ON ENERGY DECAY AND ASYMPTOTIC BEHAVIOR OF SOLUTIONS TO THE LINEAR AND NONLINEAR DISSIPATIVE WAVE EQUATIONS

by

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I certify that this thesis satisfies all the requirements as a thesis for the degree of Master of Science.

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This is to certify that we have read this thesis and that in our opinion it is fully adequate, in scope and quality as a thesis for the degree of Master of Science.

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ABSTRACT

ON ENERGY DECAY AND ASYMPTOTIC BEHAVIOR OF SOLUTIONS TO THE LINEAR AND NONLINEAR DISSIPATIVE WAVE EQUATIONS

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In this thesis we survey energy decay and asymptotic behavior of solutions to the Cauchy problems for the linear and nonlinear wave equations with dissipative terms.

The thesis consists of four chapters. In the first chapter, necessary definitions and theorems are given. In the second chapter, it is shown that the solution of the Cauchy problem to the linear wave equation with damping with variable coefficient decays to zero polynomially as t tends to infinity. In the third chapter, it is proved that the solution to the Cauchy problem for the nonlinear dissipative wave equation decays to zero as t tends to infinity. In the last chapter, a short discussion of the advantages of the strengthened multipliers method used in the thesis is given.

Keywords: Nonlinear Wave Equation, Dissipative Term, Asymptotic Behavior, Decay Rate of Energy.

ÖZET

LİNEER VE LİNEER OLMAYAN SÖNÜM TERİMLİ DALGA DENKLEMLERİNİN ENERJİ AZALIMI VE ASİMTOTİK DAVRANIŞI

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Bu tezde, lineer ve lineer olmayan sönüm terimli dalga denklemleri için verilen Cauchy problemlerinin enerji azalım ve asimtotik davranışları ele alınmıştır.

Tez dört bölümden oluşmaktadır. Birinci bölümde, verilen problemlere ilişkin önceki çalışmalara değinilmekte ve tez için gerekli olan tanımlar ve teoremler verilmektedir. İkinci bölümde ise lineer ve değişken katsayılı sönüm terimli dalga denklemi için verilen Cauchy probleminin çözümünün polinom hızıyla sıfıra yaklaştığı gösterilmektedir. Üçüncü bölümde ise lineer olmayan sönüm terimli dalga denklemi için verilen Cauchy probleminin çözümünün polinom hızıyla sıfıra yaklaştığı gösterilmektedir. Sonuç bölümünde ise kuvvetlendirilmiş çarpanlar yönteminin faydalarından kısaca bahsedilmiştir.

Anahtar Kelimeler: Dalga Denklemi, Sönüm Terimi, Asimtotik Davranış, Enerji Azalım Oranı.

To My Family,

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CHAPTER 1

INRODUCTION AND PRELIMINARIES

1.1 Introduction

The energy decay and asymptotic behavior of solutions to the Cauchy problems for nonlinear wave equations with dissipative terms are important questions and a key starting point for many open problems.

In this thesis we study the long time behavior of solutions of the Cauchy problems for the linear wave equation with variable damping term (see equation (2.1)) and nonlinear wave equation with variable damping and absorbtion terms (see equation (3.1)). In both cases the variable coefficient $a(x) \sim a_0|x|^{-\alpha}$, where a_0 and $\alpha \in [0, 1)$. In the former case decay estimates of the energy and the L^2 norm of solutions are established. In the latter case decay estimates of the energy, the L^2 and L^{p+1} norms of solutions are established. Here p is the exponent of the absorbtion term satisfying $1 < p < \frac{n+2}{n+1}$ and $n \geq 3$ is the space dimension. The decay rates of solutions of these problems depend on solutions of the Poisson equation (see equation (2.24)).

The thesis is organized as follows. In the next two sections we give some definitions, basic facts and inequalities. In Chapter 2, firstly the main problem to the Cauchy problem for the dissipative wave equations is introduced and previous studies regarding the energy decay and asymptotic behavior of various dissipative wave equations are given. Next the main results of this chapter are presented.

Chapter 3 is devoted to the investigation of decay estimates and asymptotic behavior of solutions to the Cauchy problem for the nonlinear dissipative wave equations.

In Chapter 4 the advantages of the method used in the thesis are briefly discussed.

1.2 Definitions and basic facts

In this section we will give some definitions, notations and general facts.

Definition 1.1 *Let Ω be an open, connected set (domain) in $\mathbb{R}^n$ and $1 \leq p$ be a real number. We denote by $L^p(\Omega)$ the class of all measurable functions u, defined on Ω for which*

$$\int_\Omega |u(x)|^p dx < \infty.$$

$L^p(\Omega)$ is a Banach space with the norm

$$||u||_{L^p} := \Big(\int_\Omega |u(x)|^p dx \Big)^{1/p} < \infty.$$

For $p = \infty$, $L^\infty(\Omega)$ is a Banach space with the norm

$$||u||_\infty = ess \sup_{x \in \Omega} |u(x)|.$$

For $p = 2$, $L^2(\Omega)$ is a Hilbert space with the inner product

$$(u, v) := \int_\Omega u(x)v(x)dx$$

for all u and $v \in L^2(\Omega)$.

Definition 1.2 *Let $\Omega \subset \mathbb{R}^n$ be an open, connected set and let $\Phi : \Omega \to \mathbb{R}$ be a function. Support of Φ can be defined as*

$$supp\{\Phi\} = \overline{\{x \in \Omega : \Phi(x) \neq 0\}}.$$

Definition 1.3 *Let $\Omega \subset \mathbb{R}^n$ be an open, connected set. A function $u : \Omega \to \mathbb{R}$ is said to be locally integrable if for every compact set $K \subset \Omega$,*

$$\int_K |u(x)|dx < \infty.$$

We denote by $L^1_{loc}(\Omega)$ the space of locally integrable functions defined on Ω.

Definition 1.4 *$C_0^\infty(\Omega)$ is the space of infinitely differentiable functions with compact support. This space is also known as the space of all test functions defined on Ω.*

Definition 1.5 *Let $x \in \mathbb{R}^n$ with coordinates $x = (x_1, \cdots, x_n)$. A multi-index is an n-tuple $\alpha = (\alpha_1, \cdots, \alpha_n)$ ($\alpha_i \in \mathbb{N}$). If we set*

$$|\alpha| = \sum_{i=1}^{n} \alpha_i,$$

then

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots x_n^{\alpha_n}}$$

represents the α-th order partial differentiation operator.

Definition 1.6 *Let Ω be a domain. Suppose $u, v \in L^1_{loc}(\Omega)$, and α is a multi-index. We say that v is the α^{th}-weak partial derivative of u, written*

$$D^\alpha u = v,$$

provided

$$\int_\Omega u D^\alpha \phi dx = (-1)^{|\alpha|} \int_\Omega v\phi dx$$

for all test functions $\phi \in C_0^\infty(\Omega)$.

Definition 1.7 *A sequence of functions $\{\phi_m\}$ in $C_0^\infty(\Omega)$ is said to converge to 0 if there exists a fixed compact set $K \subset \Omega$ such that supp $(\phi_m) \subset K$ for all m and $\{\phi_m\}$ and all its*

derivatives converge uniformly to zero on K.

Definition 1.8 *A linear functional T from* $C_0^\infty(\Omega)$ *to* $\mathbb{R}$ *is said to be a distribution, or generalized function if whenever* $\phi_m \to 0$ *in* C_0^∞*, we have* $T(\phi_m) \to 0$*.*

Definition 1.9 *The space* $C([0,T],X)$ *consists of all continuous functions* $u : [0,T] \to X$ *such that*

$$\|u\|_{C([0,T],X)} \equiv \max_{0\le t\le T} \|u(.,t)\|_X < \infty.$$

Definition 1.10 *Let k be a non-negative integer and let* $1 \le p \le \infty$*. Then we define* $W^{k,p}(\Omega)$ *to be set of all distributions* $u \in L^P(\Omega)$ *such that* $D^\alpha u \in L^P(\Omega)$ *for* $|\alpha| \le k$*.*

In $W^{k,p}(\Omega)$, we define a norm by

$$\|u\|_{k,p} := \Big(\sum_{|\alpha|\le k} \|D^\alpha u\|_p^p \Big)^{1/p} \quad if \quad 1 \le p < \infty$$

and

$$\|u\|_{k,\infty} := \max_{0\le|\alpha|\le k} \|D^\alpha u\|_\infty \quad if \quad p = \infty.$$

For $p = 2$ we define an inner product by

$$(u,v)_k := \sum_{|\alpha|\le k} \int_\Omega D^\alpha u(x) D^\alpha v(x) dx.$$

We also use the notation $H^k(\Omega)$ for $W^{k,2}(\Omega)$ and $L^2(\Omega)$ for $W^{0,2}(\Omega)$.

Definition 1.11 *By* $W_0^{k,p}(\Omega)$ *we denote the closure of* $C_0^\infty(\Omega)$ *in* $W^{k,p}(\Omega)$*.*

This means that $u \in W_0^{k,p}(\Omega)$ if and only if there exist functions $u_m \in C_0^\infty(\Omega)$ such that $u_m \to u \in W_0^{k,p}(\Omega)$.

Theorem 1.12 *(The divergence theorem) Let* Ω *be a bounded open subset of* $\mathbb{R}^n$ *with smooth (or piecewise smooth) boundary* $\partial\Omega$*. Let* $\mathbf{F}$ *be a smooth vector field defined in* $\mathbb{R}^n$ *and let* $\mathbf{n}$ *be the unit outward-pointing normal of* $\partial\Omega$*. Then*

$$\int_{\partial\Omega} \mathbf{F} \cdot \mathbf{n} ds = \int_{\Omega} \nabla \cdot \mathbf{F} dx.$$

1.3 Inequalities

In this section we give some basic inequalities which will be employed in the thesis.

Cauchy's inequality.

$$ab \leq \frac{a^2}{2} + \frac{b^2}{2}, \quad \forall a,\ b \in \mathbb{R}.$$

Cauchy's inequality with ϵ**.**

$$ab \leq \epsilon a^2 + \frac{b^2}{4\epsilon}, \ \ \forall a, b \in \mathbb{R},\ \forall \epsilon > 0.$$

Young's inequality. Let $1 < p, q < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q} \ \ (a, b > 0).$$

Cauchy-Schwarz's Inequality. Let H be a Hilbert space associated with the inner product $(.,.)$ and norm $||u|| = (u, u)^{1/2}$. Then

$$|(u, v)| \leq ||u||.||v||, \ \ \forall u, v \in H.$$

Hölder's Inequality. Assume $1 \leq p, q \leq \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then if $u \in L^p(\Omega)$, $v \in L^q(\Omega)$, we have

$$\int_{\Omega} |uv| dx \leq ||u||_{L^p} ||v||_{L^q}.$$

Gagliardo-Nirenberg's inequality.

$$||D^j u||_p \leq C||D^\mu u||_r^\theta ||u||_q^{1-\theta} \quad \forall u \in \mathbb{C}_0^\mu(\mathbb{R}^n),$$

where $0 \leq \frac{j}{\mu} \leq \theta < 1$, $C = C(n, \mu, j, q, r, \theta) > 0$ are constants, and

$$\frac{1}{p} = \frac{j}{n} + \theta(\frac{1}{r} - \frac{\mu}{n}) + (1-\theta)\frac{1}{q}.$$

CHAPTER 2

DECAY ESTIMATES FOR THE LINEAR WAVE EQUATIONS

2.1 Introduction

Here we consider the following initial value problem for the linear wave equation with variable coefficient dissipative term:

$$u_{tt} - \Delta u + a(x)u_t = 0, \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}^n, \tag{2.1}$$

$$u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x), \quad x \in \mathbb{R}^n, \tag{2.2}$$

where the unknown function $u(t, x) : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$ ($n \geq 3$ integer) denotes the displacement and the coefficient a is a decaying positive radial C^1 function which denotes the friction or potential. The initial conditions $u_0(x)$ and $u_1(x)$ are compactly supported functions from the energy space:

$$\begin{aligned} &u_0 \in H^1(\mathbb{R}^n), \quad u_1 \in L^2(\mathbb{R}^n), \\ &\operatorname{supp} u_i \subset B(K) \equiv \{x \in \mathbb{R}^n : |x| \leq K\}, \quad i = 0, 1. \end{aligned} \tag{2.3}$$

The last expression implies that

$$u_0(x) = 0 \text{ and } u_1(x) = 0 \text{ for } |x| > K$$

for some $K > 0$.

The existence of a solution u to (2.1)-(2.2) with the regularity

$$u \in C((0,\infty), H^1(\mathbb{R}^n)), \quad u_t \in C((0,\infty), L^2(\mathbb{R}^n)), \tag{2.4}$$

and compact support

$$u(t,x) = 0 \ \text{ for } \ |x| > t + K \tag{2.5}$$

can be obtained by the finite speed of propagation property (see [32] or [24]).

Our goal, in this chapter, is to review the results obtained in [22], where the authors have found multipliers that can capture the asymptotic behavior of solutions to problem (2.1)-(2.2). The idea used here to strengthen the L^2 estimates for (2.1)-(2.2) comes from the so called diffusion phenomenon for dissipative wave equations with constant potential $a(x) = 1$ (see [5], [6] and [7]) and the references therein. The diffusion phenomenon for wave equations with damping is crucial since it suggests considering multipliers related to a parabolic equation; these multipliers yield almost optimal decay estimates.

It can be seen from the previous studies that the multiplier method has been very effective in various problems concerning exponentially small solutions of dissipative wave equations. We refer to [1], [2], [3] and references therein. Applications to polynomially decaying solutions, however, have been less successful. A typical example is the Cauchy problem for the wave equation with a constant damping (i. e. $a(x)$= constant) whose solutions have L^2 norms decaying like $t^{-n/4}$(see [4]). In such a case or for the time dependent potential $a(t)$, the Fourier integral transform remains powerful tool and yields optimal results ([12], [13] and [9, 10, 11]). In particular, all multipliers derived from the scaling invariance of the wave operator ($x \cdot \nabla u$ and similar Morawetz multipliers) yield dimension independent estimates of the L^2 norm, such as $O(t^{-1/2})$ for all $n \geq 2$. Below we present more examples suggesting that the multiplier method

based on symmetries of the wave operator is not very suitable for dissipative wave equations. (Possible exceptions are $n = 1$ and $n = 2$).

On the other hand, the Fourier integral transform becomes useless when the damping is a space dependent potential $a(x)$. The alternative is a multiplier method.

The asymptotic behavior of the L^2 norm and energy is well understood when $a = a_0$ is a positive constant. Matsumura [4], using the Fourier analysis, has shown that the solution of (2.1)-(2.2) satisfies the estimate

$$\int u^2 dx \le C_0 t^{-\frac{n}{2}},$$
$$\int (u_t^2 + |\nabla u|^2) dx \le C_0 t^{-\frac{n}{2}-1},$$

where C_0 is a positive constant depending on the initial data.

When $a = a(t)$ is a function of time, the authors (see [9, 10, 11], [12] and [13]) have applied the Fourier integral transform and obtained a series of sharp estimates for (2.1)-(2.2), including

$$\int u^2 dx \le C_0 t^{-(1-k)\frac{n}{2}}, \tag{2.6}$$
$$\int (u_t^2 + |\nabla u|^2) dx \le C_0 t^{-(1-k)(\frac{n}{2}+1)},$$

for $a(t) = a_0(1+t)^k$ and $k \in (-1, 1)$, where a_0 and C_0 are positive constants. Let us point out a striking phenomenon: The damping coefficient weakens but the decay rates increase as $k \to -1$. Here the absence of x in the coefficient a allows a partial Fourier transform of u and a WKB-representation of u by Fourier multipliers. This approach also generalizes Strichartz' estimates [14].

When $a = a(x)$ Fourier techniques become cumbersome as they involve localizations in both frequency space and extended phase space $(0, \infty) \times \mathbb{R}^n$. A simple alternative is the multiplier method. Below we discuss several results found by non-trivial

applications of this method.

