ISTANBUL TECHNICAL UNIVERSITY ★ GRADUATE SCHOOL OF SCIENCE ENGINEERING AND TECHNOLOGY

TRANSFORMATIONS OF NON-CLASSICALLY CORRELATED STATES

Ph.D. THESIS

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Department of Physics Engineering

Physic Engineering Programme

JANUARY 2019

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İSTANBUL TEKNİK ÜNİVERSİTESİ ★ FEN BİLİMLERİ ENSTİTÜSÜ

KLASİK OLMAYAN İLİNTİLİ DURUMLARIN DÖNÜŞÜMLERİ

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Gökhan TORUN, a Ph.D. student of ITU Graduate School of Science Engineering and Technology with ID number 509122116, successfully defended the thesis entitled "TRANSFORMATIONS OF NON-CLASSICALLY CORRELATED STATES", which he prepared after fulfilling the requirements specified in the associated legislations, before the jury whose signatures are below.

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To my grandmother Kibar Torun who passed away when I was in Nottingham,

FOREWORD

I would like to thank to my supervisor Ali Yıldız for his guidance and support. Honestly, could I reach this point without him? ... the answer is certainly, No! His wisdom and his counsel are invaluable, and he has made a great contribution to not only my academic career but also my life. When I got a new result for a problem that we have been interested in, many times, I have made a bet with him that the result is correct this time, and I have lost almost all of them–I had also promised food to him. I could not have imagined having a better supervisor for my doctorate. I am tremendously grateful to him for his patience and trust. He has been an example to me, both scientifically and personally. I hope our relationship not only within the context of physics but also beyond physics goes on forever.

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One of the things that made the time more fun during my stay in Nottingham is definitely the football matches I played with friends from the mathematics department. "It's time to say goodbye to one of our own. Sadly Gokhan will be making his last appearance for Maths Postgrad football on Friday 14th December at David Ross. Come along to see him smack one over the fence then turn round and give us all a big grin for the final time. Sign up on the Doodle poll to be part of the great man's testimonial. Guaranteed goals and guaranteed tears." This is the last email I got for my testimonial which I scored five goals, yes, you read right, five! I would like to thank all these great guys from the mathematics department at the University of Nottingham. I will miss Maths Postgraduate football matches and my friends there.

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January 2019 Gökhan TORUN

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ABBREVIATIONS

DIO	**:** Dephasing-covariant Incoherent Operation
FO	**:** Free Operations
IO	**:** Incoherent Operation
LOCC	**:** Local Operations and Classical Communcation
LU	**:** Local Unitary
MIO	**:** Maximal Incoherent Operation
OSBP	**:** One Successful Branch Protocol
PIO	**:** Physical Incoherent Operation
PM	**:** Projective Measurement
POVM	**:** Positive Operator Valued Measure
SIO	**:** Strictly Incoherent Operation
SP	**:** Set of Permutations

SYMBOLS

$\mathcal{H}$	:	Hilbert space
I	:	Identity operator
$\otimes$	:	Tensor product
$\mathcal{S}$	:	System of interest
$\mathcal{I}$	:	Incoherent states
$\mathcal{F}$	:	Free operations
Δ	:	Completely dephasing map
$\mathcal{D}$	:	Quantum discord

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TRANSFORMATIONS OF NON-CLASSICALLY CORRELATED STATES

SUMMARY

This thesis is about the deterministic and probabilistic transformations of (some of) the non-classically correlated states. Non-classical correlations which will be discussed here, entanglement, quantum coherence, and quantum discord, are used as a resource in many information processing tasks. In order to increase the success probability of the information processing tasks, it is rather important to convert a non-classically correlated state into a better one in the same class by local quantum operations.

In Ch. 2 two explicit protocols for the deterministic transformations of coherent states under incoherent operations are presented. First, a permutation-based protocol which provides a method for the single-step transformation of d-dimensional coherent states is proposed. Generalized solutions of this protocol for some special cases of d-level systems are obtained. Then, an alternative protocol is presented where we use d'-level ($d' < d$) subspace solutions of the permutation-based protocol to achieve the complete transformation as a sequence of coherent state transformations.

In Ch. 3, in a similar manner, deterministic transformations of pure bipartite entangled states is discussed. The results, obtained for the deterministic transformations of coherent states under incoherent operations, can easily be adapted to d-level bipartite entangled pure state transformations. Various illustrative examples both to demonstrate the adaptability of our results to the deterministic transformations of entanglement and to enhance the intelligibility of our method are given. One of the most important things to bear in mind is that majorization is essential for these deterministic transformations.

The problem of distilling as much entanglement (coherence) as possible from a given pure state by means of a probabilistic protocol using local operations and classical communication (incoherent operations) is of fundamental importance. Instead of aiming at a single output state, however, one can consider a discrete class of states as targets, namely that formed by all maximally entangled states (coherent states) of any possible local dimension q. In this sense, in Ch. 4, an optimal probabilistic protocol to distill quantum coherence is presented. Inspired by a specific entanglement distillation protocol, our main result yields a strictly incoherent operation that produces one of a family of maximally coherent states of variable dimension from any pure quantum state. We also expand this protocol to the case where it is possible, for some initial states, to avert any waste of resources as far as the output states are concerned, by exploiting an additional transformation into a suitable intermediate state. These results provide practical schemes for efficient quantum resource manipulation.

It is well known that quantum states that can be transformed into each other by local unitary transformations are equal from the information theoretic point of view. This defines equivalence classes of states and allows one to write any state with the minimal number of parameters called the canonical form of the state. In this context,

in Ch. 5, we define the equivalence classes of local measurements for three-qubit entanglement manipulation such that local operations which transform states from one equivalence class into another with the same probability are equivalent. This equivalence relation allows one to write the operators with the minimal number of parameters, which we call canonical operators, and hence the use of the canonical operators simplifies the optimal manipulation of quantum states. The canonical local operators for the concentration of three-qubit Greenberger-Horne-Zeilinger states is used, and the optimal concentration protocols in terms of the unitary invariants of quantum states, namely, the bipartite concurrences and the three-tangle are obtained. Then, an explicit protocol for the deterministic transformations of three-qubit pure states by local operations and classical communications is proposed.

Quantum discord can be used as a resource for some information processing tasks. In this context, Werner states are the most studied quantum states. In Ch. 6 three procedures to obtain the separable Werner states which have nonzero discord are introduced. The general idea is simple, and the following. In the first stage, we have a classically correlated state in our hand. Then, we perform some local operations, where our primary aim is to obtain these local operations, on subsystems of the initial state. After completing the whole process, a nonzero discord Werner state is acquired.

KLASİK OLMAYAN İLİNTİLİ DURUMLARIN DÖNÜŞÜMLERİ

ÖZET

Tüm bilim tarihinde olduğu gibi, doğanın işleyiş yasaları keşfedildikten sonra bu keşifler insanlığın kullanabileceği ürünler vermeye başlamıştır. Ne var ki bilimin ve teknolojinin geldiği aşamada bilgi işlem teknolojileri klasik sistemler kullanarak ulaşabileceği sınırlara yaklaşmıştır. Artık küçülen devrelerle birlikte daha önceden dikkate alınmayan kuantum ilintiliklerin dikkate alınması zorunlu hale gelmiştir. Bu kuantum etkileri kuantum bilgisayarlarında, bilgi işleme süreçlerinde ve kriptografide devrim niteliğinde sıçramalara yol açmıştır.

Bu tez bütünü itibariyle çeşitli klasik olmayan ilintiliğe sahip durumların (*non-classically correlated states*) kendi içinde deterministik (*deterministic*) ve olasılıksal (*probabilistic*) dönüşümleri (*transformations*) ile ilgilidir. Burada ele alınacak klasik olmayan ilintilikler–dolaşıklık (*entanglement*), kuantum eşevrelilik (*quantum coherence*) ve kuantum uyuşmazlık (*quantum discord*)–birçok kuantum bilgi işleme süreçlerinde kaynak (*resource*) olarak kullanılmaktadır. Bilgi işleme süreçlerinin başarısını artırabilmek adına bu kuantum ilintililerin kuantum operasyonlar yardımıyla daha iyi performans verebilecek aynı sınıftaki başka kaynaklara dönüştürülebilmesi oldukça önemlidir.

Bu bağlamda, ilk olarak, d-seviyeli eşevreli bir başlangıç durumunun (*initial coherent state*), örneğin $\sum_{k=1}^{d} a_k e^{i\alpha_k} |k\rangle$ ($a_k \in \mathbb{R}$ ve $0 \leq \alpha_k \leq \pi$), başka bir d-seviyeli eşevreli hedef durumuna (*target coherent state*), örneğin $\sum_{k=1}^{d} b_k e^{i\beta_k} |k\rangle$ ($b_k \in \mathbb{R}$ ve $0 \leq \beta_k \leq \pi$), eşevreli olmayan operasyonlar (*incoherent operations*) üzerinden deterministik dönüşümleri çalışılmıştır. Eşevreli herhangi bir durumun katsayıları o duruma ait tüm bilgiyi içerir diyebiliriz. En genelde bu katsayılar arasında düzenli bir sıralamanın olması zorunlu değildir; ancak bu katsayıları eşevrelilik yaratmayan çeşitli birimsel (*unitary*) dönüşümler yardımıyla büyükten küçüğe (ya da küçükten büyüğe) olacak şekilde sıralamak mümkündür. Örneğin, eşevreli başlangıç durumuna ait katsayılar $a_1 \geq a_2 \geq \cdots \geq a_d \geq 0$ ve eşevreli hedef durumuna ait katsayılar $b_1 \geq b_2 \geq \cdots \geq b_d \geq 0$ şeklinde olsun. Bazı farklı iki eşevreli durumun büyükten küçüğe doğru sıralı olan katsayıları arasında *majorization* adı verilen bir kısmi sıralama (*partial order*) da mevcuttur. Şöyle ki, $\sum_{i=1}^{k} a_i^2 \leq \sum_{i=1}^{k} b_i^2$ eşitsizliği her $k = 1, 2, \ldots, d$ için sağlanır ise ($k = d$ için $\sum_{i=1}^{d} a_i^2 = \sum_{i=1}^{d} b_i^2 = 1$ olmaktadır) başlangıç ve hedef eşevreli durumlar arasında bir majorization ilişkisi mevcuttur. Daha da önemlisi, majorization koşulu sağlandığı takdirde eşevreli başlangıç durumu hedef duruma eşevreli olmayan operasyonlar yardımıyla deterministik olarak dönüştürülebilmektedir (majorization, böylesi dönüşümler için gerek ve yeter şarttır). Bu tezin ilk bölümünde bu dönüşümü gerçekleştiren iki farklı yöntem önerilmiştir. Birinci yöntem ilgili dönüşümü tek adımda başaran permütasyon odaklı bir yöntemdir. Başlangıç ve hedef durumlarının katsayıları a_x ve b_x ($x = 1, \ldots, d$) arasındaki bağıntılara göre ($a_x \geq b_x$ veya $a_x \leq b_x$) bir tablo elde edilmiştir. Bu tablo olası tüm permütasyonları içermektedir. Daha

sonra, önerdiğimiz resmedilmiş yöntem vasıtasıyla hangi permütasyon kümelerinin yeterli olduğu belirlenmiştir. Belirlenen bu permütasyon kümeleri kullanılarak ilgili dönüşümü gerçekleştiren kuantum ölçüm operatörleri ve olasılıklar elde edilmiştir. Bu yöntem kullanılarak tüm ölçüm operatörleri ve olasılıklar $d = 2,3,\ldots,6,7$ seviyeleri için elde edilmiştir. Ayrıca bu yöntem çok sayıda yüksek seviyeli başka örnekler ile de test edilmiştir. Bu ilk yönteme ait çeşitli genel çözümler de elde edilmiştir, ve bu tezde özel olarak bir tanesi sunulmuştur. İkinci yöntem d-seviyeli iki eşevreli durumun dönüşümünü birinci yöntemde olduğu gibi tek adımda yapmak yerine (belli bir adımda) peş peşe yapılan dönüşümlerle sağlamaktadır. İkinci yöntemde, birinci yöntemden faydalanılarak, d'-seviyeli ($d' < d$) dönüşümler için elde edilen sonuçlar kullanılmıştır. İkinci yöntem detaylıca açıklanırken $d' = 5$ alınmıştır. Bu durumda toplam adım sayısı $\lfloor (d+2)/4 \rfloor$ olmaktadır. İkinci yöntemde dikkat edilmesi gereken en önemli nokta peş peşe yapılan her bir ölçümde majorization koşulunun korunması gerekliliğidir. Majorization şartının her bir adımda korunabilmesi için ara durumların (*intermediate coherent states*) doğru şekilde belirlenmesi oldukça önemlidir. İlk önce her bir ölçüm için ara durumlar belirlenecek, ardından birinci yöntem ile elde edilen sonuçlar kullanılarak istenilen dönüşüm gerçekleştirilecektir. Bu yöntemin nasıl çalıştığı ve majorization koşulunun her bir adımda korunması gerektiğinin önemi bir örnek ile detaylı bir şekilde anlatılmıştır.

Eşevreli durumların dönüşümleri için literatürde çeşitli başka yöntemler de mevcuttur. Ancak bazı çalışmalardaki çözümler dolaşıklık dönüşümlerine uyarlanamamaktadır. Öte yandan, bu tezde eşevreli durumların dönüşümleri için elde ettiğimiz her iki yöntem de kolaylıkla dolaşıklık dönüşümlerine uygulanabilmektedir. Bu bağlamda, her biri d-seviyeli iki parçacıklı dolaşık saf bir başlangıç durumunun (*initial bipartite entangled pure state*), örneğin $\sum_{k=1}^{d} a_k e^{i\alpha_k} |k\rangle |k\rangle$ ($a_k \in \mathbb{R}$ ve $0 \leq \alpha_k \leq \pi$), yine her biri d-seviyeli olan iki parçacıklı dolaşık saf bir hedef durumuna (*target bipartite entangled pure state*), örneğin $\sum_{k=1}^{d} b_k e^{i\beta_k} |k\rangle |k\rangle$ ($b_k \in \mathbb{R}$ ve $0 \leq \beta_k \leq \pi$), yerel operasyonlar üzerinden deterministik dönüşümleri çalışılmıştır. Burada a_k ve b_k başlangıç ve hedef durumların Schmidt katsayıları olarak bilinmektedir. İki parçacıklı dolaşık bir durumun Schmidt katsayıları o birleşik sisteme ait tüm bilgiyi içermektedir. Bu Schmidt katsayılarını (eşevreli durumların katsayılarında da olduğu gibi) çeşitli yerel birimsel dönüşümler yardımıyla büyükten küçüğe (ya da küçükten büyüğe) olacak şekilde sıralamak mümkündür. Benzer şekilde, ilk duruma ait Schmidt katsayıları $a_1 \geq \cdots \geq a_d \geq 0$ ve son duruma ait Schmidt katsayıları ise $b_1 \geq \cdots \geq b_d \geq 0$ şeklinde olmak üzere, bunlar arasında majorization koşulu, $\sum_{i=1}^{k} a_i^2 \leq \sum_{i=1}^{k} b_i^2$ ($k = 1,2,\ldots,d$), sağlanıyor ise eşevreli durumların dönüşümleri için elde ettiğimiz çözümler $d \otimes d$ dolaşık durumların dönüşümlerini de gerçekleştirir. İlgili problem için çeşitli örnekler verilerek sunulan iki yöntemin de dolaşıklığa uygulanabilirliği gösterilmiştir.

Sonrasında, kuantum eşvrelilik damıtma (*quantum coherence distillation*) olarak bilinen, oldukça önemli, bir diğer problem farklı bir çerçeveden incelenmiştir. Bu bölümde sunulan yöntem dolaşıklığa da uygulanabilmektedir. Kuantum bilgi işleme süreçlerinde kaynak olarak kullanılan kuantum durumları bu süreçlerde farklı performans göstermektedir. Dolaşıklığı ele alacak olursak, iki parçacıklı bir kuantum durumunu oluşturan parçacıklar arasında dolaşıklık miktarı ne kadar büyükse, bilgi işleme sürecinin başarıya ulaşma olasılığı o denli yüksek olmaktadır. Örneğin, $a|1\rangle|1\rangle + b|2\rangle|2\rangle$ ($a \geq b > 0$) ile $1/\sqrt{2}(|1\rangle|1\rangle + |2\rangle|2\rangle)$–iki kubit maksimal dolaşık durum–kaynak durumlarının kuantum telenakil (*quantum teleportation*) prokolündeki başarı olasılıkları sırasıyla $2b^2$ ve 1 olmaktadır. Diğer bir ifadeyle kaynak durum

dolaşıklığı ne kadar fazla ise protokolün başarı olasığı da o denli yüksektir. Bu bağlamda, d-seviyeli eşevreli bir başlangıç durumunun, örneğin $\sum_{i=1}^{d} \psi_i |i\rangle$ ($\psi_i \in \mathbb{R}$), q-seviyeli eşevreli hedef durumlarına, $\frac{1}{\sqrt{q}} \sum_{i=1}^{q} |i\rangle$ ($q = 2, \ldots, d$), eşevreli olmayan operasyonlar üzerinden deterministik dönüşümleri çalışılmıştır. Burada $\frac{1}{\sqrt{q}} \sum_{i=1}^{q} |i\rangle$ durumu q-seviyeli maksimal eşevreli durum olarak bilinmektedir. Standart damıtma probleminde d-seviyeli eşevreli bir başlangıç durumdan d-seviyeli maksimal eşevreli durum ve $(d-1)$-seviyeli eşevreli herhangi bir son durum elde edilmektedir. Bu tezde, standart eşevrelilik (dolaşıklık) damıtma probleminden farklı olarak, tüm q-seviyeli maksimal eşevreli (dolaşık) durumlar elde edilmiştir. Eşevreliliğin hangi miktarda yitirildiği de tartışılmıştır. Sunulan yöntemde sıfırdan farklı bir olasılıkla eşevreli olmayan (benzer şekilde dolaşık olmayan (*separable*)) bir durum da, $|1\rangle$ ($|1\rangle|1\rangle$), elde edilmiştir. Ayrıca, quantum eşevrelilik ve dolaşıklık dönüşümleri için sunulan yöntemlerden (ikinci ve üçüncü bölümler) yararlanılarak $|1\rangle$ ($|1\rangle|1\rangle$) durumunu elde etme olasığı sıfırlanmıştır. Böylelikle, sunulan protokolü bir adım ileri götürerek, son aşamada elde edilen tüm durumların her bir seviyeye ($q = 2, \ldots, d$) ait maksimal eşevreli (dolaşık) durum olması sağlanmıştır. İlgili yöntem bir örnek ile tartışılmıştır.

Kuantum bilgi işleme süreçlerinde, iki parçacıklı (*bipartite*) sistemlerin dolaşıklığı kadar çok parçacıklı (*multipartite*) sistemlerin dolaşıklığı da önemli bir yer tutmaktadır. Bu bağlamda, üç parçacıklı (her biri iki-seviyeli olan) sistemlerin dolaşıklık dönüşümleri de hem deterministik hem de olasılıksal çerçeveden incelenmiştir. Üç-kubit (*three-qubit*) dolaşık durumlar, yerel işlemler ile birbirine dönüştürülemeyen, iki farklı sınıfa ayrılmaktadır: Greenberger-Horne-Zeilinger (GHZ) durumu, $\frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$, ve W durumu, $\frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle)$. Tezin beşinci bölümünde, ilk olarak, üç-kubit saf durumun optimal manipülasyonu problemi için çözümler sunulacaktır. Öncelikle üç kubite ait üniter dönüşümlerden bahsedilerek üç-kubit saf durumun kanonik halinin diğer bir formu elde edilmiştir. Bunun ardından, üç-kubit için yerel kuantum ölçüm operatörlerinin genel formu bulunmuştur. Saf GHZ durumunun elde edilmesi için bir yöntem önerilmiştir. Bu yöntem kullanılarak GHZ durumunu elde etme olasılığı ve bu dönüşümü gerçekleştirem ölçüm operatöleri üç-kubit sisteminin değişmezleri cinsinden elde edilmiştir. Bu bölüm, hem saf W hem de saf GHZ durumlarından kendi sınıfları içindeki durumlara deterministik geçişlerini sağlayan operasyonların elde edilmesiyle sonlanacaktır.

Son bölümde yıllarda oldukça popülerlik kazanmış bir diğer önemli klasik olmayan ilintililiğe–kuantum uyuşmazlık–değinilmiştir. Bir sisteme ait kuantum uyuşmazlık, ortak bilginin (*mutual information*) $I(A:B) = S(\rho_A) + S(\rho_B) - S(\rho_{AB})$ ve $J(A:B) = S(\rho_B) - S(B|A)$ iki farklı tanımı arasındaki fark olarak tanımlanmaktadır: $\mathcal{D} = I(A:B) - J(A:B)$. Klasik bilgi teorisinde ortak bilginin bu iki tanımı birbirine eşdeğerdir. Diğer bir ifadeyle, klasik bilgi teorisinde $I(A:B) - J(A:B) = 0$ olmaktadır. Ancak, kuantum mekaniğinde bir kuantum durumu üzerinde yapılacak ölçümler o kuantum durumunu değiştirdiği için kuantum enformasyon teorisinde ortak bilginin bu iki tanımı arasındaki fark genel olarak sıfırdan faklı bir değer vermektedir. Kuantum uyuşmazlık, dolaşıklığı da kapsayan bir kuantum ilintiliktir. Birçok kuantum bilgi işleme süreçlerinde, dolaşık olmayan ve sıfırdan farklı kuantum uyuşmazlığa sahip durumlar, diğer bazı kuantum uyuşmazlık miktarı sıfır olan durumlara göre daha yüksek performans göstermektedir. Bunun yanı sıra, yerel operasyonlarla dolaşıklık yaratılamayacağını biliyoruz. Diğer bir ifadeyle dolaşık olmayan kuantum durumları yerel operasyonlar altında ayrık kalmaya devam ederler. Ancak bu durum discord

için geçerli değildir. Klasik ilintiliğe sahip durumları kullanarak, yerel operasyonlarla, kuantum ilintililiğe–kuantum uyuşmazlık–sahip durumlar elde etmek mümkündür. Tezin bu bölümünde, başlangıç aşamasında sıfır kuantum uyuşmazlığa sahip olan (*zero discord*) bir durumdan sıfırdan faklı bir uyuşmazlık değerine sahip (*nonzero discord*) olan diğer bir duruma, örneğin $\rho_{W_{1/3}} = \frac{1}{3}|\psi^-\rangle\langle\psi^-| + \frac{1}{6}I_4$ ($|\psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$, I_4 ise 4×4'lük birim matris), geçişi sağlayan yerel operasyonlar üç yöntem ile sunulmuştur. Bu bölümde sunulan hesaplar, sadece tek bir durum için değil tüm dolaşık olmayan ve sıfırdan farklı kuantum uyuşmazlık değerine sahip olan Werner durumlarının elde edilmesi için uygulanabilmektedir.

