

**QUANTUM ASPECTS OF MOLECULAR CORRELATIONS
IN BIOLOGICAL CATALYSIS**



Ph.D. THESIS

Onur PUSULUK

Department of Physics Engineering

Physics Engineering Programme

AUGUST 2018

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**Thesis Advisor: Prof. Dr. Kerem CANKOÇAK
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**BİYOLOJİK KATALİZDE
MOLEKÜLER İLİNTİLERİN KUANTUM DOĞASI**

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AĞUSTOS 2018

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To my dearest wife and daughter,



FOREWORD

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August 2018

Onur PUSULUK
(B.Sc. in Mol. Biol. and Genet. & Phys. Eng.)

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ABBREVIATIONS

H	: Hydrogen
D	: Deuterium
O	: Oxygen
C	: Carbon
N	: Nitrogen
H₂O	: Water Molecule
LBHB	: Low Barrier Hydrogen Bond
SSHB	: Short and Strong Hydrogen Bond
TST	: Transition State Theory
E	: Enzyme Molecule
S	: Substrate Molecule
S[‡]	: Transition State (of Substrate Molecule)
P	: Product Molecule
ES	: Enzyme - Substrate Complex
ES[‡]	: Enzyme - Substrate Transition State Complex
EP	: Enzyme - Product Complex
DNA	: Deoxyribonucleic Acid
DNA_{pol}	: DNA Polymerase
RNA	: Ribonucleic Acid
A, A[*]	: Common Amino and Rare Imine Forms of Adenine
T, T[*]	: Common Keto and Rare Enol Forms of Thymine
G, G[*], G[#]	: Common Keto, Rare Enol, and Rare Imino-Enol Forms of Guanine
C, C[*], C[#]	: Common Amino, Rare Imino, Rare Imino-Enol Forms of Cytosine
DFT	: Density Functional Theory
MP2	: The Second Order Møller-Plesset Perturbation Theory
QTAIM	: The Quantum Theory of Atoms in Molecules
BCH	: Baker-Campbell-Hausdorff
LGKS	: Lindblad-Gorini-Kossakowski-Sudarshan
CHSH	: Clauser-Horne-Shimony-Holt
GHZ	: Greenberger-Horne-Zeilinger
SEP	: Set of Separable States
ND	: Set of Non-Distillable States
PPT	: Set of States with Positive Partial Transpose
CC	: Set of Classically Correlated States
PRO	: Set of Product States
ZD	: Set of States with Zero Discord
IC	: Set of Incoherent States
POVM	: Positive Operator-Valued Measure
CP	: Completely Positive
OSR	: Operator-Sum Representation
LOCC	: Local Operations and Classical Communication
IO	: Incoherent Operations
PV	: Cauchy Principal Value



SYMBOLS

j	: Discrete variable
x	: Continuous variable
$\mathcal{H}$	: Hilbert space
$\cdot^\dagger$	: Hermitian conjugate or complex conjugate transpose
$ \psi\rangle$	: Ket or state vector living in $\mathcal{H}$
$ j\rangle, x\rangle$	: Basis states of discrete and continuous systems
$ e\rangle$	: Energy eigenstates
$\langle\psi $	: Bra, conjugate transpose of ket, living in dual vector space $\mathcal{H}^*$
$\langle\phi \psi\rangle$	: Inner product
$ \psi\rangle\langle\phi $	: Outer product
ψ_j	: Probability amplitude that equals to $\langle j \psi\rangle$ - j^{th} component of $ \psi\rangle$
$\psi(x)$	: Wave function, $\langle x \psi\rangle$
P_j	: Quantum probability, $ \psi_j ^2$
$Q^{(j)}$	: Classical probability
$Q^{(J=j, J'=j')}$	: Joint probability of the outcomes $J = j$ and $J' = j'$
$Q^{(J=j J'=j')}$	: Conditional probability of $J = j$ given $J' = j'$
$P(x)$	: Quantum probability density, $ \psi(x) ^2$
$\delta_{jj'}$	: Kronecker delta function, $\langle j j'\rangle$
$\delta(x-x')$	: Dirac delta function of $x-x'$
$\otimes$	: Tensor product of two vectors/matrices/operators, a map from $\mathcal{H}_1$ and $\mathcal{H}_2$ into $\mathcal{H}_{1\&2}$
ρ	: Density matrix/operator or the generic quantum state that equals to $\sum_j Q^{(j)} \psi_j\rangle\langle\psi_j $
ρ^∞	: Stationary state
ρ^{th}	: Thermal state
$\text{tr}[\rho]$	: Trace super-operator that maps a matrix/operator into a real valued scalar that equals to $\sum_j (\rho)_{jj}$
$\text{tr}_j[\rho]$	: Partial trace super-operator that is performed over the degrees of freedom of the subsystem(s) j
$\hat{A}$	: Observable, self-adjoint or Hermitian operator on $\mathcal{H}$, i.e. $\hat{A}^\dagger = \hat{A}$
$\hat{J}, \hat{X}$	: Discrete and continuous observables
$\langle\hat{A}\rangle$	: Average of $\hat{A}$, $\text{tr}[\rho\hat{A}]$
$\sigma_{x/y/z}$	: Pauli sigma matrices
$\sigma_\pm$	: $(\sigma_x \pm i\sigma_y)/2$
$\mathcal{S}$	: The system of interest
$\mathcal{A}$	: The measurement apparatus
$\mathcal{E}$	: The environment
$\mathcal{B}$	: The heat bath
$\mathcal{L}_t[\rho]$	: Liouville super-operator
$\mathcal{D}_t[\rho]$	: Dissipator super-operator
$\mathcal{E}_t[\rho]$	: CP trace preserving dynamical map/super-operator, or quantum operation/channel

$\hat{A}_{j,S}, \rho_{j,S}$	: Schrödinger picture representations of operator $\hat{A}_j$ and state ρ_j
$\hat{A}_{j,H}, \rho_{j,H}$	: Heisenberg picture representations of operator $\hat{A}_j$ and state ρ_j
$\hat{A}_{j,I}, \rho_{j,I}$	: Dirac or interaction picture representations of operator $\hat{A}_j$ and state ρ_j
H	: Hamiltonian operator
H_{LS}	: Lamb shift Hamiltonian
U	: Unitary time evolution operator generated by H
$\mathbb{I}$	: Identity operator
M_j	: Measurement operators
Π_j	: Projective or von Neumann measurement operators
E_j	: POVM elements or POVM measurement operators
K_j	: Kraus operators
A_j	: Lindblad operators
B_j	: Hermitian bath operators coupled to A_j
$\omega, \omega_{j,j''}$	: Bohr frequencies of H_S
$A_j(\omega)$	: Eigenoperators of H_S
γ_j	: Dissipation rates
β	: Inverse temperature, $1/k_B T$
$\Gamma_{jj'}(\omega)$	: One-sided Fourier transforms of the bath correlation functions
τ_B	: Lifetime of the environmental excitations
τ_R	: Relaxation time scale
τ_S	: Intrinsic time scale of the system for the coherent evolution
$H(J)$	: Shannon information entropy of a variable J
$H(J, J')$	: Joint entropy of variables J and J'
$H(J J')$	: Conditional entropy of variable of J given J'
$H(J : J')$	: Classical mutual information between variables J and J'
$H(Q R)$	: Relative entropy or Kullback-Leibler distance between probability distributions Q and R
$S[\rho]$	: von Neumann entropy of the state ρ
$S[\rho \sigma]$	: von Neumann or quantum relative entropy between states ρ and σ
$S[\rho \sigma]$	: von Neumann or quantum conditional entropy between states ρ and σ
$E_E[\rho]$	: Entropy of entanglement of state ρ
$E_F[\rho]$	: Entanglement of formation of state ρ
$E_R[\rho]$	: Relative entropy of entanglement of state ρ
$C[\rho_{jj'}], C_{j:j'}$	: Concurrence between subsystems j and j'
$\tau_{j:j'}$	: Two-Tangle between subsystems j and j'
$\tau_{j:j':j''}$	: Three-Tangle between subsystems j, j', j''
$I[\rho_{jj'}], I_{j:j'}$	: Quantum mutual information between subsystems j and j'
$J_{j:j'}$	: (Antisymmetric) Quantum mutual information
$\delta_{1:2}$	: (Antisymmetric) Quantum discord
$D_R^{CC}[\rho]$	: Symmetric quantum discord or relative entropy of quantum discord
$D_G^{ZD}[\rho]$	: Geometric measure of discord
$C_R^{IC}[\rho]$	: Relative entropy of coherence
$C_{l_1}[\rho]$	: l_1 norm of coherence
$C_m^{cc}[\rho_{jj'}]$	: Correlated coherence where $m = \{R, l_1\}$

$\mathbf{X}$	: An atom or functional group involved in hydrogen bonds
$\mathbf{X}_j-\mathbf{H}\cdots\mathbf{X}_{j'}$	: Hydrogen-bonded system
r_j	: The length of the single $\mathbf{X}_j-\mathbf{H}$ covalent bond
$R_{jj'}$	: The separation between $\mathbf{X}_j$ and $\mathbf{X}_{j'}$
$\phi_{jj'}$	: The bond angle $\angle\mathbf{H}\mathbf{X}_j\mathbf{X}_{j'}$ of hydrogen bond
$\sigma_{LP}(\mathbf{X}_j)$	: The lone pair orbital of the proton-acceptor
$\sigma_{\mathbf{X}_j-\mathbf{H}}$	: The occupied bonding orbital of the proton-donating bond
$\sigma_{\mathbf{X}_j-\mathbf{H}}^*$	: The unoccupied antibonding orbital of the proton-donating bond
$a_j^\dagger$	: Proton creation operator at the location j associated with $\sigma_{\mathbf{X}_j-\mathbf{H}}$
a_j	: Proton annihilation operator at the location j associated with $\sigma_{\mathbf{X}_j-\mathbf{H}}$
$\hat{n}_j$	: Proton number operator at the location j , $a_j^\dagger a_j$
W_j	: On-site energy of proton at the location j
$J_{jj'}$	: Hopping constant responsible for the proton tunneling between the locations j and j'
$V_{jj'}$	: Inter-site interaction energy between the locations j and j'
λ	: Constant
V^C	: Coulomb potential
V^M	: Morse potential
V^{LJ}	: Lennard-Jones potential
$J_{x/y/z}$	: Heisenberg couplings
B	: Homogeneous magnetic field
b	: Inhomogeneous magnetic field
$b_{j,k}^\dagger$	: Phonon creation operator with energy $\hbar\omega_{j,k}$ at the location j associated with $\sigma_{\mathbf{X}_j-\mathbf{H}}$
$b_{j,k}$	: Phonon annihilation operator with energy $\hbar\omega_{j,k}$ at the location j associated with $\sigma_{\mathbf{X}_j-\mathbf{H}}$
$\hat{u}_j$	: Displacement of the $\mathbf{X}_j$ from its reference position
$d_{j,k}^\dagger$	: Phonon creation operator with energy $\hbar\Omega_k$ associated with the vibration of $\mathbf{X}_j$
$d_{j,k}$	: Phonon annihilation operator with energy $\hbar\Omega_k$ associated with the vibration of $\mathbf{X}_j$
μ	: Reduced mass between $\mathbf{X}_1$ and $\mathbf{X}_2$
$N_j(\omega_{j,k})$	: Average number of phonons with energy $\hbar\omega_{j,k}$
$J_j(\omega)$	: Spectral density function for j th bath
$\gamma_{jj'}(\omega), \gamma_j(\omega)$	: Half of the real part of $\Gamma_{jj'}(\omega)$ and a function of $N_j(\omega)$ and $J_j(\omega)$
$S_{jj'}(\omega), S_j(\omega)$	: Imaginary part of $\Gamma_{jj'}(\omega)$



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QUANTUM ASPECTS OF MOLECULAR CORRELATIONS IN BIOLOGICAL CATALYSIS

SUMMARY

Hydrogen bonds are of central importance in many natural phenomena in physical, chemical, and biological systems. In particular, several critical cellular processes require structural and electronic complementarity between molecules mimicking a lock and a key, and the degree of this complementarity is mainly achieved through the number and strength of intermolecular hydrogen bonds. From an information-theoretic point of view, the biomolecular complementarity described in this way is nothing but a type of correlation generated through hydrogen bonds.

There are various experimental and theoretical findings suggesting the occurrence of proton tunneling through the hydrogen bonds formed during some of the critical cellular processes, including nucleic acid base pairing and enzyme catalysis. This raises the question for a relation between the quantum correlations generated by proton tunneling events and the biological functions of the hydrogen-bonded biomolecules performed inside the cell. In this respect, the object of this thesis is not only to discover how an intermolecular interaction can generate quantum correlations via proton tunneling through hydrogen bonds, but also to investigate how an enzyme can use them to catalyze a biochemical reaction. In order to test the hypothesis that the tunneling of protons through hydrogen bonds generates intermolecular quantum correlations, we use the following plan.

First, we construct a fermionic Hamiltonian for the motion of a single proton in a hydrogen-bonded system and then convert it into a two-qubit pseudo-spin Hamiltonian using the Jordan-Wigner transformation. This pseudo-spin formalism enables us to quantify the amount of correlations generated by the proton mobility using the tools of quantum information theory. It also enables us to develop a Markovian master equation approach for the open system dynamics of the proton under the influence of the stretch vibrations of single covalent bonds involving the hydrogen when memory effects are neglected.

Second, we extend our pseudo-spin formalism to a twelve-qubit model of a single hexameric H_2O ring. After constructing a map from the actual 3-d structures of water ice to the present 2-d model of a single hexamer based on the single-hexamer proton relocation events occurring in them, we find that the violation of the ice rules in the model corresponds to the proton-disordering phase transition in hexagonal water ice. We then consider the steady-state solution of the master equation which reveals the role of vibrations in the emergence of ice rules and depends only on two parameters. We fix these parameters using i) the unit probability of pseudo-spins to be found inside the ice-rule preserving subspace as an indicator of proton-ordered phase and ii) the entropy of pseudo-spins as a measure of proton-disorder.

Using these model parameters, we show that l_1 norm of coherence of the pseudo-spin steady state captures the behaviour of coherent proton mobility observed in

low-temperature experiments that suspect a concerted tunneling of six protons. We investigate the nature and the amount of pairwise pseudo-spin correlations by means of the entanglement of formation, the concurrence, the quantum discord, the geometric discord, and the quantum mutual information of the reduced states. We conclude that individual quantum tunneling events in the bonds of water ice are classically rather than quantum correlated at low temperatures, a point that can't be addressed by the previous studies. We also show that although the quantum correlations in hydrogen bonds seem to be insignificant when compared to their classical counterparts, temperature dependence of quantum coherent proton mobility still originates from them.

Thirdly, we move on to our main hypothesis in this thesis which states that an enzyme, at least in principle, can derive its catalytic potency from the quantum correlations generated through intermolecular hydrogen bonds. To test this hypothesis, we take into account a prototypical enzyme-catalyzed nucleotide tautomerization reaction starting by the formation of two weak hydrogen bonds between an enzyme and its substrate. We describe the substrate recognition process by an induced-fit mechanism in which the initial hydrogen bonds are assumed to trigger a slow and continuous conformational change in the binding site of the enzyme. The two biomolecules are assumed to decouple from each other at the end of the binding stage after which the enzyme rapidly undergoes a subsequent conformational change to return back to its initial state.

In this respect, we use our pseudo-spin formalism to construct a time-independent three-qubit Hamiltonian for the tautomeric transitions of the nucleotide. We extend our master equation approach to include the effects of conformational dynamics of the nucleotide on the intramolecular proton dynamics. We describe the enzyme-nucleotide hydrogen bonds by the two-qubit pseudo-spin model assuming that the protons stay in the ground state of the time-dependent total pseudo-spin interaction Hamiltonian during the slow conformational change of the enzyme which is induced upon substrate binding. On the other hand, the effect of the subsequent rapid conformational change of the enzyme on the proton dynamics is modeled by a post-selection measurement on the pseudo-spins of the enzyme. The quantum correlations generated through the intermolecular hydrogen bonds at the end of binding state are found to be transferred to the nucleotide when the enzyme suddenly returns back to its initial state in this way. The resulting intramolecular correlations inside the nucleotide constitute a transition state which actually corresponds to a quantum coherent superposition of the substrate and product forms.

We analyze the open system dynamics of the nucleotide using our master equation approach starting after this transition state. The stationary solution is found to depend on only two parameters, which are estimated from the literature. In particular, we show that the nucleotide reaches an equilibrium such that both probabilities corresponding to substrate and product forms are equal to $1/2$ when the overall entanglement generated through hydrogen bonds is chosen to be maximal at the end of binding stage. The transition state has a free energy that equals to the average of the free energies of the substrate and the product forms, a value much lower than expected in biochemical reactions.

According to our results, the equilibrium probabilities are found to be unaffected when the quantum correlations generated through hydrogen bonds at the end of binding stage rapidly decay to classical correlations before the subsequent conformational change. The calculations show that the average energy is also independent from such

a decoherence. On the other hand, it does make a difference to the free energies. Although this is not a significant difference at all, the direction of the change is opposite to that which one expects. The free energy of the transition state drops further as a result of the increase in the entropy that is produced by the decoherence.

The rate enhancement the enzyme brings about in this way is found to be in the range of 22 to 27 orders of magnitudes for different amounts of intermolecular entanglement generation at the end of the binding stage. We compare this enzyme-catalyzed reaction with the water-mediated tautomerization of the nucleotide in which intermolecular quantum correlations are also generated through water-nucleotide hydrogen bonds. A maximal entanglement generation through intermolecular hydrogen bonds produces the product with an equilibrium probability of $1/4$ in this case. Moreover, it produces cationic and ionic transition states with the same equilibrium probability. Besides this, the water-mediated tautomerization requires the right orientations of the molecule which is very unlikely inside the cellular environment. Hence we conclude that the cyclic and classical conformational change of the enzyme increases the likelihood of the formation of quantum hydrogen bonds with the nucleotide and makes them a more efficient resource for the tautomerization process.



BİYOLOJİK KATALİZDE MOLEKÜLER İLİNTİLERİN KUANTUM DOĞASI

ÖZET

Hidrojen bağları fiziksel, kimyasal ve biyolojik birçok doğal olguda merkezi bir önemdedir. Her şeyden önce, suyu yaşam için ideal bir ortam yapan muhtelif fiziksel özellikleri H_2O molekülleri arasındaki hidrojen bağlarından kaynaklanır. Ayrıca, biyomoleküllere eşsiz yapısını veren protein katlanması sürecinin başat sürücülerinden birisi proteinin bir boyutlu amino asit zinciri içindeki hidrojen bağlarının enerjisidir. Biyolojik yapı ve işlev ilişkisi bağlamında hidrojen bağlarının rolü bununla da sınırlı kalmaz. Özellikle bazı hayati hücresel süreçler moleküller arasında anahtar ve kilit uyumunda bir yapısal ve elektronik tamamlayıcılık gerektirir. Bu moleküler tamamlayıcılığın esas ölçüsü de moleküller arası hidrojen bağlarının sayısı ve kuvvetidir. Kuramsal bilişim¹ bakış açısından bunun hidrojen bağları ile üretilen bir çeşit biyolojik ilintiden başka bir şey olmadığını söyleyebiliriz.

Nükleik asit baz eşleşmesi ve enzim katalizi dahil bu hayati süreçlerin bir kısmında, kurulan hidrojen bağları üzerinden proton tünellemesinin olduğunu iddia eden birçok deneysel ve teorik bulgu mevcuttur. Bunlar, proton tünellemesi olayları ile üretilen kuantum ilintiler ve hidrojen bağlı moleküllerin hücre içinde gerçekleştirdiği biyolojik işlevler arasında bir ilişki bulunması olasılığını doğurmaktadır. Bu bağlamda, sunmuş olduğumuz tezin amacı sadece moleküller arası bir etkileşimin hidrojen bağları üzerinden proton tünellemesi ile nasıl kuantum ilinti yaratacağının ortaya çıkartılması değil, ayrıca bir enzimin biyokimyasal bir tepkimeyi katalize etmek için bu ilintileri nasıl kullanabileceğinin de araştırılmasıdır.

Hidrojen bağları boyunca protonların tünellemesinin moleküller arası kuantum ilintiler yaratacağı varsayımını test etmek için ilk olarak hidrojen bağlı bir sistemde tekil proton hareketini betimleyen fermiyonik bir Hamiltonyen yazdık. Jordan-Wigner dönüşümü kullanarak bu Hamiltonyeni bir iki-kubit² spinimsi³ Hamiltonyenine çevirdik. Bu spinimsi formüllemesi bize proton hareketliliğiyle üretilen ilinti miktarını kuantum bilişim teorisinin araçları ile ölçme imkanı sağladı. Ayrıca protonun (hidrojenin dahil olduğu) tekil kovalent bağların titreşimlerinin etkisi altındaki açık sistem dinamiği için bir Markovyen “master” denklemi yaklaşımı geliştirdik.

İkinci olarak, geliştirdiğimiz spinimsi formüllemesini tekil bir H_2O halkasındaki proton hareketliliğini tarif edecek bir oniki-kubit modeli elde etmek için genişlettik. Su buzundaki mevcut üç boyutlu gerçek yapılardan modellediğimiz tekil heksamere kendilerinde gerçekleşen proton yer değiştirme olaylarına dayanarak bir gönderim oluşturduk. Bu gönderim bağlamında model içinde buz kurallarının çiğnenmesinin hegzagonal su buzunda XI/I_h proton-düzensizleşme faz değişimine karşılık geldiğini

¹“Information-theoretic” karşılığında.

²“Quantum Bit” anlamında, “qubit” karşılığında kubit.

³“Pseudo-spin” karşılığında.

gösterdik. Bunun ardından master denkleminin durağan-durum çözümünü göz önüne aldık. Titreşimlerin buz kurallarının ortaya çıkışındaki rolünü açığa çıkartan bu çözüm sadece iki parametreye bağlı bulundu. Bu parametrelerin alacağı değerleri XI/I_h faz geçişi deneysel verilerine göre sabitledik. Bunu yaparken de i) spinimsilerin buz kuralının korunduğu alt uzayda bulunma birim olasılığını protonların düzenli olduğu XI fazının bir göstergesi, ii) spinimsilerin von Neumann entropisini ise proton düzensizliğinin bir ölçüsü olarak kullandık.

Bu şekilde belirlenmiş model parametreleriyle spinimsi durağan durumunun l_1 norm cinsinden kuantum eşevreliliğinin⁴ düşük sıcaklık deneylerinde gözlenen eşevreliliği proton hareketliliğini yansıttığını gösterdik. Bu deneylerin bir kısmında altı protonun ilintili olarak birlikte tünellediği iddiası ileri sürüldüğü için, spinimsiler arasındaki ikili ilintilerin doğasını ve miktarını indirgenmiş durumların oluşum dolaşıklığı⁵, uyuşumu⁶, kuantum uyuşmazlığı⁷, geometrik uyuşmazlığı⁸ ve kuantum karşılıklı/ortak bilgisi⁹ aracılığıyla inceledik. Düşük sıcaklıkta su buzunu bağlarında gerçekleşen bireysel kuantum tünelleme olaylarının kuantum ilintili değil, klasik ilintili olduğu sonucuna ulaştık. Bu ilintilerin doğasının ne olduğu literatürde bulunan daha önceki çalışmalarla araştırılabilecek bir problem değildi. Ayrıca, bireysel hidrojen bağlarında proton tünellemesi ile üretilen kuantum ilintilerin klasik emsallerine oranla oldukça az miktarda bulunmalarına rağmen, l_1 norm ile karakterize ettiğimiz eşevreliliği proton hareketliliğinin sıcaklığa bağlı davranışını şekillendiren asıl etken olduğunu gösterdik.

İlk varsayımımızı bu şekilde test ettikten sonra, bir enzimin katalitik gücünü - en azından prensipte - moleküller arası hidrojen bağları ile üretilen kuantum ilintilerden türetecebileceği şeklinde ifade edilebilecek olan ve tezin temelini oluşturan ikinci varsayımımızın testine geçtik. Bunun için bir enzim ile onun bir nükleotit olan substratı arasında iki tane zayıf hidrojen bağı kurulmasıyla başlayan ve enzimle kataliz edilen bir prototip tautomerizasyon tepkimesini dikkate aldık. Substrat tanıma sürecini hidrojen bağlarının enzimin bağlanma bölgesinde yavaş ve sürekli bir biçimsel¹⁰ değişim tetiklediğini varsayan eyletik-uyum¹¹ mekanizması ile tarif ettik. Bağlanma aşamasının sonunda biyomoleküllerin ani bir şekilde birbirinden ayrıldığını ve enzimin kendisini başlangıç durumuna döndürecek olan hızlı bir ikinci biçimsel değişim altına girdiğini varsaydık.

Bu bağlamda, öncelikle spinimsi formülleştirmemizi kullanarak nükleotidin tautomerik geçişlerini betimleyen zamandan bağımsız bir üç-kubit Hamiltonyeni tasarladık. Nükleotidin biçimsel dinamiğinin molekül içi proton dinamiği üzerindeki etkisini de kapsayacak şekilde master denklemi yaklaşımımızı genişlettik. Enzimin substrat bağlanması ile tetiklenen yavaş biçimsel değişimi boyunca protonların her an zamana bağlı toplam spinimsi etkileşim Hamiltonyeninin taban durumda kaldığını varsayarak enzim-nükleotit hidrojen bağlarını iki-kubit spinimsi modelimiz ile tarif ettik. Öte yandan, enzimin takip eden hızlı biçimsel değişiminin proton dinamiği

⁴“Coherence” karşılığında eşevrelilik.

⁵“Entanglement of formation” karşılığında.

⁶“Concurrence” karşılığında.

⁷“Quantum discord” karşılığında.

⁸“Quantum measure of discord” karşılığında.

⁹“Quantum mutual information” karşılığında.

¹⁰“Conformational” karşılığında.

¹¹“Induced-fit” karşılığında.

üzerindeki etkisini enzimin spinimsileri üstüne uygulanan bir sonradan-seçim¹² ölçümü ile modelledik. Bu modele göre enzimi aniden başlangıç durumuna döndürdüğümüzde nükleotidin yeni durumunu substrat ve ürüne karşılık gelen hallerinin kuantum eşevreli bir üst üste binmesi¹³ olarak bulduk. Bağlanma aşamasının sonunda moleküller arası hidrojen bağları üzerinden üretilen kuantum ilintiler böylece nükleotide aktarılmış oldu.

Tepkime koordinatı diyagramında geçiş durumuna¹⁴ denk gelen bu üst üste binmenin oluşmasından başlayarak nükleotidin açık sistem dinamiğini master denklemi yaklaşımımız ile inceledik. Durağan durum çözümünün sadece iki parametreye bağlı olduğu bulduk ve bu parametreleri literatü kullanarak tahmin ettik. Özellikle, bağlanma aşamasının sonunda hidrojen bağları üzerinden üretilen dört-kubit dolaşıklığının azami olması durumunda nükleotidin substrat ve ürün hallerinde eşit olarak 1/2 olasılıkla bulunduğu bir dengeye ulaştığını gösterdik. Geçiş durumunun serbest enerjisinin substrat ve ürün hallerinin serbest enerjilerinin ortalamasına eşit olduğunu bulduk ki bu, biyokimyasal tepkimelerde beklenen asgari serbest enerji değerinin oldukça altında bir değere karşılık geliyor: bir anlamda enzimin aktivasyon enerjisini bu kadar aşağıya çekmesi tautomerizasyon tepkimesindeki enerji bariyerini tamamen ortadan kaldırmış oldu.

Eşevresizliğin¹⁵ sonuçlarımızı nasıl etkileyeceğine bakmak için bağlanma aşamasının sonunda hidrojen bağları üzerinden üretilen kuantum ilintilerin enzimin geçirdiği ikinci biçimsel değişimden önce hızlıca klasik ilintilere çöktüğü durumu da inceledik. Denge durumundaki substrat ve ürün olasılıkları bu çöküşten etkilenmedi. Hesaplamalarımız ortalama enerjinin de böyle bir eşevresizlikten bağımsız olduğunu gösterdi. Diğer taraftan, eşevresizlik serbest enerjide bir fark yarattı. Bu genel olarak anlamlı miktarda bir fark olmasa da değişimin yönü sezgisel olarak beklenenin tam tersi yönde bulundu. Eşevresizlik yüzünden meydana gelen çöküş ile üretilen entropi, geçiş durumunun zaten oldukça düşük bir değerde bulunan serbest enerjisinin daha da düşmesine sebep oldu.

Bağlanma aşamasının sonunda üretilen moleküller arası dolaşıklığın değişik miktarları için enzimin özetlediğimiz şekilde nükleotit tautomerizasyon tepkimesinde sağladığı hız artışı 10 üzeri 22 ile 27 mertebeleri arasında bulundu. Yani kuantum ilintili hidrojen bağlarının enzim katalizinde kaynak olarak kullanılması ile ele aldığımız tepkimenin olağanüstü hızlara ulaşabildiğini göstermiş olduk.

Enzimin biçim değişimlerinin bu hız artışıdaki rolünü vurgulamak için su aracılığıyla gerçekleşen tautomerizasyon tepkimesine de baktık ve her iki durumda elde ettiğimiz sonuçları karşılaştırdık. Bunun için bir ya da daha fazla su molekülünün nükleotit ile iki tane hidrojen bağı üzerinden etkileştiğini ve bu bağlarla moleküller arasında kuantum ilintileri üretilebileceğini varsaydık. Su-nükleotit hidrojen bağlarının azami miktarda moleküller arası dolaşıklık üretmesi durumunda nükleotit, substrat ve ürün hallerinde eşit olarak 1/4 olasılıkla bulunduğu bir dengeye ulaştı. Bu denge durumunda anyonik ve katyonik geçiş durumlarının da aynı olasılıkla ortaya çıktığı görüldü. Üstelik su aracılığıyla tautomerizasyonda varsaydığımız şekilde azami kuantum ilinti üretimi için moleküllerin ilk başta doğru yönelimle birbirini bulması gerekiyor ki

¹²“Post-selection” karşılığında.

¹³“Superposition” karşılığında.

¹⁴“Transition state” karşılığında.

¹⁵“Decoherence” karşılığında.

bu hücrel ortam içinde gerçekleşmesi çok da muhtemel olmayan bir durum. Bu yüzden enzimin eyetik-uyum mekanizması uyarınca geçirdiđi döngüsel ve klasik biçem değışimlerinin kuantum ilintili hidrojen bağları oluşumunu daha olası kılmakla kalmadığını, ayrıca onları tautomerizasyon sürecinde daha verimli bir kaynak yaptığı sonucuna ulaşmış olduk.



1. INTRODUCTION

Hydrogen bonding (or H-bonding) plays a key role in many natural phenomena in physical, chemical, and biological systems due to the fact that it is not only strong enough to provide stability, but also weak enough to allow flexibility. The H-bonds between H₂O molecules are responsible for various physical properties of water that make it the ideal biochemical medium [4]. The energetics of a protein's backbone H-bonds constitute a key determinant of that protein's folding which gives the protein its unique structure [5]. However, the role of H-bonding in biochemical self-organization goes beyond protein folding, e.g., the two strands of deoxyribonucleic acid (DNA) are held together by means of interbase H-bonds [6, 7]. In addition to this significance in molecular structure, H-bonding is also essential to molecular recognition. First and foremost, it is of importance for an enzyme in binding to the substrate molecule [8] and in driving the catalytic function [9, 10].

This chapter starts with a brief introduction to the quantum nature of H-bonding assuming a priori knowledge on the phenomenon of quantum tunneling. After providing the mathematical formalism of quantum mechanics in Ch. 2, we will revisit the tunneling phenomenon and construct a model for the proton tunneling through H-bonds in Ch. 3.

We go on this chapter moving on to describe some of the basic biochemical concepts of enzyme catalyzed reactions in Sec. 1.2. We particularly focus on the tautomerization reactions involving the relocations of H atoms inside a molecule. In Sec. 1.3, we overview the tautomerization of nucleotide bases in DNA to exemplify that the proton tunneling through H-bonds can promote tautomerization in biological systems.

Sec. 1.5 is devoted to the current literature on the likelihood of proton tunneling in specific H-bonded (bio)chemical systems such as solid phases of water, enzyme - substrate complexes, and double-stranded DNA. Finally, we introduce the hypothesis of the thesis and the chapter ends with a detailed outline of the thesis.

1.1 Quantum Nature of Hydrogen Bonds

The right balance between stability and flexibility in H-bonding interactions should not be independent of their mixed classical and quantum nature on which there is an extensive literature. Although non-trivial quantum effects weren't excluded even in the very first attempts to reveal the underlying physics behind H-bonding [11, and references therein] (see also [12–14]), the role of these effects in specific systems is still a controversial issue that is hotly debated as will be explained later in this chapter.

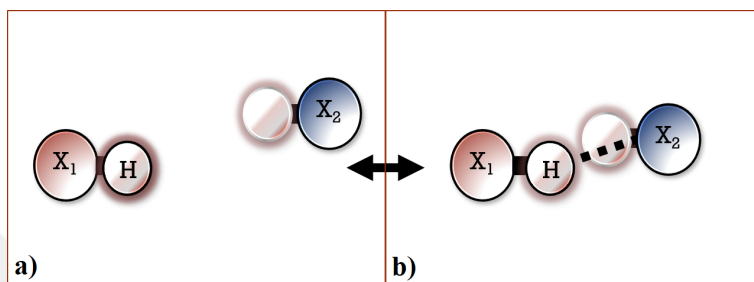


Figure 1.1 : (a) A simple depiction of the H-bonded atoms/groups based on the ball-and-stick model. Empty sphere represents the next location of H atom when it migrates from the proton-donor to the proton-acceptor by either quantum tunneling or classical hopping. (b) Elongation of the covalent bond (solid black stick) between the H atom and the proton-donor is a general indicator of a H-bond (dashed black line).

In a H-bonded system designated as $X_1-H\cdots X_2$ (Figure 1.1), X_1 is the proton-donor atom/group and X_2 is the proton-acceptor atom/group. There are three main geometric parameters defining this system: the length of the proton-donating X_1-H covalent bond (r_1), the separation between X_1 and X_2 (R_{12}) and the bond angle ($\phi_{12} \equiv \angle HX_1X_2$). Non-trivial quantum effects enter into the physics of this system in the form of electron and/or proton tunneling back and forth between donor and acceptor depending on the values of r_1 , R_{12} , and ϕ_{12} together with the chemical identities of donor, acceptor and their environment.

Electron tunneling through H-bonds occurs between the lone pair orbital of the proton-acceptor ($\sigma_{LP(X_2)}$) and the unoccupied antibonding orbital of the proton-donating bond ($\sigma_{X_1-H}^*$). This is called as the *covalency* of the $H\cdots X_2$ interaction [15]. Although the amount of charge transfer itself is generally on the order of just a few millielectrons, such a covalent contribution to the attractive energy is comparable to the electrostatic contribution in many H-bonded systems, from various phases of water [16–19] to DNA [20–22].

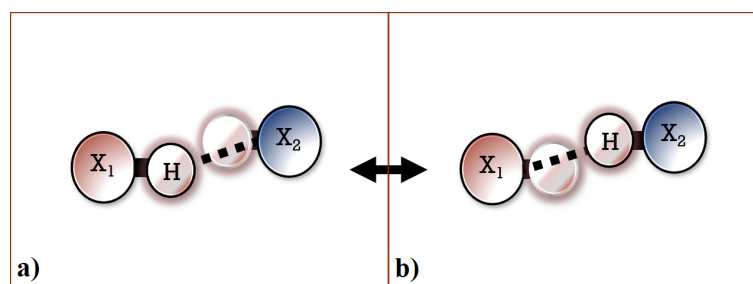


Figure 1.2 : The proton tunnels back and forth between the configurations (a) and (b) due to orbital interactions.

In addition to the charge transfer in the form of electron tunneling, the proton (nucleus) of the H atom is also likely to tunnel back and forth between the donor and acceptor as a result of intermolecular orbital interactions (Figure 1.2). As will be summarized in Sec. 1.5, there is a lot of evidence suggesting that the proton tunneling through H-bonds is of importance in several systems.

1.2 (Enzyme Catalyzed) Biochemical Reactions

In biochemistry, a catalyst molecule that increases the speed of a reaction without being consumed is called as an *enzyme* (E), while a specific reactant molecule consumed in the course of an enzyme catalyzed reaction is called as a *substrate* (S). Most enzymes are proteins and act upon more than one substrate to generate one or more *products* (P). For the sake of simplification, we consider the special case of a single substrate that is transformed into a single product in what follows.

Rate of a (bio)chemical reaction is explained by the *transition state theory* (TST) which states that the conversion of the substrate to the product depends on the formation of more reactive intermediates corresponding to saddle points on the potential energy surface (Figure 1.3). Such an intermediate molecular structure is called a *transition state* and is denoted by $S^\ddagger$.

As uncatalysed substrate requires to pass over a high energy barrier called the *activation energy* to reach the transition state, its conversion to the product is expected to be slow. The enzyme E lowers this energy to accelerate the reaction (see the energy of $ES^\ddagger$ in Figure 1.3). Such an enzymatic speed-up can be achieved in several ways [23, 24], e.g., by destabilizing the substrate, by stabilizing the transition state, or by leading the reaction into an alternative chemical pathway. However, an enzyme is

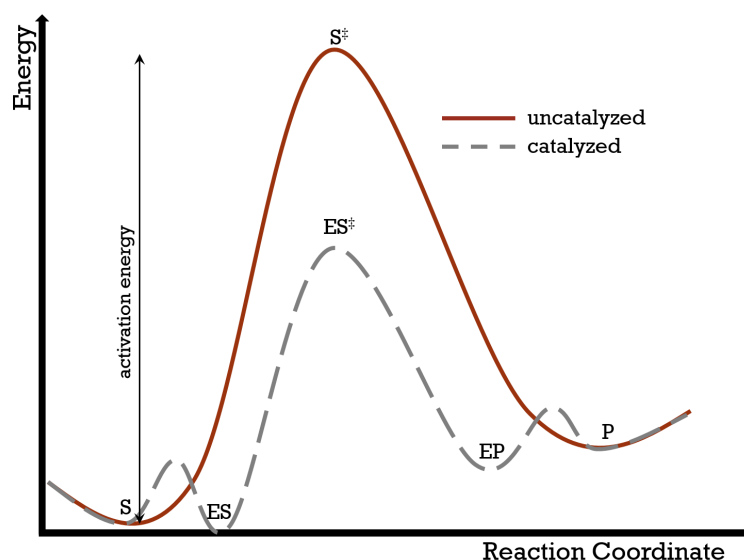


Figure 1.3 : Potential energy profile (an intersection of potential energy surface) associated to catalyzed and uncatalysed biochemical reactions. The horizontal axis which is labeled as *reaction coordinate* is an abstract coordinate and represents the progress along the reaction pathway.

not expected to remove the energy barrier completely. The residual barrier guarantees that this catalyzed reaction is sufficiently slow for the organism inside which it takes place such that the organism can survive.

An enzyme must bind its substrate and form an enzyme - substrate (ES) complex before catalyzing the reaction. There are three different mechanistic explanations for the substrate recognition, each of which can be thought as a limiting case of the most general mechanism. The earlier explanation known as the *lock and key model* states that there is a structural complementarity between the enzyme and the substrate, in the manner of a lock and key [25]. Although this model was intended to explain the high specificity observed in some enzymes, it is not consistent with TST as it attributes static structures to both the enzyme and the substrate. Also, some enzymes exhibit *catalytic promiscuity*, catalyzing chemically, and often mechanistically distinct reactions that involve different substrates.

The second and widely accepted explanation of the substrate recognition is known as the *induced-fit model* [26]. This model postpones the idea of structural complementarity proposed in the lock and key model until the end of the binding stage. It assumes that complementarity is provided by a continuous conformational change in the binding site of the enzyme triggered by the initial weak interactions. Now, there is

a growing body of evidence suggesting the presence of conformational changes upon substrate binding in many enzymes [27]. Moreover, this model appears to enhance the recognition specificity in a noisy environment [28].

Both the lock-and-key and the induced fit models attribute a single and stable conformation to an enzyme under given conditions. An alternative of these two mainstream explanations that considers the dynamical nature of enzymes is the *conformational selection model* also known as *preexisting equilibrium model*, *population selection model*, *fluctuation fit model*, or *selected fit model* [29–31]. According to this model, the enzyme exists in an equilibrium described by a conformational ensemble before substrate binding. Then, the intermolecular binding shifts this ensemble towards the conformation that is most complementary to the substrate by a selective stabilization. The conformational change upon the substrate binding is expected to be less significant in this model.

1.2.1 Tautomerization: Reactions involving intramolecular hydrogen relocation

It is possible to classify enzymes into 6 major groups depending on the type of chemical reaction they catalyze: oxidoreductases, transferases, hydrolases, lyases, isomerases, and ligases. As will be summarized in Sec. 1.5.2, so far the role of proton tunneling in enzyme catalysis has been investigated particularly focusing on the class of *oxidoreductases* that oxidize a substrate by transferring its H to an acceptor molecule through intermolecular H-bonds. These proton transfer reactions are generally restricted to carbon-hydrogen (C-H) bonds, rather than the nitrogen-hydrogen (N-H) or oxygen-hydrogen (O-H) bonds that are expected to form H-bonds strong enough to show a quantum character. The conformational changes of the oxidoreductases are usually in accordance with the conformational selection model rather than the induced-fit model. Here, to include strong H-bonds and large conformational changes, we depart from the current literature and consider the *tautomerization* reactions instead of the oxidation reactions.

In its essence, tautomerization is the relocation of the protons inside a molecule. Different configurations of the molecule that can be inter-converted to each other by this mechanism are called *tautomeric forms* of this molecule or *tautomers*. One of these

forms, the so-called *major* tautomer, predominates under physiological conditions. The remaining forms of the molecule are said to be *minor* tautomers.

The enzymes catalyzing tautomerization reactions constitute the *tautomerase superfamily*. This is a special subclass of *isomerases*, the class which alters the chemical structure of a substrate by rearranging its atoms without changing its molecular formula. Likewise the oxidoreduction reactions, the biochemical reactions catalyzed by the known members of the tautomerase superfamily generally involve a proton transport from and/or to a C atom [32]. However, neither the nature of tautomerization nor its role in biocatalysis is limited to tautomerase activity. In particular, tautomerization of nucleotides occurs generally via proton transfer between N and O atoms, and is also in charge of the initial stages of various bio-catalyzed reactions.

Exchange of protons is likely between the carbonyl and/or amino groups of nucleotides and solvent molecules. Thus, a single nucleotide can exist in many tautomeric forms due to the solvent-mediated tautomerization. Whereas the major tautomeric form of a nucleotide is the substrate of an enzyme, another enzyme may require one of the possible minor tautomeric forms of the same nucleotide, i.e., tautomeric preferences of different enzymes may be also different. As an example, to prevent point mutations, members of the transferase class of enzymes that drive DNA replication should ensure that each thymine nucleotide exists in its major tautomeric form during catalytic activity. On the other hand, the DNA repair enzyme Nei-like 1, a member of hydrolase class of enzymes, converts thymine glycol nucleotides to minor tautomeric forms during the substrate binding stage, and this tautomerization gives rise to a more efficient catalytic activity [33]. Moreover, nucleotide tautomerization also plays a direct role in a number of different functions of ribonucleic acid (RNA) enzymes [34].

In addition to the likelihood of quantum H-bonds in nucleotide tautomerization and its importance in biological functions, the induced-fit mechanism also appears to be common in nucleotide recognition [35–39]. Hence, the role of quantum H-bonds in an induced-fit enzyme catalysis can be investigated constructing a generic explanatory model inspired from tautomerization of nucleotides.

1.3 A Case Study of “Tautomerization via Hydrogen Bonding” in Nucleotides

Nucleotides have several important functions in the metabolism. They are also the fundamental subunits of nucleic acids DNA and RNA. All of them are made up from a phosphate group, a five-carbon sugar, and a unique nitrogenous base. Apart from the tens of bases existing in nature, the most common bases found in DNA are *adenine* (A), *guanine* (G), *cytosine* (C), and *thymine* (T) (Figure 1.4). DNA is composed of two antiparallel and complementary polynucleotide strands involving these bases in such a way that the opposite bases found in two strands always exist as A·T or G·C pair. Indeed, what holds the two strands together is the H-bonding between these so called *Watson-Crick pairs*, i.e., there is a double H-bond in A·T, and a triple H-bond in G·C.

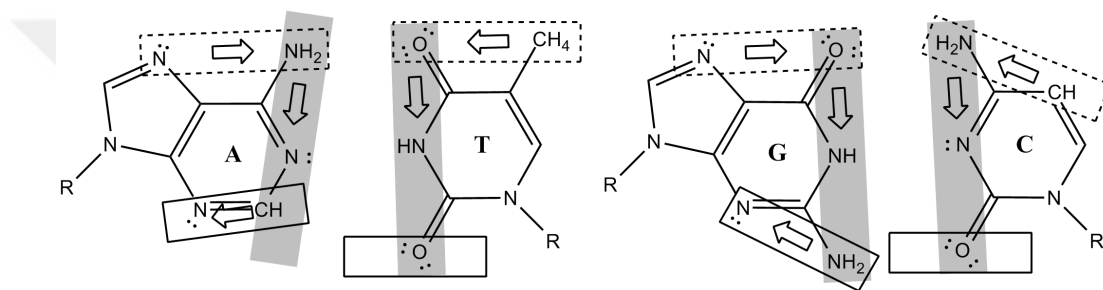


Figure 1.4 : Nucleotide bases found in DNA usually form Watson-Crick pairs connected via H-bonds over the regions indicated by filled boxes. The dotted and solid boxes are Hoogsten and sugar edges appearing respectively at major groove and minor groove sides of the double-helical structure of DNA.

Each nucleotide base has a lot of tautomeric forms [40–45], e.g., 70 different tautomeric forms can originate from a free guanine by possible single proton relocations inside the molecule. As most of the tautomers are not energetically favourable when compared to the common forms given in Figure 1.4, their occurrence in DNA is quite rare. However, if a nucleotide base in DNA exists in one of these rare forms during the DNA replication, it may change the genetic information encoded in DNA as will be explained in what follows.

When DNA undergoes replication, two strands split by the breakage of interbase H-bonds and each strand is used by enzymes as a template to duplicate the DNA, e.g., the enzyme DNA polymerase (DNApol) recognizes the nucleotide bases of one template DNA strand and finds the canonical Watson-Crick pair of each nucleotide

from the surrounding environment to synthesize a new strand complementary to the template.

However, if a rare tautomeric form exists in the template during the replication process, it may be more energetically favourable for this base to combine with a different nucleotide base than the canonical Watson-Crick pair of its common form. As an example, the rare enol form of guanine denoted by G^* in Figure 1.5 is likely to be paired with thymine during base pairing: $G^* \cdot T$ is formed by three favourable H-bonds over the region shown in Figure 1.4 whereas the interaction of G^* with cytosine, its canonical Watson-Crick pair, over the same region will no longer be attractive. Since DNA does not normally contain thymine at this location, a mutation is said to have occurred if this anomaly will not fixed by error correction mechanisms.

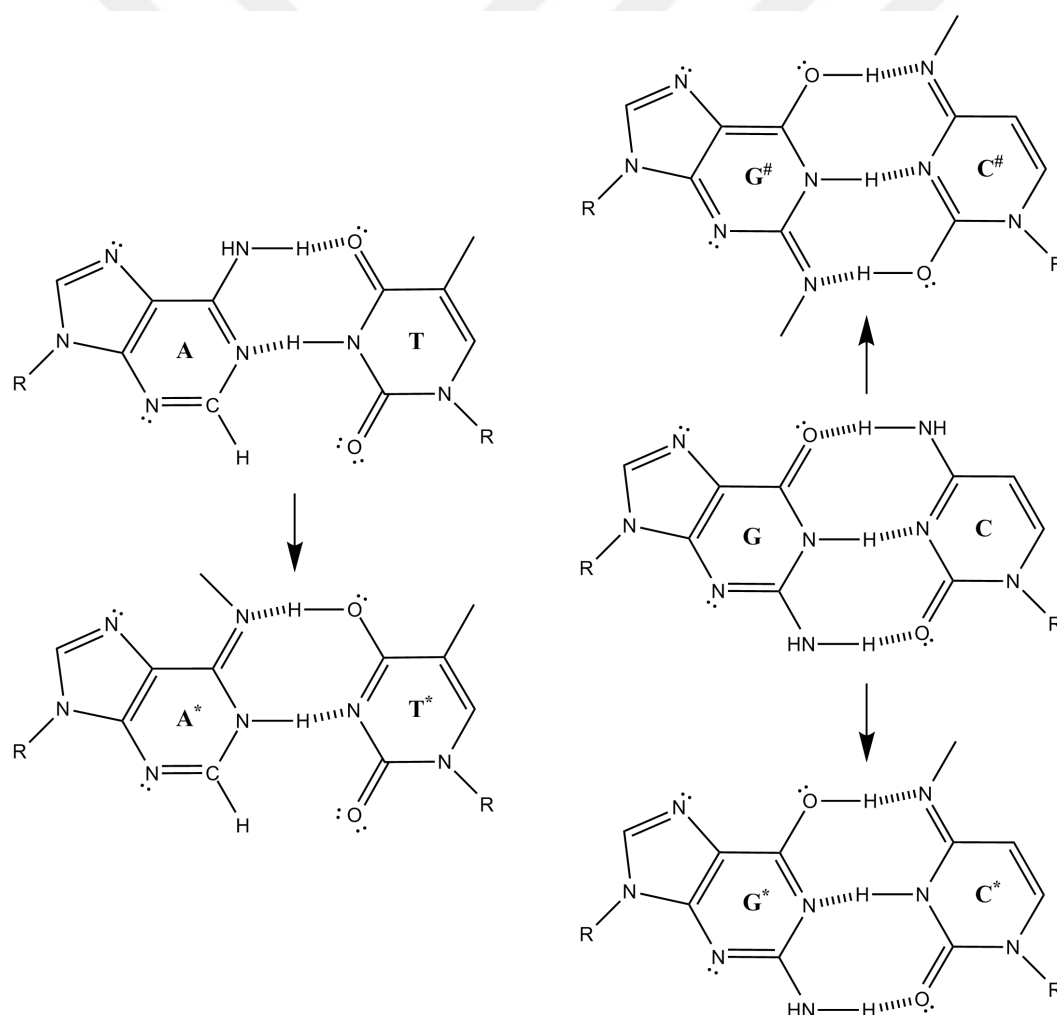


Figure 1.5 : $A \cdot T \rightarrow A^* \cdot T^*$, $G \cdot C \rightarrow G^\# \cdot C^\#$ and $G \cdot C \rightarrow G^* \cdot C^*$ tautomeric transitions due to the (correlated) tunneling of two protons in interbase H-bonds.

The appearance of rare tautomers in the double-stranded DNA requires a double proton transfer through the interbase H-bonds as shown in Figure 1.5. As the A·T pair is composed of a double H-bond, such a proton mobility in H-bonds results in the occurrence of only one rare tautomer for each nucleotide base, an imine form of adenine (configuration imi_1_n1_n9 in [41]) and an enol form of thymine (configuration 5 in [43]) respectively denoted by A* and T*. G·C pair contains a triple H-bond and so the intermolecular double proton transfer can occur in two alternative ways. An enol form of guanine (configuration G5 in [42]) denoted by G* arises from a double proton transfer through neighbouring H-bonds, whereas an imino-enol form of guanine (configuration G31 in [42]) denoted by G[#] occurs in the other case.

The proton transfer causing tautomeric shifts on both bases can happen in the form of double proton tunneling due to the quantum nature of interbase N–H···N and N–H···O bonds [46]. This possibility will be explored as the related literature is discussed in Sec. 1.5.3.

1.4 Purpose of Thesis

Structural and electronic complementarity of the biomolecules plays a key role in most of the critical cellular processes such as nucleic acid base pairing in DNA replication [6, 7] and transcription [47], receptor and ligand binding in signal transduction [48], enzyme and substrate interactions in biocatalysis [25–27, 29–31], and antibody recognition of antigens in immune response [49–51]. The degree of this complementarity is mostly related to the number and strength of non-covalent interactions including H-bonds, and determines the specificity and stability of the molecular interaction. In this respect, both the nature and the amount of intermolecular correlations created by H-bonds are of importance for living systems.

Likelihood of proton tunneling through H-bonds opens up the possibility for the creation of quantum correlations between biomolecules during these critical processes. To this extent an interesting question, which is worth pondering, is whether there may be a relation between the creation of such correlations and the biological functions fulfilled during these processes.

The object of this thesis is to discover how an intermolecular interaction can generate quantum correlations via proton tunneling through H-bonds, and also to investigate how an enzyme can use them to catalyze a biochemical reaction. We particularly intend to construct a generic model of enzyme catalyzed tautomerization inspired from the nucleotide biochemistry and to approach the intermolecular correlations generated through proton tunneling in H-bonds from the perspective of quantum information theory. The purpose of this approach is to provide a comprehensive understanding of the role of proton tunneling in biological catalysis, since correlations are routinely regarded as resource for specific tasks in the quantum information theory.

1.5 Literature Review on Proton Tunneling in Hydrogen Bonds

1.5.1 Proton tunneling in solid phases of water

Several studies examined the importance of proton tunneling in water ice phase transitions. Tunneling events through individual H-bonds were found to drive the phase transition from proton-ordered ice VIII to proton-disordered ice VII in an *ab initio* study around 100 K [52], and are also likely to be in charge of the transition from proton-disordered ice I_h to proton-ordered ice XI according to a microscopic model [53]. However, such spontaneous single proton tunnelings are opposed to the Bernal-Fowler *ice rules* [54, 55] that are predicted to be responsible for correlations between individual proton tunnelings. As a matter of fact, some recent experiments using incoherent quasielastic neutron scattering measurements on ice I_h and I_c [56], scanning tunnelling microscopy of cyclic water tetramer [57], and high precision measurements of the complex dielectric constant of ice XI [58] have successively confirmed the presence of correlated proton tunneling at low temperatures in water ice. These experimental findings suggesting the presence of proton correlations in water ice can be also supported by a diverse collection of theoretical models including one-particle density matrix analysis [59], path integral simulations [60] and lattice-based calculations [61]. However, nature of the correlations between individual proton tunneling events and their actual role in phase transitions have so far not been considered from a quantum informational perspective, an approach that stands to yield new insights.

1.5.2 Proton tunneling in enzyme-substrate complexes

Elucidation of the observed rates of several enzyme-catalyzed oxidoreduction reactions in terms of TST requires a quantum-mechanical correction for the tunneling of H-species [62–69]. However, as these observations are based on the kinetic isotope effect studies (where the H atom is replaced with its heavier isotope deuterium atom to eliminate the effects of the tunneling on reaction rate), they can not be used as a direct evidence of the role of proton tunneling in the functions of enzymes.

Besides these, protein dynamics are claimed to be in charge during the proton tunneling according to some kinetic isotope experiments [65, 66], molecular dynamic simulations [70–72], quantum mechanical molecular mechanics calculations [73–77], and qualitative quantum rate models [78].

