# COMPLETELY SIMPLE SEMIGROUPS

by

MEHMET KALKAN

THESIS SUBMITTED TO

THE GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

OF

THE ABANT İZZET BAYSAL UNIVERSITY

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE

DEGREE

OF

MASTER OF SCIENCE

IN

THE DEPARTMENT OF MATHEMATICS

DECEMBER 2007

Approval of the Graduate School of Natural and Applied Sciences.

Professor Dr. Nihat ÇELEBİ
Director

I certify that this thesis satisfies all the requirements as a thesis for the degree of Master of Science.

Professor Dr. Zafer ERCAN
Head of Department

This is to certify that we have read this thesis and that in our opinion it is fully adequate, in scope and quality as a thesis for the degree of Master of Science.

Assistant Prof. Dr. Ömür Umut
Supervisor

Examining Committee Members

Professor Dr. Cem Tezer

Assistant Prof. Dr. Ömür Umut

Assistant Prof. Dr. Tahire Özen

# ABSTRACT

COMPLETELY SIMPLE SEMIGROUPS

Kalkan, Mehmet

Master, Department of Mathematics

Supervisor: Assistant Prof. Dr. Ömür Umut

December 2007, 42 pages

In this thesis, we study on Molaei's generalized groups, completely simple semigroups and the equivalence theorems involving them. For this aim, first we present basic definitions such as regularity and Green's equivalence. Next, we deal with Molaei's generalized groups. In 1999, Molaei introduced generalized groups as a class of algebras of interest in physics, and proved some results about them. Then we consider semigroups with no proper ideals known as simple semigroups. A simple semigroup is called completely simple semigroup if it satisfies the conditions $min_L$ and $min_R$ that is to say if every nonempty set either of L-classes or of R-classes possesses a minimal member. Finally, we introduce Rees matrix semigroup to prove that the structure of the Molaei's generalized groups is equivalent to the notion of completely simple semigroups.

Keywords: *Regularity, Green's equivalence relations, Molaei's generalized groups, simple semigroups, completely simple semigroups, Rees matrix semigroups.*

# ÖZET

TAM BASIT YARIGRUPLAR

Kalkan, Mehmet

Master Tezi, Matematik Bölümü

Tez Danışman: Yrd. Doç. Dr. Ömür Umut

Aralık 2007, 42 sayfa

Bu tezde, Molaei'nin genelleştirilmiş gruplarını, tam basit yarı grupları ve bunları içeren denklik teoremlerini çalışacağız. Bunun için, önce düzenlilik ve Green denklik bağıntıları gibi temel tanımları sunacağız. Bunun akabinde Molaei'nin genelleştirilmiş gruplarına değineceğiz. 1999 yılında Molaei, genelleştirilmiş grupları bir cebir dalı olarak ve fizikle ilişkilendirerek tanımlamış ve bu gruplarla ilgili bazı sonuçları ispatlamıştır. Daha sonra basit yarıgruplar olarak bilinen ve kendisi dışında alt ideali olmayan yarıgrupları ele alacağız. Bir basit yarıgrup $min_L$ ve $min_R$ koşllarını sağlıyorsa, yani boş olamayan her kümenin L-sınıfları veya R-sınıfları minimal elemanı içerirse, tam basit yarıgrup olarak adlandırılır. Son olarak, bir Molaei genelleştirilmiş grup yapısının bir basit yarı grup kavramına denk olduğunu göstermek için Rees matris yarıgruplarını tanımlayacağız.



Anahtar Kelimeler: *Düzenlilik, Green denklik bağıntıları, Molaei'nin genelleşmiş grupları, basit yarıgrouplar, tam basit yarıgruplar ve Rees matris grupları.*

*To My Wife Cennet and My Daughter Zeynep*

## ACKNOWLEDGEMENTS

I would like to thank to my supervisor Asst. Prof. Dr. Ömür Umut for her patience, tolerance and guidance during the development of the thesis. I have learnt a lot from her. Her suggestions and evaluations have helped me make this thesis better. I would also like to thank to the other members of my committee, Professor Dr. Cem Tezer and Asst. Prof. Dr. Tahire Özen. And I would also like to thank to Assoc. Prof. Dr. Mohammad Reza Molaei and Research Asst. Dilek Buyruk for their patience and help.

As a last word, I would be indebted to my wife for her existence and valuable support. You are so precious to me.

# TABLE OF CONTENTS

# CHAPTER 1

# INTRODUCTION

Algebra is a part of mathematics which studies abstract sets with certain operations defined on them. Semigroup theory is a thriving field in modern algebra, though perhaps not a very well-known one.

The term semigroup was first coined in a French group theory textbook [1] in 1904 (though with a more stringent definition than the modern one), before being introduced to the English-speaking mathematical world by Leonard Dickson the following year [2]. There then followed three decades, during which the only semigroup theory being done was that done in near-obscurity (at least from the Western perspective) by a Russian mathematician, Anton Kazimirovich Suschkewitsch, working in the Ukraine. Suschkewitsch was essentially doing semigroup theory before the rest of the world knew that there was such a thing, thus many of this results were rediscovered by later researchers who were unaware of his achievements.

The first tentative Western steps toward semigroup theory were taken during the 1930s ([1-4]), and after the publication of a series of highly influential papers in the early 1940s ([5-7]), the subject exploded.

D.Rees [5], using the Wedderburn Theorem as a template produced a description of completely 0-simple semigroup. Rees defines a completely simple semigroup $S$ to be a semigroup satisfying [8]

(i) $S$ is simple;

(ii) Every $x$ in $S$ lies in at least one set $fSe$, where $e, f$ are idempotents;

(iii) There is a primitive idempotent $f$ under every non primitive idempotent $e$ of $S$.

Let $L, M$ be two sets, let $G$ be a group with zero, and denote by $(a)_{lm}$ the $(L, M)$ matrix over $G$ with $(l, m)$ entry a and all other entries 0. Let

$$S = \{(a)_{lm} : a \in G,\ a \neq 0, l \in L, m \in M\}$$

together with the $(L, M)$ matrix over $G$ whose entries are all 0. Let $P = (p_{ml})$ be an $(M, L)$ matrix over $G$ having at least one non-zero entry in each row and each column. Let multiplication in $S$ be defined by

$$(a)_{ij} \circ (b)_{lm} = (a)_{ij} P (b)_{lm} = (a p_{jl} b)_{im}$$

Rees then show that S, as defined, is a completely simple semigroup, and that every completely simple semigroup is isomorphic to one defined in this manner.

A. H. Clifford [6], in a paper submitted on 17 December 1940,defined what we now call a completely regular semigroup as a semigroup satisfying:
for each $a \in S$, $\exists\ e, a' \in S$ such that $ea = ae = a$ and $aa' = a'a = e$.

The D. Rees' and the A. H. Clifford's papers are notable for containing substantial theorems. Dubreil's paper is different, but in its way even more remarkable.

In the 1960s, A. H. Clifford and G. B. Preston wrote a two-volume work on the algebraic theory of semigroups which played a fundamental role in lying the basic of the theory.

In 1951, J. A. Green [9] introduced five equivalence relations on semigroup $S$.
$\mathcal{L} = \{(a, b) \in S \times S : (\exists x, y \in S^1)\ \ xa = b, yb = a\}$,
$\mathcal{R} = \{(a, b) \in S \times S : (\exists u, v \in S^1)\ \ au = b, bv = a\}$,
$\mathcal{H} = \mathcal{L} \cap \mathcal{R}$,
$\mathcal{D} = \{(a, b) \in S \times S : (\exists c \in S)\ (a, c) \in \mathcal{L}, (c, b) \in \mathcal{R}\}$,
$\mathcal{J} = \{(a, b) \in S \times S : (\exists x, y, u, v \in S^1) xay = b, ubv = a\}$.
where 1 is an identity element and

$$S^1 = \begin{cases} S & \text{if S has an identity} \\ S \cup \{1\} & \text{otherwise} \end{cases}$$

These five relations, which all reduce to the universal relation if S is a group, have proved an immensely powerful tool in examining semigroups, especially regular semigroups.

V. V. Vagner [10] (1952) and G. B. Preston [11] (1954) independently came up with the idea of an *inverse semigroup*. V. V. Vagner called them *generalized groups*, which was fortunate, for it is probably better to regard an inverse semigroup as a generalization of a group rather than a specialization of a semigroup.

M.R.Molaei [12] (1999) introduced Molaei's generalized groups as the algebraic approach to the Unified Theory. He also proved some results about these groups. The definition of Molaei's generalized groups has different meaning from the definition of Vagner.

# CHAPTER 2

# BASIC CONCEPTS

Let $S$ be a set and $\star : S \times S \to S$ a binary operation that maps each ordered pair $(x, y)$ of $S$ to an element $\star(x, y)$ of $S$. The pair $(S, \star)$ is called a groupoid. The mapping $\star$ is called a product of $(S, \star)$. We shall mostly write simply $xy$ instead of $\star(x, y)$. If we want to emphasize the place of the operation, then we often write $x \cdot y$. The element $xy(= \star(x, y))$ is the product of $x$ and $y$ in $S$.