Matsumura [15] and Uesaka [16] have established decay estimates for the energy when the coefficient has critical decay

$$a(t,x) \geq a_0(1+t+|x|)^{-1},$$

with $a_0 > 0$. Their results can be stated as

$$\int (u_t^2 + |\nabla u|^2)dx \leq C_0 t^{-\min\{2,\, a_0\}}. \tag{2.7}$$

The basic multiplier is $\{w(t)u\}_t$ with a suitable weight $w(t)$. Remarkably this method can handle the critical decay, such as

$$a(x) \geq a_0(1+|x|)^{-\alpha}, \quad \alpha \in [0,1). \tag{2.8}$$

In fact estimate (2.7) does not remain sharp for stronger dissipations; notice that the decay rate of energy should be close to $t^{-\frac{n}{2}-1}$ if α is close to zero.

Multipliers depending on t and x yield sharper estimates for (2.1)-(2.2). The equation is studied in the exterior of star-shaped domains by Nakao [17], Mochizuki and Nakazawa [18], and Ikehata and Matsuyama [19]. If $n \geq 2$ and

$$a \in L^\infty, \quad a(x) \geq a_0$$

for large x, the result of [19], which is the best possible rate in $\mathbb{R}^2$, is that the energy decays like t^{-2}. A clever integration of (2.1) with respect to t and a radial multiplier, similar to Morawetz's multiplier in [20], are the decisive ideas.

Another interesting application of the multiplier method is the weighted estimate

of Ikehata [21] for coefficients

$$a(x) = \frac{a_0}{|x|^\alpha}, \quad \alpha \in [0, 1).$$

Using an exponential multiplier, he has established the following:

$$\int e^{2a_0(2-\alpha)^{-2}\frac{|x|^{2-\alpha}}{t}}(u_t^2 + |\nabla u|^2)dx \leq C_0,$$

for $t \geq 1$. Hence u decays very fast if $|x|^{2-\alpha}/t$ is large. If this ratio is small, the result does not provide new information.

In general multiplier techniques yield weaker decay estimates than the Fourier techniques whenever the latter can be applied to (2.1)-(2.2). The discrepancy may be explained with the equal weight of u_t^2 and $|\nabla u|^2$ in the standard multiplier method, while these terms are very different at large times. In fact the so-called diffusion phenomenon is observed for $a = 1$, which means that $u \sim C_0 w$ for a constant C_0 depending on the initial data and a solution w of the diffusion equation

$$w_t - \Delta w = 0, \quad x \in \mathbb{R}^n, \quad t > 0.$$

See the review by Narazaki [5]. We can actually use a Gaussian

$$w(t, x) = t^{-\frac{n}{2}} e^{-\frac{|x|^2}{4t}}.$$

Clearly the time derivative of the Gaussian is smaller than its spatial derivatives, that is, $\|w_t\|_{L^2}$ is smaller than $\|\nabla w\|_{L^2}$.

The diffusion phenomenon for $a = a(t)$ was studied by Wirth [9]. Here the precise meaning of $u \sim C_0 w$ and the exact constant C_0 are more subtle. However, the important

fact is that $w^{-1}u \sim C_0$ for a suitable solution of the diffusion equation

$$a(t)w_t - \Delta w = 0, \quad x \in R^n, \quad t > 0.$$

In this chapter, we review the paper of Todorova and Yordanov [22], where they have strengthened the multiplier method for (2.1)-(2.2) by factoring out the asymptotic profile w and working with $w^{-1}u$. This quotient will admit more precise estimates since it will vary relatively slowly. Unexpectedly w is allowed to be an approximate solution of

$$a(t)w_t - \Delta w = 0, \quad x \in \mathbb{R}^n, \quad t > 0, \tag{2.9}$$

see formula (2.35) below. The decay rate of u will be expressed in terms of w and, implicitly, in terms of a. In fact they have derived explicit rates in several cases. These decay estimates are sharp at the limit $\alpha \to 0$ in (2.8), but we do not know whether they are sharp for all $\alpha \in [0, 1)$.

The rest of this chapter is organized as follows. In Section 2.2 the main weighted energy identities are presented. In Section 2.3 first the modified equation for $w^{-1}u$ is derived. Next the weights are chosen and then the proof of main results are given.

2.2 A Weighted Energy Identity

Let us first consider a general first-order perturbation of the wave equation:

$$u_{tt} - \Delta u + au_t + b \cdot \nabla u + cu = 0, \tag{2.10}$$

where the coefficients a, $b = (b_1, b_2, \cdots, b_n)$ and c are C^1 functions. Our aim is to derive a weighted identity for u involving three positive C^2 functions w_0, w_1 and w.

Now defining $u = w\hat{u}$ and substituting derivatives of u into (2.10), we have

$$\hat{u}_{tt} - \Delta\hat{u} + \hat{a}\hat{u}_t + \hat{b}\cdot\nabla\hat{u} + \hat{c}\hat{u} = 0 \tag{2.11}$$

with new coefficients

$$\begin{aligned} &\hat{a} = a + 2w^{-1}w_t, \quad \hat{b} = b - 2w^{-1}\nabla w, \\ &\hat{c} = w^{-1}(w_{tt} - \Delta w + aw_t + b\cdot\nabla w + cw). \end{aligned} \tag{2.12}$$

We note that here and hereafter we will use the property of finite propagation speed of solutions of (2.12) in evaluating the boundary integrals. Also for the sake of convenience, the symbol $\int$ denote integration over $\mathbb{R}^n$.

Proposition 2.1 *Let u be a solution of (2.10) with compactly supported initial data $u_0 \in H^2(\mathbb{R}^n)$ and $u_1 \in H^1(\mathbb{R}^n)$. Assume that w_0, w_1 and $w > 0$ are C^2 functions. Then the identity*

$$\frac{d}{dt}E(\hat{u}_t, \nabla\hat{u}, \hat{u}) + F(\hat{u}_t\nabla\hat{u}) + G(\hat{u}) = 0, \tag{2.13}$$

holds, where

$$E(\hat{u}_t, \nabla\hat{u}, \hat{u}) = \frac{1}{2}\int\Big[w_1(\hat{u}_t^2 + |\nabla\hat{u}|^2) + 2w_0\hat{u}_t\hat{u} + (\hat{c}w_1 - \partial_t w_0 + \hat{a}w_0)\hat{u}^2\Big]dx,$$

$$\begin{aligned} F(\hat{u}_t, \nabla\hat{u}) \;=\; & \frac{1}{2}\int\Big(-\partial_t w_1 + 2\hat{a}w_1 - 2w_0\Big)\hat{u}_t^2 dx \\ & + \int(\nabla w_1 + \hat{b}w_1)\cdot\hat{u}_t\nabla\hat{u}dx + \frac{1}{2}\int(-\partial_t w_1 + 2w_0)|\nabla\hat{u}|^2 dx, \end{aligned}$$

and

$$G(\hat{u}) = \frac{1}{2}\int[\partial_t^2 w_0 - \Delta w_0 - \partial_t(\hat{a}w_0) - \nabla\cdot(\hat{b}w_0) + 2\hat{c}w_0 - \partial_t(\hat{c}w_1)]\hat{u}^2 dx.$$

The coefficients $\hat{a}$, $\hat{b}$ and $\hat{c}$ are given by (2.12).

Proof. Multiplying equation (2.11) by $w_0\hat{u}$ and $w_1\hat{u}_t$ and integrating over $\mathbb{R}^n$ and using the divergence theorem, respectively, we obtain

$$\begin{aligned}\int w_0\hat{u}\hat{u}_{tt}\,dx \;&+\; \int \nabla\hat{u}\cdot\nabla(w_0\hat{u})\,dx + \int \hat{a}w_0\hat{u}\hat{u}_t\,dx \\ &+\; \int w_0\hat{u}\hat{b}\cdot\nabla\hat{u}\,dx + \int \hat{c}w_0\hat{u}^2\,dx = 0\end{aligned} \tag{2.14}$$

and

$$\begin{aligned}\int w_1\hat{u}_t\hat{u}_{tt}\,dx \;&+\; \int \nabla\hat{u}\cdot\nabla(w_1\hat{u}_t)\,dx + \int \hat{a}w_1\hat{u}_t^2\,dx \\ &+\; \int \hat{b}\cdot\nabla\hat{u}w_1\hat{u}_t\,dx + \int \hat{c}w_1\hat{u}\hat{u}_t\,dx = 0.\end{aligned} \tag{2.15}$$

Adding equations (2.14) and (2.15) and employing the identities

$$\begin{aligned}w_1\hat{u}_t\hat{u}_{tt} &= \frac{d}{dt}\left(\frac{w_1\hat{u}_t^2}{2}\right) - \frac{1}{2}\hat{u}_t^2\partial_t w_1, \\ w_1\hat{u}_t\Delta\hat{u} &= \nabla\cdot(w_1\hat{u}_t\nabla\hat{u}) - w_1\frac{d}{dt}\frac{|\nabla\hat{u}|^2}{2} - \hat{u}_t\nabla u\nabla w_1, \\ w_0\hat{u}\hat{u}_{tt} &= \frac{d}{dt}(w_0\hat{u}\hat{u}_t) - w_0\hat{u}_t^2, \\ \nabla w_0\cdot\hat{u}\nabla\hat{u} &= \frac{1}{2}\nabla\cdot(\hat{u}^2\nabla w_0) - \frac{1}{2}\hat{u}^2\Delta w_0\end{aligned} \tag{2.16}$$

we obtain the desired identity. □

We can apply the identity in Proposition 2.1 to the wave equation with damping (2.1). Here

$$a = a(x),\quad b = 0,\quad c = 0,$$

so the transformed coefficients (2.12) become

$$\hat{a} = a + 2w^{-1}w_t,\quad \hat{b} = -2w^{-1}\nabla w,$$

$$\hat{c} = w^{-1}(w_{tt} - \Delta w + a w_t). \tag{2.17}$$

The weighted identity for $\hat{u}$ in Proposition 2.1 simplifies if we choose $w_0 = w$. There is no simple expression for w_1 in terms of w, so w_1 will be chosen later. It is convenient to keep the most complex coefficient $\hat{c}$ and express the other coefficients in terms of w, w_1 and a. The simplified functionals are

$$\begin{aligned}
E(\hat{u}_t, \nabla\hat{u}, \hat{u}) &= \frac{1}{2}\int \Big[w_1(\hat{u}_t^2 + |\nabla\hat{u}|^2) + 2w\hat{u}_t\hat{u} + (\hat{c}w_1 - \partial_t w + aw)\hat{u}^2 \Big] dx, \\
F(\hat{u}_t, \nabla\hat{u}) &= \frac{1}{2}\int \Big(-\partial_t w_1 + 2aw_1 + 4w_1\partial_t \ln w - 2w \Big) \hat{u}_t^2 dx \\
&\quad + \int (\nabla w_1 - 2w_1 \nabla \ln w) \cdot \hat{u}_t \nabla\hat{u} dx \\
&\quad + \frac{1}{2}\int (-\partial_t w_1 + 2w)|\nabla\hat{u}|^2 dx, \\
G(\hat{u}) &= \frac{1}{2}\int [\hat{c}w - \partial_t(\hat{c}w_1)]\hat{u}^2 dx
\end{aligned}$$

which satisfy identity (2.13). However, we will show an inequality that holds for all initial data in the energy space. At the same time we need more conditions on the damping terms and weight functions w and w_1 to insure $F \geq 0$ and $G \geq 0$, so $dE/dt \leq 0$. These are given in the following proposition.

Proposition 2.2 *Let w and w_1 be positive weights and let $\hat{c}$ be defined in* (2.17)*. Assume that the conditions*

(i) $\hat{c} \geq 0, \quad \partial_t \hat{c} \leq 0,$

(ii) $-\partial_t w_1 + w \geq 0,$

(iii) $(-\partial_t w_1 + 2w)(-\partial_t w_1 + 2aw_1 + 4w_1\partial_t \ln w - 2w) \geq (\nabla w_1 - 2w_1 \nabla \ln w)^2$

are satisfied. Then for a solution u of (2.1)-(2.2) with $u_0 \in H^1(\mathbb{R}^n)$ and $u_1 \in L^2(\mathbb{R}^n)$,

$E(\hat{u}_t, \nabla\hat{u}, \hat{u})$ is a non-increasing function of time:

$$\frac{1}{2}\int\Big[w_1(\hat{u}_t^2 + |\nabla\hat{u}|^2) + 2w\hat{u}_t\hat{u} + (\hat{c}w_1 + \partial_t w + aw)\hat{u}^2\Big]dx \le E_0 \tag{2.18}$$

for all $t \ge t_0$, where $\hat{u} = w^{-1}u$ and $E_0 = E(\hat{u}_t, \nabla\hat{u}, \hat{u})|_{t=t_0}$.

Proof. If $u_0 \in H^2(\mathbb{R}^n)$ and $u_1 \in H^1(\mathbb{R}^n)$ are compactly supported, then identity (2.13) holds. Notice that conditions (*i*) and (*ii*) imply

$$\hat{c}w - \partial_t(\hat{c}w_1) = \hat{c}(w - \partial_t w_1) - (\partial_t\hat{c})w_1 \ge 0.$$

Hence $G(\hat{u}) \ge 0$. Condition (*iii*) and $-\partial_t w_1 + 2w \ge 0$, which follows from (*ii*), guarantee that the quadratic form $F(\hat{u}_t, \nabla\hat{u}) \ge 0$. Thus (2.13) yields $dE(\hat{u}_t, \nabla\hat{u}, \hat{u})/dt \le 0$ or after integration with respect to t,

$$E(\hat{u}_t, \nabla\hat{u}, \hat{u}) \le E_0$$

holds.

Let us show that the last estimate holds for all initial data in the energy space. For any compactly supported initial data $u_0 \in H^1$ and $u_1 \in L^2$, there exist compactly supported C^∞ sequences of functions $u_0^{(k)} \to u_0$ in H^1 and $u_1^{(k)} \to u_1$ in L^2. Let us denote the corresponding solutions of (2.1)-(2.2) by $u^{(k)}$ and their weighted energy functionals by $E(\hat{u}_t^{(k)}, \nabla\hat{u}^{(k)}, \hat{u}^{(k)})$. The first part of proof shows that

$$E(\hat{u}_t^{(k)}, \nabla\hat{u}^{(k)}, \hat{u}^{(k)}) \le E(\hat{u}_t^{(k)}, \nabla\hat{u}^{(k)}, \hat{u}^{(k)})|_{t=t_0}. \tag{2.19}$$

Since the weights w and w_1 are continuous functions, the weighted energy is bounded by the standard energy:

$$|E(\hat{u}_t^{(k)}, \nabla\hat{u}^{(k)}, \hat{u}^{(k)}) - E(\hat{u}_t, \nabla\hat{u}, \hat{u})| \le c(T)\Big(\|u_t^{(k)} - u_t\|_{L^2} + \|\nabla u^{(k)} - \nabla u\|_{L^2}\Big),$$

whenever $t \in [0, T]$. Also it is not difficult to verify that the weak solutions of problem (2.1)-(2.2) satisfy the energy estimate

$$\|u_t^{(k)} - u_t\|_{L^2}^2 + \|\nabla u^{(k)} - \nabla u\|_{L^2}^2 \leq c\Big(\|u_1^{(k)} - u_1\|_{L^2}^2 + \|\nabla u_0^{(k)} - \nabla u_0\|_{L^2}^2\Big).$$

Hence we have

$$|E(\hat{u}_t^{(k)}, \nabla\hat{u}^{(k)}, \hat{u}^{(k)}) - E(\hat{u}_t, \nabla\hat{u}, \hat{u})| \leq c(T)\Big(\|u_t^{(k)} - u_t\|_{L^2} + \|\nabla u_0^{(k)} - \nabla u_0\|_{L^2}\Big).$$

Passing to the limit as $k \to \infty$ we have

$$\begin{aligned} E(\hat{u}_t, \nabla\hat{u}, \hat{u}) &= \lim_{k\to\infty} E(\hat{u}_t^{(k)}, \nabla\hat{u}^{(k)}, \hat{u}^{(k)}) \\ &\leq \lim_{k\to\infty} E(\hat{u}_t^{(k)}, \nabla\hat{u}^{(k)}, \hat{u}^{(k)})|_{t=t_0} = E_0. \end{aligned} \tag{2.20}$$

This completes the proof for general initial data taken from the energy space. □

It is essential to have a positive definite weighted energy $E(\hat{u}_t, \nabla\hat{u}, \hat{u})$. The next step is to find some sufficient conditions. We will rely on an auxiliary estimate derived from the inequality $E(\hat{u}_t, \nabla\hat{u}, \hat{u}) \leq E_0$ in Proposition (2.1). In fact, neglecting a few non-negative terms of $E(\hat{u}_t, \nabla\hat{u}, \hat{u})$ and using the identity

$$2w\hat{u}\partial_t\hat{u} + \partial_t w\hat{u}^2 = \partial_t(w\hat{u}^2)$$

from (2.20) we obtain

$$\frac{1}{2}\frac{d}{dt}\int w\hat{u}^2 dx + \frac{1}{2}\int aw\hat{u}^2 dx \leq E_0. \tag{2.21}$$

This is sufficient for an upper bound on the weighted L^2 norm if a satisfies

$$a_0(1 + |x|)^{-\alpha} \leq a(x), \quad \alpha \in [0, 1). \tag{2.22}$$

Proposition 2.3 *Let w and w_1 be positive weights satisfying conditions $(i) - (iii)$ in Proposition 2.2. Assume that a satisfies (2.22). Then the following estimate holds for the solution u of problem (2.1)-(2.2):*

$$\int w\hat{u}^2 dx \leq N_0 + CE_0 t^{\alpha}$$

for $t \geq t_0$, where $N_0 = \int w\hat{u}^2 dx|_{t=t_0}$.