1. INTRODUCTION

After the discovery of the laws of nature as in the whole history of science, these discoveries push humankind forward. However, at the current stage of science and technology, information technology has approached the limits it can reach by using classical systems. Therefore, it has become imperative to take into account the intrinsic features of quantum mechanics which have not been previously considered. These quantum effects have led to revolutionary leaps in quantum computers, in information processing tasks and in cryptography.

One of the best signatures of nonclassicality in a quantum system is the existence of correlations that have no classical counterpart. Different methods for quantifying the quantum and classical parts of correlations are among the more actively studied topics of quantum information theory. Nonclassical correlations are used as resources in many quantum information processes, and hence obtaining the necessary resource from any given quantum state is of utmost importance. The success probability in fulfilling quantum information tasks depends on the properties of the nonclassical correlation which is used as a resource. This requires a well-developed theory of the local manipulation of quantum states which have nonclassical correlations.

1.1 Non-classical Correlations

Quantum states display various correlations. Of these nonclassically correlated states, entangled states are most used and best known resources. Quantum entanglement [1] has been playing a critical role in quantum information processes because of its properties, and it has been widely studied in the context of bipartite pure and mixed states. Quantum coherence [2, 3] is another core manifestation of nonclassicality. It is used as a crucial resource in many quantum information processing tasks. Quantum discord [4, 5] is one of the most popular candidates for general nonclassical correlations. Investigation of the behaviour of nonclassical correlations under local

operations is of paramount importance, and holds the key for the development of quantum computation and quantum information science.

1.2 State Transformations

Can resource states be converted to each other, and if so, what is the probability of success? Under what conditions can one state be converted to another using the free operations? If one defines necessary (and sufficient) conditions for this purpose, then, one can also ask the following question: What are the quantum operations for these transformations? These are fundamental questions considered in quantum information.

1.3 Majorization

Majorization [6, 7] is an authoritative tool in quantum information. While it can be characterized in various ways [6], we here only consider its usage in the context of state transformations. Let $p = (p_1^{\downarrow}, \ldots, p_d^{\downarrow})^T$ and $q = (q_1^{\downarrow}, \ldots, q_d^{\downarrow})^T$ be elements of $\mathbb{R}^d$ whose components are in non-increasing order, i.e., $p_1^{\downarrow} \geq p_2^{\downarrow} \geq \cdots \geq p_d^{\downarrow}$ and $q_1^{\downarrow} \geq q_2^{\downarrow} \geq \cdots \geq q_d^{\downarrow}$. Then p is majorized by q (equivalently q dominates p), written $p \prec q$, if the inequalities

$$\sum_{x=1}^{k} p_x^{\downarrow} \leq \sum_{x=1}^{k} q_x^{\downarrow}, \tag{1.1}$$

are satisfied for any $k \in [1, d-1]$ with equality holding when $k = d$.

1.4 Purpose of Thesis

One of the central problems of quantum information theory is the optimal manipulation both from a single copy and from many copies of quantum states. Optimal manipulation of a single copy of a state is of importance from the practical point of view, as experimentalists are not able to perform joint operations on multiple copies of a system in general. The object of this thesis is to provide alternative, experimentally friendly protocols for the deterministic and (optimal) probabilistic transformations of non-classically correlated states. The non-classical correlations that we are going to focus on are quantum coherence, entanglement and quantum discord.

This thesis consists of five chapters excluding the current one and the conclusion chapter, and is organised as follows. In Chapter 2 we present two explicit protocols for the deterministic transformations of coherent states via incoherent operations. Protocol

I is a permutation-based protocol which provides the single-step transformation of d-level coherent states, and protocol II is a step-by-step transformation protocol. In Chapter 3 we use the results, obtained in Chapter 2, for $d \otimes d$ entangled pure state transformations. In Chapter 4 we present a simple, practical and efficient strategy for optimal one-shot distillation of quantum coherence from pure input states of arbitrary dimension. The key advantage of our protocol lies in its ability to provide a single map to obtain all q-level ($q = 2,3,\ldots,d$) maximally coherent pure states starting from a d-level coherent input pure state. In Chapter 5 we define the equivalence classes of local measurements such that local operations which transform states from one equivalence class into another with the same probability are equivalent. This equivalence relation allows one to write the operators with the minimal number of parameters, which we call canonical operators. The use of the canonical operators simplifies the optimal manipulation of quantum states. We use the canonical local operators for the concentration of three-qubit Greenberger-Horne-Zeilinger (GHZ) states and obtain the optimal concentration protocols in terms of the unitary invariants of quantum states, namely, the bipartite concurrences and the three-tangle. Then, we propose an explicit protocol for the deterministic transformations of three-qubit pure states by local operations and classical communications. In Chapter 6 we introduce three procedures, all are fairly similar, to obtain the separable Werner states which have nonzero discord.

The various chapters of this thesis are based on the following publications or preprints.

- **Chapter 2**. G.Torun, and A.Yildiz, Deterministic transformations of coherent states under incoherent operations, *Phys. Rev. A*, 97, 052331, 2018.

- **Chapter 3**. G. Torun, and A. Yildiz, Deterministic transformations of coherent states under incoherent operations, *Phys. Rev. A*, 97, 052331, 2018, and G. Torun, and A. Yildiz, Deterministic transformations of bipartite pure states, *Physics Letters A*, 379(3), 113 -118, 2015.

- **Chapter 4**. G. Torun, L. Lami, G. Adesso and A. Yildiz, Optimal distillation of quantum coherence with reduced waste of resources, *Phys. Rev. A*, 99, 012321, 2019.

- **Chapter 5**. G.Torun, and A.Yildiz, Canonical operators and the optimal concentration of three-qubit Greenberger-Horne-Zeilinger states, *Phys. Rev. A*, 89, 032320, 2014.

- **Chapter 6**. The work presented in this part of the thesis can be thought of as an alternative method for the corresponding problem, and is planned to be published later as an article after some progress.

- **Chapter 7**. We summarize our conclusions.

2. COHERENCE TRANSFORMATIONS

In this chapter, we present two explicit protocols for the deterministic transformations of coherent states under incoherent operations. We first introduce a permutation-based protocol which provides a method for the single-step transformation of d-dimensional coherent states. We also obtain generalized solutions of this protocol for some special cases of d-level systems. Then, we present an alternative protocol where we use d'-level ($d' < d$) subspace solutions of the permutation-based protocol to achieve the complete transformation as a sequence of coherent state transformations. The material covered in this chapter is published in the paper [8][1].

2.1 Introduction

Coherence, a core manifestation of nonclassicality, is used as a crucial resource in many quantum information processing tasks, such as quantum thermodynamics [9–12], quantum metrology [13–16], and quantum algorithms [17–19]. As with all resources, coherence also needs to be quantified. In line with this goal, the relative entropy of coherence and the l_1 norm of coherence were presented in [2] for the quantification of coherence as a resource. Manipulation of coherence is also an important part of resource theory of coherence [3, 20–25]. Quantum coherence has similar features with quantum entanglement [1], the well-known fundamental resource in many quantum information processes, in the context of state-to-state transitions. Nielsen [26] used the linear-algebraic theory of majorization and obtained the necessary and sufficient conditions for a class of entanglement transformations. It was later realized that it is possible, with the aid of a completely similar approach, to achieve the deterministic transformations of coherent states under incoherent operations. The researchers [27,28] have built the counterpart of Nielsen's theorem for coherence manipulation and showed that majorization is also a key ingredient for the interconvertibility of coherent states. An optimal local conversion strategy of bipartite

[1] Gökhan Torun and Ali Yildiz, Deterministic transformations of coherent states under incoherent operations, *Phys. Rev. A*, 97, 052331, (2018).

entangled pure states was proposed by Vidal [29], which is also a generalization of Nielsen's theorem [26]. This strategy was adapted to the optimal conversion of coherent states under incoherent operations [30]. See [3] for a comprehensive review of the development of this rapidly growing research field that encompasses the characterization, quantification, manipulation, dynamical evolution, and operational application of quantum coherence.

While the interconversion of pure states under incoherent operations was studied in various papers [20, 27, 30–34], providing alternative and easily implementable protocols for the state-to-state transitions is of paramount importance in quantum resource theories. We present two explicit protocols (followed by illustrative examples) for the deterministic transformations of coherent states via incoherent operations. Protocol I is a permutation-based protocol which provides the single-step transformation of d-level coherent states, and protocol II is a step-by-step transformation protocol. We use d'-level ($d' < d$) subspace solutions of the former to construct the latter, and for $d' = 5$ the number of steps is $\lfloor (d+2)/4 \rfloor$.

The present chapter is structured as follows. We begin with a succinct summary of resource theory of coherence within the scope of the state transformations (Section 2.2). We then give the definition of majorization criteria with the explanation of its connection with coherent state manipulation (Section 2.3). We construct two explicit protocols (Subsections 2.4.1 and 2.4.2) for the deterministic transformations of coherent states under incoherent operations. We explicitly examine illustrative examples with discussion for both protocols (Subsections 2.4.1.1 and 2.4.2.1) and present solutions for some generalizable cases of the first protocol (Subsection 2.4.1.2). Finally, we conclude (Section 2.5).

2.2 Basics of the Resource Theory of Quantum Coherence

Throughout this section we give a concise summary of resource theory of coherence (not the whole theory, just some basics which are adequate to meet the requirements of our main problem).

2.2.1 Superposition

The fundamental unit of classical information used in classical computation is a *bit*–a binary digit. A bit can take one of two values where these are typically characterized as either a "0" or "1". The quantum analogue of the classical bit, i.e., the basic unit of quantum information, is *qubit* (quantum bit) where two possible states for a qubit are the states "$|0\rangle$" or "$|1\rangle$" [2]. Moreover, a pure single qubit state can be defined by a linear combination of these states, called a *superposition*:

$$|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle\,, \tag{2.1}$$

where α and β are complex numbers, and satisfy the relation $|\alpha|^2 + |\beta|^2 = 1$. A resource theory of superposition, which is a generalization of coherence theory, can be found in [35]. Additionally, a geometrical representation of the pure state space of a two-level quantum system can be illustrated as follows. One can write the state given in Eq. (2.1), choosing $\alpha = \cos(\frac{\theta}{2})$ and $\beta = e^{i\varphi}\sin(\frac{\theta}{2})$, such that

$$|\psi\rangle = \cos(\frac{\theta}{2})\,|0\rangle + e^{i\varphi}\sin(\frac{\theta}{2})\,|1\rangle\,. \tag{2.2}$$

As shown in Figure 2.1, the numbers θ and φ define a point on the three-dimensional unit sphere: the *Bloch sphere*. It is a convenient visualisation for the state of a

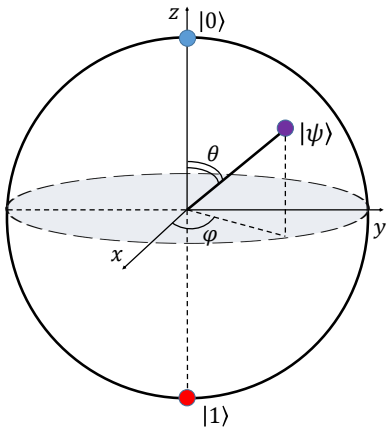

Figure 2.1 : Bloch sphere representation of a qubit.

single qubit [36]. In the following we proceed with coherence which comes from superposition principle.

[2] "$|\ \rangle$" is the standard notation, called the Dirac notation, for states in quantum mechanics

2.2.2 Coherence

The superposition principle, mentioned in the previous subsection, states that the most general form of a finite pure state is written such that

$$|\psi\rangle = \alpha_1 |1\rangle + \alpha_2 |2\rangle + \cdots + \alpha_d |d\rangle = \sum_{i=1}^{d} \alpha_i |i\rangle , \quad \Big(\sum_{i=1}^{d} |\alpha_i|^2 = 1 \Big), \tag{2.3}$$

where $\{|i\rangle\}_{i=1,2,\ldots,d}$ constitutes an orthonormal basis of the d-dimensional Hilbert space $\mathcal{H}$, i.e., $\langle i|k\rangle = \delta_{ik}$, and the coefficients $\{\alpha_i\}_{i=1,2,\ldots,d}$ are called as probability amplitudes, that is $\alpha_i = \langle i|\psi\rangle$. Here, the state (2.3) with a fixed orthonormal basis $\{|i\rangle\}_{i=1,2,\ldots,d}$ contains coherence if one has at least two nonzero coefficients. In other words, if the state[3] $\rho_{|\psi\rangle} = |\psi\rangle\langle\psi| = \sum_{i,j=1}^{d} \alpha_i \alpha_j^* |i\rangle\langle j|$ has off-diagonal entries, then this state contains coherence, and is called (pure) coherent state. On the other hand, quantum states that are diagonal with respect to a fixed orthonormal basis $\{|i\rangle\}_{i=1,2,\ldots,d}$ are defined as incoherent, and they constitute a set labeled by $\mathcal{I} \subset \mathcal{H}$ [2]. All incoherent states $\delta \in \mathcal{I}$ are of the form

$$\delta = \sum_{i=1}^{d} \delta_i |i\rangle\langle i| , \tag{2.4}$$

where $\delta_i \in [0,1]$ and $\sum_{i=1}^{d} \delta_i = 1$. One can measure the amount of coherence in a state by two well known proper measures of coherence [2]: The relative entropy of coherence

$$C_{\text{rel.ent}} = S(\rho_{\text{diag}}) - S(\rho), \tag{2.5}$$

where S is the von Neumann entropy[4] and ρ_{diag} denotes the state obtained from ρ by deleting all off-diagonal elements, and l_1 norm of coherence

$$C_{l_1} = \sum_{\substack{i,j \\ i \neq j}} |\rho_{ij}|. \tag{2.6}$$

It is easily seen that for the incoherent state given in Eq. (2.4) one has $C_{\text{rel.ent}}(\delta) = C_{l_1}(\delta) = 0$.

[3]The density matrix ρ for the pure state $|\psi\rangle$is given by $\rho_{|\psi\rangle} = |\psi\rangle\langle\psi|$, see Section 3.2 for detail.

[4]We are going to give the definition of von Neumann entropy in Subsection 6.2.1.

2.2.3 Free operations

The definition of free operations for the resource theory of coherence is not unique and different choices are being examined in the literature. A resource theory is fundamentally determined by constraints which determine the set of the freely accessible quantum operations $\mathcal{F}$. Here we present the most important classes and briefly discuss their properties.

2.2.4 Types of incoherent operations

We now recall the five types of incoherent operations. This part of this chapter is mainly adapted from the papers [2, 3, 32, 33].

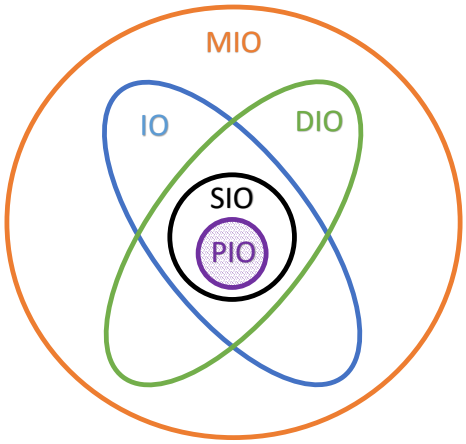

Figure 2.2 : Heuristic comparison between the five incoherent operations MIO, DIO, IO, SIO and PIO.

Physical incoherent operations (PIO): A completely positive trace-preserving map (CPTP)[5] $\mathcal{E}$ is a physical incoherent operation (PIO) if and only if it can be expressed as a convex combination of maps each having Kraus operators $\{K_i\}_{i=1,\ldots,n}$ of the form

$$K_i = U_i P_i = \sum_x e^{i\theta_x} |\pi_i(x)\rangle \langle x| P_i, \tag{2.7}$$

where the P_i form an orthogonal and complete set of incoherent projectors on primary system and π_i are permutations.

[5] A map $\mathcal{E}$ is said to be trace preserving (TP): $\mathrm{Tr}(\mathcal{E}(\rho)) = \mathrm{Tr}(\rho)$. One also requires that $\mathcal{E}$ maps positive operators to positive operators. A map $\mathcal{E}$ is said to be completely positive (CP): $\mathcal{E} \otimes I_d$ be positive for all $d \in N$, where I_d denotes the $d \otimes d$ identity operator. Thus, a map satisfying these two properties is said to be completely positive trace preserving (CPTP).

Strictly incoherent operations (SIO): A strictly incoherent operation (SIO) $\mathcal{E}$ has a Kraus operator representation $\{K_i\}_{i=1,\ldots,n}$ such that $K_i\Delta(\rho)K_i^\dagger = \Delta(K_i\rho K_i^\dagger)$ for all i, where Δ is the completely dephasing map given by

$$\rho \mapsto \Delta(\rho) = \sum_{i=1}^{d} |i\rangle\langle i|\,\rho\,|i\rangle\langle i|\,. \tag{2.8}$$

Incoherent operations (IO): The quantum operations defined by Kraus operators $\{K_i\}$ such that $\sum_i K_i^\dagger K_i = I$ and, for all i and $\rho \in \mathcal{I}$,

$$\rho \to \rho_i = \frac{K_i\rho K_i^\dagger}{\mathrm{Tr}[K_i\rho K_i^\dagger]} \in \mathcal{I}, \tag{2.9}$$

are called incoherent operation, and Kraus operators $\{K_i\}$ are called incoherent operators with the requirement $K_i\rho K_i^\dagger \subset \mathcal{I}$ for all i, i.e., incoherent states remain incoherent under incoherent operations.

Dephasing-covariant incoherent operations (DIO): We say that a quantum operation $\mathcal{E}$ is dephasing-covariant relative to the preferred subspaces if it commutes with the associated dephasing operation Δ, i.e., if

$$\mathcal{E}\circ\Delta = \Delta\circ\mathcal{E}. \tag{2.10}$$

Dephasing-covariant quantum operations are easily seen to be incohorence preserving. It suffices to note that if $\mathcal{E}$ is dephasing-covariant, then for any incoherent state $\rho \in \mathcal{I}$,

$$\mathcal{E}(\rho) = \mathcal{E}(\Delta(\rho)) = \Delta(\mathcal{E}(\rho)), \tag{2.11}$$

and therefore $\mathcal{E}(\rho)$ is invariant under dephasing and hence incoherent.

Maximal incoherent operations (MIO): A CPTP map $\mathcal{E}$ is said to be a maximal incoherent operations (MIO) if $\mathcal{E}(\rho) \in \mathcal{I}$ for every $\rho \in \mathcal{I}$.

2.3 Majorization for Coherence Transformations

Majorization is a good criterion that tells us whether one state can be transformed into another under some incoherent operations. It was shown [27, 28] that a pure coherent state

$$|\psi\rangle = \sum_{i=1}^{d} \psi_i\,|i\rangle\,, \quad \psi_i \geq \psi_{i+1} > 0, \tag{2.12}$$

can be deterministically transformed via incoherent operations to another pure coherent state

$$|\phi\rangle = \sum_{j=1}^{d} \phi_j |j\rangle, \quad \phi_j \geq \phi_{j+1} > 0, \tag{2.13}$$

if and only if the coherence vector, defined in [34], $\mu(\psi) = (\psi_1^2, \psi_2^2, \ldots, \psi_d^2)^T$ is majorized by the coherence vector $\mu(\phi) = (\phi_1^2, \phi_2^2, \ldots, \phi_d^2)^T$ [27], written $\mu(\psi) \prec \mu(\phi)$. If the following inequalities,

$$\sum_{i=1}^{k} \psi_i^2 \leq \sum_{j=1}^{k} \phi_j^2 \tag{2.14}$$

are satisfied for any $k \in [1, d-1]$ with equality holding when $k = d$, i.e., $\sum_{i=1}^{d} \psi_i^2 = \sum_{j=1}^{d} \phi_j^2 = 1$, then $\mu(\psi) \prec \mu(\phi)$.

2.4 Deterministic Transformations of Coherent States Under Incoherent Operations

In the following, we present two protocols for the deterministic transformations of the coherent state (2.12) to the coherent state (2.13) for which the majorization condition is satisfied. We reinforce them by discussing various examples.