**Definition 2.1** *A non-empty set $S$ with an associative binary operation is called a semigroup.*

**Example 2.2** *[13] $(N, \cdot)$ is a semigroup for the usual multiplication of integers, $n \cdot m$. Also $(N, +)$ is a semigroup, when $+$ is the ordinary addition of integers.*

**Example 2.3** *[13] Define $(N, *)$ by $n * m = max\{n, m\}$. $(N, *)$ is a semigroup, since*

$$\begin{aligned} n * (m * k) &= max\{n, max\{m, k\}\} = max\{n, m, k\} \\ &= max\{max\{n, m\}, k\} = (n * m) * k. \end{aligned}$$

**Lemma 2.4** *The operation $*$ in Example 2.3 is well-defined.*

**Proof.** Let $(m, n) = (k, t)$, without loss of generality, we may assume that $m \geq n$ that implies that $k = m \geq n$ and $n = t$ implies $k \geq t$.
Hence, $m*n = max\{m, n\} = m = k = max\{k, t\} = k*t$. $\square$

**Example 2.5** *[13] Consider the upper triangular integer matrices*

$$S = \{ \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} : n \geqslant 1\}.$$

*For all $m, n, k \geq 1$;*

$$\begin{aligned}
\left( \begin{bmatrix} 1 & m \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} &= \begin{bmatrix} 1 & m+n \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & (m+n)+k \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & m+(n+k) \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & m \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & n+k \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & m \\ 0 & 1 \end{bmatrix} \left( \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} \right).
\end{aligned}$$

*Then $S$ is a semigroup with the well-defined usual matrix product.*

**Example 2.6** *[13] Let $T_X$ be the set of all functions $\phi : X \to X$. Then $(T_X, \circ)$ is a semigroup, where $\circ$ is the composition of functions: $(\phi \circ \varphi)(x) = \phi(\varphi(x))$ for all $x \in X$.*

**Definition 2.7** *[14] A nonempty subset $T$ of a semigroup $S$ is a subsemigroup of $S$ if it is closed under the operation of $S$; i.e., if $a, b \in T$ then $ab \in T$.*

**Definition 2.8** *[14] A nonempty subset $T$ of a semigroup $S$ is a left ideal of $S$ if $s \in S, t \in T$ imply $st \in T$; $T$ is a right ideal if $s \in S, t \in T$ imply $ts \in T$; $T$ is an ideal if it is both a left and a right ideal. An ideal of $S$ different from $S$ is proper ideal.*

**Definition 2.9** *[14] The intersection of all ideals of a semigroup $S$, if nonempty, is the kernel of $S$.*

**Lemma 2.10** *[14] If $I$ is a simple ideal of a semigroup $S$, then $I$ is the kernel of $S$.*

**Proof.** If $J$ is an ideal of $S$, then $I \cap J \supseteq IJ$ and hence $I \cap J$ is nonempty and is thus an ideal of $S$. But $I \cap J$ is also an ideal of $I$. Simplicity of $I$ implies $I \cap J = I$ thus $I \subseteq J$. $\square$

**Definition 2.11** *[14] A subsemigroup $F$ of a semigroup $S$ is a filter of $S$ if for all $x, y \in S$, $\quad xy \in F \Rightarrow x, y \in F$.*

For any element $x$ of a semigroup $S$, let $N(x)$ (is the $\mathfrak{N}$-class of $S$ containing $x$) denote the least filter of $S$ containing $x$, and let

$$N_x = \{y \in S : N(x) = N(y)\}.$$

**Definition 2.12** *[13] Let $S$ be a semigroup. An element $x \in S$ is a left identity of $S$, if*

$$\forall y \in S : xy = y.$$

*Similarly, $x$ is a right identity of $S$, if*

$$\forall y \in S : yx = y.$$

*If $x$ is both a left and a right identity of $S$, then $x$ is called an identity of $S$.*

If a semigroup $S$ has a left identity $e$ and a right identity $f$ then $e = ef = f$, and this element is an identity.

**Example 2.13** *[15] The one element semigroup $S = \{e\}$, with $ee = e$ is called the trivial semigroup.*

**Definition 2.14** *[13] A semigroup is called a monoid, if it has an identity.*

**Lemma 2.15** *[13] A semigroup $S$ can have at most one identity. In fact, if $S$ has a left identity $x$ and a right identity $y$, then $x = y$ is an identity and there is no other identity.*

**Proof.** By the definitions, $y = xy = x$. $\square$

The identity [16] of a monoid $S$ is usually denoted by 1. But as an example we can see that the identity of the monoid $(N, +)$ is 0. For a semigroup $S$ we define a monoid $S^1$ by adjoining an identity to $S$, if $S$ does not have one:

$$S^1 = \begin{cases} S & \text{if S has an identity} \\ S \cup \{1\} & \text{otherwise} \end{cases}$$

where 1 is an identity element.

A monoid $G$ is a group, if every $x \in G$ has a (group) inverse

$$x^{-1} \in G : xx^{-1} = 1 = x^{-1}x.$$

**Definition 2.16** *Let $S$ and $T$ are semigroups. A map $\psi : S \to T$, is called a homomorphism if $\forall x, y \in S$;*

$$\psi(xy) = \psi(x)\psi(y).$$

*A one to one and onto homomorphism is called an isomorphism.*

**Definition 2.17** *[13,16] An element $x \in S$ is a left zero, if*

$$\forall y \in S : xy = x.$$

*Similarly, $x$ is a right zero, if*

$$\forall y \in S : yx = x.$$

If $S$ does not have a zero it is easy to adjoin an extra element 0. We define

$$0s = s0 = 00 = 0 \qquad \forall s \in S,$$

and we define

$$S^0 = \begin{cases} S & \text{if S has a zero element} \\ S \cup \{0\} & \text{otherwise} \end{cases}$$

and refer to $S^0$ as the semigroup obtained from $S$ by adjoining a zero element if necessary.

**Definition 2.18** *An element $e \in S$ is an idempotent, if $e^2 = e$. The set of all idempotents of $S$ is denoted by $E = E_S$.*

For any semigroup $S$, $E_S$ with the binary relation defined by

$$e \leq f \Leftrightarrow \ e = ef = fe.$$

**Lemma 2.19** *[14] For any semigroup $S$, $E_S$ is a partial ordered set.*

**Proof.** $E_S$ is reflexive; since $a \leq a$ for all $a \in E_S$,
$E_S$ is antisymmetric; since if $a \leq b \Rightarrow a = ab = ba$ and $b \leq a \Rightarrow b = ba = ab$. Hence $a = b, \quad \forall a, b \in E_S$,
$E_S$ is transitive; let $a \leq b$ and $b \leq c$ then $a \leq b \leq c \Rightarrow a \leq c \ \ \forall a, b, c \in E_S$.
Therefore, $E_S$ is a partial ordered set. $\square$

**Example 2.20** *[13] The matrix semigroup*

$$\mathbb{Z}^{2\times 2} = \{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{Z}\}.$$

*The identity matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is the identity of $\mathbb{Z}^{2\times 2}$ and the zero matrix $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is its zero.*
*This monoid has quite many idempotents. As an example, $\begin{bmatrix} -1 & -1 \\ 2 & 2 \end{bmatrix}$ is an idempotent.*

**Lemma 2.21** *[13] Let $e \in E_S$ be an idempotent. If $x\mathcal{L}e$, then $xe = x$. If $x\mathcal{R}e$, then $ex = x$.*

**Proof.** If $x\mathcal{L}e \ \Rightarrow (x, e) \in \mathcal{L}$, that is, $x = se$ for some $s \in S^1$. Therefore, since $e = ee, x = se \cdot e = xe$. Similarly;

If $y\mathcal{R}e \ \Rightarrow (y,e) \in \mathcal{R}$, that is, $y = et$ for some $t \in S^1$.

Therefore, since $e = ee, y = e \cdot et = ey$. $\square$

**Definition 2.22** *[13] A semigroup $S$ is left cancellative, if*

$$zx = zy \Rightarrow x = y,$$

*and $S$ is right cancellative,if*

$$xz = yz \Rightarrow x = y.$$

*If $S$ is both left and right cancellative, then it is cancellative.*

**Example 2.23** *[13] The matrix semigroup $\mathbb{Z}^{2\times 2}$ is not a cancellative semigroup.*

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}.$$

*But, if we take all matrices $M \in \mathbb{Z}^{2\times 2}$ with $det(M) = 1$, then this semigroup is cancellative.*

**Definition 2.24** *[16] An element $x$ of a semigroup $S$ is said to be regular if there is $y \in S$ such that $xyx = x$. If every element of $S$ is regular, then $S$ is said to be a regular semigroup.*

The notion of regularity is probably the most important bridge between semigroups and groups.