Proof. Since $|x| \leq t + K$ on the support of $\hat{u}$, (2.21) and (2.22) lead to

$$y'(t) + a_0(1 + t + K)^{-\alpha} y(t) \leq 2E_0,$$

where

$$y(t) = \int w\hat{u}^2 dx,$$

Rewriting the above inequality as

$$\frac{d}{dt}\left(e^{\frac{a_0}{1-\alpha}(1+t+K)^{1-\alpha}} y(t)\right) \leq 2E_0 e^{\frac{a_0}{1-\alpha}(1+t+K)^{1-\alpha}}$$

and integrating from t_0 to t we get

$$y(t) \leq e^{\frac{a_0}{1-\alpha}(1+t_0+K)^{1-\alpha}} e^{-\frac{a_0}{1-\alpha}(1+t+K)^{1-\alpha}} N_0 + 2E_0 e^{-\frac{a_0}{1-\alpha}(1+t+K)^{1-\alpha}} \int_{t_0}^{t} e^{\frac{a_0}{1-\alpha}(1+\tau+K)^{1-\alpha}}\, d\tau.$$

Using the estimate

$$\int_{t_0}^{t} e^{C\tau^{1-\alpha}} d\tau = \frac{1}{1-\alpha} \int_{t_0^{1-\alpha}}^{t^{1-\alpha}} e^{Cz} z^{\frac{\alpha}{1-\alpha}} dz \leq \frac{t^{\alpha} e^{Ct^{1-\alpha}}}{C(1-\alpha)}$$

on the right hand side of the above inequality gives the desired inequality. □

We can now state the conditions for positive definiteness of $E(\hat{u}_t, \nabla\hat{u}, \hat{u})$ which is a consequence of Propositions 2.2 and 2.3.

Proposition 2.4 *Assume that condition (2.22) on a holds. If w and w_1 satisfy conditions* $(i) - (iii)$ *in Proposition 2.1 and the additional conditions*

(iv) $w \le C_1 t^{-\alpha} w_1$,

(v) $\partial_t w \ge -C_1 t^{-\alpha} w$

with $C_1 > 0$, *then the solution u of (2.1)-(2.2) satisfies*

$$\begin{aligned} \int w_1(\hat{u}_t^2 + |\nabla \hat{u}^2|)dx &\le C(N_0 + E_0), \\ \int a w \hat{u}^2 dx &\le C(N_0 + E_0), \end{aligned}$$

for $t \ge t_0$, *where* E_0 *and* N_0 *are given in Propositions 2.1 and 2.2, respectively.*

Proof. First an application of Cauchy's inequality with $\epsilon > 0$ gives

$$|2w\hat{u}_t\hat{u}| \le \epsilon t^{\alpha} w \hat{u}_t^2 + \epsilon^{-1} t^{-\alpha} w \hat{u}^2.$$

Using this estimate in (2.18) we obtain

$$\int \Big[(w_1 - \epsilon t^{\alpha} w)(\hat{u}_t^2 + |\nabla \hat{u}|^2) + (\hat{c} w_1 + \partial_t w + aw - \epsilon^{-1} t^{-\alpha} w)\hat{u}^2 \Big] dx \le 2E_0.$$

By using (iv) and (v) in this inequality and after some rearrangement we get

$$\begin{aligned} (1 - \epsilon C_1) \int w_1(\hat{u}_t^2 &+ |\nabla \hat{u}|^2)dx + \int (\hat{c} w_1 + aw)\hat{u}^2 dx \\ &\le 2E_0 + (C_1 + \epsilon^{-1}) t^{-\alpha} \int w \hat{u}^2 dx. \end{aligned}$$

Let us choose $\epsilon = (2C_1)^{-1}$ and estimate the right-hand side by Proposition 2.3 to have

$$\int w_1(\hat{u}_t^2 + |\nabla \hat{u}|^2)dx + 2\int (\hat{c} w_1 + aw)\hat{u}^2 dx \le C(E_0 + N_0)$$

for $t \geq t_0$, where $C = \max\{(2 + 6C_1C), 6C_1t_0^{-\alpha}\}$. The proof is complete since $\hat{c}w_1 \geq 0$. □

Since $\hat{u} = w^{-1}u$, Proposition 2.4 shows that the L^2 norm of the original solution u satisfies

$$\int aw^{-1}u^2dx \leq C(N_0 + E_0). \tag{2.23}$$

The corresponding estimates of u_t and $|\nabla u|$ in terms of $\hat{u}_t$ and $\nabla\hat{u}$ are not immediate. In fact we need another assumption on w and w_1. The final estimates are given below.

Proposition 2.5 *Assume that conditions (2.22) and* (*i*) - (*v*) *hold. If*

(vi) $w_1w^{-3}(w_t^2 + |\nabla w|^2) \leq C_2a(x)$

holds for some $C_2 > 0$, *the solution u of problem (2.1)-(2.2) satisfies*

$$\begin{aligned} \int aw^{-1}u^2\,dx &\leq C(N_0 + E_0), \\ \int w_1w^{-2}(\hat{u}_t^2 + |\nabla\hat{u}|^2)\,dx &\leq C(N_0 + E_0) \end{aligned}$$

for all $t \geq t_0$.

Proof. The first estimate is just (2.23). Only the estimates of u_t and ∇u require proof. We notice that

$$\hat{u}_t^2 = (-w^{-2}w_tu + w^{-1}u_t)^2 \geq \frac{1}{2}w^{-2}u_t^2 - 3w^{-4}w_t^2u^2$$

and

$$|\nabla\hat{u}|^2 = (-w^{-2}\nabla wu + w^{-1}\nabla u)^2 \geq \frac{1}{2}w^{-2}|\nabla u|^2 - 3w^{-4}|\nabla w|^2u^2.$$

Thus

$$\frac{1}{2}w_1 w^{-2}(u_t^2 + |\nabla u|^2) \le w_1(\hat{u}_t^2 + |\nabla \hat{u}|^2) + 3w_1 w^{-4}(w_t^2 + |\nabla w|^2)u^2.$$

Integrating this inequality and applying the estimate in Proposition 2.4, we obtain

$$\frac{1}{2}\int w_1 w^{-2}(u_t^2 + |\nabla u|^2)dx \le C(N_0 + E_0) + 3\int w_1 w^{-4}(w_t^2 + |\nabla w|^2)u^2 dx.$$

Condition (vi) implies that the integral on the left side is bounded by

$$3C_2 \int a w^{-1} u^2 dx \le C(N_0 + E_0),$$

as a consequence of (2.23). □

Let us point out that the last result yields non-trivial estimates if the weights not only have properties $(i) - (vi)$ but also decay sufficiently fast as t and $|x|$ go to infinity. For instance, the non-decaying weights $w = 1$ and $w_1 = t$ are admissible but the estimates in Proposition 2.5 are rather weak for this choice:

$$\begin{aligned} \int a u^2 dx &\le C(N_0 + E_0), \\ \int (u_t^2 + |\nabla u|^2)dx &\le C(N_0 + E_0)t^{-1}. \end{aligned}$$

The existence of more useful weights is related with the properties of solutions of the Poisson equation in $\mathbb{R}^n$. We study this problem in the next section.

2.3 Construction of weights and decay estimates

In this section we will find non-trivial weights w and w_1 which meet conditions $(i) - (vi)$ in the previous section. For reasons discussed in the introduction, w will be an

approximate solution of equation (2.1)-(2.2)(or (2.9) and w_1 will be similar to $\int w dt$.

To establish the decay estimates of solutions u of (2.1)-(2.2), we will first study the Poisson equation

$$\Delta A(x) = a(x), \quad x \in \mathbb{R}^n, \tag{2.24}$$

where $a(x)$ is a positive radial function satisfying condition (2.22) in $n \geq 3$ dimensions. Here we will show that equation (2.24) has a radial solution $A(x)$ in the class of functions satisfying

(a1) $A(x) \geq 0$ for all x,

(a2) $A(x) = 0(|x|^{2-\alpha})$ for large $|x|$,

(a3) $m(a) = \liminf_{x\to\infty} \dfrac{a(x)A(x)}{|\nabla A(x)|^2} > 0.$

We rely on the radial symmetry of a to simplify the solution, since the Poisson equation for radial functions becomes an ODE:

$$\frac{d^2A}{dr^2} + \frac{n-1}{r}\frac{dA}{dr} = a(r), \tag{2.25}$$

where $r = |x| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$ and we write $a(r)$ instead of $a(r\omega)$, $\omega \in S^{n-1}$. Notice that we can multiply (2.25) with r^{n-1} and rewrite the left-hand side to obtain

$$\frac{d}{dr}\left(r^{n-1}\frac{dA}{dr}\right) = r^{n-1}a(r). \tag{2.26}$$

Integrating this equation on $[0, r]$, $r > 0$ we have

$$\frac{dA(r)}{dr} = r^{1-n}\int_0^r \tau^{n-1}a(\tau)d\tau. \tag{2.27}$$

Integrating the last equality once more, choosing $A(0) = 1$ and reversing the order of

integration we get

$$A(r) = 1 + \frac{1}{n-2}\int_0^r (1 - \frac{\tau^{n-2}}{r^{n-2}})\tau a(\tau)d\tau. \tag{2.28}$$

This solution satisfies $A(0) = 1$ and $A'(0) = 0$. Recall that $A(r)$ means $A(r\omega)$ with $\omega \in S^{n-1}$. We will use the above formulas for A to show the following.

Proposition 2.6 *Assume that a(x) depends on $r = |x|$ and satisfies*

$$a_0(1+|x|)^{-\alpha} \le a(x) \le a_1(1+|x|)^{-\alpha}, \ \ \alpha \in [0,1), \tag{2.29}$$

where a_0 and a_1 are positive constants. Then the solution A(x) of equation (2.25) defined in (2.28) satisfies

$$A_0(1+|x|)^{2-\alpha} \le A(x) \le A_1(1+|x|)^{2-\alpha}, \ \ x \in \mathbb{R}^n,$$
$$A_2(1+|x|)^{1-\alpha} \le |\nabla A(x)| \le A_3(1+|x|)^{1-\alpha}, \ \ x \ge 1,$$

for some $A_0, \cdots, A_3 > 0$.

Proof. To estimate A we substitute the lower and upper bounds of a into formula (2.28). Thus we have

$$1 + a_0 I(|x|) \le A(x) \le 1 + a_1 I(|x|),$$

where

$$I(r) = \frac{1}{n-2}\int_0^r \tau\Big(1 - \frac{\tau^{n-2}}{r^{n-2}}\Big)(1+\tau)^{-\alpha}d\tau, \ \ r > 0.$$

Clearly

$$I(r) \le \frac{1}{n-2}\int_0^r \tau(1+\tau)^{-\alpha}d\tau \le I_1(1+r)^{2-\alpha}$$

with some $I_1 > 0$. To derive a lower bound we use

$$\begin{aligned} I(r) &\ge \frac{1}{n-2}\int_0^r \tau\Big(1-\frac{\tau^{n-2}}{r^{n-2}}\Big)(1+\tau)^{-\alpha}d\tau \\ &\ge \frac{1}{n-2}\int_0^{r/2}\frac{1}{2}\tau(1+\tau)^{-\alpha}d\tau \\ &\ge I_0 r^2(1+r)^{-\alpha}, \end{aligned}$$

where $I_0 > 0$. Hence (2.28) becomes

$$1 + a_0 I_0 \frac{|x|^2}{(1+|x|)^\alpha} \le A(x) \le 1 + a_1 I_1 (1+|x|)^{2-\alpha}.$$

The above two-sided inequality is equivalent to the first claim in Proposition 2.6.

To show the second claim, we substitute the lower and upper bounds of $a(r)$ into expression (2.27) for $dA(r)/dr$. Thus after integration we get

$$1 + a_0 J(|x|) \le A(x) \le 1 + a_1 J(|x|),$$

where

$$J(r) = r^{1-n}\int_0^r \tau^{n-1}(1+\tau)^{-\alpha}d\tau, \quad r > 0.$$

So we estimate

$$J(r) \le \int_0^r (1+\tau)^{-\alpha}d\tau \le J_1(1+r)^{1-\alpha}.$$

The lower bound of J(r) is similar: if $r \geq 1$,

$$J(r) \geq r^{1-n}(1+r)^{-\alpha} \int_0^r \tau^{n-1} \geq J_0(1+r)^{-\alpha}$$

where $J_0 > 0$ is independent of r. The proof of Proposition 2.6 is complete.

In the next proposition we show that such solutions $A(x)$, including radial coefficients $a(x)$ behave like $|x|^{-\alpha}$ as $|x| \to \infty$. □

Proposition 2.7 *Let a be a radially symmetric function in* $C^1(\mathbb{R}^n)$, $n \geq 3$ *satisfying (2.29). Then equation (2.24) admits a solution* $A \in C^2(\mathbb{R}^n)$ *such that*

(A1) $A_0(1+|x|)^{2-\alpha} \leq A(x) \leq A_1(1+|x|)^{2-\alpha}$,

(A2) $m(a) > 0$,

where A_0 *and* A_1 *are positive constants. In the special case*

$$a(x) \sim a_2|x|^{-\alpha}, \quad |x| \to \infty, \tag{2.30}$$

with $a_2 > 0$, *equation (2.24) has a solution with the following properties:*

(A3) $A(x) \sim \frac{a_2}{(2-\alpha)(n-\alpha)}|x|^{-\alpha}$, $|x| \to \infty$,

(A4) $m(a) = \frac{n-\alpha}{2-\alpha}$.

Proof. To prove claims (*A*1) and (*A*2) we apply Proposition 2.6 to the solution *A* given by (2.28). It is clear that this solution meets conditions (*A*1)-(*A*2). Claims (*A*3) and (*A*4) follow from (2.27) and

$$a(r) = a_2 r^{-\alpha} + o(r^{-\alpha}) :$$

$$\frac{dA(r)}{dr} = r^{1-n} \int_0^r \tau^{n-1}[a_2\tau^{-\alpha} + o(\tau^{-\alpha})]d\tau, \quad r \to \infty.$$

Hence

$$\frac{dA(r)}{dr} = \frac{a_2}{n-\alpha} r^{1-\alpha} + o(r^{1-\alpha}),$$

$$A(r) = \frac{a_2}{(n-\alpha)(2-\alpha)} r^{2-\alpha} + o(r^{2-\alpha}),$$

as $r \to \infty$. Now the definition of $m(a)$ yields

$$m(a) = \frac{n-\alpha}{2-\alpha}.$$

□

We have shown that there exists a solution A with properties $(a1)$-$(a3)$. Given a small $\delta \in (0, \frac{1}{2}m(a))$ and a large $S_0 > 0$, we set

$$m = m(a) - 2\delta, \quad S(x) = (m(a) - \delta)A(x) + S_0. \tag{2.31}$$

The two weights are defined as

$$w(t,x) = t^{-m} e^{-\frac{S(x)}{t}}, \quad w_1(t,x) = \frac{3}{4}\left(\frac{6}{t} + \frac{S(x)}{t^2}\right)^{-1} w(t,x). \tag{2.32}$$

Constants like S_0, $\frac{3}{4}$ and 6 are introduced for technical reasons. However, $\delta > 0$ plays an essential role in the proof of condition (i), i. e. , $\hat{c} \geq 0$ and $\partial_t \hat{c} \leq 0$. This parameter leads to the loss of t^δ in our decay estimates.