2.4.1 Protocol I

One of the main advantages of this protocol is that it provides a single map for $|\psi\rangle \xrightarrow{\text{ico}} |\phi\rangle$ (ico stands for incoherently). Let us assume that there are Kraus operators of the form

$$K_s^i = U_s^i \left(\sqrt{p_s^i} \sum_{j=1}^{d} \frac{c_{sij}}{\psi_j} |j\rangle \langle j| \right) = U_s^i M_s^i, \tag{2.15}$$

which constitute a single map Φ_s for the transformation $|\psi\rangle\langle\psi| \to |\phi\rangle\langle\phi|$ such that

$$\Phi_s[\rho_\psi] = \sum_{i=1}^{d} K_s^i \rho_\psi K_s^{i\dagger} = \rho_\phi, \tag{2.16}$$

where $\sum_{i=1}^{d} K_s^{i\dagger} K_s^i = I_d$ and $p_s^i = \mathrm{Tr}[K_s^i \rho_\psi K_s^{i\dagger}]$. It is obvious that quantum operation Φ_s is an incoherent operation whose Kraus operators are also incoherent. In Eq. (2.15) U_s^i is the ith element of the set of permutations U_s where $s = 1, 2, \ldots, n$. A partial detailed exposition for the subscript s follows below. We express the coefficients c_{sij}

in compact form as the elements of the matrix c_s, i.e., c_{sij} is the (ij)th element of the $d \times d$ matrix c_s where

$$U_s^i(c_{si1}, c_{si2}, \ldots, c_{sid})^T = (\phi_1, \phi_2, \ldots, \phi_d)^T. \tag{2.17}$$

We may also interpret the deterministic transformation (2.16) as a d-outcome positive-operator valued measure (POVM)[6] with measurement operators

$$M_s^i = \sqrt{p_s^i} \sum_{j=1}^{d} \frac{c_{sij}}{\psi_j} |j\rangle \langle j|, \tag{2.18}$$

satisfying $\sum_{i=1}^{d} {M_s^i}^\dagger M_s^i = I_d$. The POVM measurement yields

$$M_s^i |\psi\rangle \to |\varphi_s^i\rangle = \sum_{j=1}^{d} c_{sij} |j\rangle \quad (i = 1, \ldots, d), \tag{2.19}$$

with probabilities $p_s^i = \langle\psi| {M_s^i}^\dagger M_s^i |\psi\rangle$ where $|\varphi_s^1\rangle$ is equal to $|\phi\rangle$, and $|\varphi_s^i\rangle$ $(i = 2, 3, ..., d)$ is equal to $|\phi\rangle$ up to permutation of the basis $|j\rangle$, i.e., $U_s^i |\varphi_s^i\rangle = |\phi\rangle$. The crucial point of the protocol is to find the correct set of d states $|\varphi_s^i\rangle$ obtained as a result of the measurement. In general, there are many states which are equal to the target state $|\phi\rangle$ up to permutations of the basis, and there are too many possibilities to choose sets with d elements. However, there are only few sets which satisfy the conditions on the measurement, positivity of p_s^i and $\sum_i p_s^i = 1$. For a complete set of a given case we need n sets of permutations (where $s = 1, 2, \ldots, n$) which fulfill the positivity of p_s^i. We encode the correct sets in the d unitary operations (permutations $U_s^1, U_s^2, \ldots, U_s^d$) given in the solutions. These correct sets of states and the permutation sets depend on the relations between the coefficients of the initial and final states. Hence, finding the set of permutations (finding c_{sij}) is a highly nontrivial problem, and the problem becomes exponentially difficult as the dimension becomes greater. However, we are able to propose a complete solution (a permutation-based protocol) for d-level systems. We first identify the set of permutations (SP), U_s, and using these we obtain both probabilities and a set of Kraus operators. The condition $\sum_{i=1}^{d} {K_s^i}^\dagger K_s^i = I_d$ implies that the following s sets of d equations

$$\sum_{i=1}^{d} p_s^i c_{sij}^2 = \psi_j^2 \quad (j = 1, 2, \ldots, d), \quad \Big(\sum_{i=1}^{d} p_s^i = 1, \quad p_s^i \geq 0 \Big), \tag{2.20}$$

should be satisfied. Here, the coefficients c_{sij} are given by

[6] We are going to give the definition of POVM in Subsection 3.4.

Table 2.1 : All possible permutations (or unitary transformations) for d-dimensional coherent-state transformations via incoherent operations. The table contains permutations $|x\rangle \leftrightarrow |y\rangle$ where $y > x$. For a given case of d-level systems, the relation between the coefficients of the coherent states $|\psi\rangle = \sum_{j=1}^{d} \psi_j |j\rangle$ and $|\phi\rangle = \sum_{j=1}^{d} \phi_j |j\rangle$ is either $\psi_k \geq \phi_k$ or $\psi_k \leq \phi_k$ for any $k = 2, 3, ..., d-1$ where $\psi_1 \leq \phi_1$ and $\psi_d \geq \phi_d$ is satisfied due to the majorization condition. On the one hand, if the relation is $\psi_k \leq \phi_k$ then the row in which $\psi_k \geq \phi_k$ is located is crossed out. On the other hand, if the relation is $\psi_k \geq \phi_k$ then the column in which $\psi_k \leq \phi_k$ is located is crossed out. In a table there will be a certain number, ζ, of permutations where $(d-1) \leq \zeta \leq \lfloor d/2 \rfloor \lceil d/2 \rceil$.

	$\psi_1 \leq \phi_1$	$\psi_2 \leq \phi_2$	$\psi_3 \leq \phi_3$	$\psi_4 \leq \phi_4$	$\dots$	$\psi_{d-2} \leq \phi_{d-2}$	$\psi_{d-1} \leq \phi_{d-1}$
$\psi_2 \geq \phi_2$	$\|1\rangle \leftrightarrow \|2\rangle$	-	-	-	$\dots$	-	-
$\psi_3 \geq \phi_3$	$\|1\rangle \leftrightarrow \|3\rangle$	$\|2\rangle \leftrightarrow \|3\rangle$	-	-	$\dots$	-	-
$\psi_4 \geq \phi_4$	$\|1\rangle \leftrightarrow \|4\rangle$	$\|2\rangle \leftrightarrow \|4\rangle$	$\|3\rangle \leftrightarrow \|4\rangle$	-	$\dots$	-	-
$\psi_5 \geq \phi_5$	$\|1\rangle \leftrightarrow \|5\rangle$	$\|2\rangle \leftrightarrow \|5\rangle$	$\|3\rangle \leftrightarrow \|5\rangle$	$\|4\rangle \leftrightarrow \|5\rangle$	$\dots$	-	-
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\psi_{d-2} \geq \phi_{d-2}$	$\|1\rangle \leftrightarrow \|d-2\rangle$	$\|2\rangle \leftrightarrow \|d-2\rangle$	$\|3\rangle \leftrightarrow \|d-2\rangle$	$\|4\rangle \leftrightarrow \|d-2\rangle$	$\dots$	-	-
$\psi_{d-1} \geq \phi_{d-1}$	$\|1\rangle \leftrightarrow \|d-1\rangle$	$\|2\rangle \leftrightarrow \|d-1\rangle$	$\|3\rangle \leftrightarrow \|d-1\rangle$	$\|4\rangle \leftrightarrow \|d-1\rangle$	$\dots$	$\|d-2\rangle \leftrightarrow \|d-1\rangle$	-
$\psi_d \geq \phi_d$	$\|1\rangle \leftrightarrow \|d\rangle$	$\|2\rangle \leftrightarrow \|d\rangle$	$\|3\rangle \leftrightarrow \|d\rangle$	$\|4\rangle \leftrightarrow \|d\rangle$	$\dots$	$\|d-2\rangle \leftrightarrow \|d\rangle$	$\|d-1\rangle \leftrightarrow \|d\rangle$

$$(c_{si1}, c_{si2}, \ldots, c_{sid})^T = U_s^{i\dagger}(\phi_1, \phi_2, \ldots, \phi_d)^T. \tag{2.21}$$

Hence, given the SPs (we propose a method on how to find them), the problem is reduced to solving d linear equations with d unknowns p_s^i $(i = 1, 2, \ldots, d)$, and the solutions of Eq. (2.20) determine the solutions for the Kraus operators K_s^i in Eq. (2.15) which make the transformation $\rho_\psi \to \rho_\phi$. Succinctly, the problem is to find the correct SPs in Eq. (2.17).

Although the relations $\psi_1 \leq \phi_1$ and $\psi_d \geq \phi_d$ follow from the majorization, there are 2^{d-2} $(d > 2)$ possible relations for the other coefficients of the initial and the final coherent states. The first step, the crux of our method, is to construct a table (see Table 2.1) which contains all possible permutations for a case (either $\psi_k \leq \phi_k$ or $\psi_k \geq \phi_k$ where $k = 2, 3, ..., d-1$) of d-level systems. In a table there will be a certain number ζ of permutations where $(d-1) \leq \zeta \leq \lfloor d/2 \rfloor \lceil d/2 \rceil$. The $(d-1)$ element combinations of these permutations constitute SPs together with I_d ($d \times d$ identity transformation). A single SP $(s = 1)$ is sufficient if $\zeta = d - 1$, and there are $d - 1$ single SP for d-level systems. On the other hand, if $\zeta > d - 1$, then the given case splits into subcases, and therefore, for a complete set we need $n > 1$ $(s = 1, 2, \ldots, n)$ SPs for the corresponding case, i.e., number of subcases is n. For different coefficients satisfying different relations, we need a different SP and set of Kraus operators. We label each set by the index "s" which refers to a solution for a given relation (subcase relation) between coefficients. Now, before we begin to explain our protocol, we define

$$\begin{aligned} \phi_x^2 + \phi_{x+1}^2 + \cdots + \phi_k^2 - \psi_x^2 - \psi_{x+1}^2 - \cdots - \psi_k^2 &\equiv \alpha_{x(x+1)\ldots k}, \\ \psi_l^2 + \psi_{l+1}^2 + \cdots + \psi_y^2 - \phi_l^2 - \phi_{l+1}^2 - \cdots - \phi_y^2 &\equiv \beta_{l(l+1)\ldots y}, \\ \phi_x^2 - \phi_y^2 &\equiv \gamma_{xy}, \end{aligned} \tag{2.22}$$

where $y > x$ and $\gamma_{xy} > 0$. Given the correct set of permutations, the solutions of Eq. (2.20) for the probability, corresponding to the permutation $|x\rangle \leftrightarrow |y\rangle$, turn out to be either

$$p = \frac{\alpha_{x(x+1)(x+2)\ldots k}}{\gamma_{xy}}, \tag{2.23}$$

where $k \in [x, y-1]$, or

$$p = \frac{\beta_{l(l+1)(l+2)\ldots y}}{\gamma_{xy}}, \tag{2.24}$$

where $l \in [x+1, y]$.

In our protocol, the set of permutations is obtained as follows: We first construct a table using the relations between the coefficients of the initial (source) and the final (target) states. Then,

(i) All sets of permutations contain the identity transformation, $U_s^1 = I_d$. The measurement (where the Kraus operator is K_s^1) probability corresponding to this permutation is found to be $p_s^1 = 1 - \sum_{i=2}^{d} p_s^i$.

(ii) The permutation $|v\rangle \leftrightarrow |m\rangle$ exists in all the sets of permutations if $|v\rangle \leftrightarrow |m\rangle$ is the only permutation in a column of the resulting table. The measurement probability corresponding to this permutation is found to be $(\phi_v^2 - \psi_v^2)/(\phi_v^2 - \phi_m^2) \equiv \alpha_v/\gamma_{vm}$.

(iii) The permutation $|h\rangle \leftrightarrow |k\rangle$ exists in all the sets of permutations if $|h\rangle \leftrightarrow |k\rangle$ is the only permutation in a row of the resulting table. The measurement probability corresponding to this permutation is found to be $(\psi_k^2 - \phi_k^2)/(\phi_h^2 - \phi_k^2) \equiv \beta_k/\gamma_{hk}$.

(iv) If any permutation $|u\rangle \leftrightarrow |u+1\rangle$ appears in the table then it must be an element of all the sets of permutations. The measurement probability corresponding to this permutation is either $(\psi_{u+1}^2 - \phi_{u+1}^2)/(\phi_u^2 - \phi_{u+1}^2) \equiv \beta_{u+1}/\gamma_{u(u+1)}$ or $(\phi_u^2 - \psi_u^2)/(\phi_u^2 - \phi_{u+1}^2) \equiv \alpha_u/\gamma_{u(u+1)}$.

(v) All sets of permutations contain the permutation $|1\rangle \leftrightarrow |d\rangle$ ($U_s^r = |1\rangle \leftrightarrow |d\rangle \equiv |d\rangle\langle 1| + \sum_{i=2}^{d-1} |i\rangle\langle i| + |1\rangle\langle d|$).

After applying the above steps as a first stage of our protocol, we get the set(s) of permutations of the form

$$\begin{aligned} U_s &= \{U_s^1, U_s^2, \ldots, U_s^r, U_s^{r+1}, \ldots, U_s^d\} \\ &= \{I_d, |v\rangle \leftrightarrow |m\rangle, |h\rangle \leftrightarrow |k\rangle, |u\rangle \leftrightarrow |u+1\rangle, \\ &\quad \ldots, U_s^{r-1}, |1\rangle \leftrightarrow |d\rangle, U_s^{r+1}, \ldots, U_s^d\}. \end{aligned} \tag{2.25}$$

We note that a table may contain more than one of permutations $|v\rangle \leftrightarrow |m\rangle$, $|h\rangle \leftrightarrow |k\rangle$ and $|u\rangle \leftrightarrow |u+1\rangle$. Therefore, r permutations are found after five steps described by (i)-(v). We still need a certain number, $d-r$, of permutations to complete the set U_s. In order to be able to explain how the remaining permutations, $\{U_s^{r+1}, U_s^{r+2}, \ldots, U_s^d\}$, are chosen, we give illustrative examples with pictorial representations.

Using the pictorial representation of the table of permutations, we observe that Eq. (2.20) has a solution with positive probabilities for all SPs with no intersection of permutations. Each nonintersecting set of permutations corresponds to a solution for an additional relation (subcase) between the coefficients of the states. More importantly, the collection of the solutions of Eq. (2.20) for all subcases forms the complete solution and the collection of the SPs with no intersection in the pictorial representation forms what we call the "complete set."

Hence, the problem of obtaining the correct sets of permutations (U_s) is reduced to the problem of obtaining sets of nonintersecting permutations in the pictorial representations. It is also obvious that the permutations described by (i)−(v) (I_d, $|v\rangle \leftrightarrow |m\rangle$, $|h\rangle \leftrightarrow |k\rangle$, $|u\rangle \leftrightarrow |u+1\rangle$, $|1\rangle \leftrightarrow |d\rangle$) do intersect each other and any other permutations in the pictorial representation of any table.

2.4.1.1 Examples for protocol I

We can best understand protocol I by explicitly examining some examples; we give two illustrative examples.

Example 1: Consider the case $\psi_2 \leq \phi_2$ and $\psi_3 \geq \phi_3$ for $d = 4$ where $\psi_1 \leq \phi_1$ and $\psi_4 \geq \phi_4$ follow from the majorization condition. In the beginning we construct Table 2.2 which contains all possible permutations for this case. There are four permutations

Table 2.2 : All possible permutations for the case $\psi_1 \leq \phi_1$, $\psi_2 \leq \phi_2$, $\psi_3 \geq \phi_3$ and $\psi_4 \geq \phi_4$ of $d = 4$.

	$\psi_1 \leq \phi_1$	$\psi_2 \leq \phi_2$
$\psi_3 \geq \phi_3$	$\|1\rangle \leftrightarrow \|3\rangle$	$\|2\rangle \leftrightarrow \|3\rangle$
$\psi_4 \geq \phi_4$	$\|1\rangle \leftrightarrow \|4\rangle$	$\|2\rangle \leftrightarrow \|4\rangle$

in Table 2.2. Figure 2.3 shows the pictorial representation of these. The three-element combinations of these four permutations constitute the SPs together with identity transformation I_4, where Fig. 2.4 provides the pictorial representations of all possible three-element combinations. According to our protocol, any set(s) of permutations contain the identity transformation, I_4. We have also proposed that if any permutation $|u\rangle \leftrightarrow |u+1\rangle$ appears in the table then it must be an element of all SPs, $|2\rangle \leftrightarrow |3\rangle$ belongs to this category. The permutation $|1\rangle \leftrightarrow |4\rangle$ must also be an element of all

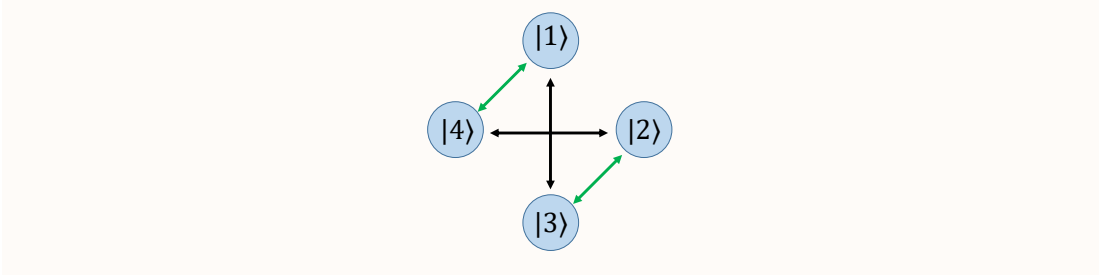

Figure 2.3 : Pictorial representation of four permutations given in Table 2.2 for the case $\psi_2 \leq \phi_2$ and $\psi_3 \geq \phi_3$ of four-level states. The permutations $|1\rangle \leftrightarrow |4\rangle$ and $|2\rangle \leftrightarrow |3\rangle$ do not intersect each other and any other permutations (the green (leftright)arrows). Therefore, $|1\rangle \leftrightarrow |4\rangle$ and $|2\rangle \leftrightarrow |3\rangle$ will be elements of any possible SPs.

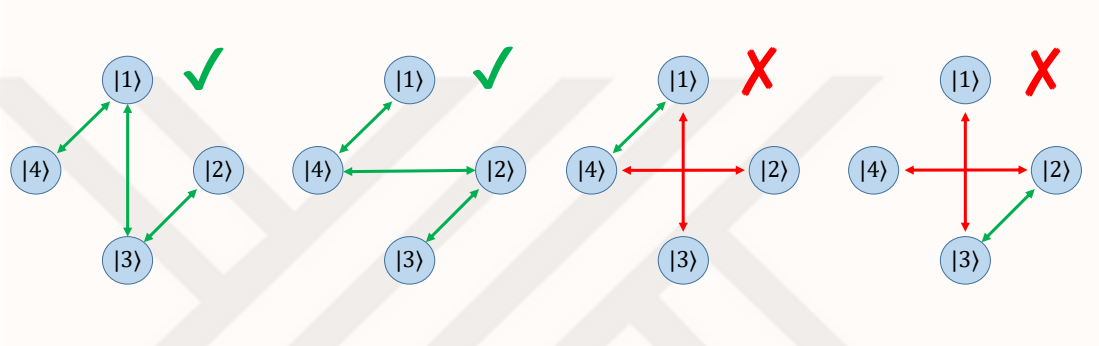

Figure 2.4 : Pictorial representations of all possible three-element combinations of the permutations given in Table 2.2. Here, the first two three-element combinations of the permutations (in green with check mark) are sufficient to constitute a complete set (together with I_4). However, last two (in red with cross mark) are not usable where these sets give negative probabilities (which is unphysical).

SPs. Thus, the SPs have been obtained such that

$$U_s = \left\{ I_4, |2\rangle \leftrightarrow |3\rangle, |1\rangle \leftrightarrow |4\rangle, U_s^4 \right\}. \tag{2.26}$$

The remaining permutation, U_s^4, is chosen among the permutations $|1\rangle \leftrightarrow |3\rangle$ and $|2\rangle \leftrightarrow |4\rangle$. We obtained that $U_1^4 = |1\rangle \leftrightarrow |3\rangle$ and $U_2^4 = |2\rangle \leftrightarrow |4\rangle$. The SPs were identified, and using these we can obtain the two sets of Kraus operators. The Kraus operators are given by

$$K_s^i = U_s^i \left(\sqrt{p_s^i} \sum_{j=1}^{4} \frac{c_{sij}}{\psi_j} |j\rangle \langle j| \right) \quad (s = 1, 2), \tag{2.27}$$

where $\sum_{i=1}^{4} K_s^{i\dagger} K_s^i = I_4$ and $p_s^i = \mathrm{Tr}[K_s^i \rho_\psi K_s^{i\dagger}]$. We express the coefficients c_{sij} in compact form as the elements of the matrix c_s , i.e., c_{sij} is the (ij)th element of the

matrix c_s where $U_s^i(c_{si1}, c_{si2}, c_{si3}, c_{si4})^T = (\phi_1, \phi_2, \phi_3, \phi_4)^T$. The matrix c_s is given by

$$c_s = \begin{pmatrix} \phi_1 & \phi_2 & \phi_3 & \phi_4 \\ \phi_1 & \phi_3 & \phi_2 & \phi_4 \\ \phi_4 & \phi_2 & \phi_3 & \phi_1 \\ c_{s41} & c_{s42} & c_{s43} & c_{s44} \end{pmatrix}. \tag{2.28}$$

The condition $\sum_{i=1}^4 K_s^{i\dagger} K_s^i = I_4$ implies that the following two sets of four linear equations

$$\sum_{i=1}^4 p_s^i c_{sij}^2 = \psi_j^2, \quad (j = 1,2,3,4), \quad (s = 1,2), \tag{2.29}$$

should be satisfied with the constraint $p_s^i \geq 0$. Solutions of the linear equations $\sum_{i=1}^4 p_1^i c_{1ij}^2 = \psi_j^2$ gives the probabilities of the first set where the matrix c_1 is given by (follows from $U_1 = \{I_4, |2\rangle \leftrightarrow |3\rangle, |1\rangle \leftrightarrow |4\rangle, |1\rangle \leftrightarrow |3\rangle\}$)

$$c_1 = \begin{pmatrix} \phi_1 & \phi_2 & \phi_3 & \phi_4 \\ \phi_1 & \phi_3 & \phi_2 & \phi_4 \\ \phi_4 & \phi_2 & \phi_3 & \phi_1 \\ \phi_3 & \phi_2 & \phi_1 & \phi_4 \end{pmatrix}. \tag{2.30}$$

Thus, probabilities of the first set are found to be

$$p_1^1 = 1 - \sum_{i=2}^4 p_1^i, \quad p_1^2 = \frac{\alpha_2}{\gamma_{23}}, \quad p_1^3 = \frac{\beta_4}{\gamma_{14}}, \quad p_1^4 = \frac{\beta_{23}}{\gamma_{13}}, \tag{2.31}$$

where $\beta_{23} \geq 0$, i.e, $\psi_2^2 + \psi_3^2 \geq \phi_2^2 + \phi_3^2$. Similarly, solutions of the linear equations $\sum_{i=1}^4 p_2^i c_{2ij}^2 = \psi_j^2$ gives the probabilities of the second set where the matrix c_2 is given by (follows from $U_2 = \{I_4, |2\rangle \leftrightarrow |3\rangle, |1\rangle \leftrightarrow |4\rangle, |2\rangle \leftrightarrow |4\rangle\}$)

$$c_2 = \begin{pmatrix} \phi_1 & \phi_2 & \phi_3 & \phi_4 \\ \phi_1 & \phi_3 & \phi_2 & \phi_4 \\ \phi_4 & \phi_2 & \phi_3 & \phi_1 \\ \phi_1 & \phi_4 & \phi_3 & \phi_2 \end{pmatrix}. \tag{2.32}$$

Then, the probabilities of the second set are found to be

$$p_2^1 = 1 - \sum_{i=2}^4 p_1^i, \quad p_2^2 = \frac{\beta_3}{\gamma_{23}}, \quad p_2^3 = \frac{\alpha_1}{\gamma_{14}}, \quad p_2^4 = \frac{\alpha_{23}}{\gamma_{24}}, \tag{2.33}$$

where $\alpha_{23} \geq 0$, i.e., $\psi_2^2 + \psi_3^2 \leq \phi_2^2 + \phi_3^2$. This case, $\psi_1 \leq \phi_1$, $\psi_2 \leq \phi_2$, $\psi_3 \geq \phi_3$ and $\psi_4 \geq \phi_4$ for $d = 4$, splits into two subcases. If the subcase relation is $\psi_2^2 + \psi_3^2 \geq \phi_2^2 + \phi_3^2$ ($\beta_{23} \geq 0$) then the first SP, U_1, must be used. If the subcase relation is $\psi_2^2 + \psi_3^2 \leq \phi_2^2 + \phi_3^2$ ($\alpha_{23} \geq 0$) then the second SP, U_2, must be used. These two SPs constitute the complete set and give the complete solution of the problem, i.e., the probabilities

and Kraus operators are found for all possible relations among the coefficients for the given case $\psi_1 \leq \phi_1$, $\psi_2 \leq \phi_2$, $\psi_3 \geq \phi_3$ and $\psi_4 \geq \phi_4$ of $d = 4$.