In a regular semigroup $S$ we have particularly useful way of looking at the equivalences $\mathcal{L}$ and $\mathcal{R}$. If $S$ is regular semigroup then $a = axa \in aS$, and similarly $a \in Sa$, $a \in SaS$. Hence in describing the Green equivalences for a regular semigroup we can drop all references to $S^1$, and assert simply that

$$a\mathcal{L}b \Leftrightarrow Sa = Sb,$$

$$a\mathcal{R}b \Leftrightarrow aS = bS,$$

$$a\mathcal{J}b \Leftrightarrow SaS = SbS.$$

**Example 2.25** *[13] All groups are regular: $x = xx^{-1}x$, where $x^{-1}$ is the group inverse of $x$.*

**Example 2.26** *[13] If $e \in E_S$ is an idempotent, then $e = eee$, and hence all idempotents of a semigroup are regular elements of $S$.*

**Definition 2.27** *[13] A semigroup $S$ is called a completely regular semigroup if there is a unary operation $a \to a^{-1}$ on $S$ such that:*

$$(a^{-1})^{-1} = a, \quad aa^{-1}a = a, \quad aa^{-1} = a^{-1}a.$$

**Example 2.28** *[11,16] Let $S = \{1, 2\}$. Then $S$ with the binary operation: $2 \cdot 2 = 2$, $2 \cdot 1 = 1 \cdot 2 = 2$, $1 \cdot 1 = 1$ is a semigroup. As it seen, the operation $\cdot$ is well-defined. Now if we define $2^{-1} = 2$, $1^{-1} = 1$, then $S$ is a completely regular semigroup.*

We say that $y \in S$ is an inverse element of $x \in S$, if $x = xyx$ and $y = yxy$ . Note that an inverse element of $x$, if such an element exists, need not be unique.

**Lemma 2.29** *[13] Each regular element $x \in S$ has an inverse element.*

**Proof.** If $x \in S$ is regular, then for some $y \in S, x = xyx$. Now, $yxy = yxy \cdot x \cdot yxy$, and so $yxy$ is also regular. Also, $x = x \cdot yxy \cdot x$ and consequently $yxy$ is an inverse element of $x$. $\square$

There is another definition in semigroup theory which is called "Generalized group" or "Inverse semigroup". This notion presented by Vagner [10]. The same structure with the name Inverse semigroups was presented by Preston [18] independently.

**Definition 2.30** *[10,17] A semigroup $S$ is called an inverse semigroup, if each $x \in S$ has a unique inverse element $x^{-1}$.*

There are other kind of semigroups which are called semigroups admitting relative inverses, and were studied first by Clifford [6].

**Definition 2.31** *[6] A semigroup $S$ is called a semigroup admitting relative inverse if for each $a \in S$ there exists $e \in S$ such that:*
*$ea = ae = a$, and there exists $b$ relative to $e$ in $S$ such that $ab = ba = e$.*

In the *Example 2.26* the semigroup $S$ is a semigroup admitting relative inverse.

**Definition 2.32** *[16] A semigroup $S$ is called a rectangular band if*

$$aba = a, \ \forall a, b \in S.$$

**Theorem 2.33** *[16] Let $S$ be a semigroup. Then the following conditions are equivalent:*
*(1) $S$ is a rectangular band;*
*(2) every element of $S$ is idempotent, and $abc = ac \ \forall a, b, c \in S$;*
*(3) there exist a left zero semigroup $L$ and a right zero semigroup $R$ such that $S \simeq L \times R$;*
*(4) $S$ is isomorphic to a semigroup of the form $A \times B$, where $A$ and $B$ are non-empty sets, and where multiplication is given by*

$$(a_1, b_1)(a_2, b_2) = (a_1, b_2).$$

**Proof.** (1) $\Rightarrow$ (2). Let $a \in S$. Then by (1) we have $a^3 = a$, and so $a^4 = a^2$. Again by (1) we have $a = a(a^2)a = a^4$. Hence $a^2 = a$ as required.

Now let $a, b, c \in S$. From (1) we have $a = aba, c = cbc$ and $b = b(ac)b$. Hence

$$ac = (aba)(cbc) = a(bacb)c = abc,$$

as required.

(2) $\Rightarrow$ (3). Choose and fix an element of $S$. Let $L = Sc$ and $R = cS$. Then, using (2), we see that, for all $x = zc$ and $y = tc$ in $L$,

$$xy = zctc = zc^2 = zc = x,$$

and so $L$ is a left zero semigroup.Similarly, R is a right zero semigroup. Define $\phi : S \rightarrow L \times R$ by

$$\phi(x) = (xc, cx) \ x \ \in S.$$

Then $\phi$ is one to one, for if $(xc, cx) = (yc, cy)$ then

$$\begin{aligned} x = x^2 = xcx \qquad & \text{by } (2) \\ = ycx = ycy = y^2 = y. \end{aligned}$$

Also, $\phi$ is onto; since $\forall (ac, cb) \in L \times R$, we may use condition (2) to see that

$$(ac, cb) = (abc, cab) = \phi(ab).$$

Finally, $\phi$ is morphism, since $\forall \ x, y \in S$,

$$\phi(xy) = (xyc, cxy) = (xc, cy) = (xcyc, cxcy) = (xc, cx)(yc, cy) = \phi(x)\phi(y).$$

(3) $\Rightarrow$ (4). Suppose that $S = L \times R$, where $L$ is a left zero semigroup and $R$ is a right zero semigroup. Then the product of two elements $(a, b)$ and $(c, d)$ in $S$ is given by

$$\text{(a,b)(c,d)} = \text{(ac,bd)} = \text{(a,d)}.$$

Thus we need only take $A = L$ and $B = R$.

(4) $\Rightarrow$ (1). Let $S = A \times B$, with the given multiplication. Then for all $a = (x, y)$ and $b = (z, t)$ in S,

$$\text{aba} = \text{(x,y)(z,t)(x,y)} = \text{(x,t)(x,y)} = \text{(x,y)} = \text{a}.$$

$\square$

# CHAPTER 3

# MOLAEI'S GENERALIZED GROUPS

**Definition 3.1** *[12,19] A non-empty set $G$ with a binary operation (i.e.; multiplication) is called Molaei's generalized group if it satisfies the following three conditions:*

*(M1) $(xy)z = x(yz)$ for all $x, y, z$ in $G$ (associative law);*

*(M2) For each $x$ in $G$ there exists a unique $z$ in $G$ such that $xz = zx = x$, we denote $z$ by $e(x)$ (existence and uniqueness of identity);*

*(M3) For each $x$ in $G$ there exists $y$ in $G$ such that $xy = yx = e(x)$ (existence of inverse).*

**Example 3.2** *[12] The set $G = \{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c,$ and $d$ are real numbers $\}$ with the operation*

$$\left( \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} e & f \\ g & h \end{bmatrix} \right) \mapsto \begin{bmatrix} a & f \\ g & d \end{bmatrix} \qquad ,$$

*is a Molaei's generalized group.*

**Lemma 3.3** *For every $x \in G$, the unique element $e(x) \in G$ such that $xe(x) = e(x)x = x$ is an idempotent.*

**Proof.** Indeed,

$$\begin{aligned} e(x)e(x) &= (xx^{-1})(xx^{-1}) = x(x^{-1}(xx^{-1})) = x((x^{-1}x)x^{-1}) \\ &= x(e(x)x^{-1}) = (xe(x))x^{-1} = xx^{-1} = e(x). \end{aligned}$$

$\square$

**Theorem 3.4** *[12] Each $x$ in a Molaei's generalized group $G$ has a unique inverse in $G$.*

**Proof.** Let $x \in G$ be given, and let $y, z \in G$ such that $yx = xy = e(x)$ and $zx = xz = e(x)$. Then

$$x = xe(x) = x(yx) = x(ye(y))x = ((xy)e(y))x = (e(x)e(y))x$$

and in a similar way

$$x = e(x)x = (xy)x = x(e(y)y)x = x(e(y)(yx)) = x(e(y)e(x)).$$

So the uniqueness of $e(x)$ shows that

$$e(x)e(y) = e(y)e(x) = e(x).$$

Hence $e(x) = e(y)$.

Similarly $e(x) = e(z)$.

Thus $y = ye(y) = ye(x) = y(xz) = (yx)z = e(x)z = z$ $\square$

**Lemma 3.5** *Let $G$ be a Molaei's generalized group and denote the inverse of $x \in G$ by $x^{-1}$. Then*

$$e(x)e(x^{-1}) = e(x^{-1})e(x).$$

**Proof.**

$$e(x)e(x^{-1}) = (xx^{-1})e(x^{-1}) = x(x^{-1}e(x^{-1})) = xx^{-1} = e(x).$$

Similarly,

$$e(x^{-1})e(x) = e(x^{-1})(x^{-1}x) = (e(x^{-1})x^{-1})x = x^{-1}x = e(x).$$

and so $e(x)e(x^{-1}) = e(x^{-1})e(x)$.

Note that we proved $e(x)e(x^{-1}) = e(x^{-1})e(x) = e(x)$, which implies that $e(x^{-1}) = e(x)$ (by (M2) and the fact that $e(x)e(x) = e(x)$). $\square$

**Definition 3.6** *[12] Let $G$ and $H$ are two Molaei's generalized groups and $\varphi : G \to H$ is a mapping then $\varphi$ is called a homomorphism if*

$$\varphi(ab) = \varphi(a)\varphi(b) \qquad \forall\ a, b \in G.$$

A one to one and onto homomorphism $\varphi : G \to H$ is called an isomorphism.