Conversely, it is also possible to find some oxidoreductase (and also lyase) enzymes appearing not to enhance the tunneling (that has taken place) to increase their reaction rates in some experimental tests using artificial catalysts [79] and realistic simulations based on path integral formulation [80–85]. Thus, the role of tunneling of H species in catalytic reactions is not fully understood yet [86–88].

The tools of the physical organic chemistry and the computational molecular biology have been widely used in this outstanding debate whether enzymes harness the proton tunneling to exert their catalytic functions. However, these tools are unable to reveal the importance of the dynamical intermolecular correlations that are generated between the enzyme and its substrate through H-bonds. On the other hand, the correlations generated through H-bonds may be used as a resource for specific tasks in the catalysis, e.g., an enzyme can extract mechanical work from such correlations. Hence, the tools of quantum information theory may shed new light on this topic as well.

1.5.3 Proton tunneling in double-stranded DNA

The nuclei of H atoms are likely to tunnel through the interbase H-bonds too. Just ten years after the first successful attempts to build models for the molecular structure of DNA [6, 7], it was pointed out in Ref. [46] that the tunneling of two protons between H-bonded DNA bases may result in certain types of genetic mutations called *point mutations* due to the tautomeric shifts on both bases (see Figure 1.5 and Sec. 1.3

for more detail). Many studies have investigated the likelihood of interbase proton tunneling to check the validity of this mutagenesis mechanism to date. The extent of a tunneling based contribution to the point mutations, on which there is a hot debate [89–99], is outside of our scope. Comparison of either the tautomer populations and mutation rates or the tautomer lifetimes and base splitting periods is not central to the study introduced in the thesis. Here, we are only interested in the relation between tunneling through H-bonds and tautomerization.

Controversial results obtained in the literature suggest that the different models give prominence to different reaction pathways of interbase proton transfer, e.g., concerted vs stepwise or synchronous vs asynchronous pathways. Here and in the following, the term *concerted* corresponds to a reaction without an intermediate, whereas the term *stepwise* refers to a reaction involving a stable intermediate. On the other hand, the (a)synchronicity is related with the final time gap between the movement of protons.

Models based on one of the most widely used and most reliable hybrid functionals¹ in density functional theory² (DFT) demonstrated that the nature of the interbase double proton transfer depends on the base from which the first H atom migrates to initialize the process [93, 94], e.g., a concerted transfer begins only if the first H atom moves from a *purine* base (A or G) to the opposite *pyrimidine* base (T or C) through the H-bond, otherwise the transfer is likely to be stepwise. In both cases, the movement of H atoms were achieved as a consequence of tunneling. On the contrary, an alternative model based on a combined method³ claimed that the $A\cdot T \rightarrow A^*\cdot T^*$ transition arises from a concerted and asynchronous proton transfer without any proton tunneling [95]. However, analysis of G·C system by the same combined method revealed a concerted and asynchronous proton transfer based reversible transition between G·C and $G^*\cdot C^*$ without ruling out the possibility of proton tunneling [96].

As each model can propose a different reaction pathway for the same process, the estimation of the tunneling rates changes depending on the chosen model as well. Double proton tunneling probability in A·T system was calculated to be of the order of 10^{-9} by a time-dependent master equation approach using a double-well potential

¹Becke, 3-parameter, Lee-Yang-Parr (B3LYP) functional.

²A computational quantum mechanical modelling method.

³This model combines theoretical investigations at the levels of the second order Møller-Plesset perturbation theory (MP2) and DFT with the quantum theory of atoms in molecules (QTAIM) analysis.

deduced by DFT [97] but the parametrization used in this study was criticized in a following paper based on MP2 [98]. In the meantime, a recent model based on 2D Marcus theory estimated the $G^{\cdot}C^{\cdot}$ population ratio to be of the order of 10^{-7} [99], which is much lower than the value of 0.0025 obtained by a previous 6D model⁴ based on DFT [92].

1.6 Hypothesis

The first hypothesis of the research presented in this thesis is that quantum correlations can be generated between H-bonded molecules as a result of the tunneling of protons. To test this hypothesis, first, we will map the motion of a single proton in a H-bonded system into a two-qubit problem and develop a master equation approach for its open system dynamics. This will provide us a useful consideration of the quantum nature of H-bonding, e.g., the amount of correlations generated by the proton mobility will be quantified by well-known measures of quantum entanglement and quantum discord. Then, we will apply this model on a hexameric H₂O ring to reproduce the experimental findings on the low-temperature proton mobility that suspect a concerted tunneling of the protons in water ice. Finally, we will explore the underlying pairwise correlations of the overall proton mobility.

Supposing that quantum correlations can be generated between H-bonded molecules as a result of the tunneling of protons, we can ask how an enzyme can harness such correlations as a resource to achieve a biological function. Our main hypothesis in this thesis is that an enzyme, at least in principle, can derive its catalytic potency from these correlations. In this respect, we will consider the formation of two H-bonds between a generic nucleotide and a prototypical enzyme that first catalyzes the tautomerization of the nucleotide to move on to a subsequent catalysis. We will incorporate the classical conformational changes of the enzyme induced in its binding site upon nucleotide binding, letting enzyme's conformation have an effect on the intermolecular proton dynamics. Finally, we will investigate the role of quantum correlations generated through intermolecular H-bonds in the tautomerization of the nucleotide.

⁴The three H atoms under investigation in this model are considered to move either linearly through the H-bonds they are participated in or nonlinearly across the neighboring two H-bonds.

1.7 Outline of Thesis

The rest of the thesis is organized as follows.

Ch. 2 is essentially a review and does not include our original works. We start this chapter with the representation of quantum physical systems as mathematical objects in Sec. 2.1. Sec. 2.2 is concerned with some of the mathematical techniques of open quantum systems theory. After presenting the basic formalism of the broader theory, we summarize the microscopic derivation of the Markovian quantum master equation in Sec. 2.2.3.2, which will be used in subsequent research chapters. Sec. 2.3 covers the fundamental notions of quantum information theory, that will be also used extensively in the rest of the thesis.

Ch. 3 is devoted to a pseudo-spin model of the proton tunneling through a single H-bond, which forms a basis for later chapters. In the light of the mathematical framework provided in Ch. 2 and based mainly on our research presented in [100], we first revisit the notion of quantumness of the H-bonding in Sec. 3.1. We then construct a two-qubit representation for a generic H-bonded system in Sec. 3.2, and develop a master equation approach to its open system dynamics in Sec. 3.3. This allows us to investigate the steady state quantum correlations generated through a generic H-bond in Sec. 3.4.

The work covered in Ch. 4 is based mainly on our research presented in [101]. This chapter starts by reviewing the structure of hexagonal water ice and its temperature dependent phase transitions in Sec. 4.1. After extending the two-qubit pseudo-spin model proposed in the previous chapter to a twelve-qubit model of a single hexameric H₂O ring in Sec. 4.2, we discuss the map between the actual 3-d structures of water ice and the present 2-d model of a single hexamer in Sec. 4.3. Our formalism enables us to quantify the coherent proton mobility around the hexagonal ring by one of the principal measures of coherence, the l_1 norm of coherence. The nature of the pairwise pseudo-spin correlations underlying the overall mobility is further investigated within this formalism in Sec. 4.4. There, we show that the classical correlations of the individual quantum tunneling events in long-time limit is sufficient to capture the behaviour of coherent proton mobility observed in low-temperature experiments. We

give a brief discussion of the future directions in Sec. 4.5, and summarize our main results in Sec. 4.6.

The work covered in Ch. 5 is based mainly on our research presented in [102]. The role of proton tunneling in biological catalysis is investigated there within the mathematical framework provided in Ch.s 2 and 3. The biological scenario under consideration is introduced in Sec. 5.1. This includes the formation of two H-bonds between a nucleotide and a prototypical enzyme that first catalyzes the tautomerization of the nucleotide to move on to a subsequent catalysis. To describe the tautomeric transitions of the nucleotide, we extend the two-qubit pseudo-spin model proposed in the Ch. 3 to a time-independent three-qubit Hamiltonian in the first part of Sec. 5.2. We continue this section extending our master equation approach to include the effects of internal conformation dynamics of the nucleotide on the intramolecular proton dynamics. We then describe the enzyme- nucleotide H-bonds using the same two-qubit pseudo-spin model and end the section explaining the role of enzyme's conformation in the intermolecular proton dynamics. The protons are assumed to stay in the ground state of the total pseudo-spin interaction Hamiltonian at every instant of time during the slow conformational change of the enzyme, while the effect of the rapid conformational change of the enzyme on the proton dynamics is modeled by a post-selection measurement on the pseudo-spins of the enzyme. The required model parameters are estimated from the literature in Sec. 5.3, and our results are summarized in Sec. 5.4. In particular, we show that quantum correlations generated through intermolecular H-bonds can be used by the enzyme to lead the nucleotide into a new transition state which has a free energy just in-between the free energies of its substrate and the product forms. We also present a more surprising result for the case in which the intermolecular quantum correlations quickly decay due to a strong decoherence.

Finally, we conclude with a short list of topics to be investigated in future works, in Ch. 6.



2. MATHEMATICAL FRAMEWORK

In this chapter, we review some of the mathematical tools of the open quantum system theory and the quantum information theory that form the basis of the subsequent research chapters.

2.1 Representation of Physical Systems

2.1.1 Isolated systems

State of an isolated quantum system is represented by *state vector* that lives in a complex inner product space called Hilbert space ($\mathcal{H}$) [103–106]. This state is also known as a *ket* and denoted by $|\psi\rangle$ according to the bra-ket notation introduced by Dirac [107]. Algebraically, it is a unit column vector of dimension n that equals to the maximum number of linearly independent states that the system of interest can occupy. The dimension n can be interpreted from a physical viewpoint as the number of states that are classically allowed to the system. Let's call such *classical states* as *basis states* in what follows, unless stated otherwise. On the other hand, physical interpretation of the unity condition requires an elaboration on the formalism.

Inner product spaces are the vector spaces in which the lengths and angles can be measured. In this respect, Hilbert space is a vector space over the complex numbers equipped with an inner product that maps two vectors into a complex number:

$$(|\psi\rangle, |\phi\rangle) = (|\psi\rangle)^\dagger (|\phi\rangle) = \langle\psi||\phi\rangle \equiv \langle\psi|\phi\rangle \in \mathbb{C}. \quad (2.1)$$

Here, $\langle\psi|$ is the conjugate transpose of the ket $|\psi\rangle$ and lives in the dual vector space $\mathcal{H}^*$. It is called as *bra* in the Dirac notation [107] and a bra-ket $\langle\cdot|\cdot\rangle$ can be regarded as a generalization of the Euclidean dot product that takes complex values.

In the case of discrete systems where n is finite or countably infinite, assume that the basis states are labeled by a discrete variable j . A generic state vector can be written

as a coherent superposition of basis states $\{|j\rangle\}$ in the form of a sum:

$$|\psi\rangle = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \vdots \\ \psi_n \end{pmatrix} = \sum_{j=1}^n \psi_j |j\rangle = \psi_1 |1\rangle + \psi_2 |2\rangle + \cdots + \psi_n |n\rangle, \quad (2.2)$$

where $\{|j\rangle\}$ constitutes an orthonormal basis, i.e., $\langle j|j'\rangle = \delta_{jj'}$. The coefficients ψ_j preceding the kets $|j\rangle$ are called as *probability amplitudes* that can be calculated by the inner product $\langle j|\psi\rangle$. When the system in state $|\psi\rangle$ is measured in the basis of $\{|j\rangle\}$, the absolute square of these amplitudes give the measurement probabilities [106, 108], i.e., the probability of finding this system in a particular state $|j\rangle$ after the measurement equals to $P_j = \psi_j^* \psi_j = |\psi_j|^2$.

Since the continuous quantum systems have uncountably infinite degrees of freedom, a continuous variable should be used to label their basis states. Let's chose x for this purpose as it is conventionally used for the position, the most widely studied observable that has a continuous spectrum. General state of such systems should be described by a coherent superposition of basis states $\{|x\rangle\}$ in the form of an integral:

$$|\psi\rangle = \int \psi(x) |x\rangle dx, \quad (2.3)$$

where the orthonormality relation is given by $\langle x|x'\rangle = \delta(x - x')$ this time and the coefficient functions $\psi(x)$ preceding the kets $|x\rangle$ are called as *wave functions* that have the same meaning with the probability amplitudes [106, 108], e.g., $\psi(x) = \langle x|\psi\rangle$ and $P(x) = |\psi(x)|^2$.

After providing a context for the probability amplitudes and wave functions briefly, we can explain the physical meaning of the unity property that a valid state vector should have. Norm of a state vector $|\psi\rangle$ can be obtained by the square root of the self inner product $\langle \psi|\psi\rangle$. Then, the unity condition implies conservation of the probability both in discrete systems:

$$1 = \langle \psi|\psi\rangle = \left(\sum_{j'} \psi_{j'}^* \langle j'| \right) \left(\sum_j \psi_j |j\rangle \right) = \sum_{j,j'} \psi_{j'}^* \psi_j \delta_{jj'} = \sum_j P_j, \quad (2.4)$$

and in continuous systems:

$$\begin{aligned}
1 &= \langle \psi | \psi \rangle \\
&= \left(\int \psi(x')^* \langle x' | dx' \right) \left(\int \psi(x) |x\rangle dx \right) \\
&= \int \int dx dx' \psi(x')^* \psi(x) \delta(x - x') \\
&= \int P(x) dx.
\end{aligned} \tag{2.5}$$

The representation of the state of a particular system depends on the dynamical variable that is chosen to be measured, e.g., the position of a particle in a box is a continuous variable, whereas the allowed energies take discrete values for the same particle.

A dynamical variable describing the basis of measurement is known as an *observable* and denoted by $\hat{A}$ in quantum mechanics. Algebraically, observables are represented by self-adjoint or *Hermitian* operators $\hat{A} = \hat{A}^\dagger$ acting on the Hilbert space and constructed by the so called spectral decomposition theorem $\hat{J} = \sum_j j |j\rangle \langle j|$ and $\hat{X} = \int dx x |x\rangle \langle x|$ [103–106]. Here, the operation denoted by a ket-bra $|\cdot\rangle \langle \cdot|$ is the *outer product* that maps the two vectors into a matrix and the quantity preceding outer product should be a real number that takes the values of the possible outcomes of a particular measurement of the associated observable. Actually, these numbers used in the spectral decomposition are the eigenvalues of the operator, while the following kets are the corresponding eigenvectors or *eigenstates*, e.g., $\hat{J}|j\rangle = j|j\rangle$ and $\hat{X}|x\rangle = x|x\rangle$. It is guaranteed by the Hermiticity of the operator that the eigenvalues representing the measurement outcomes are always real.

State vector provides a probability distribution for the value of each observable, i.e., the average of the measurement outcomes, denoted by $\langle \hat{A} \rangle$, equals to $\langle \psi | \hat{A} | \psi \rangle$ where $|\psi\rangle$ is the initial state of the system. It is straightforward to show this equality for discrete observables as:

$$\begin{aligned}
\langle \hat{J} \rangle_\psi &= \langle \psi | \hat{J} | \psi \rangle = \left(\sum_{j'} \psi_{j'}^* \langle j' | \right) \left(\sum_j j |j\rangle \langle j| \right) \left(\sum_{j''} \psi_{j''} |j''\rangle \right) \\
&= \sum_{j,j',j''} j \psi_{j'}^* \psi_{j''} \langle j' | j \rangle \langle j | j'' \rangle = \sum_{j,j',j''} j \psi_{j'}^* \psi_{j''} \delta_{jj'} \delta_{jj''} \\
&= \sum_j j P_j.
\end{aligned} \tag{2.6}$$

However, this is not the most general average encountered in quantum systems as will be explained in the following subsection.

2.1.2 Open systems

In reality, quantum systems are in interaction with other quantum systems that are not of direct interest or with their environment that contains a lot of inaccessible degrees of freedom. Also, it is not generally possible to isolate a quantum system from its surroundings in a perfect sense. Furthermore, it is the very self of the system - system or system - environment interaction that is of interest in some particular situations. Such interactions do not allow a description of the system by a single state vector as the available information is less than maximum. Instead, the system exist in an incoherent superposition of state vectors, i.e., in a statistical mixture of state vectors $\{|\psi^{(j)}\rangle\}$ with associated classical probabilities $\{Q^{(j)}\}$ as in a classical ensemble:

$$|\psi\rangle = \{P = Q^{(1)}, |\psi^{(1)}\rangle; P = Q^{(2)}, |\psi^{(2)}\rangle; \dots\}. \quad (2.7)$$

In this case, one can attribute the most general quantum statistical ensemble to the observables:

$$\langle \hat{J} \rangle = \sum_k Q^{(k)} \langle \psi^{(k)} | \hat{J} | \psi^{(k)} \rangle = \sum_{k,j} j P_j^{(k)} Q^{(k)}. \quad (2.8)$$

When the information on the system is less than maximum as is given above, it is better to move on from state vector formalism to density matrix formalism to encode the all measurement statistics by a single mathematical quantity. Although one can trace the origin of *density matrix* to the papers [109–111] published in the early days of the very first precise formulation of quantum mechanics with state vectors, its operational significance in the formalism was ignored for a very long time [112], probably because of its primary and limited consideration as *statistical matrix* in the context of quantum statistical mechanics.

Algebraically, a density matrix labeled by ρ is a Hermitian, positive-semidefinite and normalized operator that acts on the Hilbert space. It is defined as

$$\rho = \sum_j Q^{(j)} |\psi^{(j)}\rangle \langle \psi^{(j)}|, \quad (2.9)$$

for a system described by the generic ensemble of state vectors given in (2.7). Consequently, (2.8) becomes:

$$\langle \hat{J} \rangle = \text{tr}[\rho \hat{J}]. \quad (2.10)$$

Actually, the density matrix has the minimum number of parameters to calculate the grand average above. Hermiticity of this operator is satisfied by the definition and guarantees that the grand average $\langle \hat{J} \rangle$ takes a real value for all the observables.

Positivity property of the density matrix can be written as $\langle \phi | \rho | \phi \rangle \geq 0 \forall |\phi\rangle \in \mathcal{H}$. It is a consequence of the fact that each diagonal element of the density matrix in a given basis $\{|\phi\rangle\}$ is a weighted sum of the probabilities in this basis as follows:

$$\begin{aligned}
\langle \phi | \rho | \phi \rangle &= \langle \phi | \left(\sum_j Q^{(j)} |\psi^{(j)}\rangle \langle \psi^{(j)}| \right) | \phi \rangle \\
&= \sum_j Q^{(j)} \langle \phi | \left(\sum_{\phi'} \psi_{\phi'}^{(j)} |\phi'\rangle \right) \left(\sum_{\phi''} \psi_{\phi''}^{*(j)} \langle \phi''| \right) | \phi \rangle \\
&= \sum_{j, \phi', \phi''} Q^{(j)} \psi_{\phi'}^{(j)} \psi_{\phi''}^{*(j)} \underbrace{\langle \phi | \phi' \rangle}_{\delta_{\phi \phi'}} \underbrace{\langle \phi'' | \phi \rangle}_{\delta_{\phi'' \phi}} \\
&= \sum_j Q^{(j)} P_{\phi}^{(j)},
\end{aligned} \tag{2.11}$$

where $Q^{(j)}$ and $P_{\phi}^{(j)}$ are greater than or equal to zero for all the values of j and ϕ as they are probabilities corresponding to $P(|\psi\rangle = |\psi^{(j)}\rangle)$ and $P(|\psi^{(j)}\rangle = |\phi\rangle)$ respectively.

Finally, the density matrix should be normalized with respect to trace norm because of the conservation of the probability:

$$\begin{aligned}
\text{tr}[\rho] &= \sum_j \langle j | \rho | j \rangle \\
&= \sum_{j'} Q^{(j')} \left(\sum_j P_j^{(j')} \right) \\
&= \sum_{j'} Q^{(j')} = 1.
\end{aligned} \tag{2.12}$$

Here, note that $\sum_j P_j^{(j')} = 1$ is nothing but the unity property of the constituent state vectors.

2.1.2.1 Pure and mixed states

In the case of an isolated system, the density matrix simply equals to the self outer product of the single state vector $|\psi\rangle$ describing the system, i.e., $\rho = |\psi\rangle \langle \psi|$. Such density matrices are said to be *pure*, while the mixture of pure states as in the ensemble (2.9) that describes a general open system are said to be *mixed*.

The conventional way of discriminating between pure or mixed states is based on a simple criterion: $\text{tr}[\rho^2]$ equals to 1 for pure states as $\rho^2 = (|\psi\rangle\langle\psi|)^2 = |\psi\rangle\langle\psi| = \rho$, while it is less than 1 for mixed states as will be explained in what follows. However, before showing the inequality $\text{tr}[\rho^2] < 1$, it should be noticed that the density matrix is a positive-semidefinite operator which implies that it has a spectral decomposition as below:

$$\rho = \sum_j \lambda_j |j\rangle\langle j|, \quad (2.13)$$

where $\{|j\rangle\}$ constitutes an orthonormal basis and $\{\lambda_j\}$ are non-negative eigenvalues. It should be noticed that the normalization property of the density matrix requires that $\sum_j \lambda_j = 1$. Then, it is straightforward to show that

$$\begin{aligned} \text{tr}[\rho^2] &= \sum_j \langle j | \rho^2 | j \rangle \\ &= \sum_j \langle j | \left(\sum_{j'} \lambda_{j'} |j'\rangle\langle j'| \right)^2 | j \rangle \\ &= \sum_{j, j', j''} \lambda_{j'} \lambda_{j''} \langle j | j' \rangle \langle j' | j'' \rangle \langle j'' | j \rangle \\ &= \sum_j \lambda_j^2 < \left(\sum_j \lambda_j \right)^2 = 1. \end{aligned} \quad (2.14)$$

All the density matrices satisfying the condition $\text{tr}[\rho^2] \leq 1$ constitute a convex set, i.e., any convex linear combination of density matrices is also a density matrix:

$$\rho = \sum_j Q^{(j)} \rho^{(j)} \ni \sum_j Q^{(j)} = 1. \quad (2.15)$$

Physical interpretation of Eq. (2.15) is that a statistical mixture of the density matrices $\{\rho^{(j)}\}$ with probabilities $\{Q^{(j)}\}$ is also a density matrix. However, it should be emphasized that a pure state can not be decomposed into such a convex linear combination of density matrices that have different statistical properties, i.e., (2.15) holds for pure states if and only if $\rho^{(j)} = \rho$.

2.1.3 Composite systems

To study the system-system or system-environment interactions, we first need to describe the state of joint systems that are composed of several subsystems. Let's denote each subsystem of a composite system by an index, e.g., $\psi_j^{(1)}$, $|\psi_1\rangle$, $|\psi\rangle_1$,

ρ_1 , or $\hat{A}^{(1)}$, unless stated otherwise and focus on a bipartite system for the sake of simplification. Also assume that there is not correlations between two subsystems respectively being in states ρ_1 and ρ_2 . Then, the initial state of the bipartite joint system is described by

$$\rho_{12} = \rho_1 \otimes \rho_2 \quad (2.16)$$

where the operation denoted by $\otimes$ is known as *tensor product* that can be performed either between state vectors

$$|\psi_1\rangle \otimes |\psi_2\rangle = \begin{pmatrix} \psi_1^{(1)} |\psi_2\rangle \\ \psi_2^{(1)} |\psi_2\rangle \\ \vdots \\ \psi_n^{(1)} |\psi_2\rangle \end{pmatrix} = \begin{pmatrix} \psi_1^{(1)} \psi_1^{(2)} \\ \vdots \\ \psi_1^{(1)} \psi_m^{(2)} \\ \psi_2^{(1)} \psi_1^{(2)} \\ \vdots \\ \psi_2^{(1)} \psi_m^{(2)} \\ \vdots \\ \psi_n^{(1)} \psi_1^{(2)} \\ \vdots \\ \psi_n^{(1)} \psi_m^{(2)} \end{pmatrix}, \quad (2.17)$$

if both systems are pure or in general between density matrices:

$$\begin{aligned} \rho_1 \otimes \rho_2 &= \begin{pmatrix} (\rho_1)_{11} \rho_2 & \cdots & (\rho_1)_{1n} \rho_2 \\ \vdots & \ddots & \vdots \\ (\rho_1)_{n1} \rho_2 & \cdots & (\rho_1)_{nn} \rho_2 \end{pmatrix} \\ &= \begin{pmatrix} (\rho_1)_{11} (\rho_2)_{11} & \cdots & (\rho_1)_{11} (\rho_2)_{1m} & \cdots & (\rho_1)_{1n} (\rho_2)_{11} & \cdots & (\rho_1)_{1n} (\rho_2)_{1m} \\ \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\ (\rho_1)_{11} (\rho_2)_{m1} & \cdots & (\rho_1)_{11} (\rho_2)_{mm} & \cdots & (\rho_1)_{1n} (\rho_2)_{m1} & \cdots & (\rho_1)_{1n} (\rho_2)_{mm} \\ \vdots & & \vdots & \ddots & \vdots & & \vdots \\ (\rho_1)_{n1} (\rho_2)_{11} & \cdots & (\rho_1)_{n1} (\rho_2)_{1m} & \cdots & (\rho_1)_{nn} (\rho_2)_{11} & \cdots & (\rho_1)_{nn} (\rho_2)_{1m} \\ \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\ (\rho_1)_{n1} (\rho_2)_{m1} & \cdots & (\rho_1)_{n1} (\rho_2)_{mm} & \cdots & (\rho_1)_{nn} (\rho_2)_{m1} & \cdots & (\rho_1)_{nn} (\rho_2)_{mm} \end{pmatrix}. \end{aligned} \quad (2.18)$$

Algebraically, the tensor product forms a larger Hilbert space $\mathcal{H}_{12}$ of dimension $n \times m$ from two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ of dimensions n and m respectively, i.e., $\mathcal{H}_{12}$ is spanned by the basis $\{|j_1\rangle \otimes |j'_2\rangle\}$ when $\{|j_1\rangle\}$ and $\{|j'_2\rangle\}$ are orthonormal bases of $\mathcal{H}_1$ and $\mathcal{H}_2$. Note that it is usually preferred to use abbreviated notations for the tensor product omitting “ $\otimes$ ”, e.g., $|j_1\rangle |j'_2\rangle$, $|j_1, j'_2\rangle$ or $|j j'\rangle_{12}$ can be used instead of $|j_1\rangle \otimes |j'_2\rangle$.

2.1.3.1 Separable and entangled states

The pure joint states are said to be *separable* if they can be written in the form of a single tensor product of pure states of two subsystems as in (2.17). However, $\mathcal{H}_{12}$ also includes the linear combinations of these states:

$$|\psi\rangle_{12} = \sum_{j,j'} \psi_{jj'}^{(12)} |\psi_j\rangle_1 \otimes |\psi_{j'}\rangle_2, \quad (2.19)$$

some of which are not separable as they can be written in the form:

$$|\psi\rangle_{12} = \sum_j \lambda_j |j\rangle_1 \otimes |j\rangle_2 \neq \left(\sum_j \psi_j^{(1)} |j\rangle_1 \right) \otimes \left(\sum_{j'} \psi_{j'}^{(2)} |j'\rangle_2 \right), \quad (2.20)$$

where $\{|j\rangle_1\}$ and $\{|j\rangle_2\}$ are orthonormal bases that respectively span $\mathcal{H}_1$ and $\mathcal{H}_2$, $\{\lambda_j\}$ satisfies that $\lambda_j \in \mathbb{R}$, $\lambda_j \geq 0$, $\sum_j \lambda_j^2 = 1$, and there is more than one nonzero λ_j . The composite system described above exists in a coherent superposition state in such a way that the joint probability amplitudes $\{\lambda_j\} = \{\psi_{jj'}^{(12)}\}$ cannot be factorized into a product of $\{\psi_j^{(1)}\}$ and $\{\psi_{j'}^{(2)}\}$ even if the subsystems are spatially separated by long distances. This implies a correlation between subsystems that cannot be created or increased using local operations and classical communication (LOCC). Such nonlocal correlations known as *entanglement* enforces a major departure from the classical physics and the pure joint states allowing entanglement between subsystems as in equation (2.20) are said to be *entangled* [113–117].

In the case of mixed joint states, the definition of entangled states via separability can be extended as follows [118]:

$$\rho_{12} \neq \sum_j Q^{(j)} \rho_1^{(j)} \otimes \rho_2^{(j)}, \quad (2.21)$$

in which the right-hand side equals to the general form of a mixed separable state. Unlike the pure ones, mixed separable states constitute a convex set and their subsystems can be correlated in a classical or even in a quantum sense when j goes over more than one value in the right-hand side of (2.21). In particular, the mixed separable states that can be written in the form:

$$\rho_{12} = \sum_{j,j'} (\rho_{12})_{jj'} |j_1\rangle\langle j_1| \otimes |j'_2\rangle\langle j'_2|, \quad (2.22)$$

are said to be *classically correlated* if $\{|j_1\rangle\}$ and $\{|j'_2\rangle\}$ span orthonormal bases respectively in $\mathcal{H}_1$ and $\mathcal{H}_2$ [119–122]. Physically, as the orthogonality of local bases

provides distinguishability of the local states¹, the correlation contained in the joint state becomes locally accessible. These states constitute a non-convex subset of the mixed separable states. A mixture of them can possess nonclassical correlations known as *quantum discord* [119,120] when the orthogonality condition breaks down at least in one subsystem. The local indistinguishability requires the use of quantum operations in addition to LOCC and if it is achieved in the first (or second) subsystem only, the joint state is said to be *quantum-classical* (or *classical-quantum*), whereas it is said to be *quantum-quantum* in the case of an indistinguishability with respect to both subsystems [123,124].

2.1.3.2 Reduced states

The state of only one or more subsystems may be of interest in some situations, e.g., in situations in which the joint state is too complicated to be practicable for the calculations. Such situations are captured by the concept of *reduced density matrix* that was first introduced in Ref. [125]. For a partition of the composite system into subsystem 1 and subsystem 2, the reduced state of the subsystem 1 is described by

$$\rho_1 = \text{tr}_2[\rho_{12}], \quad (2.23)$$

where tr_2 is called *partial trace* over subsystem 2.

Algebraically, the partial trace is a *super-operator*, a linear map from linear operators on one Hilbert space to linear operators on another Hilbert space and is defined by

$$\text{tr}_2[\rho_{12}] = \sum_j \left(\mathbb{I}^{(1)} \otimes \langle j_2 | \right) \rho_{12} \left(\mathbb{I}^{(1)} \otimes | j_2 \rangle \right), \quad (2.24)$$

where $\mathbb{I}^{(1)}$ is the identity operator that acts on $\mathcal{H}_1$ and $\{|j_2\rangle\}$ is an orthogonal basis that spans $\mathcal{H}_2$. This operator-valued function can be thought as a generalization of the scalar-valued trace function. In this respect, it is cyclic only in the operators acting on the same Hilbert space and always commutes with the operators acting on other Hilbert spaces.

¹When two states are not orthogonal to each other, their inner product does not vanish. Hence, a system in one of them can be also found in the other with a probability that equals to absolute square of the inner product. That is to say that these two states cannot be distinguished from each other with this probability. In this respect, if two elements of a local basis are not orthogonal to each other in (2.22), e.g., $|j_1 = 1\rangle \not\perp |j_1 = 2\rangle$, the state of the associated subsystem is indistinguishable with a probability of $|\langle j_1 = 1 | j_1 = 2 \rangle|^2 \neq 0$.

Let's elaborate on the reduced states applying partial trace to the specific states introduced in the previous subsection. The reduced states of pure separable states $|\psi_{12}\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$ are also pure as expected:

$$\begin{aligned}
\rho_1 &= \text{tr}_2[|\psi_{12}\rangle\langle\psi_{12}|] \\
&= \text{tr}_2[|\psi_1\rangle\langle\psi_1| \otimes |\psi_2\rangle\langle\psi_2|] \\
&= \text{tr}_2[|\psi_1\rangle\langle\psi_1| \otimes \langle\psi_2|\psi_2\rangle] \\
&= |\psi_1\rangle\langle\psi_1|,
\end{aligned} \tag{2.25}$$

but it is not possible to assign pure states to the reduced states of pure entangled states:

$$\begin{aligned}
\rho_1 &= \sum_{j''} \langle j_2'' | \left(\sum_j \lambda_j |j_1\rangle \otimes |j_2\rangle \right) \left(\sum_{j'} \lambda_{j'} \langle j_1' | \otimes \langle j_2' | \right) |j_2''\rangle \\
&= \sum_{j,j',j''} \lambda_j \lambda_{j'} |j_1\rangle \langle j_1' | \otimes \langle j_2'' | j_2 \rangle \langle j_2' | j_2 \rangle = \sum_j \lambda_j^2 |j_1\rangle \langle j_1|,
\end{aligned} \tag{2.26}$$

since the trace of the square of (2.26) is less than 1:

$$\begin{aligned}
\text{tr}[\rho_1^2] &= \sum_{j''} \langle j_1'' | \left(\sum_j \lambda_j^2 |j_1\rangle \langle j_1| \right) \left(\sum_{j'} \lambda_{j'}^2 \langle j_1' | \langle j_1' | \right) |j_1''\rangle \\
&= \sum_{j,j',j''} \lambda_j^2 \lambda_{j'}^2 \langle j_1'' | j_1 \rangle \langle j_1 | j_1' \rangle \langle j_1' | j_1'' \rangle = \sum_j \lambda_j^4 < \left(\sum_j \lambda_j^2 \right)^2 = 1,
\end{aligned} \tag{2.27}$$

because of the presence of more than one λ_j that are greater than zero. The mixedness of the reduced state of an isolated system arises from the fact that ignorance of one subsystem eliminates its degrees of freedom and does not allow one to access the maximum information on the other subsystem. Hence, it is an indicator of the entanglement.

Conversely, taking the partial trace on one subsystem does not yield any disturbance on the other one for the classically correlated states:

$$\begin{aligned}
\rho_1 &= \sum_{j''} \langle j_2'' | \left(\sum_{j,j'} (\rho_{12})_{jj'} |j_1\rangle \langle j_1| \otimes |j_2'\rangle \langle j_2''| \right) |j_2''\rangle \\
&= \sum_{j,j'} (\rho_{12})_{jj'} |j_1\rangle \langle j_1| \otimes \left(\sum_{j''} \langle j_2'' | j_2' \rangle \langle j_2' | j_2'' \rangle \right) = \sum_{j,j'} (\rho_{12})_{jj'} |j_1\rangle \langle j_1|.
\end{aligned} \tag{2.28}$$

If its classical subsystem is discarded, the quantum subsystem of the quantum-classical (or classical-quantum) state is unperturbed:

$$\begin{aligned}\rho_1 &= \sum_{j'} \langle j'_2 | \left(\sum_j Q^{(j)} \rho_1^{(j)} \otimes |j_2\rangle \langle j_2| \right) |j'_2\rangle \\ &= \sum_j Q^{(j)} \rho_1^{(j)} \otimes \left(\sum_{j'} \langle j'_2 | j_2 \rangle \langle j_2 | j'_2 \rangle \right) = \sum_j Q^{(j)} \rho_1^{(j)},\end{aligned}\tag{2.29}$$

but not *vice versa*:

$$\begin{aligned}\rho_2 &= \sum_{j'} \langle j'_1 | \left(\sum_j Q^{(j)} \rho_1^{(j)} \otimes |j_2\rangle \langle j_2| \right) |j'_1\rangle \\ &= \sum_j Q^{(j)} \left(\sum_{j'} \langle j'_1 | \rho_1^{(j)} |j'_1\rangle \right) |j_2\rangle \langle j_2| \neq \sum_j Q^{(j)} |j_2\rangle \langle j_2|,\end{aligned}\tag{2.30}$$

as $\{\rho_1^{(j)}\}$ is a collection of indistinguishable states in the orthogonal basis $\{|j'_1\rangle\}$.

2.2 Evolution of Physical Systems

2.2.1 Closed system dynamics: Schrödinger, Heisenberg and Dirac representations

In quantum mechanics it is conventional to describe the time dependence of the state vector by the following equation of motion:

$$\frac{d}{dt} |\psi(t)\rangle = \sum_j \frac{d\psi_j}{dt} |j\rangle = -\frac{i}{\hbar} H(t) |\psi(t)\rangle,\tag{2.31}$$

which is known as *Schrödinger's equation* [126–132]. Here, the operator labeled by H is the Hamiltonian of the system and $\hbar$ is the so called Planck's constant. The solution of this equation is completely equivalent to a time-dependent unitary transformation:

$$|\psi(t)\rangle = U(t, t_0) |\psi(t_0)\rangle,\tag{2.32}$$

where the unitarity of time evolution demands the equality below:

$$U^\dagger(t, t_0) U(t, t_0) = U(t, t_0) U^\dagger(t, t_0) = \mathbb{I},\tag{2.33}$$

and ensures that the total probability does not change with time:

$$\begin{aligned}
\sum_j P_j(t) &= \langle \psi(t) | \psi(t) \rangle \\
&= (\langle \psi(t_0) | U^\dagger(t, t_0)) (U(t, t_0) | \psi(t_0) \rangle) \\
&= \langle \psi(t_0) | \left(U^\dagger(t, t_0) U(t, t_0) \right) | \psi(t_0) \rangle \\
&= \langle \psi(t_0) | \psi(t_0) \rangle \\
&= \sum_j P_j(t_0) = 1.
\end{aligned} \tag{2.34}$$

In general, the evolution operator U is defined in the form of a time ordered product

$$U(t, t_0) = T_{\leftarrow} \exp \left[-\frac{i}{\hbar} \int_{t_0}^t H(s) ds \right], \tag{2.35}$$

where $T_{\leftarrow}$ is the time-ordering operator that is required when the generators of the transformation at different times do not commute, i.e., $[H(t_1), H(t_2)] = H(t_1)H(t_2) - H(t_2)H(t_1) \neq 0$. In the case of isolated systems where H is time-independent, $U(t, t_0)$ simply becomes $\exp[-iH(t - t_0)/\hbar]$.

The description of closed system dynamics in terms of a unitary transformation can be easily extended for the mixed initial states as:

$$\begin{aligned}
\rho(t) &= \sum_j Q^{(j)} |\psi^{(j)}(t)\rangle \langle \psi^{(j)}(t)| \\
&= \sum_j Q^{(j)} U(t, t_0) |\psi^{(j)}(t_0)\rangle \langle \psi^{(j)}(t_0)| U^\dagger(t, t_0) \\
&= U(t, t_0) \left(\sum_j Q^{(j)} |\psi^{(j)}(t_0)\rangle \langle \psi^{(j)}(t_0)| \right) U^\dagger(t, t_0) \\
&= U(t, t_0) \rho(t_0) U^\dagger(t, t_0).
\end{aligned} \tag{2.36}$$

if the classical probabilities $Q^{(j)}$ are assumed to be time independent. Differentiation of the both sides of the last line in this equation results in the *quantum Liouville* or *von Neumann equation* that dictates:

$$\frac{d}{dt} \rho(t) = -\frac{i}{\hbar} [H(t), \rho(t)] = \mathcal{L}_t[\rho(t)]. \tag{2.37}$$

Here, $\mathcal{L}_t$ is the Liouville super-operator and the time dependence is carried by the probability amplitudes as is in (2.31). This representation in which only the states incorporate a dependency on time is called the *Schrödinger picture*. Another possible

and equivalent representation of the quantum dynamics is the *Heisenberg picture* where the time dependence is transferred from states to observables:

$$\begin{aligned}
\langle \hat{A} \rangle_t &= \text{tr}[\rho_S(t) \hat{A}_S] \\
&= \text{tr}\left[U(t, t_0) \rho(t_0) U^\dagger(t, t_0) \hat{A}\right] \\
&= \text{tr}\left[U(t, t_0) \rho U^\dagger(t, t_0) \hat{A}(t_0)\right] \\
&= \text{tr}\left[\rho U^\dagger(t, t_0) \hat{A}(t_0) U(t, t_0)\right] \\
&= \text{tr}[\rho_H \hat{A}_H(t)].
\end{aligned} \tag{2.38}$$

Here, the picture in use is denoted by a subscript $X = \{S, H\}$ and the differentiation of $\hat{A}_H(t) = U^\dagger(t, t_0) \hat{A}_H(t_0) U(t, t_0)$ with respect to time gives the *Heisenberg equation*:

$$\frac{d}{dt} \hat{A}_H(t) = +\frac{i}{\hbar} [H_H(t), \hat{A}_H(t)] + \frac{\partial}{\partial t} \hat{A}_H(t). \tag{2.39}$$

Note that in the special case of isolated systems, it is straightforward to show that $[H_H, U] = 0$ and subsequently, $H_H(t) = H_H(t_0) = H_S$.

Both the Schrödinger and Heisenberg pictures can be thought of to be the limiting cases of a more general representation known as the *interaction* or *Dirac picture*. This picture is particularly useful for the investigation of the dynamics of composite systems in which isolated subsystem dynamics is known/trivial and the interaction between subsystems is of importance. In this respect, assume that the Hamiltonian of the composite system can be split into two parts:

$$H_S(t) = H_{0,S} + H_{I,S}(t), \tag{2.40}$$

where the sum of the self-Hamiltonians of the subsystems labeled by $H_{0,S}$ are generally time-independent and exactly solvable, whereas the subsystem interaction is described by an interaction Hamiltonian $H_{I,S}$ that may contain an explicit time-dependence. Then, the time evolution operator $U(t, t_0)$ factorizes into $U_0(t, t_0) U_I(t, t_0)$ in such a way that $U_0(t, t_0) = \exp[-iH_{0,S}(t - t_0)/\hbar]$ and the time-dependence is carried by both of

the states and observables at the same time:

$$\begin{aligned}
\langle \hat{A} \rangle_t &= \text{tr}[\rho_S(t) \hat{A}_S] \\
&= \text{tr}\left[U(t, t_0) \rho(t_0) U^\dagger(t, t_0) \hat{A}\right] \\
&= \text{tr}\left[U(t, t_0) \rho(t_0) U^\dagger(t, t_0) \hat{A}(t_0)\right] \\
&= \text{tr}\left[U_0(t, t_0) U_I(t, t_0) \rho(t_0) U_I^\dagger(t, t_0) U_0^\dagger(t, t_0) \hat{A}(t_0)\right] \\
&= \text{tr}\left[U_I(t, t_0) \rho(t_0) U_I^\dagger(t, t_0) U_0^\dagger(t, t_0) \hat{A}(t_0) U_0(t, t_0)\right] \\
&= \text{tr}[\rho_I(t) \hat{A}_I(t)].
\end{aligned} \tag{2.41}$$

Here, $\rho_I(t) = U_I(t, t_0) \rho(t_0) U_I^\dagger(t, t_0)$ and $\hat{A}_I(t) = U_0^\dagger(t, t_0) \hat{A}(t_0) U_0(t, t_0)$. If $H_{0,S} = 0$, $U_0(t, t_0) = \mathbb{I}$ and $U_I(t, t_0) = U(t, t_0)$. Hence, one recovers the Schrödinger picture from the interaction picture, i.e., $\rho_I(t) = \rho_S(t)$ and $\hat{A}_I(t) = \hat{A}_S$. On the other hand, if $H_{I,S} = 0$, the interaction picture reduces to the Heisenberg picture, i.e., $U_I(t, t_0) = \mathbb{I}$ and $U_0(t, t_0) = U(t, t_0)$ which implies that $\rho_I(t) = \rho_H$ and $\hat{A}_I(t) = \hat{A}_H(t)$.

The von Neumann equation in the interaction picture can be obtained by the differentiation of $\rho_I(t) = U_I(t, t_0) \rho(t_0) U_I^\dagger(t, t_0)$ with respect to time:

$$\frac{d}{dt} \rho_I(t) = -\frac{i}{\hbar} [H_{I,I}(t), \rho_I(t)]. \tag{2.42}$$

It should be emphasized that $H_{I,I}(t)$ appearing in equation (2.42) equals to $U_0^\dagger(t, t_0) H_{I,S}(t) U_0(t, t_0)$ as $H_{I,S/I}(t)$ is an observable whose eigenvalues correspond to possible interaction energies that can be measured in the composite system.

2.2.2 Dynamics of quantum measurement

The unitary dynamics of closed and isolated systems described so far are reversible in the sense that it is always possible to find a unitary transformation $U(t + \Delta t, t)$ that equals to $U^\dagger(t, t_0)$ and brings the system back to its initial state as $U(t + \Delta t, t) U(t, t_0) = \mathbb{I}$. However, the reduced dynamics are not restricted to the unitary evolution $U_j(t, t_0) = (U_j^\dagger(t, t_0))^{-1}$ since

$$\begin{aligned}
\rho_1(t) &= \text{tr}_2\left[U(t, t_0) \rho_{12}(t_0) U^\dagger(t, t_0)\right] \\
&= \sum_j \langle j_2 | \left(U(t, t_0) \rho_{12}(t_0) U^\dagger(t, t_0) \right) | j_2 \rangle \\
&= \sum_{j, j', j''} \langle j_2 | U(t, t_0) | j'_2 \rangle \langle j'_2 | \rho_{12}(t_0) | j''_2 \rangle \langle j''_2 | U^\dagger(t, t_0) | j_2 \rangle \\
&\neq U_1(t, t_0) \rho_1(t_0) U_1^\dagger(t, t_0),
\end{aligned} \tag{2.43}$$

as long as $\rho_{12}(t_0) \neq \rho_1(t_0) \otimes \rho_2(t_0)$ and $U(t, t_0) \neq U_1(t, t_0) \otimes U_2(t, t_0)$. That is to say, the reduced dynamics are reversible only if the subsystems are uncorrelated not only at t_0 , but also at any time t as they never interact. Thus, the irreversibility generally enters into the play when the reduced dynamics become of interest in an interacting composite system.

2.2.2.1 Irreversibility of measurement in the context of decoherence

The prime example of irreversible quantum dynamics is the quantum measurement during which the measurement apparatus $\mathcal{A}$ becomes quantum correlated to the system of interest $\mathcal{S}$ at time $t_1 > t_0$:

$$\begin{aligned} |\psi(t_0)\rangle_{\mathcal{SA}} &= |\psi\rangle_{\mathcal{S}} \otimes |a_0\rangle_{\mathcal{A}} = \left(\sum_j \psi_j(t_0) |j\rangle_{\mathcal{S}} \right) \otimes |a_0\rangle_{\mathcal{A}} \\ &\rightarrow |\psi(t_1)\rangle_{\mathcal{SA}} = \sum_j (\psi_j(t_0) |j\rangle_{\mathcal{S}} \otimes |a_j\rangle_{\mathcal{A}}). \end{aligned} \quad (2.44)$$

The unitary transformations are sufficient to generate such system - apparatus correlations [106]. To show this, let's assume that the system - apparatus interaction is governed by the interaction Hamiltonian:

$$H_{\mathcal{SA}} = g \hat{J}^{(\mathcal{S})} \otimes \hat{B}^{(\mathcal{A})} = g \left(\sum_{j=0}^{N-1} j |j\rangle \langle j|_{\mathcal{S}} \right) \otimes \left(\sum_{j'=0}^{N-1} j' |b_{j'}\rangle \langle b_{j'}|_{\mathcal{A}} \right), \quad (2.45)$$

where g is the coupling strength and $\hat{B}^{(\mathcal{A})}$ is a shift² operator in the basis $\{|a_j\rangle_{\mathcal{A}}\}$ ³.

That is to say,

$$\begin{aligned} |b_j\rangle_{\mathcal{A}} &= \frac{1}{\sqrt{N}} \sum_{j'=0}^{N-1} \exp \left[+ \frac{2\pi i}{N} j j' \right] |a_{j'}\rangle_{\mathcal{A}}, \\ |a_j\rangle_{\mathcal{A}} &= \frac{1}{\sqrt{N}} \sum_{j'=0}^{N-1} \exp \left[- \frac{2\pi i}{N} j j' \right] |b_{j'}\rangle_{\mathcal{A}}. \end{aligned} \quad (2.46)$$

²A shift in a given basis means a conversion from one of the elements of this basis to another one.

³The conditional dynamics given in (2.44) requires a coupling of $\hat{J}^{(\mathcal{S})}$ to a shift operator in the basis $\{|a_j\rangle_{\mathcal{A}}\}$ as in (2.45)

If $g(t_1 - t_0)/\hbar$ is adjusted to be a multiple of $2\pi/N$, the evolution given in (2.44) can be obtained using (2.45) in the interaction picture as below:

$$\begin{aligned}
|\psi(t_1)\rangle_{\mathcal{SA}} &= e^{-\frac{iH_{\mathcal{SA}}(t_1-t_0)}{\hbar}} \left(\sum_j \psi_j(t_0) |j\rangle_S \otimes |a_0\rangle_{\mathcal{A}} \right) \\
&= \sum_j \psi_j(t_0) |j\rangle_S \otimes \left(\frac{1}{\sqrt{N}} \sum_{j'} e^{-\frac{igjj'(t_1-t_0)}{\hbar}} |b_{j'}\rangle_{\mathcal{A}} \right) \\
&= \sum_j \psi_j(t_0) |j\rangle_S \otimes |a_j\rangle_{\mathcal{A}}.
\end{aligned} \tag{2.47}$$

However, what happens then is beyond the limits of unitary transformations: as is already discussed in the beginning of the chapter, the measurement of a single quantum system existing in a coherent superposition of the basis states yields the outcomes j with probabilities $|\psi_j(t_0)|^2$ at time $t_2 > t_1$:

$$|\psi(t_1)\rangle_{\mathcal{SA}} \rightarrow \{P_j(t_2) = |\psi_j(t_0)|^2, |\psi(t_2)\rangle_{\mathcal{SA}} = |j\rangle_S \otimes |a_j\rangle_{\mathcal{A}}\}. \tag{2.48}$$

Actually, this means that the joint system - apparatus state evolves from a coherent superposition to an incoherent superposition at the end of the measurement process:

$$\begin{aligned}
\rho_{\mathcal{SA}}(t_1) &= \sum_{j,j'} \psi_j(t_0) \psi_{j'}^*(t_0) |j\rangle\langle j'|_S \otimes |a_j\rangle\langle a_{j'}|_{\mathcal{A}} \\
&\rightarrow \rho_{\mathcal{SA}}(t_2) = \sum_j |\psi_j(t_0)|^2 |j\rangle\langle j|_S \otimes |a_j\rangle\langle a_j|_{\mathcal{A}}.
\end{aligned} \tag{2.49}$$

The transition from quantum to classical in this way is called the *decoherence* [133–141] as it is the loss of quantum coherence in essence. As will be explained below, the underlying mechanism behind the decoherence requires the inclusion of a very large number of hidden degrees of freedom into the measurement scheme described above. These additional degrees of freedom can belong either to an external environment or to the measurement apparatus itself. Let's collect them into a single partition and denote this new subsystem by $\mathcal{E}$. The actual unitary evolution should be generated by an extension of the interaction Hamiltonian (2.45) to the following form

$$H_{\mathcal{SA}\mathcal{E}} = g \hat{J}^{(S)} \otimes \hat{B}^{(A)} \otimes \hat{C}^{(\mathcal{E})}, \tag{2.50}$$

and the resulting conditional dynamics should spread the quantum correlations given in (2.44) and (2.47) over the whole $\mathcal{SAE}$ composite system:

$$\begin{aligned}
|\psi(t_0)\rangle_{\mathcal{SAE}} &= \left(\sum_j \psi_j(t_0) |j\rangle_{\mathcal{S}} \right) \otimes |a_0\rangle_{\mathcal{A}} \otimes |\epsilon_0\rangle_{\mathcal{E}} \\
\rightarrow |\psi(t_1)\rangle_{\mathcal{SAE}} &= \sum_j (\psi_j(t_0) |j\rangle_{\mathcal{S}} \otimes |a_j\rangle_{\mathcal{A}}) \otimes |\epsilon_0\rangle_{\mathcal{E}} \\
\rightarrow |\psi(t_2)\rangle_{\mathcal{SAE}} &= \sum_j (\psi_j(t_0) |j\rangle_{\mathcal{S}} \otimes |a_j\rangle_{\mathcal{A}} \otimes |\epsilon_j\rangle_{\mathcal{E}}).
\end{aligned} \tag{2.51}$$

As $\mathcal{E}$ has an enormous number of degrees of freedom by the definition, neither the quantum correlations of $\mathcal{SA}$ that rapidly leak into $\mathcal{E}$ are easily retrievable nor the degrees of the freedom of $\mathcal{E}$ are experimentally accessible. In this respect, it is the compulsory ignorance of subsystem $\mathcal{E}$ that makes the measurement process irreversible:

$$\begin{aligned}
\rho_{\mathcal{SA}}(t_2) &= \text{tr}_{\mathcal{E}}[\rho_{\mathcal{SAE}}(t_2)] \\
&= \sum_{j,j',j''} \psi_j(t_0) \psi_{j'}^*(t_0) |j\rangle\langle j'|_{\mathcal{S}} \otimes |a_j\rangle\langle a_{j'}|_{\mathcal{A}} \otimes \langle \epsilon_{j''} | \epsilon_j \rangle \langle \epsilon_{j'} | \epsilon_{j''} \rangle_{\mathcal{E}} \\
&= \sum_j |\psi_j(t_0)|^2 |j\rangle\langle j|_{\mathcal{S}} \otimes |a_j\rangle\langle a_j|_{\mathcal{A}}.
\end{aligned} \tag{2.52}$$

2.2.2.2 Positive operator-valued measure formalism

It is straightforward to generalize the measurement dynamics summarized in the previous subsection as follows. Let's first rewrite (2.51) in a more compact way:

$$\begin{aligned}
|\psi(t_2)\rangle_{\mathcal{SAE}} &= U_{\mathcal{SAE}}(t_2, t_0) (|\psi(t_0)\rangle_{\mathcal{S}} \otimes |a_0\rangle_{\mathcal{A}} \otimes |\epsilon_0\rangle_{\mathcal{E}}) \\
&= \sum_j (M_j |\psi(t_0)\rangle_{\mathcal{S}} \otimes |a_j\rangle_{\mathcal{A}} \otimes |\epsilon_j\rangle_{\mathcal{E}}),
\end{aligned} \tag{2.53}$$

where M_j are called the *measurement operators* that act only on $\mathcal{H}_{\mathcal{S}}$. Conservation of the probability that is satisfied by the unitarity of the global dynamics demands a completeness relation between measurement operators:

$$\begin{aligned}
1 &= \langle \psi(t_2) | \psi(t_2) \rangle_{\mathcal{SAE}} \\
&= \sum_{j,j'} \langle \psi(t_0) | M_{j'}^\dagger M_j | \psi(t_0) \rangle_{\mathcal{S}} \otimes \langle a_{j'} | a_j \rangle_{\mathcal{A}} \otimes \langle \epsilon_{j'} | \epsilon_j \rangle_{\mathcal{E}} \\
&= \langle \psi(t_0) |_{\mathcal{S}} \underbrace{\sum_j M_j^\dagger M_j}_{\mathbb{I}} | \psi(t_0) \rangle_{\mathcal{S}}
\end{aligned} \tag{2.54}$$

since $\{|a_j\rangle_{\mathcal{A}}\}$ and $\{|\epsilon_j\rangle_{\mathcal{E}}\}$ are orthonormal bases and the system of interest is assumed to be initially isolated from the rest, i.e., $\langle \psi(t_0) | \psi(t_0) \rangle_{\mathcal{S}} = 1$. Then, the probability that

result j occurs is given by

$$P_j = \langle \psi(t_0) | M_j^\dagger M_j | \psi(t_0) \rangle_S, \quad (2.55)$$

while the corresponding post-measurement state is

$$|\psi(t_2)\rangle_S = \frac{1}{\sqrt{P_j}} M_j |\psi(t_0)\rangle_S. \quad (2.56)$$

The most common measurement operators are projection operators as given in (2.51), i.e., $M_j = \Pi_j = |j\rangle\langle j|_S$ such that $\Pi_j \Pi_{j'} = \delta_{jj'} \Pi_j$ as $\{|j\rangle_S\}$ is an orthonormal basis. This kind of operators are known as *projective* or *von Neumann measurements*, and constitute just a special class of the general quantum measurements. To move on to the general measurements, let's define a complete set of positive operators on $\mathcal{H}_S$ as follows:

$$E_j = M_j^\dagger M_j. \quad (2.57)$$

It is easy to verify both the positivity of E_j and the completeness of $\{E_j\}$. The former is a consequence of the fact that the probabilities should be greater than or equal to zero, i.e., $\langle \psi(t_0) | E_j | \psi(t_0) \rangle_S = P_j \geq 0$, while the latter is related to the conservation of probability during the course of time evolution: $\langle \psi(t_2) | \psi(t_2) \rangle_{S\mathcal{A}\mathcal{E}}$ equals to 1 if and only if $\sum_j E_j = \sum_j M_j^\dagger M_j = \mathbb{I}$.