Example 2: Consider the case $\psi_a \leq \phi_a$ ($a = 1,4,5,7,8$) and $\psi_b \geq \phi_b$ ($b = 2,3,6,9$) for $d = 9$. In Table 2.3, there are ten permutations, some eight-element combinations of

Table 2.3 : All possible permutations for the case $\psi_a \leq \phi_a$ ($a = 1,4,5,7,8$) and $\psi_b \geq \phi_b$ ($b = 2,3,6,9$) of $d = 9$.

	$\psi_1 \leq \phi_1$	$\psi_4 \leq \phi_4$	$\psi_5 \leq \phi_5$	$\psi_7 \leq \phi_7$	$\psi_8 \leq \phi_8$
$\psi_2 \geq \phi_2$	$\|1\rangle \leftrightarrow \|2\rangle$	-	-	-	-
$\psi_3 \geq \phi_3$	$\|1\rangle \leftrightarrow \|3\rangle$	-	-	-	-
$\psi_6 \geq \phi_6$	$\|1\rangle \leftrightarrow \|6\rangle$	$\|4\rangle \leftrightarrow \|6\rangle$	$\|5\rangle \leftrightarrow \|6\rangle$	-	-
$\psi_9 \geq \phi_9$	$\|1\rangle \leftrightarrow \|9\rangle$	$\|4\rangle \leftrightarrow \|9\rangle$	$\|5\rangle \leftrightarrow \|9\rangle$	$\|7\rangle \leftrightarrow \|9\rangle$	$\|8\rangle \leftrightarrow \|9\rangle$

which, together with the identity transformation, constitute the permutations U_s^i ($i = 1,\ldots,9$). In our protocol, the SPs are obtained in five steps described in (i)−(v) plus an additional step (using nonintersecting permutations in the pictorial representations). Let us consider each step in turn. First, all SPs contain the identity transformation; $U_s^1 = I_9$. Second, the permutation $|v\rangle \leftrightarrow |m\rangle$ exists in all the SPs if $|v\rangle \leftrightarrow |m\rangle$ is the only permutation in any column of Table 2.3; $U_s^2 = |7\rangle \leftrightarrow |9\rangle$ and $U_s^3 = |8\rangle \leftrightarrow |9\rangle$. Third, the permutation $|h\rangle \leftrightarrow |k\rangle$ exists in all the SPs if $|h\rangle \leftrightarrow |k\rangle$ is the only permutation in any row of Table 2.3; $U_s^4 = |1\rangle \leftrightarrow |2\rangle$ and $U_s^5 = |1\rangle \leftrightarrow |3\rangle$. Fourth, if any permutation $|u\rangle \leftrightarrow |u+1\rangle$ appears in the table then it must be an element of all the SPs; $U_s^6 = |5\rangle \leftrightarrow |6\rangle$. Fifth, all SPs contain the permutation $|1\rangle \leftrightarrow |d\rangle$; $U_s^7 = |1\rangle \leftrightarrow |9\rangle$. Thus, after five steps, the point we have reached is that

$$\begin{aligned} U_s = \big\{ & I_9, |7\rangle \leftrightarrow |9\rangle, |8\rangle \leftrightarrow |9\rangle, |1\rangle \leftrightarrow |2\rangle, |1\rangle \leftrightarrow |3\rangle, \\ & |5\rangle \leftrightarrow |6\rangle, |1\rangle \leftrightarrow |9\rangle, U_s^8, U_s^9 \big\}. \end{aligned} \tag{2.34}$$

The SPs given by Eq. (2.34) satisfies our interesting result−any pairs of permutations in a SP do not intersect each other in the pictorial representation as seen in Fig. 2.5.

We still need two more permutations, U_s^8 and U_s^9, and these will be selected from among four remaining permutations, $|1\rangle \leftrightarrow |6\rangle$, $|4\rangle \leftrightarrow |6\rangle$, $|4\rangle \leftrightarrow |9\rangle$ and $|5\rangle \leftrightarrow |9\rangle$. The final step is to determine the correct remaining permutations for SPs to constitute a complete set. The result we have obtained is that three SPs ($s = 1,2,3$) among six combinations are sufficient [see Fig. 2.6] to constitute a complete set, i.e., for a complete solution, three SPs are required which equally means that three subcases

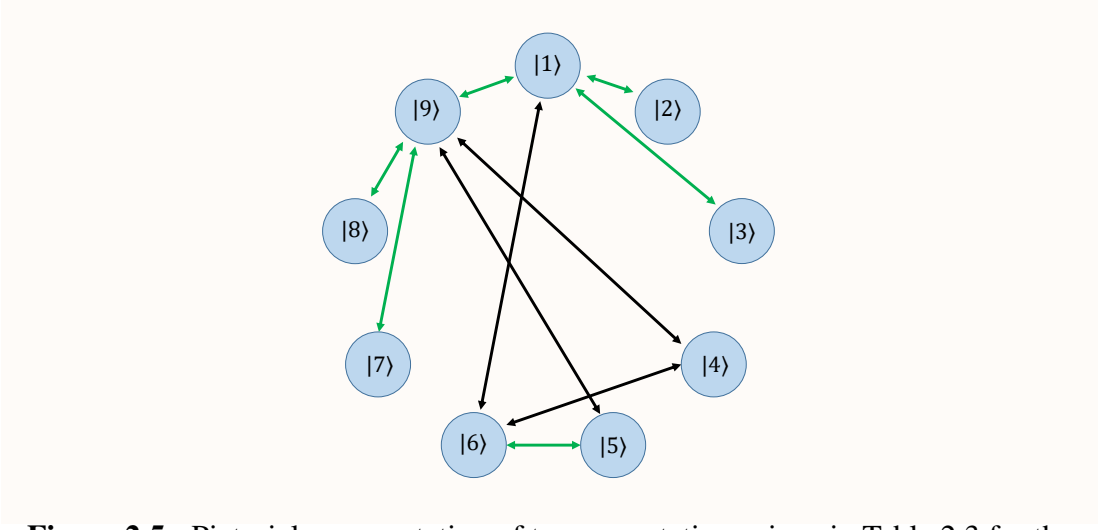

Figure 2.5 : Pictorial representation of ten permutations given in Table 2.3 for the case $\psi_a \leq \phi_a$ ($a = 1,4,5,7,8$) and $\psi_b \geq \phi_b$ ($b = 2,3,6,9$) of $d = 9$. The permutations $|7\rangle \leftrightarrow |9\rangle$, $|8\rangle \leftrightarrow |9\rangle$, $|1\rangle \leftrightarrow |2\rangle$, $|1\rangle \leftrightarrow |3\rangle$, $|5\rangle \leftrightarrow |6\rangle$, and $|1\rangle \leftrightarrow |9\rangle$ do not intersect each other and any other permutations.

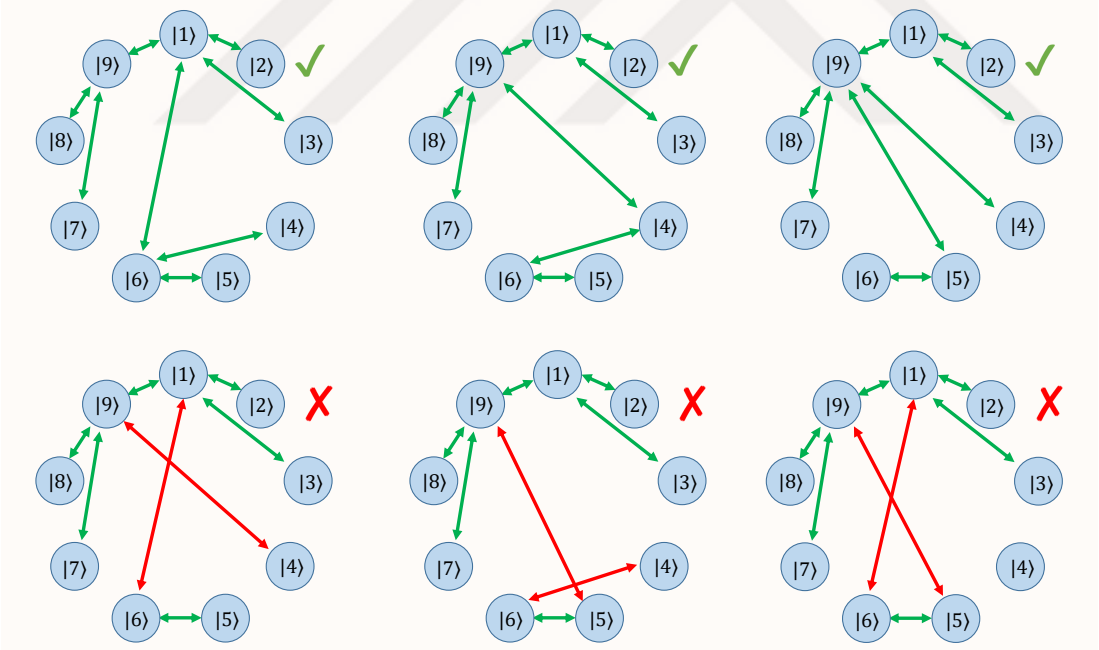

Figure 2.6 : Pictorial representations of the SPs (also I_9 is an element of all SPs) for the case $\psi_a \leq \phi_a$ ($a = 1,4,5,7,8$) and $\psi_b \geq \phi_b$ ($b = 2,3,6,9$) of $d = 9$. Here, the first three eight-element combinations of the permutations (in green with check mark) are sufficient to constitute a complete set. However, last three eight-element combinations of the permutations (in red with cross mark) are not usable while these give negative probabilities.

exist for the given case. Thus, two remaining permutations for these three SPs are found to be

$$\begin{aligned}
\left(U_1^8, U_1^9\right) &= \left(|1\rangle \leftrightarrow |6\rangle, |4\rangle \leftrightarrow |6\rangle\right), \\
\left(U_2^8, U_2^9\right) &= \left(|4\rangle \leftrightarrow |9\rangle, |4\rangle \leftrightarrow |6\rangle\right), \\
\left(U_3^8, U_3^9\right) &= \left(|4\rangle \leftrightarrow |9\rangle, |5\rangle \leftrightarrow |9\rangle\right).
\end{aligned} \tag{2.35}$$

The SPs were identified [combine Eq. (2.34) and Eq. (2.35)], and using these we obtain the sets of Kraus operators. The Kraus operators are given by

$$K_s^i = U_s^i\left(\sqrt{p_s^i}\sum_{j=1}^{9}\frac{c_{sij}}{\psi_j}|j\rangle\langle j|\right) \quad (s=1,2,3), \tag{2.36}$$

where $(c_{si1}, c_{si2}, c_{si3}, \ldots, c_{si9})^T = (U_s^i)^\dagger(\phi_1, \phi_2, \phi_3, \ldots, \phi_9)^T$ and $p_s^i = \mathrm{Tr}[K_s^i \rho_\psi K_s^{i\dagger}]$. The matrix c_s is given by

$$c_s = \begin{pmatrix}
\phi_1 & \phi_2 & \phi_3 & \phi_4 & \phi_5 & \phi_6 & \phi_7 & \phi_8 & \phi_9 \\
\phi_1 & \phi_2 & \phi_3 & \phi_4 & \phi_5 & \phi_6 & \phi_9 & \phi_8 & \phi_7 \\
\phi_1 & \phi_2 & \phi_3 & \phi_4 & \phi_5 & \phi_6 & \phi_7 & \phi_9 & \phi_8 \\
\phi_2 & \phi_1 & \phi_3 & \phi_4 & \phi_5 & \phi_6 & \phi_7 & \phi_8 & \phi_9 \\
\phi_3 & \phi_2 & \phi_1 & \phi_4 & \phi_5 & \phi_6 & \phi_7 & \phi_8 & \phi_9 \\
\phi_1 & \phi_2 & \phi_3 & \phi_4 & \phi_6 & \phi_5 & \phi_7 & \phi_8 & \phi_9 \\
\phi_9 & \phi_2 & \phi_3 & \phi_4 & \phi_5 & \phi_6 & \phi_7 & \phi_8 & \phi_1 \\
c_{s81} & c_{s82} & c_{s83} & c_{s84} & c_{s85} & c_{s86} & c_{s87} & c_{s88} & c_{s89} \\
c_{s91} & c_{s92} & c_{s93} & c_{s94} & c_{s95} & c_{s96} & c_{s97} & c_{s98} & c_{s99}
\end{pmatrix}. \tag{2.37}$$

The condition $\sum_{i=1}^{9} K_s^{i\dagger} K_s^i = I_9$ implies the following linear equations

$$\sum_{i=1}^{9} p_s^i c_{sij}^2 = \psi_j^2, \quad p_s^i \geq 0, \quad (j=1,2,\ldots,9), \tag{2.38}$$

whose solutions for p_s^i give the probabilities for each set. The probabilities corresponding to the first SP U_1 (solutions of the linear equations $\sum_{i=1}^{9} p_1^i c_{1ij}^2 = \psi_j^2$) are found to be

$$\begin{aligned}
&p_1^1 = 1 - \sum_{i=2}^{9} p_s^i, \quad p_1^2 = \frac{\alpha_7}{\gamma_{79}}, \quad p_1^3 = \frac{\alpha_8}{\gamma_{89}}, \quad p_1^4 = \frac{\beta_2}{\gamma_{12}}, \\
&p_1^5 = \frac{\beta_3}{\gamma_{13}} \quad p_1^6 = \frac{\alpha_5}{\gamma_{56}}, \quad p_1^7 = \frac{\beta_{789}}{\gamma_{19}}, \quad p_1^8 = \frac{\beta_{456}}{\gamma_{16}}, \quad p_1^9 = \frac{\alpha_4}{\gamma_{46}},
\end{aligned} \tag{2.39}$$

where $\beta_{456} \geq 0$. The probabilities corresponding to the second SP U_2 (solutions of the linear equations $\sum_{i=1}^{9} p_2^i c_{2ij}^2 = \psi_j^2$) are found to be

$$\begin{aligned}
&p_2^1 = 1 - \sum_{i=2}^{9} p_s^i, \quad p_2^2 = \frac{\alpha_7}{\gamma_{79}}, \quad p_2^3 = \frac{\alpha_8}{\gamma_{89}}, \quad p_2^4 = \frac{\beta_2}{\gamma_{12}}, \\
&p_2^5 = \frac{\beta_3}{\gamma_{13}} \quad p_2^6 = \frac{\alpha_5}{\gamma_{56}}, \quad p_2^7 = \frac{\alpha_{123}}{\gamma_{19}}, \quad p_2^8 = \frac{\alpha_{456}}{\gamma_{49}}, \quad p_2^9 = \frac{\beta_{56}}{\gamma_{46}},
\end{aligned} \tag{2.40}$$

where $\alpha_{456} \geq 0$ and $\beta_{56} \geq 0$. The probabilities corresponding to the third SP U_3 (solutions of the linear equations $\sum_{i=1}^{9} p_3^i c_{3ij}^2 = \psi_j^2$) are found to be

$$\begin{aligned} p_3^1 &= 1 - \sum_{i=2}^{9} p_s^i, \quad p_3^2 = \frac{\alpha_7}{\gamma_{79}}, \quad p_3^3 = \frac{\alpha_8}{\gamma_{89}}, \quad p_3^4 = \frac{\beta_2}{\gamma_{12}}, \\ p_3^5 &= \frac{\beta_3}{\gamma_{13}} \quad p_3^6 = \frac{\beta_6}{\gamma_{56}}, \quad p_3^7 = \frac{\alpha_{123}}{\gamma_{19}}, \quad p_3^8 = \frac{\alpha_4}{\gamma_{49}}, \quad p_3^9 = \frac{\alpha_{56}}{\gamma_{59}}, \end{aligned} \tag{2.41}$$

where $\alpha_{56} \geq 0$. We note that the probabilities p_s^1, p_s^2, p_s^3, p_s^4 and p_s^5 turn out to be of the form

$$p_s^1 = 1 - \sum_{i=2}^{9} p_s^i, \quad p_s^2 = \frac{\alpha_7}{\gamma_{79}}, \quad p_s^3 = \frac{\alpha_8}{\gamma_{89}}, \quad p_s^4 = \frac{\beta_2}{\gamma_{12}}, \quad p_s^5 = \frac{\beta_3}{\gamma_{13}} \quad (s = 1,2,3), \tag{2.42}$$

as we stated in (i), (ii) and (iii). We obtained that there are three SPs for this given case. Then, if the subcase is $\psi_4^2 + \psi_5^2 + \psi_6^2 \geq \phi_4^2 + \phi_5^2 + \phi_6^2$ ($\beta_{456} \geq 0$) then the SP U_1 will be used. If the subcase is $\psi_4^2 + \psi_5^2 + \psi_6^2 \leq \phi_4^2 + \phi_5^2 + \phi_6^2$ and $\psi_5^2 + \psi_6^2 \geq \phi_5^2 + \phi_6^2$ ($\alpha_{456} \geq 0$ and $\beta_{56} \geq 0$) then the SP U_2 will be used. If the subcase is $\psi_5^2 + \psi_6^2 \leq \phi_5^2 + \phi_6^2$ ($\alpha_{56} \geq 0$) then the SP U_3 will be used. Thus, these three SPs ($s = 1,2,3$) form a complete set for the given case−using one of these three SPs one can transform nine-level coherent states, whose coefficients have the relations $\psi_a \leq \phi_a$ ($a = 1,4,5,7,8$) and $\psi_b \geq \phi_b$ ($b = 2,3,6,9$), via incoherent operations.

Note that we applied protocol I, to check and verify its validity, to all possible d'-level ($d' = 2,3,4,5,6,7$) sources and target coherent states (and also some special cases of higher level systems) and obtained the explicit solutions, i.e., sets of permutations, probabilities, and Kraus operators, in line with the above discussions. Hence, we conclude that the protocol we have presented solves the problem of single-step deterministic transformations of coherent states via incoherent operations.

2.4.1.2 Examples of generalized solutions using protocol I

Although finding the SPs (and hence finding probabilities) is easy, constructing a general form of probabilities is a highly non-trivial problem, and the problem becomes exponentially difficult as the dimension gets greater. However, we are able to extrapolate a complete solution for some special cases of d-level systems. As an example, we consider the case $\psi_1 \leq \phi_1$, $\psi_2 \geq \phi_2$, $\psi_3 \geq \phi_3$, $\ldots$, $\psi_{k-1} \geq \phi_{k-1}$, $\psi_k \geq \phi_k$, $\psi_{k+1} \leq \phi_{k+1}$, $\psi_{k+2} \leq \phi_{k+2}$, $\ldots$, $\psi_{d-2} \leq \phi_{d-2}$, $\psi_{d-1} \leq \phi_{d-1}$, $\psi_d \geq \phi_d$ (see Fig. (2.7))and construct the table for the possible permutations as listed in Table 2.4.

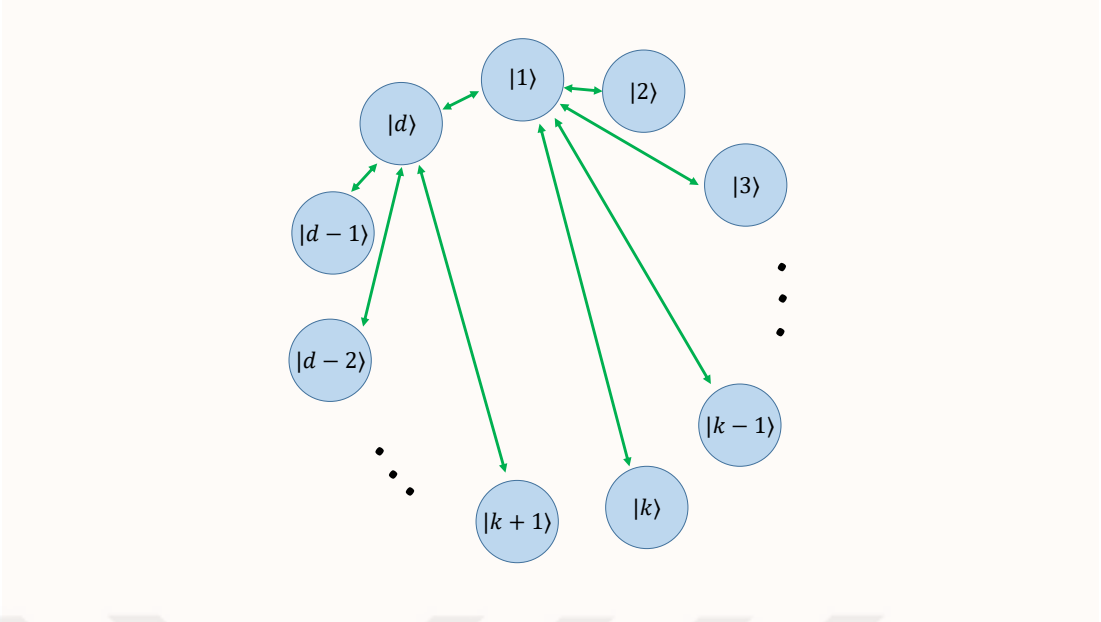

Figure 2.7 : Pictorial representation of permutations for the case $\psi_1 \leq \phi_1$, $\psi_2 \geq \phi_2$, $\psi_3 \geq \phi_3, \ldots, \psi_{k-1} \geq \phi_{k-1}$, $\psi_k \geq \phi_k$, $\psi_{k+1} \leq \phi_{k+1}$, $\psi_{k+2} \leq \phi_{k+2}, \ldots, \psi_{d-2} \leq \phi_{d-2}$, $\psi_{d-1} \leq \phi_{d-1}$, $\psi_d \geq \phi_d$. of d-level systems.