**Lemma 3.7** *[17] If $G$ is a Molaei's generalized group and $a \in G$, then*

$$aG = e(a)G, \text{ where } aG = \{ag : g \in G\}.$$

**Proof.** $aG = e(a)aG \subseteq e(a)G$, and $e(a)G = aa^{-1}G \subseteq aG$, where $a \in G$ and $a^{-1}$ is the inverse of $a$. Similarly $Ga = Gae(a) \subseteq Ge(a)$, and $Ge(a) = Ga^{-1}a \subseteq Ga$ so $Ga = Ge(a)$, where $a \in G$ □

**Lemma 3.8** *[17] If $G$ is a Molaei's generalized group, then for all $a \in G$,*

$$GaG = G.$$

**Proof.** If $a \in G$, then $GaG \subseteq G$. Conversely if $x \in G$, then $e(x) = e(e(x)ae(x))$. We have $x \in Ge(x)G = Ge(e(x)ae(x))G = Ge(x)ae(x)G \subseteq GaG$.
Thus $G = GaG$. □

**Lemma 3.9** *[17] If $e(b)G \cap e(a)G \neq \emptyset$, then $e(b)G = e(a)G$, where $a, b \in G$.*

**Proof.** Let $c \in e(b)G \cap e(a)G$. Then there exist $f$, and $d \in G$ such that $c = e(b)d = e(a)f$. So $e(a)c = e(a)e(a)f = e(a)f = c$. Previous lemma implies that there exist $z$, and $t \in G$ such that $e(a) = zct$. Hence $e(a) = ze(a)ct$. Thus $e(a) = (e(a))^3 = e(a)ze(a)cte(a)$. If $x = e(a)ze(a)$ and $y = te(a)$, then $e(a) = xcy$, $e(a)x = xe(a)$, and $ye(a) = y$. Let $g = cyx$, then $e(a)g = cyx = g$, and $ge(a) = cyxe(a) = cye(a)x = cyx = g$. So $e(a) = e(e(a)) = g$. Thus $e(a)G \subseteq cG \subseteq e(a)G$. Hence $e(a)G = cG$. Similarly $e(b)G = cG$.
Thus $e(a)G = e(b)G$. □

There is an important propertie of Molaei's generalized groups;
Let $G$ is a Molaei's generalized group, then:
If $xy = yx$, then $e(x) = e(y)$ where $x, y \in G$. So if $xy = yx$, the $G$ is an Abelian group.

# CHAPTER 4

# COMPLETELY SIMPLE SEMIGROUPS

**Definition 4.1** *[14] A semigroup $S$ is left simple if $S$ is its only left ideal, $S$ is right simple if $S$ is its only right ideal and $S$ is simple if $S$ is its only both-sided ideal.*

It means that a semigroup $S$ is said to be simple if it has no proper two-sided ideals.

**Theorem 4.2** *[15] The following conditions are equivalent for a semigroup $S$:*

*(i) $S$ is simple;*

*(ii) $SaS = S, \quad \forall\, a \in S$;*

*(iii) For any $a, b \in S$ there exist $s, t \in S$ such that $sat = b$;*

*(iv) $S$ has a single $\mathcal{J}$-class.*

**Proof.** $(i) \Rightarrow (ii)$ It is easy to see that $SaS$ is an ideal of $S$. Since $S$ is simple we must have $SaS = S$.

$(ii) \Rightarrow (iii)$ This is obvious.

$(iii) \Rightarrow (iv)$ Let $a, b \in S$ be arbitrary. From $(iii)$ it follows that there are $s, t \in S$ such that $sat = b$, and also $u, v \in S$ such that $abv = a$. Hence $a\mathcal{J}b$, and $\mathcal{J} = S \times S$.

$(iv) \Rightarrow (i)$ Let I be an ideal of $S$, and let $a \in I$ and $b \in S$ be arbitrary. By $(iv)$, we have $a\mathcal{J}b$; in particular, there are $s, t \in S^1$ such that $sat = b$. Since $a \in I$, it follows that $b \in I$ too, and so $I = S$. □

**Example 4.3** *[15] Every group $G$ is simple. Indeed, for arbitrary $a, b \in G$ we have*

$$baa^{-1} = b,$$

*and so the condition (iii) of the above theorem holds.*

**Example 4.4** *[15] Every rectangular band $S = I \times \Lambda$ is simple. Indeed, for any $(i, \lambda), (j, \mu) \in S$ we have*

$$(j, \mu)(i, \lambda)(j, \lambda) = (j, \mu).$$

**Definition 4.5** *[15] A (left, right or two-sided) ideal $I$ of a semigroup $S$ is said to be minimal if it contains no other (left, right or two-sided) ideals of $S$.*

Every finite semigroup has minimal left, right and two-sided ideals.
An infinite semigroup need not have any, e.g., $N$.
Also a semigroup may have several minimal left or right ideals, e.g., a semigroup of left or right zeros. However, a semigroup has at most one minimal two-sided ideal. Indeed if $I$ and $J$ are two such ideals then $I \cap J$ is also an ideal, and is contained in both $I$ and $J$, so that we have $I = I \cap J = J$.

**Theorem 4.6** *[15] If $I$ is the minimal two-sided ideal of a semigroup $S$ then $I$ is a simple semigroup.*

**Proof.** Let $a \in I$ be arbitrary. Since $I$ is an ideal, we have $IaI$ is an ideal. Also $IaI \subseteq I$ since $a \in I$. Minimality of $I$ implies that $IaI = I$, and hence $I$ is simple by Theorem 4.2. □

If a semigroup $S$ has a zero then it cannot be simple, because $\{0\}$ is a proper ideal.

**Definition 4.7** *[15] A semigroup $S$ with zero is said to be 0-simple if $S$ and $\{0\}$ are its only ideals, where $S^2 \neq \{0\}$.*

**Definition 4.8** *[15] A (left, right or two-sided) ideal $I$ of a semigroup $S$ with zero is said to be minimal if $I$ and $\{0\}$ (where $I \neq 0$)are the only (left, right or two-sided) ideals of $S$ contained in $I$.*

**Proposition 4.9** *[16] A semigroup $S$ is 0-simple if and only if $SaS = S$ for every $a \neq 0$ in $S$, that is, if and only if for every $a, b \in S \setminus \{0\}$ there exists $x, y \in S$ such that $xay = b$.*

**Proof.** Suppose, first that $S$ is 0-simple. Then $S^2$ is an ideal of $S$. By definition $S^2 \neq \{0\}$. So, $S^2 = S$. This follows that $S^3 = S^2.S = S.S = S$. Let $0 \neq a \in S$. Then the subset $SaS$ of $S$ is an ideal. Hence, either $SaS = S$ or $SaS = \{0\}$. If $SaS = \{0\}$ then the set $I = \{x \in S : SxS = \{0\}\}$ contains the non-zero element $a$. Since $I$ is an ideal of $S$ we have $I = S$. Thus, $SxS = \{0\}$ for every $x \in S$. But, this implies that $S^3 = \{0\}$, which contradicts to fact that $S^3 = S$. Therefore $SaS = S$.

Conversely, suppose $SaS = S$ for all $a \neq 0$ in $S$. Then certainly $S^2 \neq \{0\}$. If $A$ is an ideal of $S$ containing a non-zero element $a$ then

$$S = SaS \subseteq SAS \subseteq A,$$

and so $A = S$. Thus $S$ is 0-simple. □

**Proposition 4.10** *[16] If $M$ is a 0-minimal ideal of $S$ then either $M^2 = \{0\}$ or $M$ is a 0-simple semigroup.*

**Proof.** Since $M^2$ is an ideal of $S$ contained in $M$, we must have either $M^2 = M$ or $M^2 = \{0\}$. Suppose that $M^2 = M$. Then $M^3 = M$. If $0 \neq a \in M$ then $S^1 a S^1$ is a non-zero ideal of $S$ and contained in $M$. Therefore $S^1 a S^1 = M$.
Hence $MaM \subseteq S^1 a S^1 = M = M^3 = M(S^1 a S^1)M = (MS^1)a(S^1 M) \subseteq MaM$, and also in fact $MaM = M$.
Thus, $M$ is 0-simple by Proposition 4.9. □

**Definition 4.11** *[16] A semigroup $S$ is said to be completely 0-simple if it is 0-simple and if it possesses 0-minimal left and right ideals.*

**Definition 4.12** *[20] A semigroup $S$ is said to be completely simple semigroup if it satisfies the following conditions:*

*(C1) $SaS = S, \forall a \in S$; ($SaS = \{xay : x, y \in S\}$.)*

*(C2) If $e, f \in S$ are idempotents such that $ef = fe = e$ then $e = f$.*

**Definition 4.13** *[17] A non-empty subset of a completely simple semigroup $G$ is called a completely simple subsemigroup if it with the binary operation of $G$ is a completely simple semigroup.*