First we collect a few useful properties of m and S.

Lemma 2.8 *Define m and S by (2.31) and assume that $m(a)$ and A have properties $(a1) - (a3)$. There exists $S_0 > 0$ such that*

(S1) $\Delta S(x) = (m + \delta)a(x)$ *for all x,*

(S2) $S(x) = O(|x|^{2-\alpha})$ *for large $|x|$,*

(S3) $\left(1 - \frac{\delta}{2m(a)}\right) a(x)S(x) - |\nabla S(x)|^2 \geq 0$ *for all x.*

Proof. Property $(S\,1)$ follows from (2.31) and $\Delta A = a$. Property $(S\,2)$ is a restatement of $(a2)$. To verify $(S\,3)$, we fix $\epsilon > 0$ and use $(a3)$ to obtain the inequality

$$\frac{1+\epsilon}{m(a)}\, a(x)A(x) - |\nabla A(x)|^2 \geq 0$$

for sufficiently large $|x|$. Multiplying through with $(m(a)-\delta)^2$, we have

$$\frac{(1+\epsilon)(m(a-\delta)}{m(a)} a(x)\tilde{S}(x) - |\nabla \tilde{S}(x)|^2 \geq 0$$

with $\tilde{S}(x) = (m(a)-\delta)A(x)$. We now choose $\epsilon = \delta/(2(m(a)-\delta))$. Inequality $(S\,3)$ holds with $S(x) = \tilde{S}(x)$ if $|x|$ is large. Clearly $S(x) = \tilde{S}(x) + S_0$ satisfies $(S\,3)$ for all x if S_0 is large constant depending on δ.

Next we calculate the first and second order derivatives of w:

$$\begin{aligned} w_t &= \Big(-\frac{m}{t} + \frac{S(x)}{t^2}\Big) w, \qquad w_{tt} = \Big(-\frac{m}{t} + \frac{S(x)}{t^2}\Big)^2 w + \Big(-\frac{m}{t^2} + \frac{2S(x)}{t^3}\Big) w, \\ \nabla w &= -\frac{\nabla S(x)}{t} w, \qquad \Delta w = \Big(-\frac{\Delta S(x)}{t} + \frac{|\nabla S(x)|^2}{t^2}\Big) w. \end{aligned} \tag{2.33}$$

□

Thus we can express the coefficient $\hat{c}$ in terms of m and S.

Lemma 2.9 *Let w and w_1 be defined by (2.32) and $\hat{c}$ be defined at (2.17). Then*

$$\begin{aligned} \hat{c} &= \frac{\Delta S - ma}{t} + \frac{aS - |\nabla S|^2}{t^2} \\ &+ \Big(-\frac{m}{t} + \frac{S(x)}{t^2}\Big)^2 + \frac{m}{t^2} - \frac{2S(x)}{t^3}. \end{aligned}$$

Proof. We substitute formulas (2.33) for $w_{tt}, \Delta w$, and w_t into (2.17). □

Finally we state a simple but useful upper bound on $\partial_t w_1 / w_1$.

Lemma 2.10 *Let w_1 be defined by (2.32). Then*

$$\frac{\partial_t w_1(t,x)}{w_1(t,x)} \leq \frac{-m+1}{t} + \frac{4}{3}\frac{S(x)}{t^2}.$$

Proof. We calculate $\partial_t w_1 / w_1 = \partial_t \ln w_1$ using definition (2.32):

$$\frac{\partial_t w_1(t,x)}{w_1(t,x)} = \frac{-m}{t} + \frac{S(x)}{t^2} + \Big(\frac{6}{t} + \frac{2S(x)}{t^2}\Big)\Big(6 + \frac{S(x)}{t}\Big)^{-1}.$$

Hence

$$\frac{\partial_t w_1(t,x)}{w_1(t,x)} \leq \frac{-m}{t} + \frac{S(x)}{t^2} + \Big(\frac{6}{t} + \frac{2S(x)}{t^2}\Big)\frac{1}{6} = \frac{-m+1}{t} + \frac{4}{3}\frac{S(x)}{t^2}.$$

□

We can now confirm that our choice of weights leads to conditions $(i) - (iv)$

Proposition 2.11 *Assume that A satisfies* $(a1)$-$(a3)$. *Let w and* w_1 *be defined in (2.32) with m and S defined at (2.31). Then conditions* (i)-(vi) *in Section* 2.2 *hold for sufficiently large* $t \geq t_0$.

Proof.

(i) The formula for $\hat{c}$ in Lemma 2.9 and Lemma 2.10 show that

$$\begin{aligned} \hat{c} &\geq \frac{\delta a(x)}{t} + \frac{\delta_1 a(x) S(x) - C_\delta}{t^2} - \frac{2S(x)}{t^3} \\ &= \frac{\delta a(x) - C_\delta/t}{t} + \frac{S(x)(\delta_1 a(x) - 2/t)}{t^2}, \end{aligned}$$

where $\delta_1 = \delta/(2m(a))$ and C_δ is determined by condition $(S3)$. Clearly we can assume that $|x| \leq t + K$ in these estimates. From $a(x) \geq a_0(1+|x|)^{-\alpha}$, with $\alpha \in [0,1)$, we conclude that $\hat{c} > 0$ for sufficiently large time t.

The proof of $\partial_t \hat{c} < 0$ is similar, since $-t\partial_t \hat{c}$ and $\hat{c}$ are two polynomials of t^{-1} with the same leading coefficient $\delta a(x)$.

(ii) Using definitions (2.32) and Lemma 2.10, we find that

$$\begin{aligned} -\partial_t w_1 &= \left(-\frac{\partial_t w_1}{w_1} + \frac{w}{w_1} \right) w_1 \\ &\geq \left(\frac{m-1}{t} - \frac{4}{3}\frac{S(x)}{t^2} + \frac{8}{t} + \frac{4}{3}\frac{S(x)}{t^2} \right) w_1 \\ &\geq 0. \end{aligned}$$

(iii) We can estimate the first factor on the left side by condition (*ii*):

$$-\partial_t w_1 + 2w = (-\partial_t w_1 + w) + w \geq w.$$

The second factor needs more work. We rewrite

$$-\partial_t w_1 + 2aw_1 + 4w_1 \partial_t \ln w - 2w = \left(-\frac{\partial_t w_1}{w_1} + 2a \right) w_1 + \left(4\frac{\partial_t w}{w} - 2\frac{w}{w_1} \right) w_1$$

and apply Lemma 2.10 to estimate $-\partial_t w_1 / w_1$:

$$\begin{aligned} -\partial_t w_1 + 2aw_1 + 4w_1 \partial_t \ln w - 2w &\geq \left(\frac{m-1}{t} - \frac{4}{3}\frac{S(x)}{t^2} + 2a(x) \right) w_1 \\ &+ \left(-\frac{4m}{t} + \frac{4S(x)}{t^2} - \frac{16}{t} - \frac{8}{3}\frac{S(x)}{t^2} \right) w_1. \end{aligned}$$

Regrouping the terms, we obtain

$$\begin{aligned} \partial_t w_1 + 2aw_1 + 4w_1 \partial_t \ln w - 2w &\geq \left(-\frac{3m+17}{t} + 2a(x) \right) w_1 \\ &\geq a(x)\, w_1, \end{aligned}$$

where the latter inequality follows from

$$a(x) \geq a_0(1 + t + K)^{-\alpha} \geq (3m + 17)t^{-1}$$

whenever $|x| \leq t + K$ and t is sufficiently large. Thus we can multiply the lower bounds of the two factors:

$$\begin{aligned}(-\partial_t w_1 + 2w)(-\partial_t w_1 + 2aw_1 \quad &+ \quad 4w_1\partial_t \ln w - 2w) \geq w \cdot aw_1 \\ &= \quad \frac{4a(x)(6t + S(x))}{3t^2} w_1^2 \\ &\geq \quad \frac{4}{3}\frac{a(x)S(x)}{t^2} w_1^2,\end{aligned} \tag{2.34}$$

where we have used the fact that

$$w/w_1 = 4(6t + S(x))/(3t^2).$$

It remains to bound the right side of inequality (*iii*). Using

$$\nabla w_1/w_1 = \nabla \ln w_1$$

we compute

$$\begin{aligned}\nabla w_1 - 2w_1\nabla \ln w \quad &= \quad (\nabla \ln w_1 - 2\nabla \ln w)\, w_1 \\ &= \quad \left(-\frac{\nabla S(x)}{6t + S(x)} + \frac{\nabla S(x)}{t}\right) w_1 \\ &= \quad \frac{(5t + S(x))\nabla S(x)}{t(6t + S(x))}\, w_1\end{aligned}$$

and

$$|\nabla w_1 - 2w_1\nabla \ln w|^2 = \frac{(5t + S(x))^2 |\nabla S(x)|^2}{t^2(6t + S(x))^2}\, w_1^2.$$

Condition $(S3)$ shows that

$$|\nabla S(x)|^2 \leq a(x)S(x),$$

so the above expression can be compared with the lower bound (2.34):

$$\begin{aligned}\frac{(5t+S(x))^2|\nabla S(x)|^2}{t^2(6t+S(x))^2}\, w_1^2 &\leq \frac{a(x)S(x)(5t+S(x))^2}{t^2(6t+S(x))^2}\, w_1^2 \\ &\leq \frac{a(x)S(x))}{t^2} w_1^2.\end{aligned}$$

Hence condition (iii) holds.

(iv) From property $(S2)$ in Lemma 2.8 we have

$$\begin{aligned}\frac{w(t,x)}{w_1(t,x)} &= \frac{4}{3}\Big(\frac{6t+S(x)}{t^2}\Big) \\ &\leq \frac{4}{3}\Big(\frac{6t+C|x|^{2-\alpha}}{t^2}\Big) \\ &\leq \frac{4}{3}\Big(\frac{6}{t}+\frac{C_0}{t^\alpha}\Big) \leq Ct^{-\alpha}\end{aligned}$$

for $|x| \leq t + K$ for t is sufficiently large. This is equivalent to condition (iv).

(v) The expression for w_t in (2.33) and property $(S2)$ imply this condition. We know that

$$\partial_t w = \Big(-\frac{m}{t}+\frac{S(x)}{t^2}\Big) w.$$

This implies that

$$\begin{aligned}-\partial_t w &= \Big(\frac{m}{t}-\frac{S(x)}{t^2}\Big) w \leq \Big(\frac{m}{t}+\frac{S(x)}{t^2}\Big) w \\ &\leq \Big(\frac{m}{t}+\frac{C}{t^\alpha}\Big) \leq C_0 t^{-\alpha}\, w.\end{aligned}$$

(vi) It is convenient to rewrite

$$\frac{w_1}{w}((\partial_t w)^2) + |\nabla w|^2) \le Ca(x),$$

and express the condition in terms of m and S:

$$\frac{t^2}{6t + S(x)}\Big(\Big(-\frac{m}{t} + \frac{S(x)}{t^2}\Big)^2 + \frac{|\nabla S(x)|^2}{t^2}\Big) \le Ca(x).$$

This inequality is true, since $S(x) = O(|x|^{2-\alpha})$ and $|\nabla S(x)|^2 \le a(x)S(x)$.

□

The main results of this chapter are the following.

Theorem 2.12 *Assume that conditions (2.22) and* $(a1)$-$(a3)$ *hold. Then for every* $\delta > 0$ *the solution of (2.1)-(2.2) satisfies*

$$\int e^{(m(a)-\delta)\frac{A(x)}{t}} a(x)u^2 dx \le C_\delta(\|\nabla u_0\|^2_{L^2} + \|u_1\|^2_{L^2})t^{\delta - m(a)},$$

and

$$\int e^{(m(a)-\delta)\frac{A(x)}{t}} (u_t^2 + |\nabla u|^2)dx \le C_\delta(\|\nabla u_0\|^2_{L^2} + \|u_1\|^2_{L^2})t^{\delta - m(a)-1}$$

for all $t \ge 1$. *The constant* C_δ *depends also on* K, a, *and* n.

Proof. We apply Proposition 2.5 with the weights defined by (2.31) and (2.32)to obtain

$$\int e^{(m(a)-\delta)\frac{A(x)}{t}} a(x)u^2 dx \le ct^{2\delta - m(a)},$$

$$\int e^{(m(a)-\delta)\frac{A(x)}{t}} \Big(\frac{1}{t} + \frac{A(x)}{t^2}\Big)^{-1} (u_t^2 + |\nabla u|^2)dx \le ct^{2\delta - m(a)},$$

where $t \ge t_0$ and c is a constant depending on E_0 and N_0. To simplify the second

estimate, we notice that

$$\left(\frac{1}{t}+\frac{A(x)}{t^2}\right)^{-1}=t\left(1+\frac{A(x)}{t}\right)^{-1}\geq cte^{-\delta\frac{A(x)}{t}}.$$

□

The estimates in Theorem 2.12 follow with a loss of decay 2δ and large $t\geq t_0$. To obtain the final form we increase the constant c_δ, if necessary, and replace 2δ by δ.

Hence $m(a)$ determines the decay rate and A(x) determines the actual support of solutions. (Solutions decay exponentially in the region $A(x)>t$.) Moreover, $m(a)$ is invariant under scaling $a\to Ca$. Definition $(a3)$ shows that $m(a)$ is determined by the behavior of $a(x)$ as $|x|\to\infty$. Additional assumptions on a will probably allow $\delta=0$.

Corollary 2.13 *Assume that a is radial C^1 function satisfying condition (2.29). Then for every $\delta>0$, the solution of (2.1)-(2.2) satisfies*

$$\int e^{A_0(2-\alpha+\delta)^{-2}\frac{|x|^{2-\alpha}}{t}}u^2dx\leq C_\delta t^{\delta+\frac{\alpha}{2-\alpha}-m(a)},$$

and

$$\int e^{A_0(2-\alpha+\delta)^{-2}\frac{|x|^{2-\alpha}}{t}}(u_t^2+|\nabla u|^2)dx\leq C_\delta t^{\delta-m(a)-1},$$

where $t\geq 1$.

Proof. The weighted energy estimate is immediate. To obtain the weighted L^2 estimate we recall

$$A_0(1+|x|)^{2-\alpha}\leq A(x)\leq A_1(1+|x|)^{2-\alpha}.$$

Thus $a(x)\geq c(A(x))^{-\frac{\alpha}{2-\alpha}}$ with some $c>0$. We have the lower bound

$$a(x)\geq ct^{-\frac{\alpha}{2-\alpha}}\left(\frac{A(x)}{t}\right)^{-\frac{\alpha}{2-\alpha}}\geq ct^{-\frac{\alpha}{2-\alpha}}e^{-\delta\frac{A(x)}{t}},$$

which completes the proof. □

Corollary 2.14 *Assume that a is radial C^1-function satisfying conditions (2.29) and (2.30). Then for every $\delta > 0$ the solution of (2.1)-(2.2) satisfies*

$$\int e^{a_2(2-\alpha+\delta)^{-2}\frac{|x|^{2-\alpha}}{t}} u^2 dx \leq C_\delta t^{\delta-\frac{n-2\alpha}{2-\alpha}},$$

and

$$\int e^{a_2(2-\alpha+\delta)^{-2}\frac{|x|^{2-\alpha}}{t}} (u_t^2 + |\nabla u|^2) dx \leq C_\delta t^{\delta-\frac{n-\alpha}{2-\alpha}-1},$$

where $t \geq 1$.

Proof. The proof is based on Corollary 2.13 and equality $(A4)$ from Proposition 2.7. □

We should note that the above estimates predict slower decay rate than estimates (2.6) with $k = -\alpha$. No conclusion can be drawn until we know the sharpness of Corollary 2.14.