In this respect, a complete set of positive operators on $\mathcal{H}_S$ is sufficient to describe the measurement statistics. The general quantum measurements described by such sets are known as the *Positive Operator-Valued Measure (POVM) measurements* and the operators E_j are called the *POVM elements* associated with the measurement.

2.2.3 Open system dynamics

After having briefly sketched the irreversible dynamics of quantum measurement, let us move on to fundamental equations describing the dynamics of a generic open quantum system in terms of super-operators acting on the reduced state ρ_S . These super-operators are defined by $\mathcal{E}_t[\rho_S] = \rho_S \mapsto \rho'_S$ as below:

$$\begin{array}{ccc} \rho_{S\mathcal{E}}(t_0) = \rho_S(t_0) \otimes \rho_{\mathcal{E}}(t_0) & \xrightarrow{\text{unitary evolution}} & \rho_{S\mathcal{E}}(t) = U_{S\mathcal{E}}(t, t_0) \rho_{S\mathcal{E}}(t_0) U_{S\mathcal{E}}^\dagger(t, t_0) \\ \text{tr}_{\mathcal{E}} \downarrow & & \text{tr}_{\mathcal{E}} \downarrow \\ \rho_S(t_0) & \xrightarrow{\text{dynamical map}} & \rho_S(t) = \mathcal{E}_t[\rho_S(t_0)] \end{array} \quad (2.58)$$

and known by different names such as *dynamical maps*, *quantum operations*, *quantum channels* and so on. In this thesis, we will use each of these terms alternatively depending on the context.

Recall that a density matrix should be a Hermitian, positive-semidefinite and normalized operator to represent the state of a physical system. Hence, a dynamical map that transforms a density matrix $\rho_S(t_0)$ at one time to a density matrix $\rho_S(t)$ at another time should preserve the Hermiticity, positivity, and trace to represent a physical process [142–147]. If the initial state is completely uncorrelated, i.e., $\rho_{SE}(t_0) = \rho_S(t_0) \otimes \rho_E(t_0)$, the dynamical map should not only be positive, but also be *linear* and *completely positive* (CP). Linearity requirement arises from the fact that the normalized weights in an incoherent mixture should be time-independent:

$$\begin{aligned} \rho_S(t_0) &= \sum_j Q^{(j)} \rho_S^{(j)} \\ \rightarrow \rho_S(t) &= \mathcal{E}_t \left[\sum_j Q^{(j)} \rho_S^{(j)} \right] = \sum_j Q^{(j)} \mathcal{E}_t \left[\rho_S^{(j)} \right], \end{aligned} \quad (2.59)$$

whereas complete positivity requirement is an extension of the positivity of the map while going to a larger Hilbert space, i.e., if $\mathcal{I}_A$ is the identity map acting on the states of an auxiliary system called *ancilla*, then the map $\mathcal{E}_t \otimes \mathcal{I}_A$ acting on the joint states should be positive. That is to say, the physicality of the SA composite system is preserved during the course of the dynamics. However, the reduced dynamics of the system of interest need not to be linear or CP in general, e.g., for entangled [148–151] or discordant [152–158] initial system - environment states.

2.2.3.1 Operator-sum representation by Kraus operators

If a dynamical map is CP, then it can be written in the following form known as the *operator-sum representation* (OSR)

$$\rho_S(t) = \mathcal{E}_t[\rho_S(t_0)] = \sum_j K_j(t, t_0) \rho_S(t_0) K_j^\dagger(t, t_0), \quad (2.60)$$

where the operators $K_j(t, t_0)$ are called the *Kraus operators* [145]. To derive this form, let's revisit Eq. (2.43) with the assumptions that (i) the initial state $\rho_{SE}(t_0)$ of the composite system factorizes into $\rho_S(t_0) \otimes \rho_E(t_0)$ and (ii) the initial state $\rho_E(t_0)$ of the

environment is diagonal in the basis $\{|j'_\mathcal{E}\rangle\}$:

$$\begin{aligned}
\rho_S(t) &= \text{tr}_\mathcal{E} \left[U_{S\mathcal{E}}(t, t_0) \rho_{S\mathcal{E}}(t_0) U_{S\mathcal{E}}^\dagger(t, t_0) \right] \\
&= \sum_j \langle j_\mathcal{E} | \left(U_{S\mathcal{E}}(t, t_0) \rho_S(t_0) \otimes \rho_\mathcal{E}(t_0) U_{S\mathcal{E}}^\dagger(t, t_0) \right) | j_\mathcal{E} \rangle \\
&= \sum_{j, j', j''} \langle j_\mathcal{E} | U_{S\mathcal{E}}(t, t_0) | j'_\mathcal{E} \rangle \langle j'_\mathcal{E} | \rho_S(t_0) \otimes \rho_\mathcal{E}(t_0) | j''_\mathcal{E} \rangle \langle j''_\mathcal{E} | U_{S\mathcal{E}}^\dagger(t, t_0) | j_\mathcal{E} \rangle \quad (2.61) \\
&= \sum_{j, j'} \langle j'_\mathcal{E} | \rho_\mathcal{E}(t_0) | j'_\mathcal{E} \rangle \left(\langle j_\mathcal{E} | U_{S\mathcal{E}}(t, t_0) | j'_\mathcal{E} \rangle \rho_S(t_0) \langle j'_\mathcal{E} | U_{S\mathcal{E}}(t, t_0) | j_\mathcal{E} \rangle \right) \\
&= \sum_j K_j(t, t_0) \rho_S(t_0) K_j^\dagger(t, t_0).
\end{aligned}$$

Above, the Kraus operators are defined as

$$K_j(t, t_0) = \sum_{j'} \sqrt{\langle j'_\mathcal{E} | \rho_\mathcal{E}(t_0) | j'_\mathcal{E} \rangle} \langle j_\mathcal{E} | U_{S\mathcal{E}}(t, t_0) | j'_\mathcal{E} \rangle. \quad (2.62)$$

Since the trace should be preserved by the quantum operation, the Kraus operators should satisfy the completeness relation:

$$1 = \text{tr}[\rho_S(t)] = \text{tr} \left[\underbrace{\sum_j K_j^\dagger(t, t_0) K_j(t, t_0)}_{\mathbb{I}} \rho_S(t_0) \right]. \quad (2.63)$$

In this respect, the partial trace and measurements are two special examples of quantum operations. Kraus operators are the multiples of identity operator in the former, i.e., $K_j(t, t_0) = \lambda_j \mathbb{I}_S$ such that $0 \leq \lambda_j \leq 1$ and $\sum_j \lambda_j = 1$, whereas they equal to the measurement operators M_j in the latter. Also, OSR can cover the unitary evolution as a special case in which only one operator is present in the sum, i.e., $K_{j=1} = U_S$ and $K_{j \neq 1} = 0$. More concrete examples can be found in quantum information and quantum computation textbooks such as [159–162].

Unitary transformations form a group, a set that is closed under multiplication with an identity and inverse for each element such that (i) $U(t_2, t_0) = U(t_2, t_1)U(t_1, t_0)$, (ii) $\lim_{\Delta t \rightarrow \infty} [U(t_0 + \Delta t, t_0)] = \mathbb{I}$, and (iii) $U^{-1}(t, t_0) = U^\dagger(t, t_0)$. If its elements satisfy (i) and (ii) without fulfilling (iii), the set is referred to be a semigroup instead of a group. As (iii) is expected not to be hold in an irreversible process, the whole set of transformations corresponding to quantum operations should form a semigroup at the most. However, to form a semigroup by the Kraus operators that ensure the property (i), the state $\rho_{S\mathcal{E}}$ of the composite system should be assumed to factorize

into $\rho_S \otimes \rho_E$ not only at time t_0 as we said before, but also at any time during the course of the dynamics. Otherwise, the dynamical map cannot be divisible, i.e., $K_j(t_2, t_0) \neq K_j(t_2, t_1) K_j(t_1, t_0)$.

2.2.3.2 Representation by Markovian master equation in Lindblad form

Quantum dynamical semigroups can be represented in an exponential form,

$$\mathcal{E}_t[\rho_S] = \exp[\mathcal{L}_t[\rho_S] t], \quad (2.64)$$

where $\mathcal{L}_t$, the generator of the semigroup, is the generalization of Liouville super-operator introduced in equation (2.37). Hence, by differentiating the equation (2.64) with respect to time, the dynamical map between the reduced states of the system of interest can be described by an equation of motion in terms of the *quantum master equation in the Lindblad-Gorini-Kossakowski-Sudarshan (LGKS) form* [163, 164]:

$$\frac{d}{dt}\rho_S(t) = \mathcal{L}_t[\rho_S(t)] \quad (2.65)$$

$$= -\frac{i}{\hbar}[H_S + H'_S, \rho_S(t)] + \sum_{j=0}^{N \times N} \gamma_j (A_j \rho_S(t) A_j^\dagger - \frac{1}{2}\{A_j^\dagger A_j, \rho_S(t)\}), \quad (2.66)$$

where H_S is the self-Hamiltonian of S , H'_S is an effective Hamiltonian describing the unitary dynamics induced by $\mathcal{E}$ on S , A_j are the *Lindblad operators* representing the coupling between S and $\mathcal{E}$, γ_j are nonnegative numbers corresponding to dissipation rates due to the influence of $\mathcal{E}$, and $\{X, Y\}$ denotes an anticommutator that equals to $XY + YX$. N is the dimension of $\mathcal{H}_S$ and there are at most $N \times N$ independent Lindblad operators acting on S . In general, the partition between the unitary and dissipative dynamics induced by $\mathcal{E}$ is not unique, i.e., the dynamics generated by (2.65) are invariant to the inhomogeneous transformations $A_j \rightarrow A_j + a_j \mathbb{I}$ and $H'_S \rightarrow H'_S + \frac{\hbar}{2i} \sum_j \gamma_j (a_j^* A_j - a_j A_j^\dagger) + b \mathbb{I}$ where a_j are complex numbers while b is real.

Microscopic derivation

A unique and consistent decomposition of environment-induced unitary and dissipative dynamics is obtained by a microscopic derivation of the semigroup generator in the weak system - environment coupling limit [165]. Many textbooks about the open quantum systems, like [166–168], cover this derivation. In what follows, we will particularly follow chapter 3 of [166].

Our starting point is the assumption that the environment is a *reservoir*, that is to say, it has an infinite number of degrees of freedom to lead to an irreversible dynamics on S . We further assume that this reservoir exists in a stationary state such as the thermal equilibrium state, i.e., $\rho = \rho^\infty = \rho^{\text{th}}$. This property is referred by the term *heat bath*. Hence, we will prefer $\mathcal{B}$ rather than $\mathcal{E}$ to denote the environment after this point.

We focus on the total Hamiltonian that is of the following form in the Schrödinger picture:

$$H_{SB} = H_S \otimes \mathbb{I}_B + \mathbb{I}_S \otimes H_B + H_I, \quad (2.67)$$

where the self-Hamiltonian of the system of interest has the eigendecomposition:

$$H_S = \sum_j e_j |e_j\rangle \langle e_j|_S, \quad (2.68)$$

and the interaction Hamiltonian can be decomposed into the general form:

$$H_I = \sum_j A_j^{(S)} \otimes B_j^{(B)}, \quad (2.69)$$

with $A_j^{(S)}$ and $B_j^{(B)}$ are Hermitian operators. To easily switch between the Schrödinger and interaction pictures when it is necessary, let's rewrite the interaction Hamiltonian in terms of the eigenoperators of H_S :

$$H_I = \sum_j \tilde{A}_j \otimes B_j, \quad (2.70)$$

where the superscripts on the local operators are omitted and the eigenoperator relation is defined as:

$$[H_S, \tilde{A}_j] = \lambda_j \tilde{A}_j. \quad (2.71)$$

Using the Baker-Campbell-Hausdorff (BCH) formula given by

$$e^X A e^{-X} = A + [X, A] + \frac{1}{2!} [X, [X, A]] + \frac{1}{3!} [X, [X, [X, A]]] + \dots, \quad (2.72)$$

it is straightforward to move on to the interaction picture representation of the eigenoperators as below:

$$\begin{aligned} \tilde{A}_{j,I} &= e^{\frac{it}{\hbar} H_S} \tilde{A}_{j,S} e^{-\frac{it}{\hbar} H_S} \\ &= \tilde{A}_{j,S} + \left(\frac{it}{\hbar} \right) [H_S, \tilde{A}_{j,S}] + \frac{1}{2!} \left(\frac{it}{\hbar} \right)^2 [H_S, [H_S, \tilde{A}_{j,S}]] + \dots \\ &= \tilde{A}_{j,S} + \left(\frac{i\lambda_j t}{\hbar} \right) \tilde{A}_{j,S} + \frac{1}{2!} \left(\frac{i\lambda_j t}{\hbar} \right)^2 \tilde{A}_{j,S} + \dots \\ &= e^{\frac{i\lambda_j t}{\hbar}} \tilde{A}_{j,S}. \end{aligned} \quad (2.73)$$

In the meantime, the Schrödinger picture eigenoperators of H_S are constructed as follows:

$$\tilde{A}_j = A_j(\omega) = \sum_{\omega_{j'j''}=\omega} |e_{j''}\rangle\langle e_{j''}|A_j|e_{j'}\rangle\langle e_{j'}|, \quad (2.74)$$

where the sum is taken over all $\{j', j''\}$ pairs such that the corresponding Bohr frequency of H_S , i.e., $\omega_{j'j''} \equiv (e_{j'} - e_{j''})/\hbar$, equals to ω . Note that a second consecutive sum over ω returns A_j back to itself:

$$\sum_{\omega} A_j(\omega) = A_j. \quad (2.75)$$

Since A_j are Hermitian, $A_j^\dagger(\omega) = A_j(-\omega)$ by the definition:

$$\begin{aligned} A_j^\dagger(\omega) &= \left(\sum_{\omega} |e_{j''}\rangle\langle e_{j''}|A_j|e_{j'}\rangle\langle e_{j'}| \right)^\dagger = \sum_{\omega} |e_{j'}\rangle\langle e_{j'}|A_j|e_{j''}\rangle\langle e_{j''}| \\ &= A_j(-\omega). \end{aligned} \quad (2.76)$$

Then, it is easy to check the eigenoperator relations:

$$[H_S, A_j(\omega)] = -\hbar\omega A_j(\omega), \quad (2.77)$$

$$[H_S, A_j^\dagger(\omega)] = +\hbar\omega A_j^\dagger(\omega), \quad (2.78)$$

and consequently the interaction picture representations:

$$A_{j,I}(\omega) = e^{-i\omega t} A_j(\omega), \quad (2.79)$$

$$A_{j,I}^\dagger(\omega) = e^{+i\omega t} A_j^\dagger(\omega). \quad (2.80)$$

As B_j are Hermitian, the substitution of (2.75) into (2.69) yields:

$$H_{I,S} = \sum_{\omega,j} A_j(\omega) \otimes B_j, \quad (2.81)$$

$$= \sum_{\omega,j} A_j^\dagger(\omega) \otimes B_j^\dagger, \quad (2.82)$$

and its interaction picture counterpart turns out to be in the particularly simple form:

$$H_{I,I} = \sum_{\omega,j} e^{-i\omega t} A_j(\omega) \otimes B_j(t), \quad (2.83)$$

$$= \sum_{\omega,j} e^{+i\omega t} A_j^\dagger(\omega) \otimes B_j^\dagger(t), \quad (2.84)$$

where $B_j(t) = B_{j,I} = e^{\frac{it}{\hbar}H_B} B_j e^{-\frac{it}{\hbar}H_B} = B_j^\dagger(t)$.

Now, we can elaborate on derivation of the quantum master equation by integrating both sides of the interaction picture von Neumann equation (2.42) from $t_0 = 0$ to t and leaving the time-dependent total density matrix alone:

$$\rho_{SB,I}(t) = \rho_{SB,I}(0) - \frac{i}{\hbar} \int_0^t dt' [H_{I,I}(t'), \rho_{SB,I}(t')]. \quad (2.85)$$

This formal solution can be substituted back into (2.42) to obtain the following integro-differential equation up to second order in $H_{I,I}$:

$$\frac{d}{dt} \rho_{SB,I}(t) = -\frac{i}{\hbar} [H_{I,I}(t), \rho_{SB,I}(0)] - \frac{1}{\hbar^2} \int_0^t dt' [H_{I,I}(t), [H_{I,I}(t'), \rho_{SB,I}(t')]], \quad (2.86)$$

whose partial trace over bath degrees of freedom gives a master equation for the reduced density matrix $\rho_{S,I}$:

$$\begin{aligned} \frac{d}{dt} \rho_{S,I}(t) = & -\frac{i}{\hbar} \text{tr}_{\mathcal{B}} [H_{I,I}(t), \rho_{SB,I}(0)] \\ & - \frac{1}{\hbar^2} \int_0^t dt' \text{tr}_{\mathcal{B}} [H_{I,I}(t), [H_{I,I}(t'), \rho_{SB,I}(t')]]]. \end{aligned} \quad (2.87)$$

To get rid of the dependence of the right hand side of (2.87) on $\rho_{SB,I}$, we will make the *Born approximation* which assumes a weak coupling between $\mathcal{S}$ and $\mathcal{B}$ in such a way that the interaction has a negligible effect on the states of $\mathcal{B}$, i.e., $\rho_{SE,I}(t) \simeq \rho_{S,I}(t) \otimes \rho_{B,I}(0)$. This will enable us to rewrite the first term in the right hand side of (2.87) using (2.83) as:

$$\begin{aligned} \text{tr}_{\mathcal{B}} [H_{I,I}(t), \rho_{SB,I}(0)] &= \sum_{\omega,j} e^{-i\omega t} \text{tr}_{\mathcal{B}} [A_j(\omega) \otimes B_j(t), \rho_{S,I}(0) \otimes \rho_{B,I}(0)] \\ &= \sum_{\omega,j} e^{-i\omega t} [A_j(\omega), \rho_{S,I}(0)] \otimes \text{tr}_{\mathcal{B}} [B_j(t), \rho_{B,I}(0)], \end{aligned} \quad (2.88)$$

which vanishes because of the assumption on the initial state of $\mathcal{B}$ that is previously made, i.e., $\rho_{\mathcal{B}}(0) = \rho_{\mathcal{B}}^{\text{th}} = e^{-\beta H_{\mathcal{B}}} / \text{tr}_{\mathcal{B}} [e^{-\beta H_{\mathcal{B}}}]$ where β is the inverse temperature. This can be verified using the cyclic property of the trace and the commutativity of $H_{\mathcal{B}}$ and $\rho_{\mathcal{B}}^{\text{th}}$:

$$\begin{aligned} \text{tr}_{\mathcal{B}} [B_j(t), \rho_{B,I}(0)] &= \frac{\text{tr}_{\mathcal{B}} [e^{\frac{it}{\hbar} H_{\mathcal{B}}} B_j e^{-\frac{it}{\hbar} H_{\mathcal{B}}}, e^{-\beta H_{\mathcal{B}}}] }{\text{tr}_{\mathcal{B}} [e^{-\beta H_{\mathcal{B}}}] } \\ &= \frac{\text{tr}_{\mathcal{B}} [B_j, e^{-\beta H_{\mathcal{B}}}] }{\text{tr}_{\mathcal{B}} [e^{-\beta H_{\mathcal{B}}}] } = 0. \end{aligned} \quad (2.89)$$

The master equation introduced in (2.87) consequently becomes

$$\frac{d}{dt}\rho_{S,I}(t) = -\frac{1}{\hbar^2} \int_0^t dt' \text{tr}_B \left[[H_{I,I}(t), [H_{I,I}(t'), \rho_{S,I}(t') \otimes \rho_B^{\text{th}}]] \right]. \quad (2.90)$$

As this equation is not local in time, the dynamical map generated by it is not divisible. In this respect, in order for the dynamical map to form a semigroup, we will make a further simplification on the master equation by performing the *Markov approximation*. The first step of this approximation is the replacement of $\rho_{S,I}(t')$ by $\rho_{S,I}(t)$ in the right hand side of (2.90). That is to say, the change in the reduced state at one time does not depend on the reduced states at other times. The corresponding master equation is known as the *Redfield equation* [166, 169] that can be written as below:

$$\begin{aligned} \frac{d}{dt}\rho_{S,I}(t) &= -\frac{1}{\hbar^2} \sum_{j,j'} \int_0^t dt' \text{tr}_B \left[[A_j(t) \otimes B_j(t), [A_{j'}(t') \otimes B_{j'}(t'), \rho_{S,I}(t) \otimes \rho_B^{\text{th}}]] \right] \\ &= +\frac{1}{\hbar^2} \sum_{j,j'} \int_0^t dt' \left([A_{j'}(t') \rho_{S,I}(t), A_j(t)] \otimes \text{tr}_B [B_j(t) B_{j'}(t') \rho_B^{\text{th}}] \right. \\ &\quad \left. - [\rho_{S,I}(t) A_{j'}(t'), A_j(t)] \otimes \text{tr}_B [B_{j'}(t') B_j(t) \rho_B^{\text{th}}] \right), \end{aligned} \quad (2.91)$$

where $\text{tr}_B [B_j(t) B_{j'}(t') \rho_B^{\text{th}}] = \langle B_j(t) B_{j'}(t') \rangle_{\text{th}}$ are the bath correlation functions⁴ encoding the memory of the bath in the form of environmental excitations. Let's define $A_j(t)$ and $A_{j'}(t')$ respectively through (2.84) and (2.83) inside the first commutator of the last line above, whereas define the same operators respectively through (2.83) and (2.84) inside the subsequent commutator. Then the Redfield equation becomes:

$$\begin{aligned} \frac{d}{dt}\rho_{S,I}(t) &= \sum_{j,j'} \sum_{\omega, \omega'} \left(e^{i(\omega - \omega')t} \Gamma_{jj'}(\omega') [A_{j'}(\omega') \rho_{S,I}(t), A_j^\dagger(\omega)] \right. \\ &\quad \left. - e^{i(\omega' - \omega)t} \Gamma_{j'j}^*(\omega') [\rho_{S,I}(t) A_j^\dagger(\omega'), A_j(\omega)] \right), \end{aligned} \quad (2.92)$$

where $\Gamma_{jj'}(\omega')$ are defined as

$$\begin{aligned} \Gamma_{jj'}(\omega') &= \frac{1}{\hbar^2} \int_0^t d\tau e^{i\omega'\tau} \langle B_j(t) B_{j'}(t - \tau) \rangle_{\text{th}} \\ &= \frac{1}{\hbar^2} \int_0^t d\tau e^{i\omega'\tau} \langle B_j(\tau) B_{j'}(0) \rangle_{\text{th}}, \end{aligned} \quad (2.93)$$

with τ is the retarded time that equals to $t - t'$. The second line in this definition arises from the fact that bath correlation functions are expected to be homogenous in time since the bath is in a stationary state.

⁴Note that $\text{tr}_B [B_{j'}(t') B_j(t) \rho_B^{\text{th}}] = \text{tr}_B [(B_j(t) B_{j'}(t') \rho_B^{\text{th}})^\dagger] = (\text{tr}_B [B_j(t) B_{j'}(t') \rho_B^{\text{th}}])^\dagger$

If the bath correlation functions decay over a time scale which is resolved in the reduced dynamics, we cannot neglect the influence of the past states. Hence, to carry out the Markov approximation, we must ensure that lifetime of the environmental excitations labeled by τ_B is sufficiently small when compared to the relaxation time τ_R over which the system's reduced state appreciably changes due to the environment. This can be exactly achievable only for a bath involving a continuum of frequencies as well as infinitely large degrees of freedom. Mathematically, it is effectively equivalent to let the upper limit of the integral in (2.93) go to infinity:

$$\Gamma_{jj'}(\omega') = \frac{1}{\hbar^2} \int_0^\infty d\tau e^{i\omega'\tau} \langle B_j(\tau) B_{j'}(0) \rangle_{\text{th}}, \quad (2.94)$$

by which $\Gamma_{jj'}(\omega')$ turn out to be the one-sided Fourier transforms of the bath correlation functions. Substituting them back into (2.92), we arrive at the *Born-Markov master equation*:

$$\frac{d}{dt} \rho_{S,I}(t) = \sum_{j,j'} \sum_{\omega, \omega'} e^{i(\omega - \omega')t} \Gamma_{jj'}(\omega') [A_{j'}(\omega') \rho_{S,I}(t), A_j^\dagger(\omega)] + \text{h.c.}, \quad (2.95)$$

where $|\omega - \omega'|^{-1}$ for $\omega \neq \omega'$ defines an intrinsic time scale τ_S for the coherent evolution of $\mathcal{S}$. In the case of $\tau_S \ll \tau_R$, the secular terms corresponding to $\omega \neq \omega'$ rapidly oscillate during the time scale over which a noticeable change is observed in the system's reduced state. If we neglect these terms in (2.95) using the *secular* or *rotating wave approximation*, we obtain:

$$\frac{d}{dt} \rho_{S,I}(t) = \sum_{\omega} \sum_{j,j'} \Gamma_{jj'}(\omega) [A_{j'}(\omega) \rho_{S,I}(t), A_j^\dagger(\omega)] + \text{h.c.} \quad (2.96)$$

To end up with a master equation in the LGKS form (2.65) with a unique partition, we decompose the Fourier transforms as follows:

$$\Gamma_{jj'}(\omega) = \frac{1}{2} \gamma_{jj'}(\omega) + i S_{jj'}(\omega), \quad (2.97)$$

where the coefficients in the real part form a positive matrix for a fixed ω :

$$\gamma_{jj'}(\omega) = \Gamma_{jj'}(\omega) + \Gamma_{j'j}^*(\omega) = \frac{1}{\hbar^2} \int_{-\infty}^\infty d\tau e^{i\omega\tau} \langle B_j(\tau) B_{j'}(0) \rangle_{\text{th}}, \quad (2.98)$$

while the coefficients in the imaginary part form a Hermitian matrix:

$$S_{jj'}(\omega) = \frac{1}{2i} \left(\Gamma_{jj'}(\omega) - \Gamma_{j'j}^*(\omega) \right) = -i \left(\Gamma_{jj'}(\omega) - \frac{\gamma_{jj'}(\omega)}{2} \right). \quad (2.99)$$

Introducing this partition in (2.96), we arrive at the master equation of the form:

$$\frac{d}{dt}\rho_{S,I}(t) = -\frac{i}{\hbar}[\hbar H_{LS}, \rho_{S,I}(t)] + \mathcal{D}_t[\rho_{S,I}(t)]. \quad (2.100)$$

where H_{LS} is known as the *Lamb shift Hamiltonian* that equals to:

$$H_{LS} = \sum_{\omega} \sum_{j,j'} S_{jj'}(\omega) A_j^{\dagger}(\omega) A_{j'}(\omega), \quad (2.101)$$

and $\mathcal{D}_t$ is called the *dissipator* defined as:

$$\begin{aligned} \mathcal{D}_t[\rho_{S,I}(t)] = & \sum_{\omega} \sum_{j,j'} \gamma_{jj'}(\omega) (A_{j'}(\omega) \rho_{S,I}(t) A_j^{\dagger}(\omega) \\ & - \frac{1}{2} \{A_j^{\dagger}(\omega) A_{j'}(\omega), \rho_{S,I}(t)\}). \end{aligned} \quad (2.102)$$

The Markovian master equation can be cast into the following simple form transforming (2.100) back into the Schrödinger picture:

$$\frac{d}{dt}\rho_{S,S}(t) = -\frac{i}{\hbar}[H_S + \hbar H_{LS}, \rho_{S,S}(t)] + \mathcal{D}_t[\rho_{S,S}(t)], \quad (2.103)$$

where the Lamb shift Hamiltonian provides the bath-induced unitary dynamics and the dissipator super-operator is responsible for the bath-induced dissipative dynamics. If we diagonalize the matrices introduced in (2.98), we obtain the LGKS form given in (2.65) with the unique partition of unitary and dissipative dynamics.

2.3 Quantification of Correlations in Physical Systems

In this section, we will briefly summarize some fundamental notions used in the quantum information theory to quantify correlations. A comprehensive survey of the most widely adopted measures of correlations can be found in Refs. [170–175].

The realization that the quantumness of a physical system is a measurable quantity was first started by the *Bell's inequalities* formulated in 1964 [176]. A simple and experimentally tested subclass of these equations known as *Clauser-Horne-Shimony-Holt (CHSH) inequality* [177–181] considers four different local observables in a bipartite composite system in such a way that (i) the subsystems share the observables equally and (ii) each observable has only two possible measurement outcomes, $+1$ or -1 . Let's denote these observables respectively by $\hat{Q}^{(1)}$, $\hat{R}^{(1)}$, $\hat{S}^{(2)}$, and $\hat{T}^{(2)}$ and focus on the quantity defined as:

$$\hat{E}^{(12)} = (\hat{Q}^{(1)} + \hat{R}^{(1)}) \otimes \hat{S}^{(2)} + (\hat{Q}^{(1)} - \hat{R}^{(1)}) \otimes \hat{T}^{(2)}. \quad (2.104)$$

If the measurement outcomes have definite values existing independently of observation, as is assumed in classical physics, the expectation value of $\hat{E}^{(12)}$ is found to be bounded from above by 2:

$$\begin{aligned}\langle \hat{E}^{(12)} \rangle &= \sum_{q,r,s,t} P(Q=q, R=r, S=s, T=t) \underbrace{((q+r)s + (q-r)t)}_{\leq 2} \\ &\leq \sum_{q,r,s,t} P(Q=q, R=r, S=s, T=t) \times 2 \\ &= 2,\end{aligned}\tag{2.105}$$

since $(q+r)$ equals to ± 2 while $(q-r)$ is zero and vice versa. Furthermore, if local measurements on one subsystem do not effect the other subsystem, as is also expected in classical systems, the equation (2.105) becomes:

$$\begin{aligned}\langle \hat{E}^{(12)} \rangle &= \sum_{q,r,s,t} P(q,r,s,t) qs + \sum_{q,r,s,t} P(q,r,s,t) rs \\ &\quad + \sum_{q,r,s,t} P(q,r,s,t) qt - \sum_{q,r,s,t} P(q,r,s,t) rt \\ &= \langle \hat{Q}^{(1)} \otimes \hat{S}^{(2)} \rangle + \langle \hat{R}^{(1)} \otimes \hat{S}^{(2)} \rangle + \langle \hat{Q}^{(1)} \otimes \hat{T}^{(2)} \rangle - \langle \hat{R}^{(1)} \otimes \hat{T}^{(2)} \rangle \leq 2.\end{aligned}\tag{2.106}$$

The last line above is what is called the CHSH inequality. We can check the validity of this inequality for quantum systems. To start with, we need to introduce the fundamental unit of quantum information which is the counterpart of the binary digit or *bit* of classical information [182, 183], namely, the *quantum bit* or *qubit* [184] with the conventional definition below:

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.\tag{2.107}$$

Then, we need to introduce some single-qubit quantum operations known as *Pauli sigma matrices*:

$$\sigma_x = \begin{pmatrix} 0 & +1 \\ +1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ +i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix}.\tag{2.108}$$

Now, we can go on checking the validity of CHSH inequality in quantum regime as follows. Suppose that we have a two-qubit quantum system that exists in an entangled state of the form:

$$|\psi_{12}\rangle = \alpha|01\rangle_{12} - \sqrt{1-\alpha^2}|10\rangle_{12},\tag{2.109}$$

where $0 < \alpha < 1$.⁵ Assuming that the local observables under consideration are given as below:

$$\hat{Q}^{(1)} = \sigma_x^{(1)}, \quad \hat{R}^{(1)} = \sigma_z^{(1)}, \quad \hat{S}^{(2)} = \frac{-\sigma_z^{(2)} - \sigma_x^{(2)}}{\sqrt{2}}, \quad \hat{T}^{(2)} = \frac{\sigma_z^{(2)} - \sigma_x^{(2)}}{\sqrt{2}}, \quad (2.110)$$

it is straightforward to show that

$$\begin{aligned} \langle \hat{Q}^{(1)} \otimes \hat{S}^{(2)} \rangle &= \alpha \sqrt{2(1 - \alpha^2)}, & \langle \hat{R}^{(1)} \otimes \hat{S}^{(2)} \rangle &= +1/\sqrt{2}, \\ \langle \hat{Q}^{(1)} \otimes \hat{T}^{(2)} \rangle &= \alpha \sqrt{2(1 - \alpha^2)}, & \langle \hat{R}^{(1)} \otimes \hat{T}^{(2)} \rangle &= -1/\sqrt{2}. \end{aligned} \quad (2.111)$$

By gathering these four equations together, we obtain:

$$\langle \hat{Q}^{(1)} \hat{S}^{(2)} \rangle + \langle \hat{R}^{(1)} \hat{S}^{(2)} \rangle + \langle \hat{Q}^{(1)} \hat{T}^{(2)} \rangle - \langle \hat{R}^{(1)} \hat{T}^{(2)} \rangle = \sqrt{2} (1 + 2\alpha \sqrt{1 - \alpha^2}) \quad (2.112)$$

where the right hand side reaches a maximum value of $2\sqrt{2}$ when the average is taken over the state⁶:

$$|\psi_{12}\rangle = \frac{1}{\sqrt{2}} (|01\rangle_{12} \pm |10\rangle_{12}). \quad (2.113)$$

Since this upper bound is larger than the one given in (2.106), the entangled states of the form (2.109) are found to be likely to violate the CHSH inequality, and so the locality assumption used in the derivation of the inequality. In this respect, entangled states violating the CHSH inequality are called to be *nonlocal* and the maximally nonlocal states given in (2.113) are particularly known as the *Bell states*.

Some entangled states of the form (2.109) do not violate the CHSH inequality as shown in Figure 2.1. However, an entanglement measure (see below) is expected to be positive for all entangled states, while being zero only on the separable states⁷. Hence, the extent to which the quantum state violates this inequality cannot be directly used as a measure of entanglement.

⁵Note that when α approaches to 1 or 0, the qubits become disentangled.

⁶The value of α that maximizes (2.112) is found to be $\pm 1/\sqrt{2}$ by differentiating the right hand side with respect to α and equating it to zero.

⁷In fact, an entanglement measure should be constant and minimal on separable states, but it is convenient to set this constant to zero.

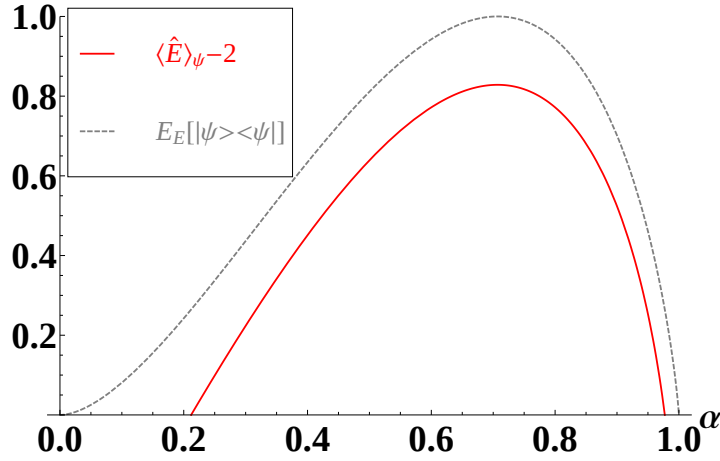


Figure 2.1 : Red curve is the extent to which the quantum state $|\psi_{12}\rangle$ given in (2.109) violates the CHSH inequality, while the gray curve is the entropy of entanglement that equals to the von Neumann entropy of the reduced state as described in (2.116).

2.3.1 Elementary measures of quantum entanglement

The most important property of an entanglement measure E , which was first proposed in Ref. [185], is that it can not increase under LOCC, i.e.,

$$\begin{aligned} E[\rho] &\geq (E \circ \mathcal{E}_l)[\rho] = E\left[\sum_j (K_j \rho K_j^\dagger)\right], \\ &= E\left[\left(\sum_j \bigotimes_k \hat{A}_j^{(k)} \rho \bigotimes_k \hat{A}_j^{\dagger(k)}\right)\right]. \end{aligned} \quad (2.114)$$

Above, k denotes the subsystems and $\hat{A}_j^{(k)}$ are called the *separable operations*, that is to say, local operations which are correlated in a classical way.⁸

Most of the entanglement measures satisfy a stronger version of the monotonicity condition (2.114) which requires that the measure should not to increase on average under LOCC, i.e.,

$$E[\rho] \geq \sum_j P_j E\left[\left(K_j \rho K_j^\dagger\right) / P_j\right], \quad (2.115)$$

where the Kraus operators K_j represent the LOCC protocol and the corresponding probabilities P_j equal to $\text{tr}[K_j \rho K_j^\dagger]$. In this respect, the unique entanglement measure for the pure bipartite states, e.g., for the states given in (2.109), is the *entropy of*

⁸Note that the Kraus operators K_j describing a LOCC protocol can be always decomposed into the form $\hat{A}_j^{(1)} \otimes \hat{A}_j^{(2)} \otimes \cdots \otimes \hat{A}_j^{(N)}$ but not vice versa. Hence, the LOCC operations constitute a subclass of separable operations. However, the separable operations possess a significant amount of LOCC-ality.

entanglement [186, 187] which is defined as

$$E_E[|\psi\rangle\langle\psi|] = (S \circ \text{tr}_{1(2)})[|\psi\rangle\langle\psi|], \quad (2.116)$$

where S is the *von Neumann entropy* [110] :

$$S[\rho] = -\text{tr}[\rho \log_2 \rho]. \quad (2.117)$$

Substituting (2.13) into (2.117) as first done in Ref. [184], the von Neumann entropy can be regarded as the quantum analogous of the *information entropy*⁹:

$$S[\rho] = -\sum_j \lambda_j \log_2 \lambda_j, \quad (2.118)$$

which means that it gives not only the amount of disorder, uncertainty, or unpredictability of the statistical ensemble represented by ρ , but also the average amount of quantum information contained in the (sub)system existing in the state ρ .

The entropy of entanglement E_E is a good measure of the pure bipartite entanglement. It vanishes for separable states¹⁰, attains a maximum value for the Bell states and imposes a unique order on pure bipartite states according to their entanglement content (see Figure 2.1). This is consistent with the partial ordering on the entangled states dictated by *majorization* [188]. Algebraically, if ρ_{12} is majorized by ρ'_{12} ($\rho_{12} \prec \rho'_{12}$), i.e., if $\sum \lambda_j^\downarrow(\rho_1) \leq \sum \lambda_j^\downarrow(\rho'_1)$,¹¹ then $E_E[\rho_{12}] = S[\rho_1] \geq S[\rho'_1] = E_E[\rho'_{12}]$.

Furthermore, the entropy of entanglement can be extended to the mixed states by means of a convex roof. This extension is known as the *entanglement of formation* [189] that is defined as

$$E_F[\rho] = \min\left\{\sum_j Q^{(j)} E_E[|\psi_j\rangle\langle\psi_j|]\right\}, \quad (2.119)$$

where the minimum is taken over all the possible pure state decompositions of ρ that realize $\rho = \sum_j Q^{(j)} |\psi_j\rangle\langle\psi_j|$. Operationally, E_F quantifies how many Bell states are required per a copy of ρ while preparing many copies of ρ using a specific LOCC

⁹The Shannon information entropy of a discrete random variable J , that takes the value of j with a probability $Q^{(j)}$, is defined as $H(J) = -\sum Q^{(j)} \log_2 Q^{(j)}$ and measures the information content of a message produced by J in bits. See Refs. [182, 183] for the details.

¹⁰ $E_E = -\lim_{\lambda_j \rightarrow 1} \lambda_j \log_2 \lambda_j = 0$ since (i) the reduced states in separated pure systems are pure as well and (ii) the spectral decomposition of a pure state includes only one non-zero eigenvalue.

¹¹The down arrows appearing in the superscripts denote that the eigenvalues λ_j are rearranged in a non-increasing order in such way that $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \dots \geq \lambda_N$.

protocol. Although it is hard to compute it for a general state, even numerically, an explicit formula in the form of a binary entropy can be derived in the case of two-qubit systems [190–192]:

$$E_F[\rho] = -f \log_2(f) - (1-f) \log_2(1-f), \quad (2.120)$$

where $f = f(C) = (1 + \sqrt{1 - C^2})/2$ and C is an entanglement monotone, called *concurrence*. This entanglement monotone is defined as

$$C[\rho] = \max\{0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}\}. \quad (2.121)$$

where λ_j are the eigenvalues of the operator $\rho(\sigma_y \otimes \sigma_y) \rho^*(\sigma_y \otimes \sigma_y)$ in decreasing order.

In addition to providing an explicit formula for the entanglement of formation, the concurrence can also be used to reveal one of the most fundamental properties of entanglement known as the *monogamy of entanglement* [193–195], which imposes a trade-off between the amount of entanglement between different subsystems in a composite system. In the case of pure three-qubit systems, this trade-off can be described by the inequality:

$$C_{1:2}^2 + C_{1:3}^2 \leq 4 \det \rho_1 \equiv C_{1:23}^2, \quad (2.122)$$

where $C_{1:j}^2$ denotes $C^2[\rho_{1j}]$ and $C_{1:23}^2$ can be regarded as a concurrence of the subsystems 1 with the rest.¹² It is also possible to extend this inequality to the mixed states by means of a convex roof.:

$$C_{1:2}^2 + C_{1:3}^2 \leq \min\left\{\sum_j Q^{(j)} C_{1:23}^2[|\psi_j\rangle\langle\psi_j|]\right\}, \quad (2.123)$$

where the minimum is taken over all possible pure state decompositions of ρ_{123} that realize $\rho_{123} = \sum_j Q^{(j)} |\psi_j\rangle\langle\psi_j|$. Squares of the concurrence used in this inequality are also known as *two-tangles* and are denoted by $\tau_{j:j'}$. In this respect, the difference between both sides of the inequality is called the *three-tangle* or *residual tangle*, i.e., $\tau_{1:2:3} = \tau_{1:23} - \tau_{1:2} - \tau_{1:3}$. This new measure can be used to distinguish two different

¹²This is possible since only two of the four dimensions in $\mathcal{H}_{23}$ are required for the derivation of the inequality (see Ref. [193] for the details.)

entanglement classes for pure tripartite systems that are incomparable in the context of majorization.

The first three-qubit entanglement class is known as *GHZ class* due to the its maximally entangled state introduced by Greenberger, Horne, and Zeilinger [196, 197]:

$$|GHZ\rangle_{123} = \frac{1}{\sqrt{2}} (|000\rangle + |111\rangle), \quad (2.124)$$

while the second class is known as *W class* due to the its maximally entangled state introduced in Ref. [198]:

$$|W\rangle_{123} = \frac{1}{\sqrt{3}} (|001\rangle + |010\rangle + |100\rangle). \quad (2.125)$$

For the GHZ state, $\tau_{j:j'} = 0$ while $\tau_{1:2:3} = 1$. That is to say, the qubits in each pair are classically correlated but not entangled, while the three qubits have a genuine tripartite entanglement. For the W state, $\tau_{j:j'} = 8/9$ and $\tau_{1:2:3} = 0$ which follows that any pair retains a maximal amount of bipartite entanglement when the third qubit is discarded but there is no genuine tripartite entanglement (that can be measured by the tree-tangle) in the whole system.

Besides the convex roof measures summarized so far, there are also some geometric measures E_D^Σ based on the minimization of a distance $D(\rho||\sigma_j)$ between the given state ρ and the states σ_j in a set Σ . To define an entanglement measure, D should be monotonous and Σ should be closed under LOCC operations, but D does not need to be a metric in general.

The *von Neumann* or *quantum relative entropy* [185, 199] is one of the most important distances used in quantum information theory [200, 201] since it plays a key role in the quantification and unification of the correlations as will be detailed later on this section. It is the quantum analogue of *relative entropy* or *Kullback-Leibler distance*¹³ in essence [185]:

$$S[\rho||\sigma] = -\text{tr}[\rho (\log_2 \rho - \log_2 \sigma)]. \quad (2.126)$$

and quantifies the distinguishability of input states¹⁴ [203]. If D is chosen to be the quantum relative entropy and Σ is chosen to be the set of all separable states (SEP),

¹³It is defined as $H(Q||R) = \sum Q^{(J=j)} \log_2 Q^{(J=j)} / R^{(J=j)}$ and quantifies the similarity between two probability distributions $\{Q^{(J=j)}\}$ and $\{R^{(J=j)}\}$ associated with the same discrete random variable J that takes the possible values $\{j\}$. See Ref. [202] for the details.

¹⁴Note that $S[\rho||\rho] = 0$ and $S[\rho||\rho^\perp] = \infty$ where $\rho = |\psi\rangle\langle\psi|$, $\rho^\perp = |\psi^\perp\rangle\langle\psi^\perp|$, and $\langle\psi|\psi^\perp\rangle = 0$.

resulting distance-based entanglement measure

$$E_R \equiv E_R^{\text{SEP}} = \min_{\sigma \in \text{SEP}} S[\rho || \sigma] \quad (2.127)$$

is called the *relative entropy of entanglement* [185, 199]. To obtain a good distance-based entanglement measure using the quantum relative entropy, the set Σ can be taken as the set of non-distillable states (ND) [204] or the set of states with positive partial transpose (PPT) [205] as well. Since ND and PPT are larger than SEP, the relative entropic measures E_R^{ND} and E_R^{PPT} can be evaluated easier than E_R^{SEP} .

2.3.2 Elementary measures of quantum discord

Relative entropic measures can be also used to quantify the general correlations beyond entanglement if the constraint that forces Σ to be closed under LOCC is removed and it is allowed to be a proper set depending on the context, e.g., the set of classically correlated states (CC) for the quantification of *symmetric quantum discord* [206]

$$D_R^{\text{CC}} = \min_{\chi \in \text{CC}} S[\rho || \chi], \quad (2.128)$$

or the set of product states (PRO) for the quantification of total correlations [206]

$$I_R^{\text{PRO}} = \min_{\pi \in \text{PRO}} S[\rho || \pi]. \quad (2.129)$$

Let's elaborate on the latter for the case of a bipartite system in what follows. The closest product state π_{12} of a given state ρ_{12} is formed by the tensor product of its reduces states¹⁵, i.e., $S(\rho_{12} || \pi_{12})$ is minimum when $\pi_{12} = \text{tr}_2[\rho_{12}] \otimes \text{tr}_1[\rho_{12}] = \rho_1 \otimes \rho_2$. Then, it is straightforward to show that

$$\begin{aligned} S[\rho_{12} || \pi_{12}] &= -S[\rho_{12}] - \text{tr}[\rho_{12} \log_2[\rho_1 \otimes \rho_2]] \\ &= -S[\rho_{12}] - \text{tr}[\rho_{12} (\log_2 \rho_1 \otimes \mathbb{I}_2)] - \text{tr}[\rho_{12} (\mathbb{I}_1 \otimes \log_2 \rho_2)] \\ &= -S[\rho_{12}] - \text{tr}[\rho_1 \log_2 \rho_1] - \text{tr}[\rho_2 \log_2 \rho_2] \\ &= -S[\rho_{12}] + S[\rho_1] + S[\rho_2] = I[\rho_{12}] \equiv I_{1:2}, \end{aligned} \quad (2.130)$$

where $I[\rho_{12}]$ or $I_{1:2}$ is called the *quantum mutual information* as it is the quantum analogous of classical mutual information¹⁶ that quantifies correlations between two

¹⁵See Lemma 1 in Ref. [206] for the details.

¹⁶In classical information theory [207], the mutual information is defined for a pair of discrete random variables J and J' as $H(J : J') = H(J) + H(J') - H(J, J')$ where $H(J, J') = -\sum_{j, j'} Q^{(J=j, J'=j')} \log_2 Q^{(J=j, J'=j')}$ is the *joint entropy* and $Q^{(J=j, J'=j')}$ is the probability that the outcomes $J = j$ and $J' = j'$ occur at the same time.

variables in terms of the amount of information about one variable that can be acquired by determining the value of the other. In the meantime, there is an alternative definition of the classical mutual information¹⁷ that can be generalized to quantum systems as:

$$J_{1:2} = S[\rho_1] - S[\rho_1 | \{M_j^{(2)}\}] = S[\rho_1] - \sum_j P_j S[\rho_{1|M_j^{(2)}}], \quad (2.131)$$

where $S[\rho_1 | \{M_j^{(2)}\}]$ is the *von Neumann* or *quantum conditional entropy* of the first subsystem given the complete measurement $\{M_j^{(2)}\}$ on the second subsystem and $\rho_{1|M_j^{(2)}} = \text{tr}_2[M_j \rho_{12} M_j^\dagger] / P_j$ are post-measurement states of the first subsystem with corresponding probabilities $P_j = \text{tr}_2[M_j^\dagger M_j \rho_{12}]$.

Note that, as opposed to the classical case, the state of the second system cannot be determined until measured and this local measurement is expected to disturb the pre-measurement state of both subsystems. There are many different local measurements that can be performed on the second subsystem in contrast with the classical case. Furthermore, quantum correlations contained in the joint system ρ_{12} cannot be extracted by local measurements. Hence, the equivalence between the different definitions of the mutual information¹⁸ breaks down in quantum domain: $I_{1:2}$ measures the total correlations between subsystems, whereas $J_{1:2}$ quantifies only classical correlations [119]. The original definition of quantum discord [120] is grounded in the difference between these two different quantum generalizations of the mutual information:

$$\begin{aligned} \delta_{1:2} &= I_{1:2} - J_{1:2} \\ &= S[\rho_2] + \sum_j P_j S[\rho_{1|M_j^{(2)}}] - S[\rho_{12}] \end{aligned} \quad (2.132)$$

and in essence, it is the measure of nonclassical correlations which includes entanglement as a subset.

¹⁷ $H(J : J') = H(J) - H(J|J')$ where $H(J|J') = \sum_{j'} Q^{(J'=j')} H(J|J' = j')$ is the *conditional entropy* that is the average of entropies $H(J|J' = j') = -\sum_j Q^{(J=j|J'=j')} \log_2 Q^{(J=j|J'=j')}$ and $Q^{(J=j|J'=j')}$ is the *conditional probability* that equals to $Q^{(J=j, J'=j')} / Q^{(J'=j')}$ according to Bayes rule. See Refs. [207, 208] for more information.

¹⁸It is straightforward to show this equivalency using the Bayes rule as follows. $Q^{(J=j, J'=j')} = Q^{(J'=j')} Q^{(J=j|J'=j')}$ implies that $H(J, J') = H(J') + H(J|J')$ and consequently $H(J : J') = -H(J, J') + H(J) + H(J') = H(J) - H(J|J')$.

To make $J_{1:2}$ independent of the measurements, it is convenient to maximize it over all states:

$$J_{1:2} = \max_{\{M_j^{(2)}\}} \{S[\rho_1] - \sum_j P_j S[\rho_{1|M_j^{(2)}}]\}, \quad (2.133)$$

which is equivalent to minimize the quantum discord:

$$\delta_{1:2} = \min_{\{M_j^{(2)}\}} \{S[\rho_2] + \sum_j P_j S[\rho_{1|M_j^{(2)}}] - S[\rho_{12}]\}. \quad (2.134)$$

This minimization results in the extraction of maximum information about the first system by the measurement on the second system as the measurement that minimizes $S[\rho_1|\{M_j^{(2)}\}]$ is the one that disturbs the state minimally. If $M_j^{(2)}$ is chosen to be $\Pi_j^{(2)}$ as is in the original definition given in Ref. [120], one can find a measurement that does not disturb the systems given in a quantum-classical state and so gives a vanishing discord. Hence, as opposed to the relative entropic measure D_R^{CC} given in (2.128), this mutual information based measure of the discord is not symmetric, i.e., $\delta_{1:2} \neq \delta_{2:1}$, as it considers the classicality of only one subsystem.

Since any analytic expression is unknown for either its relative entropic measure D_R^{CC} or its mutual information based measure $\delta_{1:2}$, it is hard to evaluate the quantum discord in a generic system. However, an explicit formula of another distance-based measure known as the *geometric measure of discord* is available for a generic two-qubit system [209]. This measure quantifies the distance between two states ρ and ρ' using the metric *squared Hilbert-Schmidt norm*, $\|\rho - \rho'\|^2 = \text{tr}[\rho - \rho']^2$, instead of the relative entropy and minimizes the distance of a given state ρ_{12} to the set of states with zero discord (ZD)¹⁹:

$$D_G^{\text{ZD}} = \min_{\chi \in \text{ZD}} \{\|\rho_{12} - \chi\|^2\}. \quad (2.135)$$

A generic two-qubit the state can be written in the *Bloch representation*:

$$\rho_{12} = \frac{1}{4} \left(\mathbb{I}_1 \otimes \mathbb{I}_2 + \sum_{j=1}^3 x_j \mathbb{I}_1 \otimes \sigma_j^{(2)} + \sum_{j=1}^3 y_j \sigma_j^{(1)} \otimes \mathbb{I}_2 + \sum_{j,j'=1}^3 T_{jj'} \sigma_j^{(1)} \otimes \sigma_{j'}^{(2)} \right) \quad (2.136)$$

¹⁹Note that ZD is the set of quantum-classical states for $D_{G,1:2}^{\text{ZD}}$, while it is the set of classical-quantum states for $D_{G,2:1}^{\text{ZD}}$.

where σ_j stands for the Pauli sigma matrices²⁰. Using this representation, the equation (2.135) can be reduced to:

$$D_{G,1:2}^{\text{ZD}} = \frac{1}{4} (\|\vec{x}\|^2 + \|T\|^2 - k_{\max}), \quad (2.137)$$

$$D_{G,2:1}^{\text{ZD}} = \frac{1}{4} (\|\vec{y}\|^2 + \|T\|^2 - l_{\max}), \quad (2.138)$$

where $\vec{r}$, $r = \{x, y\}$, equals to (r_1, r_2, r_3) with $x_j = \text{tr}[\rho_{12}(\mathbb{I}_1 \otimes \sigma_j^{(2)})]$ and $y_j = \text{tr}[\rho_{12}(\sigma_j^{(1)} \otimes \mathbb{I}_2)]$, $\|T\|^2 = \text{tr}[T^T T]$ with $T = T_{jj'} |j\rangle\langle j'|$, $k_{\max}$ and $l_{\max}$ are the largest eigenvalues of the matrices $K = \vec{x}\vec{x}^T + T^T T$ and $L = \vec{y}\vec{y}^T + T T^T$ respectively [209]. These eigenvalues can be analytically evaluated as well [210].