The Kraus operators are given by

$$K^i = U^i\left(\sqrt{p^i}\sum_{j=1}^{d}\frac{c_{ij}}{\psi_j}\left|j\right\rangle\left\langle j\right|\right) \quad (i=1,\ldots,d), \tag{2.43}$$

where $\sum_{i=1}^{d} K^{i\dagger}K^i = I_d$ and $p^i = Tr(K^i\rho_\psi K^{i\dagger})$. A single SP, for the complete solution, is sufficient for this case where it is found to be

$$\begin{aligned} U = \Big\{ & I_d, |k+1\rangle \leftrightarrow |d\rangle, |k+2\rangle \leftrightarrow |d\rangle, |k+3\rangle \leftrightarrow |d\rangle, \ldots, |d-2\rangle \leftrightarrow |d\rangle, \\ & |d-1\rangle \leftrightarrow |d\rangle, |1\rangle \leftrightarrow |2\rangle, |1\rangle \leftrightarrow |3\rangle, |1\rangle \leftrightarrow |4\rangle, \ldots, |1\rangle \leftrightarrow |k-2\rangle, \\ & |1\rangle \leftrightarrow |k-1\rangle, |1\rangle \leftrightarrow |k\rangle, |1\rangle \leftrightarrow |d\rangle \Big\}. \end{aligned} \tag{2.44}$$

Table 2.4 : All possible permutations for the case $\psi_1 \leq \phi_1$, $\psi_2 \geq \phi_2$, $\psi_3 \geq \phi_3, \ldots, \psi_{k-1} \geq \phi_{k-1}$, $\psi_k \geq \phi_k$, $\psi_{k+1} \leq \phi_{k+1}$, $\psi_{k+2} \leq \phi_{k+2}, \ldots, \psi_{d-1} \leq \phi_{d-1}$, $\psi_d \geq \phi_d$. There are $d-1$ permutations for this case, and a single set of permutations is sufficient for complete solution.

	$\psi_1 \leq \phi_1$	$\psi_{k+1} \leq \phi_{k+1}$	$\psi_{k+2} \leq \phi_{k+2}$	$\ldots$	$\psi_{d-1} \leq \phi_{d-1}$
$\psi_2 \geq \phi_2$	$\|1\rangle \leftrightarrow \|2\rangle$	-	-	$\ldots$	-
$\psi_3 \geq \phi_3$	$\|1\rangle \leftrightarrow \|3\rangle$	-	-	$\ldots$	-
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\psi_{k-1} \geq \phi_{k-1}$	$\|1\rangle \leftrightarrow \|k-1\rangle$	-	-	$\ldots$	-
$\psi_k \geq \phi_k$	$\|1\rangle \leftrightarrow \|k\rangle$	-	-	$\ldots$	-
$\psi_d \geq \phi_d$	$\|1\rangle \leftrightarrow \|d\rangle$	$\|k+1\rangle \leftrightarrow \|d\rangle$	$\|k+2\rangle \leftrightarrow \|d\rangle$	$\ldots$	$\|d-1\rangle \leftrightarrow \|d\rangle$

The probabilities are found to be

$$\begin{aligned}\{p^i\}^{i=1,2,\ldots,d} = \Big\{ p^1 = 1-\sum_{i=2}^{d} p^i, \quad p^2 = \frac{\alpha_{k+1}}{\gamma_{(k+1)d}}, \quad p^3 = \frac{\alpha_{k+2}}{\gamma_{(k+2)d}}, \\ p^4 = \frac{\alpha_{k+3}}{\gamma_{(k+3)d}}, \quad \ldots, \quad p^{d-k-1} = \frac{\alpha_{d-2}}{\gamma_{(d-2)d}}, \\ p^{d-k} = \frac{\alpha_{d-1}}{\gamma_{(d-1)d}}, \quad p^{d-k+1} = \frac{\beta_2}{\gamma_{12}}, \quad p^{d-k+2} = \frac{\beta_3}{\gamma_{13}}, \quad \ldots, \\ p^{d-2} = \frac{\beta_{k-1}}{\gamma_{1(k-1)}}, \quad p^{d-1} = \frac{\beta_k}{\gamma_{1k}}, \quad p^d = \frac{\alpha_{12\ldots k}}{\gamma_{1d}} \Big\},\end{aligned} \tag{2.45}$$

where $\sum_{j=1}^{k} \phi_j^2 \geq \sum_{j=1}^{k} \psi_j^2$ (or equivalently $\alpha_{12\ldots k} \geq 0$) follows from majorization. The resulting table consists of a single row for the case $\psi_1 \leq \phi_1$, $\psi_2 \leq \phi_2$, $\psi_3 \leq \phi_3$, $\ldots$, $\psi_{d-1} \leq \phi_{d-1}$, $\psi_d \geq \phi_d$. The set of permutations and the probabilities in that case turn out to be

$$\begin{aligned}U = \big\{ I_d, |1\rangle \leftrightarrow |d\rangle, |2\rangle \leftrightarrow |d\rangle, |3\rangle \leftrightarrow |d\rangle, \ldots, \\ |d-3\rangle \leftrightarrow |d\rangle, |d-2\rangle \leftrightarrow |d\rangle, |d-1\rangle \leftrightarrow |d\rangle \big\},\end{aligned} \tag{2.46}$$

$$p^1 = 1-\sum_{i=2}^{d} p^i, \quad \{p^i\}^{i=2,\ldots,d} = \frac{\alpha_{i-1}}{\gamma_{(i-1)d}} = \frac{\phi_{i-1}^2 - \psi_{i-1}^2}{\phi_{i-1}^2 - \phi_d^2}, \tag{2.47}$$

respectively. Similarly, the resulting table consists of a single column for the case $\psi_1 \leq \phi_1$, $\psi_2 \geq \phi_2$, $\psi_3 \geq \phi_3$, $\ldots$, $\psi_{d-1} \geq \phi_{d-1}$, $\psi_d \geq \phi_d$. The set of permutations and the probabilities in that case turn out to be

$$\begin{aligned}U = \big\{ I_d, |1\rangle \leftrightarrow |2\rangle, |1\rangle \leftrightarrow |3\rangle, |1\rangle \leftrightarrow |4\rangle, \ldots, \\ |1\rangle \leftrightarrow |d-2\rangle, |1\rangle \leftrightarrow |d-1\rangle, |1\rangle \leftrightarrow |d\rangle \big\},\end{aligned} \tag{2.48}$$

$$p^1 = 1-\sum_{i=2}^{d} p^i, \quad \{p^i\}^{i=2,\ldots,d} = \frac{\beta_i}{\gamma_{1i}} = \frac{\psi_i^2 - \phi_i^2}{\phi_1^2 - \phi_i^2}, \tag{2.49}$$

respectively. Besides these, there are also different cases which are generalizable. If a table of permutations, for instance, has $d-1$ permutations then for such cases probabilities can easily be found while a single SP is sufficient. Also, the number of a single SP is $d-1$ for d-level systems where the other $2^{d-2}-d+1$ ($d>3$) cases split into subcases, and therefore, the complete solutions for these cases require more than one SP. By using protocol-I, one can obtain all SPs of a complete set.

2.4.2 Protocol II

We now present another protocol for coherence transformations similar to the one for the LOCC deterministic transformations of bipartite entangled pure states [37]. We use d'-dimensional subspace solutions ($d' < d$), obtained by protocol I, to achieve the transformation $|\psi\rangle\langle\psi| \to |\phi\rangle\langle\phi|$ as a sequence of transformations of coherent states, that is, $|\psi\rangle\langle\psi| \to |\eta_1\rangle\langle\eta_1| \to \cdots \to |\eta_k\rangle\langle\eta_k| \to |\phi\rangle\langle\phi|$, where $|\eta_i\rangle$ are intermediate coherent states satisfying the majorization conditions $\mu(\psi) \prec \mu(\eta_1) \prec \ldots \prec \mu(\eta_k) \prec \mu(\phi)$. We stress that the most crucial point of the sequence of coherent states transformations is to preserve the majorization condition for the entire transformation. In each step of $|\psi\rangle\langle\psi| \to |\phi\rangle\langle\phi|$ we aim to obtain some coefficients of the final coherent state $|\phi\rangle$ using d'-level subspace solutions (we move closer to the final coherent state step-by-step).

In the following protocol, we use five-level ($d' = 5$) subspace solutions and transform four coefficients of the existing coherent state to the coefficients of the final coherent state starting from smaller ones. This protocol is implementable for transforming any coefficients satisfying the majorization condition given by the inequalities $\sum_{j=1}^{k} {\psi_j}^2 \leq \sum_{j=1}^{k} {\phi_j}^2$, and an equivalent form of the majorization condition is given by

$$\sum_{j=k+1}^{d} \psi_j^2 \geq \sum_{j=k+1}^{d} \phi_j^2, \tag{2.50}$$

where $k = 1, 2, \ldots, d-1$ and $\sum_{j=1}^{d} \psi_j^2 = \sum_{j=1}^{d} \phi_j^2 = 1$. We have coherent states $|\psi\rangle$, $|\eta_1\rangle$, $|\eta_2\rangle$, $\ldots$, $|\eta_k\rangle$ and $|\phi\rangle$ such that

$$\begin{aligned}
\mu(\psi) &= \left(\psi_1^2, \ldots, \psi_{d-5}^2, \psi_{d-4}^2, \psi_{d-3}^2, \psi_{d-2}^2, \psi_{d-1}^2, \psi_d^2\right)^T, \\
\mu(\eta_1) &= \left(\psi_1^2, \ldots, \psi_{d-6}^2, \psi_{d-5}^2, \phi_{d-4}'^2, \phi_{d-3}^2, \phi_{d-2}^2, \phi_{d-1}^2, \phi_d^2\right)^T, \\
\mu(\eta_2) &= \left(\psi_1^2, \ldots, \psi_{d-9}^2, \phi_{d-8}'^2, \phi_{d-7}^2, \phi_{d-6}^2, \ldots, \phi_{d-1}^2, \phi_d^2\right)^T, \\
&\vdots \\
\mu(\eta_k) &= \left(\psi_1^2, \ldots, \psi_{m-1}^2, \psi_m^2, \phi_{m+1}'^2, \phi_{m+2}^2, \ldots, \phi_{d-1}^2, \phi_d^2\right)^T, \\
\mu(\phi) &= \left(\phi_1^2, \phi_2^2, \phi_3^2, \ldots, \phi_{d-3}^2, \phi_{d-2}^2, \phi_{d-1}^2, \phi_d^2\right)^T,
\end{aligned} \tag{2.51}$$

where $1 \leq m \leq 4$. Here, for the first step of the entire transformation, we have $\sum_{j=d-4}^{d} \psi_j^2 = \sum_{j=d-3}^{d} \phi_j^2 + \phi_{d-4}'^2$, and the majorization condition implies those $\sum_{j=d-3}^{d} \psi_j^2 \geq \sum_{j=d-3}^{d} \phi_j^2$ and $\psi_{d-4}^2 \leq \phi_{d-4}'^2$. Additionally, if one starts to transform

smaller coefficients, there is a probability of obtaining the relations $\phi_l'^2 > \phi_1^2$ and $\phi_l'^2 > \psi_k^2$ between the coefficients of the coherent states where $k = 1,\ldots,(l-1)$ and $l = (d-4),(d-8),\ldots,(m+1)$, which may violate the majorization condition, and no further deterministic transformation to the final state would be possible. Therefore, while we construct protocol II, we impose that the intermediate states satisfy $\phi_l'^2 \leq \psi_{l-1}^2$ in any step of the entire transformation. In the following, however, we are going to discuss the cases by an explicit example where the intermediate states satisfy $\phi_l'^2 > \phi_1^2$ and $\phi_l'^2 > \psi_{l-1}^2$ or more generally, $\phi_l'^2 \geq \psi_x^2$ $[x = 1,2,\ldots,(l-1)]$, and we explain how one can overcome this problem. Thus, in each step of the entire transformation the majorization condition is preserved, i.e., $\mu(\psi) \prec \mu(\eta_1) \prec \ldots \prec \mu(\eta_k) \prec \mu(\phi)$, under the assumption $\phi_l'^2 \leq \psi_{l-1}^2$.

This protocol consists of deterministic transformations of d'-dimensional (consider $d' = 5$) subspace of the initial state (2.12) in each step (and for the last step $2 \leq d' \leq 5$). We give the explicit solutions for the first two transformations, $|\psi\rangle\langle\psi| \to |\eta_1\rangle\langle\eta_1|$ and $|\eta_1\rangle\langle\eta_1| \to |\eta_2\rangle\langle\eta_2|$, and the last transformation, $|\eta_k\rangle\langle\eta_k| \to |\phi\rangle\langle\phi|$, for illustrative purposes, and other steps of complete transformation can be treated similarly. The initial coherent state $|\psi\rangle$ given in Eq. (2.12) can be written in the form

$$|\psi\rangle = \sum_{j=1}^{d-5} \psi_j |j\rangle + \eta_1 |\chi\rangle, \tag{2.52}$$

where $|\chi\rangle = \sum_{i=d-4}^{d} \chi_i |i\rangle$ is a normalized coherent state ($\sum_{i=d-4}^{d} {\chi_i}^2 = 1$). Here, we have $\eta_1\chi_d = \psi_d$, $\eta_1\chi_{d-1} = \psi_{d-1}$, $\eta_1\chi_{d-2} = \psi_{d-2}$, $\eta_1\chi_{d-3} = \psi_{d-3}$, and $\eta_1\chi_{d-4} = \psi_{d-4}$. The intermediate coherent states are given by

$$|\eta_l\rangle = \sum_{j=1}^{d-4l-1} \psi_j |j\rangle + \phi'_{d-4l} |d-4l\rangle + \sum_{j=d-4l+1}^{d} \phi_j |j\rangle, \tag{2.53}$$

where

$$\phi'_{d-4l} = \sqrt{\psi_{d-4l}^2 + \sum_{j=d-4l+1}^{d} \left(\psi_j^2 - \phi_j^2\right)}, \tag{2.54}$$

with $1 \leq l \leq \lfloor (d-2)/4 \rfloor$, and we assume $\phi'_{d-4l} \leq \psi_{d-4l-1}$. In each step, four more coefficients of the target state $|\phi\rangle$ are obtained, i.e., the smallest $4l$ coefficients of $|\eta_l\rangle$ and $|\phi\rangle$ are equal. For the first transformation $|\psi\rangle\langle\psi| \to |\eta_1\rangle\langle\eta_1|$, the pure coherent state $|\eta_1\rangle$ is of the form

$$|\eta_1\rangle = \sum_{j=1}^{d-5} \psi_j |j\rangle + \eta_1 |\tilde{\chi}\rangle, \tag{2.55}$$

where $|\tilde{\chi}\rangle = \sum_{i=d-4}^{d} \tilde{\chi}_i |i\rangle$ is a normalized coherent state and $\sum_{j=1}^{d-5} {\psi_j}^2 + {\eta_1}^2 = 1$. We have $\eta_1 \tilde{\chi}_d = \phi_d$, $\eta_1 \tilde{\chi}_{d-1} = \phi_{d-1}$, $\eta_1 \tilde{\chi}_{d-2} = \phi_{d-2}$, $\eta_1 \tilde{\chi}_{d-3} = \phi_{d-3}$, and $\eta_1 \tilde{\chi}_{d-4} = \phi'_{d-4}$, and the assumption is $\phi'_{d-4} \leq \psi_{d-5}$ [therefore $\mu(\chi) \prec \mu(\tilde{\chi})$]. The Kraus operators are given by

$$K_{i1} = \sum_{j=1}^{d-5} \sqrt{p_i} |j\rangle \langle j| \oplus \tilde{K}_{i1}, \tag{2.56}$$

where $\sum_{i=1}^{5} K_{i1}^{\dagger} K_{i1} = I_d$ and $\sum_{i=1}^{5} \tilde{K}_{i1}^{\dagger} \tilde{K}_{i1} = I_5$. Here, the Kraus operators $\tilde{K}_{i1}$ are written as

$$\tilde{K}_{i1} = U_{i1} \Big(\sqrt{p_i} \sum_{j=d-4}^{d} \frac{z_{ij}}{\chi_j} |j\rangle \langle j| \Big), \tag{2.57}$$

where $\tilde{K}_{i1} |\chi\rangle = \sqrt{p_i} |\tilde{\chi}\rangle$. Both $|\chi\rangle$ and $|\tilde{\chi}\rangle$ are five-level states, so we use solutions for $d' = 5$. There are eight possible cases for coefficients of coherent states except the lowest and greatest ones, and the set of Kraus operators $\{\tilde{K}_{i1}\}_{i=1,\ldots,5}$ is chosen accordingly. For instance, let us consider the case $\chi_{d-3} \geq \tilde{\chi}_{d-3}$, $\chi_{d-2} \leq \tilde{\chi}_{d-2}$, and

Table 2.5 : All possible permutations for the case $\chi_{d-3} \geq \tilde{\chi}_{d-3}$, $\chi_{d-2} \leq \tilde{\chi}_{d-2}$ and $\chi_{d-1} \leq \tilde{\chi}_{d-1}$ of five-level coherent states $|\chi\rangle$ and $|\tilde{\chi}\rangle$, where $\chi_{d-4} \leq \tilde{\chi}_{d-4}$ and $\chi_d \geq \tilde{\chi}_d$ follow from the majorization condition.

	$\chi_{d-4} \leq \tilde{\chi}_{d-4}$	$\chi_{d-2} \leq \tilde{\chi}_{d-2}$	$\chi_{d-1} \leq \tilde{\chi}_{d-1}$
$\chi_{d-3} \geq \tilde{\chi}_{d-3}$	$\|d-4\rangle \leftrightarrow \|d-3\rangle$	-	-
$\chi_d \geq \tilde{\chi}_d$	$\|d-4\rangle \leftrightarrow \|d\rangle$	$\|d-2\rangle \leftrightarrow \|d\rangle$	$\|d-1\rangle \leftrightarrow \|d\rangle$

$\chi_{d-1} \leq \tilde{\chi}_{d-1}$ where $\chi_{d-4} \leq \tilde{\chi}_{d-4}$ and $\chi_d \geq \tilde{\chi}_d$ follow from the majorization condition. In Table 2.5, there are four permutations, and these constitute the SP together with identity transformation I_5, i.e., a single set is sufficient for the complete solution. Figure 2.8(a) provides the pictorial representation of the permutations given in Table 2.5. The set of permutations is obtained as $\{U_{i1}\}_{i=1,\ldots,5} = \big\{ I_5, |d-4\rangle \leftrightarrow |d\rangle, |d-1\rangle \leftrightarrow |d\rangle, |d-2\rangle \leftrightarrow |d\rangle, |d-4\rangle \leftrightarrow |d-3\rangle \big\}$, where $I_5 = \sum_{u=d-4}^{d} |u\rangle \langle u|$. Then, if we perform

Kraus operators, we obtain

$$
\begin{aligned}
K_{i1}|\psi\rangle &= \left(\sum_{j=1}^{d-5}\sqrt{p_i}|j\rangle\langle j| \oplus U_{i1}\left(\sqrt{p_i}\sum_{j=d-4}^{d}\frac{z_{ij}}{\chi_j}|j\rangle\langle j|\right)\right) \\
&\quad\times\left(\sum_{k=1}^{d-5}\psi_k|k\rangle+\eta_1\sum_{k=d-4}^{d}\chi_k|k\rangle\right) \\
&= \sqrt{p_i}\Bigg(\sum_{j=1}^{d-5}\sum_{k=1}^{d-5}\psi_k|j\rangle\langle j\mid k\rangle+\sum_{j=1}^{d-5}\sum_{k=d-4}^{d}\psi_k|j\rangle\langle j\mid k\rangle \\
&\quad+U_{i1}\sum_{j=d-4}^{d}\sum_{k=1}^{d-5}\frac{z_{ij}}{\chi_j}\psi_k|j\rangle\langle j\mid k\rangle+U_{i1}\eta_1\sum_{j=d-4}^{d}\sum_{k=d-4}^{d}\frac{z_{ij}}{\chi_j}\chi_k|j\rangle\langle j\mid k\rangle\Bigg) \\
&= \sqrt{p_i}\Bigg(\sum_{j=1}^{d-5}\sum_{k=1}^{d-5}\psi_k|j\rangle\langle j\mid k\rangle+U_{i1}\eta_1\sum_{j=d-4}^{d}\sum_{k=d-4}^{d}\frac{z_{ij}}{\chi_j}\chi_k|j\rangle\langle j\mid k\rangle\Bigg) \\
&= \sqrt{p_i}\Bigg(\sum_{j=1}^{d-5}\psi_j|j\rangle+\eta_1U_{i1}\sum_{j=d-4}^{d}z_{ij}|j\rangle\Bigg) \\
&= \sqrt{p_i}|\eta_1\rangle\,.
\end{aligned}
\tag{2.58}
$$

Then, from $K_{i1}|\psi\rangle=\sqrt{p_i}|\eta_1\rangle$ it follows

$$
\sum_{i=1}^{5}K_{i1}|\psi\rangle\langle\psi|K_{i1}^{\dagger}=\sum_{i=1}^{5}p_i|\eta_1\rangle\langle\eta_1|=|\eta_1\rangle\langle\eta_1|\,. \tag{2.59}
$$

The condition $\sum_{i=1}^{5}\tilde{K}_{i1}^{\dagger}\tilde{K}_{i1}=I_5$ implies the following linear equations

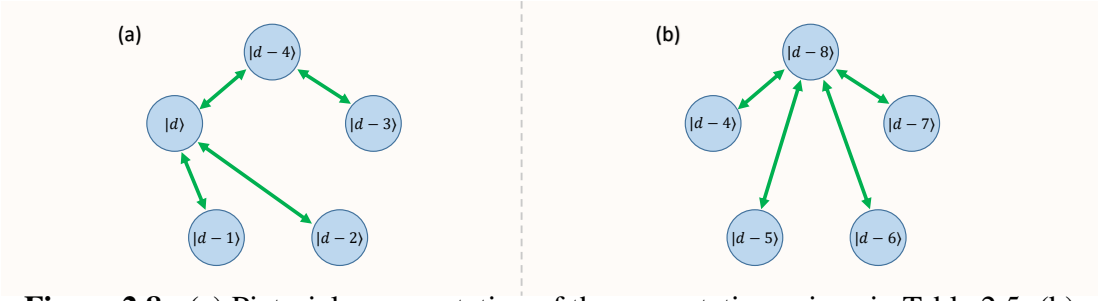

Figure 2.8 : (a) Pictorial representation of the permutations given in Table 2.5. (b) Pictorial representations of the permutations given in Table 2.6.