Another important consequence of Theorem 2.12 is that all norms under consideration, restricted to $\{x : A(x) \geq t^{1+\epsilon}\}$ with $\epsilon > 0$, decay exponentially as $t \to \infty$. Such regions are inside the cone $\{x : |x| \leq t + K\}$ in many cases, including coefficients (2.29) and $\epsilon \in (0, 1-\alpha)$.

Corollary 2.15 *Assume that conditions (2.22) and $(a1)$-$(a3)$ hold. Then for every δ and $\epsilon > 0$, the solution of (2.1)-(2.2) satisfies*

$$\int_{A(x)\geq t^{1+\epsilon}} (u^2 + u_t^2 + |\nabla u|^2) dx \leq C_\delta e^{-(m(a)-\delta)t^\epsilon},$$

where $t \geq 1$.

Proof. We add the two estimates in Corollary 2.13, or Corollary 2.14, and restrict the integration to the set $\{x \in \mathbb{R}^n : A(x) \geq t^{1+\epsilon}\}$:

$$\int_{A(x)\geq t^{1+\epsilon}} e^{(m(a)-\delta)\frac{A(x)}{t}}(u^2 + u_t^2 + |\nabla u|^2)dx \leq c_\delta t^c,$$

where c depends on α and n. From $\frac{A(x)}{t} \geq t^\epsilon$ we find that

$$\int_{A(x)\geq t^{1+\epsilon}} (u^2 + u_t^2 + |\nabla u|^2)dx \leq c_\delta e^{-(m(a)-\delta)t^\epsilon},$$

since t^ϵ can be included in the exponential term after a slight increase of δ. Notice that the region of integration is inside the support of u if $0 < \epsilon < 1 - \alpha$. □

The proof of Theorem 2.12 is split into Sections $2 - 4$. The crucial step is derive an equation for $w^{-1}u$, where w is an approximate solution of (2.9). In fact we choose

$$w(t, x) = t^{-m_1}e^{-m_2\frac{A(x)}{t}}, \tag{2.35}$$

with suitable parameters m_1, $m_2 < m(a)$. We estimate $w^{-1}u$ by a standard multiplier method using $w_1(w^{-1}u)_t + w_0(w^{-1}u)$. The weights w_0 and w_1 are also functions of A and $m(a)$. A good choice is

$$w_0 = w, \quad w_1 = \frac{m_3 w}{\frac{m_4}{t} + \frac{w_t}{w}}$$

with some constant m_3 and m_4. It turns out that $w^{-1}u$ is asymptotically much simpler than u itself, which explains why this approach works.

CHAPTER 3

DECAY ESTIMATES FOR THE NONLINEAR WAVE EQUATIONS

3.1 Introduction

This chapter is devoted to investigate the energy decay properties of solutions to the semilinear wave equations with variable damping

$$u_{tt} - \Delta u + a(x)u_t + |u|^{p-1}u = 0, \quad x \in \mathbb{R}^n,\ t > 0 \tag{3.1}$$

together with compactly supported initial data (2.2) satisfying (2.3), where $n \geq 3$ is an integer and p is a real number satisfying $1 < p < (n+2)/(n-2)$. The potential a is a radial C^1 function of x satisfying condition (2.29). The terms $a(x)u_t$ and $|u|^{p-1}u$ in the equation, where $u(t,x) : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$ is the solution of (2.2)-(3.1), are the damping with variable coefficient and absorbtion, respectively.

It is well known that under the above conditions, problem (2.2)-(3.1) possesses a unique global solution u satisfying (2.4) and (2.5)(see [23] and [24]).

Let us recall the definition of energy associated with a solution of (2.2)-(3.1):

$$E(t) = \frac{1}{2}\int (u_t^2 + |\nabla u|^2)dx + \frac{1}{p+1}\int |u|^{p+1}dx, \tag{3.2}$$

which satisfies

$$E(t) = E(0) - \int_0^t \int a\, u_t(x,s)^2 dxds, \tag{3.3}$$

where

$$E(0) = \frac{1}{2} \int (u_1^2 + |\nabla u_0|^2) dx + \frac{1}{p+1} \int |u_0|^{p+1} dx.$$

Let us recall that identity (3.3) is obtained by multiplying equation (3.1)by u_t and integrating over $(0,t) \times \mathbb{R}^n$ and using (2.5). This quantity decays in time as a result of the conservation laws. The decay rate, however, is not evident from such conservation laws.

We are also interested in the asymptotic behavior of the L^2, L^{p+1} and energy norms of u:

$$\|u\|_{L^2}, \ \|u\|_{L^{p+1}} \ \text{ and } \ \|u_t\|_{L^2} + \|\nabla u\|_{L^2},$$

respectively.

Identity (3.2) implies that the energy $E(t)$ associated with problem (2.2)-(3.1) is a non-increasing function of t and hence it is a natural question whether the energy decays to zero as t goes to infinity and if so, how fast it decays. Of course a lower bound on a will be necessary , since the solutions are asymptotically free if a decays very fast. It is known that the energy of u approaches a non-zero constant as $t \to \infty$ if $a(x) = O(|x|^{-1-\delta})$ with some $\delta > 0$ (see [8]).

There are many papers where the decay problems of solutions u to the Cauchy problem for the damped wave equations with absorption and potential $a(x) = 1$ have been treated. Due to the diffusion phenomena the behavior of u as $t \to \infty$ is expected to be the same as the corresponding heat equation

$$u_t - \Delta u + |u|^{p-1} u = 0, \quad x \in \mathbb{R}^n, \ t > 0. \tag{3.4}$$

Kawashima, Nakao and Ono in [25] have considered the decay problem for large exponent p of the absorption where the decay rate is expected to be as that of the linear

heat equation. The authors of [25], by using the energy method combined with L^p-L^q estimates, have proved this only for very large p, namely for

$$p > 1 + \frac{4}{n} \text{ and } 1 \le n \le 3.$$

They have shown that if

$$1 + \frac{4}{n} < p < \frac{n+2}{n-2} \text{ for } n = 3 \text{ or } 1 + \frac{4}{n} < p < \infty \text{ for } n = 1, 2,$$

the solution u decays as

$$\|u\|_{L^2} = O(t^{-\frac{n}{2}(\frac{1}{r}-\frac{1}{2})})$$

when the initial data

$$(u_0, u_1) \in (H^1 \cap L^r(R^n)) \times (L^2 \cap L^r(\mathbb{R}^n)) \text{ for } 1 \le r \le 2.$$

Let us mention here the paper [26] where the authors have considered the equation with a forcing nonlinearity

$$u_{tt} - \Delta u + u_t = |u|^p, \quad x \in \mathbb{R}^n \quad t > 0$$

in the supercritical case, namely $p > 1 + \frac{2}{n}$ and they have proved that any small initial data solutions are global and their decay rate is like the decay rate of the linear heat equation

$$\|Du\|_{L^2} = O(t^{-\frac{n}{4}-\frac{1}{2}}),$$

where $D = (\frac{\partial}{\partial t}, \nabla_x)$.

Based on [25], Karch [27] have shown that the asymptotic profile of the solution u is the Gauss kernel. Again the above result have been proved only for very large p and small n namely

$$p > 1 + \frac{4}{n} \text{ and } 1 \leq n \leq 3.$$

Only in dimension $n = 1$ in a very recent paper [28] the authors have succeeded to close the gap and have proved that the asymptotic profile of u for the supercritical case

$$p > 1 + \frac{2}{n} \text{ and } n = 1$$

is the Gauss kernel. The decay rate of the solution in subcritical case, namely with small exponent

$$1 < p < 1 + \frac{2}{n}$$

of the absorption term was expected to be the decay of the corresponding nonlinear heat equation (3.4). In a very recent paper [29] Nishihara and Zhao have proved this for the exponentially decaying data as $x \to \infty$, by using the weighted energy method with convenient weights similar to the weights in [26]. They have shown that

$$(\|u\|_{L^{p+1}}, \|\nabla u\|_{L^2}) = O(t^{-\frac{1}{p-1}+\frac{n}{2(p+1)}}, t^{-\frac{1}{p-1}+\frac{n}{4}-\frac{1}{2}}),$$

for $1 < p < 1 + \frac{2}{n}$, which coincides with the decay results of Escobedo and Kavian [30] for the corresponding norms of the semi-linear heat equation.

In this chapter, based on the paper [31], we consider the decay problems of solutions for the nonlinear damped wave equation (2.2)-(3.1) with potential $a(x)$ in both cases supercritical and subcritical. Of course the critical number as well as the decay rates are quite different than the case of constant potential $a(x) = 1$. Moreover we observe a new effect which does not show up in the case of constant potential. Unlike the decay problems for constant potential were there are two different decay rate regions

dependent on the sizes of the exponent of nonlinearity. The decay rate for the nonlinear problem is closely related with the decay rate of the linear damped wave equation in the region of large exponents of the nonlinearity. This problem is quite delicate and deserves separate consideration.

We should note that in deriving the decay estimates of solutions of (2.2)-(3.1) the same approach, as in the previous chapter, will be used and it relies on three components:

1. Finding an approximate solution of (3.1).

2. Deriving a modified equation for $w^{-1}u$ and weighted identities.

3. Choosing weights to derive the optimal decay rate for u.

Unexpectedly w is allowed to be even an approximate solution of the inequality

$$w_{tt} + a(x)w_t - \Delta w \geq -f(t,x)w \tag{3.5}$$

with a suitable function f decaying sufficiently fast at infinity. In fact the approximate solution of inequality (3.5) can be chosen as

$$w(t,x) = t^{-m}e^{-\beta\frac{A(x)}{t}} \tag{3.6}$$

with suitable parameters m, β. Here A is a positive radial solution of the Poisson equation (2.24) satisfying conditions $(a1)$-$(a3)$. The choice of weights and of the function f in (3.5) is delicate. It turns out that the sharpest estimates for high exponents p are obtained when $f = 0$; low exponents p require $f > 0$.

3.2 A Weighted Energy Identity

We derive an identity for first-order perturbations of the semi-linear wave equation. Similar identities will hold for all second-order hyperbolic equations. Our nonlinear equation has the form

$$v_{tt} - \Delta v + a v_t + b.\nabla v + cv + h|v|^{p-1}v = 0 \tag{3.7}$$

with coefficients a, $b = (b_1, b_2, ..., b_n)$, c and $h \geq 0$ in $C^1((0,\infty)\times\mathbb{R}^n)$. It is convenient to assume that

$$v(t,x) = 0 \text{ if } |x| > t + R \tag{3.8}$$

for some $R > 0$, and

$$v \in C((0,\infty), H^2(\mathbb{R}^n)) \cap C^1((0,\infty), H^1(\mathbb{R}^n)).$$

These conditions are simple but not optimal; later we use a standard approximation argument to establish the result for data in the energy space. Another direction for generalization is to allow data decaying exponentially fast at infinity. Given w and w_1 in $C^2((0,\infty)\times\mathbb{R}^n)$, we form the multiplier $w_1 v_t + wv$. To derive the modified equation for v, we substitute $u = vw$ into equation (3.1). This yields a first-order perturbation of the semi-linear wave equation:

$$v_{tt} - \Delta v + \hat{a} v_t + \hat{b}\cdot\nabla v + \hat{c} v + h|v|^{p-1}v = 0, \tag{3.9}$$

where the coefficients are given by

$$\begin{aligned} &\hat{a} = a + 2w^{-1}w_t, \quad \hat{b} = b - 2w^{-1}\nabla w, \\ &\hat{c} = w^{-1}(w_{tt} - \Delta w + aw_t + b \cdot \nabla w + cw) \\ &h = w^{p-1} \end{aligned} \tag{3.10}$$

Proposition 3.1 *Let $v \in C((0,\infty), H^2(\mathbb{R}^n)) \cap C^1((0,\infty), H^1(\mathbb{R}^n))$ be a solution of (3.7) with compact support (3.8) and $h \geq 0$. For any pair of C^2functions w_1 and w, we have the equality*

$$\frac{d}{dt}E(v_t, \nabla v, v) + F(v_t, \nabla v) + G(v) + H(v) = 0. \tag{3.11}$$

where

$$\begin{aligned} E(v_t, \nabla v, v) &= \frac{1}{2}\int [w_1(v_t^2 + |\nabla v|^2) + 2wv_t v + (\hat{c}w_1 - w_t + \hat{a}w)v^2 \\ &+ \frac{1}{p+1}\int w_1 h|v|^{p+1}dx, \\ F(v_t, \nabla v) &= \frac{1}{2}\int (-(w_1)_t + 2\hat{a}w_1 - 2w)v_t^2 dx + \int (\nabla w_1 + \hat{b}w_1)\cdot v_t \nabla v dx \\ &+ \frac{1}{2}\int (-(w_1)_t + 2w)|\nabla v|^2 dx, \\ G(v) &= \frac{1}{2}\int [w_{tt} - \Delta w - (\hat{a}w)_t - \nabla\cdot(w\hat{b}) + 2\hat{c}w - (\hat{c}w_1)_t]v^2 dx, \\ H(v) &= \int \left(wh - \frac{1}{p+1}(w_1 h)_t\right)|v|^{p+1}dx. \end{aligned}$$

A simple consequence is that $E(t)$ is bounded above if

$$F(t) + G(t) + H(t) \geq -f(t) \text{ with } f \in L^1.$$

Next we relax the regularity conditions on v showing that the same result holds for all v in the energy space.

Proof. Multiplying equation (3.9) by wv and $w_1 v_t$ and integrating over $\mathbb{R}^n$ and using the divergence theorem, respectively, we obtain

$$\begin{aligned} \int wvv_{tt}\,dx &+ \int \nabla v \cdot \nabla(wv)\,dx + \int \hat{a} wvv_t\,dx \\ &+ \int wv\hat{b}\cdot\nabla v\,dx + \int \hat{c} wv^2\,dx + \int h|v|^{p+1}\,dx = 0 \end{aligned} \tag{3.12}$$

and

$$\begin{aligned} \int w_1 v_t v_{tt}\,dx &+ \int \nabla v \cdot \nabla(w_1 v_t)\,dx + \int \hat{a} w_1 v_t^2\,dx \\ &+ \int \hat{b}\cdot w_1 v_t \nabla v\,dx + \int \hat{c} w_1 v v_t\,dx + \int h|v|^{p-1} v v_t\,dx = 0. \end{aligned} \tag{3.13}$$

Adding the above equations and using the identity

$$h|v|^{p-1}vv_t = \frac{1}{p+1}\frac{d}{dt}h|v|^{p+1} - \frac{1}{p+1}h_t|v|^{p+1}$$

as well as identities (2.16) we obtain identity (3.11). □

We can apply the identity in Proposition 3.1 to the wave equation with damping (3.1). Here

$$a = a(x), \quad b = 0, \quad c = 0.$$

So the transformed coefficients are

$$\begin{aligned} \hat{a} &= a + 2w^{-1}w_t, \quad \hat{b} = -2w^{-1}\nabla w, \\ \hat{c} &= w^{-1}(w_{tt} - \Delta w + aw_t), \quad h = w^{p-1}. \end{aligned} \tag{3.14}$$

Proposition 3.2 *Let u be a solution of (3.1)-(2.2) with compactly supported data*

$$u_0 \in H^1, \;\; u_1 \in L^2$$

and define $v = w^{-1}u$, *where* $w > 0$ *is a* C^2*function. Assume that w and* $\hat{c}$ *is given at (3.14). Then*

$$\begin{aligned}
E(v_t, \nabla v, v) &= \frac{1}{2}\int [w_1(v_t^2 + |\nabla v|^2) + 2wv_t v + (\hat{c}w_1 + w_t + \hat{a}w)v^2]dx \\
&+ \frac{1}{p+1}\int w_1 w^{p-1}|v|^{p+1}dx, \\
F(v_t, \nabla v) &= \frac{1}{2}\int (-(w_1)_t + 2\hat{a}w_1 + 4w_1(\ln w)_t - 2w)v_t^2 dx \\
&+ \int (\nabla w_1 - 2w_1\nabla \ln w)\cdot v_t \nabla v dx + \frac{1}{2}\int (-(w_1)_t + 2w)|\nabla v|^2 dx, \\
G(v) &= \frac{1}{2}\int [\hat{c}w - (\hat{c}w_1)_t]v^2 dx, \\
H(v) &= \int (w^p - \frac{1}{p+1}(w_1 w^{p-1})_t)|v|^{p+1}dx.
\end{aligned}$$

Moreover, the condition

$$F(v_t, \nabla v) + G(v) + H(v) \geq -f(t)$$

with a non-negative function $f \in L^1(\mathbf{R}_+)$ *implies that the weighted energy of v satisfies*

$$E(v_t, \nabla v, v) \leq E_0 + \int_{t_0}^{t} f(s)ds, \;\; t \geq t_0,$$

where $E_0 = E(v_t, \nabla v, v)|_{t=t_0}$.