2.3.3 Elementary measures of quantum coherence

Quantum correlations arise from the superposition principle, that is to say, from the fact that quantum systems can exist in superposition states. The degree of superposition in a generic state ρ with respect to a given orthogonal basis $\{|j\rangle\}$ is also the essence of *quantum coherence*. Hence, a quantitative connection is expected to be found between quantum coherence and quantum correlations. In what follows, we will try to justify this intuition summarizing the related literature.

To start with, we need to introduce the *incoherent states* [211] that are defined as

$$\varsigma = \sum_{\vec{j}} Q^{(\vec{j})} |\vec{j}\rangle\langle\vec{j}|, \quad (2.139)$$

where $|\vec{j}\rangle = |j_1\rangle \otimes |j_2\rangle \otimes \cdots |j_N\rangle$ with $|j_k\rangle$ is the initially fixed local basis of the k th subsystem²¹. Then, we need to introduce the notion of *incoherent operations* (IO) [211] under which the set of incoherent states (IC) is closed.

Now, we can move on to quantification of quantum coherence. Coherence measures should fulfill some set of requirements similar to the ones included in the definition of entanglement measures, e.g., they should be monotonous under incoherent operations or they should reach a maximum for the *maximally coherent states*²². One of the most

²⁰ $\sigma_1 = \sigma_x$, $\sigma_2 = \sigma_y$, $\sigma_3 = \sigma_z$.

²¹ The choice of the reference basis in (2.139) changes depending on the system under investigation in general.

²² Maximally coherent states are of the form $|\psi_N\rangle = \frac{1}{\sqrt{N}} \sum_{\vec{j}} |\vec{j}\rangle$. It is straightforward to show that any coherent state can be obtained by means of incoherent operations using them as resource.

widely used coherence measures that satisfy all the requirements [211] is the *relative entropy of coherence* [212]:

$$\begin{aligned} C_R^{\text{IC}}[\rho] &= \min_{\varsigma \in \text{IC}} S[\rho || \varsigma] \\ &= S[\rho_d] - S[\rho], \end{aligned} \quad (2.140)$$

where ρ_d is the diagonal part of the density matrix and can be regarded as the result of dephasing due to the following incoherent operation:

$$\rho_d = \sum_j \Pi_j \rho \Pi_j. \quad (2.141)$$

Using the relative entropy of coherence together with the other relative entropy based geometric measures of quantumness, we can reveal the connection between quantum coherence and quantum correlations. First of all, as was shown in Ref. [213], the local coherence in a composite system can be converted to bipartite entanglement using incoherent operations since it is always possible to find a bipartite incoherent operation $\mathcal{E}_{\text{IO}}$ that gives

$$(E_R^{\text{SEP}} \circ \mathcal{E}_{\text{IO}})[\rho_1 \otimes |0\rangle\langle 0|_2] = C_R^{\text{IC}}[\rho_1] \quad (2.142)$$

as long as the dimension of the first subsystem is not allowed to be larger than the dimension of the other one. Secondly, this relation can be easily extended to the scenarios in which the entanglement is replaced with the asymmetric quantum discord, that is to say, bipartite incoherent operations can create quantum discord utilizing the local coherence as a resource [214]. And finally, if the relative entropy of coherence becomes basis independent by being minimized over all local unitary transformations $\vec{U}$ that are of the form $U_1 \otimes U_2 \otimes \cdots \otimes U_N$, it then ends up being equal to the relative entropy of quantum discord [215] as follows:

$$\begin{aligned} C_R^{\text{free}}[\rho] &= \min_{\vec{U}} C_R^{\text{IC}}[\vec{U} \rho \vec{U}^\dagger] \\ &= \min_{\vec{U}} \{ S[\sum_{\vec{j}} \langle \vec{j} | \vec{U} \rho \vec{U}^\dagger | \vec{j} \rangle | \vec{j} \rangle \langle \vec{j} |] - S[\vec{U} \rho \vec{U}^\dagger] \} \\ &= \min_{|\vec{j}'\rangle = \vec{U}^\dagger |\vec{j}\rangle} \{ S[\sum_{\vec{j}} \langle \vec{j}' | \rho | \vec{j}' \rangle | \vec{j}' \rangle \langle \vec{j}' |] \} - S[\rho] \\ &= \min_{|\vec{j}'\rangle} \{ S[\sum_{\vec{j}'} \langle \vec{j}' | \rho | \vec{j}' \rangle | \vec{j}' \rangle \langle \vec{j}' |] \} - S[\rho] = D_R^{\text{CC}}[\rho] \end{aligned} \quad (2.143)$$

where the equality in the last line is a result of the theorem 2 given in Ref. [206]. It means that basis free quantum coherence is nothing but the symmetric quantum discord in terms of the relative entropy.

To elaborate on the connection between quantum coherence and quantum correlations, we can introduce the notion of *correlated coherence* [216] which is defined for a bipartite system as below:

$$C_m^{cc}[\rho_{12}] = C_m[\rho_{12}] - C_m[\rho_1] - C_m[\rho_2], \quad (2.144)$$

where the subscript m denotes the particular measure of the coherence in use. Original definition of the correlated coherence includes the l_1 *norm of coherence* that is another proper measure of coherence²³ as shown in Ref. [211] and is defined by

$$C_{l_1}[\rho] = \sum_{j \neq j'} |\langle j|\rho|j'\rangle|. \quad (2.145)$$

Correlated coherence provides a natural framework in which quantum correlations can be interpreted as specialized forms of quantum coherence. As an example, the amount of entanglement found in a bipartite state ρ_{12} can be measured by means of the minimization of the coherence contained in its *unitarily symmetric extensions*²⁴ ρ_{1324} [216]:

$$E[\rho_{12}] = \min_{\rho_{1324}} C_{l_1}^{cc}[\rho_{1324}]. \quad (2.146)$$

where the reference basis is chosen as tensor product of the eigenbases of the reduced states $\rho_1 = \text{tr}_2[\rho_{12}]$ and $\rho_2 = \text{tr}_1[\rho_{12}]$. Moreover, if $C_{l_1}^{cc}$ is replaced with C_R^{cc} in (2.146), the lower and upper bounds of the measure (2.146) turns out to be E_R and E_F respectively [217]. Besides this, if a discordlike correlation is constructed for the bipartite coherence using an analogy based on the replacement of (i) $S[\rho_{12}]$ with $C_R^{IC}[\rho]$, (ii) $I[\rho_{jj'}]$ with $C_R^{cc}[\rho_{jj'}]$, (iii) $M_j = \Pi_j$ with IO²⁵ in (2.132), then the asymmetric quantum discord can be retrieved by the minimization of this discordlike correlation over all possible reference bases [218].

²³It is important to emphasize that unlike the l_1 norm of coherence, squared Hilbert-Schmidt norm which is used in the definition of geometric discord does not constitute a proper measure of coherence.

²⁴An extension ρ_{1324} of the state ρ_{12} is said to be unitarily symmetric only if it is possible to find some local unitary operations U_{13} and U_{24} such that $\rho_{1324} = (U_{13} \otimes U_{24}) (U_{SWAP} \rho_{1324} U_{SWAP}^\dagger) (U_{13}^\dagger \otimes U_{24}^\dagger)$ with $U_{SWAP}|j_{13}j_{34}\rangle = |j'_{13}j_{34}\rangle$.

²⁵ Π_j are particularly replaced with the local rank-one *projective physically incoherent operations* in Ref. [218].



3. PROTON MOTION IN A HYDROGEN-BONDED SYSTEM

We start this chapter with reconsideration of the quantum nature of H-bonds in the light of mathematical framework provided in Ch. 2. To construct an information-theoretic model, we map the proton motion in a H-bonded system into a problem of pseudo-spins in Sec. 3.2. Then we extend this pseudo-spin model to open system dynamics in Sec. 3.3. To do so, both (i) the stretch vibrations of proton-donating (covalent) bonds and (ii) the oscillations of donor-acceptor separation are taken into account in the form of phonon baths. Finally, we end the chapter by presenting the interpretation of the steady state quantum correlations generated through a generic H-bond in Sec. 3.4.

3.1 Quantum Nature of Hydrogen Bonds - Revisited

After being equipped with the basic tools of the quantum information theory in the previous chapter, we are ready to revisit the notion of *quantumness of the H-bonding* introduced in Sec. 1.1. Before doing so, recall that we denote a H-bonded system by $X_1-H\cdots X_2$ and define it by means of the geometric parameters $r_1(t) \equiv d(X_1, H)$, $R_{12}(t) \equiv d(X_1, X_2)$ and $\phi_{12}(t) \equiv \angle HX_1X_2$ (see Figure 1.1).

Let's first focus on the single covalent bond between two arbitrary atoms labeled by X and Y. The ground state of this bond can be written as a coherent quantum superposition [13, 219]:

$$\begin{aligned} |X-Y\rangle &= \alpha(a|X\cdot\rangle \otimes |\cdot Y\rangle + b|X\cdot\rangle \otimes |\cdot Y\rangle) + \beta|(X\cdot)^-\rangle \otimes |Y^+\rangle + \gamma|X^+\rangle \otimes |(\cdot Y)^-\rangle \\ &\equiv \alpha|X\cdot - \cdot Y\rangle + \beta|(X\cdot)^-, Y^+\rangle + \gamma|X^+, (\cdot Y)^-\rangle, \end{aligned} \quad (3.1)$$

where all the probability amplitudes depend on distance $r \equiv d(X, Y)$ and obeys normalization conditions $|a|^2 + |b|^2 = 1$ and $|\alpha|^2 + |\beta|^2 + |\gamma|^2 = 1$, the black and gray dots stand for the electrons participating in the chemical bond, the former term $|X\cdot - \cdot Y\rangle$ represents the share of two electrons in the bonding molecular orbital σ_{X-Y} and the subsequent two terms are responsible for the partial ionic character of the bond.

When the second atom labeled by Y is a hydrogen, the last ionic term in (3.1) can be ignored because of the weak electron affinity of this atom [13, 219]. The X–H molecule is then able to form a weak H-bond with a second electronegative atom.

Consider a H-bond between two arbitrary atoms X_1 and X_2 . To minimize the overall energy, the amounts of neutral and ionic contributions to the ground state of the X_1 –H bond are initially rearranged in the presence of X_2 as follows:

$$|\psi(t_i = 0)\rangle = \left(\alpha(r_1)|X_1 \cdots H\rangle + \beta(r_1)|(X_1 \cdots)^-, H^+\rangle \right) \otimes |\cdots X_2\rangle \quad (3.2)$$

$$\begin{aligned} &\rightarrow \left(\alpha'(r_1, R_{12}, \phi_{12})|X_1 \cdots H\rangle + \beta'(r_1, R_{12}, \phi_{12})|(X_1 \cdots)^-, H^+\rangle \right) \otimes |\cdots X_2\rangle \\ &= \alpha'(r_1, R_{12}, \phi_{12})|X_1 \cdots H\rangle \otimes |\cdots X_2\rangle + \beta'(r_1, R_{12}, \phi_{12})|(X_1 \cdots)^-, H^+, \cdots X_2\rangle \\ &\equiv |X_1 - H \equiv X_2\rangle, \end{aligned} \quad (3.3)$$

where $r_1(t) > r_1(0)$ and $|\alpha'(r_1, R_{12}, \phi_{12})|^2 < |\alpha(r_1)|^2$ for $t > 0$ in general.

As the changes in the amplitudes described above are only induced by electrostatic interactions, the state transition from (3.2) to (3.3) is completely classical and is expected to occur for every proton-donor and -acceptor pair. Besides this, the electron lone pair orbital of the proton-acceptor ($\sigma_{LP(X_2)}$) and the unoccupied antibonding orbital of the proton-donating bond ($\sigma_{X_1-H}^*$) may overlap in the ground state of the whole system, which in turn results in a charge transfer from $\sigma_{LP(X_2)}$ to $\sigma_{X_1-H}^*$ in the form of electron tunneling. This electron delocalization occurring in the ground state not only weakens and elongates X_1 –H bond, but also gives a partially covalent character to the interaction $H \cdots X_2$:

$$\begin{aligned} |X_1 - H \equiv X_2\rangle &\rightarrow \alpha_-(r_1, R_{12}, \phi_{12})|X_1 \cdots H\rangle \otimes |\cdots X_2\rangle \\ &\quad + \delta_-(r_1, R_{12}, \phi_{12})|(X_1 \cdots \dot{\cdot} H)^-, (\cdots X_2)^+\rangle \\ &\quad + \beta_-(r_1, R_{12}, \phi_{12})|(X_1 \cdots)^-, H^+, \cdots X_2\rangle \\ &\equiv |(X_1 - H) - X_2\rangle, \end{aligned} \quad (3.4)$$

where $|\alpha_-|^2 + |\delta_-|^2 + |\beta_-|^2 = 1$. Here, we depict the presence of an electron in the antibonding orbital $\sigma_{X_1-H}^*$ by a dot above a second line between X_1 and H.

The extent of the covalency of H-bonds is not limited to the lone pair electrons of the proton-acceptor. The proton which constitutes the nucleus of the H atom is also likely to be delocalized between X_1 and X_2 due to the orbital interactions in the ground

state [13,219]:

$$\begin{aligned}
|X_1-H \cdots X_2\rangle \rightarrow & \alpha_+(r_1, R_{12}, \phi_{12}) |X_1 \cdots H\rangle \otimes |\cdots X_2\rangle \\
& + \delta_+(r_1, R_{12}, \phi_{12}) |(X_1 \cdots)^-\rangle \otimes |(H \cdots X_2)^+\rangle \\
& + \beta_+(r_1, R_{12}, \phi_{12}) |(X_1 \cdots)^-, H^+, \cdots X_2\rangle \\
\equiv & |X_1-(H)-X_2\rangle,
\end{aligned} \tag{3.5}$$

where $|\alpha_+|^2 + |\delta_+|^2 + |\beta_+|^2 = 1$. This is observed especially in so called *low barrier hydrogen bonds*¹ (LBHBs) and *short and strong hydrogen bonds*² (SSHBs).

Now, we can move on to quantum nature of H-bond in relation to the proton tunneling. To refine this relation, we first neglect the ionic character of the X_1-H bond taking β_+ in (3.5) to be zero. In this way, we eliminate the quantumness generated by this single covalent bond. This enables us to rewrite the ground state in the presence of proton delocalization as follows:

$$\begin{aligned}
|X_1-(H)-X_2\rangle &= \alpha_+ |X_1 \cdots H\rangle \otimes |\cdots X_2\rangle + \delta_+ |(X_1 \cdots)^-\rangle \otimes |(H \cdots X_2)^+\rangle \\
&\equiv \alpha_+ |X_1-H \cdots X_2\rangle + \delta_+ |X_1 \cdots H-X_2\rangle,
\end{aligned} \tag{3.6}$$

where we omit the dependence of amplitudes on the geometric parameters for the sake of simplicity of notation.

Since the two basis states superposed above are distinguishable in the sense that they differ from each other in the location of H atom, we have

$$\langle X_1-H \cdots X_2 | X_1 \cdots H-X_2 \rangle = 0. \tag{3.7}$$

This orthogonality relation implies that the smallest Hilbert space $\mathcal{H}_{HB}$ in which the states $|X_1-H \cdots X_2\rangle$ and $|X_1 \cdots H-X_2\rangle$ can be placed is 2-dimensional. That is to say, these states can be respectively taken to be $|0\rangle$ and $|1\rangle$ in any minimalistic quantum mechanical model of the proton mobility in H-bond. In this respect, there are two different kinds of effective Hamiltonian techniques in the literature each of

¹Potential energy exerted on the shared proton in a H-bond can be characterized by a double-well (see Figure 3.1). In this context, a H-bond is classified as a LBHB when the energy barrier between the two wells is comparable to the vibrational zero-point energy and as a consequence, the proton is able to delocalize between X_1 and X_2 . This barrier decreases by the donor-acceptor distances R_{12} , e.g., $O-H \cdots O$ bond turns out to be a LBHB when R_{OO} is about 2.5 Angstroms ($\text{\AA}$) [220]. The existence of such bonds in enzymes and their functional role in the catalysis are still controversial [221–233].

²For the extremely short donor-acceptor separations, e.g., for $R_{OO} \lesssim 2.4 \text{ \AA}$ in a $O-H \cdots O$ system, the energy barrier vanishes and the potential energy reduces to a single-well. Such H-bonds are said to be SSHBs [229].

which describes the H-bonding by a two-level quantum system. These techniques are known as the *spin-boson models* [234–242] and the *diabatic state models* [243–250] correspondingly. However, they are capable to capture the quantumness of H-bonding only in terms of the quantum coherence since higher-dimensional Hilbert spaces are required to define the quantum correlations such as the entanglement and discord.

On the other hand, Eq. (3.6) turns out to be nothing but an entangled state if the dimension of Hilbert space $\mathcal{H}_{HB}$ is increased from 2 to 4 by taking $|X_1-H\cdots X_2\rangle = |10\rangle$ and $|X_1\cdots H-X_2\rangle = |01\rangle$. Unlike the two-state representations mentioned above, this two-qubit representation enables us to study quantum correlations generated by the proton motion through a H-bond using the tools of quantum information theory summarized in Sec. 2.3. In this respect, we will derive a two-qubit representation for the state of a H-bonded system in what follows.

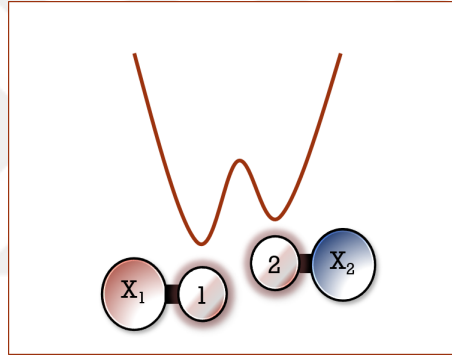


Figure 3.1 : Red curve describes the potential energy exerted on the shared proton in a H-bond. Proton is allowed to reside at enumerated locations that are associated with the bonding orbitals σ_{X_j-H} , $j = \{1, 2\}$ and correspond to the local minimums of this double-well potential.

3.2 Two-Qubit Representation of a Hydrogen-Bonded System

The locations in between the proton tunnels in a H-bond can be regarded as the vertices of an interaction graph in such a way that each vertex $j = \{1, 2\}$ is associated with the bonding orbital σ_{X_j-H} (Figure 3.1). Then the proton is expected to move from one vertex to another in accordance with the Hamiltonian

$$H_{HB} = \sum_{j=1}^2 W_j \hat{n}_j - J_{12}(a_1^\dagger a_2 + a_1 a_2^\dagger) + V_{12} \hat{n}_1 \hat{n}_2 + \lambda \mathbb{I}_{12}, \quad (3.8)$$

where $\hat{n}_j$ is the *proton number operator* that equals to $a_j^\dagger a_j$ and gives the number of protons residing at vertex j , while $a_j^\dagger$ and a_j are the *proton creation* and *annihilation*

operators that respectively creates and annihilates a proton at this vertex. As the proton is a *fermion*, the creation and annihilation operators should obey the following anticommutation relations:

$$\{a_j, a_{j'}\} = \{a_j^\dagger, a_{j'}^\dagger\} = 0, \{a_j, a_{j'}^\dagger\} = \delta_{jj'}. \quad (3.9)$$

Besides these operators, the *on-site energy* W_j represents the total potential felt by a proton when it resides at vertex j , the *hopping constant* J_{12} is responsible for the proton tunneling between two vertices, the *inter-site interaction energy* V_{12} imposes a penalty on the case in which there is one proton at each vertex, and the constant λ stands for the total intermolecular interactions between X_1 and X_2 .

Each of these coefficients included in Eq. (3.8) exhibits a different functional dependence on the geometric parameters. W_j can be decomposed into two effective potentials W_j^{cov} and W_j^{el} respectively describing the single covalent bond with the adjacent atom/molecule X_j and the electrostatic attraction to the opposite atom/molecule $X_{j'}$. The former can be chosen to be a Morse potential³ which depends on r_j as below:

$$W_j^{cov}(r_j) = V_{X_j, H}^M = D_j^M \left(e^{-2a_j^M(r_j - r_j^{eq})} - 2e^{-a_j^M(r_j - r_j^{eq})} \right), \quad (3.10)$$

where r_j^{eq} is the equilibrium X_j-H bond length in the absence of $X_{j'}$, D_j^M is the binding energy that equals to the depth of the well and a_j^M is the decay constant that controls the width of the well. The latter potential in the decomposition of W_j is nothing but the Coulomb potential that is an inverse function of $|\vec{R}_{jj'} - \vec{r}_j|$:

$$W_j^{el}(r_j, R_{jj'}, \phi_{jj'}) = V_{H, X_{j'}}^C = k_e \frac{q(H)q(X_{j'})}{\sqrt{R_{jj'}^2 + r_j^2 - 2R_{jj'}r_j \cos(\phi_{jj'})}}, \quad (3.11)$$

where k_e is the Coulomb constant and q is the charge density. Likewise the W_j^{el} , the inter-site interaction energy has an electrostatic nature too, i.e., $V_{12} = V_{H_1, H_2}^C = k_e q(H_1)q(H_2)/d(H_1, H_2)$.

³Note that this is a convenient description of the covalent bond in an isolated H-bonded system $X_1-H \cdots X_2$. In a more general situation at which the X_1 and X_2 are embedded inside complex structures such as biomolecules, it may be better to replace Morse potentials with the eigenvalues of the self-Hamiltonians of individual molecules.

It is in accordance with the directionality of H-bonding to attribute the following functional form⁴ to the hopping coefficient J_{12} :

$$J_{12}(r_1, R_{12}, \phi_{12}) = \Delta_0^J \cos(\phi_{12}) \frac{R_{12} - r_1 \cos(\phi_{12})}{\sqrt{R_{12}^2 + r_1^2 - 2R_{12}r_1 \cos(\phi_{12})}} e^{-b^J(R_{12}-R_0^J)}, \quad (3.12)$$

where Δ_0^J , b^J , and R_0^J can take different values⁵ depending on the chemical identities of X_1 , X_2 and the environment that surrounds them. For fixed values of r_1 and R_{12} , J_{12} reaches a maximum value of $\Delta_0^J e^{-b^J(R_{12}-R_0^J)}$ when ϕ_{12} equals to zero and vanishes when ϕ_{12} approaches to $\pi/2$. Hence, the proton tunneling is more likely to occur in linear H-bonds. In this sense, this form gives a proper directionality to H-bonding.

Finally, the total intermolecular interaction between X_1 and X_2 is expected to be a combination of Pauli repulsion, Van der Waals interaction, and London dispersion which can be represented by a Lennard-Jones potential:

$$\lambda(R_{12}) = V_{X_1, X_2}^{\text{LJ}} = 4\epsilon^{\text{LJ}} \left(\left(\frac{\sigma^{\text{LJ}}}{R_{12}} \right)^{12} - \left(\frac{\sigma^{\text{LJ}}}{R_{12}} \right)^6 \right), \quad (3.13)$$

where ϵ^{LJ} and σ^{LJ} are standard Lennard-Jones parameters that also depend on the chemical identities of X_1 , X_2 and the environment that surrounds them.

To obtain a two-qubit representation using the fermionic Hamiltonian (3.8), we introduce the following Jordan-Wigner transformation for a_j , $a_j^\dagger$, and $\hat{n}_j$

$$\begin{aligned} a_j &= \exp \left(-i\pi \sum_{k=1}^{j-1} \sigma_-^{(k)} \sigma_+^{(k)} \right) \sigma_+^{(j)}, \\ a_j^\dagger &= \exp \left(+i\pi \sum_{k=1}^{j-1} \sigma_-^{(k)} \sigma_+^{(k)} \right) \sigma_-^{(j)}, \\ \hat{n}_j &= \sigma_-^{(j)} \sigma_+^{(j)}, \end{aligned} \quad (3.14)$$

where $|1_j\rangle = |X_j - \text{H}\rangle$, $|0_j\rangle = |X_j \cdots\rangle$, $\sigma_-^{(j)} = |1_j\rangle\langle 0_j|$ and $\sigma_+^{(j)} = |0_j\rangle\langle 1_j|$ with the convention for Pauli z operator that $\sigma_z^{(j)} = |0_j\rangle\langle 0_j| - |1_j\rangle\langle 1_j|$. This transformation is somewhat different from the standard Jordan-Wigner transformation used in electron transport phenomena since the annihilation of a proton at vertex j is an energy

⁴This form was first used to describe the constant that couples the states $|X_1 - \text{H} \cdots X_2\rangle$ and $|X_1 \cdots \text{H} - X_2\rangle$ in a diabatic state model proposed in Refs. [249, 250].

⁵Although the same form can be used to describe both J_{12} and the coupling constant given in Refs. [249, 250], the parameters Δ_0^J and b^J that define the energy and length time scales respectively, should be fixed to different values here since the model under investigation is quite different than a diabatic state model.

increasing process whereas the creation of an electron at site j increases the energy in electron transport models.

After writing (3.14) in terms of Pauli operators given in (2.108) and substituting it into (3.8), we end up with the two-qubit Hamiltonian

$$H_{HB} = J_x \sigma_x^{(1)} \otimes \sigma_x^{(2)} + J_y \sigma_y^{(1)} \otimes \sigma_y^{(2)} + J_z \sigma_z^{(1)} \otimes \sigma_z^{(2)} + (B+b) \sigma_z^{(1)} + (B-b) \sigma_z^{(2)} + \tilde{\lambda}, \quad (3.15)$$

where $J_x = J_y = -J_{12}/2$, $J_z = V_{12}/4$, $B = -(W_1 + W_2 + V_{12})/4$, $b = -(W_1 - W_2)/4$, and $\tilde{\lambda} = \lambda + (2W_1 + 2W_2 + V_{12})/4$. Actually, this is nothing but a well-studied spin dimer Hamiltonian known as *Heisenberg XXZ model* under one homogeneous magnetic field B and one inhomogeneous magnetic field b . Since the homogeneous magnetic field B covers all the classical interactions occurring in H-bond, it is an indicator of *classicality*. On the other hand, the inhomogeneous magnetic field b quantifies the asymmetry of H-bond and vanishes for symmetric bonds, i.e., for $W_1 = W_2$.

Eigensystem of the *pseudo-spin* Hamiltonian (3.15) can be written in an increasing order of the eigenvalues as below:

$$\begin{cases} e_1 = -J_z - 2\sqrt{b^2 + J_x^2} + \tilde{\lambda}, & |e_1\rangle = (\eta_-|01\rangle + |10\rangle)/(1 + \eta_-^2)^{\frac{1}{2}}, \\ e_2 = -J_z + 2\sqrt{b^2 + J_x^2} + \tilde{\lambda}, & |e_2\rangle = (\eta_+|01\rangle + |10\rangle)/(1 + \eta_+^2)^{\frac{1}{2}}, \\ e_3 = -2B + J_z + \tilde{\lambda}, & |e_3\rangle = |11\rangle, \\ e_4 = +2B + J_z + \tilde{\lambda}, & |e_4\rangle = |00\rangle, \end{cases} \quad (3.16)$$

where $\eta_{\pm} = (b \pm \sqrt{b^2 + J_x^2})/J_x$. The ground state $|e_1\rangle$ is just the state described in Eq. 3.6. Also, the ground state energy e_1 goes to the expected energy of a classical bond, i.e., $W_1 + \lambda$, when J_x vanishes. Moreover, this energy seems to be suitable for energy decomposition analysis as it equals to $W_1 + \lambda + J_{12}$ for symmetric H-bonds.

3.3 Open System Dynamics

A generic environment model for the motion of a proton through a H-bond should include at least three kinds of vibrations each of which is associated with the oscillation of one of the three geometric parameters r_j , $\theta_{jj'}$, and $R_{jj'}$. Charge fluctuations in the surrounding molecules may drive periodic oscillations of r_j that correspond to the X_j -H *stretching modes*. On the other hand, changes in $\theta_{jj'}$ are related to the rotational vibration modes such as *bending*, *rocking*, *wagging*, *twisting* and *libration* modes.

Such rotational vibrations may occur due to the collisions with solvent molecules for the H-bonds in solution or as a result of the intramolecular protein dynamics for the bio-molecular H-bonds. Besides these, determinant of the equilibrium value of $R_{jj'}$ and its recurring deviations may change depending on the system under investigation. As an example, it is hard to draw the underlying mechanism behind the observed values of $R_{jj'}$ for the intermolecular H-bonds in biological systems.

3.3.1 Local proton-phonon coupling

Here and in the following, we will consider the X_j –H stretch vibrations by taking them into account as two independent thermal baths existing around the vertices and having the individual self-Hamiltonians⁶

$$H_B^{(j)} = \sum_k \hbar \omega_{j,k} b_{j,k}^\dagger b_{j,k}, \quad (3.17)$$

where $b_{j,k}^\dagger$ and $b_{j,k}$ are the operators that respectively create and annihilate a phonon with energy $\hbar \omega_{j,k}$ at the vertex j and obey the following commutation relations:

$$[b_{j,k}, b_{j,k'}] = [b_{j,k}^\dagger, b_{j,k'}^\dagger] = 0, [b_{j,k}, b_{j,k'}^\dagger] = \delta_{kk'}. \quad (3.18)$$

The allowed proton positions, or equivalently the pseudo-spin states of proton are assumed to be linearly coupled to phonons through

$$H_I^l = \sum_j \hat{n}_j \sum_k \left(g_{j,k} b_{j,k}^\dagger + g_{j,k}^* b_{j,k} \right) \propto \sum_j \sigma_z^{(j)} \sum_k \left(g_{j,k} b_{j,k}^\dagger + g_{j,k}^* b_{j,k} \right). \quad (3.19)$$

The local proton-phonon interaction described above is expected to induce entanglement between the pseudo-spins and phonons. In the case of $J_x = 0$, it is possible to separate the dynamics of the pseudo-spins from each other. Then, the entanglement generated by (3.19) leads to a pure dephasing process for each pseudo-spin as described in Ref. [251]. In the next subsection, we will discuss the case of non-vanishing inter-spin coupling by assuming that the Born-Markov approximation (see Sec. 2.2.3.2) is justified, that is to say, the state of pseudo-spins varies over a time scale much longer than the lifetime of X_j –H stretch vibrations.

⁶Actually, X_j –H stretch vibrations should be anharmonic as the Morse oscillator (MO) is a better representation of the X_j –H bond than the quantum harmonic oscillator (qHO). However, the lowest-energy levels of MO can be approximated by qHO. For the sake of simplification, we restrict our analysis to these low-energy harmonic oscillations of X_j –H bond.

3.3.1.1 Representation by Markovian master equation

Bath operators in interaction picture

Proton-phonon interaction Hamiltonian (3.19) is in accordance with the general form given in (2.69), i.e., local operators $A_j = \sigma_z^{(j)}$ and $B_j = \sum_k (g_{j,k} b_{j,k}^\dagger + g_{j,k}^* b_{j,k})$ are both Hermitian. If we switch into the interaction picture, the bath operators B_j become

$$\begin{aligned} B_j(t) &= e^X B_j e^{-X} \\ &= B_j + [X, B_j] + [X, [X, B_j]]/2! + [X, [X, [X, B_j]]]/3! + \dots \end{aligned} \quad (3.20)$$

where $X = iH_B^{(j)} t/\hbar$. To evaluate the commutators above, first we need to find the commutators $[H_B^{(j)}, b_{j,k'}^\dagger]$ and $[H_B^{(j)}, b_{j,k'}]$. In this respect, the relations (3.18) imply

$$\begin{aligned} [H_B^{(j)}, b_{j,k'}^\dagger] &= \sum_k \hbar \omega_{j,k} [b_{j,k}^\dagger b_{j,k}, b_{j,k'}^\dagger] \\ &= \sum_k \hbar \omega_{j,k} \left(b_{j,k}^\dagger [b_{j,k}, b_{j,k'}^\dagger] + [b_{j,k}^\dagger, b_{j,k'}^\dagger] b_{j,k} \right) \\ &= \sum_k \hbar \omega_{j,k} b_{j,k}^\dagger \delta_{k,k'} = +\hbar \omega_{j,k'} b_{j,k'}^\dagger, \end{aligned} \quad (3.21)$$

$$\begin{aligned} [H_B^{(j)}, b_{j,k'}] &= \sum_k \hbar \omega_{j,k} [b_{j,k}^\dagger b_{j,k}, b_{j,k'}] \\ &= \sum_k \hbar \omega_{j,k} \left(b_{j,k}^\dagger [b_{j,k}, b_{j,k'}] + [b_{j,k}^\dagger, b_{j,k'}] b_{j,k} \right) \\ &= \sum_k \hbar \omega_{j,k} (-\delta_{k,k'}) b_{j,k} = -\hbar \omega_{j,k'} b_{j,k'}. \end{aligned} \quad (3.22)$$

Then, it is easy to calculate the commutators in (3.20) after writing them in terms of (3.21) and (3.22):

$$\begin{aligned} [X, B_j] &= (it/\hbar) [H_B^{(j)}, B_j] \\ &= \sum_k (it/\hbar) \left(g_{j,k}^* [H_B^{(j)}, b_{j,k}] + g_{j,k} [H_B^{(j)}, b_{j,k}^\dagger] \right) \\ &= \sum_k (i\omega_{j,k} t) \left(-g_{j,k}^* b_{j,k} + g_{j,k} b_{j,k}^\dagger \right), \end{aligned} \quad (3.23)$$

$$\begin{aligned} [X, [X, B_j]] &= (it/\hbar) [H_B^{(j)}, [X, B_j]] \\ &= \sum_k (i^2 \omega_{j,k} t^2 / \hbar) \left(-g_{j,k}^* [H_B^{(j)}, b_{j,k}] + g_{j,k} [H_B^{(j)}, b_{j,k}^\dagger] \right) \\ &= \sum_k (i^2 \omega_{j,k}^2 t^2) \left(+g_{j,k}^* b_{j,k} + g_{j,k} b_{j,k}^\dagger \right), \end{aligned} \quad (3.24)$$

$$\begin{aligned}
[X, [X, [X, B_j]]] &= (it/\hbar)[H_B^{(j)}, [X, [X, B_j]]] \\
&= \sum_k (i^3 \omega_{j,k}^2 t^3 / \hbar) \left(+g_{j,k}^* [H_B^{(j)}, b_{j,k}] + g_{j,k} [H_B^{(j)}, b_{j,k}^\dagger] \right) \\
&= \sum_k (i^3 \omega_{j,k}^3 t^3) \left(-g_{j,k}^* b_{j,k} + g_{j,k} b_{j,k}^\dagger \right).
\end{aligned} \tag{3.25}$$

By substituting these commutators into (3.20) and collecting the terms involving $b_{j,k}$ and $b_{j,k}^\dagger$ together, we end up with the interaction picture operators below:

$$\begin{aligned}
B_j(t) &= \sum_k \sum_{l=0}^{\infty} \frac{(-i \omega_{j,k} t)^l}{l!} g_{j,k}^* b_{j,k} + \sum_k \sum_{l=0}^{\infty} \frac{(+i \omega_{j,k} t)^l}{l!} g_{j,k} b_{j,k}^\dagger \\
&= \sum_k g_{j,k}^* e^{-i \omega_{j,k} t} b_{j,k} + g_{j,k} e^{+i \omega_{j,k} t} b_{j,k}^\dagger.
\end{aligned} \tag{3.26}$$

Thermal bath correlation function and dissipation rates

To calculate the bath correlation function $\langle B_j^\dagger(t) B_j(0) \rangle_{\text{th}}$, we will use the following thermal expectations:

$$\begin{aligned}
\langle b_{j,k} b_{j,k} \rangle_{\text{th}} &= 0, & \langle b_{j,k}^\dagger b_{j,k} \rangle_{\text{th}} &= N_j(\omega_{j,k}), \\
\langle b_{j,k} b_{j,k}^\dagger \rangle_{\text{th}} &= 1 + N_j(\omega_{j,k}), & \langle b_{j,k}^\dagger b_{j,k}^\dagger \rangle_{\text{th}} &= 0.
\end{aligned} \tag{3.27}$$

where $N_j(\omega_{j,k})$ is the *average number of phonons* with energy $\hbar\omega_{j,k}$ for the *Bose-Einstein statistics* and equals to $1/(e^{\beta\hbar\omega_{j,k}} - 1)$. Let's show only one of these equalities, e.g.,

$$\begin{aligned}
\langle b_{j,k}^\dagger b_{j,k} \rangle_{\text{th}} &= \text{tr}_{\mathcal{B}}[\rho_j^{\text{th}} b_{j,k}^\dagger b_{j,k}] \\
&= \text{tr}_{\mathcal{B}}[e^{-\beta H_B^{(j)}} b_{j,k}^\dagger b_{j,k}] / \text{tr}_{\mathcal{B}}[e^{-\beta H_B^{(j)}}] \\
&= N_j(\omega_{j,k}).
\end{aligned} \tag{3.28}$$

By using the series expansion $\sum_{j=0}^{\infty} x^j = \frac{1}{1-x}$ for $|x| < 1$, it is straightforward to evaluate the denominator in the second line:

$$\begin{aligned}
\text{tr}[e^{-\beta H_B^{(j)}}] &= \text{tr}[e^{-\beta \sum_k \hbar\omega_{j,k} b_{j,k}^\dagger b_{j,k}}] = \text{tr}\left[\prod_k e^{-\beta \hbar\omega_{j,k} b_{j,k}^\dagger b_{j,k}}\right] \\
&= \sum_{n_1, n_2, \dots} \langle n_1, n_2, \dots | \prod_k e^{-\beta \hbar\omega_{j,k} b_{j,k}^\dagger b_{j,k}} | n_1, n_2, \dots \rangle \\
&= \sum_{n_1} \langle n_1 | e^{-\beta \hbar\omega_{j,1} b_{j,1}^\dagger b_{j,1}} | n_1 \rangle \sum_{n_2} \langle n_2 | e^{-\beta \hbar\omega_{j,2} b_{j,2}^\dagger b_{j,2}} | n_2 \rangle \dots \\
&= \sum_{n_1} \left(e^{-\beta \hbar\omega_{j,1}} \right)^{n_1} \sum_{n_2} \left(e^{-\beta \hbar\omega_{j,2}} \right)^{n_2} \dots \\
&= \prod_k \frac{1}{(1 - e^{-\beta \hbar\omega_{j,k}})}.
\end{aligned} \tag{3.29}$$

If we also use the series expansion $\sum_{j=0}^{\infty} j x^j = \frac{x}{(x-1)^2}$ for $|x| < 1$, the numerator in the second line of (3.28) is found to be:

$$\begin{aligned}
\text{tr}[e^{-\beta H_B^{(j)}} b_{j,k}^\dagger b_{j,k}] &= \text{tr}[e^{-\beta \sum_{k'} \hbar \omega_{j,k'} b_{j,k'}^\dagger b_{j,k'}} b_{j,k}^\dagger b_{j,k}] = \text{tr}[\prod_{k'} e^{-\beta \hbar \omega_{j,k'} b_{j,k'}^\dagger b_{j,k'}} b_{j,k}^\dagger b_{j,k}] \\
&= \sum_{n_1, n_2, \dots} \langle n_1, n_2, \dots | \prod_{k'} e^{-\beta \hbar \omega_{j,k'} b_{j,k'}^\dagger b_{j,k'}} b_{j,k}^\dagger b_{j,k} | n_1, n_2, \dots \rangle \\
&= \sum_{n_k} \langle n_k | e^{-\beta \hbar \omega_{j,k} b_{j,k}^\dagger b_{j,k}} b_{j,k}^\dagger b_{j,k} | n_k \rangle \prod_{k' \neq k} \sum_{n_{k'}} \langle n_{k'} | e^{-\beta \hbar \omega_{j,k'} b_{j,k'}^\dagger b_{j,k'}} | n_{k'} \rangle \\
&= \sum_{n_k} n_k \left(e^{-\beta \hbar \omega_{j,k}} \right)^{n_k} \prod_{k' \neq k} \sum_{n_{k'}} \left(e^{-\beta \hbar \omega_{j,k'}} \right)^{n_{k'}} \\
&= \frac{e^{-\beta \hbar \omega_{j,k}}}{(e^{-\beta \hbar \omega_{j,k}} - 1)^2} \prod_{k' \neq k} \frac{1}{(1 - e^{-\beta \hbar \omega_{j,k'}})}. \tag{3.30}
\end{aligned}$$

By substituting (3.29) and (3.30) back into (3.28), we end up with

$$\begin{aligned}
\langle b_{j,k}^\dagger b_{j,k} \rangle_{\text{th}} &= \frac{e^{-\beta \hbar \omega_{j,k}}}{(e^{-\beta \hbar \omega_{j,k}} - 1)^2} \prod_{k' \neq k} \frac{1}{(1 - e^{-\beta \hbar \omega_{j,k'}})} / \prod_{k'} \frac{1}{(1 - e^{-\beta \hbar \omega_{j,k'}})} \\
&= \frac{e^{-\beta \hbar \omega_{j,k}}}{(e^{-\beta \hbar \omega_{j,k}} - 1)^2} / \frac{1}{(1 - e^{-\beta \hbar \omega_{j,k}})} = \frac{1}{(e^{\beta \hbar \omega_{j,k}} - 1)} = N_j(\omega_{j,k}). \tag{3.31}
\end{aligned}$$

Now, we can move on to calculate the correlation function of interaction picture bath operators using (3.26) and (3.27). For independent baths, it equals to:

$$\begin{aligned}
\langle B_j^\dagger(t) B_{j'}(0) \rangle_{\text{th}} &= \sum_{k, k'} \left\langle (g_{j,k}^* e^{-i\omega_{j,k}t} b_{j,k} + g_{j,k} e^{+i\omega_{j,k}t} b_{j,k}^\dagger) (g_{j',k'} b_{j',k'}^\dagger + g_{j',k'}^* b_{j',k'}) \right\rangle_{\text{th}} \\
&= \sum_{k, k'} (g_{j,k} g_{j',k'}^* e^{+i\omega_{j,k}t} \langle b_{j,k}^\dagger, b_{j',k'} \rangle_{\text{th}} + g_{j,k}^* g_{j',k'} e^{-i\omega_{j,k}t} \langle b_{j,k}, b_{j',k'}^\dagger \rangle_{\text{th}} \\
&\quad + g_{j,k} g_{j',k'} e^{+i\omega_{j,k}t} \langle b_{j,k}^\dagger, b_{j',k'}^\dagger \rangle_{\text{th}} + g_{j,k}^* g_{j',k'}^* e^{-i\omega_{j,k}t} \langle b_{j,k}, b_{j',k'} \rangle_{\text{th}}) \\
&= \sum_k |g_{j,k}|^2 \left(e^{-i\omega_{j,k}t} (1 + N_j(\omega_{j,k})) + e^{+i\omega_{j,k}t} N_j(\omega_{j,k}) \right) \delta_{jj'}. \tag{3.32}
\end{aligned}$$

Dissipation rates, half of the real part of one-sided Fourier transforms of $\langle B_j^\dagger(t)B_j(0) \rangle_{\text{th}}$, can be calculated by putting (3.32) into (2.98)

$$\begin{aligned}
\gamma_{jj'}(\omega) &= \frac{1}{\hbar^2} \delta_{jj'} \sum_k |g_{j,k}|^2 \left((1 + N_j(\omega_{j,k})) \int_{-\infty}^{\infty} d\tau e^{i(\omega - \omega_{j,k})\tau} \right. \\
&\quad \left. + N_j(\omega_{j,k}) \int_{-\infty}^{\infty} d\tau e^{i(\omega + \omega_{j,k})\tau} \right) \\
&= \frac{1}{\hbar^2} \delta_{jj'} \sum_k |g_{j,k}|^2 \left(2\pi\delta(\omega - \omega_{j,k}) (1 + N_j(\omega_{j,k})) + 2\pi\delta(\omega + \omega_{j,k}) N_j(\omega_{j,k}) \right) \\
&= \frac{2}{\hbar} \delta_{jj'} \int_0^{\infty} d\omega' J_j(\omega') \left((1 + N_j(\omega')) \delta(\omega - \omega') + N_j(\omega') \delta(\omega + \omega') \right) \\
&= \frac{2}{\hbar} \delta_{jj'} \begin{cases} J_j(\omega) (1 + N_j(\omega)) & \text{for } 0 < \omega < \infty \\ J_j(-\omega) N_j(-\omega) & \text{for } -\infty < \omega < 0 \end{cases} \quad (3.33) \\
&\equiv \delta_{jj'} \gamma_j(\omega),
\end{aligned}$$

where the sum over the absolute square of the discrete coupling constants $g_{j,k}$ is replaced by an integral over a continuous function $J_j(\omega)$ that is defined as $\pi/\hbar \sum_k |g_{j,k}|^2 \delta(\omega - \omega_{j,k})$. Here, $J_j(\omega)$ is called the *spectral density function* and encapsulates all the effects of the j th bath on the associated pseudo-spin.

Note that $-N_j(-\omega)$ equals to $1 + N_j(\omega)$. Hence, if $J_j(\omega)$ is an odd function, $\gamma_{jj}(\omega) \equiv \gamma_j(\omega)$ turns out to be $2/\hbar J_j(\omega) (1 + N_j(\omega))$ for all values of ω . Also note that $\gamma_{jj'}(\omega)$ is reduced to $\gamma_j(\omega)$ above because each pseudo-spin is associated to an independent environment. This is expected for the imaginary part of one-sided Fourier transforms of $\langle B_j^\dagger(t)B_j(0) \rangle_{\text{th}}$ (see Eq. (2.99)) too, i.e., $S_{jj'}(\omega) = \delta_{jj'} S_{jj}(\omega) \equiv S_j(\omega)$.

Lamb shift Hamiltonian and dissipator

To start analyzing the open system dynamics of pseudo-spins, eigenoperators of the self-Hamiltonian H_{HB} should be calculated using Eq. (2.74) with $A_j = \sigma_z^{(j)}$. Since H_{HB} has 4 non-degenerate energy levels as given in (3.16), there are $\binom{4}{2} = 12$ different transitions in the system. Each possible nonzero value of Bohr frequency ω corresponds to one of these transitions. However, an interaction with the environment does not need to give rise to a transition always. Hence, to account for such situations where no transition is enabled, ω can take one more value that equals to zero.

Only 3 of the 13 values of ω correspond to non-zero eigenoperators as follows:

$$\begin{aligned}
A_j(0) &= -|e_3\rangle\langle e_3| + |e_4\rangle\langle e_4| + (-1)^j \frac{b}{\sqrt{b^2 + J_x^2}} (|e_1\rangle\langle e_1| - |e_2\rangle\langle e_2|), \\
A_j(\omega_{12}) &= (-1)^{j-1} \frac{J_x}{\sqrt{b^2 + J_x^2}} |e_2\rangle\langle e_1|, \\
A_j(\omega_{21}) &= (-1)^{j-1} \frac{J_x}{\sqrt{b^2 + J_x^2}} |e_1\rangle\langle e_2|.
\end{aligned} \tag{3.34}$$

Note that in the case of $J_x = 0$, all the eigenoperators with a non-zero Bohr frequency vanish. Consequently, there will be no term left in the dissipator related with an energy transfer between pseudo-spins and their local environments. This means that only information exchange will occur due to the interaction described by (3.19) and consequently we will get two independent dephasing processes as mentioned before.

In order to find out what happens when $J_x \neq 0$, we will put the eigenoperators given in (3.34) into the Lamb shift Hamiltonian (2.101) and the dissipator (2.102). Then, the Lamb shift Hamiltonian H_{LS} becomes $H_{LS}^{(1)} + H_{LS}^{(2)}$ such that

$$\begin{aligned}
H_{LS}^{(j)} &= S_j^0 (|e_3\rangle\langle e_3| + |e_4\rangle\langle e_4|) + \left((1 - \zeta) S_j^0 + \zeta S_j^{1,2} \right) |e_1\rangle\langle e_1| \\
&\quad + \left((1 - \zeta) S_j^0 + \zeta S_j^{2,1} \right) |e_2\rangle\langle e_2|
\end{aligned} \tag{3.35}$$

where $\zeta = J_x^2/(b^2 + J_x^2)$, $S_j^0 = S_j(0)$, and $S_j^{j',j''} = S_j(\omega_{j'j''})$. Similarly, the dissipator $\mathcal{D}(\rho)$ is decomposed into two dissipators each of which takes the following form

$$\begin{aligned}
\mathcal{D}^{(j)}[\rho] &= -2\gamma_j^0 (\wp_{3,4}|e_3\rangle\langle e_4| + \wp_{4,3}|e_4\rangle\langle e_3|) \\
&\quad - \frac{1}{2} ((2 - \zeta + (-1)^j 2\sqrt{1 - \zeta}) \gamma_j^0 + \zeta \gamma_j^{1,2}) (\wp_{3,1}|e_3\rangle\langle e_1| + \wp_{1,3}|e_1\rangle\langle e_3|) \\
&\quad - \frac{1}{2} ((2 - \zeta - (-1)^j 2\sqrt{1 - \zeta}) \gamma_j^0 + \zeta \gamma_j^{1,2}) (\wp_{4,1}|e_4\rangle\langle e_1| + \wp_{1,4}|e_1\rangle\langle e_4|) \\
&\quad - \frac{1}{2} ((2 - \zeta - (-1)^j 2\sqrt{1 - \zeta}) \gamma_j^0 + \zeta \gamma_j^{2,1}) (\wp_{3,2}|e_3\rangle\langle e_2| + \wp_{2,3}|e_2\rangle\langle e_3|) \\
&\quad - \frac{1}{2} ((2 - \zeta + (-1)^j 2\sqrt{1 - \zeta}) \gamma_j^0 + \zeta \gamma_j^{2,1}) (\wp_{4,2}|e_4\rangle\langle e_2| + \wp_{2,4}|e_2\rangle\langle e_4|) \\
&\quad - \frac{1}{2} (4(1 - \zeta) \gamma_j^0 + \zeta \gamma_j^{1,2} + \zeta \gamma_j^{2,1}) (\wp_{1,2}|e_1\rangle\langle e_2| + \wp_{2,1}|e_2\rangle\langle e_1|) \\
&\quad - \zeta (\gamma_j^{1,2} \wp_{1,1} - \gamma_j^{2,1} \wp_{2,2}) |e_1\rangle\langle e_1| \\
&\quad + \zeta (\gamma_j^{1,2} \wp_{1,1} - \gamma_j^{2,1} \wp_{2,2}) |e_2\rangle\langle e_2|
\end{aligned} \tag{3.36}$$

with $\gamma_j^0 = \gamma_j(0)$, $\gamma_j^{j',j''} = \gamma_j(\omega_{j'j''})$, and $\wp_{j,j'} = \wp_{j,j'}(t) \equiv \langle e_j | \rho(t) | e_{j'} \rangle$ are the elements of the pseudo-spin density matrix in energy eigenbasis $\{|e_j\rangle\}$.