$$
\sum_{i=1}^{5}p_iz_{ij}^2=\chi_j^2,\quad (j=(d-4),(d-3),\ldots,d), \tag{2.60}
$$

whose solutions for p_i give the probabilities where $(z_{i(d-4)},z_{i(d-3)},\ldots,z_{id})^T=U_{i1}^{\dagger}(\phi'_{d-4},\phi_{d-3},\phi_{d-2},\phi_{d-1},\phi_d)^T$. We denote z_{ij} as the (ij)th element of the matrix z given by

$$
z=\begin{pmatrix}
\phi'_{d-4} & \phi_{d-3} & \phi_{d-2} & \phi_{d-1} & \phi_d \\
\phi_d & \phi_{d-3} & \phi_{d-2} & \phi_{d-1} & \phi'_{d-4} \\
\phi'_{d-4} & \phi_{d-3} & \phi_{d-2} & \phi_d & \phi_{d-1} \\
\phi'_{d-4} & \phi_{d-3} & \phi_d & \phi_{d-1} & \phi_{d-2} \\
\phi_{d-3} & \phi'_{d-4} & \phi_{d-2} & \phi_{d-1} & \phi_d
\end{pmatrix}. \tag{2.61}
$$

The probabilities are found to be

$$p_1 = 1 - \sum_{i=1}^{4} p_i, \quad p_2 = \frac{\tilde{\chi}_{d-4}^2 + \tilde{\chi}_{d-3}^2 - \chi_{d-4}^2 - \chi_{d-3}^2}{\tilde{\chi}_{d-4}^2 - \tilde{\chi}_{d}^2},$$
$$p_3 = \frac{\tilde{\chi}_{d-1}^2 - \chi_{d-1}^2}{\tilde{\chi}_{d-1}^2 - \tilde{\chi}_{d}^2}, \quad p_4 = \frac{\tilde{\chi}_{d-2}^2 - \chi_{d-2}^2}{\tilde{\chi}_{d-2}^2 - \tilde{\chi}_{d}^2}, \quad p_5 = \frac{\chi_{d-3}^2 - \tilde{\chi}_{d-3}^2}{\tilde{\chi}_{d-4}^2 - \tilde{\chi}_{d-3}^2}, \tag{2.62}$$

where $\tilde{\chi}_{d-4}^2 + \tilde{\chi}_{d-3}^2 \geq \chi_{d-4}^2 + \chi_{d-3}^2$ due to the majorization condition. Thus, we obtain the state $|\eta_1\rangle\langle\eta_1|$. Now, in the next step, we obtain the transformation $|\eta_1\rangle\langle\eta_1| \rightarrow |\eta_2\rangle\langle\eta_2|$. The pure coherent state $|\eta_1\rangle$ is written in the form

$$\begin{aligned} |\eta_1\rangle &= \sum_{j=1}^{d-5} \psi_j |j\rangle + \phi'_{d-4} |d-4\rangle + \sum_{j=d-3}^{d} \phi_j |j\rangle \\ &= \sum_{j=1}^{d-9} \psi_j |j\rangle + \eta_2 |\varphi\rangle + \sum_{j=d-3}^{d} \phi_j |j\rangle, \end{aligned} \tag{2.63}$$

where $|\varphi\rangle = \sum_{j=d-8}^{d-4} \varphi_j |j\rangle$ is a normalized coherent state ($\sum_{i=d-8}^{d-4} {\varphi_i}^2 = 1$), $\eta_2\varphi_{d-8} = \psi_{d-8}$, $\eta_2\varphi_{d-7} = \psi_{d-7}$, $\eta_2\varphi_{d-6} = \psi_{d-6}$, $\eta_2\varphi_{d-5} = \psi_{d-5}$, and $\eta_2\varphi_{d-4} = \phi'_{d-4}$. The second intermediate coherent state is given by

$$|\eta_2\rangle = \sum_{j=1}^{d-9} \psi_j |j\rangle + \eta_2 |\tilde{\varphi}\rangle + \sum_{j=d-3}^{d} \phi_j |j\rangle, \tag{2.64}$$

where $|\tilde{\varphi}\rangle = \sum_{j=d-8}^{d-4} \tilde{\varphi}_j |j\rangle$ is a normalized coherent state ($\sum_{i=d-8}^{d-4} \tilde{\varphi}_i^2 = 1$), $\eta_2\tilde{\varphi}_{d-7} = \phi_{d-7}$, $\eta_2\tilde{\varphi}_{d-6} = \phi_{d-6}$, $\eta_2\tilde{\varphi}_{d-5} = \phi_{d-5}$, $\eta_2\tilde{\varphi}_{d-4} = \phi_{d-4}$, and $\eta_2\tilde{\varphi}_{d-8} = \phi'_{d-8}$ (and the assumption is $\phi'_{d-8} \leq \psi_{d-9}$). Thus, we have $\mu(\varphi) \prec \mu(\tilde{\varphi})$. The Kraus operators are given by

$$K_{i2} = \sum_{j=1}^{d-9} \sqrt{p_i} |j\rangle\langle j| \oplus \tilde{K}_{i2} \oplus \sum_{j=d-3}^{d} \sqrt{p_i} |j\rangle\langle j|, \tag{2.65}$$

where $\sum_{i=1}^{5} K_{i2}^\dagger K_{i2} = I_d$ and $\sum_{i=1}^{5} \tilde{K}_{i2}^\dagger \tilde{K}_{i2} = I_5$. Here, the Kraus operators $\{\tilde{K}_{i2}\}_{i=1,\ldots,5}$ are given by

$$\tilde{K}_{i2} = U_{i2}\Big(\sqrt{p_i} \sum_{j=d-8}^{d-4} \frac{z'_{ij}}{\varphi_j} |j\rangle\langle j|\Big), \tag{2.66}$$

where they are chosen according to the eight possible cases of five-level systems to make the transformation $\tilde{K}_{i2}|\varphi\rangle = \sqrt{p_i}|\tilde{\varphi}\rangle$. For instance, let us consider the case $\varphi_{d-7} \geq \tilde{\varphi}_{d-7}$, $\varphi_{d-6} \geq \tilde{\varphi}_{d-6}$ and $\varphi_{d-5} \geq \tilde{\varphi}_{d-5}$. In Table 2.6, there are four permutations, and these, together with identity transformation I_5, constitute the set of permutations (a single set is sufficient). Figure 2.8(b) provides the pictorial

Table 2.6 : All possible permutations for the case $\varphi_{d-7} \geq \tilde{\varphi}_{d-7}$, $\varphi_{d-6} \geq \tilde{\varphi}_{d-6}$, and $\varphi_{d-5} \geq \tilde{\varphi}_{d-5}$ of five-level coherent states $|\varphi\rangle$ and $|\tilde{\varphi}\rangle$ where $\varphi_{d-8} \leq \tilde{\varphi}_{d-8}$ and $\varphi_{d-4} \geq \tilde{\varphi}_{d-4}$ follow from the majorization condition.

	$\varphi_{d-8} \leq \tilde{\varphi}_{d-8}$
$\varphi_{d-7} \geq \tilde{\varphi}_{d-7}$	$\|d-8\rangle \leftrightarrow \|d-7\rangle$
$\varphi_{d-6} \geq \tilde{\varphi}_{d-6}$	$\|d-8\rangle \leftrightarrow \|d-6\rangle$
$\varphi_{d-5} \geq \tilde{\varphi}_{d-5}$	$\|d-8\rangle \leftrightarrow \|d-5\rangle$
$\varphi_{d-4} \geq \tilde{\varphi}_{d-4}$	$\|d-8\rangle \leftrightarrow \|d-4\rangle$

representation of the permutations given in Table 2.6. The SP is obtained as $\{U_{i1}\}_{i=1,\ldots,5} = \{I_5, |d-8\rangle \leftrightarrow |d-7\rangle, |d-8\rangle \leftrightarrow |d-6\rangle, |d-8\rangle \leftrightarrow |d-5\rangle, |d-8\rangle \leftrightarrow |d-4\rangle\}$ where $I_5 = \sum_{u=d-8}^{d-4} |u\rangle\langle u|$. Then, from $K_{i2}|\eta_1\rangle = \sqrt{p_i}|\eta_2\rangle$ it follows

$$\sum_{i=1}^{5} K_{i2}|\eta_1\rangle\langle\eta_1|K_{i2}^{\dagger} = \sum_{i=1}^{5} p_i|\eta_2\rangle\langle\eta_2| = |\eta_2\rangle\langle\eta_2|. \tag{2.67}$$

The condition $\sum_{i=1}^{5} \tilde{K}_{i2}^{\dagger}\tilde{K}_{i2} = I_5$ implies the following linear equations

$$\sum_{i=1}^{5} p_i z_{ij}^{\prime 2} = \varphi_j^2, \quad (j = (d-8), (d-7), \ldots, (d-4)), \tag{2.68}$$

whose solutions for p_i give the probabilities. We denote z'_{ij} as the (ij)th element of the matrix z' given by

$$z' = \begin{pmatrix} \phi'_{d-8} & \phi_{d-7} & \phi_{d-6} & \phi_{d-5} & \phi_{d-4} \\ \phi_{d-7} & \phi'_{d-8} & \phi_{d-6} & \phi_{d-5} & \phi_{d-4} \\ \phi_{d-6} & \phi_{d-7} & \phi'_{d-8} & \phi_{d-5} & \phi_{d-4} \\ \phi_{d-5} & \phi_{d-7} & \phi_{d-6} & \phi'_{d-8} & \phi_{d-4} \\ \phi_{d-4} & \phi_{d-7} & \phi_{d-6} & \phi_{d-5} & \phi'_{d-8} \end{pmatrix}. \tag{2.69}$$

The probabilities are found to be

$$\begin{aligned} p_1 &= 1 - \sum_{i=1}^{4} q_i,\ p_2 = \frac{\varphi_{d-7}^2 - \tilde{\varphi}_{d-7}^2}{\tilde{\varphi}_{d-8}^2 - \tilde{\varphi}_{d-7}^2},\ p_3 = \frac{\varphi_{d-6}^2 - \tilde{\varphi}_{d-6}^2}{\tilde{\varphi}_{d-8}^2 - \tilde{\varphi}_{d-6}^2}, \\ p_4 &= \frac{\varphi_{d-5}^2 - \tilde{\varphi}_{d-5}^2}{\tilde{\varphi}_{d-8}^2 - \tilde{\varphi}_{d-5}^2},\ p_5 = \frac{\varphi_{d-4}^2 - \tilde{\varphi}_{d-4}^2}{\tilde{\varphi}_{d-8}^2 - \tilde{\varphi}_{d-4}^2}, \end{aligned} \tag{2.70}$$

Thus, we obtained the state $|\eta_2\rangle\langle\eta_2|$ where

$$|\eta_2\rangle = \sum_{j=1}^{d-9} \psi_j|j\rangle + \phi'_{d-8}|d-8\rangle + \sum_{j=d-7}^{d} \phi_j|j\rangle. \tag{2.71}$$

For the next to the last transformation of the entire transformation we have the coherent states given by

$$|\eta_{k-1}\rangle \;=\; \sum_{j=1}^{m} \psi_j|j\rangle + \eta_k \sum_{j=m+1}^{m+5} \omega_j|j\rangle + \sum_{j=m+6}^{d} \phi_j|j\rangle, \tag{2.72}$$

and

$$|\eta_k\rangle \;=\; \sum_{j=1}^{m} \psi_j |j\rangle + \eta_k \sum_{j=m+1}^{m+5} \tilde{\omega}_j |j\rangle + \sum_{j=m+6}^{d} \phi_j |j\rangle\,, \tag{2.73}$$

where $1 \leq m \leq 4$, $\eta_k \omega_{m+1} = \psi_{m+1}$, $\eta_k \omega_{m+2} = \psi_{m+2}$, $\eta_k \omega_{m+3} = \psi_{m+3}$, $\eta_k \omega_{m+4} = \psi_{m+4}$, and $\eta_k \omega_{m+5} = \phi'_{m+5}$. Since our aim is the obtain four more coefficients of the target state $|\phi\rangle$, the coefficients are chosen to be $\eta_k \tilde{\omega}_{m+2} = \phi_{m+2}$, $\eta_k \tilde{\omega}_{m+3} = \phi_{m+3}$, $\eta_k \tilde{\omega}_{m+4} = \phi_{m+4}$, $\eta_k \tilde{\omega}_{m+5} = \phi_{m+5}$, and $\eta_k \tilde{\omega}_{m+1} = \phi'_{m+1}$ (and the assumption is $\phi'_{m+1} \leq \psi_m$). Thus, we obtain the intermediate state transformations, i.e., $|\eta_{k-1}\rangle\langle\eta_{k-1}| \rightarrow |\eta_k\rangle\langle\eta_k|$. The last transformation of the entire transformation is $|\eta_k\rangle\langle\eta_k| \rightarrow |\phi\rangle\langle\phi|$. The state $|\eta_k\rangle$ is written as

$$|\eta_k\rangle = \sum_{j=1}^{m} \psi_j |j\rangle + \phi'_{m+1} |m+1\rangle + \sum_{j=m+2}^{d} \phi_j |j\rangle\,, \tag{2.74}$$

where $1 \leq m \leq 4$, and the final state is $|\phi\rangle = \sum_{j=1}^{d} \phi_j |j\rangle$. The last transformation is effectively $(m+1)$-dimensional transformation. Let us consider, for instance, the case $m = 2$. Then we have

$$|\eta_k\rangle = \psi_1 |1\rangle + \psi_2 |2\rangle + \phi'_3 |3\rangle + \sum_{j=4}^{d} \phi_j |j\rangle\,, \tag{2.75}$$

and the Kraus operators are given by

$$\begin{aligned} K_{i(k+1)} \;&=\; \tilde{K}_{i(k+1)} \oplus \sum_{j=4}^{d} \sqrt{p_i}\, |j\rangle\langle j| \\ &=\; U_{i(k+1)} \sqrt{p_i} \Big(\frac{z_{i1}}{\psi_1} |1\rangle\langle 1| + \frac{z_{i2}}{\psi_2} |2\rangle\langle 2| + \frac{z_{i3}}{\phi'_3} |3\rangle\langle 3| \Big) \\ &\quad \oplus \sum_{j=4}^{d} \sqrt{p_i}\, |j\rangle\langle j|\,, \end{aligned} \tag{2.76}$$

where $\sum_{i=1}^{3} K^{\dagger}_{i(k+1)} K_{i(k+1)} = I_d$ and $\sum_{i=1}^{3} \tilde{K}^{\dagger}_{i(k+1)} \tilde{K}_{i(k+1)} = I_3$. Here, we have $\phi'_3 \geq \phi_3$, and one of the inequality relations $\psi_2 \leq (\geq) \phi_2$ can be possible, and the set of Kraus operators $\{\tilde{K}_{i(k+1)}\}$ is chosen accordingly. For instance, for the case $\psi_2 \leq \phi_2$, the set of unitary permutations is obtained as $\{U_{i(k+1)}\}_{i=1,2,3} = \big\{ I_3, |2\rangle \leftrightarrow |3\rangle, |1\rangle \leftrightarrow |3\rangle \big\}$. The condition $\sum_{i=1}^{3} \tilde{K}^{\dagger}_{i(k+1)} \tilde{K}_{i(k+1)} = I_3$ implies the following linear equations

$$\sum_{i=1}^{3} p_i z_{ij}^2 = \psi_j^2, \quad (j = 1,2), \quad \sum_{i=1}^{3} p_i z_{i3}^2 = \phi_3'^2, \tag{2.77}$$

whose solutions for p_i give the probabilities. We denote z_{ij} as the (ij)th element of the matrix z given by

$$z = \begin{pmatrix} \phi_1 & \phi_2 & \phi_3 \\ \phi_1 & \phi_3 & \phi_2 \\ \phi_3 & \phi_2 & \phi_1 \end{pmatrix}. \tag{2.78}$$

The probabilities are found to be $p_1 = 1 - p_2 - p_3$, $p_2 = (\phi_2^2 - \psi_2^2)/(\phi_2^2 - \phi_3^2)$ and $p_3 = (\phi_1^2 - \psi_1^2)/(\phi_1^2 - \phi_3^2)$. Then, from $K_{i(k+1)} |\eta_k\rangle = \sqrt{p_i} |\phi\rangle$ it follows

$$\sum_{i=1}^{3} K_{i(k+1)} |\eta_k\rangle \langle\eta_k| K_{i(k+1)}^{\dagger} = \sum_{i=1}^{3} p_i |\phi\rangle \langle\phi| = |\phi\rangle \langle\phi|, \tag{2.79}$$

where $k+1 = \lfloor (d+2)/4 \rfloor$. Thus, we obtain the d-level pure coherent states transformations by cascading a sequence of coherence transformations, each corresponding to a single incoherent operation $\Phi[\cdot]$. The sequence of pure coherent states transformations can be summarized as

$$\begin{aligned} |\psi\rangle \langle\psi| &\to \sum_{f=1}^{m+1} K_{f(k+1)} \left(\sum_{r=1}^{5} K_{rk} \left(\cdots \sum_{i=1}^{5} K_{i1} |\psi\rangle \langle\psi| K_{i1}^{\dagger} \cdots \right) K_{rk}^{\dagger} \right) K_{f(k+1)}^{\dagger} \\ &= \Phi_{(k+1)} \Bigg[\underbrace{\Phi_{(k)} \Bigg[\cdots \Phi_{(3)} \Bigg[\Phi_{(2)} \Big[\underbrace{\Phi_{(1)}(|\psi\rangle \langle\psi|)}_{=|\eta_1\rangle\langle\eta_1|} \Big] \Bigg] \cdots \Bigg]}_{=|\eta_2\rangle\langle\eta_2|} \Bigg] \\ &\qquad \underbrace{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}_{=|\eta_k\rangle\langle\eta_k|} \\ &= |\phi\rangle \langle\phi|, \end{aligned} \tag{2.80}$$

where $1 \leq m \leq 4$ and $k+1 = \lfloor (d+2)/4 \rfloor$. Consequently, we obtain the entire transformation $|\psi\rangle \langle\psi| \to |\phi\rangle \langle\phi|$ as $|\psi\rangle \langle\psi| \to |\eta_1\rangle \langle\eta_1| \to \ldots \to |\eta_k\rangle \langle\eta_k| \to |\phi\rangle \langle\phi|$ in $\lfloor (d+2)/4 \rfloor$ steps via the set of incoherent operations $\{\Phi_{(i)}\}_{i=1,2,\ldots,\lfloor (d+2)/4 \rfloor}$. It should be noted that the number of steps can be further reduced by using subspace solutions of d'-level where $d' > 5$ (actually, number of steps is $\lfloor (d+d'-3)/(d'-1) \rfloor$).