**Theorem 4.14** *[17] A non-empty subset $H$ of a completely simple semigroup $G$ is a completely simple subsemigroup if and only if for all $a$ and $b$ in $H$, $ab^{-1} \in H$.*

**Proof.** If $H$ is a completely simple subsemigroup and $a, b \in H$, then the definition of completely simple subsemigroup implies that: $b^{-1} \in H$ and $ab^{-1} \in H$. Conversely, if $H \neq \emptyset$ and $a, b \in H$ then we have:
$bb^{-1} = e(b) \in H$, $e(b)b^{-1} = b^{-1} \in H$, and $ab = a(b^{-1})^{-1} \in H$. $\square$

**Theorem 4.15** *[17] Let $a \in G$ and $\psi : G \to H$ be a homomorphism. Then the kernel of $\psi$ at $a$ which is denoted by ker $\psi(a) := \{x \in G : \psi(x) = \psi(e(a))\}$ is a completely simple subsemigroup of $G$. Moreover $\psi$ is one-to-one if and only if ker $\psi(a) = \{e(a)\}$ for all $a \in G$.*

**Proof.** Let $x, y \in ker\ \psi(a)$. Then

$$\begin{aligned} \psi(xy^{-1}) &= \psi(x)(\psi(y))^{-1} = \psi(e(a))(\psi(e(a)))^{-1} \\ &= \psi(e(a))\psi(e(a)^{-1}) = \psi(e(a))\psi(e(a)) = \psi(e(a)). \end{aligned}$$

So $xy^{-1} \in ker\ \psi(a)$.
Now let $\psi$ be a one-to-one homomorphism, then $ker\ \psi(a) = \{e(a)\}$ for all $a \in G$. Conversely if $\psi(x) = \psi(y)$, then $\psi(x)\psi(y^{-1}) = \psi(y)\psi(y^{-1})$. So $\psi(xy^{-1}) = \psi(e(y))$. Since $ker\ \psi(y) = \{e(y)\}$ we have

$$xy^{-1} = e(y) = e(y^{-1}). \tag{4.1}$$

Similarly $\psi(y^{-1}x) = \psi(e(y))$. So

$$y^{-1}x = e(y) = e(y^{-1}). \tag{4.2}$$

Hence $x = ((y^{-1})^{-1}) = y$ by (4.1) and (4.2). $\square$

We now give some properties of completely simple semigroup.

If $G$ and $G'$ are two completely simple semigroups and $\psi : G \to G'$ is a homomorphism, then :

**1.** $\psi(e(a)) = e(\psi(a))$ is an identity element in $G'$ for all $a \in G$;

**2.** $\psi(a^{-1}) = (\psi(a))^{-1}$, for all $a \in G$;

**3.** If $K$ is a completely simple subsemigroup of $G$, then $\psi(K)$ is a completely simple subsemigroup of $G'$;

**4.** If $D$ is a completely simple subsemigroup of $G'$ and $\psi^{-1}(D) \neq \emptyset$, then $\psi^{-1}(D)$ is a completely simple subsemigroup of $G$.

# CHAPTER 5

# REES MATRIX SEMIGROUPS

Let $S$ be a semigroup, then for all elements of $S$;

$$x \leq y \Leftrightarrow \exists\ e \in E_S : x = ey.$$

Here, the relation $\leq$ is;

a) Reflexive, since $x = (xx^{-1})x$, where $xx^{-1} \in E_S$;

b) Antisymmetric, since if $x = ey$ and $y = fx$, then $x = ey = eey = ex$, and so $x = ey = efx = fex = fx = y$;

c) Transitive, since if $x = ey$ and $y = fz$, then also $x = ey = efz$, where $e, f \in E_S$.

Therefore $\leq$ is partial order.

If we restrict $\leq$ onto $E_S$ we get the partial order of idempotents. Indeed, if $e \leq f$ then there exists $g \in E_S$ such that $e = gf$, and here $e = gff = ef = fe$ as required.

**Proposition 5.1** *[14] The following conditions on an element $a$ of a semigroup $S$ are equivalent.*

*i)* *$a$ is completely regular.*

*ii)* *$a$ has an inverse with which it commutes.*

*iii)* $a \in a^2Sa^2$.

*iv)* $a \in Sa^2 \cap a^2S$.

*v)* *$a$ is contained in a subgroup of $S$.*

**Theorem 5.2** *[14] The following conditions on a semigroup $S$ are equivalent.*

*i)* *$S$ is completely regular.*

*ii)* *For every $a \in S,\ a \in aSa^2$.*

*iii)* *$S$ is a union of disjoint groups.*

*iv)* *Every $\mathfrak{H}-$class is a group.*

*v)* *Every $\mathfrak{N}$-class is simple and completely regular.*

*vi)* *Every left and every right ideal of $S$ is completely semiprime.*

**Proof.** i) implies ii). This follows from Proposition 5.1.

ii) implies iii). By Proposition 5.1, it suffices to show that $\forall\ a \in S,\ a\ a^2S$. There exists $x \in S$ such that $a = axa^2$ and $y \in S$ such that $xa = (xa)y(xa)^2$. Consequently

$$\begin{aligned} a = \ & axa^2 = a(xa)a = a(xa)y(xa)^2a = (axayx)(axa^2) = axayxa = axay(xa) \\ & = (axay)(xa)y(xa)^2 = ay(xa)^2 = (ayx)a(xa) = (ayx)(axa^2)(xa) \\ & = (ay)(xa)^2(axa) = a(axa) = a^2xa \end{aligned}$$

so that $a \in a^2S$.

iii) implies iv). Let $H$ be an $\mathfrak{H}$-class. For any $a \in H,\ a$ belongs to a maximal subgroup $G_e$. But then $a\mathfrak{H}e$, so that $G_e = H$ since any two distinct maximal subgroups are disjoint.

iv) implies v). Since $\mathfrak{H} \subseteq \mathfrak{N}$, each $\mathfrak{N}$-class is a union of groups. By [14, IV.1.4], each $\mathfrak{N}$-class is completely regular. Also by Proposition 5.1, for every $a \in S,\ a = a^2xa^2$ for some $x \in S$ so that

$$a = a^2xa(a^2xa^2) \in Sa^2S,$$

which [14, II.4.5] implies that every $\mathfrak{N}$-class is simple.

v) implies vi). If $L$ is a left ideal of $S$ and $a^2 \in L$, then since $S$ is clearly completely regular, we have $a = xa^2 \in L$ and $L$ is completely semiprime; the situation is analogous for every right ideals.

vi) implies i). For any $a \in S$, $Sa^2$ is a left ideal and $a^4 \in Sa^2$. But then $a \in Sa^2$; similarly $a \in a^2S$, which by Proposition 5.1 implies that $S$ is completely regular. $\square$

**Lemma 5.3** *[14] A regular semigroup $S$ all of whose idempotents are primitive is completely regular subgroups given by*

$$G_e = aSa \qquad (e \in E_S, a \in G_e).$$

**Proof.** Let $a \in S$. Then $a = axa$ and $a^2 = a^2ya^2$ for some$x, y \in S$. Let $e = ax$ and $f = a^2yax$,

$$f^2 = (a^2yax)(a^2yax) = a^2y(axa)ayax = (a^2y)^2ax = (a^2y)ax = f,$$
$$ef = (ax)(a^2yax) = (axa)(ayax) = a^2yax = f,$$
$$fe = (a^2yax)(ax) = a^2yax = f,$$

so that $e, f \in E_S$ and $e \geq f$. Therefore, we have $e = f \Rightarrow ax = a^2yax$, which, after multiplication by $a$ on the right, we get $a = a^2ya \in a^2Sa$. Since $a$ is arbitrary in $S$, this implies that $S$ is completely regular semigroup by Theorem 5.2.
Let $e \in E_S$ and $a \in G_e$. If $x \in G_e$, then

$$x = exe = (aa^{-1})x(a^{-1}a) \in aSa,$$

and thus $G_e \subseteq aSa$. Conversely, let $x = aya$. Then $x \in G_f$ for some $f \in E_S$ so that

$$ef = e(xx^{-1}) = e(aya)x^{-1} = xx^{-1} = f$$

and dually $fe = f$ so that $e \geq f$. By hypothesis, we must have $e = f$ and hence $x \in G_e$. Since $x \in aSa$ is arbitrary, then $aSa \subseteq G_e$. $\square$

**Theorem 5.4** *[14] The following conditions on a semigroup $S$ are equivalent.*

*i)* $S$ *is a completely simple semigroup.*

*ii)* $S$ *is completely regular and simple.*

*iii)* $S$ *is regular and all its idempotents are primitive.*

*iv)* $S$ *is regular and weakly cancellative.*

*v)* $S$ *is regular and for any* $a, x \in S$, $a = axa \Rightarrow x = xax$.

