}+P_{-(k+4)})x_0+P_{-(k+2)}x_{-1}x_0+P_{-(k+3)}}
\end{aligned}$$

which completes the proof and the induction.

**Proof.** **(Second proof)** We will show the proof of this situation in a different way, so let's take $n = 0, 1, 2, ....$

$$\begin{aligned}
n &= 0 \quad \Longrightarrow \quad x_1 = -\frac{1}{2x_0 - x_0x_{-1} + 1}, \\
n &= 1 \quad \Longrightarrow \quad x_2 = \frac{2x_0 - x_0x_{-1} + 1}{x_0x_{-1} - 3x_0 + 1}, \\
n &= 2 \quad \Longrightarrow \quad x_3 = \frac{-x_0x_{-1} + 3x_0 - 1}{x_0 - x_0x_{-1} + 4}, \\
n &= 3 \quad \Longrightarrow \quad x_4 = -\frac{x_0 - x_0x_{-1} + 4}{9x_0 - 4x_0x_{-1} + 3}, \\
n &= 4 \quad \Longrightarrow \quad x_5 = \frac{9x_0 - 4x_0x_{-1} + 3}{3x_0x_{-1} - 10x_0 + 6}, \\
n &= 5 \quad \Longrightarrow \quad x_6 = -\frac{3x_0x_{-1} - 10x_0 + 6}{9x_0 - 6x_0x_{-1} + 16}, \\
n &= 6 \quad \Longrightarrow \quad x_7 = -\frac{9x_0 - 6x_0x_{-1} + 16}{38x_0 - 16x_0x_{-1} + 7}, \\
n &= 7 \quad \Longrightarrow \quad x_8 = \frac{38x_0 - 16x_0x_{-1} + 7}{7x_0x_{-1} - 30x_0 + 31}, \\
n &= 8 \quad \Longrightarrow \quad x_9 = -\frac{7x_0x_{-1} - 30x_0 + 31}{55x_0 - 31x_0x_{-1} + 61}, \\
n &= 9 \quad \Longrightarrow \quad x_{10} = -\frac{55x_0 - 31x_0x_{-1} + 61}{153x_0 - 61x_0x_{-1} + 6}, \\
& \qquad\qquad\qquad \vdots
\end{aligned}$$

According to the third order recurrence relation (4.4), the proof is completed.

**Theorem 4.2** *The Eq.(4.1) has unique positive equilibrium point $\overline{x} = \alpha$ and $\alpha$ is repeller.*

**Proof.** The equilibrium points of Eq.(4.1) is the real root of following equation

$$\overline{x} = \frac{1}{\overline{x}\left(\overline{x}-2\right)-1}$$

After simplification, one can get the of Eq.(4.1). The cubic equation is

$$\overline{x}^3 - 2\overline{x}^2 - x - 1 = 0. \tag{4.6}$$

Therefore, the unique real equilibrium point of Eq.(4.6) is $\overline{x} = \alpha$.

Let $f : I^2 \to I$ be a continuous function defined by, and let $I$ be an interval of real numbers

$$f\left(x, y\right) = \frac{1}{x\left(y-2\right)-1}$$

Consequently, it follows that partial derivatives

$$\begin{aligned}
\frac{\partial f\left(x, y\right)}{\partial x} &= \frac{-(y-2)}{\left(x\left(y-2\right)-1\right)^2}, \\
\frac{\partial f\left(x, y\right)}{\partial y} &= \frac{-x}{\left(x\left(y-2\right)-1\right)^2}.
\end{aligned}$$

Then, we get

$$\begin{aligned}
\frac{\partial f\left(\overline{x}, \overline{x}\right)}{\partial x} &= \frac{-(y-2)}{\left(x\left(y-2\right)-1\right)^2} \\
&= \frac{-(\overline{x}-2)}{\left(\overline{x}\left(\overline{x}-2\right)-1\right)^2} \\
&= -\overline{x}-1 \\
&= -\alpha - 1 \\
\frac{\partial f\left(\overline{x}, \overline{x}\right)}{\partial y} &= \frac{-x}{\left(x\left(y-2\right)-1\right)^2} \\
&= \frac{-\overline{x}}{\left(\overline{x}\left(\overline{x}-2\right)-1\right)^2} \\
&= -\alpha^3
\end{aligned}$$