Exact solution of the master equation

When we substitute (3.15), (3.35) and (3.36) into (2.103), we end up with a master equation for the system of our interest. It is straightforward to solve this master equation analytically. Exact solution in the energy eigenbasis is as follows:

$$\begin{aligned}
\rho_{1,1}(t) &= \frac{\tilde{\gamma}_{2,1}}{\tilde{\gamma}_{1,2} + \tilde{\gamma}_{2,1}} (\rho_{1,1}(0) + \rho_{2,2}(0)) \\
&\quad + e^{-\zeta(\tilde{\gamma}_{1,2} + \tilde{\gamma}_{2,1})t} \left(\frac{\tilde{\gamma}_{1,2}}{\tilde{\gamma}_{1,2} + \tilde{\gamma}_{2,1}} \rho_{1,1}(0) - \frac{\tilde{\gamma}_{2,1}}{\tilde{\gamma}_{1,2} + \tilde{\gamma}_{2,1}} \rho_{2,2}(0) \right), \\
\rho_{1,2}(t) &= e^{-i\zeta(\tilde{S}_{1,2} - \tilde{S}_{2,1} + \frac{\omega_{12}}{\zeta})t} e^{-\frac{\zeta}{2} \left(4\frac{1-\zeta}{\zeta} \tilde{\gamma}_0 + \tilde{\gamma}_{1,2} + \tilde{\gamma}_{2,1} \right)t} \rho_{1,2}(0), \\
\rho_{1,3}(t) &= e^{+i\zeta(\tilde{S}_0 - \tilde{S}_{1,2} + \frac{\omega_{31}}{\zeta})t} e^{-\frac{\zeta}{2} \left(\frac{2-\zeta}{\zeta} \tilde{\gamma}_0 - \frac{2\sqrt{1-\zeta}}{\zeta} \tilde{\gamma}_0 + \tilde{\gamma}_{1,2} \right)t} \rho_{1,3}(0), \\
\rho_{1,4}(t) &= e^{+i\zeta(\tilde{S}_0 - \tilde{S}_{1,2} + \frac{\omega_{41}}{\zeta})t} e^{-\frac{\zeta}{2} \left(\frac{2-\zeta}{\zeta} \tilde{\gamma}_0 + \frac{2\sqrt{1-\zeta}}{\zeta} \tilde{\gamma}_0 + \tilde{\gamma}_{1,2} \right)t} \rho_{1,4}(0), \\
\rho_{2,2}(t) &= \frac{\tilde{\gamma}_{1,2}}{\tilde{\gamma}_{1,2} + \tilde{\gamma}_{2,1}} (\rho_{1,1}(0) + \rho_{2,2}(0)) \\
&\quad - e^{-\zeta(\tilde{\gamma}_{1,2} + \tilde{\gamma}_{2,1})t} \left(\frac{\tilde{\gamma}_{1,2}}{\tilde{\gamma}_{1,2} + \tilde{\gamma}_{2,1}} \rho_{1,1}(0) - \frac{\tilde{\gamma}_{2,1}}{\tilde{\gamma}_{1,2} + \tilde{\gamma}_{2,1}} \rho_{2,2}(0) \right), \\
\rho_{2,3}(t) &= e^{+i\zeta(\tilde{S}_0 - \tilde{S}_{2,1} + \frac{\omega_{32}}{\zeta})t} e^{-\frac{\zeta}{2} \left(\frac{2-\zeta}{\zeta} \tilde{\gamma}_0 + \frac{2\sqrt{1-\zeta}}{\zeta} \tilde{\gamma}_0 + \tilde{\gamma}_{2,1} \right)t} \rho_{2,3}(0), \\
\rho_{2,4}(t) &= e^{+i\zeta(\tilde{S}_0 - \tilde{S}_{2,1} + \frac{\omega_{42}}{\zeta})t} e^{-\frac{\zeta}{2} \left(\frac{2-\zeta}{\zeta} \tilde{\gamma}_0 - \frac{2\sqrt{1-\zeta}}{\zeta} \tilde{\gamma}_0 + \tilde{\gamma}_{2,1} \right)t} \rho_{2,3}(0), \\
\rho_{3,3}(t) &= \rho_{3,3}(0), \\
\rho_{3,4}(t) &= e^{-i\omega_{34}t} e^{-2\tilde{\gamma}_0 t} \rho_{3,4}(0), \\
\rho_{4,4}(t) &= \rho_{4,4}(0),
\end{aligned} \tag{3.37}$$

where $\tilde{S}_0 = S_1^0 + S_2^0$, $\tilde{S}_{j',j''} = S_1^{j',j''} + S_2^{j',j''}$, $\tilde{\gamma}_0 = \gamma_1^0 + \gamma_2^0$, $\tilde{\gamma}_{j',j''} = \gamma_1^{j',j''} + \gamma_2^{j',j''}$, and $\tilde{\gamma}_0 = \gamma_1^0 - \gamma_2^0$. To elaborate on this calculation for an initial state that is likely to occur in real systems, e.g., for $|10\rangle\langle 10|$, $|01\rangle\langle 01|$, $|e_1\rangle\langle e_1|$ or ρ^{th} , we don't need to evaluate all the coefficients in (3.37). Indeed, only $\tilde{\gamma}_{1,2}/(\tilde{\gamma}_{1,2} + \tilde{\gamma}_{2,1})$, $\tilde{\gamma}_{2,1}/(\tilde{\gamma}_{1,2} + \tilde{\gamma}_{2,1})$, $\tilde{\gamma}_{1,2} + \tilde{\gamma}_{2,1}$, $\tilde{\gamma}_0$, and $\tilde{S}_{1,2} - \tilde{S}_{2,1}$ are necessary for such initial states. We can evaluate first four of them for non-identical baths at the same temperature, e.g, for $J_1(\omega) \neq J_2(\omega)$ but $N_1(\omega) = N_2(\omega) = N(\omega)$ by using Eq. (3.33) together with the fact that $\omega_{12} = -\omega_{21} < 0$:

$$\frac{\tilde{\gamma}_{1,2}}{\tilde{\gamma}_{1,2} + \tilde{\gamma}_{2,1}} = \frac{N(\omega_{21})}{N(\omega_{21}) + (1 + N(\omega_{21}))} = \frac{1}{1 + e^{+\beta(e_2 - e_1)}} = \frac{e^{-\beta e_2}}{e^{-\beta e_2} + e^{-\beta e_1}}, \tag{3.38}$$

$$\frac{\tilde{\gamma}_{2,1}}{\tilde{\gamma}_{1,2} + \tilde{\gamma}_{2,1}} = \frac{(1 + N(\omega_{21}))}{N(\omega_{21}) + (1 + N(\omega_{21}))} = \frac{e^{+\beta(e_2 - e_1)}}{1 + e^{+\beta(e_2 - e_1)}} = \frac{e^{-\beta e_1}}{e^{-\beta e_2} + e^{-\beta e_1}}, \tag{3.39}$$

$$\begin{aligned}
\tilde{\gamma}_{1,2} + \tilde{\gamma}_{2,1} &= \frac{2}{\hbar} (J_1(\omega_{21}) + J_2(\omega_{21})) \left((1 + N(\omega_{21})) + N(\omega_{21}) \right) \\
&= \frac{2}{\hbar} (J_1(\omega_{21}) + J_2(\omega_{21})) \coth \left(\frac{\beta \hbar \omega_{21}}{2} \right),
\end{aligned} \tag{3.40}$$

$$\tilde{\gamma}_0 = \frac{1}{2} \lim_{\omega_{21} \rightarrow 0} (\tilde{\gamma}_{1,2} + \tilde{\gamma}_{2,1}). \tag{3.41}$$

The last coefficient can be evaluated using (2.99) and (3.33) as follows:

$$\begin{aligned}
\tilde{S}_{1,2} - \tilde{S}_{2,1} &= \sum_j S_j(\omega_{12}) - S_j(\omega_{21}) \\
&= \sum_j +\frac{i}{2} (\gamma_j(\omega_{12}) - \gamma_j(\omega_{21})) - i(\Gamma_j(\omega_{12}) - \Gamma_j(\omega_{21})) \\
&= \sum_j -\frac{i}{\hbar} J_j(\omega_{21}) - \frac{i}{\hbar^2} \sum_k |g_{j,k}|^2 I(\omega_{j,k}) \\
&= \sum_j -\frac{i}{\hbar} J_j(\omega_{21}) - \frac{i}{\pi \hbar} \int_0^\infty d\omega' J_j(\omega') I(\omega') \\
&= -\frac{2\omega_{21}}{\pi \hbar} \text{PV} \int_0^\infty d\omega' \frac{J_1(\omega') + J_2(\omega')}{\omega_{21}^2 - \omega'^2} (2N(\omega') + 1) \\
&= -\frac{2\omega_{21}}{\pi \hbar} \text{PV} \int_0^\infty d\omega' \frac{J_1(\omega') + J_2(\omega')}{\omega_{21}^2 - \omega'^2} \coth \left(\frac{\beta \hbar \omega'}{2} \right),
\end{aligned} \tag{3.42}$$

where PV stands for the *Cauchy principal value* and the function $I(\omega)$ appearing in the second and third lines equals to

$$\begin{aligned}
I(\omega) &= + (1 + N(\omega)) \int_0^\infty d\tau e^{+i(\omega_{12} - \omega)\tau} + N(\omega) \int_0^\infty d\tau e^{+i(\omega_{12} + \omega)\tau} \\
&\quad - (1 + N(\omega)) \int_0^\infty d\tau e^{+i(\omega_{21} - \omega)\tau} - N(\omega) \int_0^\infty d\tau e^{+i(\omega_{21} + \omega)\tau} \\
&= + (1 + N(\omega)) \int_0^\infty d\tau e^{-i(\omega_{21} + \omega)\tau} + N(\omega) \int_0^\infty d\tau e^{-i(\omega_{21} - \omega)\tau} \\
&\quad - (1 + N(\omega)) \int_0^\infty d\tau e^{+i(\omega_{21} - \omega)\tau} - N(\omega) \int_0^\infty d\tau e^{+i(\omega_{21} + \omega)\tau} \\
&= + (1 + N(\omega)) \left(\pi \delta(\omega_{21} + \omega) - i \text{PV} \frac{1}{\omega_{21} + \omega} \right) \\
&\quad + N(\omega) \left(\pi \delta(\omega_{21} - \omega) - i \text{PV} \frac{1}{\omega_{21} - \omega} \right) \\
&\quad - (1 + N(\omega)) \left(\pi \delta(\omega_{21} - \omega) + i \text{PV} \frac{1}{\omega_{21} - \omega} \right) \\
&\quad - N(\omega) \left(\pi \delta(\omega_{21} + \omega) + i \text{PV} \frac{1}{\omega_{21} + \omega} \right) \\
&= -i \text{PV} (2N(\omega) + 1) \frac{2\omega_{21}}{\omega_{21}^2 - \omega^2} + \pi \delta(\omega_{21} + \omega) - \pi \delta(\omega_{21} - \omega).
\end{aligned} \tag{3.43}$$

$\wp_{3,3}$ and $\wp_{4,4}$ are found to be constants of the open system dynamics in (3.37). Besides this, $\wp_{1,1}$ and $\wp_{2,2}$ seem to go to nonzero constant values too in the asymptotic limit. On the other hand, all the other elements of density matrix vanish when t goes to infinity. Let's show it more clearly by checking the stationary state that is obtained by taking the left-hand side of master equation (2.103) as zero:

$$\begin{aligned}
\rho^\infty &= \wp_{3,3}(0)|e_3\rangle\langle e_3| + \wp_{4,4}(0)|e_4\rangle\langle e_4| \\
&\quad + \frac{1 - \wp_{3,3}(0) - \wp_{4,4}(0)}{\tilde{\gamma}_{1,2} + \tilde{\gamma}_{2,1}} (\tilde{\gamma}_{2,1}|e_1\rangle\langle e_1| + \tilde{\gamma}_{1,2}|e_2\rangle\langle e_2|) \\
&= \wp_{3,3}(0)|e_3\rangle\langle e_3| + \wp_{4,4}(0)|e_4\rangle\langle e_4| \\
&\quad + \frac{1 - \wp_{3,3}(0) - \wp_{4,4}(0)}{e^{-\beta e_1} + e^{-\beta e_2}} (e^{-\beta e_1}|e_1\rangle\langle e_1| + e^{-\beta e_2}|e_2\rangle\langle e_2|).
\end{aligned} \tag{3.44}$$

For the initial states satisfying $\wp_{3,3}(0) = e^{-\beta e_3} / \sum_i e^{-\beta e_i}$ and $\wp_{4,4}(0) = e^{-\beta e_4} / \sum_i e^{-\beta e_i}$, this stationary state turns out to be the thermal state. However, it doesn't mean that thermalization is the underlying mechanism for this result. Actually, a partial dephasing appears to be in charge: environment washes out all the coherence in the basis of $\{|e_3\rangle, |e_4\rangle\}$, while it imposes a detailed balance between $|e_1\rangle$ and $|e_2\rangle$. As none of the eigenoperators of H_{HB} that corresponds to a transition from and/or to e_3 or e_4 survives in (3.34), environment can only exchange information with these two energy levels and this results in a partial dephasing in the associated energy eigenstates. On the other hand, same environment can exchange heat with the remaining energy levels since there are non-zero eigenoperators for these transitions and so, it equilibrates energy eigenstates $|e_1\rangle$ and $|e_2\rangle$.

3.3.2 Nonlocal proton-phonon coupling

We will extend the open system dynamics to include the oscillations of donor-acceptor separation R_{12} in what follows. Assume that $\hat{u}_j$ is the displacement of the X_j from its reference position. Then the deviation of R_{12} from its equilibrium value R_{12}^{eq} can be defined as $\Delta R_{12} = \hat{u}_2 - \hat{u}_1$. By considering this, let's expand the hopping constant J_{12} given in (3.12) about the point $R_{12} = R_{12}^{\text{eq}}$:

$$\begin{aligned}
J_{12}(r_1, R_{12}, \phi_{12}) &\approx J_{12}(r_1, R_{12}^{\text{eq}}, \phi_{12}) + \left. \frac{\partial J_{12}}{\partial R_{12}} \right|_{R_{12}=R_{12}^{\text{eq}}} (\hat{u}_2 - \hat{u}_1) \\
&\equiv J_{12}^{(0)}(R_{12}^{\text{eq}}) + J_{12}^{(1)}(R_{12}^{\text{eq}})(\hat{u}_2 - \hat{u}_1).
\end{aligned} \tag{3.45}$$

To reduce in complexity and extent, we assume that X_1 is stationary ($\hat{u}_1 = 0$) and the mass of X_2 equals to the reduced mass between X_1 and X_2 ($m_{X_2} = \mu$). Then, we switch into the second-quantization representation of $\hat{u}_2$ replacing it with $\sum_k \sqrt{\hbar/(2\mu\Omega_k)}(d_{2,k}^\dagger + d_{2,k})$ where Ω_k are the frequencies of the oscillation of R_{12} , and $d_{2,k}^\dagger$ and $d_{2,k}$ are respectively the phonon creation and annihilation operators associated with the vibration of X_2 . After this replacement, substitution of (3.45) into (3.15) causes the transformation $H_{HB} \rightarrow H_{HB} + H_I^{nl}$ where the value of parameter J_x in H_{HB} turns out to be $-J_{12}^{(0)}(R_{12}^{eq})/2$ and the H_I^{nl} is a non-local proton-phonon interaction described by

$$H_I^{nl} = \left(\sigma_x^{(1)} \otimes \sigma_x^{(2)} + \sigma_y^{(1)} \otimes \sigma_y^{(2)} \right) \sum_k h_k (d_{2,k}^\dagger + d_{2,k}) \quad (3.46)$$

$$\equiv A_3 \otimes B_3,$$

with h_k equals to $-J_{12}^{(1)}(R_{12}^{eq})\sqrt{\hbar/(8\mu\Omega_k)}$. This new proton-phonon interaction requires to entail the calculation of 3 more eigenoperators of H_{HB} :

$$A_3(0) = -\frac{2J_x}{\sqrt{b^2 + J_x^2}}(|e_1\rangle\langle e_1| - |e_2\rangle\langle e_2|),$$

$$A_3(\omega_{12}) = -\frac{2b}{\sqrt{b^2 + J_x^2}}|e_2\rangle\langle e_1|, \quad (3.47)$$

$$A_3(\omega_{21}) = -\frac{2b}{\sqrt{b^2 + J_x^2}}|e_1\rangle\langle e_2|,$$

that give rise to the emergence of the following Lamb-shift Hamiltonian in addition to the ones given in (3.35):

$$H_{LS}^{(3)} = 4 \left((1 - \zeta)S_3^{1,2} + \zeta S_3^0 \right) |e_1\rangle\langle e_1| + 4 \left((1 - \zeta)S_3^{2,1} + \zeta S_3^0 \right) |e_2\rangle\langle e_2|, \quad (3.48)$$

and the following dissipator in addition to the ones given in (3.36):

$$\begin{aligned} \mathcal{D}^{(3)}[\rho] = & -2(\zeta\gamma_3^0 + (1 - \zeta)\gamma_3^{1,2})(\wp_{3,1}|e_3\rangle\langle e_1| + \wp_{1,3}|e_1\rangle\langle e_3|) \\ & -2(\zeta\gamma_3^0 + (1 - \zeta)\gamma_3^{1,2})(\wp_{4,1}|e_4\rangle\langle e_1| + \wp_{1,4}|e_1\rangle\langle e_4|) \\ & -2(\zeta\gamma_3^0 + (1 - \zeta)\gamma_3^{2,1})(\wp_{3,2}|e_3\rangle\langle e_2| + \wp_{2,3}|e_2\rangle\langle e_3|) \\ & -2(\zeta\gamma_3^0 + (1 - \zeta)\gamma_3^{2,1})(\wp_{4,2}|e_4\rangle\langle e_2| + \wp_{2,4}|e_2\rangle\langle e_4|) \\ & -2(4\zeta\gamma_3^0 + (1 - \zeta)\gamma_3^{1,2} + (1 - \zeta)\gamma_3^{2,1})(\wp_{1,2}|e_1\rangle\langle e_2| + \wp_{2,1}|e_2\rangle\langle e_1|) \\ & -4(1 - \zeta)(\gamma_3^{1,2}\wp_{1,1} - \gamma_3^{2,1}\wp_{2,2})|e_1\rangle\langle e_1| \\ & +4(1 - \zeta)(\gamma_3^{1,2}\wp_{1,1} - \gamma_3^{2,1}\wp_{2,2})|e_2\rangle\langle e_2|. \end{aligned} \quad (3.49)$$

Inclusion of these additional terms into the master equation changes the exact solution from (3.37) to:

$$\begin{aligned}
\rho_{1,1}(t) &= \frac{\tilde{\gamma}_{2,1}}{\tilde{\gamma}_{1,2} + \tilde{\gamma}_{2,1}} (\rho_{1,1}(0) + \rho_{2,2}(0)) \\
&\quad + e^{-\zeta(\tilde{\gamma}_{1,2} + \tilde{\gamma}_{2,1})t} \left(\frac{\tilde{\gamma}_{1,2}}{\tilde{\gamma}_{1,2} + \tilde{\gamma}_{2,1}} \rho_{1,1}(0) - \frac{\tilde{\gamma}_{2,1}}{\tilde{\gamma}_{1,2} + \tilde{\gamma}_{2,1}} \rho_{2,2}(0) \right), \\
\rho_{1,2}(t) &= e^{-i\zeta(\tilde{S}_{1,2} - \tilde{S}_{2,1} + \frac{\omega_{12}}{\zeta})t} e^{-\frac{\zeta}{2} \left(4\frac{1-\zeta}{\zeta} \tilde{\gamma}_0 + \tilde{\gamma}_{1,2} + \tilde{\gamma}_{2,1} \right)t} \rho_{1,2}(0), \\
\rho_{1,3}(t) &= e^{+i\zeta(\tilde{S}_0 - \tilde{S}_{1,2} + \frac{\omega_{31}}{\zeta})t} e^{-\frac{\zeta}{2} \left(\frac{2-\zeta}{\zeta} \tilde{\gamma}_0 - \frac{2\sqrt{1-\zeta}}{\zeta} (\tilde{\gamma}_0 + \tilde{\gamma}_0) + \tilde{\gamma}_{1,2} \right)t} \rho_{1,3}(0), \\
\rho_{1,4}(t) &= e^{+i\zeta(\tilde{S}_0 - \tilde{S}_{1,2} + \frac{\omega_{41}}{\zeta})t} e^{-\frac{\zeta}{2} \left(\frac{2-\zeta}{\zeta} \tilde{\gamma}_0 + \frac{2\sqrt{1-\zeta}}{\zeta} (\tilde{\gamma}_0 - \tilde{\gamma}_0) + \tilde{\gamma}_{1,2} \right)t} \rho_{1,4}(0), \\
\rho_{2,2}(t) &= \frac{\tilde{\gamma}_{1,2}}{\tilde{\gamma}_{1,2} + \tilde{\gamma}_{2,1}} (\rho_{1,1}(0) + \rho_{2,2}(0)) \\
&\quad - e^{-\zeta(\tilde{\gamma}_{1,2} + \tilde{\gamma}_{2,1})t} \left(\frac{\tilde{\gamma}_{1,2}}{\tilde{\gamma}_{1,2} + \tilde{\gamma}_{2,1}} \rho_{1,1}(0) - \frac{\tilde{\gamma}_{2,1}}{\tilde{\gamma}_{1,2} + \tilde{\gamma}_{2,1}} \rho_{2,2}(0) \right), \\
\rho_{2,3}(t) &= e^{+i\zeta(\tilde{S}_0 - \tilde{S}_{2,1} + \frac{\omega_{32}}{\zeta})t} e^{-\frac{\zeta}{2} \left(\frac{2-\zeta}{\zeta} \tilde{\gamma}_0 + \frac{2\sqrt{1-\zeta}}{\zeta} (\tilde{\gamma}_0 - \tilde{\gamma}_0) + \tilde{\gamma}_{2,1} \right)t} \rho_{2,3}(0), \\
\rho_{2,4}(t) &= e^{+i\zeta(\tilde{S}_0 - \tilde{S}_{2,1} + \frac{\omega_{42}}{\zeta})t} e^{-\frac{\zeta}{2} \left(\frac{2-\zeta}{\zeta} \tilde{\gamma}_0 - \frac{2\sqrt{1-\zeta}}{\zeta} (\tilde{\gamma}_0 + \tilde{\gamma}_0) + \tilde{\gamma}_{2,1} \right)t} \rho_{2,3}(0), \\
\rho_{3,3}(t) &= \rho_{3,3}(0), \\
\rho_{3,4}(t) &= e^{-i\omega_{34}t} e^{-2\tilde{\gamma}_0 t} \rho_{3,4}(0), \\
\rho_{4,4}(t) &= \rho_{4,4}(0),
\end{aligned} \tag{3.50}$$

where $\tilde{S}_0 = \tilde{S}_0 - 4S_3^0$, $\tilde{S}_{j',j''} = \tilde{S}_{j',j''} + (4b^2/J_x^2)S_3^{j',j''}$, $\tilde{\gamma}_0 = \tilde{\gamma}_0 + (4J_x^2/b^2)\gamma_3^0$, $\tilde{\gamma}_{j',j''} = \tilde{\gamma}_{j',j''} + (4b^2/J_x^2)\gamma_3^{j',j''}$, and $\tilde{\gamma}_0 = (2J_x^2\sqrt{b^2 + J_x^2}/b^3)\gamma_3^0$. On the other hand, if any temperature difference is not allowed among the three phonon baths, stationary state remains same as below:

$$\begin{aligned}
\rho^\infty &= \rho_{3,3}(0)|e_3\rangle\langle e_3| + \rho_{4,4}(0)|e_4\rangle\langle e_4| \\
&\quad + \frac{1 - \rho_{3,3}(0) - \rho_{4,4}(0)}{\tilde{\gamma}_{1,2} + \tilde{\gamma}_{2,1}} (\tilde{\gamma}_{2,1}|e_1\rangle\langle e_1| + \tilde{\gamma}_{1,2}|e_2\rangle\langle e_2|) \\
&= \rho_{3,3}(0)|e_3\rangle\langle e_3| + \rho_{4,4}(0)|e_4\rangle\langle e_4| \\
&\quad + \frac{1 - \rho_{3,3}(0) - \rho_{4,4}(0)}{e^{-\beta e_1} + e^{-\beta e_2}} (e^{-\beta e_1}|e_1\rangle\langle e_1| + e^{-\beta e_2}|e_2\rangle\langle e_2|),
\end{aligned} \tag{3.51}$$

since $\tilde{\gamma}_{1,2}/(\tilde{\gamma}_{1,2} + \tilde{\gamma}_{2,1})$ and $\tilde{\gamma}_{2,1}/(\tilde{\gamma}_{1,2} + \tilde{\gamma}_{2,1})$ respectively reduce to $\tilde{\gamma}_{1,2}/(\tilde{\gamma}_{1,2} + \tilde{\gamma}_{2,1})$ and $\tilde{\gamma}_{2,1}/(\tilde{\gamma}_{1,2} + \tilde{\gamma}_{2,1})$ as can be shown using

$$\tilde{\gamma}_{1,2} = \frac{2}{\hbar} (J_x^2 J_1(\omega_{21}) + J_x^2 J_2(\omega_{21}) + 4b^2 J_3(\omega_{21})) N(\omega_{21}), \tag{3.52}$$

$$\tilde{\gamma}_{2,1} = \frac{2}{\hbar} (J_x^2 J_1(\omega_{21}) + J_x^2 J_2(\omega_{21}) + 4b^2 J_3(\omega_{21})) (1 + N(\omega_{21})). \tag{3.53}$$

3.3.3 Spectral density function

Asymptotic solutions (3.44) and (3.51) are independent of the spectral density functions. However, as the coefficients given in (3.40), (3.41), (3.42), (3.53) and (3.52) depend on $J_j(\omega)$, so do the exact solutions (3.37) and (3.50). Thus, to return back to the exact solution of each master equation after checking its attractors, we need to elaborate on the form of these functions.

In phenomenological models, spectral densities are often taken to be *Ohmic*, that is to say, they are assumed to scale linearly at low frequencies [166]. Also, to justify the re-normalization of the physical parameters due to the interaction with the high frequency modes of the environment, a cutoff frequency ω_c being much larger than the system transition frequencies is generally introduced into them either in the form of an exponential cutoff function:

$$J(\omega) = \kappa \omega e^{-\omega/\omega_c}, \quad (3.54)$$

or in the form of a Lorentz-Drude cutoff function:

$$J(\omega) = \gamma \omega \frac{\omega_c^2}{\omega_c^2 + \omega^2}, \quad (3.55)$$

where κ and γ are frequency-independent coupling constants.

True open system dynamics, on the contrary, may require more realistic forms of $J(\omega)$, e.g., $J(\omega)$ may scale sub-linearly or super-linearly at low frequencies or it may be a combination of more than one spectral density with different cutoff frequencies. It is possible to construct a realistic spectral density function based on the molecular dynamics simulations or the density functional theory calculations. However, the present work aims to construct a simple and physically insightful model with the minimum number of parameters that can be estimated from comparisons of the predictions of the model with experimental results. Hence, we will not specify the exact form of the spectral density functions in what follows and restrict our analysis to steady state solutions of the master equations under consideration.

3.4 Quantum Correlations in Steady State

Suppose that the proton residing at the site $j = 2$ in Figure 3.1 moves to the site $j = 1$ by either classical hopping or quantum tunneling between the initial and final times t_i and

t_f . In the case of thermally activated classical hopping, it leaves the site $j = 2$ at $t = t_i$, then enters into the exterior space between the sites, and finally reaches the site $j = 1$ at $t = t_f$. Since we do not take into account its presence in the exterior space between the sites, it disappears at the site $j = 2$ at $t = t_i$ and reappears at the site $j = 1$ at $t = t_f$ in our model, i.e., $\rho(t) = |00\rangle\langle 00| = |e_4\rangle\langle e_4|$ where $t_i < t < t_f$. Conversely, it will never enter into the exterior space between the sites in the course of its motion during tunneling, but it will be delocalized between both sites, i.e., $\rho(t) = |\psi\rangle\langle\psi|$ where $|\psi\rangle = \alpha|10\rangle + \beta|01\rangle$ and $t_i < t < t_f$. Note that both $|e_1\rangle$ and $|e_2\rangle$ are such coherent superposition states. In this case, the proton is found at the first site with a probability of $|\alpha|^2$ if its location is measured at any time between t_i and t_f . The decoherence taking place at the end of quantum tunneling, at $t = t_f$, converts the coherent superposition $|\psi\rangle\langle\psi|$ into incoherent superposition $|\alpha|^2|10\rangle\langle 10| + |\beta|^2|01\rangle\langle 01|$ removing the off-diagonal elements from the density matrix.

The unidirectional tunneling process described above can be captured by a snapshot of the state within our pseudo-spin formalism, that is to say, we can detect it by evaluating the off-diagonal elements of the pseudo-spin density matrix at a single time. The advantages of our approach go beyond this. Consider the static structures of pseudo-spins, e.g., the ground state during closed-system dynamics or the thermal state during open-system dynamics. When such a stationary state equals to $|\psi\rangle\langle\psi|$, it means that the proton under consideration delocalizes between the first and second sites over a long time period. This can be regarded as a quantum tunneling of the proton back and forth between these two sites since the state never collapses onto the basis states, which represent the proton localizations at their respective sites. In contrast, we cannot claim a back and forth tunneling event when the steady state is found to be $|\alpha|^2|10\rangle\langle 10| + |\beta|^2|01\rangle\langle 01|$. Hence, in addition to unidirectional proton tunneling events in the short time limit, the bidirectional proton tunneling events in the long time limit are also described by well-defined density matrices.

The steady state given in Eq.s (3.44) and (3.51) is actually an incoherent superposition (or a mixture) of the energy eigenstates $\{|e_m\rangle\}$, i.e., it can be written in the form of $\sum_m Q_m |e_m\rangle\langle e_m|$. Some of these energy eigenstates, $|e_1\rangle$ and $|e_2\rangle$, are coherent superpositions of the computational basis states $|01\rangle$ and $|10\rangle$. Consequently, the steady state obtained in our calculations describes a back and forth proton tunneling

with a probability of $Q_1 + Q_2 = 1 - \wp_{3,3}(0) - \wp_{4,4}(0)$, which goes to unity when the initial state lives in the subspace spanned by $\{|01\rangle, |10\rangle\}$. This means that the steady state quantum correlations originate from back and forth quantum tunneling events.





4. VALIDATION OF THE MODEL IN WATER ICE

This chapter starts in Sec. 4.1 with a brief introduction to the structure of hexagonal water ice and its temperature dependent phase transitions. The next section is devoted to extending the two-qubit model of H-bonding proposed in the previous chapter to the low-temperature proton mobility in a single hexameric H_2O ring. Sec. 4.3 discusses the connection between the actual 3-d structures of water ice and the present 2-d model of a single hexamer. It also explains how the changes related to the violation of the ice rules in a single ring can be used to probe the proton-disordering process in the water ice. At the end of this section, we give the operational meanings of the all measures that we use within our formalism. In particular, the ability to capture the behaviour of coherent proton mobility observed in the experiments comes naturally to the l_1 norm of coherence of the stationary state of the extended master equation. The next section continues evaluating the pairwise pseudo-spin correlations underlying the overall mobility. Finally in Sec.s 4.5 and 4.6, the chapter ends with a discussion of the future directions and concluding remarks.

4.1 Introduction

Dominant phase of the water ice on Earth is *hexagonal ice*, or simply ice I_h . Water molecules in this phase obeys the Bernal-Fowler *ice rules* [54, 55]: (i) each oxygen atom which is in fixed position has four hydrogen atoms in its neighborhood in such a way that (ii) two of the hydrogens are linked to the oxygen via covalent bonds, while the remaining ones are linked to the same oxygen via H-bonds. From the viewpoint of hydrogens, two oxygens are expected to be found around each hydrogen such that the hydrogen has two equally likely positions between these neighboring oxygens. Violation of the ice rules due to an oxygen covalently bonded to less or more than two hydrogens is called as an *ionic defect*, whereas presence of two hydrogens between two neighboring oxygens is said to be a *Bjerrum defect*.

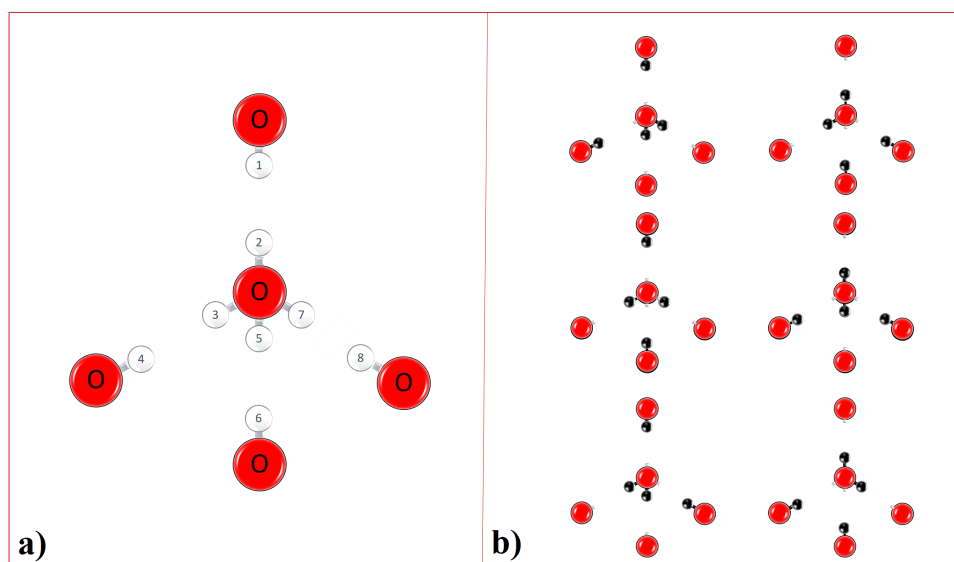


Figure 4.1 : a) A single H_2O tetrahedron in the hexagonal ice. Enumerated locations are for the protons. b) According to the ice rules, only four protons can exist inside this tetrahedron with only six possible configurations.

These rules allow six different proton configurations for each H_2O tetrahedron as shown in Figure 4.1-b. All the tetrahedrons do not need to possess the same configuration globally in ice I_h . In this respect, ice I_h is said to be a proton-disordered phase.

Basic structure of ice I_h can be visualized as a hexameric box whose planes are either chair-form or boat-form hexamers. Its unit cell belongs to the space group $P6_2/mmc$. During cooling, this unit cell undergoes a phase transition into an orthorhombic structure (with space group $Cmc2_1$). In the meantime, the protons rearrange themselves to reach the same configuration in the all H_2O tetrahedrons, i.e., only one of the six configurations in Figure 4.1-b is observed in this phase. Equivalently, H_2O hexamers sharing the same 3-d form and the same orientation switch into a common proton configuration. Hence, unlike the previous phase ice I_h , this new phase is globally proton-ordered and called as ice XI.

Actually, it isn't easy to observe I_h/XI transition as protons are expected to become classically immobile around 100 – 110 K where a glass transformation occurs [1–3]. First calorimetric measurements [252, 253] overcame this problem by using alkali hydroxides as catalyzer and caught the transition at 72 K. According to recent complex dielectric constant measurements of pure water ice reported in [1], hexagonal ice undergoes a phase transition from proton-disordered ice I_h to proton-ordered ice

XI at 58.9 K, whereas a reverse transformation occurs at 73.4 K¹. Since a charge movement should increase the imaginary part of the dielectric constant ϵ'' , these phase transitions were determined by detecting the anomalies in the cooling and warming curves of $\epsilon''(T)$. Whereas a clear peak was observed in the warming curve of $d\epsilon''/dT$ at 73.4 K, a discontinuous change occurred in the slope of cooling curve of $d\epsilon''/dT$ at 58.9 K. The former was reflected in the warming curve of $\epsilon''(T)$ as a continuous increase followed by a change in the slope, while the latter corresponded to a smooth change in the slope of the cooling curve of $\epsilon''(T)$. See Figure 3 in Reference [1] for more details.

Role of the proton tunneling during the proton ordering phase transition in hexagonal ice was first brought into question in a microscopic model [53], in which the motion of the protons was first mapped into a pseudo-spin model, as we do, but then converted to a gauge theory problem. This description allowed the authors to characterize the ordered and disordered phases respectively by confined and deconfined behaviours of ionic defects in the ground state of the system. It was then found that the phase transition under consideration is possible only if the protons tunnel through H-bonds with a rate greater than a critical value (see Figure 6-b in [53]). However, if the proton tunneling really takes place there, the individual tunneling events should be correlated not to violate the ice rules. As a matter of fact, the presence of such correlations in both phases were predicted at low-temperatures based on the results of incoherent quasielastic neutron scattering measurements [56], high precision dielectric constant measurements [58], path integral simulations [60], and lattice-based calculations [61].

In particular, dielectric constant measurements that determined the phase transition temperatures in pure water ice [1] were extended down to 5 K in Reference [58], and an anomaly was observed in the cooling and warming curves of $\epsilon''(T)$ in the form of a minimum around 20 K. This anomaly was interpreted as an evidence of concerted six-proton tunneling as follows. The increase in the imaginary part of the dielectric constant during cooling below 20 K indicates an increase in the proton mobility. However, the monotonic behaviour of the real part of the dielectric constant observed in the same data implies that this proton mobility should preserve the dipole moment

¹These forward and reversed phase transition temperatures were respectively reported to increase to 63.7 K and 78.2 K for D₂O ice [1].

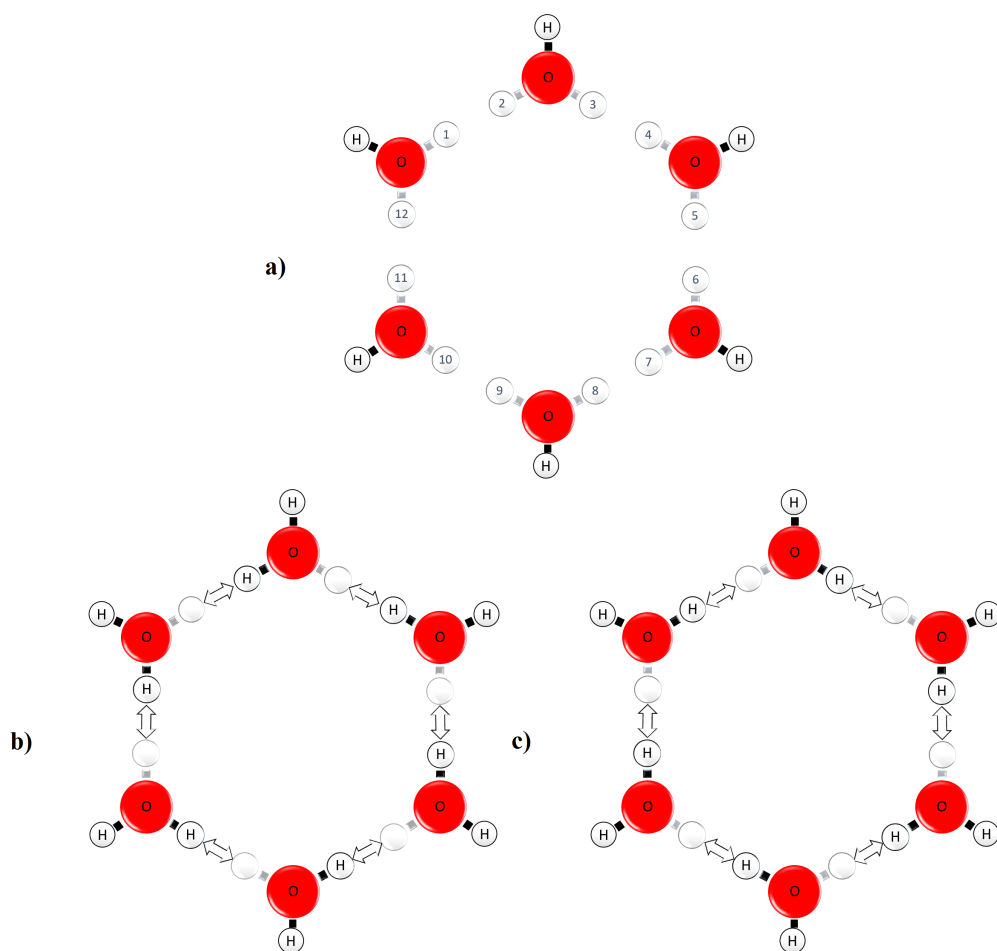


Figure 4.2 : (a) The hexagonal lattice. It is made of OH^- ions in fixed positions relative to each other. Edges can be interpreted as H-bonds. Protons are allowed to live on the enumerated locations. Ice rules are satisfied only by 2 of the six-proton configurations in (b) and (c).

distributions. That is to say, all the H_2O hexamers sharing the same 3-d form and the same orientation simultaneously switch from their common proton configuration to another one through a collective motion of the six protons in such a way that the total dipole moment distributions remain same. Disappearance of this anomaly in the repeat measurements on heavy ice suggests that the collective proton mobility under consideration is actually a back and forth tunneling of protons in groups of six. See Figures 2 and 3 in [58] for details.

4.2 Construction of the Model

Although their unit cells belong to different space groups, the basic structures of both ice I_h and XI can be visualized as a hexameric box whose planes are either chair-form or boat-form 3-d hexamers. To reduce the complexity, we restrict our model to a 2-d

hexagonal ring with a hydroxyl ion (OH^-) resides in each vertex, as shown in Figure 4.2-a. Rigid rotations of the vertices are not taken into account because of the high energy cost. Each edge linking two vertices represents a H-bond and includes two equally likely locations for H^+ ions. These locations (enumerated in Figure 4.2-a) can be regarded as a crystal lattice in which H^+ ions, or protons, move according to the Hamiltonian

$$H_{\text{Hex}} = \sum_{j=1}^{12} W_j n_j - \sum_{j=1}^{12} J_{j,j+1} (a_j^\dagger a_{j+1} + a_j a_{j+1}^\dagger) + \sum_{j=1}^{12} V_{j,j+1} n_j n_{j+1} + \lambda, \quad (4.1)$$

where subscripts are in mod 12, $n_j = a_j^\dagger a_j$ is the proton number operator at lattice site j , $a_j^\dagger$ and a_j are respectively proton creation and annihilation operators that obey the anticommutation relations given in (3.9).

Actually, this Hamiltonian is an extension of Eq. (3.8) introduced in the previous chapter. On-site energy W_j can be taken as the total potential felt by a proton at j th site, i.e., the sum of a Morse potential describing the single (covalent) bond with the adjacent OH^- ion and a Coulomb potential representing the electrostatic attraction to the opposite OH^- ion. $J_{j,j+1}$ stands for the orbital interactions which causes proton tunneling. Its expected dependence on the geometry given in (3.12) implies that $J_{j,j+1}^{(\text{edge})} \gg J_{j,j+1}^{(\text{vertex})}$ where the former is the intermolecular proton tunneling coefficient for the neighbouring sites occupying the same edge, while the latter is the intramolecular proton tunneling coefficient for the successive sites close to the same vertex. As $J_{j,j+1}^{(\text{edge})}$ should already be quite small compared to other coefficients, we can neglect $J_{j,j+1}^{(\text{vertex})}$. This assumption guarantees the absence of any quantum correlation between the motions of protons through individual H-bonds in the isolated hexamer.

Any configuration other than (b) or (c) given in Figure 4.2 violates the ice rules. $V_{j,j+1}$ is introduced to penalize two-proton cases associated with these violations. Edge sharing sites and vertex sharing sites have different penalty coefficients as well as different tunneling coefficients. Presence of two protons on the same edge corresponds to a Bjerrum defect, whereas occupation of both sites near the same vertex corresponds to an ionic defect. As Bjerrum defects require more energy, $V_{j,j+1}^{(\text{edge})} \gg V_{j,j+1}^{(\text{vertex})}$. Finally, λ is a constant responsible for the total intermolecular interactions between vertices, such as Pauli repulsion, Van der Waals interaction, and London dispersion.

Symmetry of the lattice provides that $W_j = W$, $J_{j,j+1}^{(\text{edge})} = J$, $V_{j,j+1}^{(\text{edge})} = V_{\text{inter}}$, and $V_{j,j+1}^{(\text{vertex})} = V_{\text{intra}}$. Applying the Jordan-Wigner transformation for a_j , $a_j^\dagger$, and n_j in Eq. (4.1) as described in Eq. (3.14), we obtain the following pseudo-spin Hamiltonian that preserves the anti-commutation relations (given in Eq. (3.9)),

$$\begin{aligned}
H_{\text{Hex}} = & + \sum_{j=1}^6 J_x \left(\sigma_x^{(2j-1)} \otimes \sigma_x^{(2j)} + \sigma_y^{(2j-1)} \otimes \sigma_y^{(2j)} \right) \\
& + \sum_{j=1}^6 J_z^{(\text{inter})} \left(\sigma_z^{(2j-1)} \otimes \sigma_z^{(2j)} \right) \\
& + \sum_{j=1}^6 J_z^{(\text{intra})} \left(\sigma_z^{(2j)} \otimes \sigma_z^{(2j+1)} \right) \\
& + \sum_{j=1}^{12} B \sigma_z^{(j)} + \tilde{\lambda},
\end{aligned} \tag{4.2}$$

where the superscripts of the Pauli matrices are in mod 12, $J_x = -J/2$, $J_z^{(\text{inter})} = V_{\text{inter}}/4$, $J_z^{(\text{intra})} = V_{\text{intra}}/4$, $B = -(2W + V_{\text{inter}} + V_{\text{intra}})/4$, and $\tilde{\lambda} = \lambda + 6W + 3(V_{\text{inter}} + V_{\text{intra}})/2$.

So far, we have focused only on nearest-neighbor interactions and neglected the further interactions with other neighbors, including the concerted tunneling of the six protons arising from the collective overlap of the orbitals. Unlike such long-range intra-ring interactions, the long-range inter-ring interactions are expected to be non-negligible in real water ice. The electrostatic and topological interactions with adjacent rings should at least have significant effects on the parameters J_x , $J_z^{(\text{inter})}$, $J_z^{(\text{intra})}$, B , and $\tilde{\lambda}$. We assume that the effects of other rings on each of these parameters can be approximated respectively by a single averaged effect. In what follows J_x , $J_z^{(\text{inter})}$, $J_z^{(\text{intra})}$, B , and $\tilde{\lambda}$ are recounted as effective parameters that include mean-field averages.

4.2.1 Asymptotic limit of the open system dynamics

Parameters of the pseudo-spin Hamiltonian (4.2) can be determined using the tools of quantum chemistry taking into account all the details of the electronic structure of water ice. It is also possible to construct a realistic spectral density function based on the molecular dynamics simulations or the density functional theory calculations. However, the present thesis aims to construct a simple and physically insightful model with the minimum number of parameters that can be estimated from comparisons of the predictions of the model with experimental results. To fulfill this aim, our first

task is to consider a minimalistic environment model consisting of just the periodic oscillations associated with the lengths of proton-donating bonds. In what follows we will show that the steady state of the corresponding master equation depends on only two parameters in contrast to previous multi-parameter models and the quantum information theoretic analysis of this state is enough to give a quantitative description of the experimental data.

The O–H stretch vibrations can be incorporated into our model as independent thermal baths around lattice sites with individual self-Hamiltonians given in Eq. (3.17). We assume that the equilibrium positions of the protons are linearly coupled to the positions of the phonons through the interaction Hamiltonian (3.19) and describe the evolution of the state of 12 pseudo-spins using a Markovian master equation as it was done for only 2 pseudo-spins in Sec. 3.3.1.

Before elaborating on the steady state solution of the master equation, let us first examine the role of memory effects in open system dynamics. The Born-Markov approximation can be justified only if the state of pseudo-spins varies over a time scale much longer than the lifetime of the environmental excitations. The vibration of the O–H bond in O–H $\cdots$ O systems has a period of $\simeq 10$ fs, which corresponds to a stretch harmonic frequency of $\simeq 3500$ cm $^{-1}$. Unlike the short-lived (≈ 1 ps) H-bonds in liquid water [254], H-bonds survive sufficiently long in ice I $_h$ and the jump time of protons in these bonds is larger than tens of fs, e.g., is equal to 3.7 ps at 5 K [56]. So, we can describe the picosecond evolution of the pseudo-spins' state ρ on the basis of a Markovian master equation in the Lindblad form.

The normal modes of lattice vibrations are complicated in real water ice. The lattice sites, especially the pair of sites sharing the same vertex, are so close to each other that it is expected to find correlations between them. Presence of the correlations between the individual baths may result in the emergence of quantum correlations between the motions of protons in the course of open system dynamics. However, we would like to restrict our analysis to the importance of classical correlations between the quantum tunneling events on the overall proton mobility. Hence, we ignore the correlations between the individual baths as well as the Hamiltonian parameter $J_{j,j+1}^{(\text{vertex})}$ and assume that each pseudo-spin is associated with an independent environment as in Sec. 3.3.1, i.e., the one-sided Fourier transformation of the thermal bath correlation

function equals to $\Gamma_{jj'} = \Gamma_{jj} \delta_{jj'}$. Furthermore, we focus on a symmetric lattice at a constant temperature T that makes these individual baths identical, so $\Gamma_{jj} = \Gamma$.

The interaction with the environment chosen in this way does not bring the system into a thermal equilibrium in general. On the contrary, it divides 2^{12} -dimensional Hilbert space $\mathcal{H}$ into subspaces $\mathcal{H}_{\mathcal{J}}$ each of which is independently invariant under $\{A_j\}_{j=1}^{12}$ operators². In the asymptotic limit, it provides a detailed balance only inside these subspaces as below:

$$\rho_{\infty} = \sum_{\mathcal{J}} \frac{Q^{(\mathcal{J})}}{Z^{(\mathcal{J})}} \sum_{|\epsilon_m\rangle \in \mathcal{H}_{\mathcal{J}}} e^{-\beta \epsilon_m} |\epsilon_m\rangle \langle \epsilon_m|, \quad (4.3)$$

where

$$H_{Hex} |\epsilon_m\rangle = \epsilon_m |\epsilon_m\rangle, \quad (4.4)$$

$$Q^{(\mathcal{J})} = \sum_{|\epsilon_m\rangle \in \mathcal{H}_{\mathcal{J}}} \langle \epsilon_m | \rho(t=0) | \epsilon_m \rangle, \quad (4.5)$$

$$Z^{(\mathcal{J})} = \sum_{|\epsilon_m\rangle \in \mathcal{H}_{\mathcal{J}}} e^{-\beta \epsilon_m}, \quad (4.6)$$

with $\beta = 1/k_B T$.

One of the subspaces $\mathcal{H}_{\mathcal{J}}$ consists of two special kinds of pseudo-spin states that will be mentioned in the next section, i.e., 2 pseudo-spin states obeying the ice rules and 62 pseudo-spin states corresponding to ionic defects. These state sets will be assumed to map to the proton-ordered and disordered phases of the hexagonal water ice respectively. Hence, we label their union by $\mathcal{H}_{ice}$. If the initial state $\rho(t=0)$ lives only in this 64-dimensional subspace, the state of the pseudo-spins that relax to equilibrium still stays inside the same subspace as follows

$$\rho_{\infty}^{ice} = \sum_{|\epsilon_m\rangle \in \mathcal{H}_{ice}} e^{-\beta \epsilon_m} |\epsilon_m\rangle \langle \epsilon_m| \quad / \quad \sum_{|\epsilon_m\rangle \in \mathcal{H}_{ice}} e^{-\beta \epsilon_m}. \quad (4.7)$$

It is straightforward to show that the athermal attractor above essentially depends on two free parameters, J_x and $J_z^{(intra)}$ as all the energy eigenvalues of subspace $\mathcal{H}_{ice}$ have a common functional dependence on the remaining coefficients included in (4.2).

²Note that A_j are the Hermitian operators coupled to the environment, i.e., Pauli z operators as introduced in Eq. (3.19)

4.3 Implementation of the Model

4.3.1 Extension to the physical system

Before estimating the values of J_x and $J_z^{(\text{intra})}$ using the experimental results found in the literature, we first find a reasonable map between the actual 3-d structures of water ice and the present 2-d model of a single hexamer. If we extended the model by connecting multiple hexamers in 3-d as in the water ice, we would first replace OH^- ions with O atoms in the vertices and increase the number of equally likely locations for protons close to each vertex from two to four, i.e., the local constraints on a single hexamer would change. Keeping this difference in mind, we infer the ordering dynamics in multi-hexamer real structures from the underlying proton dynamics in single hexamers and construct the map between the model and physical system based on the single-hexamer proton relocation events occurring in them.

As mentioned in Sec. 4.1, both of the ice I_h and XI obey the ice rules, but hexagonal rings of ice XI possess a global proton order which is absent in ice I_h , i.e., H_2O hexamers sharing the same 3-d form and the same orientation have also the same proton configuration. This proton order can't be preserved in the presence of proton relocation unless each of these hexamers simultaneously switches into another proton configuration through a collective motion of the six protons. Thus, not only each of the 3-d hexamers, but also the whole ice XI crystal being constituted by them is allowed to be found only in two different configurations. The switch between these two configurations can be mapped to the transition between $|01\rangle^{\otimes 6}$ and $|10\rangle^{\otimes 6}$ pseudo-spin states (respectively depicted in Figures 4.2-b and -c) as both of them require concerted six-proton relocation.

$|01\rangle^{\otimes 6}$ and $|10\rangle^{\otimes 6}$ pseudo-spin states span the whole subspace in which the single hexamer satisfies the ice rules. In a sense, we assume that this ice rule preserving subspace in the model corresponds to the proton-ordered phase in hexagonal water ice.

Ice I_h is composed of 3-d hexamers that also fulfill the ice rules but do not show global correlation. Proton configuration of these hexamers can be achieved from the ones in ice XI by the proton relocation events occurring through H-bonds and keeping the proton number in each hexamer fixed at six. In our model, similar proton relocation events bring the states living inside the ice rule preserving subspace into

another subspace spanned by 62 pseudo-spins representing the ionic defects. Thus, this defective pseudo-spin subspace can be assumed to coincide with the proton-disordered phase in hexagonal water ice.

4.3.2 Model parameters

In our model, each basis state represents a different configuration of the protons. As explained in the previous subsection, the ice rule preserving subspace spanned by $\{|01\rangle^{\otimes 6}, |10\rangle^{\otimes 6}\}$ can be mapped to the XI phase of water ice based on the proton relocation dynamics. Thus, the change in the probability of pseudo-spins to be found inside this subspace can be used as an indicator of the proton-disordering phase transition, e.g., it should be close to unity in XI phase and show a decrease during the transition to I_h phase. We will denote this probability by Q_{BF} in what follows.

As opposed to experimental data on $\epsilon''(T)$, no hysteresis is expected between the warming and cooling curves of $Q_{BF}(T)$ since we focus on the steady state solution of the master equation and do not allow the system under consideration to be driven out of equilibrium due to quantum fluctuations. Moreover, as our model is restricted to a single hexamer, a sharp discontinuity in $Q_{BF}(T)$ is unlikely to occur during phase transition. Hence, instead of a first order phase transition, we anticipate observing a smooth change in the slope of $Q_{BF}(T)$ similar to that of the cooling curve of $\epsilon''(T)$ shown in Figure 3-a in [1] and in Figure 2 in [58]. However, even this smooth change in our finite size system can be treated as evidence of a proton-disordering phase transition since it reflects the true microscopic mechanism driving the proton-disordering process.

In this respect, we fixed our free parameters J_x and $J_z^{(\text{intra})}$ respectively to -1 meV and $+10$ meV to reproduce the expected trend of $Q_{BF}(T)$ using the steady state ρ_{∞}^{ice} as shown in Figure 4.3. The state of pseudo-spins stays inside the ice rule preserving subspace with a probability close to unity while $T < 58.9$ K. The states outside this subspace violate the ice rules and gradually become available between the blue and red lines. Above the temperature of ≈ 73.4 , $Q_{BF}(T)$ shows a sharp decline that corresponds to an ever-increasing population of ice rule violating states.

The behaviour of the von Neumann entropy of $\rho_{\infty}^{ice}(T)$ supports the arguments above for the fixed values of J_x and $J_z^{(\text{intra})}$. The von Neumann entropy defined in (2.117)

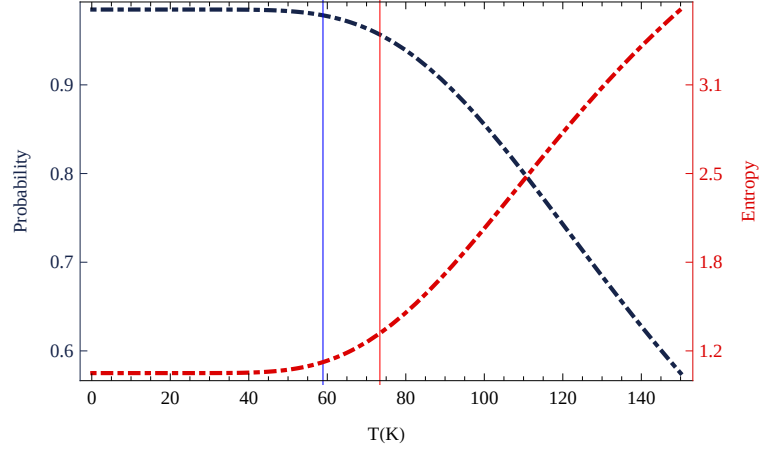


Figure 4.3 : Estimation of the parameters based on the predictions of the model with experiments. $XI \rightarrow I_h$ and $I_h \rightarrow XI$ phase transitions of hexagonal ice respectively occur at 58.9 K and 73.4 K [1]. Vertical solid lines coloured blue and red pinpoint these temperatures. The dashed dark blue curve is $Q_{BF}(T)$, the probability of pseudo-spins to be found inside the ice rule preserving subspace spanned by $\{|01\rangle^{\otimes 6}, |10\rangle^{\otimes 6}\}$ when they are prepared in the steady state ρ_{∞}^{ice} . We set J_x and $J_z^{(intra)}$ respectively to -1 meV and $+10$ meV to observe the violation of ice rules around the phase transition temperatures. The dashed dark red curve shows the temperature dependence of the von Neumann entropy of ρ_{∞}^{ice} .

measures the amount of disorder, uncertainty, or unpredictability of a generic quantum state. Since each basis state of pseudo-spins represents a different proton configuration in our model, $S[\rho_{\infty}^{ice}(T)]$ directly quantifies the proton-disorder. According to Figure 4.3, it remains close to unity until 58.9 K. Note that the pseudo-spins live inside the subspace spanned by $\{|01\rangle^{\otimes 6}, |10\rangle^{\otimes 6}\}$ in the same temperature range³. Then this unit disorder is possible only if the pseudo-spins can exist in only two orthogonal states living inside this subspace with an equal probability⁴, i.e., the configurations given in Figures 4.2-b and -c or two of their coherent superpositions orthogonal to each other are equally likely for the protons. $S[\rho_{\infty}^{ice}(T)]$ rises slowly with further increases of temperature until 73.4 K. Hence, a smooth change occurs in the proton-disorder around 58.9 – 73.4 K. After that, the slope of $S[\rho_{\infty}^{ice}(T)]$ is approximately constant, which shows a rapid increase in the proton disorder.