Proof. We apply the following corollary to equation (3.9) with $\hat{a}, \hat{b}, \hat{c}$ and h defined at (3.14). □

Corollary 3.3 *Let* $v \in C((0,\infty), H^1(\mathbb{R}^n)) \cap C^1((0,\infty), L^2(\mathbb{R}^n))$ *be a solution of (3.7) with compact support (3.8) and* $h \geq 0$. *Define the functionals* F, G, H, *and* E *as in Proposition (3.1). If there exists a non-negative function* $f \in L^1(\mathbb{R}_+)$, *such that*

$$F(v_t, \nabla v) + G(v) + H(v) \geq -f(t), \quad t \geq t_0,$$

then the weighted energy of v satisfies

$$E(v_t, \nabla v, v) \leq E_0 + \int_{t_0}^{t} f(s)ds, \quad t \geq t_0,$$

where $E_0 = E(v_t, \nabla v, v)|_{t=t_0}$.

Proof. It is simple to prove if $v_0 \in H^2$ and $v_1 \in H^1$. If we have $v_0 \in H^1$ and $v_1 \in L^2$ we find compactly supported regular sequences $v_0^{(k)} \to v_0$ in H^1 , $v_1^{(k)} \to v_1$ in L^2 and denote the corresponding solutions of (3.7) by $v^{(k)}$. Notice that $E(v_t, \nabla v, v)$ is controlled by the energy norm:

$$\begin{aligned} & |E(v_t^{(k)}, \nabla v^{(k)}, v^{(k)}) - E(v_t^{(l)}, \nabla v^{(l)}, v^{(l)})| \\ \leq \quad & c(T, M)(\|v_t^{(k)} - v_t^{(l)}\|_{L^2}^2 + \|\nabla v^{(k)} - \nabla v^{(l)}\|_{L^2}^2) \end{aligned}$$

whenever $t \in [0, T]$ and $\|v_t^{(k)}\|_{L^2} + \|\nabla v^{(k)}\|_{L^2} \leq M$, $\|v_t^{(l)}\|_{L^2} + \|\nabla v^{(l)}\|_{L^2} \leq M$. The latter conditions hold with

$$M = c(T)(\|v_t(0,.)\|_{L^2} + \|\nabla v(0,.)\|_{L^2} + \|v_0\|_{L^{p+1}}^{(p+1)/2});$$

in fact $h \geq 0$ implies the following estimate for all j and $t \in [0, T]$:

$$\|v_t^{(j)}\|_{L^2} + \|\nabla v^{(j)}\|_{L^2} \leq c(T)(\|v_t^{(j)}\|_{L^2} + \|\nabla v_0^{(j)}\|_{L^2} + \|v_0^{(j)}\|_{L^{p+1}}^{(p+1)/2}).$$

The energy norm of solutions to problem (3.1)-(2.2) is a locally Lipschitz function of the initial data if $p \in [1, (n+2)/(n-2))$ (see [23] or [24]). Hence

$$\begin{aligned} & |E(v_t^{(n)}, \nabla v^{(n)}, v^{(n)}) - E(v_t^{(k)}, \nabla v^{(k)}, v^{(k)})| \\ \leq \quad & c(T, M)(\|v_1^{(n)} - v_1^{(k)}\|_{L^2}^2 + \|\nabla v_0^{(n)} - \nabla v_0^{(k)}\|_{L^2}^2), \end{aligned}$$

which yields

$$\begin{aligned} E(v_t, \nabla v, v) &= \lim_{n\to\infty} E(v_t^{(n)}, \nabla v^{(n)}, v^{(n)}) \\ &\leq \lim_{n\to\infty} E(v_t^{(n)}, \nabla v^{(n)}, v^{(n)})|_{t=t_0} + \int_{t_0}^{t} f(s)\, ds \\ &= E_0 + \int_{t_0}^{t} f(s)\, ds. \end{aligned}$$

□

We will apply the estimate in Corollary (3.3) to $v = w^{-1}u$, where u is a solution of (3.1)-(2.2) and $w > 0$ is a smooth solution of the differential inequality

$$w_{tt} - \Delta w + a(x)w_t \geq \hat{c}^- w \tag{3.15}$$

with a suitable $\hat{c}^-$ decaying sufficiently fast at infinity. The idea is to work with a new function v which is asymptotically much simpler than u, [22]. The sharpest estimates for high exponents p are obtained when $\hat{c}^- = 0$; low exponents p require certain $\hat{c}^- > 0$, see Section 5.

3.3 Weighted energy estimates

In this section we show that Proposition 3.1 applies to all positive weights w_1 and w satisfying eight differential inequalities. Hence the $E(v_t, \nabla v, v)$ is a bounded function of time for such weights. Another problem is to show that the weighted energy is positive definite, so it can be used for decay estimates of v and the original solution $u = wv$.

Proposition 3.4 *Assume that $w_1 > 0$ and $w > 0$ are C^1 functions, such that*

(i) $\hat{c}w - (\hat{c}w_1)_t \geq \hat{c}^- w$,

where

$$\int_{t_0}^{\infty} \int w^{-1} |\hat{c}^-|^{\frac{p+1}{p-1}} \, dx \, dt < \infty,$$

(ii) $-(w_1)_t + w \geq 0$,

(iii) $(-(w_1)_t + 2w)(-(w_1)_t + 2a(x)w_1 + 4w_1(\ln w)_t - 2w) \geq (\nabla w_1 - 2w_1 \nabla \ln w)^2$,

(iv) $(p+1)w^p - (w_1 w^{p-1})_t \geq w^p$.

Let u be a solution of (3.1)-(2.2) in the energy space and define $v = w^{-1}u$. Then

$$F(v_t, \nabla v) + G(v) + H(v) \geq -C \int w^{-1} |\hat{c}^-|^{\frac{p+1}{p-1}} \, dx;$$

see the definitions of these functionals in Proposition 3.1. Thus the weighted energy $E(v_t, \nabla v, v)$ satisfies

$$\begin{aligned} E_0 + c_0 &\geq \frac{1}{2} \int [w_1(v_t^2 + |\nabla v|^2) + 2wv_t v + (\hat{c}w_1 + w_t + aw)v^2] dx \\ &+ \frac{1}{p+1} \int w_1 w^{p-1} |v|^{p+1} \, dx \end{aligned}$$

for all $t \geq t_0$, where

$$E_0 = E(v_t, \nabla v, v)|_{t=t_0}, \quad c_0 = C \int_{t_0}^{\infty} \int w^{-1} |\hat{c}^-|^{\frac{p+1}{p-1}} \, dx \, dt.$$

Proof. From conditions (*ii*) and (*iii*) that $F(v_t, \nabla v) \geq 0$ as a nonnegative quadratic form of v_t and ∇v. To deduce a lower bound on $G(v)$, we use condition (*i*) and Young's inequality with exponents $(p+1)/2$ and $(p+1)/(p-1)$:

$$\begin{aligned} G(v) &= \frac{1}{2} \int [\hat{c} w - (\hat{c} w_1)_t] v^2 \, dx \geq \frac{1}{2} \int \hat{c}^- w v^2 \, dx \\ &\geq -\int w^p |v|^{p+1} \, dx - C \int w^{-1} |\hat{c}^-|^{\frac{p+1}{p-1}} \, dx dt. \end{aligned}$$

Condition (*iv*) yields

$$H(v) \geq \int w^p |v|^{p+1} dx,$$

so we have

$$G(v) + H(v) \geq -C \int w^{-1} |\hat{c}^-|^{\frac{p+1}{p-1}} \, dx \, dt.$$

We complete the proof we just apply Proposition 3.1. □

It is important to have a positive definite weighted energy $E(v_t, \nabla v, v)$. Thus we need a suitable upper bound on the indefinite term $2wvv_t$ under conditions $(i) - (iv)$. We will essentially rely on assumption (2.29) concerning the decay rate of a. Unfortunately there is an additional condition (v) on w_1 and w to guarantee that $\hat{c} w_1 v^2$ is dominated by the positive nonlinearity $w_1 w^{p-1} |v|^{p+1}$.

Proposition 3.5 *Assume that conditions $(i) - (iv)$ in Proposition 3.4 hold for w_1 and w and condition (2.29) holds for $a(x)$. If*

(v) $\hat{c} \geq \hat{c}^-$ *and* $\sup_{t\geq t_0} \int w_1 w^{-2} |\hat{c}^-|^{\frac{p+1}{p-1}}\, dx < \infty$,

then

$$\int wv^2 dx \leq G_0 + C(E_0 + 1)t^\alpha$$

for $t \geq t_0$, *where* $G_0 = \int wv^2 dx|_{t=t_0}$.

Proof. Notice that if we drop the derivatives in $E(v_t, \nabla v, v) \leq E_0 + c_0$ from Proposition 3.4, the result is

$$\begin{aligned} & \frac{d}{dt}\int wv^2 dx + \int awv^2 dx \\ \leq\ & 2(E_0 + c_0) + \int |\hat{c}^-| w_1 v^2 dx - \frac{2}{p+1}\int w_1 w^{p-1}|v|^{p+1}\, dx. \end{aligned}$$

We use Young's inequality

$$|\hat{c}^-| w_1 v^2 \leq \frac{1}{p+1} w_1 w^{p-1}|v|^{p+1} + C w_1 w^{-2} |\hat{c}^-|^{\frac{p+1}{p-1}}$$

and introduce

$$b_0 = C \sup_{t\geq t_0} \int w_1 w^{-2} |\hat{c}^-|^{\frac{p+1}{p-1}}\, dx.$$

Thus we obtain

$$\frac{d}{dt}\int wv^2\, dx + \int a(x) wv^2 \leq 2(E_0 + c_0) + b_0, \quad t \geq t_0.$$

We can now derive an estimate of the weighted L^2norm of v. From $|x| \leq t + R$ and condition (2.29) on a, we get

$$\frac{d}{dt}\int wv^2\, dx + \frac{a_0}{(1 + t + R)^\alpha}\int wv^2\, dx \leq C(E_0 + 1).$$

This inequality is equivalent to

$$\frac{d}{dt}\left(e^{\frac{a_0}{1-\alpha}(1+t+R)^{1-\alpha}}\int wv^2\,dx\right) \le C(E_0+1)e^{\frac{a_0}{1-\alpha}(1+t+R)^{1-\alpha}}.$$

To complete the proof we integrate on $[t_0, t]$ and apply the following estimate:

$$\int_{t_0}^{t} e^{C\tau^{1-\alpha}}\,d\tau = \frac{1}{1-\alpha}\int_{t_0^{1-\alpha}}^{t^{1-\alpha}} e^{Cz}z^{\frac{\alpha}{1-\alpha}}dz \le \frac{t^\alpha e^{Ct^{1-\alpha}}}{C(1-\alpha)}.$$

□

Combination of the last two results yields a weighted estimate of v_t and ∇v. Now recall the inequality of Proposition 3.4:

$$\begin{aligned} 2(E_0+c_0) &\ge \int [w_1(v_t^2+|\nabla v|^2)+2wv_tv+(\hat{c}w_1+w_t+aw)v^2]dx \\ &+ \frac{2}{p+1}\int w_1w^{p-1}|v|^{p+1}\,dx. \end{aligned}$$

This result and $|2v_tv| \le \epsilon t^\alpha wv_t^2 + \epsilon^{-1}t^{-\alpha}wv^2$, where $\epsilon \in (0,1)$ imply

$$\begin{aligned} &\int (w_1-\epsilon t^\alpha w)(v_t^2+|\nabla v|^2)\,dx + \int awv^2\,dx \\ \le\ & 2(E_0+c_0) + \int (\epsilon^{-1}t^{-\alpha}w - w_t)v^2\,dx \\ +\ & \int |\hat{c}^-|w_1v^2\,dx - \frac{2}{p+1}\int w_1w^{p-1}|v|^{p+1}\,dx. \end{aligned}$$

Applying the argument of Proposition 3.5, we find that

$$\begin{aligned} &\int (w_1-\epsilon t^\alpha w)(v_t^2+|\nabla v|^2)\,dx + \int awv^2\,dx \\ +\ & \frac{1}{p+1}\int w_1w^{p-1}|v|^{p+1}\,dx \\ \le\ & 2(E_0+c_0)+b_0+\int (\epsilon^{-1}t^{-\alpha}w - w_t)v^2\,dx. \end{aligned} \tag{3.16}$$

The left side is positive and the right side is bounded by a constant if the weights w_1 and w meet two additional conditions.

Proposition 3.6 *Assume that a satisfies (2.29). If the weights w_1 and w satisfy conditions* (*i*) – (*iv*) *in Proposition 3.4, condition* (*v*) *in Proposition 3.5, and*

(vi) $w \le c_1 t^{-\alpha} w_1$,

(vii) $|w_t| \le c_1 t^{-\alpha} w$,

we have

$$\begin{aligned} \int [w_1(v_t^2 + |\nabla v|^2)\, dx &\le C(G_0 + E_0 + 1), \\ \int awv^2 &\le C(G_0 + E_0 + 1), \\ \int w_1 w^{p-1} |v|^{p+1}\, dx &\le C(G_0 + E_0 + 1), \end{aligned}$$

for all $t \ge t_0$.

Proof. Estimate (3.16) and Proposition 3.5 show that $w_1 - \epsilon t^{\alpha} w \ge w_1/2$ and

$$\int (\epsilon^{-1} t^{-\alpha} w - w_t) v^2\, dx \le C \int wv^2\, dx$$

are sufficient to establish the claim. These hold if we choose $\epsilon = (2c_1)^{-1}$ with c_1 given in condition (*vi*) and use condition (*vii*), respectively. □

Let us restate Proposition 3.5 and Proposition 3.6 in terms of u. Using $v = w^{-1}u$ we readily obtain

$$\begin{aligned} \int w^{-1} u^2\, dx &\le G_0 + C(E_0 + 1) t^{\alpha}, \\ \int aw^{-1} u^2\, dx &\le C(G_0 + E_0 + 1), \\ \int w_1 w^{-2} |u|^{p+1} dx &\le C(G_0 + E_0 + 1). \end{aligned}$$

However, we need a final assumption on w_1 and w to deduce such estimates for u_t and ∇u. From $v = w^{-1}u$ we find that

$$v_t^2 = (-w^{-2}w_t u + w^{-1}u_t)^2 \geq \frac{1}{2}w^{-2}u_t^2 - 3w^{-4}w_t^2 u^2$$

and

$$|\nabla v|^2 = (-w^{-2}\nabla w u + w^{-1}\nabla u)^2 \geq \frac{1}{2}w^{-2}|\nabla u|^2 - 3w^{-4}|\nabla w|^2 u^2.$$

Proposition 3.6 and these inequalities imply

$$\frac{1}{2}\int w_1 w^{-2}(u_t^2 + |\nabla u|^2)\,dx \leq C(G_0 + E_0 + 1) + 3\int w_1 w^{-4}(w_t^2 + |\nabla w|^2)u^2\,dx.$$

If the weights satisfy $w_1 w^{-4}(w_t^2 + |\nabla w|^2) \leq c_2 a(x) w^{-1}$, we can apply Proposition 3.5 to the right side. The final estimates are given below.