2.4.2.1 Explicit example for protocol II with discussion

In protocol II we consider deterministic transformations of coherent states by transforming the four smallest nonequal coefficients step-by-step. This procedure requires that the condition $\phi_l'^2 \leq \psi_{l-1}^2$ is satisfied in each step of the entire transformation to preserve the majorization condition for intermediate states. However, the condition $\phi_l'^2 \leq \psi_{l-1}^2$ is not satisfied for some sources and the target states. In all these cases, it is always possible to find intermediate states where lower dimensional transformations can be used. For illustrative purposes, we start with a six-dimensional initial pure coherent state $|\psi\rangle$ and six-dimensional final pure coherent state $|\phi\rangle$ such that

$$\mu(\psi) = \frac{1}{53}(11, 11, 8, 8, 8, 7)^T, \tag{2.81}$$

$$\mu(\phi) = \frac{1}{53}(12,12,10,9,6,4)^T, \tag{2.82}$$

respectively. It is obvious that $\mu(\psi) \prec \mu(\phi)$. For the first step of the entire transformation $|\psi\rangle\langle\psi| \to |\phi\rangle\langle\phi|$, if one starts to transform smaller coefficients, using five-level solutions, then for an intermediate coherent state $|\eta_1'\rangle$ we have

$$\mu(\eta_1') = \frac{1}{53}(11,13,10,9,6,4)^T, \tag{2.83}$$

where the last four coefficients of coherent states $|\phi\rangle$ and $|\eta_1'\rangle$ are the same, and also first coefficients of coherent states $|\psi\rangle$ and $|\eta_1'\rangle$ are the same. Since the majorization condition, $\mu(\eta_1') \prec \mu(\phi)$, is not satisfied, the deterministic transformation $|\eta_1'\rangle\langle\eta_1'| \to |\phi\rangle\langle\phi|$ is not possible via incoherent operations. This is why we stress that our protocol consists of the intermediate coherent states for which $\phi_l'^2 \leq \psi_{l-1}^2$ in any step of the entire transformation. On the other hand, for the first step of the complete transformation, if one starts to transform greater coefficients then for an intermediate coherent state $|\eta_1''\rangle$ we have

$$\mu(\eta_1'') = \frac{1}{53}(12,12,10,9,3,7)^T, \tag{2.84}$$

where the first four coefficients of coherent states $|\phi\rangle$ and $|\eta_1''\rangle$ are the same, and also the last coefficients of coherent states $|\psi\rangle$ and $|\eta_1''\rangle$ are the same. Since the majorization condition, $\mu(\eta_1'') \prec \mu(\phi)$, is not satisfied, the deterministic transformation $|\eta_1''\rangle\langle\eta_1''| \to |\phi\rangle\langle\phi|$ is not possible via incoherent operations. At first glance the transformation $|\psi\rangle\langle\psi| \to |\phi\rangle\langle\phi|$ looks unachievable by protocol II. However, it is always possible to choose intermediate states−intermediate states are not unique−which do not violate the majorization condition. For the first step of the complete transformation we consider the pure coherent state $|\eta_1\rangle$ as an intermediate state for which we have

$$\mu(\eta_1) = \frac{1}{53}(12,12,10,8,7,4)^T, \tag{2.85}$$

where four coefficients (first three and the last) of coherent states $|\phi\rangle$ and $|\eta_1\rangle$ are the same, and also the fourth coefficients of coherent states $|\psi\rangle$ and $|\eta_1\rangle$ are the same. It is obviously seen that the majorization condition is satisfied, $\mu(\psi) \prec \mu(\eta_1) \prec \mu(\phi)$. Then, under the Kraus operators given by

$$K_{11} = U_{11}\sqrt{p_1}\ \mathrm{diag}\Big(\sqrt{\frac{12}{11}}, \sqrt{\frac{12}{11}}, \sqrt{\frac{5}{4}}, 1, \sqrt{\frac{7}{8}}, \sqrt{\frac{4}{7}}\Big), \tag{2.86}$$

$$K_{21} = U_{21}\sqrt{p_2}\ \mathrm{diag}\Big(\sqrt{\frac{4}{11}}, \sqrt{\frac{12}{11}}, \sqrt{\frac{5}{4}}, 1, \sqrt{\frac{7}{8}}, \sqrt{\frac{12}{7}}\Big), \tag{2.87}$$

$$K_{31} = U_{31}\sqrt{p_3}\ \mathrm{diag}\Big(\sqrt{\frac{12}{11}}, \sqrt{\frac{4}{11}}, \sqrt{\frac{5}{4}}, 1, \sqrt{\frac{7}{8}}, \sqrt{\frac{12}{7}}\Big), \tag{2.88}$$

$$K_{41} = U_{41}\sqrt{p_4}\ \mathrm{diag}\Big(\sqrt{\frac{12}{11}}, \sqrt{\frac{12}{11}}, \sqrt{\frac{7}{8}}, 1, \sqrt{\frac{5}{4}}, \sqrt{\frac{4}{7}}\Big), \tag{2.89}$$

$$K_{51} = U_{51}\sqrt{p_5}\ \mathrm{diag}\Big(\sqrt{\frac{12}{11}}, \sqrt{\frac{12}{11}}, \sqrt{\frac{1}{2}}, 1, \sqrt{\frac{7}{8}}, \sqrt{\frac{10}{7}}\Big), \tag{2.90}$$

we obtain the transformation $|\psi\rangle\langle\psi| \to |\eta_1\rangle\langle\eta_1|$, i.e., $\sum_{i=1}^{5} K_{i1}|\psi\rangle\langle\psi|K_{i1}^{\dagger} = |\eta_1\rangle\langle\eta_1|$, where $p_1 = 1/4,\ p_2 = p_3 = 1/8,\ p_4 = 1/3,\ p_5 = 1/6,\ U_{11} = I_6,\ U_{21} = |1\rangle \leftrightarrow |6\rangle$, $U_{31} = |2\rangle \leftrightarrow |6\rangle$, $U_{41} = |3\rangle \leftrightarrow |5\rangle$, $U_{51} = |3\rangle \leftrightarrow |6\rangle$. We now obtain the transformation $|\eta_1\rangle\langle\eta_1| \to |\phi\rangle\langle\phi|$, where $|\eta_1\rangle$ and $|\phi\rangle$ can be written in the form

$$|\eta_1\rangle = \sqrt{\frac{38}{53}}|\chi\rangle + \sqrt{\frac{15}{53}}\Big(\sqrt{\frac{8}{15}}|4\rangle + \sqrt{\frac{7}{15}}|5\rangle\Big), \tag{2.91}$$

$$|\phi\rangle = \sqrt{\frac{38}{53}}|\chi\rangle + \sqrt{\frac{15}{53}}\Big(\sqrt{\frac{9}{15}}|4\rangle + \sqrt{\frac{6}{15}}|5\rangle\Big), \tag{2.92}$$

respectively. Here, $|\chi\rangle = \sqrt{\frac{6}{19}}|1\rangle + \sqrt{\frac{6}{19}}|2\rangle + \sqrt{\frac{5}{19}}|3\rangle + \sqrt{\frac{2}{19}}|6\rangle$ is a normalized coherent state. Then, under the Kraus operators given by

$$K_{12} = U_{12}\sqrt{p_1}\ \mathrm{diag}\Big(1, 1, 1, \frac{3}{2\sqrt{2}}, \sqrt{\frac{6}{7}}, 1\Big), \tag{2.93}$$

$$K_{22} = U_{22}\sqrt{p_2}\ \mathrm{diag}\Big(1, 1, 1, \frac{\sqrt{3}}{2}, \frac{3}{\sqrt{7}}, 1\Big), \tag{2.94}$$

we obtain the transformation $|\eta_1\rangle\langle\eta_1| \to |\phi\rangle\langle\phi|$, i.e., $\sum_{i=1}^{2} K_{i2}|\eta_1\rangle\langle\eta_1|K_{i2}^{\dagger} = |\phi\rangle\langle\phi|$, where $p_1 = 2/3$, $p_2 = 1/3$, $U_{12} = I_6$, $U_{22} = |4\rangle \leftrightarrow |5\rangle$, and $\sum_{i=1}^{2} K_{i2}^{\dagger}K_{i2} = I_6$. Thus, we obtain the entire transformation $|\psi\rangle\langle\psi| \to |\phi\rangle\langle\phi|$ as $|\psi\rangle\langle\psi| \to |\eta_1\rangle\langle\eta_1| \to |\phi\rangle\langle\phi|$, where $\mu(\psi) \prec \mu(\eta_1) \prec \mu(\phi)$. As an example of the nonuniqueness of the intermediate states we may consider another coherent state for which

$$\mu(\tilde{\eta}_1) = \frac{1}{53}(12, 12, 7, 9, 6, 7)^T. \tag{2.95}$$

satisfying $\mu(\psi) \prec \mu(\tilde{\eta}_1) \prec \mu(\phi)$. Hence, it is always possible to find intermediate states satisfying the majorization condition $\mu(\psi) \prec \mu(\eta_1) \prec \ldots \prec \mu(\eta_k) \prec \mu(\phi)$ and make the transformations $|\psi\rangle\langle\psi| \to |\eta_1\rangle\langle\eta_1| \to \ldots \to |\eta_k\rangle\langle\eta_k| \to |\phi\rangle\langle\phi|$ using the Kraus operators which can be obtained using protocol I.

2.5 Conclusion

In this chapter, we presented two explicit protocols for the deterministic transformations of coherent states under incoherent operations. We first presented a permutation-based protocol (protocol I) which reduces the problem of d-level deterministic coherence transformations to solving d linear equations for d unknowns (probabilities). We gave two illustrative examples to make the protocol clearer. One of the most significant points of protocol I is that it provides a single map, that is, $\Phi_s[\rho_\psi] = \rho_\phi$. Using protocol I, one can easily find the sets of permutations, probabilities and Kraus operators for d-level systems. We then presented generalized solutions for some source and target states using protocol I. We also presented an alternative protocol (protocol II) where we use lower dimensional ($d' < d$) solutions of the permutation-based protocol to obtain the complete transformation as a sequence of coherent-state transformations. In each step of the complete transformation, we obtained at least $d' - 1$ coefficients of the final coherent state $|\phi\rangle$ using d'-dimensional subspace solutions. Thus, we obtained the d-level pure coherent state transformations by cascading a sequence of coherence transformations with the set of incoherent operations $\{\Phi_{(i)}\}_{i=1,2,\ldots,\lfloor (d+d'-3)/(d'-1)\rfloor}$. We discussed an example for protocol II.

3. BIPARTITE ENTANGLEMENT TRANSFORMATIONS

In this chapter, we are interested in the deterministic transformations of pure bipartite entangled states. The results, obtained in Chapter 2, can easily be adapted to d-level bipartite entangled pure state transformations. The material covered in this chapter is published in the papers [8, 37][1].

3.1 Introduction

Quantum entanglement is used as a resource in quantum information processes such as teleportation [38], dense coding [39] and quantum-key distribution [40]. The success probability in fulfilling quantum information tasks depends on the properties of the entangled state which is used as a resource [41,42]. This requires a deep understanding of the transformation properties of entangled states under local operations and classical communication (LOCC). One line of research is the interconvertability of multi-qubit states, and the optimal [43–49] and deterministic [50–52] transformations of some classes of multi-qubit states have been widely studied. Another line of research is the interconversion of bipartite entangled states [53, 54]. In particular, Bennett et al. [55] showed that the entanglement in any pure state of a bipartite system can be concentrated by LOCC into maximally entangled states, and conversely, an arbitrary partly entangled state of a bipartite system can be prepared by LOCC using maximally entangled states as the only source of entanglement. Vidal [29] obtained the maximum transformation probability of $d \otimes d$ pure states in terms of the Schmidt coefficients and explicitly constructed a local protocol. Lo and Popescu [56] showed that any general transformation between bipartite pure states using LOCC can be performed with one-way classical communications only, and one way communication is more powerful than those without communications. Chau et al. [57] presented necessary and sufficient conditions for the probabilistic transformations of quantum states using local

[1] Gökhan Torun and Ali Yildiz, Deterministic transformations of coherent states under incoherent operations, *Phys. Rev. A*, 97, 052331, (2018), Gokhan Torun and Ali Yildiz, Deterministic transformations of bipartite pure states, *Physics Letters A*, 379(3), 113 -118, (2015).

operations (without classical communication) only. Jonathan and Plenio [58] used a minimal set of entanglement monotones, and presented an optimal local strategy for entanglement concentration.

Deterministic transformation of a state by LOCC is of fundamental importance in quantum information theory, because if a state $|\Psi\rangle$ can be transformed into another state $|\Phi\rangle$ with unit probability by LOCC, then the information tasks that can be performed by using the state $|\Psi\rangle$ can also be performed by using the state $|\Phi\rangle$. He and Bergou [59] showed that classical communication is necessary in realizing the deterministic transformations of a single bipartite entangled state. Nielsen [26] used the algebraic theory of majorization and obtained the necessary and sufficient condition for the deterministic transformations of bipartite pure states. The same condition was derived by using the method of areas [60]. Roa et al. [61] proposed a method for the probabilistic transformation of bipartite pure states based on local overlap modification and they obtained the deterministic transformation as a special case for states satisfying the majorization condition. Majorization condition states that a state in Schmidt form

$$|\Psi\rangle = \sum_{j=1}^{d} \psi_j |j\rangle |j\rangle, \quad \big(\psi_j \geq \psi_{j+1} > 0\big), \tag{3.1}$$

can be transformed into another state

$$|\Phi\rangle = \sum_{j=1}^{d} \phi_j |j\rangle |j\rangle, \quad \big(\phi_j \geq \phi_{j+1} \geq 0\big), \tag{3.2}$$

by LOCC with unit probability if and only if $\lambda(\Psi)$ is majorized by $\lambda(\Phi)$, written $\lambda(\Psi) \prec \lambda(\Phi)$, where $\lambda(\Psi)$ and $\lambda(\Phi)$ denote the vectors of decreasingly ordered eigenvalues of the reduced density matrices ρ_A^{Ψ} and ρ_A^{Φ}, respectively. Majorization condition implies that the transformation $|\Psi\rangle \rightarrow |\Phi\rangle$ can be obtained with unit probability if and only if the inequality

$$\sum_{j=k}^{d} \phi_j^2 \leq \sum_{j=k}^{d} \psi_j^2 \tag{3.3}$$

is satisfied for any k $(1 \leq k \leq d)$, with equality holding when $k = 1$.

However, deterministic transformations of states by a single measurement get more complicated as the dimension increases, since the construction of the doubly stochastic matrices[2], measurement operators and unitary operators gets more complicated

[2]A doubly stochastic matrix $A = (a_{ij})$ has the following property: $\sum_i a_{ij} = \sum_j a_{ij} = 1$ for all i and j where $a_{ij} \geq 0$.

[36]. Hence simple and explicit protocols for the deterministic transformations of quantum states are of paramount importance. If the explicit solutions for the deterministic transformations of states in higher-dimensional space are known, then deterministic transformations of states in lower-dimensional space can be obtained as the special cases of higher-dimensional solutions. However, the protocols for explicit transformations of states in higher-dimensional space using the lower-dimensional solutions still need to be developed.

The content of the present chapter is organised as follows. We begin with a succinct summary of resource theory of entanglement within the scope of the bipartite state transformations: Entanglement (Section 3.2), Schmidt Decomposition (Section 3.3), Positive Operator Valued Measure (Section 3.4). Then, we show, with various examples, that the results obtained in the previous chapter can be easily adapted to entanglement (Section 3.5). Finally, we conclude (Section 3.6).

3.2 Entanglement

An isolated quantum system which is in a pure state is described by its state vector $|\psi\rangle$. This is a normalised vector in a Hilbert space $\mathcal{H}_d$ which is associated with the quantum system. To a pure state, which we have up to now described by the normalised state vector $|\psi\rangle$ we can also unambiguously assign the operator

$$\rho = |\psi\rangle\langle\psi|. \tag{3.4}$$

It is called the density operator of a pure state or often also the density matrix. The following properties of the density operator for pure states can be read off directly:

- ρ is positive, $\langle\varphi|\rho|\varphi\rangle \geq 0, \forall|\varphi\rangle$,
- $tr(\rho) = \langle\varphi|\varphi\rangle = 1$,

which also generalize the mixed states.

A bipartite pure state $|\psi\rangle_{AB}$ is entangled if it cannot be written such that

$$|\psi\rangle_{AB} = |\alpha\rangle_A \otimes |\beta\rangle_B, \tag{3.5}$$

for any choices of states $|\alpha\rangle_A$ and $|\beta\rangle_B$. Otherwise, in the case when a bipartite pure state $|\psi\rangle_{AB}$ can be written in the form (3.5), it is called separable. In case of mixed

states, a density matrix ρ_{AB} defined on a tensor product Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$ is entangled if it cannot be written in the form

$$\rho = \sum_k p_i \rho_A^i \otimes \rho_B^i, \quad \Big(p_i \geq 0, \quad \sum_i p_i = 1 \Big). \tag{3.6}$$

3.3 Schmidt Decomposition

Let $|\varphi_i\rangle_A$ $(i = 1, \ldots, d_A)$ and $|\varphi_j\rangle_B$ $(j = 1, \ldots, d_B)$ be two arbitrary local bases of system $\mathcal{S}_A$ and $\mathcal{S}_B$, respectively. Any pure state $|\psi\rangle_{AB}$ in the product Hilbert space $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ with $\dim\mathcal{H}_A = d_A$ and $\dim\mathcal{H}_B = d_B$ can be expressed such that

$$|\psi\rangle_{AB} = \sum_{i,j} d_{ij} \, |\varphi_i\rangle_A \otimes |\varphi_j\rangle_B . \tag{3.7}$$

One can always find local orthonormal vectors $\{|u_i\rangle_A\}_{i=1,2,\ldots,d}$ and $\{|w_i\rangle_B\}_{i=1,2,\ldots,d}$ on the first and second subsystem, respectively. Then, the state $|\psi\rangle_{AB}$ can be represented such that

$$|\psi\rangle_{AB} = \sum_{i=1}^{d} \sqrt{\lambda_i} \, |u_i\rangle_A \, |w_i\rangle_B , \tag{3.8}$$

where $d = \min\{d_A, d_B\}$. Here, $\{\lambda_i \geq 0\}_{i=1,2,\ldots,d}$ are known as Schmidt coefficients and satisfy $\sum_{i=1}^{d} \lambda_i = 1$. The representation given by Eq. (3.8) known as *Schmidt decomposition*. All information on the entanglement of a bipartite state is encoded in the Schmidt coefficients. In other words, if there is only one non-vanishing Schmidt coefficient, then the state $|\psi\rangle_{AB}$ is separable. The state $|\psi\rangle_{AB}$ is entangled when at least two Schmidt coefficients are different from zero [62].

3.4 Positive Operator Valued Measure

The measurement operators which leave the Schmidt form invariant can be taken such that

$$M_i = \sum_{j=1}^{d} \mu_{ij} \, |j\rangle \langle j| , \quad E_i \equiv M_i^\dagger M_i, \sum_{i=1}^{d} E_i = I_d. \tag{3.9}$$

The operators E_i are known as the POVM$-$positive operator valued measure$-$elements associated with the measurement. The complete set of $\{E_i\}$ are known as a POVM[3](for details [36]).

[3]We define measurement operators (and therefore POVM elements) in diagonal form. However, in general, it is not necessary to be diagonal, i.e., the most general measurement operators are given by $M^{(i)} = \sum_{j,k=1} \mu_{jk}^{(i)} \, |j\rangle \langle k|$.

3.5 Deterministic Transformations of Bipartite Pure States

The initial state given in Eq. (3.1) can be transformed to the final state given in Eq. (3.2) by performing a d-outcome measurement on one of the particles with the operators defined by Eq. (2.18), where the parameters c_{sij} are the same as found by protocol I (see Sec. 2.4.1). The POVM measurement yields one of the states

$$M_s^i |\Psi\rangle \to |\Phi_s^i\rangle = \sum_{j=1}^{d} c_{sij} |j\rangle |j\rangle \quad (i = 1, \ldots, d), \tag{3.10}$$

with probabilities $p_s^i = \langle\Psi| M_s^{i\dagger} M_s^i |\Psi\rangle$. Depending on the result of the measurement one of the the unitary transformations U_s^i is applied on both particles to obtain the target state, i.e., $(U_s^i \otimes U_s^i) |\Phi_s^i\rangle = |\Phi\rangle$.

3.5.1 Examples for the entanglement transformations

We now give four illustrative examples both to demonstrate the adaptability of our results to the deterministic transformations of entanglement and to enhance the intelligibility of our method. The first three examples will be solved with protocol I (see Subsection 2.4.1) and the last one will be solved with protocol II (see Subsection 2.4.2).

We first consider the complete solution of three-level systems. There are two possible cases for $d = 3$: Either $\psi_2 \leq \phi_2$ or $\psi_2 \geq \phi_2$ may exist where $\psi_1 \leq \phi_1$ and $\psi_3 \geq \phi_3$ follow from the majorization condition.

Example 1: The case $\psi_1 \leq \phi_1,\ \psi_2 \leq \phi_2,\ \psi_3 \geq \phi_3$. We first construct Table 3.1. The

Table 3.1 : All possible permutations for the case $\psi_1 \leq \phi_1$, $\psi_2 \leq \phi_2$, and $\psi_3 \geq \phi_3$ of $d = 3$.

	$\psi_1 \leq \phi_1$	$\psi_2 \leq \phi_2$
$\psi_3 \geq \phi_3$	$\|1\rangle \leftrightarrow \|3\rangle$	$\|2\rangle \leftrightarrow \|3\rangle$

permutations in Table 3.1 constitute the SP together with identity transformation I_3 (while there are $d-1$ permutations in Table 3.1, a single SP is sufficient). Figure 3.1(a) provides the pictorial representation of permutations given in Table 3.1. The SP

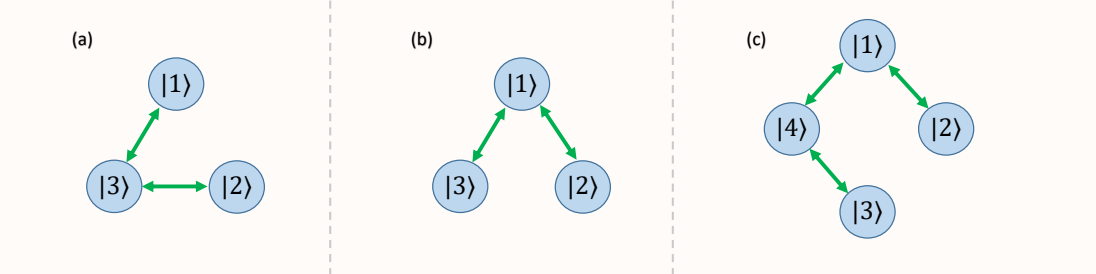

Figure 3.1 : (a) Pictorial representation of the permutations given in Table 3.1. (b) Pictorial representation of the permutations given in Table 3.2. (c) Pictorial representation of the permutations given in Table 3.3

is then obtained such that

$$U = \left\{U^1, U^2, U^3\right\} = \left\{I_3, |2\rangle \leftrightarrow |3\rangle, |1\rangle \leftrightarrow |3\rangle\right\}. \tag{3.11}$$

The measurement stage begins after completing the first and most important step−obtaining the correct set of permutations U^i. A generalized three-outcome measurement with the measurement operators,

$$M^i = \sqrt{p^i} \sum_{j=1}^{3} \frac{c_{ij}}{\psi_j} |j\rangle \langle j|, \quad (\sum_{i=1}^{3} M^{i\dagger} M^i = I_3), \tag{3.12}$$

is performed on one of the parties of the initial state $|\Psi\rangle = \sum_{j=1}^{3} \psi_j |j\rangle |j\rangle$. Here, c_{ij} is the (ij)th element of the matrix c where $(c_{i1}, c_{i2}, c_{i3})^T = (U^i)^\dagger (\phi_1, \phi_2, \phi_3)^T$. The matrix c is given by

$$c = \begin{pmatrix} \phi_1 & \phi_2 & \phi_3 \\ \phi_1 & \phi_3 & \phi_2 \\ \phi_3 & \phi_2 & \phi_1 \end{pmatrix}. \tag{3.13}$$

The state after the measurement turns out to be one of the states

$$M^i |\Psi\rangle \rightarrow |\Phi^i\rangle = \sum_{j=1}^{3} c_{ij} |j\rangle |j\rangle \quad (i = 1, 2, 3), \tag{3.14}$$

with probabilities $p^i = \langle\Psi| M^{i\dagger} M^i |\Psi\rangle$. The state $|\Phi^1\rangle$ is already the target state $|\Phi\rangle = \sum_{j=1}^{3} \phi_j |j\rangle |j\rangle$, and the states $|\Phi^2\rangle$ and $|\Phi^3\rangle$ can be transformed to the target state by local unitary transformations (permutations) $|2\rangle \leftrightarrow |3\rangle$ and $|1\rangle \leftrightarrow |3\rangle$, respectively. In other words, the POVM measurement yields one of the states

$$\begin{aligned} |\Phi^1\rangle &= \sum_{j=1}^{3} \phi_j |j\rangle |j\rangle, \\ |\Phi^2\rangle &= \phi_1 |1\rangle |1\rangle + \phi_3 |2\rangle |2\rangle + \phi_2 |3\rangle |3\rangle, \\ |\Phi^3\rangle &= \phi_3 |1\rangle |1\rangle + \phi_2 |2\rangle |2\rangle + \phi_1 |3\rangle |3\rangle, \end{aligned} \tag{3.15}$$

with probabilities p^i where $(U^2 \otimes U^2)|\Phi^2\rangle = |\Phi\rangle$, $(U^3 \otimes U^3)|\Phi^3\rangle = |\Phi\rangle$ and the permutations U^i are given in Eq. (3.11). Also, the condition $\sum_{i=1}^{3} M^{i\dagger} M^i = I_3$ implies that the following three linear equations

$$\sum_{i=1}^{3} p^i c_{ij}^2 = \psi_j^2, \quad (j = 1,2,3), \tag{3.16}$$

should be satisfied, and solutions of these linear equations give the probabilities. Thus, probabilities are found to be

$$p^1 = 1 - p^2 - p^3, \quad p^2 = \frac{\phi_2^2 - \psi_2^2}{\phi_2^2 - \phi_3^2}, \quad p^3 = \frac{\phi_1^2 - \psi_1^2}{\phi_1^2 - \phi_3^2}. \tag{3.17}$$

Example 2: The case $\psi_1 \leq \phi_1,\ \psi_2 \geq \phi_2,\ \psi_3 \geq \phi_3$. The permutations in Table 3.2 constitute the SP together with identity transformation I_3 (a single SP is sufficient). Figure 3.1(b) provides the pictorial representation of permutations given in Table 3.2.