and the linearized equation of Eq.(4.1) about $\overline{x} = \alpha$ is

$$y_{n+1} = (-\alpha - 1)y_n + \left(-\alpha^3\right) y_{n-1}$$

its equal to

$$y_{n+1} + (\alpha + 1)y_n + \alpha^3 y_{n-1} = 0.$$

As a result, the associated characteristic equation is

$\lambda^2 + (\alpha + 1)\lambda + \alpha^3 = 0.$

The roots of the characteristic equations

$$\lambda_1 = -\frac{1}{2}\alpha + \frac{1}{2}\sqrt{2\alpha + \alpha^2 - 4\alpha^3 + 1} - \frac{1}{2} \tag{4.7}$$

$$\lambda_2 = -\frac{1}{2}\alpha - \frac{1}{2}\sqrt{2\alpha + \alpha^2 - 4\alpha^3 + 1} - \frac{1}{2}$$

And numerically

$|\lambda_1| = |\lambda_2| \approx 4.0644 > 1.$

According to the Theorem (2.1), the equilibrium point $\overline{x} = \alpha$ is repeller point. The proof of the theorem is completed.

### 4.1.2 The Difference Equation (4.2)

In this subsection, we consider the Eq.(4.2), that is

$$x_{n+1} = \frac{-1}{x_n\left(x_{n-1} + 2\right) - 1},$$

and analyze the dynamical behavior of solutions Eq.(4.2).

**Theorem 4.3** *Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of the Eq.(4.2). Then, for $n = 0, 1, 2, ...,$ the form of solutions $\{x_n\}_{n=-1}^{\infty}$ is given by*

$$x_n = \frac{-P_{-n}x_{-1}x_0 + (P_{-(n+1)} + P_{-(n+2)})x_0 - P_{-(n+1)}}{P_{-(n+1)}x_{-1}x_0 - (P_{-(n+2)} + P_{-(n+3)})x_0 + P_{-(n+2)}} \tag{4.8}$$

*where $P_n$ is the nth third-order Pell number and the initial conditions $x_{-1}$, $x_0 \in \mathbb{R} - F_2$, with $F$ is the forbidden set of Eq.(4.2) given by*

$$F_2 = \bigcup_{n=-1}^{\infty} \left\{(x_{-1}, x_0) : P_{-(n+1)}x_{-1}x_0 - (P_{-(n+2)} + P_{-(n+3)})x_0 + P_{-(n+2)} = 0\right\}.$$

**Proof.** This theorem will be demonstrated by induction on $k$. For $k$=0, from Eq.(4.2),

$$x_1 = -\frac{1}{2x_0 + x_0 x_{-1} - 1} = \frac{-P_{-n}x_{-1}x_0 + (P_{-(n+1)} + P_{-(n+2)})x_0 - P_{-(n+1)}}{P_{-(n+1)}x_{-1}x_0 - (P_{-(n+2)} + P_{-(n+3)})x_0 + P_{-(n+2)}},$$

Let us assume that the equality is true for $n = k$, that is

$$x_k = \frac{-P_{-k}x_{-1}x_0 + (P_{-(k+1)} + P_{-(k+2)})x_0 - P_{-(k+1)}}{P_{-(k+1)}x_{-1}x_0 - (P_{-(k+2)} + P_{-(k+3)})x_0 + P_{-(k+2)}},$$

$$x_{k-1} = \frac{-P_{-(k-1)}x_{-1}x_0 + (P_{-k} + P_{-(k+1)})x_0 - P_{-k}}{P_{-k}x_{-1}x_0 - (P_{-(k+1)} + P_{-(k+2)})x_0 + P_{-(k+1)}},$$