Proposition 3.7 *Assume that a satisfies (2.29) and w_1 and w satisfy* (*i*) – (*vii*) *and*

(viii) $w_1 w^{-3}(w_t^2 + |\nabla w|^2) \leq c_2 a(x)$

where c_2 is a constant. Then the solution u of (3.1)-(2.2) satisfies

$$\begin{aligned} \int a w^{-1}u^2\,dx &\leq C(G_0 + E_0 + 1), \\ \int w_1 w^{-2}(u_t^2 + |\nabla u|^2)\,dx &\leq C(G_0 + E_0 + 1) \\ \int w_1 w^{-2}|u|^{p+1}\,dx &\leq C(G_0 + E_0 + 1), \end{aligned}$$

for $t \geq t_0$. Here C depends only on the equation and weights.

The above result yields non-trivial estimates if there exist weights with properties (*i*) – (*viii*) which decay sufficiently fast as t and $|x|$ go to infinity. Such weights w_1 and w are already constructed for the linear equation in [22]. Here we find a new pair of weights

adapter to the nonlinear term in problem (3.1)-(2.2). Both weights are defined in terms of certain positive solutions to the Laplace equation in $\mathbb{R}^n$.We provide more details in the next section.

3.4 Existence of weights

We rely on results from [22] and Proposition 2.7 for the existence of weights satisfying all conditions $(i) - (viii)$. Given parameters $S_0 > 0$ and $\beta > 0$, we introduce

$$S(x) = \beta A(x) + S_0, \quad x \in \mathbb{R}^n,$$

where $A(x)$ is the function constructed in Proposition 2.7, and define the family of weights

$$w(t,x) = t^{-m} e^{-\frac{S(x)}{t}}, \quad w_1(t,x) = \frac{3}{4}\Big(\frac{6}{t} + \frac{S(x)}{t^2}\Big)^{-1} w(t,x), \tag{3.17}$$

where $m > 0$ is an additional parameter. The two numbers β and m will depend on n, p and a. Constants like S_0, $\frac{3}{4}$ and 6 are introduced for technical reasons; S_0 is a large number depending on a and n (see [22]). It is important to mention that m determines the implicit decay rates as $t \to \infty$ in Proposition 3.7. Similarly, S determines the decay of these norms restricted to the region $\{x : S(x) \geq t^{1+\epsilon}\}, \quad \epsilon > 0$.

The following result is established in [22].

Proposition 3.8 *Let w and w_1 be defined in (3.17). Then conditions* (ii), (iii), *and* $(vi) - (viii)$ *on w hold for sufficiently large $t \geq t_0$.*

The proof is straightforward so it will not be repeated here. Instead we verify the remaining conditions (i), (iv), and (v). Actually the condition (iv) holds for all positive m and β in (3.17).

Proposition 3.9 *Let w and w_1 be defined in (3.17). Then condition* (*iv*) *holds for sufficiently large $t \geq t_0$.*

Proof. Condition (*iv*) is $(p+1)w^p - (w_1 w^{p-1})_t \geq w^p$ or $pw^p - (w_1 w^{p-1})_t \geq 0$. To show this we use condition (*ii*) and two simple computations:

$$-(w_1)_t + w \geq 0, \quad \frac{w_t}{w} = -\frac{m}{t} + \frac{S(x)}{t^2}, \quad \frac{w_1}{w} = \frac{3}{4}\Big(\frac{6}{t} + \frac{S(x)}{t^2}\Big)^{-1}.$$

Hence

$$-\frac{(w_1)_t}{w} \geq -1, \quad \frac{w_1}{w}\frac{w_t}{w} \leq \frac{3}{4}.$$

We obtain

$$\begin{aligned} pw^p - (w_1 w^{p-1})_t &= pw^p - (w_1)_t w^{p-1} - w_1(p-1)w^{p-2}w_t \\ &= w^p\Big(p - \frac{(w-1)_t}{w} - (p-1)\frac{w_1}{w}\frac{w_t}{w}\Big) \\ &\geq w^p\Big(p - 1 - \frac{3}{4}(p-1)\Big) \geq 0. \end{aligned}$$

The final two conditions (*i*) and (*v*) are related to the asymptotic behavior of w as a solution to (3.5). We choose $\hat{c}^-$ so that m is maximized. It turns out that the cases of low and high exponents p are very different.

Let us compute the first and second order derivatives of w :

$$\begin{aligned} w_t &= \Big(-\frac{m}{t} + \frac{S(x)}{t^2}\Big)w, \quad w_{tt} = \Big(-\frac{m}{t} + \frac{S(x)}{t^2}\Big)^2 w + \Big(\frac{m}{t^2} - \frac{2S(x)}{t^3}\Big)w, \\ \nabla w &= -\frac{\nabla S(x)}{t}w, \quad \Delta w = \Big(-\frac{\Delta S(x))}{t} + \frac{|\nabla S(x))|^2}{t^2}\Big)w. \end{aligned} \tag{3.18}$$

Thus the coefficient $\hat{c}$ in (3.14) is expressed as

$$\begin{aligned} \hat{c} &= \frac{\Delta S - ma}{t} + \frac{aS - |\nabla S|^2}{t^2} \\ &+ \left(-\frac{m}{t} + \frac{S(x)}{t^2} \right)^2 + \frac{m}{t^2} - \frac{2S(x)}{t^3}. \end{aligned} \tag{3.19}$$

We can now choose m and β so that conditions (i) and (v) are satisfied. Our first choice of parameters m and β will be optimal for supercritical exponents p. This choice is the following. Recall the definition of $m(a)$ in Proposition 2.7 and let $\delta > 0$ be an arbitrary small number. We set

$$m = m(a) - 2\delta, \quad \beta = m(a) - \delta. \tag{3.20}$$

□

Proposition 3.10 *Let w and w_1 be defined in (3.17) and m and β are defined in (3.20). Then conditions (i) and (v) hold for sufficiently large $t \geq t_0$.*

Proof. Assume that we have shown $\hat{c} \geq 0$ and $\hat{c}_t \leq 0$ for large $t \geq t_0$. Then we can take $\hat{c}^- = 0$. Thus the integrals in conditions (i) and (v) are trivially bounded. Moreover, from these assumptions and condition (ii) we find

$$\hat{c}w - (\hat{c}w_1)_t = \hat{c}(w - (w_1)_t) - \hat{c}_t w_1 \geq 0.$$

Thus the inequality in condition (i) also holds. It remains to check $\hat{c} \geq 0$ and $\hat{c}_t \leq 0$. The crucial lower bounds are

$$\Delta S(x) \geq (m + \delta)V(x), \quad (1 - k\delta)a(x)S(x) \geq |\nabla S(x)|^2$$

for large $|x|$, with $k = (1 + m(a))^{-1}$; the latter bound follows from Proposition 2.7 and our definition of $S(x)$. From equality (3.19) we obtain

$$\begin{aligned} \hat{c} &\geq \frac{\delta a(x)}{t} + \frac{k\delta a(x)S(x) - C}{t^2} - \frac{2S(x)}{t^3} \\ &= \frac{\delta a(x) - Ct^{-1}}{t} + \frac{S(x)\,(k\delta a(x) - 2t^{-1})}{t^2}, \end{aligned}$$

where C is a constant depending on a. Notice that we can consider only $|x| \leq t + R$ in these estimates. Using $a(x) \geq a_0(1 + |x|)^{-\alpha}$, with $\alpha \in [0, 1)$, we see that $\hat{c} > 0$ for sufficiently large time t. The proof of $\hat{c}_t < 0$ is similar.

Our second choice of parameters m and β is optimal only for subcritical exponents p. We again fix a small $\delta > 0$ and set

$$\begin{aligned} m &= \frac{2}{p-1} + \min\left\{\frac{\alpha}{2}\frac{p+1}{p-1} - \frac{n}{2}, 0\right\} - \delta, \\ \beta &= m(a) - \delta. \end{aligned}$$

□

Proposition 3.11 *Let w and w_1 be defined in (3.17) and m and β are defined in (3.21). Then conditions* (*i*) *and* (*v*) *hold for sufficiently large $t \geq t_0$.*

Proof. We need estimates of $\hat{c}$ and $\hat{c}_t$ from below and above, respectively. □

Lemma 3.12 *There exists positive constants k_i, $i = 1, 2, 3, 4$, such that*

$$\begin{aligned} \hat{c}(t, x) &\geq -k_1 t^{-1}(1 + |x|)^{-\alpha} + k_2 t^{-2}(1 + |x|)^{2-2\alpha}, \\ t\hat{c}_t(t, x) &\leq k_3 t^{-1}(1 + |x|)^{-\alpha} - k_4 t^{-2}(1 + |x|)^{2-2\alpha}. \end{aligned}$$

Let us postpone the proof and apply the first estimate. We see that the lower bound of $\hat{c}$ in condition (*v*) can be chosen as

$$\hat{c}^-(t,x) = \begin{cases} -k_1 t^{-1}(1+|x|)^{-\alpha}, & \text{if } 1+|x| \le kt^{1/2}, \\ 0, & \text{if } 1+|x| > kt^{1/2} \end{cases}$$

where $k = \sqrt{\frac{k_1}{k_2}}$. To verify the integral condition in (v), we use the estimates

$$w^{-1}(t,x)w_1(t,x) \le Ct \text{ and } w(t,x) \ge Ct^m$$

on the support of $\hat{c}^-$. The condition to check becomes

$$\int w_1 w^{-2} |\hat{c}^-|^{\frac{p+1}{p-1}}\, dx \le Ct^{m+1-\frac{p+1}{p-1}} \int_{|x|\le kt^{1/2}} (1+|x|)^{-\alpha\frac{p+1}{p-1}}\, dx.$$

Notice that for every $\delta > 0$ we have

$$\int_{|x|\le kt^{1/2}} (1+|x|)^{-\alpha\frac{p+1}{p-1}}\, dx \le \begin{cases} C_\delta t^{\frac{n}{2}-\frac{\alpha}{2}\frac{p+1}{p-1}+\delta}, & \text{if } \alpha\frac{p+1}{p-1} \le n, \\ C, & \text{if } \alpha\frac{p+1}{p-1} > n \end{cases}$$

which gives

$$\int w_1 w^{-2} |\hat{c}^-|^{\frac{p+1}{p-1}}\, dx \le \begin{cases} C_\delta t^{m+1+\frac{n}{2}-(1+\frac{\alpha}{2})\frac{p+1}{p-1}+\delta}, & \text{if } \alpha\frac{p+1}{p-1} \le n, \\ Ct^{m+1-\frac{p+1}{p-1}}, & \text{if } \alpha\frac{p+1}{p-1} > n. \end{cases}$$

Clearly the right sides are bounded functions of $t \ge t_0$ if m is defined as (3.21). The condition for integrability (i) leads to similar estimates; we can show that

$$\int_{t_0}^{\infty} \int w^{-1} |\hat{c}^-|^{\frac{p+1}{p-1}}\, dx\, dt < \infty$$

for the same m and β defined in (3.21).

Finally we can verify the inequality in condition (i) by using

$$t^{-1}w_1 \le Cw, \quad |(w_1)_t| \le Cw$$

together with

$$\hat{c} \geq \hat{c}^{-}, \quad t\hat{c}_t \leq \hat{c}^{-}$$

from Lemma 3.12. The resulting lower bound is

$$\begin{aligned} \hat{c}w - (\hat{c}w_1)_t &= \hat{c}w - t\hat{c}_t \cdot t^{-1}w_1 - C(w_1)_t \\ &\geq C\hat{c}^{-}w. \end{aligned}$$

Thus conditions (i) and (v) hold if Lemma 3.12 holds. To complete the proof of Proposition 3.11 we will show the auxiliary result.

Proof. The main difference with Proposition 3.10 is that m can be larger than $m(a)$. Hence a negative lower bound of the form

$$\Delta S(x) - ma(x) \geq (m(a) - m - \delta)a(x) \geq -\frac{1}{2}k_1(1 + |x|)^{-\alpha}$$

can not be improved. However, β guarantees a positive bound for large $|x|$ on the difference

$$a(x)S(x) - |\nabla S(x)|^2 \geq C\delta a(x)S(x) - C \geq 2k_2(1 + |x|)^{2-2\alpha} - C,$$

where $k_2 > 0$. These two estimates and formula (3.10) yield

$$\begin{aligned} \hat{c} &\geq -\frac{1}{2}k_1 t^{-1}(1 + |x|)^{-\alpha} + 2k_2 t^{-2}(1 + |x|)^{2-2\alpha} - Ct^{-2} - 2t^{-3}S(x) \\ &\geq -k_1 t^{-1}(1 + |x|)^{-\alpha} + k_2 t^{-2}(1 + |x|)^{2-2\alpha} \end{aligned}$$

for sufficiently large $t \geq t_0$. Thus the inequality for $\hat{c}$ in Lemma 3.12 is established.

The estimate of $t\hat{c}_t$ is very similar, since $\hat{c}$ is a polynomial of t^{-1}. There are more terms in $t\hat{c}_t$ but its leading terms are just opposite to the corresponding terms in $\hat{c}$. We

can readily bound all small terms of $t\hat{c}_t$ using $S(x) \le C(1+|x|)^{2-\alpha}$. Such estimates will complete the proof of Lemma 3.12. □

3.5 Main theorem and its corollaries

We use the weights constructed in the previous section to derive these results from the conditional results in Proposition 3.7.

Theorem 3.13 *Let $1 < p < (n+2)/(n-2)$ and assume that a satisfies (2.29). Then the solution of (2.2)-(3.1) satisfies*

$$\begin{aligned} \int e^{(m(a)-\delta)\frac{A(x)}{t}} u^2 \, dx &\le C_\delta t^{\delta+\frac{\alpha}{2-\alpha}-m(a)}, \\ \int e^{(m(a)-\delta)\frac{A(x)}{t}} (u_t^2 + |\nabla u|^2) \, dx &\le C_\delta t^{\delta-1-m(a)}, \\ \int e^{(m(a)-\delta)\frac{A(x)}{t}} |u|^{p+1} \, dx &\le C_\delta t^{\delta-1-m(a)}, \end{aligned}$$

where $\delta > 0$ is an arbitrary small number and $t \ge 1$. The constant C_δ depends on δ and the initial data u_0 and u_1.

This decay rate is optimal only for large exponent p is close to 1.The next result may be sharper when the exponent p is close to 1.

Proof. We choose weights w and w_1 defined in (3.17) with parameters $m = m(a) - 2\delta$ and $\beta = m(V) - \delta$, [see (3.20)]. Then Proposition 3.7 yields the following weighted estimates:

$$\int e^{(m(a)-\delta)\frac{A(x)}{t}} a(x) u^2 \, dx \le ct^{2\delta-m(a)},$$

$$\int e^{(m(a)-\delta)\frac{A(x)}{t}} \left(\frac{1}{t} + \frac{A(x)}{t^2}\right)^{-1} (u_t^2 + |\nabla u|^2) \, dx \le Ct^{2\delta-m(a)},$$

$$\int e^{(m(a)-\delta)\frac{A(x)}{t}}\Big(\frac{1}{t}+\frac{A(x)}{t^2}\Big)^{-1}|u|^{p+1}\,dx \;\;\leq\;\; Ct^{2\delta-m(a)},$$

where $t \geq t_0$ and C is a constant depending on E_0 and G_0. To simplify these, we recall

$$A_0(1+|x|)^{2-\alpha} \leq A(x) \leq A_1(1+|x|)^{2-\alpha}.$$

Thus $a(x) \geq C(A(x))^{-\frac{\alpha}{2-\alpha}}$ with some $C > 0$. We have the lower bounds

$$a(x) \geq Ct^{-\frac{\alpha}{2-\alpha}}\Big(\frac{A(x)}{t}\Big)^{-\frac{\alpha}{2-\alpha}} \geq Ct^{-\frac{\alpha}{2-\alpha}}e^{-\delta\frac{A(x)}{t}}$$

and

$$\Big(\frac{1}{t}+\frac{A(x)}{t^2}\Big)^{-1} = t\Big(1+\frac{A(x)}{t}\Big)^{-1} \geq Cte^{-\delta\frac{A(x)}{t}}.$$

□

The estimates in Theorem 3.13 follow with a loss of decay 2δ and large $t \geq t_0$. To obtain the final form we increase the constant c_δ, if necessary, and replace 2δ by δ.