**Proof.** i) implies ii). Let $e$ be a primitive idempotent of $S$ and $a$ be any element of $S$. Then by simplicity of $S$, we have $a = uev$ and $e = x(ea^3e)y$ for some $u, v, x, y \in S$. Letting $f = evaeyexeaue$, we obtain

$$f^2 = evaeyexea(ueev)aeyexeaue = evaeye(xea^3ey)exeaue = evaeyexeaue = f$$

and evidently $f \leq e$ so that $f = e$. Hence

$$a = ufv = (uev)aeyexea(uev) = a^2(eyexe)a^2 \in a^2Sa^2$$

and $a$ is completely regular by Proposition 5.1.

ii) implies iii). Let $e, f \in E_S$ such that $e \leq f$. Then simplicity of $S$, $f = xey$ for some $x, y \in S$. Letting $a = fxf$ and $b = fyf$, we obtain

$$aeb = (fxf)e(fyf) = f(xey)f = f. \tag{5.1}$$

For $a'$ satisfying the equalities $a = aa'a,\ aa' = a'a$, by (5.1) we have

$$f = aeb = aa'aeb = aa'f = a'af = a'a = aa' \tag{5.2}$$

and hence by using (5.1) and (5.2), we obtain

$$f = (a'a)(a'a) = a'(aa')a = a'fa = a'(aeb)a = (a'a)(eba) = feba = eba.$$

But then $e = ef = f$ which implies that $f$ is primitive.

iii) implies iv). Suppose that $ax = xb$ and $xa = xb$ by Lemma 5.3, $S$ is completely regular, so $a\ inG_e$ and $b\ inG_f$ for some $e, f \in E_S$. By Lemma 5.3, $axa \in G_e$ and $bxb \in G_f$. Hence $axa = bxa = bxb$ implies that $e = f$. Letting $y = exe$, we see by Lemma 5.3 that $y \in G_e$, and also that

$$ay = a(exe) = (ae)xe = axe = bxe = (be)xe = b(exe) = by$$

which definition of $G_e$ implies $a = b$, establishing that $S$ is weakly cancellative.

iv) implies v). If $a = axa$, then $ax = a(xax)$ and $xa = (xax)a$, which by weak cancellation implies that $x = xax$.

v) implies i). If $e, f \in E_S$ and $e \leq f$, then $e = efe$ so by hypothesis we also have $f = fef = e$ and every idempotent is primitive. By Lemma 5.3, $S$ is completely regular. Suppose that $S$ is not simple. Then by Theorem 5.1, $S$ has more than one $\mathfrak{N}$-class, and in particular it must have two $\mathfrak{N}$-classes $N_e > N_f$ with $e, f \in E_S$. Letting $x \in eN_fe$, we have $x = eae \in G_g$ for some $a \in N_f$ and $g \in E_{N_f}$ whence

$$eg = exx^{-1} = e(eae)x^{-1} = (eae)x^{-1} = g$$

and similarly $ge = g$. Hence $g < e$ since $g \in N_f$, contradicting the fact that all idempotents of $S$ are primitive. $\square$

**Lemma 5.5** *[14] Let $S$ be a completely simple semigroup and let $e, f \in E_S$. Then the following statements hold.*

*i)* *For any $a, b \in S$, $ab \in G_e \Rightarrow aSb \subseteq G_e$.*

*ii)* *$ef = e \Rightarrow fe = f$.*

*iii)* *$ef = f \Rightarrow fe = e$.*

**Proof.** Note that $S$ satisfies the hypothesis of Lemma 5.3 in view of Theorem 5.4.

i) Let $a \in G_g,\ b \in G_h,\ ab \in G_e$. Then Lemma 5.3 implies $aba \in G_g$ and $bab \in G_h$. Hence $a = (aba)u,\ b = v(bab)$ for some $u \in G_g,\ v \in G_h$. For any $x \in S$, using Lemma 5.3, we obtain

$$axb = (aba)uxv(bab) = (ab)(auxvb)(ab) \in G_e$$

since $ab \in G_e$. Since $x$ is arbitrary, we have $aSb \subseteq G_e$.

ii) If $ef = e$, then $fe \in E_S$ and also $fe = fef \in G_f$ by Lemma 5.3. But then $fe = f$.

iii) If $ef = f$, then $fe \in E_S$ and also $fe = efe \in G_e$ by Lemma 5.3. But then $fe = e$. $\square$

**Definition 5.6** *[15,17] Suppose that a semigroup $D$ , and two arbitrary index sets $I$ and $\Lambda$ are given. If $P : \Lambda \times I \to D$ is a mapping, then $I \times D \times \Lambda$ with the product*

$$(i, a, \lambda)(j, b, \mu) = (i, ap_{\lambda j}b, \mu) \tag{5.3}$$

*is a semigroup group. This semigroup is called Rees matrix semigroup (over D with respect to P) and is denoted by $\mathcal{M}[D; I, \Lambda; P]$.*

**Lemma 5.7** *The operation (5.3) is associative.*

**Proof.** Since

$$(i, a, \lambda)((j, b, \mu)(k, c, \eta)) = (i, a, \lambda)(j, bp_{\mu k}c, \eta) = (i, ap_{\lambda j}bp_{\mu k}c, \eta)$$

and also

$$((i, a, \lambda)(j, b, \mu))(k, c, \eta) = (i, ap_{\lambda j}b, \mu)(k, c, \eta) = (i, ap_{\lambda j}bp_{\mu k}c, \eta)$$

then

$$(i, a, \lambda)((j, b, \mu)(k, c, \eta)) = ((i, a, \lambda)(j, b, \mu))(k, c, \eta)$$

implies that (5.3) is associative. $\square$

**Lemma 5.8** *The operation (5.3) is well-defined.*

**Proof.** Let $x = (i, a, \lambda), y = (j, b, \mu)$ and $x_1 = (i_1, a_1, \lambda_1), y_1 = (j_1, b_1, \mu_1)$. Now let $(x, y) = (x_1, y_1)$ that is $x = x_1$ and $y = y_1$. Then we have $i = i_1,\ a = a_1,\ \lambda = \lambda_1,\ j = j_1,\ b = b_1$ and $\mu = \mu_1$. This implies $p_{\lambda j} = P(\lambda, j) = P(\lambda_1, j_1) = p_{\lambda_1 j_1}$, since $P$ is a well-defined mapping. Also, D is a semigroup with well-defined binary operation. That implies $ap_{\lambda j}b = a_1 p_{\lambda_1 j_1} b_1$.

Hence $(i, ap_{\lambda j}b, \mu) = (i_1, a_1 p_{\lambda_1 j_1} b_1, \mu_1)$ that $xy = x_1 y_1$. $\square$

**Definition 5.9 (14)** *Let $e$ be an idempotent of a semigroup $S$. Then*

$$G_e = \{a \in S : a = ea = ae,\ e = aa' = a'a \text{ for some } a \in S\}$$
$$= \{a \in S : a \in eS \cap Se,\ e \in aS \cap Sa\}$$

*where $e$ is an identity element.*

**Definition 5.10** *[15] Suppose that $D$ is a semigroup ( not containing zero) and $I$ and $\Lambda$ are two arbitrary index sets, and let $P = (p_{\lambda i})$ be a $\Lambda \times I$ matrix with entries from $D \cup \{0\}$, then $S = (I \times D \times \Lambda) \cup \{0\}$ with the product*

$$(i, a, \lambda)(j, b, \mu) = \begin{cases} (i, ap_{\lambda j}b, \mu) & \text{if} \quad p_{\lambda j} \neq 0 \\ 0 & \text{otherwise} \end{cases} \tag{5.4}$$

$$0(i, a, \lambda) = (i, a, \lambda)0 = 00 = 0$$

*is a semigroup group, which is called Rees matrix semigroup with zero, and is denoted by $\mathcal{M}^0[D; I, \Lambda; P]$.*

**Lemma 5.11** *[16] Let $G$ be a group, then $S = (I \times G \times \Lambda) \cup \{0\}$ is a completely 0-simple semigroup.*

**Proof.** It is easy to see that the composition (5.4) is associative, but it is probably more illuminating to observe that $S \setminus \{0\}$ is in one to one correspondence with the set $I \times \Lambda$ matrices $(a)_{i\lambda}$, $(a \in G)$, where $(a)_{i\lambda}$ denotes the matrix with entry $a$ in the $(i, \lambda)$ position and zeros elsewhere. Since $(0)_{i\lambda}$ is independent of $i$ and $\lambda$ we may simply write it as 0. Thus the correspondence extends to a correspondence between $S$ and the set

$$G = \{(a)_{i\lambda} : a \in G^0, i \in I, \lambda \in \Lambda\}$$

We can easily verify that

$$(a)_{i\lambda} P (b)_{j\mu} = (ap_{\lambda j}b)_{i\mu},$$

where the juxtaposition on the left denote the matrix multiplication in the usual sense. Thus the composition (5.4) in $S$ corresponds in $G$ to the evidently associative composition $\circ$ given by

$$(a)_{i\lambda} \circ (b)_{j\mu} = (a)_{i\lambda} P (b)_{j\mu},$$

and so is itself associative.