³So, the steady state has at most two nonzero eigenvalues $Q^{(1)}$ and $(1 - Q^{(1)})$ when $T < 58.9$ K, i.e., $\rho_{\infty}^{ice} = Q^{(1)}|\epsilon_1\rangle\langle\epsilon_1| + (1 - Q^{(1)})|\epsilon_2\rangle\langle\epsilon_2|$ such that $\{|\epsilon_1\rangle, |\epsilon_2\rangle\}$ is any orthogonal state pair living inside ice rule preserving subspace.

⁴ $S[\rho_{\infty}^{ice}] = -Q^{(1)}\log_2 Q^{(1)} - (1 - Q^{(1)})\log_2(1 - Q^{(1)}) = 1$ iff $Q^{(1)} = 1/2$.

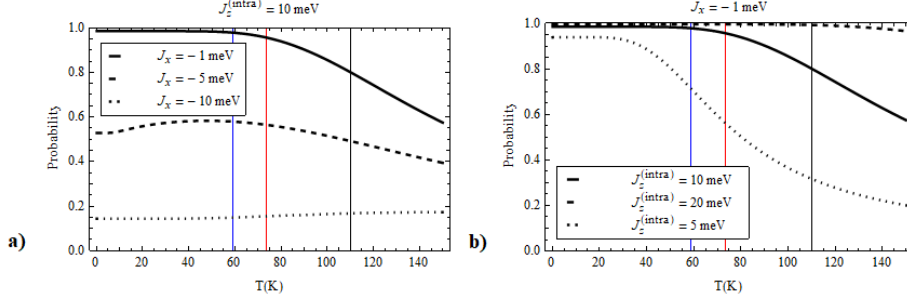


Figure 4.4 : Sensitivity of the behaviour of probability $Q_{BF}(T)$ to the fixed values of free parameters J_x (a) and $J_z^{(intra)}$ (b).

Estimation of the free parameters may still look arbitrary at first glance. However, if we decrease the value of J_x , $Q_{BF}(T)$ cannot get close to unity at any temperature (Figure 4.4-a), which means that the ice rules are always violated and the system can never end up in XI phase. On the contrary, a change in the value of $J_z^{(intra)}$ sets the temperature at which the ice rules begin to be violated, apart from the experimentally determined phase transition temperatures (Figure 4.4-b). Hence, the expected temperature dependence of Q_{BF} exhibits a sensitivity to our free parameters, i.e., deviations from the fixed values are much smaller than the energy of a H-bond, and rule out any prediction preventing the appearance of a slow decline in $Q_{BF}(T)$ from unity around 58.9 – 73.4 K.

Note that the values of $J_x = -1$ meV and $J_z^{(intra)} = +10$ meV are fixed in this way, and do not only stand for the bare coefficients of a single hexamer but also have contributions from the mean-field averages of the effects of the surrounding hexamers.

As mentioned before, it was found in [53] that the phase transition under consideration is possible only if the protons tunnel through H-bonds with a rate greater than a critical value (see Figure 6-b in [53]). Since the protons are expected to become classically immobile around 100 – 110 K where a glass transformation occurs [1–3], this is a reasonable claim. Our predictions shown in Figure 4.4-a suggest that it may be also unlikely for the hexagonal water ice to end up in XI phase unless the tunneling rate is less than another critical value. Thus, further investigation of the quantum aspects of the proton mobility in I_h /XI transition using quantum information theory may yield new knowledge about this topic.

4.3.3 Operational meanings of the measures in use

The l_1 norm of coherence defined in (2.145) quantifies the quantum coherent proton mobility described by the pseudo-spins' state when the proton number is fixed as is in (4.7). That such an operational meaning is possible can be seen by an example. The configurations respectively depicted in Figures 4.2-b and -c are represented by the basis states $|b\rangle \equiv |01\rangle^{\otimes 6}$ and $|c\rangle \equiv |10\rangle^{\otimes 6}$. A transition from one of these configuration to the other requires simultaneous relocations of the six protons to their adjacent empty sites in H-bonds. If this motion takes place in the form of concerted tunneling of six protons, the collective proton delocalization responsible for the concerted tunneling process is described by a quantum coherent superposition⁵ $|\psi\rangle = \alpha|b\rangle + \beta|c\rangle$ with $|\alpha|^2 + |\beta|^2 = 1$. The extent of the quantum character of this motion of the protons is reflected in the density matrix $\rho = |\psi\rangle\langle\psi|$ by the off-diagonal elements $\langle b|\rho|c\rangle = \alpha\beta^*$ and $\langle c|\rho|b\rangle = \alpha^*\beta$. $C_{l_1}[\rho]$ equals to the sum of the absolute values of these elements and is maximum when $\alpha = \beta = 1/\sqrt{2}$. This corresponds to a quantum state in which the protons can be found in each of the two configurations with a probability of 1/2 if their locations are measured.

If the stationary state of the open system is found to be such a coherent superposition, the delocalization described by $|\psi\rangle$ is found to last over a sufficiently long time period that allows us to regard it as a back and forth quantum tunneling event. Hence, $C_{l_1}[\rho_\infty^{ice}]$ quantifies the back and forth proton tunneling events occurring in the steady state⁶.

On the other hand, the relative entropy of coherence defined in (2.140) measures the distinguishability of a density matrix with a modified copy in which the off-diagonal elements are removed by a full dephasing process, i.e., $C_R^{IC}[\rho] = -|\alpha|^2 \log_2 |\alpha|^2 - (1 - |\alpha|^2) \log_2 (1 - |\alpha|^2)$ for $\rho = |\psi\rangle\langle\psi|$. This means that whereas $C_{l_1}[\rho_\infty^{ice}]$ takes into account distinct tunneling pathways independently of each other, $C_R^{IC}[\rho_\infty^{ice}]$ does

⁵Each of the computational basis states $|m\rangle$ represents a different proton configuration in our formalism, by definition. A proton that tunnels through the adjacent sites in a H-bond should be delocalized between these sites at every instant of time until it reaches the opposite site, and this delocalization is described by a coherent superposition of the basis states that represent the proton localizations at the respective sites. Similarly, if six protons simultaneously tunnel from one configuration to another, the collective proton delocalization responsible for this concerted tunneling process is described by the coherent superposition of the basis states that represent the corresponding configurations.

⁶Note that the proton number is fixed in (4.7)

not discriminate between these pathways that rearrange the proton configuration and highlights the overall nonclassical mobility, hence showing a holistic picture.

Now, let's move on from the interpretation of the measures of quantum coherence to the interpretation of the measures of classical and quantum correlations summarized in Sec. 2.3. As each edge in Figure 4.2-a represents a H-bond, the entanglement of formation defined in (2.120) quantifies the entanglement generated by proton tunneling through a H-bond for the reduced state of pseudo-spins lying on the corresponding edge. Besides this, non-zero entanglement of formation between a pseudo-spin pair lying on different edges indicates the inter-bond entanglement between the protons that belong to corresponding H-bonds. The trade-off between intra-bond and inter-bond entanglements can then be investigated using the concurrence as in (2.123).

Additionally, classical correlations generated by classical proton hopping through a H-bond can be quantified by the mutual information $J_{j:j'}$ (see Eq. (2.133)) between the pseudo-spins j and j' lying on the same edge that corresponds to this bond. Moreover, classical correlations between a pair of protons that belong to different H-bonds in the H₂O hexamer can be captured by $J_{j:j'}$ using the reduced state $\rho_{jj'}$ of pseudo-spins j and j' that lie on the edges corresponding to these H-bonds. Then, it becomes possible to quantify the intra- and inter-bond quantum correlations beyond entanglement calculating the $\delta_{j:j'}$ and $D_{G,j:j'}^{ZD}$ (respectively given in (2.134) and (2.137)) for pseudo-spin pairs.

4.4 Results

To provide a first insight into the quantum aspects of proton mobility in a hexameric H₂O loop, we apply the l_1 norm and relative entropy of coherence on ρ_∞^{ice} as shown in Figure 4.5. $C_R^{IC}[\rho_\infty^{ice}]$, which quantifies the distinguishability of ρ_∞^{ice} from its completely decohered version, remains constant throughout the XI phase and steadily decreases with increasing temperature. Hence, there is a rise in the loss of collective quantumness in the global proton mobility starting with the transition from XI phase to I_h phase which lasts thereafter. Conversely, $C_{l_1}[\rho_\infty^{ice}]$, which is the sum of quantum coherences in individual transitions between proton configuration pairs, shows a different behaviour with respect to temperature. It increases with the XI→I_h phase transition and reaches a peak around glass transition. Actually, this behaviour seems to be consistent with the

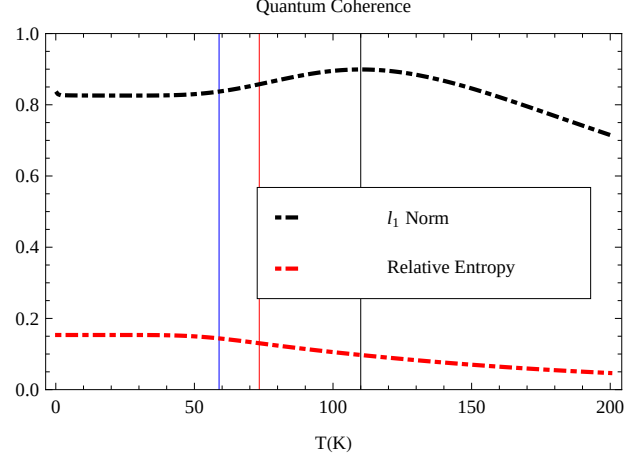


Figure 4.5 : Coherent quantum effects on proton mobility in a H₂O hexamer. Vertical solid lines coloured blue and red indicate the phase transition temperatures [1], whereas the vertical black line is anchored to the temperature of the glass transition at which proton mobility is expected to diminish [1–3].

experimental data related to real proton mobility between 60 – 110 K (see Figure 3-a in [1] and Figure 2 in [58]).

The deviation of the temperature dependence of $C_{l_1}[\rho_{\infty}^{ice}]$ from the experimental data below 60 K is related to the finite size of our model. Although the increase in $\epsilon''(T)$ during the cooling from 20 K to 5 K arises from the increasing number of protons involved in the correlated six-proton tunneling events [58], our results are limited to a single hexamer including only six protons. The loss of similarity between the curves of $C_{l_1}[\rho_{\infty}^{ice}(T)]$ and $\epsilon''(T)$ above 110 K also originates from the restrictions on our model. The rise in $\epsilon''(T)$ after the glass transition [1,58] is likely stem from thermally activated proton hopping, which is outside our scope. Note that the classical motion of the protons enters into our model in the form of incoherent superpositions of pseudo-spins that are generated from coherent superpositions as a result of decoherence.

On the other hand, the uptick in $C_{l_1}[\rho_{\infty}^{ice}]$ between the blue and red solid lines in Figure 4.5 offers fresh insights about the importance of coherent proton mobility on the proton ordering dynamics [53], i.e., although the tunneling coefficient is fixed initially, there is an increase in the amount of coherence generated by tunneling events during the phase transition from proton-disordered ice I_h to proton-ordered ice XI. Also, pinning of $C_{l_1}[\rho_{\infty}^{ice}(T)]$ at a nonzero value below 60 K eliminates the possibility that the pseudo-spins exist in a maximal mixture of the basis states $|01\rangle^{\otimes 6}$

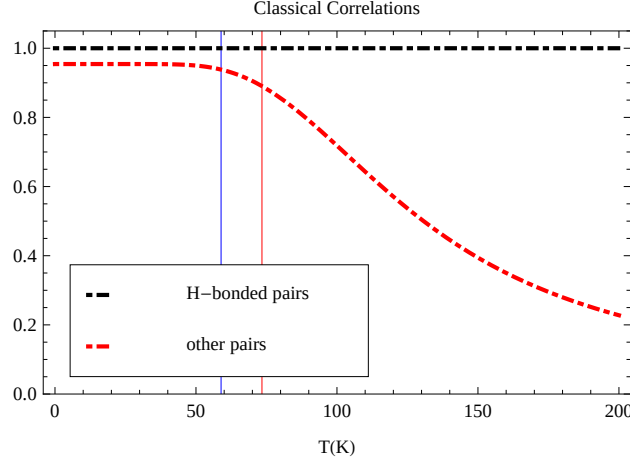


Figure 4.6 : Pairwise classical correlations. The dashed black curve corresponds to the correlations in a single H-bond. Correlations between two protons, each of which belongs to a different H-bond, is shown by the dashed dark red curve.

and $|10\rangle^{\otimes 6}$. Hence, the presence of unit disorder below 60 K in Figure 4.4-a should come from a maximal mixture of two orthogonal superpositions of these basis states, i.e., $\rho_{\infty}^{ice} = 1/2|\psi_{+}\rangle\langle\psi_{+}| + 1/2|\psi_{-}\rangle\langle\psi_{-}|$ where $|\psi_{\pm}\rangle$ are two coherent superpositions such as $(|01\rangle^{\otimes 6} \pm |10\rangle^{\otimes 6})/\sqrt{2}$. Note that each of the superposition states involved in this mixture can be interpreted as a concerted tunneling of six protons back and forth between the configurations represented by the states $|01\rangle^{\otimes 6}$ and $|10\rangle^{\otimes 6}$. This observation is in accordance with the dielectric anomaly measured in the form of a minimum around 20 K [58] where we suspect a concerted quantum tunneling of six protons could be occurring in each hexamer.

A deeper understanding of the overall behaviour of $C_{I_1}[\rho_{\infty}^{ice}(T)]$ requires investigation of the classical and quantum correlations between pseudo-spin pairs respectively shown in Figures 4.6 and -4.7. Only classical correlations appear between the pseudo-spins lying on different edges according to these figures. This is actually what we expect to see as we prevent the formation of quantum correlations between these pseudo-spin pairs by setting the Hamiltonian parameter $J_{j,j+1}^{(\text{vertex})}$ to zero and taking the O–H stretch vibrations as independent from each other. What is unexpected about these results is that the probability change observed in Figure 4.3 resembles the temperature-dependent behaviour of the classical correlations in Figure 4.6, where the mutual information J of the reduced state of the corresponding pseudo-spins is constant around unity throughout the XI phase and starts falling down during the XI/I_h

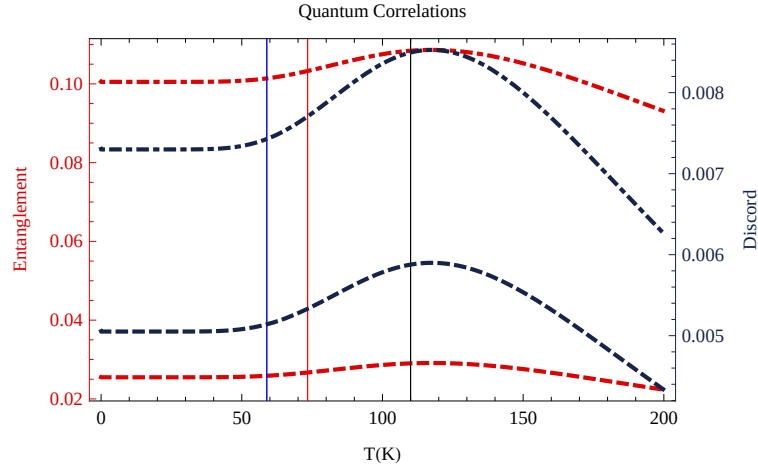


Figure 4.7 : Pairwise quantum correlations in a single H-bond. There is no quantum correlation found between proton pairs belonging to different H-bonds. The dot-dashed dark red curve is the concurrence between pseudo-spins lying on the same edge, while the dashed red curve is the entanglement of formation between them. Their quantum correlations beyond entanglement are measured by quantum discord and its geometric measure, which are displayed respectively as dot-dashed and dashed dark blue curves.

transition. This means that there is an approximately maximal amount of classical correlations between the motions of two different protons that belong to different H-bonds below 60 K.

Beside this, classical correlations between edge sharing pseudo-spins seem to be fixed at unity independent of the temperature. Hence, classical correlations generated by the proton motion in an individual H-bond are invariant under any change in temperature. On the other hand, regardless of the measure that is used to quantify quantum correlations between edge sharing pseudo-spin pairs, quantum correlations generated by proton tunneling through individual H-bonds are found to be quite low. However, each curve in Figure 4.7 shows a similarity with coherent proton dynamics described by $C_{I_1}[\rho_{\infty}^{ice}(T)]$ in Figure 4.5. Thus, although the quantum correlations in H-bonds seem to be insignificant when compared to their classical counterparts, temperature dependence of quantum coherent proton mobility still originates from them.

Based on these observations, we surmise that all the individual proton tunnelings throughout six H-bonds found in a single hexamer become classically correlated at

low temperatures. These correlations start to weaken during the phase transition from proton-ordered phase ice XI to proton-disordered phase ice I_h. At the same time, quantum correlations between H-bonded atoms become stronger, reaching a maximum during the glass transition around 110 – 120 K.

4.5 Discussions

The effect of O–H stretch vibrations is incorporated into the model in the form of Holstein-type local phonon-proton couplings. Such local interactions can originate from the second quantization of the small site displacements after expanding on-site energies W_j around some reference set of coordinates. However, not only on-site energies, but also transfer integrals $J_{j,j+1}^{(\text{edge})}$ are likely to be perturbed by the vibrations. Moreover, perturbations of the latter should have a different kind of nature, which can be described by non-local phonon-proton couplings well-known as Peierls-type interaction. Unlike the Holstein-type interaction, this interaction isn't necessarily destructive and can facilitate proton tunneling between sites. In fact, the fluctuations of the O–H bond lengths are expected to have a significant effect on the proton dynamics in real water ice structures. A natural direction to pursue future work is a new quantum master equation approach to open system dynamics of the protons in the presence of both local and non-local phonon couplings. Such a mixed Holstein-Peierls model will reveal the competition between the local and non-local phonon couplings that may be key to understanding this system.

Our first attempts to develop a mixed Holstein-Peierls model using a two qubit model system in Sec. 3.3.2 show that although nonlocal couplings change the dynamics of the system, the static structure of the final steady state remains same. Hence, the bond fluctuations appear not to affect the predictions of the present model unless we do not move on to probe the true proton dynamics around the hexagonal ring in real-time.

The simple pseudo-spin model approach can be readily employed together with the quantum chemical techniques treating electrons and protons quantum mechanically. This additional technique will allow us to give a realistic estimate of both the parameters of the self-Hamiltonian and the form of the spectral density function. It

will then be possible to probe the true proton dynamics around the hexagonal ring in real-time. This offers a new opportunity that previous studies didn't yield.

4.6 Conclusion

We expended the pseudo-spin model introduced in Ch. 3 to investigate both the possibility and the nature of concerted six-proton tunneling in a hexameric H_2O employing the tools of quantum information theory. We demonstrated that the static structure of the final steady state of the chosen master equation depends on only two parameters and the quantum information theoretic analysis of this state is sufficient to give a quantitative description of experimental data.

The role of the external environment on the concerted six-proton tunneling was clearly unveiled as this tunneling process was not imposed by the self-Hamiltonian but emerged naturally in the long-time limit of the low-temperature dynamics of the open system. Thus, phonon-assistance was found to be central in driving the concerted proton tunneling up to the temperature of the phase transition from ice XI to ice I_h . Moreover, it was found to be associated with the emergence of ice rules governing the arrangement of atoms in water ice.

Remaining within the framework of the pseudo-spin model enabled us to approach the correlation problem using the tools of quantum information theory. In turn, we inferred that the l_1 norm of coherence is sufficient to capture the behaviour of coherent proton mobility observed in experiments [1, 58]. We also discriminated between quantum and classical correlations in concerted proton tunneling. It was found that the correlations between six proton tunneling events are not inherently quantum in character. Instead, individual tunneling events were allowed to be classically correlated only. Low rates and strong correlations were observed for quantum tunneling events below a critical temperature corresponding to phase transition. Beyond this critical temperature, simulations showed a weakening in the correlations, but an increase in rates. Overall this induces a total increment in the coherent proton mobility until achieving a full proton disorder.

The finding sheds light on the nature of correlations between the individual tunneling events that can't be addressed by previous studies. This also paves the way to

investigating the proton dynamics in real-time to advance our understanding of many-proton tunneling in water ice.



5. APPLICATION TO AN INDUCED-FIT ENZYME CATALYSIS

This chapter starts in Sec. 5.1 with a presentation of the whole biological scenario that we consider. There, we examine a generic molecular recognition event in which two H-bonds are formed between a nucleotide and a multifunctional enzyme that requires the tautomerization of the nucleotide to execute a different biological function which is outside the scope of this thesis. Consistent with the induced-fit model, we allow a significant conformational change in the binding site of the enzyme. Finally, we compare the time-scales of the enzyme's subsequent conformational transitions occurring during the catalysis of nucleotide tautomerization.

In the beginning of Sec. 5.2, we describe the tautomeric transitions of the nucleotide extending the two-qubit model of H-bonding proposed in Ch. 3 to a global time-independent Hamiltonian. We continue this section giving the details of our master equation approach in Sec. 5.2.2. Throughout this subsection, we also explain the effects of internal nucleotide dynamics on the intramolecular proton dynamics. We clarify the role of enzyme's conformation in the intermolecular proton dynamics at the end of Sec. 5.2.

We estimate numerical values for the model parameters in Sec. 5.3. We give the details of the sequence of state transformations and the solution of the master equation for spontaneous, water-mediated and enzyme-catalyzed reactions in Sec. 5.4. In particular, we show that classical changes induced in the binding site of the enzyme spreads the quantum correlations among all of the four H-bonded atoms thanks to the directionality of H-bonds. If the enzyme rapidly returns to its initial state after the binding stage, the nucleotide ends in a new transition state corresponding to a quantum superposition. Open quantum system dynamics can then naturally drive the reaction in the forward direction from the substrate to the product without needing any additional catalytic activity. We find that in this scenario the enzyme lowers the activation energy

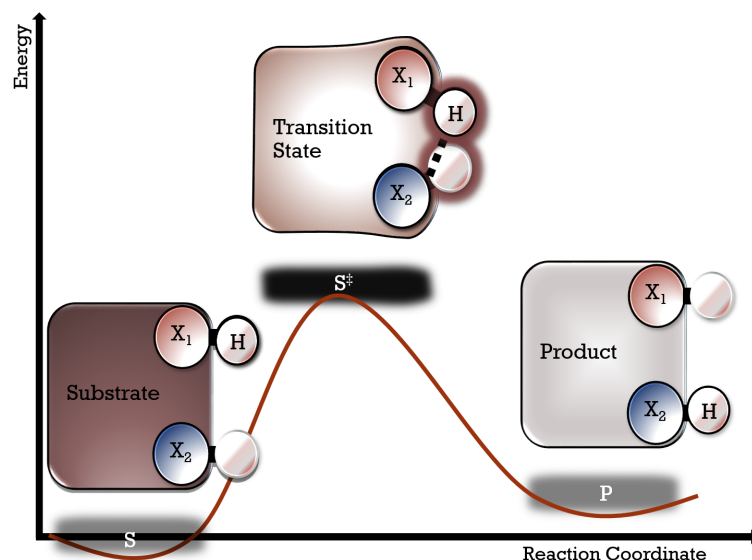


Figure 5.1 : A substrate (S), transition state structure ($S^\ddagger$), and product (P) in a generic tautomerization reaction. S and P are constitutional isomers. Although the intermolecular conversion from S to P is nothing more than the movement of a proton from X_1 to X_2 , a more reactive intermediate ($S^\ddagger$) is involved in the reaction. Since $S^\ddagger$ corresponds to a saddle point on the potential energy surface, the spontaneous tautomerization occurs very slowly.

so much that there is no energy barrier left in the tautomerization, even if the quantum correlations quickly decay.

5.1 Biological scenario

We consider a generic nucleotide existing in its major tautomeric form as the substrate (S) in a putative tautomerization event as shown in Figure 5.1. Here, S is converted to a product (P) that corresponds to a minor tautomeric form of the molecule originating from the relocation of a proton from one electronegative atom/group (X_1) to another (X_2) like oxygen or nitrogen. A direct relocation due to the tunneling is not possible because of the large bond angle $\phi_{12} \equiv \angle HX_1X_2$. Thus, the molecule undergoes a conformational change resulting in an unstable intermediate structure denoted by $S^\ddagger$. This new conformation of the molecule allows a bond angle smaller than $\pi/2$ that facilitates orbital interactions and proton tunneling between X_1 and X_2 . However, as $S^\ddagger$ corresponds to the highest potential energy along the reaction coordinate, this tautomerization reaction is not likely to occur on its own.

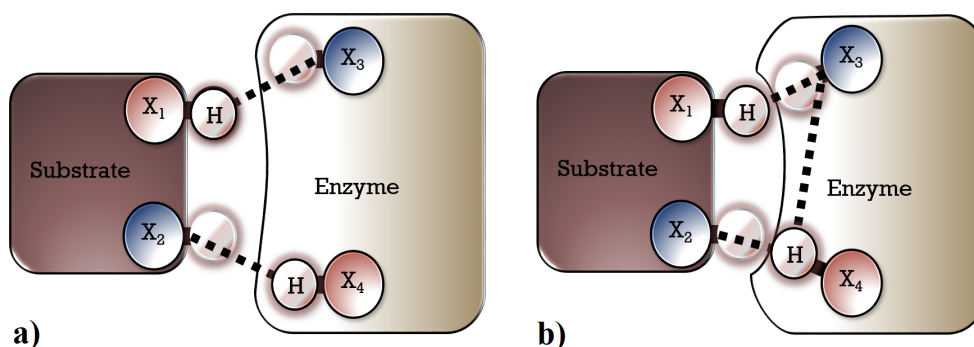


Figure 5.2 : The generic induced-fit model under consideration. (a) Hypothetical interaction where two intermolecular H-bonds are formed between an enzyme and the substrate. (b) Changes occur in the binding site of the enzyme in accordance with the induced-fit model. Note that not only ϕ_{13} and ϕ_{42} , but also ϕ_{43} becomes smaller.

In aqueous solution, water molecules can mediate the interconversion from S to P in a two-step reaction consisting of subsequent protonation and deprotonation processes. If a proton-donating water molecule interacts with X₂ first, a cation intermediate $S^{\ddagger}_{+}$ precedes the P. Otherwise, the water-mediated reaction involves the formation of an anion intermediate $S^{\ddagger}_{-}$. However, P is expected to be rapidly converted back to S in both cases. Hence, the probability of P in the equilibrium should not be high enough for an enzyme that requires P to initialize its catalytic function.

Conversely, consider now the enzyme-catalyzed tautomerization during a generic induced-fit recognition event where two H-bonds are formed between the enzyme (E) and S (Figure 5.2). The atoms/molecules X₃ and X₄ that participate in this intermolecular interaction are continuously tilted by the interaction until the bond angles ϕ_{13} and ϕ_{42} reach to the values that maximize the strength of intermolecular H-bonds. We assume here that (i) the binding site of E returns back to its initial configuration at end of the first catalytic stage and (ii) the following second catalytic stage cannot be initialized unless S is converted to P before that moment.

We can comprehend time-scales associated with the conformational transitions of the enzyme's binding site by using the following analogy. We visualize an imaginary spring that connects the unoccupied and the occupied proton locations near X₃ and X₄ in the binding site (see Figure 5.2-a). The first conformational change that occurs according to the induced-fit mechanism (Figure 5.2-b) corresponds to the compression of the spring, whereas the second conformational change that returns the enzyme back to its native conformation corresponds to the extension of the spring. Note that

there is a close relationship between the first conformational change and the binding energy of intermolecular H-bonds: the initial binding energy of weak intermolecular H-bonds triggers the first conformational change that in turn increases the strength and the binding energy of H-bonds. Thus, the binding energy should be visualized as an external force exerted on the spring that is responsible for the compression of the spring from its equilibrium length, as it increases with compression. However, the deformation of the enzyme's conformation stops at the end of the binding stage, which means that the compression of the spring should come to an end at this point. This can be provided only by a restoring force that applies to the spring and quantifies the enzyme's tendency to return back to its initial state. This restoring force should be initially smaller but increases faster than the external force during compression. That is to say, the spring under consideration is a nonlinear spring that hardens as it is compressed. In this case, the compression of the spring constantly decelerates and finally stops when the restoring force becomes equal to the external force.

The enzyme - substrate complex does not become frozen after the binding stage. Instead, the molecules decouple from each other and the enzyme undergoes the second conformational change. This corresponds to the point when the external force exerted on the spring disappears and the spring immediately starts to extend to reach its equilibrium length. The extension motion is rapid compared to the preceding compression as the net force that drives the process is not the difference between the external and restoring forces but the restoring force itself. Thus, it is plausible to describe the conformational change that decouples the binding site of the enzyme from the substrate and converts it back to the initial state on a time-scale τ_2 that is much smaller than time-scale τ_1 over which the enzyme - substrate complex reaches the optimum intermolecular binding energy.

5.2 Physical model

5.2.1 Tautomeric transitions due to the motion of protons

Here, each of the locations allowed for the proton in Figure 5.1 is regarded as a pseudo-spin and motion of the proton between these locations is described by the Hamiltonian $H_{HB}^{(12)}$ given in 3.15. Then, we collect all the other degrees of

freedom apart from the two pseudo-spins in the molecular structure into a single macromolecular configuration C and label its state as $|\zeta\rangle_C$. That is to say, the whole state $|N\rangle$ representing the nucleotide's tautomeric form $N = \{S, P, S_-^\ddagger, S_+^\ddagger, \dots\}$ is taken to be the product of its configuration state $|\zeta_N\rangle_C$ and its pseudo-spin state $|\psi_N\rangle_{12}$.

Let's first consider the states of S and P . To prevent a direct transition between these forms, both ϕ_{12} in S and ϕ_{21} in P are taken to be $\pi/2$. Then the pseudo-spin coupling constant J_x in $H_{HB}^{(12)}$ (that equals to minus half of the hopping constant J_{12} given in (3.12)) vanishes in either form, i.e., $|\psi_S\rangle_{12} = |10\rangle_{12}$ and $|\psi_P\rangle_{12} = |01\rangle_{12}$. In other words, the H-bonding interaction between X_1 and X_2 does not have a quantum character. When one of the forms is converted to the other, the energy of this classical interaction also changes. However, no significant difference in the molecular structure is expected due to this change. Thus, we can assign the same configuration state $|G\rangle_C$ to both forms: $|S\rangle = |G\rangle_C \otimes |10\rangle_{12}$ and $|P\rangle = |G\rangle_C \otimes |01\rangle_{12}$.

Similarly, neither the protonation of X_2 nor the deprotonation of X_1 is likely to alter the rest of the molecular structure. Hence, the ionic transition states $S_-^\ddagger$ and $S_+^\ddagger$ can respectively be described by $|G\rangle_C \otimes |00\rangle_{12}$ and $|G\rangle_C \otimes |11\rangle_{12}$. Conversely, a more energetic configuration, $|E\rangle_C$, should be assigned to the neutral transition state $S^\ddagger$ as the proton tunneling between X_1 and X_2 requires a bond angle less than $\pi/2$ as is given in Figure 5.1. Then we end up with $|S^\ddagger\rangle = |E\rangle_C \otimes |e_1\rangle_{12}$ where $|e_1\rangle_{12}$ is of the form given in the first line of Eq. (3.16).

Note that the ground and excited states of the molecular configuration, $|G\rangle_C$ and $|E\rangle_C$, which we can visualize them like macromolecular logic qubits $|0\rangle_L$ and $|1\rangle_L$, need not include any quantum degree of freedom.

At this point, it is possible to describe tautomerization processes as transitions between energy levels of a single Hamiltonian constructed as:

$$\begin{aligned} H_N = & E_g |G\rangle_C \langle G| + E_e |E\rangle_C \langle E| \\ & + |G\rangle_C \langle G| \otimes H_{HB}^{(12)}(r, R, \phi = \pi/2) \\ & + |E\rangle_C \langle E| \otimes H_{HB}^{(12)}(r', R', \phi' < \pi/2), \end{aligned} \quad (5.1)$$

with an eigensystem:

$$\begin{cases} \varepsilon_1 = E_g + W_1 + \lambda_{12}, & |\varepsilon_1\rangle = |G\rangle_C \otimes |10\rangle_{12} \equiv |S\rangle, \\ \varepsilon_2 = E_g + W_2 + \lambda_{12}, & |\varepsilon_2\rangle = |G\rangle_C \otimes |01\rangle_{12} \equiv |P\rangle, \\ \varepsilon_3 = E_e + \varepsilon_-, & |\varepsilon_3\rangle = |E\rangle_C \otimes |\varepsilon_-\rangle_{12} \equiv |S^\ddagger\rangle, \\ \varepsilon_4 = E_e + \varepsilon_+, & |\varepsilon_4\rangle = |E\rangle_C \otimes |\varepsilon_+\rangle_{12} \equiv |\tilde{S}^\ddagger\rangle, \\ \varepsilon_5 = E_g + \lambda_{12}, & |\varepsilon_5\rangle = |G\rangle_C \otimes |00\rangle_{12} \equiv |S_-^\ddagger\rangle, \\ \varepsilon_6 = E_g + V_{12} + \lambda_{12}, & |\varepsilon_6\rangle = |G\rangle_C \otimes |11\rangle_{12} \equiv |S_+^\ddagger\rangle, \\ \varepsilon_7 = E_e + \lambda'_{12}, & |\varepsilon_7\rangle = |E\rangle_C \otimes |00\rangle_{12} \equiv |\tilde{S}_-^\ddagger\rangle, \\ \varepsilon_8 = E_e + V'_{12} + \lambda'_{12}, & |\varepsilon_8\rangle = |E\rangle_C \otimes |11\rangle_{12} \equiv |\tilde{S}_+^\ddagger\rangle, \end{cases} \quad (5.2)$$

where $\varepsilon_- = e_1(r', R', \phi' < \pi/2)$ and $\varepsilon_+ = e_2(r', R', \phi' < \pi/2)$ (see (3.16) for the details.) Also, $\tilde{S}^\ddagger$, $\tilde{S}_-^\ddagger$, and $\tilde{S}_+^\ddagger$ are the first possible excitations from $S^\ddagger$, $S_-^\ddagger$ and $S_+^\ddagger$ respectively.

In this way, we represent S, P, and all the possible transition states between them by different eigenstates of a single Hamiltonian¹. Non-experts may be unfamiliar with the use of a time-independent Hamiltonian to describe the time evolution between the different tautomeric forms of the nucleotide. However, this type of modeling, which assigns a global time-independent Hamiltonian for a transition from an initial to a final state is quite common in quantum thermodynamics (see Sec. II in [255] and references 27–44 therein.)

5.2.2 Open quantum system dynamics

The configuration of atoms/groups and the pseudo-spins are expected to be coupled to different phonon environments in different ways. For example, the configuration states are likely to exchange energy with a heat bath B in a reversible manner

$$H_{BC} = \sum_k g_k |G\rangle_C \langle E| \otimes b_k^\dagger + g_k^* |E\rangle_C \langle G| \otimes b_k, \quad (5.3)$$

where $b_k^\dagger$ and b_k are phonon creation and annihilation operators associated with the k th oscillator mode of the bath. These bath operators can be related to the rotational vibration modes such as bending or libration modes that change the orientation of X_1 and X_2 . Such rotational vibrations may occur due to the collisions with solvent

¹This may seem like an oversimplification, since each of these different molecules is actually corresponding to the ground state of a separate Hamiltonian with many electronic levels. However, it is just a projection of all the extrema of the potential energy surfaces of the nucleotide onto a single energy spectrum, representing a unification of all the tautomeric forms along possible reaction coordinates from S to P into a hypothetical generic molecule. Such a generic model shares a common feature with important conceptual tools such as the reaction coordinate diagrams widely used in chemistry.

molecules or as a result of the intramolecular conformation dynamics. As X_1 and X_2 are covalently bonded to the rest of the molecule, these vibrations should be dependent on the orientation of the whole other atoms/groups close to them.

However, dominant environmental effect on the proton should be originated from the X_j –H stretch vibrations that aren't expected to affect rest of the molecule in a significant way. Charge fluctuations in the surrounding molecules may drive these oscillations and we can incorporate them into our model by coupling the position of the proton linearly to the equilibrium positions of phonons through

$$H_{\tilde{B}S} = \sum_{j=\{1,2\}} \sigma_z^{(j)} \otimes \sum_k \tilde{g}_{k,j} (\tilde{b}_{k,j}^\dagger + \tilde{b}_{k,j}). \quad (5.4)$$

where $\tilde{b}_{k,j}^\dagger$ and $\tilde{b}_{k,j}$ are phonon creation and annihilation operators of a second heat bath $\tilde{B}$ and they are associated with the k th oscillator mode at the j th proton location. $H_{\tilde{B}S}$ guarantees that the interaction of the pseudo-spins with this second heat bath $\tilde{B}$ destroys the quantum correlations between them and has no further effect on the configuration. Note that this interaction Hamiltonian is the one that we previously investigated for two pseudo-spins in Sec. 3.3.1 and for twelve pseudo-spins in Sec. 4.2.

As the magnitudes of E_g and E_e are significantly higher than the other components in the eigenenergies given in Eq. (5.2), the evolution of configuration and pseudo-spins in the nucleotide can be assumed to be separated in such a way that the former is not affected by the latter, but not vice versa. In this respect, we can solve a master equation for the configuration and use this solution in the evolution of the pseudo-spins. Here, the evolution of either the pseudo-spin or the configuration state is described using the Markovian master equation in the Lindblad form (2.103), which is one of the key elements of the theory of quantum thermodynamics [256].

While solving the master equation for the pseudo-spin state, ω and $A_j(\omega)$ are respectively taken to be the Bohr frequencies and the Schrödinger picture eigenoperators of the pseudo-spin Hamiltonian H_{HB} in 3.15, i.e., $\omega = (e_{j'} - e_{j''})/\hbar$ and $A_j(\omega) = \sum_{j'j''} |e_{j''}\rangle \langle e_{j'}| A_j |e_{j'}\rangle \langle e_{j''}|$ where A_j are the Pauli z operators in accordance with Eq. 5.4. Also, the coefficients $S_{jj'}(\omega)$ and $\gamma_{jj'}(\omega)$ are defined as the imaginary part and one half of the real part of the one-sided Fourier transformation of the thermal correlation function of heat bath $\tilde{B}$.

Parameters of the pseudo-spin Hamiltonian H_{HB} used in this solution should change depending on the configuration state in line with Eq. 5.1. To do so, if the configuration is in the state $|G\rangle_C$, ω and $A_j(\omega)$ are evaluated using the eigensystem of H_{HB} that is calculated with a vanishing hopping constant J_x . This makes the dynamics of the pseudo-spins fully separable from each other and corresponds to a pure two-qubit dephasing process. Conversely, a non-vanishing J_x is taken into account when the configuration is in the state $|E\rangle_C$. This couples the dynamics of the pseudo-spins to each other and leads to an evolution that brings the pseudo-spin states to a detailed balance only between $|e_1\rangle_{12}$ and $|e_2\rangle_{12}$ in the stationary state. See Sec. 3.3.1 for details.

On the other hand, the evolution of configuration state should be independent from the pseudo-spin state. Hence, the master equation 2.103 is solved for the configuration state using the Bohr frequencies and eigenoperators of an effective Hamiltonian $H_C \approx E_g|G\rangle_C\langle G| + E_e|E\rangle_C\langle E|$. However, the eigenoperators $A_j(\omega)$ cannot be directly constructed from H_C and H_{BC} . The interaction Hamiltonian (5.3) is first decomposed into the form given in (2.69). Moreover, the coefficients $S_{jj'}(\omega)$ and $\gamma_{jj'}(\omega)$ are defined for the heat bath B this time. This solution describes a thermalization process.

5.2.3 Enzyme's conformational changes during the induced-fit mechanism

Two allowed proton locations in the binding site of E are also regarded as pseudo-spins, e.g., $|\psi_E(t_0)\rangle_{34} = |01\rangle_{34}$. The formation of a H-bond between two atoms/molecules X_j and $X_{j'}$ in the biological scenario under consideration (see Figure 5.1) is then described as the coupling of the corresponding pseudo-spins by the Hamiltonian $H_{HB}^{(jj')}$.

Furthermore, we take into account the conformation of the binding site of E in a similar way and denote it by C' : $|\zeta_E(t_0)\rangle_{C'} = |G'\rangle_{C'}$. However, unlike the nucleotide's conformation C , C' is assumed to change in a classical and gradual way because of the intermolecular interaction so that its pseudo-spins are continuously tilted. Thus, the bond angles ϕ_{13} and ϕ_{42} become smaller and the resultant decrease in the energy of intermolecular H-bonds strengthens the binding interaction.

In accordance with the biological scenario under consideration (see Sec. 5.1), the time-scale τ_1 of the binding interaction due to the induced-fit mechanism is supposed

to be much larger than τ_2 , the time-scale over which E undergoes a conformational change that converts C' back to the initial state $|G'\rangle_{C'}$.

As the interaction Hamiltonian changes slowly in time relative to the conformational change that follows, the instantaneous eigenstates of the binding interaction are assumed to evolve independently. Hence, the enzyme - substrate complex (ES) is considered to be in the corresponding ground state at every instant of time until it reaches the maximum complementarity between E and S. On the other hand, as E rapidly returns back to its initial state after this point due to the relation $\tau_2 \ll \tau_1$, the reverse conformational change is supposed to be driven by a sudden post-selection measurement on the pseudo-spins of E which accompanies the loss of energy from its configuration C' to the heat bath B .

5.3 Model Parameters

Bath descriptions in our model can be made to include realistic correlation functions that allow us to probe the actual dynamics in real-time. It is also possible to work in the equilibrium without specifying the bath correlation functions. In fact, as will be shown in the following section, the steady state of the chosen master equation requires the extraction of only two parameters, $E_e - E_g$ and $\epsilon_+ - \epsilon_-$ from the biochemical data.

$E_e - E_g$ can be estimated using the activation energy values of the nucleotide tautomerization processes found in the literature. As an example, the energy barrier for the intramolecular single proton transfer on the Watson-Crick edge is calculated in the gas phase as 41.90 and 47.42 kcal/mol for thymine [43], 34.3 kcal/mol for guanine [44], and 45.6 kcal/mol for adenine [45]. To be consistent with these values, we set $E_e - E_g$ to 42 kcal/mol when required in what follows.

Since $\epsilon_+ - \epsilon_-$ equals to $((W_1 - W_2)^2 + 4J_{12}^2)^{1/2}$, we write $\epsilon_+ - \epsilon_- \geq W_2 - W_1 = \epsilon_2 - \epsilon_1$. That is to say, the value of $\epsilon_+ - \epsilon_-$ is bounded below by the energy difference between the S and P tautomers of the nucleotide. This difference is calculated as 13.08 and 18.75 kcal/mol for thymine [43], 1.3 kcal/mol for guanine [44], and 12.6 kcal/mol for adenine [45] in the tautomerization processes given in the previous paragraph. Hence, the ratio of $(E_e - E_g)/(\epsilon_2 - \epsilon_1)$ is equal to 3.20336 and 2.52907 for thymine, 26.3846 for guanine, and 3.61905 for adenine. Excluding the guanine because of the apparent

incompatibility with the others, the mean of this ratio is found to be 3.11716. Then we can substitute $1/3 \times (E_e - E_g) = 14$ kcal/mol for $\varepsilon_+ - \varepsilon_-$ since it should be just above the value of $\varepsilon_2 - \varepsilon_1$ for relatively small values of J_{12} .

Unless stated otherwise, we generate numerical predictions from the model using the values given above. To explore the sensitivity of the predictions to these values, we squeeze the energy levels of the Hamiltonian given in (??) when it is necessary. In this respect, we decrease the value of $E_e - E_g$ down to 34 kcal/mol, while increasing the value of $\varepsilon_+ - \varepsilon_-$ up to 19 kcal/mol.

5.4 Results and Discussions

5.4.1 Uncatalysed Reaction

In the absence of enzymes that catalyze the reaction, the interactions between the nucleotide and two heat baths B and $\tilde{B}$ are responsible for the spontaneous tautomerization $|S\rangle \leftrightarrow |S^\ddagger\rangle \rightarrow |P\rangle$. Such a tautomerization can occur only if each of the interactions H_N , H_{BC} , and $H_{\tilde{B}S}$ comes to the fore one by one in the order given in what follows: the initialization of the transformation from $|S\rangle$ to $|P\rangle$ requires the excitation of configuration C due to an energy absorption from the heat bath B through H_{BC} . The resultant state $|E\rangle_C \otimes |10\rangle_{12}$ is a coherent superposition of $|\varepsilon_3\rangle = |S^\ddagger\rangle$ and $|\varepsilon_4\rangle$. Immediately after this excitation, the self-Hamiltonian of the molecule drives the evolution towards $S^\ddagger$ as $\varepsilon_3 < \varepsilon_4$. The transformation from this transition state to $|P\rangle$ depends on (i) the destruction of the quantum correlations of the pseudo-spins through $H_{\tilde{B}S}$ and (ii) the loss of energy from the configuration atoms/groups through H_{BC} . These two interactions in arbitrary order lead to a mixture of $|S\rangle$ and $|P\rangle$.

As the configuration state represents the whole atoms/groups in nucleotide (except for the proton of the H atom whose locations are regarded as pseudo-spins), the occurrence of its excitation is expected to be quite rare. Since the spontaneous inter-conversion of tautomers requires this excitation to be initialized, its occurrence is expected to be rare also. We show this mathematically here.

The master equation approach introduced above relies on the assumption that the configuration energy E_g or E_e dominates the total energy of the nucleotide (see Sec. 5.2.2). This allows us to first investigate the configuration state dynamics alone and

then explore the pseudo-spin state dynamics for the predetermined configurations in time.

The initial state in an uncatalysed reaction is $|S\rangle = |G\rangle_C \otimes |10\rangle_{12}$. The steady-state solution of the master equation of the configuration state does not depend on the initial state and equals $Q^{(g)}|G\rangle\langle G|_C + Q^{(e)}|E\rangle\langle E|_C$ where $Q^{(g/e)} = e^{-\beta E_{g/e}} / (e^{-\beta E_g} + e^{-\beta E_e})$. When the configuration state is fixed to $|G\rangle_C$, there isn't any coupling between the pseudo-spins and so the interaction H_{BC} leads to decoherence. Hence, the initial pseudo-spin state $|\psi(t_0)\rangle_{12} = |10\rangle_{12}$ that does not carry any quantum coherence remains the same during the open system dynamics described by Eq. 2.103. On the other hand, there is a non-vanishing coupling between the pseudo-spins when the configuration state is fixed to $|E\rangle_C$. As long as the initial pseudo-spin state $|\psi(t_0)\rangle_{12}$ lives in a subspace spanned by $\{|01\rangle, |10\rangle\}$, H_{BC} drives the evolution to a detailed balance between $|\varepsilon_-\rangle_{12}$ and $|\varepsilon_+\rangle_{12}$ as in (3.44) according to the chosen master equation, i.e. the pseudo-spins' state relaxes to the athermal attractor state $\rho_{12}^{eq} = Q^{(-)}|\varepsilon_-\rangle\langle\varepsilon_-| + Q^{(+)}|\varepsilon_+\rangle\langle\varepsilon_+|$ where $Q^{(\pm)} = e^{-\beta\varepsilon_{\pm}} / (e^{-\beta\varepsilon_+} + e^{-\beta\varepsilon_-})$. Hence, the open system dynamics bring the state $|S\rangle$ towards

$$\begin{aligned}\rho_N^{eq} &= Q^{(g)}|G\rangle\langle G|_C \otimes |10\rangle\langle 10|_{12} + Q^{(e)}|E\rangle\langle E|_C \otimes \rho_{12}^{eq} \\ &= Q^{(g)}|S\rangle\langle S| + Q^{(e)}Q^{(-)}|S^{\ddagger}\rangle\langle S^{\ddagger}| + Q^{(e)}Q^{(+)}|\tilde{S}^{\ddagger}\rangle\langle \tilde{S}^{\ddagger}|,\end{aligned}\tag{5.5}$$

in the stationary state. It means that we will observe the molecule in the substrate state with a probability of $Q(S) = Q^{(g)}$ at equilibrium. This probability is approximately equal to unity at physiological temperatures for the choice of $E_e - E_g = 42$ kcal/mol. Similarly, the equilibrium probability of the occurrence of transition state becomes $Q(S^{\ddagger}) = Q^{(e)}Q^{(-)}$. This probability is equal to 2.84×10^{-30} for $E_e - E_g = 42$ kcal/mol and $\varepsilon_+ - \varepsilon_- = 14$ kcal/mol at $T = 37.5^\circ\text{C}$. Increasing the temperature enhances it slightly but never exceeds the order of 10^{-30} at physiological temperatures. On the other hand, the molecule cannot be converted to the product state in this approximation, even if the environment reaches extremely high temperatures as the final state ρ_N^{eq} is orthogonal to the $|P\rangle$, i.e., $Q(P) = \text{tr}[\rho_N^{eq} |P\rangle\langle P|] = 0$.

5.4.2 Water-Mediated Reaction

Before moving on to the role of quantum H-bonds in enzyme-catalyzed tautomerization, we investigate first this role in water-mediated tautomerization. To do so, we take

into account the formation of two H-bonds between the nucleotide and one or more water molecules. Allowed proton locations on the opposite sides of pseudo-spins 1 and 2 are also regarded as pseudo-spins and respectively labeled by $\bar{1}$ and $\bar{2}$.

We assume that the intermolecular H-bonds described by $H_{HB}^{(1\bar{1})}$ and $H_{HB}^{(2\bar{2})}$ generate quantum correlations between the nucleotide and the water molecule(s) as:

$$|\psi\rangle_{1\bar{1}2\bar{2}} = (\alpha|10\rangle + \sqrt{1-\alpha^2}|01\rangle)_{1\bar{1}} \otimes (\beta|01\rangle + \sqrt{1-\beta^2}|10\rangle)_{2\bar{2}}. \quad (5.6)$$

We consider the open system dynamics just after the generation of these intermolecular correlations. First, we eliminate the water's degrees of freedom $\bar{1}$ and $\bar{2}$ by taking a partial trace over them and focus on the reduced system dynamics of the nucleotide. In this case, the initial state of the nucleotide is an incoherent superposition state:

$$\begin{aligned} |\psi(t_0)\rangle_N = |G\rangle_C \otimes \{ & \alpha^2\beta^2, |10\rangle; (1-\alpha^2)\beta^2, |00\rangle; \\ & \alpha^2(1-\beta^2), |11\rangle; (1-\alpha^2)(1-\beta^2), |01\rangle \}_{12}. \end{aligned} \quad (5.7)$$

Open system dynamics of this incoherent superposition ends up in the following state:

$$\begin{aligned} \rho_{N|H_2O}^{eq} = & \alpha^2\beta^2 Q^{(g)} |S\rangle\langle S| \\ & + (1-\alpha^2)(1-\beta^2) Q^{(g)} |P\rangle\langle P| \\ & + (1-\alpha^2)\beta^2 Q^{(g)} |S_-^\ddagger\rangle\langle S_-^\ddagger| + \alpha^2(1-\beta^2) Q^{(g)} |S_+^\ddagger\rangle\langle S_+^\ddagger| \\ & + (1-\alpha^2)\beta^2 Q^{(e)} |\tilde{S}_-^\ddagger\rangle\langle \tilde{S}_-^\ddagger| + \alpha^2(1-\beta^2) Q^{(e)} |\tilde{S}_+^\ddagger\rangle\langle \tilde{S}_+^\ddagger| \\ & + (1-\alpha^2-\beta^2) Q^{(e)} (Q^{(-)} |S^\ddagger\rangle\langle S^\ddagger| + Q^{(+)} |\tilde{S}^\ddagger\rangle\langle \tilde{S}^\ddagger|). \end{aligned} \quad (5.8)$$

If the intermolecular H-bonds generate maximal entanglement between the nucleotide and the water molecule(s), i.e., $\alpha = \beta = 1/\sqrt{2}$, the nucleotide reaches an equilibrium such that $Q(S) = Q(S_-^\ddagger) = Q(S_+^\ddagger) = Q(P) = Q^{(g)}/4$ that equals to 0.25 for the chosen model parameters. The balance between these four tautomers can be driven through S or P by changing the values of α and β corresponding to a decrease in the amount of total entanglement generated through the H-bonds.

The values of α and β depend on the geometric parameters of the interaction which is totally random, i.e., although the tautomeric conversion of S to P may be mediated by means of the quantumness of water - nucleotide H-bonds, there is a very little chance for the molecules in aqueous solution to show the right orientations that provide the required values of α and β .

5.4.3 Enzyme Catalyzed Reaction

Before the interaction, the nucleotide and the enzyme both exist in their ground states:

$$\begin{aligned} |\psi(t_0)\rangle_{\text{NE}} &= |S\rangle \otimes |E\rangle \\ &= |G\rangle_C |10\rangle_{12} \otimes |G'\rangle_{C'} |01\rangle_{34}. \end{aligned} \quad (5.9)$$

The pseudo-spin term of this state can be thought as the ground state of the interaction Hamiltonian $H_I = H_{HB}^{(13)} + H_{HB}^{(24)}$ with large pseudo-spin separations and vanishing hopping constants. When R_{13} and R_{24} become sufficiently small, a small amount of entanglement is generated through the weak $X_1\text{--H}\cdots X_3$ and $X_2\cdots\text{H--}X_4$ bonds:

$$\begin{aligned} |\psi(t_1)\rangle_{\text{NE}} &= |G\rangle_C \otimes |G'\rangle_{C'} \otimes (\alpha|10\rangle + \sqrt{1-\alpha^2}|01\rangle)_{13} \\ &\quad \otimes (\beta|01\rangle + \sqrt{1-\beta^2}|10\rangle)_{24}. \end{aligned} \quad (5.10)$$

This weak interaction then induces a classical and gradual conformational change in the binding site of E according to the induced-fit mechanism and the bond angles ϕ_{13} and ϕ_{42} become smaller. As explained in Sec.s 5.1 and 5.2.3, this process occurs on a relatively large time-scale τ_1 and the pseudo-spins stay in the ground state of the time-dependent interaction Hamiltonian H_I at every instant of time. Hence, this conformational motion not only strengthens the binding interaction in ES complex due to a decrease in the energy, it also increases the quantum correlations of the intermolecular H-bonds.