Theorem 3.14 *Let $1 < p < (n+2)/(n-2)$ and assume that a satisfies (2.29). Then the solution of (3.1)-(2.2) satisfies*

$$\begin{aligned}
\int e^{(m(a)-\delta)\frac{A(x)}{t}}u^2\,dx &\leq C_\delta t^{\delta+\frac{\alpha}{2-\alpha}-\frac{2}{p-1}-min\{\frac{\alpha}{2}\frac{p+1}{p-1}-\frac{n}{2},0\}} \\
\int e^{(m(a)-\delta)\frac{A(x)}{t}}(u_t^2+|\nabla u|^2)\,dx &\leq C_\delta t^{\delta-\frac{p+1}{p-1}-min\{\frac{\alpha}{2}\frac{p+1}{p-1}-\frac{n}{2},0\}} \\
\int e^{(m(a)-\delta)\frac{A(x)}{t}}|u|^{p+1}dx &\leq C_\delta t^{\delta-\frac{p+1}{p-1}-min\{\frac{\alpha}{2}\frac{p+1}{p-1}-\frac{n}{2},0\}}
\end{aligned}$$

where $\delta > 0$ is an arbitrary small number and $t \geq 1$. The constant C_δ depends on δ and on the initial data u_0 and u_1.

Proof. By using the weights w and w_1 with the second set of parameters defined in (3.21), the result is again a simple consequence of Proposition 3.7 and the lower bounds on $a(x)$ and $\left(\frac{1}{t} + \frac{A(x)}{t^2}\right)^{-1}$. □

An important observation for both theorems is the following: the average decay rate of all norms under consideration in the region $A(x) \geq t^{1+\epsilon}$, $\epsilon > 0$, is exponential. In fact we can add the three estimates in Theorem 3.13 or Theorem 3.14 to obtain the following.

Corollary 3.15 *Let $1 < p < (n+2)/(n-2)$ and assume that a satisfies (2.29). Then the solution of (3.1)-(2.2) satisfies the estimate*

$$\int_{A(x)\geq t^{1+\epsilon}} (u^2 + {u_t}^2 + |\nabla u|^2 + |u|^{p+1})\, dx \leq C_\delta e^{-(m(a)-\delta)t^\epsilon}$$

where C_δ is a positive constant which depends on α, p, and n.

Proof. We add the three estimates in Theorem 3.13, or Theorem 3.14, and restrict the integration to $\{x : A(x) \geq t^{1+\epsilon}\}$ to obtain

$$\int_{A(x)\geq t^{1+\epsilon}} e^{(m(a)-\delta)\frac{A(x)}{t}} (u^2 + u_t^2 + |\nabla u|^2 + |u|^{p+1})\, dx \leq C_\delta t^c,$$

where C depends on α, p,and n. From $A(x)/t \geq t^\epsilon$ we have that

$$\int_{A(x)\geq t^{1+\epsilon}} (u^2 + u_t^2 + |\nabla u|^2 + |u|^{p+1})\, dx \leq C_\delta t^C e^{-(m(a)-\delta)t^\epsilon}.$$

This completes the proof, since t^C can be included in the exponential term after a slight increase of δ. Notice that the region of integration is inside the support of u if $0 < \epsilon < 1 - \alpha$. □

Remark 3.16 *Another interesting observation is that the decay of $||u||_{p+1}^{p+1}$ is the same as the decay of the energy $||u||_2^2 + ||\nabla u||_2^2$ for all solution u of problem (2.2)-(3.1). This*

estimate is much better than the estimate that can be derived by the standard interpolation inequality, namely the Gagliardo-Nirenberg inequality, and the independent decay estimates of $\|u\|_2$ *and* $\|\nabla u\|_2$*. All other norms* $\|u\|_s$ *with* $2 < s < \frac{n+2}{n-2}$ *and* $s \neq p+1$ *will have a decay rate lower then the decay rate of the energy. The exact calculation of the decay of these norms may be done by using the decay of* $\|u\|_2$ *and* $\|\nabla u\|_2$ *and Gagliardo-Nireberg inequality.*

In the case of $m(a) = \frac{n-\alpha}{2-\alpha}$ we derive from Theorem 3.13 the following explicit estimates.

Corollary 3.17 *Let a be a potential. Assume that*

$$p_1(n,\alpha) := 1 + \frac{2(4-\alpha^2)}{(n-\alpha)(4-\alpha)} \leq p < \frac{n+2}{n-2}.$$

Then the solution of (3.1)-(2.2) satisfies the following estimates:

$$\begin{aligned} \int e^{a_0(2-\alpha+\delta)^{-2}\frac{|x|^{2-\alpha}}{t}} u^2 dx &\leq C_\delta t^{\delta-\frac{n-2\alpha}{2-\alpha}}, \\ \int e^{a_0(2-\alpha+\delta)^{-2}\frac{|x|^{2-\alpha}}{t}} (u_t^2 + |\nabla u|^2) dx &\leq C_\delta t^{\delta-1-\frac{n-\alpha}{2-\alpha}}, \\ \int e^{a_0(2-\alpha+\delta)^{-2}\frac{|x|^{2-\alpha}}{t}} |u|^{p+1} dx &\leq C_\delta t^{\delta-1-\frac{n-\alpha}{2-\alpha}}, \end{aligned}$$

where $t \geq 1$*. Here* $\delta > 0$ *is an arbitrary small number and* C_δ *depends on* δ *and on the initial data* u_0 *and* u_1*.*

Corollary 3.17 shows that for the supercritical region

$$p_1(n,\alpha) \leq p < \frac{n+2}{n-2}$$

the decay of the solution of nonlinear problem coincides with the decay of the solution of corresponding linear problem, see [22] for comparison.

Remark 3.18 *Let us mention that when the potential $a(x) = 1$ it can be shown that $\delta = 0$ in the above decay estimates. In this case we get the optimal decay, that was derived by Matsumura by using the Fourier technique.Moreover, unlike Matsumura's method our approach can work in exterior domains as well.*

For the subcritical region $1 < p < p_1(n, \alpha)$, Theorem 3.14 yields much faster decay rates.

Corollary 3.19 *Let a be a potential described with (2.29) Assume that*

$$1 < p < p_1(n, \alpha).$$

Then the solution of (3.1)-(2.2) satisfies the following estimates:

$$\begin{aligned}
\int e^{a_0(2-\alpha+\delta)^{-2}\frac{|x|^{2-\alpha}}{t}} u^2 \, dx &\leq C_\delta t^{\delta+\frac{\alpha}{2-\alpha}-\frac{2}{p-1}-\min\{\frac{\alpha}{2}\frac{p+1}{p-1}-\frac{n}{2},0\}}, \\
\int e^{a_0(2-\alpha+\delta)^{-2}\frac{|x|^{2-\alpha}}{t}} (u_t^2 + |\nabla u|^2) \, dx &\leq C_\delta t^{\delta-\frac{p+1}{p-1}-\min\{\frac{\alpha}{2}\frac{p+1}{p-1}-\frac{n}{2},0\}}, \\
\int e^{a_0(2-\alpha+\delta)^{-2}\frac{|x|^{2-\alpha}}{t}} |u|^{p+1} \, dx &\leq C_\delta t^{\delta-\frac{p+1}{p-1}-\min\{\frac{\alpha}{2}\frac{p+1}{p-1}-\frac{n}{2},0\}},
\end{aligned}$$

where $\delta > 0$ is an arbitrary small number and $t \geq 1$. Here c_δ depends on δ and on the data u_0 and u_1.

Remark 3.20 *The subcritical region $1 < p < p_1(n, \alpha)$ is further divided into two subregions with completely different decay rates as follows.*

For exponent p very close to 1 namely $1 < p < 1 + \frac{2\alpha}{n-\alpha} := p_2(n, \alpha)$ the decay rates are

$$(\|u\|_{L^{p+1}}, \|\nabla u\|_{L^2} + \|u_t\|_{L^2}) = O(t^{-\frac{1}{p-1}+\delta}, t^{-\frac{p+1}{2(p-1)}+\delta}),$$

where δ is a small number. In this region the decay rate is independent of α.

For the second subcritical region, namely the region of medium exponents p,

$$p_2(n,\alpha) < p < p_1(n,\alpha),$$

the decay rate is

$$(\|u\|_{L^{p+1}}, \|\nabla u\|_{L^2} + \|u\|_{tL^2}) = O(t^{-(1+\frac{\alpha}{2})\frac{1}{p-1}+\frac{n}{2(p+1)}+\delta}, t^{-(1+\frac{\alpha}{2})\frac{p+1}{2(p-1)}+\frac{n}{4}+\delta}),$$

where δ is an arbitrary small number.

Remark 3.21 *Let us mention that in the case of constant potential in (3.1)-(2.2)* ($\alpha = 0$) *the second critical exponent* $p_2(n,\alpha) = 1 + \frac{2\alpha}{n-\alpha}$ *is equal to* 1 *which explains why in the case of constant potential there are only two different regions with respect to the decay rate of the solution of (3.1)-(2.2). In the case of constant potential* ($\alpha = 0$) *the first critical exponent* $p_1(n,\alpha) = 1 + \frac{2(4-\alpha^2)}{(n-\alpha)(4-\alpha)}$ *becomes the Fujita critical exponent* $1 + \frac{2}{n}$.

Remark 3.22 *A byproduct of the above corollaries is that the gap between the supercritical and subcritical regions is closed for any n if the potential a(x) is constant* ($\alpha = 0$).

CHAPTER 4

CONCLUSIONS

In the thesis we have surveyed the papers [22] and [26] which are about the weighted energy decay estimates leading to asymptotic behavior of solutions to the Cauchy problems for the dissipative linear and nonlinear wave equations (2.1) and (3.1), respectively.

In those papers using the fact that asymptotic behavior of linear or nonlinear dissipative wave equations is the same as the asymptotic behavior of linear heat (or diffusion) equations as $t \to \infty$(so called diffusion phenomenon), the authors established suitable multipliers (weight functions). On the other hand, their ([26] and [22]) results are restricted only for radial coefficients. In case of this restriction, this way of construction of multipliers is a key starting point and can be applied various equations for this type because the coefficients of the damping term is dependent on the space variable and it is difficult to apply the Fourier integral transform (used by many authors) even though it is easily applied to equations with coefficients are functions of time only.

Also we should note that the way of construction of multipliers of [26](or [22]) has been applied to a dissipative hyperbolic wave equation with arbitrary variable coefficients (see [32]).

As a consequence we believe that this method is general enough and will be employed to investigate decay estimates and asymptotic behavior of various dissipative wave equations with space variable coefficients.

REFERENCES

[1] B. Dehman, P. Grard, *Stabilization for the nonlinear Klein Gordon equation with critical exponent*, Preprint (2002).

[2] B. Dehman; G. Lebeau; E. Zuazua, *Stabilization and control for the subcritical semilinear wave equation*, Ann. Sci. cole Norm. Sup. **36** (2003), no. 4, 525 - 551.

[3] E. Zuazua, *Exponential decay for the semilinear wave equation with localized damping in unbounded domains*, J. Pures Appl. **70** (1991), no. 4, 513 - 529.

[4] A. Matsumura, *On the asymptotic behavior of solutions of semi-linear wave equations*, Publ. Res. Inst. Math. Sci. **12** (1976/77), no. 1, 169 - 189.

[5] T. Narazaki, *$L^p - L^q$ estimates for damped wave equations and their applications to semi-linear problem*, J. Math. Soc. Japan **56** (2004), 585 - 626.

[6] K. Nishihara, *Asymptotic behavior of solutions of quasilinear hyperbolic equations with linear damping*, J. Diff. Eq., **137** (1997), 384 - 395.

[7] H. Yang and A. Milani, *On the diffusion phenomenon of quasilinear hyperbolic waves in low space dimensions*, Bull. Sci. Math., **124** (2004), no. 1, 101 - 124.

[8] K. Mochizuki, *Scattering theory for wave equations with dissipative terms*, Publ. Res. Inst. Math. Sci. **12** (1976/77), no. 2, 383 - 390.

[9] J. Wirth, *Solution representations for a wave equation with weak dissipation*, Math. Methods Appl. Sci. **27** (2004), no. 1, 101 - 124.

[10] J. Wirth, *Wave equations with time-dependent dissipation I. Non-effective dissipation*, J. Diff. Eq. **222**, (2006), 487 -514.

[11] J. Wirth, *Wave equations with time-dependent dissipation II. Effective dissipation*, J. Diff. Eq. (to appear)

[12] M. Reissig, *$L^p - L^q$ decay estimates for wave equations with time-dependent coefficients*, J. Nonlinear Math. Phys. **11** (2004), no. 4, 534 - 548.

[13] M. Reissig and J. Wirth, *$L^p - L^q$ estimates for wave equations with monotone time-dependent dissipation*, Proceedings of the RIMS Symposium on Mathematical Models of Phenomena and Evolution Equations (to appear).

[14] R. Strichartz, *A priori estimates for the wave equation and some applications*, J. Funct. Anal. **5** (1970), 218 - 235.

[15] A. Matsumura, *Energy decay of solutions of dissipative wave equations*, Proc. Japan Acad. Ser. A Math. Sci. **53** (1977), no. 7, 232 - 236.

[16] H. Uesaka, *The total energy decay of solutions for the wave equation with a dissipative term*, J. Math. Kyoto Univ. **20** (1980), no. 1, 57 - 65.

[17] M. Nakao, *Energy decay for the linear and semilinear wave equations in exterior domains with some localized dissipations*, Math. Z. **238** (2001), no. 4, 781 - 797.

[18] K. Mochizuki and H. Nakazawa, *Energy decay and asymptotic behavior of solutions to the wave equations with linear dissipation*, Publ. Res. Inst. Math. Sci. **32** (1996), no. 3, 401 - 414.

[19] R. Ikehata, T. Matsuyama, *Fast decay solutions for linear wave equations with dissipation localized near infinity in an exterior domain*, Sci. Math. Japon. **55**(2002), 33 - 42.

[20] C. Morawetz, *The decay of solutions of the exterior initial-boundary problem for the wave equation*, Comm. Pure Appl. Math. Sci. **14** (1961), 561 - 568.

[21] R. Ikehata, *Some remarks on the wave equation with potential type damping coefficients*, Int. J. Pure Appl. Math. **21** (2005), no. 1, 19 - 24.

[22] G. Todorova, B. Yordanov, *Weighted L^2 estimates for dissipative wave equations with variable coefficients*, (to appear).

[23] W. Strauss, *Nonlinear wave equations. CBMS Regional Conference Series in Mathematics,* **73** Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1989. x + 91 pp. ISBN: 0-8218-0725-0.

[24] J. Shatah, M.Struwe, *Regularity results for nonlinear wave equations*, Ann. of Math. (2) **138** (1993), no. 3, 503–518.

[25] S. Kavashima, M. Nakao, and K. Ono, *On the decay property of solutions to the Cauchy problem of the semilinear wave equation with a dissipative term*, J. Math. Soc. Japan **47** (1995), no. 4, 617 - 653.

[26] G. Todorova and B. Yordanov, *Critical exponent for a nonlinear wave equation with damping*, J. Diff. Eq. **174** (2001), no. 2, 464-489.

[27] G. Karch, *Selfsimilar profiles in large time asymptotics of solutions to damped wave equations*, Studia Math. **143** (2000), no. 2, 175 - 197.

[28] N. Hayashi, E. Kaikina, and P. Naumkin, *Damped wave equation with super critical nonlinearities*, Differential Integral Equations **17** (2004), no. 5-6, 637 - 652.

[29] K. Nishihara and H. Zhao, *Decay properties of solutions to the Cauchy problem for the damped wave equation with absorption*, J. Math. Anal. Appl. **313** (2006), no. 2, 598 - 610.

[30] M. Escobedo and O. Kavian, *Asymptotic behaviour of positive solutions of a nonlinear heat equation*, Houston J. Math. **14** (1988), no. 1, 39-50.

[31] G. Todorova and B. Yordanov, *Nonlinear dissipative wave equations with potential*, Control methods in PDE-dynamical systems, Contemp. Math., American Math. Soc., Providence, RI 426(2007), 317–337.

[32] P. Radu, Todorova and B. Yordanov, *Decay estimates for wave equations with variable coefficients*, Trans. Amer. Math. Soc. (in press).