To verify that $S$ is 0-simple, note that for any two non-zero elements $(i, a, \lambda)$ and $(j, b, \mu)$ of $S$ we may, by the regularity of $P$, choose $\nu \in \Lambda$ and $k \in I$ such that $p_{\nu i} \neq 0,\ p_{\lambda k} \neq 0$, and then easily show that

$$(j, a^{-1}p_{\nu i}^{-1}, \nu)(i, a, \lambda)(k, p_{\lambda k}^{-1}b, \mu) = (j, b, \mu).$$

Hence by Proposition 4.9, $S$ is 0-simple.

To complete the proof that $S$ is completely 0-simple, we must first identify its idempotents. the non-zero element $(i, a, \lambda)$ is idempotent if and only if

$$(i, a, \lambda) = (i, a, \lambda)(i, a, \lambda) = (i, ap_{\lambda i}a, \lambda),$$

that is, if and only if $p_{\lambda i} \neq 0$ and $a = p_{\lambda i}^{-1}$. If we now take two non-zero idempotents $e = (i, p_{\lambda i}^{-1}, \lambda)$ and $f = (j, p_{\mu j}^{-1}, \mu)$, then $e \leq f$ if and only if $ef = fe = e$, that is, if and only if

$$(i, p_{\lambda i}^{-1}p_{\lambda j}p_{\mu j}^{-1}, \mu) = (j, p_{\mu j}^{-1}p_{\mu i}p_{\lambda i}^{-1}, \lambda) = (i, ap_{\lambda i}a, \lambda),$$

that is, if and only if$j = i$ and $\lambda = \mu$, that is, if and only if $e = f$. the conclusion is that every idempotent is primitive. Certainly there exists a primitive idempotent, and so $S$ is completely simple. □

The semigroup constructed in accordance with this recipe will, as in Clifford and Preston [21], be denoted by $\mathcal{M}^0[G; I, \Lambda; P]$, and will be called the $I \times \Lambda$ Rees matrix semigroup over the 0-group $G^0$ with the regular sandwich matrix $P$.

**Theorem 5.12** *[12,17] If $G$ is a Molaei's generalized group, then $G$ is isomorphic to a Rees matrix semigroup.*

**Proof.** We choose an element $a \in G$ and denote $e(a)$ by 1. There exist two sets $I \subseteq e(G)$ and $\Lambda \subseteq e(G)$ such that

(i) $\{aG : a \in e(G)\} = \{iG : i \in I\}$, and if $i_1 \neq i_2$ then $i_1G \cap i_2G = \emptyset$, where $i_1, i_2 \in G$;

(ii) $\{Gb : b \in G\} = \{Gl : l \in \Lambda\}$, and if $l_1 \neq l_2$ then $Gl_1 \cap Gl_2 = \emptyset$, where $l_1, l_2 \in \Lambda$;

(iii) $1 \in I \cap \Lambda$.

For given $(i, \lambda) \in I \times \Lambda$ let $r_i = i1$ and $q_\lambda = 1\lambda$. Then $f_{r_i} : (1G \cap G1) \to (iG \cap G1)$ defined by $f_{r_i}(x) = r_i x$ is an one to one mapping. To see this we have

$1(1r_i) = (11)r_i = 1r_i$ and $(1r_i)1 = 1(r_i1) = 1r_i$. So $e(1r_i) = 1$. Now if $r_ix_1 = r_ix_2$, then $1r_ix_1 = 1r_ix_2$. Hence $(1r_i)^{-1}(1r_i)x_1 = (1r_i)^{-1}(1r_i)x_2$. Therefore $1x_1 = 1x_2$. Thus $x_1 = x_2$. So $f_{r_i}$ is a one to one mapping. If $y \in iG \cap G1$, then $f_{r_i}((1r_i)^{-1}1y) = y$. Because if $z = r_i(1r_i)^{-1}1y$, then $1z = 1r_i(1r_i)^{-1}1y$. So $1z = 1y$. Hence $1iz' = 1iy'$, where $z = iz'$ and $y = iy'$. Therefore $(i1i)y' = (i1i)z'$. So $(i1i)^{-1}(i1i)z' = (i1i)^{-1}(i1i)y'$. Hence $iz' = iy'$. Thus $z = y$. So $f_{r_i}$ is also an onto mapping. Lets show that the mapping $f^{(i,q_\lambda)} : (iG \cap G1) \to (iG \cap G\lambda)$ defined by $f^{(i,q_\lambda)}(x) = xq_\lambda$ is a one to one and onto mapping.

And also we have $1(q_\lambda 1) = (1q_\lambda)1 = ((11)\lambda)1 = q_\lambda 1$ and $(q_\lambda 1)1 = q_\lambda(11) = q_\lambda 1$. So $e(q_\lambda 1) = 1$. Now if $x_1q_\lambda = x_2q_\lambda$, then $x_1q_\lambda 1 = x_2q_\lambda 1$. Hence $x_1(q_\lambda 1)(q_\lambda 1)^{-1} = x_2(q_\lambda 1)(q_\lambda 1)^{-1} \Rightarrow x_11 = x_21$. Thus $x_1 = x_2$. So $f^{(i,q_\lambda)}(x)$ is a one to one mapping. If $y \in (iG \cap G\lambda)$,then $f^{(i,q_\lambda)}(y.1(q_\lambda 1)^{-1}) = y$. Because if $k = y.1(q_\lambda 1)^{-1}q_\lambda$, then $k1 = y.1(q_\lambda 1)^{-1}q_\lambda 1$. So $k1 = y1$. Hence $k'\lambda 1 = y'\lambda 1$, where $k = k'\lambda$ and $y = y'\lambda$. Therefore, $k'(\lambda 1\lambda) = y'(\lambda 1\lambda) \Rightarrow k'(\lambda 1\lambda)(\lambda 1\lambda)^{-1} = y'(\lambda 1\lambda)(\lambda 1\lambda)^{-1} \Rightarrow k'\lambda = y'\lambda$. Thus $k = y$. So $f^{(i,q_\lambda)}$ is an onto mapping.

Lemmas 3.7 and 3.9 show that $\{iG : i \in I\}$ is a partition for $G$. Similarly one can show that $\{G\lambda : \lambda \in \Lambda\}$ is a partition for $G$. So $\{iG \cap G\lambda : i \in I \text{ and } \lambda \in \Lambda\}$ is a partition for $G$. This fact with the matter which $f^{(i,q_\lambda)} o f_{r_i}$ is a one to one and onto mapping show that each $a \in G$ has a unique expression $r_ibq_\lambda$, where $b \in 1G \cap G1$.

If $p : \Lambda \times I \to (1G \cap G1)$ defined by $p(\lambda, i) = q_\lambda r_i$, then straightforward calculation shows that $\Psi : I \times (1G \cap G1) \times \Lambda \to G$ defined by $\Psi(i, b, \lambda) = r_i b q_\lambda$ is an isomorphism between Rees matrix semigroup $\mathcal{M}[(1G \cap G1); I, \Lambda; P]$ and $G$. $\square$

**Lemma 5.13** *[20] Let $G$ be a group and let $S = \mathcal{M}[G; I, \Lambda; P]$. Then $S$ is completely simple semigroup and*

$$E_S = \{(i, p_{\lambda i}^{-1}, \lambda) \in S : \quad i \in I, \lambda \in \Lambda\}.$$

**Proof.** For any $a = (i, g, \lambda),\ \ b = (j, h, \mu) \in S$, we have

$$a = (i, gh^{-1}p_{\lambda j}^{-1}, \lambda)(j, h, \mu)(i, gp_{\lambda i}^{-1}, \lambda) \ \in SbS$$

and similarly

$$b = (j, hg^{-1}p_{\mu i}^{-1}, \mu)(i, g, \lambda)(j, hp_{\mu j}^{-1}, \mu) \ \in SaS.$$

It follows that $S$ must be simple. Furthermore

$$a^2 = (i, g, \lambda)(i, g, \lambda)$$

so that $a$ ia an idempotent if and only if $g = p_{\lambda i}^{-1}$. Now consider any two idempotents $e = (i, p_{\lambda i}^{-1}, \lambda)$ and $f = (j, p_{\mu j}^{-1}, \mu)$ with $e \leq f$. Then $e = ef = fe$ so that

$$(i, p_{\lambda i}^{-1}, \lambda) = (i, p_{\lambda i}^{-1} p_{\lambda j} p_{\mu j}^{-1}, \mu) = (j, p_{\mu j}^{-1} p_{\mu i} p_{\lambda i}^{-1}, \lambda).$$

Hence $i = j,\ \lambda = \mu$, and so $e = f$. Thus all idempotents in $S$ are primitive, there fore $S$ is completely simple semigroup. $\square$

**Theorem 5.14** *(Rees Theorem) [14] Let $G$ be a group and let $S$ be a completely regular simple semigroup; fix $g \in E_S$, and let $G = G_g$,*

$$I = \{e \in E_S : eg = e,\} \quad \Lambda = \{f \in E_S : gf = f\},$$

*$P = (p_{fe})$ where $p_{fe} = fe$. Then the mapping $\chi$ defined by*

$$\chi : a \to (e, gag, f) \qquad (a \in S)$$

*where* $ag \in G_e,\ ga \in G_f$, *is an isomorphism of* $S$ *onto* $T = \mathcal{M}[G; I, \Lambda; P]$.