In the meantime, the angle ϕ_{43} and the inter-spin separation R_{43} both change during the binding stage (see Figure 5.2). Hence, the intramolecular interaction between X_3 and X_4 gains a quantum character as well. In this respect, the total pseudo-spin interaction Hamiltonian $H_I(t)$ should be taken as $H_{HB}^{(13)} + H_{HB}^{(24)} + H_{HB}^{(34)}$ when $t_2 > t_1$.

When the interaction-induced conformational change stops at time $t_3 > t_2$, the binding site of the enzyme should end up in its highest excited state $|E'\rangle$, whereas the pseudo-spins should be in the ground state of $H_I(t_3)$:

$$\begin{aligned} |\psi(t_3)\rangle_{\text{NE}} &= |G\rangle_C \otimes |E'\rangle_{C'} \otimes (a|0011\rangle + b|0101\rangle + c|0110\rangle \\ &\quad + d|1001\rangle + e|1010\rangle + f|1100\rangle)_{1234} \\ &\equiv |ES\rangle. \end{aligned} \quad (5.11)$$

It follows from the state of the reaction intermediate ES that the entanglement generated at the end of the binding stage is shared among all of the four pseudo-spins.

Next, we show that the four-qubit entanglement generated in this way can be transferred to S when the configuration and pseudo-spins of E subsequently return their initial states, which can enable the conversion of S to P. We will also consider the case in which the four-qubit entanglement of the reaction intermediate ES decays rapidly before the next conformational change of the enzyme.

5.4.3.1 Quantum correlated hydrogen bonds

We assume without any loss of generality that the quantum correlations of each three H-bonds in Figure 5.2-b is sustained until E undergoes a conformational change that converts it back to the initial state. In other words, τ_2 is assumed to be small compared to the decoherence time τ_D enforced by the heat bath $\tilde{B}$. This is quite reasonable as the H-bonded atoms/groups are partially isolated from their environment until the detachment of the ES complex.

As explained in Sec.s 5.1 and 5.2.3, the second conformation motion of interest drives C' from $|E'\rangle$ to $|G'\rangle$ and is governed by the interaction Hamiltonian $H_{BC'}$. During the course of this relatively fast motion, the pseudo-spins of E should detach from the pseudo-spins of S and return back to the initial product state $|01\rangle_{34}$. We describe this conformation-induced process by a sudden post-selection measurement $M = \mathbb{I}_{12} \otimes |01\rangle_{34}\langle 01|$ and find that some of the four-qubit entanglement of the reaction intermediate ES can be transferred to the pseudo-spins of S when M converts the pseudo-spins of E back to their initial state:

$$\begin{aligned} M + H_{BC'} : |ES\rangle &\rightarrow |\psi(t_4)\rangle_{NE} = |G\rangle_C \otimes (b'|01\rangle + d'|10\rangle)_{12} \otimes |E\rangle \\ &= (b'|P\rangle + d'|S\rangle) \otimes |E\rangle \\ &\equiv |ES^\ddagger\rangle, \end{aligned} \quad (5.12)$$

where $b' = b/\sqrt{b^2 + d^2}$ and $d' = \sqrt{1 - b'^2}$. We can analyze the open system dynamics of the nucleotide using the master equation approach introduced in Sec. 5.2.2, starting at $t = t_4$. The stationary solution becomes

$$\begin{aligned} \rho_{N|qHB}^{eq} &= d'^2 Q^{(g)} |G\rangle\langle G|_C \otimes |10\rangle\langle 10|_{12} + b'^2 Q^{(g)} |G\rangle\langle G|_C \otimes |01\rangle\langle 01|_{12} \\ &\quad + Q^{(e)} |E\rangle\langle E|_C \otimes \rho_{12}^{eq} \\ &= d'^2 Q^{(g)} |S\rangle\langle S| + b'^2 Q^{(g)} |P\rangle\langle P| + Q^{(e)} Q^{(-)} |S^\ddagger\rangle\langle S^\ddagger| + Q^{(e)} Q^{(+)} |\tilde{S}^\ddagger\rangle\langle \tilde{S}^\ddagger|. \end{aligned} \quad (5.13)$$

This means that the enzyme converts the nucleotide into the product state with a probability of $Q(P) = b'^2 Q^{(g)}$ at equilibrium. Also, neither $Q(S^\ddagger)$ nor $Q(\tilde{S}^\ddagger)$ changes when compared to equilibrium probabilities in the uncatalysed case.

b' depends on the strength of the H-bonds formed in the enzyme-substrate interaction, as does the efficiency of the catalysis. If the three H-bonds shown in Figure 5.2-b generate a maximal W-type four-qubit entanglement², i.e., $a = b = c = d = e = f = 1/\sqrt{6}$, the nucleotide reaches an equilibrium such that $Q(S) = Q(P) = Q^{(g)}/2$ that equals 0.5 for our chosen model parameters.

We can compare this result with another obtained in water-mediated tautomerization. When the amounts of entanglement generated through each water - nucleotide H-bond are chosen to be maximal, the nucleotide reaches an equilibrium with $Q(S) = Q(P) = 0.25$. Here, when the overall four-qubit entanglement is chosen to be maximal, the pairwise entanglement generated through each enzyme - nucleotide H-bond is not maximal and the equilibrium probabilities corresponding to both tautomers are twice as high as 0.25. Moreover, the ionic transition states $S_\pm^\ddagger$ do not emerge unlike the water-mediated case in which $Q(S_\pm^\ddagger) = 0.25$. In the meantime, the equilibrium probability of the neutral transition state is still kept at the order of 10^{-30} at physiological temperatures.

In this context, not only the conformational change induced in the enzyme increases the likelihood of the formation of quantum H-bonds with the nucleotide, but also the subsequent conformational change that brings the enzyme back to its initial state makes quantum H-bonds a more efficient resource for the tautomerization process.

5.4.3.2 Classically correlated hydrogen bonds

The assumption of $\tau_2 < \tau_D$ isn't necessary to derive our insight into the possible role of proton tunneling in an induced-fit model of enzyme catalysis. Once the quantum correlations are created by proton tunneling events through H-bonds, the enzyme can use them as a resource to convert S to P by the same mechanism under consideration even if these correlations rapidly decay to classical correlations during decoherence.

²This is the extension of maximal W-type three-qubit entanglement given in (2.125) to four qubits.

In the case of $\tau_2 > \tau_D$, the final state of the pseudo-spins of ES at the end of the binding stage should be a classically correlated state rather than a four-qubit entangled state:

$$\begin{aligned} \rho_{1234}(t_3) = & a^2|0011\rangle\langle 0011| + b^2|0101\rangle\langle 0101| + c^2|0110\rangle\langle 0110| \\ & + d^2|1001\rangle\langle 1001| + e^2|1010\rangle\langle 1010| + f^2|1100\rangle\langle 1100|. \end{aligned} \quad (5.14)$$

After the pseudo-spins of E are detached from the pseudo-spins of S and returned back to their initial state by the post-selection measurement induced by the second conformational change, the nucleotide ends up in a mixture of S and P as below:

$$\begin{aligned} \rho_{NE}(t_4) = & |G\rangle\langle G|_C \otimes (d'^2|10\rangle\langle 10|_{12} + b'^2|01\rangle\langle 01|_{12}) \otimes |E\rangle\langle E| \\ = & (d'^2|S\rangle\langle S| + b'^2|P\rangle\langle P|) \otimes |E\rangle\langle E| \\ \equiv & |ES^\ddagger\rangle\langle ES^\ddagger|, \end{aligned} \quad (5.15)$$

where b' and d' are the re-normalized b and d , as before. If we feed master equation 2.103 with this incoherent superposition state instead of the coherent superposition state $|\psi(t_4)\rangle_N$ given in 5.12, the steady state solution $\rho_{N|cHB}^{eq}$ obtained is the same density matrix with $\rho_{N|qHB}^{eq}$ in 5.13. Hence, the equilibrium probabilities remain the same in both cases. Our conclusion still holds even if the quantum correlations of H-bonds found in the reaction intermediate ES are converted to classical correlations by strong decoherence just at the beginning of the catalytic stage.

5.4.3.3 Activation energy

A state transformation from ρ to ρ' is said to be thermodynamically favorable only if the free energy F goes down, i.e., $\Delta F = F[\rho'] - F[\rho] \leq 0$. Biological systems can carry out thermodynamically unfavorable state transformations by coupling them to favorable ones. However, even thermodynamically favorable biochemical state transformations generally involve the formation of transition states with higher free energies than initial states. Enzymes lower these activation barriers but are not expected to remove them completely. The difference in the free energy between the transition and initial (or final) states guarantees that this catalyzed transformation is sufficiently slow for the organism inside which it takes place.

It is worth remembering that the tautomerization process of interest is not assumed to be the ultimate catalytic function of the multifunctional enzyme, but rather supports a secondary catalytic function required for the ultimate one. In this respect, an interesting question to ask is whether all enzyme-catalyzed reactions have a

rate-limiting step between the initial and final states. In other words, can an enzyme provide the formation of a transition state with a free energy lower than the free energy of the initial (or final) state in specific cases? The simple model investigated in this study points to a surprising possible answer to this question in what follows.

The free energy of the nucleotide can be calculated by:

$$F[\rho] = \langle E \rangle - RTS[\rho] = \text{tr}[\rho H_N] + RT \text{tr}[\rho \log_2 \rho] \quad (5.16)$$

where S is the von Neumann entropy defined in (2.117) (see Ref. [257]) and R is the universal gas constant, 1.987 cal/(mol K). As the states of S, P, and $S^\ddagger$ are pure states, i.e., as $\text{tr}[\rho^2] = 1$ for these tautomers, their von Neumann entropies vanishes. These states are also eigenstates of H_N which means that average energies taken over them are nothing but the corresponding eigenenergies. Hence, we end up with

$$F[|S\rangle\langle S|] = \varepsilon_1 < F[|P\rangle\langle P|] = \varepsilon_2 < F[|S^\ddagger\rangle\langle S^\ddagger|] = \varepsilon_3. \quad (5.17)$$

We consider the free energy of the transition state $ES^\ddagger$ that appears after the binding stage of enzyme catalysis and study the case in which quantum correlations that are generated through the H-bonds of ES is preserved until the start of the catalytic stage, i.e., $\tau_2 < \tau_D$. $|ES^\ddagger\rangle$ is separable (see Eq. 5.12) and the reduced state of the nucleotide equals $|ES^\ddagger\rangle_N = |G\rangle_C \otimes (b'|01\rangle + d'|10\rangle)_{12}$ where $d' = \sqrt{1 - b'^2}$. The expected value of the energy over this state is $\text{tr}[|ES^\ddagger\rangle\langle ES^\ddagger|_N H_N] = b'^2 \varepsilon_2 + (1 - b'^2) \varepsilon_1$. As it is a pure state, its von Neumann entropy vanishes and its free energy becomes

$$F[|ES^\ddagger\rangle\langle ES^\ddagger|_N] = b'^2 \varepsilon_2 + (1 - b'^2) \varepsilon_1, \quad (5.18)$$

which corresponds to a free energy just in-between the free energy of S and P, a value much lower than expected in biochemical reactions (Figure 5.3).

In the case of $\tau_2 \not< \tau_D$, $|ES^\ddagger\rangle$ is separable again (see Eq. 5.15) but the reduced state of the nucleotide equals $|ES^\ddagger\rangle\langle ES^\ddagger|_N = |G\rangle\langle G|_C \otimes ((1 - b'^2)|10\rangle\langle 10|_{12} + b'^2|01\rangle\langle 01|_{12})$. As $\text{tr}[|ES^\ddagger\rangle\langle ES^\ddagger|_N H_N]$ is also equal to $b'^2 \varepsilon_2 + (1 - b'^2) \varepsilon_1$, the expected value of the energy between S and P along the reaction coordinate is not affected by decoherence. However, unlike the previous case, the reduced state of the nucleotide is not a pure state, which means that its von Neumann entropy does not vanish: $S[|ES^\ddagger\rangle\langle ES^\ddagger|_N] = -b'^2 \log_2 b'^2 - (1 - b'^2) \log_2 (1 - b'^2) > 0$. The free energy then drops further by the

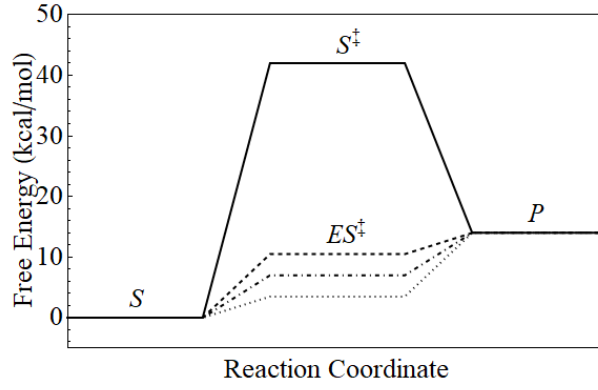


Figure 5.3 : Alternative reaction pathways from S to P. Uncatalysed S (solid line) requires a high activation energy to reach $S^\ddagger$. Here, $E_e - E_g = 42$ kcal/mol, $\epsilon_+ - \epsilon_- = 14$ kcal/mol and $F[|S\rangle\langle S|]$ is set to zero as explained in the text. E is found to lower the activation energy by leading the reaction into an alternative chemical pathway ($ES^\ddagger$). Quantum correlations generated through H-bonds enable E to lower the activation energy down to a value in-between the free energies of S and P. $T = 37.5^\circ\text{C}$, $\tau_2 < \tau_D$ and b'^2 is fixed to $3/4$, $2/4$ and $1/4$ respectively for the dashed, dot-dashed, and dotted lines.

loss of quantumness of correlations in the H-bonds as below:

$$F[|ES^\ddagger\rangle\langle ES^\ddagger|_N] = b'^2\epsilon_2 + (1 - b'^2)\epsilon_1 + RT(b'^2 \log_2 b'^2 + (1 - b'^2) \log_2 (1 - b'^2)). \quad (5.19)$$

However, this entropy-based decrease is not likely to be significant as $RT < 1$ kcal/mol at physiological temperatures and $S[|ES^\ddagger\rangle\langle ES^\ddagger|_N] \leq 1$. Thus, $F[|ES^\ddagger\rangle\langle ES^\ddagger|_N]$ is expected to be in-between $F[|S\rangle\langle S|]$ and $F[|P\rangle\langle P|]$ once again.

We can numerically compare the activation energies in the catalyzed and uncatalyzed reactions using the model parameters that we chose above (Figure 5.3). Let's fix $F[|S\rangle\langle S|]$ to zero for simplicity. To do so, both of the conformation and pseudo-spin terms of ϵ_1 are equated to zero, i.e. $E_g = 0$ and $W_1 + \lambda_{12} = 0$. Furthermore, we assume without any loss of generality that $((W_1 - W_2)^2 + 4J_{12}^2)^{1/2} \approx W_2 - W_1$. Then we end up with $F[|P\rangle\langle P|] \approx 14$ kcal/mol and $F[|S^\ddagger\rangle\langle S^\ddagger|] \approx 42$ kcal/mol. When we take $T = 37.5^\circ\text{C}$ and $b'^2 = 1/2$ (or equivalently $b = d = 1/\sqrt{6}$) as before, $F[|ES^\ddagger\rangle\langle ES^\ddagger|_N]$ becomes 7 kcal/mol if $\tau_2 \leq \tau_D$, and 6.38 kcal/mol otherwise. When b'^2 is raised to $3/4$, these values change to 10.5 and 10 kcal/mol respectively. On the other hand, a decrease in b'^2 to $1/4$ also lowers $F[|ES^\ddagger\rangle\langle ES^\ddagger|_N]$ to 3.5 for $\tau_2 < \tau_D$ and 3 kcal/mol for $\tau_2 > \tau_D$. Hence the rate enhancement the enzyme brings about is in the range of 22 to 27 orders of magnitude for $1/4 \leq b'^2 \leq 3/4$ and $T = 37.5^\circ\text{C}$, based on the first-order

rate equation $k = \frac{k_B T}{h} e^{\frac{-\Delta F^\ddagger}{RT}}$, where k_B is the Boltzmann constant and h is the Planck constant. The lower limit of this rate enhancement can be decreased down to 5×10^{19} by increasing the upper limit of b'^2 up to unity. Both limits can be pulled down by squeezing the energy levels of the Hamiltonian given in (5.1), e.g. when $1/4 \leq b'^2 < 1$, $T = 37.5^\circ\text{C}$, $E_e - E_g = 34$ kcal/mol, and $\varepsilon_+ - \varepsilon_- = 19$ kcal/mol, the enzyme leads to a rate enhancement in the range of 10 to 20 orders of magnitude. Note that although this enhancement is more reasonable when compared to the experimentally estimated upper limits, it does not change our main proposition that the free energy of the transition state is smaller than the free energy of the product in the enzyme catalyzed reaction.

5.5 Conclusions

We showed that the quantum correlations generated through H-bonds can be used as a resource in an induced-fit mechanism that leads generic nucleotide tautomerization into an alternative chemical pathway. This increases the occurrence of the minor tautomeric form P even if the conversion of the major tautomeric form S to P isn't possible in the absence of the enzyme. Moreover, the advantages of this scenario go beyond that. As the new transition state that emerges in this pathway corresponds to the quantum superposition of S and P, the free energy for the new transition state is found to be between the individual free energies of S and P. Hence, the enzyme provides an alternative reaction pathway with a dramatically smaller activation energy (Figure 5.3). Moreover, such a reactive pathway doesn't require anything more than a passive transformation of the enzyme's configuration in the catalytic stage.



6. CONCLUSIONS

We've developed two closely related models, which form a distinct but coherent body of work. The two models are related through the pseudo-spin formalism they share, that is introduced in Ch. 3. This formalism is similar to the spin-boson formalism [234–242], which is widely used in the literature. In fact, the only difference is that instead of representing the H-bond by one qubit as in spin-boson formalism, we use two qubits. This new representation based on the extension of the Hilbert space two times enables us to describe the H-bond in terms of quantum correlations between H-bonded atoms/molecules. However, the well-studied spin-boson formalism is limited to investigating only the quantum coherence in the H-bonded system.

Quantum correlations are better suited than quantum coherence to the investigation of the quantum aspects of H-bonding interactions between biomolecules. Biomolecular interactions that occur during critical cellular processes require structural and electronic complementarity between molecules, in the manner of a lock and key. The degree of this complementarity is a type of biomolecular correlation, which is particularly achieved through the number and strength of H-bonds. Hence, our pseudo-spin model, which describes H-bonding in terms of quantum correlations, is closer to the contemporary paradigm in molecular biology than spin-boson formalism.

Moreover, the local system-environment interaction that we use in Sec. 3.3.1, the coupling of the position of the proton to the positions of the phonons, is one of the standard interaction types commonly used in spin-boson formalism. To this extent even though our formalism is based on established literature, it goes beyond it in both its approach and conclusions.

The first model that we've developed is presented in Ch. 4, which is based mainly on our research in [101]. It makes a significant contribution, we believe, to our understanding of the mechanisms of proton tunneling in water ice. This remains an open question with wide ranging ramifications for physics, biology and chemistry, and

we are excited that by this model we shed new light on this topic of interest to these respective communities.

Previous studies of bonding in water ice have taken an approach using the tools of quantum chemistry [52, 59, 60] and lattice gauge theory [53, 61]. In this model, we adopt the tools of quantum information theory to encompass these previous works and approaches. We construct a simple and physically insightful open system model. We demonstrate that the steady state solution of the chosen master equation depends on only two parameters and its quantum information theoretic analysis is enough to give a quantitative description of the experimental data.

We reveal the role of the vibrations in the emergence of ice rules (see [54, 55]) and concerted six-proton tunneling (see [56, 58, 60, 61]) and find that individual quantum tunneling events in the bonds of water ice are classically rather than quantum correlated at low temperatures, a point that can't be addressed by the previous studies. Beside these findings and despite the simplicity of the present model, our work also paves the way to investigating the proton dynamics in real-time to advance our understanding of many-proton tunneling in water ice. This is important for many natural phenomena in physical, chemical, and biological systems.

The second model that we've developed is presented in Ch. 5, which is based mainly on our research in [102]. It sheds light from a radically new perspective on the role of tunneling in enzyme catalysis and its relevance to the protein dynamics. The topic is indeed an old subject that has been studied widely using molecular dynamics [70–77] or path integral [80–85] type models. However, the previous studies have generally considered the biological proton transfer reactions that involve breaking a carbon-hydrogen bond and regarded the protein dynamics in the form of conformational selection model (see [29–31]). Here, we investigate the proton motion through quantum H-bonds formed between electronegative atoms and take the protein dynamics into account in the form of induced-fit model (see [26–28, 35–39]).

Unlike the previous studies, we are also capable to reveal an underestimated relationship between H-bonding and the biological function: most of the weak interactions, like H-bonds, generate correlations between bio-macromolecules in living systems. We approach the problem of bonding using the tools of quantum information

theory and quantum thermodynamics where the correlations are routinely regarded as a resource for specific tasks. To this extent we strongly believe this approach and the insights this yields to be novel.

Although we considered a prototypical enzyme and a substrate in our model, our results and methodology represent an original contribution to the field. Our results reveal an exciting relation between classical conformational changes of biomolecules and the quantum correlations generated through H-bonds. In particular, we found that cyclic conformational changes induced in the binding site of the enzyme can increase the intermolecular quantum correlations between the H-bonded atoms, and transfer them into the intramolecular atoms of the substrate. We also showed that the destruction of this intramolecular quantumness due to the environment is the actual process that drives an otherwise impossible reaction in the forward direction from the substrate to the product. All of these insights are highly novel.

In addition to proposing these new nontrivial roles for proton tunneling in enzyme function, our model indicates that there might be something amiss in the general understanding of how an enzyme can lower the activation energy. To our own surprise we found that the enzyme leads the substrate into a reaction pathway in which there is no need to exceed the energy of the product in any of the steps. It is our view that this finding warrants further verification and constitutes an innovative facet of our work.

6.1 Future Directions

6.1.1 Fewer assumptions, more hypotheses

There are two essential assumptions made in setting up the model that we've developed in Ch. 5. First, the changes in the geometric parameters ϕ_{43} and R_{43} during the induced-fit mechanism are assumed to give a quantum character to the intramolecular interaction between X_3 and X_4 . This is, in fact, what is responsible for the generation of i) four-qubit entanglement in (5.11) or ii) four-qubit classical correlation in (5.14) at the end of the binding stage. Second, the subsequent rapid conformational change of the enzyme is assumed to have a strong influence on the pseudo-spins of the enzymes, which can be described by a sudden post selection measurement.

Suppose that the first assumption mentioned above is not satisfied in the scenario of $\tau_2 < \tau_D$ presented in Sec. 5.4.3.1. Then, when the first conformational change induced by the pseudo-spin interaction stops at time t_3 , the state of the reaction intermediate ES becomes:

$$\begin{aligned} |\psi(t_3)\rangle_{\text{NE}} = & |G\rangle_C \otimes |E'\rangle_{C'} \otimes (\alpha'|10\rangle + \sqrt{1-\alpha'^2}|01\rangle)_{13} \\ & \otimes (\beta'|01\rangle + \sqrt{1-\beta'^2}|10\rangle)_{24}. \end{aligned} \quad (6.1)$$

After the subsequent rapid conformational change that drives C' from $|E'\rangle$ to $|G'\rangle$ and induces the sudden post-selection measurement $M = \mathbb{I}_{12} \otimes |01\rangle_{34}\langle 01|$ on the pseudo-spins, the nucleotide returns back to its initial state $|\psi(t_4)\rangle_N = |G\rangle_C \otimes |10\rangle_{34} = |S\rangle$. It also ends up in the same final state for the case of $\tau_2 > \tau_D$ in which the quantum correlations generated through H-bonds rapidly decay to classical correlations. Hence it seems impossible for the enzyme to transform the substrate to the product without generating a four-qubit correlation before time t_3 .

At this point, we can avoid making the second assumption mentioned above as well. In this case, both protons shared in the intermolecular H-bonds may be left either in the substrate or in the enzyme after the detachment of the complex, which causes the formation of ionic tautomers as in the water-mediated reaction in Sec. 5.4.2.

An interesting question to ask is whether there exist alternative reaction pathways in which the enzyme transforms the substrate to the product without requiring the two assumptions mentioned above. In this respect, we are planning to explore such alternative pathways using more tools of quantum information theory in future work.

6.1.1.1 Case 1: Formation of a synergy between hydrogen bonds

First, we intend to investigate the possibility that the catalytic site of the enzyme is able to do classically correlated intramolecular operations either in the substrate or in the binding site of the enzyme, a finite set of which creates a synergy between the two intermolecular H-bonds as follows:

$$|\psi(t_1)\rangle_{1324} = (\alpha|10\rangle + \sqrt{1-\alpha^2}|01\rangle)_{13} \otimes (\beta|01\rangle + \sqrt{1-\beta^2}|10\rangle)_{24} \quad (6.2)$$

$\downarrow$ LOCC

$$a|10\rangle_{12}|01\rangle_{34} + \sqrt{1-a^2}|01\rangle_{12}|10\rangle_{34} = |\psi(t_3)\rangle_{1234}. \quad (6.3)$$

In this way, the enzyme may prevent any protonation or deprotonation process. However, the transformation from (6.2) to (6.3) is achievable under LOCCs only if the majorization condition proposed in [188] is satisfied, namely that if the partial trace of $|\psi(t_1)\rangle_{1324}$ over the pseudo-spin degrees of freedom of one molecule is majorized by the reduced state which is obtained in a similar way using $|\psi(t_3)\rangle_{1234}$. To illustrate this in more detail, let's consider two weak intermolecular H-bonds at time t_1 , e.g., take $\alpha = \sqrt{0.9}$ and $\beta = \sqrt{0.8}$ in Eq. (6.2). Then, the enzyme can create a synergy between the individual H-bonds at time t_3 as shown in Eq. (6.3) if either a^2 or $1 - a^2$ is equal or greater than the product of $\alpha^2\beta^2 = 0.72$.

Appearance of such a correlation between the protons at time t_3 provides an alternative reaction pathway for the conversion of the substrate to the product. However, unlike the present scenario in Ch. 5, this LOCC-based new scenario may demand a high amount of energy depending on which LOCCs are carried out. Moreover, it is not straightforward to find the biochemical processes corresponding to a given set of LOCCs as there is no unique unitary decomposition of a Kraus operator in an extended Hilbert space. Thus, this future work is likely to require to develop an optimization method like the genetic algorithm that we had previously developed to find the optimum quantum circuit decomposition of a given quantum operation using a given set of biochemically realizable elements in our earlier work in [258, 259].

6.1.1.2 Case 2: Temporary use of fresh hydrogen bonds

Formation of anionic and cationic tautomers may be unavoidable under the LOCCs that are allowed to be carried out by the catalytic site of the enzyme. We are also planning to pursue future work addressing such more restricted situations in which the enzyme is at best capable of modifying the electronegativity of the individual pseudo-spins in a classically correlated way as below:

$$\begin{aligned}
 |\psi(t_1)\rangle_{1324} &= (\alpha|10\rangle + \sqrt{1-\alpha^2}|01\rangle)_{13} \otimes (\beta|01\rangle + \sqrt{1-\beta^2}|10\rangle)_{24} \\
 &\downarrow \text{LOCC} \\
 |\psi(t_3)\rangle_{1234} &= a_S|10\rangle|01\rangle + a_+|11\rangle|00\rangle + a_-|00\rangle|11\rangle + a_P|01\rangle|10\rangle.
 \end{aligned} \tag{6.4}$$

In the context of this scenario, the best strategy for the enzyme seems to be maximization of the probability $P(P) = |a_P|^2$ by decreasing the remaining probabilities as much as possible. However, the enzyme may not be able to reach the upper

bound on $P(P)$ using LOCCs. To examine what can be done in this case, let's take $\alpha = \sqrt{0.9}$, $\beta = \sqrt{0.8}$, $P_{\min}(S) = |a_S|^2 = 0.01$, $P_{\min}(S_-^\pm) = |a_-|^2 = 0.095$, and $P_{\min}(S_+^\pm) = |a_+|^2 = 0.095$. These parameters make the two states given in Eq. (6.4) become incomparable by the majorization relation. Thus, the classically correlated intramolecular operations can not catalyze the transformation between these states.

In order to make the transformation (6.4) achievable with the parameters given above, we can utilize the phenomenon known as *catalytic transformation* [260] in the resource theory of entanglement as follows. Suppose that the catalytic site of the enzyme has an additional H-bond described by the state $|\psi\rangle_{56} = \sqrt{0.8}|10\rangle_{56} + \sqrt{0.2}|01\rangle_{56}$. Then, the enzyme moves one of the atoms/groups involved in this bond, let's say X_5 , to a location very close to the pseudo-spins of the substrate, whereas the other atom/group (X_6) remains near the pseudo-spins of the enzyme. After that, quantum correlations generated through this extra bond can be used without being consumed as below:

$$\begin{aligned}
& (\sqrt{0.9}|10\rangle + \sqrt{0.1}|01\rangle)_{13} \otimes (\sqrt{0.8}|01\rangle + \sqrt{0.2}|10\rangle)_{24} \otimes |\psi\rangle_{56} \\
& \quad \downarrow \text{LOCC} \\
& (\sqrt{0.01}|10\rangle|01\rangle + \sqrt{0.095}|11\rangle|00\rangle + \sqrt{0.095}|00\rangle|11\rangle + \sqrt{0.8}|01\rangle|10\rangle)_{1234} \otimes |\psi\rangle_{56}.
\end{aligned} \tag{6.5}$$

Namely, nonlocal distribution of the quantum correlations found in fresh H-bonds during classically correlated intramolecular operations can be treated as a catalytic action to convert the substrate to the product. This may open an avenue for research for new roles of H-bonding in biocatalysis based on its quantum nature.

6.1.2 Focus on single-shot quantum thermodynamics

Under some thermodynamical process, a particular state transformation of a microscopic, quantum, or highly correlated system occurs only if the *thermo-majorization criteria* given in [261] is satisfied. This criteria requires an operation on the states just before checking the usual majorization relation between their eigenvalue spectrums. This special operation, namely *Gibbs-rescaling* [262], transforms a thermal Gibbs state into a uniform state so that it becomes an attractor of a doubly stochastic matrix by which a gradual thermalization process can be described.

Thermo-majorization criteria implies an extension of the traditional second law of thermodynamics discussed in Sec. 5.4.3.3 into a family of second laws in which the

von Neumann entropy is replaced with a family of entropies known as *smooth Rényi entropies* [263, 264]. All of the smooth Rényi entropies converge to a particular one which corresponds to the von Neumann entropy only in the limit of many identical and independent subsystems [262, 263]. That is to say that a state transformation favorable according to the traditional second law may be forbidden beyond this limit, which is also called *single-shot regime*.

On the other hand, a transition which is forbidden according to the thermo-majorization criteria may be catalyzed by using a resource which returns back into its initial state [263], in a similar way as in the case of catalytic transformations [260] used in the resource theory of entanglement. To this extent, one of our long-term aims is to shift the focus of our investigation of H-bonding to the single-shot regime using the recent tools of the resource theory of thermodynamics [261–277]. This may offer some fresh insights into the role of proton tunneling in enzyme catalysis.

6.1.3 Proton tunneling in DNA

Although we've focused so far only on a generic enzyme-catalyzed biochemical state transformation from a single substrate to a single product, the information-theoretic approach adopted until this point also enables us to investigate another generic enzyme activity in which two substrates are brought together to form a binary complex by means of H-bonds. First and foremost, the enzyme-catalyzed pairing of adenine and thymine nucleobases through two H-bonds during DNA replication is a natural direction to pursue future work. We had previously put forward the idea of entanglement swapping during DNA base pairing in our earlier work in [258, 259], but haven't got a chance before to set up a test-bed to try out this idea.

In this respect, we are planning to construct a model for an enzyme that has one distinct binding site for each of its substrates, at where the molecular recognition of the substrate occurs by the same induced-fit mechanism considered in Ch. 5, i.e., H-bonds between a substrate and the corresponding binding site trigger a classical change in the conformation of this binding site, which generates an intermolecular four-qubit entanglement. We will then be able to explore the possibility for the enzyme to harnesses two four-qubit entanglements generated in this way i) to detect the

complementarity of two substrates and/or ii) to enhance the forthcoming correlations between them in the binary complex.

This model is likely to require initial correlations between the pseudo-spins and/or the conformations of two binding sites. We are currently trying to extend the model presented in Ch. 5 in this direction to build a prototype for the investigation of the role of proton tunneling in DNA replication and protein synthesis.

6.1.4 Proton tunneling in Protein Synthesis

So far in this thesis, we have neglected the partial ionic characters of the proton-donating covalent bonds to investigate the quantum entanglement that originates in a H-bonded system only from tunneling of the protons of the H atoms (see the transition from (3.5) to (3.6) in Sec. 3.1 for the details). However, we found in [100] that although the partial ionic characters of the proton-donating covalent bonds decrease the entanglement generated in such a system, they increase the resource value of H-bonds in molecular recognition, i.e., we particularly showed that a biomolecule, at least in principle, can discriminate between $3^n - 1$ agonist ligands using the entanglement shared in n intermolecular H-bonds even it is restricted to recognize its ligands using the same set of proton-acceptors and donors residing on its binding site. If we neglected the partial ionic characters of the proton-donating covalent bonds, the discernible ligand number would decrease to $2^n - 1$ in the same molecular recognition scenario.

Also, we argued in [100] that progenitors of modern aminoacyl-tRNA synthetase (aaRS) enzymes may have utilized the quantum entanglement shared in intermolecular H-bonds in a similar way as in our scenario. aaRSs are the vital elements of protein synthesis as they provide a physical basis for the connection between the receipt encoded by nucleic acids and the product made by amino acids. Hence, there may have been a place for the utilization of quantum entanglement in the evolutionary history of protein synthesis. In this respect, we are planning to extend our pseudo-spin formalism presented in Sec. 3.2 to include the partial ionic characters of the proton-donating covalent bonds. We will then be able to explore the utilization of quantum entanglement in the molecular recognition events encountered in both

the contemporary protein synthesis and its hypothetical primordial ancestor living in pre-DNA world in more detail using this extended pseudo-spin formalism.





REFERENCES

- [1] **Yen, F. and Chi, Z.** (2015). Proton ordering dynamics of H₂O ice, *Physical Chemistry Chemical Physics*, 17(19), 12458–12461, <https://doi.org/10.1039/C5CP01529D>.
- [2] **Wooldridge, P.J. and Devlin, J.P.** (1988). Proton trapping and defect energetics in ice from FT-IR monitoring of photoinduced isotopic exchange of isolated D₂O, *The Journal of Chemical Physics*, 88(5), 3086–3091, <https://doi.org/10.1063/1.453952>.
- [3] **Suga, H.** (1997). A facet of recent ice sciences, *Thermochimica Acta*, 300(1), 117–126, [https://doi.org/10.1016/S0040-6031\(96\)03121-8](https://doi.org/10.1016/S0040-6031(96)03121-8).
- [4] **Buckingham, A.D., Del Bene, J.E. and McDowell, S.A.C.** (2008). The hydrogen bond, *Chemical Physics Letters*, 463(1–3), 1–10, <https://doi.org/10.1016/j.cplett.2008.06.060>.
- [5] **Rose, G.D., Fleming, P.J., Banavar, J.R. and Maritan, A.** (2006). A backbone-based theory of protein folding, *Proceedings of the National Academy of Sciences of the United States of America*, 103(45), 16623–16633, <https://doi.org/10.1073/pnas.0606843103>.
- [6] **Watson, J.D. and Crick, F.H.C.** (1953). Molecular structure of nucleic acids: a structure for deoxyribose nucleic acid, *Nature*, 171(4356), 737–739, <https://doi.org/10.1038/171737a0>.
- [7] **Watson, J.D. and Crick, F.H.C.** (1953). Genetical implications of the structure of deoxyribonucleic acid, *Nature*, 171(4361), 964–967, <https://doi.org/10.1038/171964b0>.
- [8] **Helms, V. and Wade, R.** (1995). Thermodynamics of water mediating protein-ligand interactions in Cytochrome P450cam: A Molecular Dynamics Study, *Biophysical Journal*, 69(3), 810–824, [https://doi.org/10.1016/S0006-3495\(95\)79955-6](https://doi.org/10.1016/S0006-3495(95)79955-6).
- [9] **Groenhof, G., Lensink, M., Berendsen, H. and Mark, A.** (2002). Signal transduction in the Photoactive Yellow Protein. II. Proton transfer initiates conformational changes, *Proteins*, 48(2), 212–219, <https://doi.org/10.1002/prot.10135>.
- [10] **Agmon, N.** (2005). Proton pathways in Green Fluorescence Protein, *Biophysical Journal*, 88(4), 2452–2461, <https://doi.org/10.1529/biophysj.104.055541>.

- [11] **Sokolov, N.D.** (1959). On the quantum theory of the hydrogen bond, *D. Hadži and H.W. Thompson, editors, Hydrogen bonding: Papers presented at the Symposium on Hydrogen Bonding held at Ljubljana, 29 July - 3 August 1957*, Symposium Publications Division, Pergamon Press, pp.385–392, <https://www.elsevier.com/books/hydrogen-bonding/hadzi/978-0-08-009140-2>.
- [12] **Coulson, C.A.** (1959). The Hydrogen bond, *D. Hadži and H.W. Thompson, editors, Hydrogen bonding: Papers presented at the Symposium on Hydrogen Bonding held at Ljubljana, 29 July - 3 August 1957*, Symposium Publications Division, Pergamon Press, pp.339–360, <https://www.elsevier.com/books/hydrogen-bonding/hadzi/978-0-08-009140-2>.
- [13] **Pauling, L.** (1960). *The nature of the chemical bond and the structure of molecules and crystals: An introduction to modern structural chemistry*, Cornell University Press, 3rd edition.
- [14] **Bratož, S.** (1967). Electronic Theories of Hydrogen Bonding, *Advances in Quantum Chemistry*, 3, 209–237, [https://doi.org/10.1016/S0065-3276\(08\)60090-8](https://doi.org/10.1016/S0065-3276(08)60090-8).
- [15] **Grabowski, S.J.** (2011). What Is the Covalency of Hydrogen Bonding?, *Chemical Reviews*, 111(4), 2597–2625, <https://doi.org/10.1021/cr800346f>.
- [16] **Isaacs, E., Shukla, A., Pltazman, P.M., Hamann, D.R., Barbiellini, B. and Tulk, C.A.** (1999). Covalency of the hydrogen bond in ice: a direct x-ray measurement, *Physical Review Letters*, 82(3), 600–603, <https://doi.org/10.1103/PhysRevLett.82.600>.
- [17] **Khaliullin, R.Z., Bell, A.T. and Head-Gordon, M.** (2009). Electron donation in the water - water hydrogen bond, *Chemistry - A European Journal*, 15(4), 851–855, <https://doi.org/10.1002/chem.200802107>.
- [18] **Elgabarty, H., Khaliullin, R.Z. and Kühne, T.D.** (2015). Covalency of hydrogen bonds in liquid water can be probed by proton nuclear magnetic resonance experiments, *Nature Communications*, 6, 8318, <https://doi.org/10.1038/ncomms9318>.
- [19] **Wang, B., Jiang, W., Dai, X., Gao, Y., Wang, Z. and R.-Q., Z.** (2016). Molecular orbital analysis of the hydrogen bonded water dimer, *Scientific Reports*, (6), 22099, <https://doi.org/10.1038/srep22099>.
- [20] **Fonseca Guerra, C., Bickelhaupt, F.M., Snijders, J.G. and Baerends, E.J.** (1999). The nature of the hydrogen bond in DNA base pairs: The role of charge transfer and resonance assistance, *Chemistry - A European Journal*, 5(12), 3581–3593, [https://doi.org/10.1002/\(SICI\)1521-3765\(19991203\)5:12<3581::AID-CHEM3581>3.0.CO;2-Y](https://doi.org/10.1002/(SICI)1521-3765(19991203)5:12<3581::AID-CHEM3581>3.0.CO;2-Y).

- [21] **Fonseca Guerra, C., Bickelhaupt, F.M., Snijders, J.G. and Baerends, E.J.** (2000). Hydrogen bonding in DNA base pairs: Reconciliation of theory and experiment, *Journal of The American Chemical Society*, 122(17), 411–128, <https://doi.org/10.1021/ja993262d>.
- [22] **van der Wijst, T., Fonseca Guerra, C., Swart, M. and Bickelhaupt, F.M.** (2006). Performance of various density functionals for the hydrogen bonds in DNA base pairs, *Chemical Physics Letters*, 426(4–6), 415–421, <https://doi.org/10.1016/j.cplett.2006.06.057>.
- [23] **Fersht, A.** (1999). *Structure and mechanism in protein science: a guide to enzyme catalysis and protein folding*, W.H. Freeman, 2nd edition.
- [24] **Garcia-Viloca, M., Gao, J., Karplus, M. and Truhlar, D.G.** (2004). How enzymes work: analysis by modern rate theory and computer simulations, *Science*, 303(5655), 186–195, <https://doi.org/10.1126/science.1088172>.
- [25] **Fischer, E.** (1894). “Einfluss der configuration auf die wirkung der enzyme” (Influence of configuration on the action of enzymes), *Berichte der deutschen chemischen Gesellschaft zu Berlin (in German)*, 27(3), 2984–2993, <https://doi.org/10.1002/cber.18940270364>.
- [26] **Koshland, D.E.** (1958). Application of a theory of enzyme specificity to protein synthesis, *Proceedings of the National Academy of Sciences of the United States of America*, 44(2), 98–104, <https://doi.org/10.1073/pnas.44.2.98>.
- [27] **Hammes, G.G.** (2002). Multiple conformational changes in enzyme catalysis, *Biochemistry*, 41(26), 8221–8228, <https://doi.org/10.1021/bi0260839>.
- [28] **Savir, Y. and Tlusty, T.** (2007). Conformational proofreading: the impact of conformational changes on the specificity of molecular recognition, *PLoS ONE*, 2(5), e468, <https://doi.org/10.1371/journal.pone.0000468>.
- [29] **Straub, F.B. and Szabolcsi, G.** (1964). “O dinamicseszkij aspektah szstrukturü fermentov” (On the dynamic aspects of protein structure), **A.E. Braunstein**, editor, *Molecular Biology, Problems and Perspectives (in Russian)*, Nauka, Moscow: Izdat, pp.182–187.
- [30] **Ma, B., Kumar, S. and Tsai, C. J. and Nussinov, R.** (1999). Folding funnels and binding mechanisms, *Protein Engineering*, 12(9), 713–720, <https://doi.org/10.1093/protein/12.9.713>.
- [31] **Ma, B., Shatsky, M., Wolfson, H.J. and Nussinov, R.** (2002). Multiple diverse ligands binding at a single protein site: a matter of pre-existing populations, *Protein Science*, 11(2), 184–197, <https://doi.org/10.1110/ps.21302>.

- [32] **Poelarends, G.J., Veetil, V.P. and Whitman, C.P.** (2008). The chemical versatility of the beta-alpha-beta fold: catalytic promiscuity and divergent evolution in the tautomerase superfamily, *Cellular and Molecular Life Sciences*, 65(22), 3606–3618, <https://doi.org/10.1007/s00018-008-8285-x>.
- [33] **Zhu, C., Lu, L., Zhang, J., Yue, Z., Song, J., Zong, S., Liu, M., Stovicek, O., Gao, Y.Q. and Yi, C.** (2016). Tautomerization-dependent recognition and excision of oxidation damage in base-excision DNA repair, *Proceedings of the National Academy of Sciences of the United States of America*, 113(28), 7792–7797, <https://doi.org/10.1073/pnas.1604591113>.
- [34] **Singh, V., Fedeles, B.I. and Essigmann, J.M.** (2015). Role of tautomerism in RNA biochemistry, *RNA*, 21(1), 1–13, <https://doi.org/10.1261/rna.048371.114>.
- [35] **Zahurancik, W.C., Klein, S.J. and Suo, Z.** (2013). Kinetic mechanism of DNA polymerization catalyzed by human DNA polymerase ϵ , *Biochemistry*, 52(40), 7041–7049, <https://doi.org/10.1021/bi400803v>.
- [36] **Miller, B.I., A., P.C. and Wu, E.Y.** (2014). Molecular dynamics study of the opening mechanism for DNA polymerase I, *PLoS Computational Biology*, 10(12), e1003961, <https://doi.org/10.1371/journal.pcbi.1003961>.
- [37] **Moscato, B., Swain, M. and Loria, J.P.** (2015). Induced fit in the selection of correct versus incorrect nucleotides by DNA polymerase β , *Biochemistry*, 55(2), 382–395, <https://doi.org/10.1021/acs.biochem.5b01213>.
- [38] **Evans, G.W., Hohlbein, J., Craggs, T., Aigrain, L. and Kapanidis, A.N.** (2015). Real-time single-molecule studies of the motions of DNA polymerase fingers illuminate DNA synthesis mechanisms, *Nucleic Acids Research*, 43(12), 5998–6008, <https://doi.org/10.1093/nar/gkv547>.
- [39] **Batra, V.K., Beard, W.A., Pedersen, L.C. and Wilson, S.H.** (2016). Switch to standard view structures of DNA polymerase mispaired DNA termini transitioning to pre-catalytic complexes support an induced-fit fidelity mechanism, *Structure*, 24(11), 1863–1875, <http://dx.doi.org/10.1016/j.str.2016.08.006>.
- [40] **Alparone, A., Millefiori, A. and Millefiori, S.** (2005). Non-planarity and solvent effects on structural and polarizability properties of cytosine tautomers, *Chemical Physics*, 312(1–3), 261–274, <https://doi.org/10.1016/j.chemphys.2004.12.005>.
- [41] **Fonseca Guerra, C., Bickelhaupt, F.M., Saha, S. and Wang, F.** (2006). Adenine tautomers: Relative stabilities, ionization energies, and mismatch with cytosine, *The Journal of Physical Chemistry A*, 110(11), 4012–4020, <https://doi.org/10.1021/jp057275r>.

- [42] **Liang, W., Li, H., Hu, X. and Han, S.** (2006). Systematic theoretical investigations on all of the tautomers of guanine: From both dynamics and thermodynamics viewpoint, *Chemical Physics*, 328(1–3), 93–102, <http://dx.doi.org/10.1016/j.chemphys.2006.06.025>.
- [43] **Fan, J.C., Shang, Z.C., Liang, J., Liu, X.H. and Jin, H.** (2010). Systematic theoretical investigations on the tautomers of thymine in gas phase and solution, *Journal of Molecular Structure: THEOCHEM*, 939(1–3), 106–111, <https://doi.org/10.1016/j.theochem.2009.09.047>.
- [44] **Li, H., Zhang, L. and Fan, X.** (2014). Metal counterion modulated single proton transfer process in guanine base, *Computational and Theoretical Chemistry*, 1032, 90–96, <https://doi.org/10.1016/j.comptc.2014.01.026>.
- [45] **Li, H., Zhang, L., Zhou, H., Wang, Y. and Fan, X.** (2015). Theoretical studies on the single proton transfer process in adenine base, *Journal of Physical Organic Chemistry*, 28(12), 755–760, <https://doi.org/10.1002/poc.3479>.
- [46] **Löwdin, P.O.** (1963). Proton Tunneling in DNA and its Biological Implications, *Reviews of Modern Physics*, 35(3), 724–732, <https://doi.org/10.1103/RevModPhys.35.724>.
- [47] **Jacob, F. and Monod, J.** (1961). Genetic regulatory mechanisms in the synthesis of proteins, *Journal of Molecular Biology*, 3(3), 318–356, [https://doi.org/10.1016/S0022-2836\(61\)80072-7](https://doi.org/10.1016/S0022-2836(61)80072-7).
- [48] **Rodbell, M.** (1980). The role of hormone receptors and GTP-regulatory proteins in membrane transduction, *Nature*, 284(5751), 17–22, <https://doi.org/10.1038/284017a0>.
- [49] **Ehrlich, P.** (1891). “Experimentelle Untersuchungen über Immunität. II” (Experimental studies on immunity II), *Dtsch med Wochenschr (in German)*, 17(44), 1218–1219, <https://doi.org/10.1055/s-0029-1206825>.
- [50] **Ehrlich, P.** (1987). “Die Wertbemessung ddes Diphterieheilserums and deren theoretisch Grundlagen” (The value assessment of the diphtheria healing serum and its theoretical fundamentals), *Klinisches Jahrbuch (in German)*, 6, 299–326, <https://www.pei.de/EN/institute/paul-ehrlich/publications/publications-of-paul-ehrlich-node.html>.
- [51] **Pauling, L., Campbell, D.H. and Pressman, D.** (1943). The nature of the forces between antigen and antibody and of the precipitation reaction, *Physiological Reviews*, 23(3), 203–219, <https://doi.org/10.1152/physrev.1943.23.3.203>.
- [52] **Benoit, M., M.D. and Parrinello, M.** (1998). Tunnelling and zero-point motion in high-pressure ice, *Nature*, 392(6673), 258–261, <https://doi.org/10.1038/32609>.

- [53] **Castro Neto, A.H., Pujol, P. and Fradkin, E.** (2006). Ice: A strongly correlated proton system, *Physical Review B*, 74(2-1), 024302, <https://doi.org/10.1103/PhysRevB.74.024302>.
- [54] **Bernal, J.D. and Fowler, R.H.** (1933). A theory of water and ionic solution with particular reference to hydrogen and hydroxyl ions, *The Journal of Chemical Physics*, 1(8), 515–548, <https://doi.org/10.1063/1.1749327>.
- [55] **Pauling, L.** (1935). The structure and entropy of ice and of other crystals with some randomness of atomic arrangement, *Journal of The American Chemical Society*, 57(12), 2680–2684, <https://doi.org/10.1021/ja01315a102>.
- [56] **Bove, L.E., Klotz, S., Paciaroni, A. and Sacchetti, F.** (2009). Anomalous proton dynamics in ice at low temperatures, *Physical Review Letters*, 103(16), 165901, <https://doi.org/10.1103/PhysRevLett.103.165901>.
- [57] **Meng, X., Guo, J., Peng, J., Chen, J., Wang, Z., Shi, J.R., Li, Z.Z., Wang, E.G. and Jiang, Y.** (2015). Direct visualization of concerted proton tunnelling in a water nanocluster, *Nature Physics*, 11(3), 235–239, <https://doi.org/10.1038/nphys3225>.
- [58] **Yen, F. and Gao, T.** (2015). Dielectric Anomaly in Ice near 20 K: Evidence of Macroscopic Quantum Phenomena, *The Journal of Physical Chemistry Letters*, 6(14), 2822–2825, <https://doi.org/10.1021/acs.jpcllett.5b00797>.
- [59] **Lin, L., Morrone, J.A. and Car, R.** (2011). Correlated tunneling in hydrogen bonds, *Journal of Statistical Physics*, 145(2), 365–384, <https://doi.org/10.1007/s10955-011-0320-x>.
- [60] **Drechsel-Grau, C. and Marx, D.** (2014). Quantum simulation of collective proton tunneling in hexagonal ice crystals, *Physical Review Letters*, 112(14), 148302, <https://doi.org/10.1103/PhysRevLett.112.148302>.
- [61] **Benton, O., Sikora, O. and Shannon, N.** (2016). Classical and quantum theories of proton disorder in hexagonal water ice, *Physical Review B*, 93(12), 125143, <https://doi.org/10.1103/PhysRevB.93.125143>.
- [62] **Cha, Y., M.C. and J.P., K.** (1989). Hydrogen tunneling in enzyme reactions, *Science*, 243(4896), 1325–1330, <https://doi.org/10.1126/science.2646716>.
- [63] **Nesheim, J.C. and Lipscomb, J.D.** (1996). Large kinetic isotope effects in methane oxidation catalyzed by methane monooxygenase: evidence for C-H bond cleavage in a reaction cycle intermediate, *Biochemistry*, 35(31), 10240–10247, <https://doi.org/10.1021/bi960596w>.

- [64] **Whittaker, M.M., Ballou, D.P. and Whittaker, J.W.** (1998). Kinetic isotope effects as probes of the mechanism of galactose oxidase, *Biochemistry*, 37(23), 8426–8436, <https://doi.org/10.1021/bi980328t>.
- [65] **Basran, J., Sutcliffe, M.J. and Scrutton, N.S.** (1999). Enzymatic H-transfer requires vibration-driven extreme tunneling, *Biochemistry*, 38(10), 3218–3222, <https://doi.org/10.1021/bi982719d>.
- [66] **Kohen, A., Cannio, R., Bartolucci, S. and Klinman, J.** (1999). Enzyme dynamics and hydrogen tunneling in a thermophilic alcohol dehydrogenase, *Nature*, 399(6735), 496–499, <https://doi.org/10.1038/20981>.
- [67] **Harris, R., Meskys, R., Sutcliffe, M.J. and Scrutton, N.S.** (2000). Kinetic studies of the mechanism of carbon-hydrogen bond breakage by the heterotetrameric sarcosine oxidase of *Arthrobacter* sp. 1-IN, *Biochemistry*, 39(6), 1189–1198, <https://doi.org/10.1021/bi991941v>.
- [68] **Allemann, R. and Scrutton, N.** (2009). *Quantum tunnelling in enzyme-catalysed reactions*, The Royal Society of Chemistry, Cambridge, <https://doi.org/10.1039/9781847559975>.
- [69] **Sen, A. and Kohen, A.** (2010). Enzymatic tunneling and kinetic isotope effects: chemistry at the crossroads, *Journal of Physical Organic Chemistry*, 23(7), 613–619, <https://doi.org/10.1002/poc.1633>.
- [70] **Borgis, D. and Hynes, J.T.** (1991). Molecular dynamics simulation for a model nonadiabatic proton transfer reaction in solution, *The Journal of Chemical Physics*, 94(5), 3619–3628, <https://doi.org/10.1063/1.459733>.
- [71] **Antoniou, D. and Schwartz, S.** (1997). Large kinetic isotope effects in enzymatic proton transfer and the role of substrate oscillations, *Proceedings of the National Academy of Sciences of the United States of America*, 94(23), 12360–12365, <http://www.pnas.org/content/94/23/12360.full>.
- [72] **Antoniou, D. and Schwartz, S.** (2001). Internal enzyme motions as a source of catalytic activity: rate-promoting vibrations and hydrogen tunneling, *The Journal of Physical Chemistry B*, 105(23), 5553–5558, <https://doi.org/10.1021/jp004547b>.
- [73] **Masgrau, L., Roujeinikova, A., Johannissen, L.O., Hothi, P., Basran, J., Ranaghan, K.E., Mulholland, A., Sutcliffe, M.J., Scrutton, N.S. and Leys, D.** (2006). Atomic description of an enzyme reaction dominated by proton tunneling, *Science*, 312(5771), 237–241, <https://doi.org/10.1126/science.1126002>.
- [74] **Sutcliffe, M.J., Masgrau, L., Roujeinikova, A., Johannissen, K.E., Hothi, P., Basran, J., Ranaghan, K.E., Mulholland, A.J., Leys, D. and Scrutton, N.J.** (2006). Hydrogen tunnelling in enzyme-catalysed H-transfer reactions: flavoprotein and quinoprotein systems, *Philosophical Transactions of The Royal Society B*, 361(1472), 1375–1386, <https://doi.org/10.1098/rstb.2006.1878>.