We must therefore demonstrate that it is true for $k+1$. Considering that (4.2) and (4.8), we have $x_{k+1}$ as follows

$$\begin{aligned}
x_{k+1} &= \frac{-1}{x_k\left(x_{k-1}+2\right)-1} \\
&= \frac{-1}{\frac{(P_{-(k+1)}+P_{-(k+2)})x_0-P_{-k}x_{-1}x_0-P_{-(k+1)}}{P_{-(k+1)}x_{-1}x_0-(P_{-(k+2)}+P_{-(k+3)})x_0+P_{-(k+2)}}\left(\frac{(P_{-k}+P_{-(k+1)})x_0-P_{-(k-1)}x_{-1}x_0-P_{-k}}{P_{-k}x_{-1}x_0-(P_{-(k+1)}+P_{-(k+2)})x_0+P_{-(k+1)}}+2\right)-1} \\
&= -\frac{x_0P_{-k-2}+x_0P_{-k-3}-P_{-k-2}-x_0P_{-k-1}x_{-1}}{x_0P_{-k}-P_{-k}-x_0P_{-k-1}-3x_0P_{-k-2}-x_0P_{-k-3}+2P_{-k-1}+P_{-k-2}+2x_0P_{-k}x_{-1}+x_0P_{-k-1}x_{-1}-x_0P_{1-k}x_{-1}} \\
&= \frac{(P_{-(k+2)}+P_{-(k+3)})x_0-P_{-(k+1)}x_{-1}x_0-P_{-(k+2)}}{(P_{-(k+3)}+P_{-(k+4)})x_0-P_{-(k+2)}x_{-1}x_0-P_{-(k+3)}}
\end{aligned}$$

the proof is completed.

**Theorem 4.4** *The Eq.(4.2) has unique positive equilibrium point $\overline{x} = d$ and $d$ is repeller.*

**Proof.** The Equilibrium point of Eq.(4.2) is satisfy the equation

$$\overline{x} = \frac{-1}{\overline{x}\left(\overline{x}+2\right)-1}$$

After simplification, the cubic equation of Eq.(4.2)

$$\overline{x}^3 + 2\overline{x}^2 - x + 1 = 0. \tag{4.9}$$

Therefore, the unique real equilibrium point of Eq.(4.9) is $\overline{x} = d$.

Let $f$ be a function and $I$ be an interval of real numbers, $f : I^2 \to I$ is a function that is continuous and is defined by

$$f\left(x, y\right) = \frac{-1}{x\left(y+2\right)-1}.$$

Consequently, it follows that

$$\begin{aligned}
\frac{\partial f\left(x,y\right)}{\partial x} &= \frac{\left(y+2\right)}{\left(x\left(y+2\right)-1\right)^2}, \\
\frac{\partial f\left(x,y\right)}{\partial y} &= \frac{x}{\left(x\left(y+2\right)-1\right)^2}.
\end{aligned}$$

Then, we get

$$\begin{aligned}
\frac{\partial f\left(\overline{x},\overline{x}\right)}{\partial x} &= \frac{(y+2)}{\left(x\left(y+2\right)-1\right)^2}\\
&= \frac{(\overline{x}+2)}{\left(\overline{x}\left(\overline{x}+2\right)-1\right)^2}\\
&= d-1\\
\frac{\partial f\left(\overline{x},\overline{x}\right)}{\partial y} &= \frac{x}{\left(x\left(y+2\right)-1\right)^2}\\
&= \frac{\overline{x}}{\left(\overline{x}\left(\overline{x}+2\right)-1\right)^2}\\
&= d^3
\end{aligned}$$

and the linearized equation of Eq.(3.1) about $\overline{x}=d$ is

$$y_{n+1}=(\alpha-1)y_n+d^3y_{n-1}$$

its equal to

$$y_{n+1}-(d-1)y_n-d^3y_{n-1}=0.$$

Then, it is clearly seen that the characteristic equation

$$x^2-(d-1)x-d^3=0.$$

The characteristic equation's roots

$$\lambda_1=\frac{1}{2}d+\frac{1}{2}\sqrt{-2d+d^2+4d^3+1}-\frac{1}{2}$$

$$\lambda_2=\frac{1}{2}d-\frac{1}{2}\sqrt{-2d+d^2+4d^3+1}-\frac{1}{2}$$

Solution is numerically

$$|\lambda_1|=|\lambda_2|\approx 4.0644>1$$

According to the Theorem (2.1), the equilibrium point $\overline{x}=d$ is repeller point. The proof of the theorem is completed.

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