**Proof.** 1. For any $f \in \Lambda,\ e \in I$, we have

$$p_{fe} = fe = (gf)(eg) \in gSg = G_g$$

by Lemma 5.3 and Theorem 5.4 so that $P$ is indeed a matrix over $G$.

Also $gag \in G_g = G$ for any $a \in S$.

Let $a$ be an element of S. If $ag \in G_e$, then $u(ag) = e$ for some $u \in G_e$, and we obtain

$$eg = u(ag)g = u(ag) = e,$$

which shows that $e \in I$. Similarly we see that $ga \in G_f$ implies $f \in \Lambda$. Thus $\chi$ maps $S$ into $T$.

2. Let $a, a' \in S$ with $ag \in G_e,\ ga \in G_f,\ a'g \in G_{e'},\ ga' \in G_{f'}$; then by Lemma 5.5, $aa'g \in G_e,\ gaa' \in G_{f'}$, and we obtain

$$\begin{aligned}(a\chi)(a'\chi) &= (e, gag, f)(e', ga'g, f') = (e, (gag)(fe')(ga'g), f')\\ &= (e, (ga)(gf)(e'g)(a'g), f') = (e, [(ga)f][e'(a'g)], f')\\ &= (e, (ga)(a'g), f') = (e, g(aa')g, f') = (aa')\chi,\end{aligned}$$

and thus $\chi$ is a homomorphism.

3. With the same notation, suppose that $a\chi = a'\chi$. Then

$$(e, gag, f) = (e', ga'g, f')$$

so that $e = e',\ gag = ga'g,\ f = f'$. Consequently

$$ga = (ga)f = ga(gf) = (ga'g)f = (ga')(gf) = ga'(gf') = (ga')f' = ga'$$

and dually $ag = a'g$.

By Theorem 5.4, $S$ is weakly cancellative, so $ag = a'g,\ ga = ga' \Rightarrow a = a'$.

Thus $\chi$ is one-to-one.

4. Let $(e, x, f) \in T$. Then$eg = e$ and $gf = f$, which by Lemma 5.5 ii) and iii) imply $ge = g = fg$.

Let $a = exf$. Then

$$e(ag)e = e(exf)ge = (exf)ge = ag$$

which by Lemma 5.5 i) implies $ag \in G_e$. Similarly we obtain $ga \in G_f$.
Also

$$gag = g(exf)g = (fe)x(fg) = gxg = x.$$

Consequently $a\chi = (e, x, f)$ which proves that $\chi$ is onto.

Therefore $\chi$ is an isomorphism of $S$ onto $T$. $\square$

**Lemma 5.15** *[22] Rees matrix semigroup is a generalized group.*

**Proof.** In Lemma 5.7, we show that the operation (5.3) is associative.
Thus $\mathcal{M}[G; I, \Lambda; P]$ is a semi group.

We claim that $\mathcal{M}[G; I, \Lambda; P]$ is a generalized group.
Now to prove that $\mathcal{M}[G; I, \Lambda; P]$ satisfies (M2), let $(i, a, \lambda) \in \mathcal{M}[G; I, \Lambda; P]$ and consider the element $(i, p_{\lambda i}^{-1}, \lambda) \in \mathcal{M}[G; I, \Lambda; P]$.
We claim that $(i, p_{\lambda i}^{-1}, \lambda)$ is a unique element $e \in \mathcal{M}[G; I, \Lambda; P]$ such that

$$(i, a, \lambda)(e) = (e)(i, a, \lambda) = (i, a, \lambda).$$

First $(i, a, \lambda)(i, p_{\lambda i}^{-1}, \lambda) = (i, ap_{\lambda i}p_{\lambda i}^{-1}, \lambda) = (i, a, \lambda)$
and $(i, p_{\lambda i}^{-1}, \lambda)(i, a, \lambda) = (i, ap_{\lambda i}^{-1}p_{\lambda i}, \lambda) = (i, a, \lambda)$.
Next, suppose that $(i, a, \lambda)(j, b, \mu) = (i, a, \lambda)$ for some $(j, b, \mu) \in \mathcal{M}[G; I, \Lambda; P]$.
Then $(i, ap_{\lambda j}b, \mu) = (i, a, \lambda)$, and so $(\mu = \lambda)$ and $(ap_{\lambda j}b) = a$, implying $(b = p_{\lambda j}^{-1})$. Similarly, $(j, b, \mu)(i, a, \lambda) = (i, a, \lambda)$ implies $j = i$. It follows that $(j, b, \mu) = (i, p_{\lambda i}^{-1}, \lambda)$ for every $(i, a, \lambda) \in \mathcal{M}[G; I, \Lambda; P]$.
To prove that $\mathcal{M}[G; I, \Lambda; P]$ satisfies (M3), let $(i, a, \lambda) \in \mathcal{M}[G; I, \Lambda; P]$. and consider the element $(i, p_{\lambda i}^{-1}a^{-1}p_{\lambda i}^{-1}, \lambda) \in \mathcal{M}[G; I, \Lambda; P]$.
It is immediate by the definition of the operation (5.3) that

$$(i,a,\lambda)(i,p_{\lambda i}^{-1}a^{-1}p_{\lambda i}^{-1},\lambda)=(i,p_{\lambda i}^{-1}a^{-1}p_{\lambda i}^{-1},\lambda)(i,a,\lambda)=(i,p_{\lambda i}^{-1},\lambda)=e((i,a,\lambda)).$$

Hence $\mathcal{M}[G;I,\Lambda;P]$ is a generalized group. $\square$

**Theorem 5.16** *[22] Let $S$ be a semigroup. Then the following are equivalent:*

*(1) $S$ is* isomorphic to some $\mathcal{M}[G;I,\Lambda;P]$.

*(2) $S$ is* completely simple semigroup.

*(3) $S$ is a* Molaei's generalized group.

**Proof.**

(1) implies (2).

This follows from Lemma 5.13.

(2) implies (1).

This follows from Theorem 5.4 and Theorem 5.14

(3) implies (1).

This follows from Theorem 5.12

(1) implies (3).

This follows from Theorem 5.15

## REFERENCES


[1] J.A. de Se'gurer, *The'orie des groupes Finis: E'lements de la The'orie des Groupes Abstraits*, Paris, Gauyhier-Villars, (1904).

[2] L.E.Dickson, *On semi-groups and the general isomorphism between infinite groups*, Transactions of the American Mathematical Society, 6 (1905), 205-208.

[3] A.H.Clifford, *A system arising from a weakened set of group postulates*, Ann.Math. (2), 34 (1933), 865-871.

[4] A.H.Clifford, *Arithmetic and ideal theory of abstract multiplication*, Bull.Amer.Math.Soc., 40 (1934), 326-330.

[5] D.Rees, *On semi-groups*, Proc.Cam.Phil.Soc., 36 (1940), 387-400.

[6] A.H.Clifford, *Semigroups admitting relative inverses*, Ann.Math (2), 42 (1941), 1037-1049.

[7] P.Dubrail, *Contribution a' la the'orie des demi-groupes*, Me'in.Acad.Sci.Inst. France (2), 63 (1941), 1-52.

[8] J.M. Howie , *Semigroup, Past, Present and Future*, Proceeding of the Int. Conference on Algebra and Its Applications, (2002).

[9] J.A. Green, *On the structure of semigroups*, Ann.of Math. 54 (1951), 163-172.

[10] V.V.Vagner, *Generalized groups*, Dokl.Akad.Nauk SSSR 84 (1952), 1119-1122 (Russian).

[11] G.B.Preston, *Inverse semi-groups*, J.London Math.Soc. 29 (1954).

[12] M.R. Molaei, *Generalized groups*, Bulet. Inst. Politehn. Iaşi Sect. I 45(49) (1999), 21-24.

[13] Tero Harju, *Lecture notes on semigroups*, University of Turku (1996).

[14] M. Petrich, *Introduction to semigroups*, Merrill Publishing Company, (1973).

[15] Nikola Ruskuc, *Semigroups*, (2003)

[16] J.M. Howie, *Fundamentals of semigroup theory*, Clarendon Press, Oxford, (1995).

[17] M.R. Molaei, *Completely Simple Semigroups and Their Topological Viewpoints* , (2000).

[18] Preston G.B., *Inverse Semigroups*, Journal of the London Mathematical Society, 29, (1954), 396-403.

[19] Molaei M.R., Tahmoresi A.: *Connected Generalized Groups*, General Mathematics, Vol. 12, 1, (2004), 13-22.

[20] Mario Petrich, Norman R. Reilly, *Completely Regular Semigroup*, Wiley-Interscience and Canadian Mathematics Series of Monographs and Texts, (1999).

[21] Clifford A. H. and Preston G. B., *The algebraic theory of semigroups*, Vol. 1, Math. Survey of the American Math. Society, no. 7. Providence, R. I., (1961).

[22] Araujo J., Konieczny J., *Molaei's Generalized Groups are Completely Simple Semigroups*, Buletinul Institutului Polithnic Din Iasi, 48 (52), (2002), 1-5.