- [75] **Hammes-Schiffer, S. and Watney, J.B.** (2006). Hydride transfer catalysed by *Escherichia coli* and *Bacillus subtilis* dihydrofolate reductase: coupled motions and distal mutations, *Philosophical Transactions of The Royal Society B*, 361(1472), 1365–1373, <https://doi.org/10.1098/rstb.2006.1869>.
- [76] **Glowacki, D.R., Harvey, J.N. and Mulholland, A.J.** (2012). Taking Ockham's razor to enzyme dynamics and catalysis, *Nature Chemistry*, 4(3), 169–176, <https://doi.org/10.1038/nchem.1244>.
- [77] **Klinman, J.P. and Kohen, A.** (2013). Hydrogen tunneling links protein dynamics to enzyme catalysis, *Annual Review of Biochemistry*, 82, 471–496, <https://doi.org/10.1146/annurev-biochem-051710-133623>.
- [78] **Jevtic, S. and Anders, J.** (2017). A qualitative quantum rate model for hydrogen transfer in soybean lipoxygenase, *The Journal of Chemical Physics*, 147(11), 114108, <https://doi.org/10.1063/1.4998941>.
- [79] **Doll, K., Bender, B. and Finke, R.** (2003). The first experimental test of the hypothesis that enzymes have evolved to enhance hydrogen tunneling, *Journal of The American Chemical Society*, 125(36), 10877–10884, <https://doi.org/10.1021/ja030120h>.
- [80] **Hwang, J., Chu, Z., Yadav, A. and Warshel, A.** (1991). Simulations of quantum-mechanical corrections for rate constants of hydride-transfer reactions in enzymes and solutions, *The Journal of Physical Chemistry*, 95(22), 8445–8448, <https://doi.org/10.1021/j100175a009>.
- [81] **Hwang, J. and Warshel, A.** (1996). How important are quantum mechanical nuclear motions in enzyme catalysis?, *Journal of The American Chemical Society*, 118(47), 11745–11751, <https://doi.org/10.1021/ja962007f>.
- [82] **Olsson, M., Siegbahn, P. and Warshel, A.** (2004). Simulations of the large kinetic isotope effect and the temperature dependence of the hydrogen atom transfer in lipoxygenase, *Journal of The American Chemical Society*, 126(9), 2820–2828, <https://doi.org/10.1021/ja037233l>.
- [83] **Olsson, M., Marvi, J. and Warshel, A.** (2006). Transition state theory can be used in studies of enzyme catalysis: lessons from simulations of tunnelling and dynamical effects in lipoxygenase and other systems, *Philosophical Transactions of The Royal Society B*, 361(1472), 1417–1432, <https://doi.org/10.1098/rstb.2006.1880>.
- [84] **Williams, I.** (2010). Quantum catalysis? A comment on tunnelling contributions for catalysed and uncatalysed reactions, *Journal of Physical Organic Chemistry*, 23(7), 685–689, <https://doi.org/10.1002/poc.1658>.
- [85] **Bothma, J.P., Gilmore, J.B. and McKenzie, R.H.** (2010). The role of quantum effects in proton transfer reactions in enzymes: quantum tunneling in a

- noisy environment?, *New Journal of Physics*, 12(5), 055002, <https://doi.org/10.1088/1367-2630/12/5/055002>.
- [86] **Ball, P.** (2004). Enzymes: By chance, or by design?, *Nature*, 431(7007), 396–397, <https://doi.org/10.1038/431396a>.
- [87] **Dutton, P.L., Munro, A.W., Scrutton, N.J. and Sutcliffe, M.J.** (2006). Introduction. Quantum catalysis in enzymes: beyond the transition state theory paradigm, *Philosophical Transactions of The Royal Society B*, 361(1472), 1293–1294, <https://doi.org/10.1098/rstb.2006.1879>.
- [88] **Brookes, J.C.** (2017). Quantum effects in biology: golden rule in enzymes, olfaction, photosynthesis and magnetodetection, *Proceedings of The Royal Society A*, 473(2201), 20160822, <https://doi.org/10.1098/rspa.2016.0822>.
- [89] **Villani, G.** (2005). Theoretical investigation of hydrogen transfer mechanism in the Adenine-Tymine base pair, *Chemical Physics*, 316(1–3), 1–8, <http://dx.doi.org/10.1016/j.chemphys.2005.04.030>.
- [90] **Villani, G.** (2006). Theoretical investigation of hydrogen transfer mechanism in the Guanine-Cytosine base pair, *Chemical Physics*, 324(2–3), 438–446, <http://dx.doi.org/10.1016/j.chemphys.2005.11.006>.
- [91] **Villani, G.** (2007). Theoretical investigation of the coupling between the hydrogen transfer and the base pair opening in the Adenine-Tymine system, *Chemical Physics*, 336(2–3), 143–149, <https://doi.org/10.1016/j.chemphys.2007.06.005>.
- [92] **Villani, G.** (2008). A time-dependent quantum dynamics investigation of the Guanine-Cytosine system: A six-dimensional model, *The Journal of Chemical Physics*, 128(11), 114306, <http://dx.doi.org/10.1063/1.2890040>.
- [93] **Villani, G.** (2010). Theoretical investigation of the hydrogen atom transfer in the Adenine-Tymine base pair and its coupling with electronic rearrangement. Concerted vs. stepwise mechanism, *Physical Chemistry Chemical Physics*, 12(11), 2664–2669, <https://doi.org/10.1039/B917672A>.
- [94] **Villani, G.** (2010). Theoretical investigation of the hydrogen atom transfer in the Cytosine-Guanine base pair and its coupling with electronic rearrangement. Concerted vs. stepwise mechanism, *Journal of Physical Chemistry B*, 114(29), 9653–9662, <https://doi.org/10.1021/jp102457s>.
- [95] **Brovarets, O.O. and Hovorun, D.M.** (2014). Can tautomerization of the A·T Watson-Crick base pair via double proton transfer provoke point mutations during DNA replication? A comprehensive QM and QTAIM analysis, *Journal of Biomolecular Structure and Dynamics*, 32(1), 127–154, <http://dx.doi.org/10.1080/07391102.2012.755795>.

- [96] **Brovarets, O.O. and Hovorun, D.M.** (2014). Why the tautomerization of the G·C Watson-Crick base pair via the DPT does not cause point mutations during DNA replication? QM and QTAIM comprehensive analysis, *Journal of Biomolecular Structure and Dynamics*, 32(9), 1474–1499, <http://dx.doi.org/10.1080/07391102.2013.822829>.
- [97] **Godbeer, A.D., Al-Khalili, J.S. and Stevenson, P.D.** (2015). Modelling proton tunnelling in the adenine-thymine base pair, *Physical Chemistry Chemical Physics*, 17(19), 13034–13044, <https://doi.org/10.1039/c5cp00472a>.
- [98] **Brovarets, O.O. and Hovorun, D.M.** (2015). Proton tunneling in the A·T Watson-Crick DNA base pair: myth or reality?, *Journal of Biomolecular Structure and Dynamics*, 33(12), 2716–2720, <http://dx.doi.org/10.1080/07391102.2015.1092886>.
- [99] **Turaeva, N. and Brown-Kennerly, V.** (2015). Marcus model of spontaneous point mutation in DNA, *Chemical Physics*, 461, 106–110, <http://dx.doi.org/10.1016/j.chemphys.2015.09.005>.
- [100] **Pusuluk, O., Torun, G. and Deliduman, C.** (2018). Quantum entanglement shared in H-bonds and its usage in molecular recognition, *Modern Physics Letters B*, 32(26), 1850308, <https://doi.org/10.1142/S0217984918503086>.
- [101] **Pusuluk, O., Farrow, T., Deliduman, C. and Vedral, V.** (2018). Emergence of correlated proton tunneling in water ice, *arXiv/quant-ph*, <https://arxiv.org/abs/1703.01335>.
- [102] **Pusuluk, O., Farrow, T., Deliduman, C., Burnett, K. and Vedral, V.** (2018). Proton tunneling in hydrogen bonds and its implications in an induced-fit model of enzyme catalysis, *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 474(2218), 20180037, <http://dx.doi.org/10.1098/rspa.2018.0037>.
- [103] **von Neumann, J.** (1927). “Mathematische Begründung der Quantenmechanik”, *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse (in German)*, 1927, 1–57, <http://eudml.org/doc/59215>.
- [104] **von Neumann, J.** (1930). “Allgemeine Eigenwerttheorie Hermitescher Funktionaloperatoren”, *Mathematische Annalen (in German)*, 102, 49–131, <http://eudml.org/doc/159371>.
- [105] **Dirac, P.A.M.** (1930). *The Principles of Quantum Mechanics*, Clarendon Press, Oxford.
- [106] **von Neumann, J.** (1932). *Mathematische Grundlagen der Quantenmechanik*, J. Springer, Berlin, (English translation: **Beyer, R. T.** (1955). *Mathematical foundations of quantum mechanics*, Princeton University Press.).

- [107] **Dirac, P.A.M.** (1939). A new notation for quantum mechanics, *Mathematical Proceedings of the Cambridge Philosophical Society*, 35(3), 416–418, <https://doi.org/10.1017/S0305004100021162>.
- [108] **Born, M.** (1926). “Zur Quantenmechanik der Stoßvorgänge”, *Zeitschrift für Physik (in German)*, 37(12), 863–867, <https://doi.org/10.1007/BF01397477>, (English translation: **Wheeler, J. A. and Zurek, W. H.** (1983). On the quantum mechanics of collisions, *Wheeler, J. A. and Zurek, W. H., editors, Quantum theory and measurement*, Princeton University Press, pp.52–55.).
- [109] **von Neumann, J.** (1927). “Wahrscheinlichkeitstheoretischer Aufbau der Quantenmechanik”, *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse (in German)*, 1927, 245–272, <http://eudml.org/doc/59230>.
- [110] **von Neumann, J.** (1927). “Thermodynamik quantenmechanischer Gesamtheiten”, *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse (in German)*, 1927, 273–291, <http://eudml.org/doc/59231>.
- [111] **Landau, L.** (1927). “Das Dämpfungsproblem in der Wellenmechanik”, *Zeitschrift für Physik (in German)*, 45(5–6), 430–441, <https://doi.org/10.1007/BF01343064>, (English translation: **Haar, D. Ter** (1965). The damping problem in wave mechanics, *Haar, D. Ter, editor, Collected papers of L.D. Landau*, Pergamon Press and Gordon & Breach, pp.8–18, <https://doi.org/10.1016/B978-0-08-010586-4.50007-9>).
- [112] **Fano, U.** (1957). Description of states in quantum mechanics by density matrix and operator techniques, *Review of Modern Physics*, 29(1), 74–93, <https://doi.org/10.1103/RevModPhys.29.74>.
- [113] **Einstein, A., Podolsky, B. and Rosen, N.** (1935). Can quantum-mechanical description of physical reality be considered complete?, *Physical Review*, 47(10), 777–780, <https://doi.org/10.1103/PhysRev.47.777>.
- [114] **Schrödinger, E.** (1935). “Die gegenwärtige Situation in der Quantenmechanik”, *Die Naturwissenschaften (in German)*, 23(48), 807–812, <https://doi.org/10.1007/BF01491891>, (English translation: **Trimmer, J. D.** (1980). The present situation in quantum mechanics: A translation of Schrödinger’s “cat paradox” paper, *Proceedings of the American Philosophical Society*, 124(5), 323–338, <http://www.jstor.org/stable/986572>).
- [115] **Schrödinger, E.** (1935). “Die gegenwärtige Situation in der Quantenmechanik”, *Die Naturwissenschaften (in German)*, 23(49), 823–828, <https://doi.org/10.1007/BF01491914>, (English translation:

- Trimmer, J. D.** (1980). The present situation in quantum mechanics: A translation of Schrödinger's "cat paradox" paper, *Proceedings of the American Philosophical Society*, 124(5), 323–338, <http://www.jstor.org/stable/986572>).
- [116] **Schrödinger, E.** (1935). "Die gegenwärtige Situation in der Quantenmechanik", *Die Naturwissenschaften (in German)*, 23(50), 844–849, <https://doi.org/10.1007/BF01491987>, (English translation: **Trimmer, J. D.** (1980). The present situation in quantum mechanics: A translation of Schrödinger's "cat paradox" paper, *Proceedings of the American Philosophical Society*, 124(5), 323–338, <http://www.jstor.org/stable/986572>).
- [117] **Schrödinger, E.** (1935). Discussion of probability relations between separated systems, *Mathematical Proceedings of the Cambridge Philosophical Society*, 31(4), 555–563, <https://doi.org/10.1017/S0305004100013554>.
- [118] **Werner, R.F.** (1989). Quantum states with Einstein-Podolsky-Rosen correlations admitting a hidden-variable model, *Physical Review A*, 40(8), 4277–4281, <https://doi.org/10.1103/PhysRevA.40.4277>.
- [119] **Henderson, L. and Vedral, V.** (2001). Classical, quantum and total correlations, *Journal of Physics A: Mathematical and General*, 34(35), 6899–6905, <https://doi.org/10.1088/0305-4470/34/35/315>.
- [120] **Ollivier, H. and Zurek, W.H.** (2001). Quantum discord: A measure of the quantumness of correlations, *Physical Review Letters*, 88(1), 017901, <https://doi.org/10.1103/PhysRevLett.88.017901>.
- [121] **Oppenheim, J., Horodecki, M., Horodecki, P. and Horodecki, R.** (2002). Thermodynamical approach to quantifying quantum correlations, *Physical Review Letters*, 89(18), 180402, <https://doi.org/10.1103/PhysRevLett.89.180402>.
- [122] **Li, N. and Luo, S.** (2008). Classical states versus separable states, *Physical Review A*, 78(2), 024303.
- [123] **Horodecki, M., Horodecki, P., Horodecki, R., Oppenheim, J., Sen(De), A., Sen, U. and Synak-Radtke, B.** (2005). Local versus nonlocal information in quantum-information theory: Formalism and phenomena, *Physical Review A*, 71(6), 062307, <https://doi.org/10.1103/PhysRevA.71.062307>.
- [124] **Piani, M., Horodecki, P. and Horodecki, R.** (2008). No-local-broadcasting theorem for multipartite quantum correlations, *Physical Review Letters*, 100(9), 090502, <https://doi.org/10.1103/PhysRevLett.100.090502>.
- [125] **Dirac, P.A.M.** (1930). Note on exchange phenomena in the Thomas Atom, *Mathematical Proceedings of the Cambridge Philosophical Society*, 26(3), 376–385, <https://doi.org/10.1017/S0305004100016108>.

- [126] **Schrödinger, E.** (1926). “Quantisierung als Eigenwertproblem”, *Annalen der Physik (in German)*, 384(4), 361–376, <https://doi.org/10.1002/andp.19263840404>.
- [127] **Schrödinger, E.** (1926). “Quantisierung als Eigenwertproblem”, *Annalen der Physik (in German)*, 384(6), 489–527, <https://doi.org/10.1002/andp.19263840602>.
- [128] **Schrödinger, E.** (1926). “Über das Verhältnis der Heisenberg-Born-Jordanschen Quantenmechanik zu der meinem”, *Annalen der Physik (in German)*, 384(8), 734–756, <https://doi.org/10.1002/andp.19263840804>.
- [129] **Schrödinger, E.** (1926). “Quantisierung als Eigenwertproblem”, *Annalen der Physik (in German)*, 385(13), 437–490, <https://doi.org/10.1002/andp.19263851302>.
- [130] **Schrödinger, E.** (1926). “Quantisierung als Eigenwertproblem”, *Annalen der Physik (in German)*, 386(18), 109–139, <https://doi.org/10.1002/andp.19263861802>.
- [131] **Schrödinger, E.** (1926). “Der stetige Übergang von der Mikro- zur Makromechanik”, *Die Naturwissenschaften (in German)*, 14(28), 664–666, <https://doi.org/10.1007/BF01507634>.
- [132] **Schrödinger, E.** (1926). An undulatory theory of the mechanics of atoms and molecules, *Physical Review*, 28(6), 1049–1070.
- [133] **Zeh, H.D.** (1970). On the interpretation of measurement in quantum theory, *Foundations of Physics*, 1(1), 69–76, <https://doi.org/10.1007/BF00708656>.
- [134] **Zeh, H.D.** (1973). Toward a quantum theory of observation, *Foundations of Physics*, 3(1), 109–116, <https://doi.org/10.1007/BF00708603>.
- [135] **Zurek, W.H.** (1981). Pointer basis of quantum apparatus: Into what mixture does the wave packet collapse?, *Physica Review D*, 24(6), 1516–1525, <https://doi.org/10.1103/PhysRevD.24.1516>.
- [136] **Zurek, W.H.** (1982). Environment-induced superselection rules, *Physical Review D*, 26(8), 1862–1880, <https://doi.org/10.1103/PhysRevD.26.1862>.
- [137] **Joos, E. and Zeh, H.D.** (1985). The emergence of classical properties through interaction with the environment, *Zeitschrift für Physik B Condensed Matter*, 59(2), 223–243, <https://doi.org/10.1007/BF01725541>.
- [138] **Zurek, W.H.** (1991). Decoherence and the transition from quantum to classical, *Physics Today*, 44(10), 36–44, <https://doi.org/10.1063/1.881293>, updated version is available at [quant-ph/0306072](https://arxiv.org/abs/quant-ph/0306072).

- [139] **Zurek, W.H.** (1993). Preferred states, predictability, classicality and the environment-induced decoherence, *Progress of Theoretical Physics*, 89(2), 281–312, <https://doi.org/10.1143/ptp/89.2.281>.
- [140] **Zurek, W.H.** (2003). Decoherence, einselection, and the quantum origins of the classical, *Reviews of Modern Physics*, 75(3), 715–775, <https://doi.org/10.1103/RevModPhys.75.715>.
- [141] **Schlosshauer, M.** (2005). Decoherence, the measurement problem, and interpretations of quantum mechanics, *Reviews of Modern Physics*, 76(4), 1267–1305, <https://doi.org/10.1103/RevModPhys.76.1267>.
- [142] **Sudarshan, E.C.G., Mathews, P.M. and Rau, J.** (1961). Stochastic dynamics of quantum-mechanical systems, *Physical Review*, 121(3), 920–924, <https://doi.org/10.1103/PhysRev.121.920>.
- [143] **Jordan, T.F. and Sudarshan, E.C.G.** (1961). Dynamical mappings of density operators in quantum mechanics, *Journal of Mathematical Physics*, 2(6), 772–775, <https://doi.org/10.1063/1.1724221>.
- [144] **Davies, E.B. and Lewis, J.T.** (1970). An operational approach to quantum probability, *Communications in Mathematical Physics*, 17(3), 239–260, <https://doi.org/10.1007/BF01647093>.
- [145] **Kraus, K.** (1971). General state changes in quantum theory, *Annals of Physics*, 64(2), 311–335.
- [146] **Jamiołkowski, A.** (1972). Linear transformations which preserve trace and positive semidefiniteness of operators, *Reports on Mathematical Physics*, 3(4), 275–278, [https://doi.org/10.1016/0034-4877\(72\)90011-0](https://doi.org/10.1016/0034-4877(72)90011-0).
- [147] **Lindblad, G.** (1975). Completely positive maps and entropy inequalities, *Communications in Mathematical Physics*, 40(2), 147–151.
- [148] **Pechukas, P.** (1994). Reduced dynamics need not be completely positive, *Physical Review Letters*, 73(8), 1060–1062, <https://doi.org/10.1103/PhysRevLett.73.1060>.
- [149] **Pechukas, P.** (1995). Pechukas Replies:, *Physical Review Letters*, 75(16), 3021–3021, <https://doi.org/10.1103/PhysRevLett.75.3021>.
- [150] **Jordan, T.F., Shaji, A. and Sudarshan, E.C.G.** (2004). Dynamics of initially entangled open quantum systems, *Physical Review A*, 70(5), 052110, <https://doi.org/10.1103/PhysRevA.70.052110>.
- [151] **Carteret, H.A., Terno, D.R. and Żfyczkowski, K.** (2008). Dynamics beyond completely positive maps: Some properties and applications, *Physical Review A*, 77(4), 042113, <https://doi.org/10.1103/PhysRevA.77.042113>.

- [152] **Rodríguez-Rosario, C.A., Modi, K., Kuah, A., Shaji, A. and Sudarshan, E.C.G.** (2008). Completely positive maps and classical correlations, *Journal of Physics A: Mathematical and Theoretical*, 41(20), 205301, <https://doi.org/10.1088/1751-8113/41/20/205301>.
- [153] **Shabani, A. and Lidar, D.A.** (2009). Vanishing quantum discord is necessary and sufficient for completely positive maps, *Physical Review Letters*, 102(10), 100402, <https://doi.org/10.1103/PhysRevLett.102.100402>.
- [154] **Modi, K., Rodríguez-Rosario, C.A. and Aspuru-Guzik, A.** (2012). Positivity in the presence of initial system-environment correlation, *Physical Review A*, 86(6), 064102, <https://doi.org/10.1103/PhysRevA.86.064102>.
- [155] **Broducth, A., Datta, A., Modi, K., Rivas, A. and Rodríguez-Rosario, C.A.** (2013). Vanishing quantum discord is not necessary for completely positive maps, *Physical Review A*, 87(4), 042301, <https://doi.org/10.1103/PhysRevA.87.042301>.
- [156] **Buscemi, F.** (2014). Complete positivity, markovianity, and the quantum data-processing inequality, in the presence of initial system-environment correlations, *Physical Review Letters*, 113(14), 140502, <https://doi.org/10.1103/PhysRevLett.113.140502>.
- [157] **Dominy, J.M., Shabani, A. and Lidar, D.A.** (2016). A general framework for complete positivity, *Quantum Information Processing*, 15(1), 465–494.
- [158] **Shabani, A. and Lidar, D.A.** (2016). Erratum: Vanishing quantum discord is necessary and sufficient for completely positive maps [Phys. Rev. Lett. 102, 100402 (2009)], *Physical Review Letters*, 116(4), 049901, <https://doi.org/10.1103/PhysRevLett.116.049901>.
- [159] **Benenti, G., Casati, G. and Strini, G.** (2007). *Principles of Quantum Computation And Information: (Volume II) Basic Tools And Special Topics*, World Scientific Publishing Co., Inc., NJ, https://doi.org/10.1142/9789812708458_0001.
- [160] **Nielsen, M.A. and Chuang, I.L.** (2010). *Quantum Computation and Quantum Information: 10th Anniversary Edition*, Cambridge University Press, 10th edition, <https://doi.org/10.1017/CBO9780511976667>.
- [161] **Wilde, M.M.** (2013). *Quantum Information Theory*, Cambridge University Press, <https://doi.org/10.1017/CBO9781139525343>.
- [162] **Preskill, J.** (2015). *Lecture Notes for Physics 229: Quantum Information and Computation*, CreateSpace Independent Publishing Platform, <https://books.google.com.tr/books?id=MIV8rQEACAAJ>.
- [163] **Lindblad, G.** (1976). On the generators of quantum dynamical semigroups, *Communications in Mathematical Physics*, 48(2), 119–130, <https://doi.org/10.1007/BF016084990>.

- [164] **Gorini, V., Kossakowski, A. and Sudarshan, E.C.G.** (1976). Completely positive dynamical semigroups of N -level systems, *Journal of Mathematical Physics*, 17(5), 821–825, <https://doi.org/10.1063/1.522979>.
- [165] **Davies, E.B.** (1974). Markovian master equations, *Communications in Mathematical Physics*, 39(2), 91–110, <https://doi.org/10.1007/BF01608389>.
- [166] **Breuer, H. and Petruccione, F.** (2002). *The Theory of Open Quantum Systems*, Oxford University Press, <https://doi.org/10.1093/acprof:oso/9780199213900.001.0001>.
- [167] **Weiss, U.** (2008). *Quantum Dissipative Systems*, Series in Modern Condensed Matter Physics - Vol. 13, World Scientific Publishing Co. Pte. Ltd., 3 edition, <https://doi.org/10.1142/9789814374927>.
- [168] **Rivas, A. and Huelga, S.F.** (2012). *Open Quantum Systems: An Introduction*, SpringerBriefs in Physics, Springer, <https://doi.org/10.1007/978-3-642-23354-8>.
- [169] **Refield, A.G.** (1957). On the theory of relaxation processes, *IBM Journal of Research and Development*, 1(1), 19–31, <https://doi.org/10.1147/rd.11.0019>.
- [170] **Plenio, M.B. and Virmani, S.** (2007). An introduction to entanglement measures, *Quantum Information & Computation*, 7(1), 1–51, <https://arxiv.org/abs/quant-ph/0504163v3>.
- [171] **Horodecki, R., Horodecki, P., Horodecki, M. and Horodecki, K.** (2009). Quantum entanglement, *Reviews of Modern Physics*, 81(2), 865–942, <https://doi.org/10.1103/RevModPhys.81.865>.
- [172] **Modi, K., Brodutch, A., Cable, H., Paterek, T. and Vedral, V.** (2012). The classical-quantum boundary for correlations: Discord and related measures, *Reviews of Modern Physics*, 84(4), 1655–1707, <https://doi.org/10.1103/RevModPhys.84.1655>.
- [173] **Spehner, D.** (2014). Quantum correlations and distinguishability of quantum states, *Journal of Mathematical Physics*, 55(7), 075211, <https://doi.org/10.1063/1.4885832>.
- [174] **Adesso, G., Bromley, T.R. and Cianciaruso, M.** (2016). Measures and applications of quantum correlations, *Journal of Physics A: Mathematical and Theoretical*, 49(47), 473001, <https://doi.org/10.1088/1751-8113/49/47/473001>.
- [175] **Hu, M.L., Hu, X., Wang, J.C., Peng, Y., Zhang, Y.R. and Fan, H.** (2017). Quantum coherence and quantum correlations, *arXiv/quant-ph*, 1703.01852, <https://arxiv.org/abs/1703.01852v2>.

- [176] **Bell, J.S.** (1964). On the Einstein-Podolsky-Rosen paradox, *Physics*, 1(3), 195–200, (Reprinted in **Bell, J. S.** (1987). *Speakable and Unsayable in Quantum Mechanics*, Cambridge University Press, pp. 14–21. <https://doi.org/10.1017/CBO9780511815676.004>).
- [177] **Clauser, J.F., Horne, M.A., Shimony, A. and Holt, R.A.** (1969). Proposed experiment to test local hidden-variable theories, *Physical Review Letters*, 23(15), 880–884, <https://doi.org/10.1103/PhysRevLett.23.880>.
- [178] **Freedman, S.J. and Clauser, J.F.** (1972). Experimental test of local hidden-variable theories, *Physical Review Letters*, 28(14), 938–941, <https://doi.org/10.1103/PhysRevLett.28.938>.
- [179] **Aspect, A., Grangier, P. and Roger, G.** (1981). Experimental tests of realistic local theories via Bell’s theorem, *Physical Review Letters*, 47(7), 460–463, <https://doi.org/10.1103/PhysRevLett.47.460>.
- [180] **Aspect, A., Dalibard, J. and Roger, G.** (1982). Experimental test of Bell’s inequalities using time-varying analyzers, *Physical Review Letters*, 49(25), 1804–1807, <https://doi.org/10.1103/PhysRevLett.49.1804>.
- [181] **Aspect, A., Grangier, P. and Roger, G.** (1982). Experimental realization of Einstein-Podolsky-Rosen-Bohm gedankenexperiment: A new violation of Bell’s inequalities, *Physical Review Letters*, 49(2), 91–94, <https://doi.org/10.1103/PhysRevLett.49.91>.
- [182] **Shannon, C.E.** (1948). A mathematical theory of communication, *The Bell System Technical Journal*, 27(3), 379–423, <https://doi.org/10.1002/j.1538-7305.1948.tb01338.x>.
- [183] **Shannon, C.E.** (1948). A mathematical theory of communication, *The Bell System Technical Journal*, 27(4), 623–656, <https://doi.org/10.1002/j.1538-7305.1948.tb00917.x>.
- [184] **Schumacher, B.** (1995). Quantum coding, *Physical Review A*, 51(4), 2738–2747, <https://doi.org/10.1103/PhysRevA.51.2738>.
- [185] **Vedral, V., Plenio, M.B., Rippin, M.A. and Knight, P.L.** (1997). Quantifying entanglement, *Physical Review Letters*, 78(12), 2275–2279, <https://doi.org/10.1103/PhysRevLett.78.2275>.
- [186] **Bennett, C.H., Bernstein, H.J., Popescu, S. and Schumacher, B.** (1996). Concentrating partial entanglement by local operations, *Physical Review A*, 53(4), 2046–2052, <https://doi.org/10.1103/PhysRevA.53.2046>.
- [187] **Popescu, S. and Rohrlich, D.** (1997). Thermodynamics and the measure of entanglement, *Physical Review A*, 56(5), R3319–R3321, <https://doi.org/10.1103/PhysRevA.56.R3319>.

- [188] **Nielsen, M.A.** (1999). Conditions for a class of entanglement transformations, *Physical Review Letters*, 83(2), 436–439, <https://doi.org/10.1103/PhysRevLett.83.436>.
- [189] **Bennett, C.H., DiVincenzo, D.P., Smolin, J.A. and Wootters, W.K.** (1996). Mixed-state entanglement and quantum error correction, *Physical Review A*, 54(5), 3824–3851, <https://doi.org/10.1103/PhysRevA.54.3824>.
- [190] **Hill, S. and Wootters, W.K.** (1997). Entanglement of a pair of quantum bits, *Physical Review Letters*, 78(26), 5022–5025, <https://doi.org/10.1103/PhysRevLett.78.5022>.
- [191] **Wootters, W.K.** (1998). Entanglement of formation of an arbitrary state of two qubits, *Physical Review Letters*, 80(10), 2245–2248, <https://doi.org/10.1103/PhysRevLett.80.2245>.
- [192] **Wootters, W.K.** (2001). Entanglement of formation and concurrence, *Quantum Information & Computation*, 1(1), 27–44, <http://dl.acm.org/citation.cfm?id=2011326.2011329>.
- [193] **Coffman, V., Kundu, J. and Wootters, W.K.** (2000). Distributed entanglement, *Physical Review A*, 61(5), 052306, <https://doi.org/10.1103/PhysRevA.61.052306>.
- [194] **Koashi, M. and Winter, A.** (2004). Monogamy of quantum entanglement and other correlations, *Physical Review A*, 69(2), 022309, <https://doi.org/10.1103/PhysRevA.69.022309>.
- [195] **de Oliveira, T.R., Cornelio, M.F. and Fanchini, F.F.** (2014). Monogamy of entanglement of formation, *Physical Review A*, 89(3), 034303, <https://doi.org/10.1103/PhysRevA.89.034303>.
- [196] **Greenberger, D.M., Horne, M.A. and Zeilinger, A.** (1989). Going Beyond Bell's Theorem, *M. Kafatos, editor, Bell's Theorem, Quantum Theory and Conceptions of the Universe*, Springer Netherlands, pp.69–72, https://doi.org/10.1007/978-94-017-0849-4_10.
- [197] **Bouwmeester, D., Pan, J.W., Daniell, M., Weinfurter, H. and Zeilinger, A.** (1999). Observation of three-photon Greenberger-Horne-Zeilinger entanglement, *Physical Review Letters*, 82(7), 1345–1349, <https://doi.org/10.1103/PhysRevLett.82.1345>.
- [198] **Dür, W., Vidal, G. and Cirac, J.I.** (2000). Three qubits can be entangled in two inequivalent ways, *Physical Review A*, 62(6), 062314, <https://doi.org/10.1103/PhysRevA.62.062314>.
- [199] **Vedral, V. and Plenio, M.B.** (1998). Entanglement measures and purification procedures, *Physical Review A*, 57(3), 1619–1633, <https://doi.org/10.1103/PhysRevA.57.1619>.

- [200] **Schumacher, B. and Westmoreland, M.D.** Relative entropy in quantum information theory, *S.J. Lomonaco and H.E. Brandt, editors, Quantum Computation and Information: AMS Special Session Quantum Computation and Information, January 19-21, 2000, Washington*, Series in Contemporary mathematics - American Mathematical Society.
- [201] **Vedral, V.** (2002). The role of relative entropy in quantum information theory, *Reviews of Modern Physics*, 74(1), 197–234, <https://doi.org/10.1103/RevModPhys.74.197>.
- [202] **Kullback, S. and Leibler, R.A.** (1951). On Information and Sufficiency, *The Annals of Mathematical Statistics*, 22(1), 79–86, <http://www.jstor.org/stable/2236703>.
- [203] **Vedral, V., Plenio, M.B., Jacobs, K. and Knight, P.L.** (1997). Statistical inference, distinguishability of quantum states, and quantum entanglement, *Physical Review A*, 56(6), 4452–4455, <https://doi.org/10.1103/PhysRevA.56.4452>.
- [204] **Vedral, V.** (1999). On bound entanglement assisted distillation, *Physics Letters A*, 262(2), 121–124, [https://doi.org/10.1016/S0375-9601\(99\)00686-6](https://doi.org/10.1016/S0375-9601(99)00686-6).
- [205] **Rains, E.M.** (2001). A semidefinite program for distillable entanglement, *IEEE Transactions On Information Theory*, 47(7), 2921–2933, <https://doi.org/10.1109/18.959270>.
- [206] **Modi, K., Paterek, T., Son, W., Vedral, V. and Williamson, M.** (2010). Unified view of quantum and classical correlations, *Physical Review Letters*, 104(8), 080501, <https://doi.org/10.1103/PhysRevLett.104.080501>.
- [207] **Cover, T.M. and Thomas, J.A.** (2006). *Elements of Information Theory*, John Wiley & Sons, Inc., 2nd edition, <https://doi.org/10.1002/047174882X>.
- [208] **Vedral, V.** (2006). *Introduction to Quantum Information Science*, Oxford University Press, <https://doi.org/10.1093/acprof:oso/9780199215706.001.0001>.
- [209] **Dakić, B., Vedral, V. and Brukner, Č.** (2010). Necessary and sufficient condition for nonzero quantum discord, *Physical Review Letters*, 105(19), 190502, <https://doi.org/10.1103/PhysRevLett.105.190502>.
- [210] **Girolami, D. and Adesso, G.** (2012). Observable measure of bipartite quantum correlations, *Physical Review Letters*, 108(15), 150403, <https://doi.org/10.1103/PhysRevLett.108.150403>.
- [211] **Baumgratz, T., Cramer, M. and Plenio, M.B.** (2014). Quantifying coherence, *Physical Review Letters*, 113(14), 140401, <https://doi.org/10.1103/PhysRevLett.113.140401>.

- [212] **Åberg, J.** (2006). Quantifying superposition, *arXiv/quant-ph*, 0612146, <https://arxiv.org/abs/quant-ph/0612146v1>.
- [213] **Streltsov, A., Singh, U., Dhar, H.S., Bera, M.N. and Adesso, G.** (2015). Measuring quantum coherence with entanglement, *Physical Review Letters*, 115(2), 020403, <https://doi.org/10.1103/PhysRevLett.115.020403>.
- [214] **Ma, J., Yadin, B., Girolami, D., Vedral, V. and Gu, M.** (2016). Converting coherence to quantum correlations, *Physical Review Letters*, 116(16), 160407, <https://doi.org/10.1103/PhysRevLett.116.160407>.
- [215] **Yao, Y., Xiao, X., Ge, L. and Sun, C.P.** (2015). Quantum coherence in multipartite systems, *Physical Review A*, 92(2), 022112, <https://doi.org/10.1103/PhysRevA.92.022112>.
- [216] **Tan, K.C., Kwon, H., Park, C.Y. and Jeong, H.** (2016). Unified view of quantum correlations and quantum coherence, *Physical Review A*, 94(2), 022329, <https://doi.org/10.1103/PhysRevA.94.022329>.
- [217] **Wang, X.L., Yue, Q.L., Yu, C., Gao, F. and Qin, S.J.** (2017). Relating quantum coherence and correlations with entropy-based measures, *arXiv/quant-ph*, 1703.00648, <https://arxiv.org/abs/1703.00648v1>.
- [218] **Guo, Y. and Goswami, S.** (2017). Discordlike correlation of bipartite coherence, *Physical Review A*, 95(6), 062340, <https://doi.org/10.1103/PhysRevA.95.062340>.
- [219] **Pauling, L.** (1931). The nature of the chemical bond. Application of results obtained from the quantum mechanics and from a theory of paramagnetic susceptibility to the structure of molecules, *Journal of the American Chemical Society*, 53(4), 1367–1400, <https://doi.org/10.1021/ja01355a027>.
- [220] **Hibbert, F. and Emsley, J.**, (1990). Hydrogen Bonding and Chemical Reactivity, volume 26 of *Advances in Physical Organic Chemistry*, Academic Press, pp.255–379, [https://doi.org/10.1016/S0065-3160\(08\)60047-7](https://doi.org/10.1016/S0065-3160(08)60047-7).
- [221] **Gerlt, J.A. and Gassman, P.G.** (1993). Understanding the rates of certain enzyme-catalyzed reactions: Proton abstraction from carbon acids, acyl transfer reactions, and displacement reactions of phosphodiester, *Biochemistry*, 32(45), 11943–11952, <https://doi.org/10.1021/bi00096a001>.
- [222] **Gerlt, J.A. and Gassman, P.G.** (1993). An explanation for rapid enzyme-catalyzed proton abstraction from carbon acids: importance of late transition states in concerted mechanisms, *Journal of the American Chemical Society*, 115(24), 11552–11568, <https://doi.org/10.1021/ja00077a062>.

- [223] **Cleland, W.W. and Kreevoy, M.M.** (1994). Low-barrier hydrogen bonds and enzymic catalysis, *Science*, 264(5167), 1887–1890, <https://doi.org/10.1126/science.8009219>.
- [224] **Frey, P.A., Whitt, S.A. and Tobin, J.B.** (1994). A low-barrier hydrogen bond in the catalytic triad of serine proteases, *Science*, 264(5167), 1927–1930, <https://doi.org/10.1126/science.7661899>.
- [225] **Warshel, A., Papazyan, A. and Kollman, P.A.** (1995). On low-barrier hydrogen bonds and enzyme catalysis, *Science*, 269(5220), 102–106, <https://doi.org/10.1126/science.7661987>.
- [226] **Warshel, A. and Papazyan, A.** (1996). Energy considerations show that low-barrier hydrogen bonds do not offer a catalytic advantage over ordinary hydrogen bonds, *Proceedings of the National Academy of Sciences*, 93(24), 13665–13670, <http://www.pnas.org/content/93/24/13665.abstract>.
- [227] **Gerlt, J.A., Kreevoy, M.M., Cleland, W. and Frey, P.A.** (1997). Understanding enzymic catalysis: the importance of short, strong hydrogen bonds, *Chemistry & Biology*, 4(4), 259–267, [https://doi.org/10.1016/S1074-5521\(97\)90069-7](https://doi.org/10.1016/S1074-5521(97)90069-7).
- [228] **Cleland, W.W., Frey, P.A. and Gerlt, J.A.** (1998). The Low Barrier Hydrogen Bond in Enzymatic Catalysis, *Journal of Biological Chemistry*, 273(40), 25529–25532, <https://doi.org/10.1074/jbc.273.40.25529>.
- [229] **Schiøtt, B., Iversen, B.B., Madsen, G.K.H., Larsen, F.K. and Bruice, T.C.** (1998). On the electronic nature of low-barrier hydrogen bonds in enzymatic reactions, *Proceedings of the National Academy of Sciences*, 95(22), 12799–12802, <https://doi.org/10.1073/pnas.95.22.12799>.
- [230] **Schutz, C.N. and Warshel, A.** (2004). The low barrier hydrogen bond (LBHB) proposal revisited: The case of the Asp⁺ · · His pair in serine proteases, *Proteins: Structure, Function, and Bioinformatics*, 55(3), 711–723, <http://dx.doi.org/10.1002/prot.20096>.
- [231] **Warshel, A., Sharma, P.K., Kato, M., Xiang, Y., Liu, H. and Olsson, M.H.M.** (2006). Electrostatic Basis for Enzyme Catalysis, *Chemical Reviews*, 106(8), 3210–3235, <https://doi.org/10.1021/cr0503106>.
- [232] **Elias, M., Wellner, A., Goldin-Azulay, K., Chabriere, E., Vorholt, J.A., Erb, T.J. and Tawfik, D.S.** (2012). The molecular basis of phosphate discrimination in arsenate-rich environments, *Nature*, 491(7422), 134–137, <https://doi.org/10.1038/nature11517>.
- [233] **Hosur, M.V., Chitra, R., Hegde, S., Choudhury, R.R., Das, A. and Hosur, R.V.** (2013). Low-barrier hydrogen bonds in proteins, *Crystallography Reviews*, 19(1), 3–50, <https://doi.org/10.1080/0889311X.2013.771633>.

- [234] **Caldeira, A.O. and Leggett, A.J.** (1983). Quantum tunnelling in a dissipative system, *Annals of Physics*, 149(2), 374–456, [https://doi.org/10.1016/0003-4916\(83\)90202-6](https://doi.org/10.1016/0003-4916(83)90202-6).
- [235] **Skinner, J.L. and Trommsdorff, H.P.** (1988). Proton transfer in benzoic acid crystals: A chemical spin-boson problem. Theoretical analysis of nuclear magnetic resonance, neutron scattering, and optical experiments, *The Journal of Chemical Physics*, 89(2), 897–907, <https://doi.org/10.1063/1.455213>.
- [236] **Cukier, R.I. and Morillo, M.** (1990). A theoretical analysis of nuclear magnetic resonance experiments on proton transfer in benzoic acid crystals, *The Journal of Chemical Physics*, 93(4), 2364–2369, <https://doi.org/10.1063/1.459016>.
- [237] **Heuer, A. and Haeberlen, U.** (1991). The dynamics of hydrogens in double well potentials: The transition of the jump rate from the low temperature quantum-mechanical to the high temperature activated regime, *The Journal of Chemical Physics*, 95(6), 4201–4214, <https://doi.org/10.1063/1.461795>.
- [238] **Vitali, D.** (1995). Proton tunneling in symmetric H-bond. A simple microscopic model, *Chemical Physics*, 192(2), 79–88, [https://doi.org/10.1016/0301-0104\(94\)00195-G](https://doi.org/10.1016/0301-0104(94)00195-G).
- [239] **Ohta, Y., Soudackov, A.V. and Hammes-Schiffer, S.** (2006). Extended spin-boson model for nonadiabatic hydrogen tunneling in the condensed phase, *The Journal of Chemical Physics*, 125(14), 144522, <https://doi.org/10.1063/1.2354500>.
- [240] **Shi, Q., Zhu, L. and Chen, L.** (2011). Quantum rate dynamics for proton transfer reaction in a model system: Effect of the rate promoting vibrational mode, *The Journal of Chemical Physics*, 135(4), 044505, <https://doi.org/10.1063/1.3611050>.
- [241] **Gottwald, F., Karsten, S., Ivanov, S.D. and Kühn, O.** (2015). Parametrizing linear generalized Langevin dynamics from explicit molecular dynamics simulations, *The Journal of Chemical Physics*, 142(24), 244110, <https://doi.org/10.1021/10.1063/1.4922941>.
- [242] **Gottwald, F., Ivanov, S.D. and Kühn, O.** (2015). Applicability of the Caldeira - Leggett model to vibrational spectroscopy in solution, *The Journal of Physical Chemistry Letters*, 6(14), 2722–2727, <https://doi.org/10.1021/acs.jpclett.5b00718>.
- [243] **Warshel, A. and Weiss, R.M.** (1980). An empirical valence bond approach for comparing reactions in solutions and in enzymes, *Journal of the American Chemical Society*, 102(20), 6218–6226, <https://doi.org/10.1021/ja00540a008>.
- [244] **Timoneda, J.J.i. and Hynes, J.T.** (1991). Nonequilibrium free energy surfaces for hydrogen-bonded proton-transfer complexes in solution, *The Journal*

of *Physical Chemistry*, 95(25), 10431–10442, <https://doi.org/10.1021/j100178a034>.

- [245] **Sagnella, D.E. and Tuckerman, M.E.** (1998). An empirical valence bond model for proton transfer in water, *The Journal of Chemical Physics*, 108(5), 2073, <http://dx.doi.org/10.1063/1.475586>.
- [246] **Thompson, W.H. and Hynes, J.T.** (2000). Frequency shifts in the hydrogen-bonded OH stretch in halide-water clusters. The importance of charge transfer, *Journal of the American Chemical Society*, 122(26), 6278–6286, <https://doi.org/10.1021/ja993058q>.
- [247] **Voth, G.A.** (2006). Computer simulation of proton solvation and transport in aqueous and biomolecular systems, *Accounts of Chemical Research*, 39(2), 143–150, <https://doi.org/10.1021/ar0402098>.
- [248] **Sonnenberg, J.L., Wong, K.F., Voth, G.A. and Schlegel, H.B.** (2009). Distributed Gaussian Valence Bond Surface Derived from Ab Initio Calculations, *Journal of Chemical Theory and Computation*, 5(4), 949–961, <https://doi.org/10.1021/ct800477y>.
- [249] **McKenzie, R.H.** (2012). A diabatic state model for donor-hydrogen vibrational frequency shifts in hydrogen bonded complexes, *Chemical Physics Letters*, 535, 196–200, <https://doi.org/10.1016/j.cplett.2012.03.064>.
- [250] **McKenzie, R.H., Bekker, C., Athokpam, B. and Ramesh, S.G.** (2014). Effect of quantum nuclear motion on hydrogen bonding, *The Journal of Chemical Physics*, 140(17), 174508, <http://dx.doi.org/10.1063/1.4873352>.
- [251] **Palma, G.M., Suominen, K.A. and Ekert, A.K.** (1996). Quantum computers and dissipation, *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, 452(1946), 567–584, <https://doi.org/10.1098/rspa.1996.0029>.
- [252] **Tajima, Y., Matsuo, T. and Suga, H.** (1982). Phase transition in KOH-doped hexagonal ice, *Nature*, 299, 810–812, <https://doi.org/10.1038/299810a0>.
- [253] **Tajima, Y., Matsuo, T., and Suga, H.** (1984). Calorimetric study of phase transition in hexagonal ice doped with alkali hydroxides, *Journal of Physics and Chemistry of Solids*, 45(11), 1135–1144, [https://doi.org/10.1016/0022-3697\(84\)90008-8](https://doi.org/10.1016/0022-3697(84)90008-8).
- [254] **Keutsch, F.N. and Saykally, R.J.** (2001). Water clusters: Untangling the mysteries of the liquid, one molecule at a time, *Proceedings of the National Academy of Sciences*, 98(19), 10533–10540, <https://doi.org/10.1073/pnas.191266498>.
- [255] **Åberg, J.** (2018). Fully Quantum Fluctuation Theorems, *Physical Review X*, 8(1), 011019, <https://doi.org/10.1103/PhysRevX.8.011019>.

- [256] **Kosloff, R.** (2013). Quantum Thermodynamics: A Dynamical Viewpoint, *Entropy*, 15(6), 2100–2128, <https://doi.org/10.3390/e15062100>.
- [257] **Weilenmann, M., Kraemer, L., Faist, P. and Renner, R.** (2016). Axiomatic relation between thermodynamic and information-theoretic entropies, *Physical Review Letters*, 117(26), 260601, <https://doi.org/10.1103/PhysRevLett.117.260601>.
- [258] **Pusuluk, O. and Deliduman, C.** (2011). DNA replication via entanglement swapping, *arXiv/quant-ph*, 1008.4597v3, <https://arxiv.org/abs/1008.4597v3>.
- [259] **Pusuluk, O. and Deliduman, C.** (2011). Entanglement swapping model of DNA replication, *arXiv/quant-ph*, 1101.0073v3, <https://arxiv.org/abs/1101.0073v3>.
- [260] **Jonathan, D. and Plenio, M.B.** (1999). Entanglement-assisted local manipulation of pure quantum states, *Physical Review Letters*, 83(17), 3566–3569, <https://doi.org/10.1103/PhysRevLett.83.3566>.
- [261] **Horodecki, M. and Oppenheim, J.** (2013). Fundamental limitations for quantum and nanoscale thermodynamics, *Nature Communications*, 4, 2059, <https://doi.org/10.1038/ncomms3059>.
- [262] **Egloff, D., Dahlsten, O.C.O., Renner, R. and Vedral, V.** (2015). A measure of majorization emerging from single-shot statistical mechanics, *New Journal of Physics*, 17(7), 073001, <https://doi.org/10.1088/1367-2630/17/7/073001>.
- [263] **Brandão, F.G.S.L., Horodecki, M., Ng, N.H.Y., Oppenheim, J. and Wehner, S.** (2015). The second laws of quantum thermodynamics, *Proceedings of the National Academy of Sciences*, 112(11), 3275–3279, <https://doi.org/10.1073/pnas.1411728112>.
- [264] **van der Meer, R., Ng, N.H.Y. and Wehner, S.** (2017). Smoothed generalized free energies for thermodynamics, *Physical Review A*, 96(6), 062135, <https://doi.org/10.1103/PhysRevA.96.062135>.
- [265] **Brandão, F.G.S.L., Horodecki, M., Oppenheim, J., Renes, J.M. and Spekkens, R.W.** (2013). Resource theory of quantum states out of thermal equilibrium, *Physical Review Letters*, 111(25), 250404, <https://doi.org/10.1103/PhysRevLett.111.250404>.
- [266] **Åberg, J.** (2013). Truly work-like work extraction via a single-shot analysis, *Nature Communications*, 1925, <https://doi.org/10.1038/ncomms2712>.
- [267] **Yunger Halpern, N., Garner, A.J.P., Dahlsten, O.C.O. and Vedral, V.** (2015). Introducing one-shot work into fluctuation relations, *New Journal of Physics*, 17(9), 095003, <https://doi.org/10.1088/1367-2630/17/9/095003>.

- [268] **Goold, J., Huber, M., Riera, A., del Rio, L. and Skrzypczyk, P.** (2016). The role of quantum information in thermodynamics—a topical review, *Journal of Physics A: Mathematical and Theoretical*, 49(14), 143001, <https://doi.org/10.1088/1751-8113/49/14/143001>.
- [269] **Yunger Halpern, N. and Renes, J.M.** (2016). Beyond heat baths: Generalized resource theories for small-scale thermodynamics, *Physical Review E*, 93(2), 022126, <https://doi.org/10.1103/PhysRevE.93.022126>.
- [270] **del Rio, L., Hutter, A., Renner, R. and Wehner, S.** (2016). Relative thermalization, *Physical Review E*, 94(2), 022104, <https://doi.org/10.1103/PhysRevE.94.022104>.
- [271] **Alhambra, A.M., Oppenheim, J. and Perry, C.** (2016). Fluctuating states: What is the probability of a thermodynamical transition?, *Physical Review X*, 6(4), 041016, <https://doi.org/10.1103/PhysRevX.6.041016>.
- [272] **Alhambra, A.M., Masanes, L., Oppenheim, J. and Perry, C.** (2016). Fluctuating work: From quantum thermodynamical identities to a second law equality, *Physical Review X*, 6(4), 041017, <https://doi.org/10.1103/PhysRevX.6.041017>.
- [273] **Sparaciari, C., Oppenheim, J. and Fritz, T.** (2017). Resource theory for work and heat, *Physical Review A*, 96(5), 052112, <https://doi.org/10.1103/PhysRevA.96.052112>.
- [274] **Bera, M.N., Riera, A., Lewenstein, M. and Winter, A.** (2017). Generalized laws of thermodynamics in the presence of correlations, *Nature Communications*, 8(1), 2180, <https://doi.org/10.1038/s41467-017-02370-x>.
- [275] **Narasimhachar, V. and Gour, G.** (2017). Resource theory under conditioned thermal operations, *Physical Review A*, 95(1), 012313, <https://doi.org/10.1103/PhysRevA.95.012313>.
- [276] **Müller, M.P.** (2017). Correlating thermal machines and the second law at the nanoscale, *arXiv/quant-ph*, 1707.03451v2, <https://arxiv.org/abs/1707.03451>.
- [277] **Scharlau, J. and Müller, M.P.** (2018). Quantum Horn’s lemma, finite heat baths, and the third law of thermodynamics, *Quantum*, 2, 54, <https://doi.org/10.22331/q-2018-02-22-54>.



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PUBLICATIONS, PRESENTATIONS AND PATENTS ON THE THESIS:

- **Pusuluk, O.**, Torun, G., Deliduman, C., 2018. Quantum entanglement shared in hydrogen bonds and its usage as a resource in molecular recognition, (arXiv:1807.00647 [quant-ph]), published in *Modern Physics Letters B* on Aug 24, 2018.
- **Pusuluk, O.**, Farrow, T., Deliduman, C., Burnett, K., and Vedral, V., 2018. Proton tunneling in hydrogen bonds and its possible implications in an induced-fit model of enzyme catalysis, (arXiv:1703.00789 [physics.chem-ph]), published in *Proceedings of the Royal Society A* on Oct 03, 2018.
- **Pusuluk, O.**, Farrow, T., Deliduman, C., and Vedral, V., 2018. Emergence of correlated proton tunneling in water ice, (arXiv:1703.01335 [quant-ph]), submitted for publication in *Proceedings of the Royal Society A* on Aug 30, 2017.
- *Oral Representation* - **Pusuluk, O.**, Farrow, T., Deliduman, C., Burnett, K., and Vedral, V., 2017. Hidrojen bağının kuantumluluğu ve biyolojik katalizde kaynak olarak kullanımı. *the Quantum Optics and Information Meeting - 1 (Kobit - 1)*, February 2–3, 2017 Izmir Institute of Technology, Turkey.
- *Oral Representation* - **Pusuluk, O.** and Deliduman, C., 2016. Quantum Aspects of DNA Replication. *The International Conference on Quantum Science and Applications*, May 24–28, 2016 Eskisehir Osmangazi University, Turkey.

OTHER PUBLICATIONS, PRESENTATIONS AND PATENTS:

- *Unpublished Preprint* - **Pusuluk, O.** and Deliduman C., 2011. DNA Replication via Entanglement Swapping, arXiv:1008.4597 [quant-ph].
- *Unpublished Preprint* - **Pusuluk, O.** and Deliduman C., 2011. Entanglement Swapping Model of DNA Replication, arXiv:1101.0073 [quant-ph].
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