Table 3.2 : All possible permutations for the case $\psi_1 \leq \phi_1$, $\psi_2 \geq \phi_2$, and $\psi_3 \geq \phi_3$ of $d = 3$.

	$\psi_1 \leq \phi_1$
$\psi_2 \geq \phi_2$	$\|1\rangle \leftrightarrow \|2\rangle$
$\psi_3 \geq \phi_3$	$\|1\rangle \leftrightarrow \|3\rangle$

Then, the SP is obtained such that

$$U = \left\{U^1, U^2, U^3\right\} = \left\{I_3, |1\rangle \leftrightarrow |2\rangle, |1\rangle \leftrightarrow |3\rangle\right\}. \tag{3.18}$$

A generalized three-outcome measurement with the measurement operators defined by Eq. (3.12) is performed on one of the particles of the initial state $|\Psi\rangle$. In Eq. (3.12), c_{ij} is the (ij)th element of the matrix c where $(c_{i1}, c_{i2}, c_{i3})^T = (U^i)^\dagger (\phi_1, \phi_2, \phi_3)^T$. The matrix c is given by

$$c = \begin{pmatrix} \phi_1 & \phi_2 & \phi_3 \\ \phi_2 & \phi_1 & \phi_3 \\ \phi_3 & \phi_2 & \phi_1 \end{pmatrix}. \tag{3.19}$$

The state after the measurement turns out to be one of the states, given in Eq. (3.14), with probabilities $p^i = \langle\Psi| M^{i\dagger} M^i |\Psi\rangle$. The state $|\Phi^1\rangle$ is already the target state $|\Phi\rangle$, and the states $|\Phi^2\rangle$ and $|\Phi^3\rangle$ can be transformed to the target state by local unitary transformations (permutations) $|1\rangle \leftrightarrow |2\rangle$ and $|1\rangle \leftrightarrow |3\rangle$, respectively. In other words,

the POVM measurement yields one of the states

$$\begin{aligned}|\Phi^1\rangle =& \sum_{j=1}^{3} \phi_j |j\rangle |j\rangle, \\ |\Phi^2\rangle =& \phi_2 |1\rangle |1\rangle + \phi_1 |2\rangle |2\rangle + \phi_3 |3\rangle |3\rangle, \\ |\Phi^3\rangle =& \phi_3 |1\rangle |1\rangle + \phi_2 |2\rangle |2\rangle + \phi_1 |3\rangle |3\rangle,\end{aligned} \tag{3.20}$$

with probabilities p^i where $(U^2 \otimes U^2)|\Phi^2\rangle = |\Phi\rangle$, $(U^3 \otimes U^3)|\Phi^3\rangle = |\Phi\rangle$ and the permutations U^i are given in Eq. (3.18). The condition $\sum_{i=1}^{3} M^{i\dagger} M^i = I_3$ implies that the following linear equations

$$\sum_{i=1}^{3} p^i c_{ij}^2 = \psi_j^2, \quad (j = 1,2,3), \tag{3.21}$$

should be satisfied, and solutions of these linear equations give the probabilities. Thus, probabilities are found to be

$$p^1 = 1 - p^2 - p^3, \quad p^2 = \frac{\psi_2^2 - \phi_2^2}{\phi_1^2 - \phi_2^2}, \quad p^3 = \frac{\psi_3^2 - \phi_3^2}{\phi_1^2 - \phi_3^2}. \tag{3.22}$$

We eventually obtained the complete solution for $d = 3$ by solving the two possible cases.

Example 3. As a third example, we consider the case $\psi_2 \geq \phi_2$ and $\psi_3 \leq \phi_3$ for $d = 4$ where $\psi_1 \leq \phi_1$ and $\psi_4 \geq \phi_4$ follow from the majorization condition. To begin with, we construct Table 3.3. The permutations in Table 3.3 constitute the SP together with

Table 3.3 : All possible permutations for the case $\psi_1 \leq \phi_1$, $\psi_2 \geq \phi_2$, $\psi_3 \leq \phi_3$, and $\psi_4 \geq \phi_4$ of $d = 4$.

	$\psi_1 \leq \phi_1$	$\psi_3 \leq \phi_3$
$\psi_2 \geq \phi_2$	$\|1\rangle \leftrightarrow \|2\rangle$	-
$\psi_4 \geq \phi_4$	$\|1\rangle \leftrightarrow \|4\rangle$	$\|3\rangle \leftrightarrow \|4\rangle$

identity transformation I_4 (while there are $d-1$ permutations in Table 3.3, a single SP is sufficient). Figure 3.1(c) provides the pictorial representation of permutations given in Table 3.3. The SP is then obtained such that

$$\begin{aligned}U =& \left\{U^1, U^2, U^3, U^4\right\} \\ =& \left\{I_4, |1\rangle \leftrightarrow |2\rangle, |3\rangle \leftrightarrow |4\rangle, |1\rangle \leftrightarrow |4\rangle\right\}.\end{aligned} \tag{3.23}$$

A generalized four-outcome measurement with the measurement operators,

$$M^i = \sqrt{p^i} \sum_{j=1}^{4} \frac{c_{ij}}{\psi_j} |j\rangle \langle j|, \quad (\sum_{i=1}^{4} M^{i\dagger} M^i = I_4), \tag{3.24}$$

is performed on one of the particles of the initial state $|\Psi\rangle = \sum_{j=1}^{4} \psi_j |j\rangle |j\rangle$. Here, c_{ij} is the (ij)th element of the matrix c where $(c_{i1}, c_{i2}, c_{i3}, c_{i4})^T = (U^i)^\dagger (\phi_1, \phi_2, \phi_3, \phi_4)^T$. The matrix c is given by

$$c = \begin{pmatrix} \phi_1 & \phi_2 & \phi_3 & \phi_4 \\ \phi_2 & \phi_1 & \phi_3 & \phi_4 \\ \phi_1 & \phi_2 & \phi_4 & \phi_3 \\ \phi_4 & \phi_2 & \phi_3 & \phi_1 \end{pmatrix}. \tag{3.25}$$

The state after the measurement turns out to be one of the states

$$M^i |\Psi\rangle \to |\Phi^i\rangle = \sum_{j=1}^{4} c_{ij} |j\rangle |j\rangle \quad (i = 1,2,3,4), \tag{3.26}$$

with probabilities $p^i = \langle \Psi | {M^i}^\dagger M^i |\Psi\rangle$. The state $|\Phi^1\rangle$ is already the target state $|\Phi\rangle = \sum_{j=1}^{4} \phi_j |j\rangle |j\rangle$, and the states $|\Phi^2\rangle$, $|\Phi^3\rangle$ and $|\Phi^4\rangle$ can be transformed to the target state by local unitary transformations (permutations) $|1\rangle \leftrightarrow |2\rangle$, $|3\rangle \leftrightarrow |4\rangle$ and $|1\rangle \leftrightarrow |4\rangle$, respectively. Also, the condition $\sum_{i=1}^{4} {M^i}^\dagger M^i = I_4$ implies that the following linear equations

$$\sum_{i=1}^{4} p^i c_{ij}^2 = \psi_j^2, \quad (j = 1, \ldots, 4), \tag{3.27}$$

should be satisfied, and solutions of these linear equations give the probabilities. Thus, probabilities are found to be

$$\begin{aligned} p^1 &= 1 - \sum_{i=2}^{4} p^i, \quad p^2 = \frac{\psi_2^2 - \phi_2^2}{\phi_1^2 - \phi_2^2}, \quad p^3 = \frac{\phi_3^2 - \psi_3^2}{\phi_3^2 - \phi_4^2}, \\ p^4 &= \frac{\psi_3^2 + \psi_4^2 - \phi_3^2 - \phi_4^2}{\phi_1^2 - \phi_4^2}. \end{aligned} \tag{3.28}$$

Succinctly, the POVM measurement yields one of the states

$$\begin{aligned} |\Phi^1\rangle &= \sum_{j=1}^{4} \phi_j |j\rangle |j\rangle, \\ |\Phi^2\rangle &= \phi_2 |1\rangle |1\rangle + \phi_1 |2\rangle |2\rangle + \phi_3 |3\rangle |3\rangle + \phi_4 |4\rangle |4\rangle, \\ |\Phi^3\rangle &= \phi_1 |1\rangle |1\rangle + \phi_2 |2\rangle |2\rangle + \phi_4 |3\rangle |3\rangle + \phi_3 |4\rangle |4\rangle, \\ |\Phi^4\rangle &= \phi_4 |1\rangle |1\rangle + \phi_2 |2\rangle |2\rangle + \phi_3 |3\rangle |3\rangle + \phi_1 |4\rangle |4\rangle, \end{aligned} \tag{3.29}$$

with the probabilities p^i given in Eq. (3.28) where $(U^i \otimes U^i) |\Phi^i\rangle = |\Phi\rangle$ $(i = 1,2,3,4)$ and the permutations U^i are given in Eq. (3.23). Since all states obtained after the measurement are transformed to the target state by local unitary transformations

and the total probability of success is unity ($\sum_{i=1}^{4} p^i = 1$), we may conclude that the deterministic transformation $|\Psi\rangle \to |\Phi\rangle$ can be obtained by LOCC.

Example 4. We now will use the protocol II to transform a $5 \otimes 5$ entangled pure state into another one (in the first step we will use three-level solutions and then in the second step we will use two-level solutions of protocol I) for which majorization condition is satisfied (see Subsection 2.4.2 for details). Consider the state

$$|\Psi\rangle = \sqrt{\frac{3}{8}}\,|1\rangle\,|1\rangle + \sqrt{\frac{5}{24}}\,|2\rangle\,|2\rangle + \frac{1}{\sqrt{6}}\,|3\rangle\,|3\rangle + \frac{1}{\sqrt{8}}\,|4\rangle\,|4\rangle + \frac{1}{\sqrt{8}}\,|5\rangle\,|5\rangle\,, \quad (3.30)$$

as the initial one and the state

$$|\Phi\rangle = \frac{1}{\sqrt{2}}\,|1\rangle\,|1\rangle + \frac{1}{\sqrt{3}}\,|2\rangle\,|2\rangle + \frac{1}{\sqrt{12}}\,|3\rangle\,|3\rangle + \frac{1}{\sqrt{24}}\,|4\rangle\,|4\rangle + \frac{1}{\sqrt{24}}\,|5\rangle\,|5\rangle\,, \quad (3.31)$$

as the final one. Then, as protocol II required, we need to define an appropriate intermediate (or temporary) state. We choose this state such that

$$|\eta\rangle = \sqrt{\frac{3}{8}}\,|1\rangle\,|1\rangle + \frac{1}{\sqrt{3}}\,|2\rangle\,|2\rangle + \sqrt{\frac{5}{24}}\,|3\rangle\,|3\rangle + \frac{1}{\sqrt{24}}\,|4\rangle\,|4\rangle + \frac{1}{\sqrt{24}}\,|5\rangle\,|5\rangle\,, \quad (3.32)$$

where three Schmidt coefficients of the states $|\Phi\rangle$ and $|\eta\rangle$ are same, and two Schmidt coefficients of the states $|\Psi\rangle$ and $|\eta\rangle$ are same. First, using three-level solutions of protocol I, one can obtain the POVM measurement operators such that

$$\begin{aligned}
M_{11} =& (U_{11} \otimes U_{11})\sqrt{p_1}\ \mathrm{diag}\Big(1, 1, \sqrt{2}, \sqrt{\frac{1}{3}}, \sqrt{\frac{1}{3}}\Big), \\
M_{21} =& (U_{21} \otimes U_{21})\sqrt{p_2}\ \mathrm{diag}\Big(1, 1, \frac{1}{2}, \sqrt{\frac{8}{3}}, \sqrt{\frac{1}{3}}\Big), \\
M_{31} =& (U_{31} \otimes U_{31})\sqrt{p_3}\ \mathrm{diag}\Big(1, 1, \frac{1}{2}, \sqrt{\frac{1}{3}}, \sqrt{\frac{8}{3}}\Big), \quad \Big(\sum_{i=1}^{3} M_{i1}^{\dagger} M_{i1} = I\Big),
\end{aligned} \quad (3.33)$$

where $p_1 = 3/7$, $p_2 = 2/7$ and $p_3 = 2/7$. The POVM measurement operators given in Eq. (3.33) transform the state $|\Psi\rangle$ given in Eq. (3.30) into the state $|\eta\rangle$ given in Eq. (3.31) deterministically. In other words we have

$$M_{i1}\,|\Psi\rangle = \sqrt{p_i}(U_{i1} \otimes U_{i1})\,|\chi^i\rangle\,, \quad (3.34)$$

where $(U_{i1} \otimes U_{i1})\,|\chi^i\rangle = |\chi\rangle$ for $i = 1, 2, 3$, $(U \otimes U)\,|\chi\rangle = |\eta\rangle$ and $U = |2\rangle \leftrightarrow |3\rangle$. Here, the unitary transformations are found to be $U_{11} = I$, $U_{21} = |3\rangle \leftrightarrow |4\rangle$ and $U_{31} = |3\rangle \leftrightarrow |5\rangle$. We now will obtain the transformation $|\eta\rangle \to |\Phi\rangle$, where the states $|\eta\rangle$ and $|\Phi\rangle$ can be written in the form

$$|\eta\rangle = \sqrt{\frac{5}{12}}\,|\varphi\rangle + \sqrt{\frac{7}{12}}\Big(\sqrt{\frac{9}{14}}\,|1\rangle\,|1\rangle + \sqrt{\frac{5}{14}}\,|3\rangle\,|3\rangle\Big), \quad (3.35)$$

$$|\Phi\rangle = \sqrt{\frac{5}{12}}\,|\varphi\rangle + \sqrt{\frac{7}{12}}\Big(\sqrt{\frac{6}{7}}\,|1\rangle\,|1\rangle + \frac{1}{\sqrt{7}}\,|3\rangle\,|3\rangle\Big), \tag{3.36}$$

respectively. Here, $|\varphi\rangle = \sqrt{\frac{4}{5}}\,|2\rangle\,|2\rangle + \frac{1}{\sqrt{10}}\,|4\rangle\,|4\rangle + \frac{1}{\sqrt{10}}\,|5\rangle\,|5\rangle$ is a normalized entangled state. Then, under the POVM measurement operators given by

$$\begin{aligned} M_{12} =& (U_{12}\otimes U_{12})\sqrt{p_1}\ \mathrm{diag}\Big(\sqrt{\frac{4}{3}},1,\sqrt{\frac{2}{5}},1,1\Big), \\ M_{22} =& (U_{22}\otimes U_{22})\sqrt{p_2}\ \mathrm{diag}\Big(\sqrt{\frac{2}{9}},1,\sqrt{\frac{12}{5}},1,1\Big), \end{aligned} \tag{3.37}$$

we obtain the transformation $|\eta\rangle \to |\Phi\rangle$ where $p_1 = 7/10$, $p_2 = 3/10$, $U_{12} = I$, $U_{22} = |1\rangle \leftrightarrow |3\rangle$, and $\sum_{i=1}^{2} M_{i2}^{\dagger} M_{i2} = I$. In other words we have

$$M_{i2}\,|\eta\rangle = \sqrt{p_i}(U_{i2}\otimes U_{i2})\,|\Phi^i\rangle\,, \tag{3.38}$$

where $(U_{i2}\otimes U_{i2})\,|\Phi^i\rangle = |\Phi\rangle$ for $i = 1,2$, $U_{12} = I$ and $U_{22} = |1\rangle \leftrightarrow |3\rangle$. Thus, we obtain the entire transformation $|\Psi\rangle \to |\Phi\rangle$ as $|\Psi\rangle \to |\eta\rangle \to |\Phi\rangle$.

3.6 Conclusion

We adapted the results obtained in the previous chapter to the deterministic transformations of $d \otimes d$ bipartite entangled pure states. We explicitly examined various examples for both protocols.

4. OPTIMAL DISTILLATION OF COHERENCE

In this chapter, we present an optimal probabilistic protocol to distill quantum coherence. Inspired by a specific entanglement distillation protocol, our main result yields a strictly incoherent operation that produces one of a family of maximally coherent states of variable dimension from any pure quantum state. We also expand this protocol to the case where it is possible, for some initial states, to avert any waste of resources as far as the output states are concerned, by exploiting an additional transformation into a suitable intermediate state. These results provide practical schemes for efficient quantum resource manipulation. The material covered in this chapter is taken from the paper [63][1].

4.1 Introduction

Over the past three decades quantum entanglement has been identified as one of the main resources that allows us to overcome the intrinsic limits of classical information processing in a distributed setting [1]. It is therefore not surprising that entanglement manipulation is often seen as one of the fundamental tasks in the theory of quantum information. In several cases of practical interest, the goal is that of preparing a target state (e.g., maximally entangled) starting either from many i.i.d.[2] copies of the same state [64, 65], or – probabilistically – from a single copy of a known pure state [26, 29, 56]. The problem of distilling as much entanglement as possible from a given pure state by means of a probabilistic protocol using local operations and classical communication (LOCC) was considered in [58, 60]. Instead of aiming at a single output state, however, one can consider a discrete class of states as targets, namely that formed by all maximally entangled states of any possible local dimension q. The protocol given in [58, 60] always succeeds in producing one of these states, and a failure occurs only when said local dimension takes the "trivial" value $q = 1$.

[1] Gökhan Torun, Ludovico Lami, Gerardo Adesso, and Ali Yildiz, Optimal distillation of quantum coherence with reduced waste of resources, *Phys. Rev. A*, 99, 012321, (2019).

[2] i.i.d: independent identically distributed.

As entanglement of pure states is one of the manifestations of the superposition principle, one can more fundamentally regard the phenomenon of coherent superposition as a valuable resource in its own right. Quantum coherence plays in fact an essential role in applications to quantum algorithms, quantum metrology, and quantum biology [3]. To deal with this point of view, a resource theory of quantum coherence has been recently established [2, 3, 20, 66, 67]. Coherence distillation is a central task in the resource theory of quantum coherence, and is a subject of very active current investigation [2, 20, 68–72].

In this chapter, we introduce an explicit protocol for coherence distillation via a single strictly incoherent operation where we originally have a d-level coherent input state, see Fig. 4.1. This strategy is a counterpart to the entanglement distillation given in [58, 60]. One of the most significant points of this single-step strategy, when we compare it to some common distillation protocols [56, 73, 74], is that we can have any of all q-level ($q = 2, 3, \ldots, d$) maximally coherent pure states at the end of the measurement process. When compared with the previously available protocols [56], we see that the failure probability is thus relatively small, and a useful coherent state is almost always produced, unless the incoherent outcome ($q = 1$) is obtained. In particular, our protocol is optimal with respect to the distillation of d-level maximally coherent states, as the associated probability of success is maximal. We complement our analysis with a quantification of the coherence loss on average in our protocol, and comment on how and for which input states it is possible to modify our strategy, to avoid any waste of resources and always output a state with nonzero coherence.

The content of the present chapter is organised as follows. We present a protocol for achieving the optimal distillation of coherence (Section 4.2). We explicitly introduce our protocol (Subsection 4.2.1), examine coherence loss (Subsection 4.2.2), and discuss no complete waste of resources (Subsection 4.2.3). Finally, we conclude (Section 4.3).

4.2 Optimal Distillation

In the following, we introduce an explicit protocol for coherence distillation where we originally have a d-level coherent state. Our protocol, in the respect of the distillation

of d-level maximally coherent state, is optimal while the probability of success is maximal.

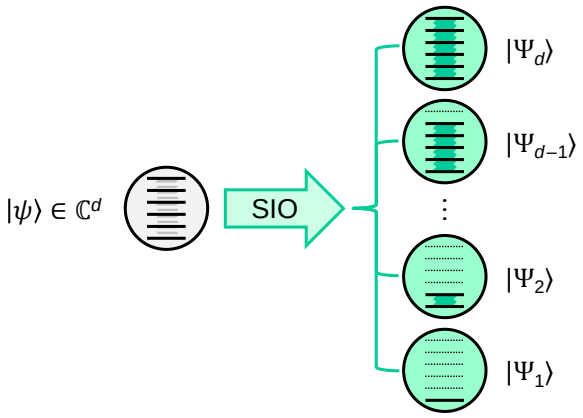

Figure 4.1 : Our strategy solves the coherence distillation problem as follows. We originally have a d-level coherent pure state $|\psi\rangle$. We perform a strictly incoherent operation (SIO) on the particle and obtain any of all q-level ($q = 2,3,\ldots,d$) maximally coherent states $|\Psi_q\rangle$, or an incoherent state ($q = 1$). The explicit quantum operation used in the protocol is described in the main text.

4.2.1 Coherence distillation

To start with, we need to recall the basis-dependent notions of incoherent and coherent states followed by incoherent operations. Quantum states that are diagonal with respect to a fixed orthonormal basis $\{|i\rangle\}_{i=1,2,\ldots,d}$ are defined as incoherent, and they constitute a set labeled by $\mathcal{I}$ [2, 3]. All incoherent states $\rho \in \mathcal{I}$ are of the form

$$\rho = \sum_{i=1}^{d} p_i |i\rangle \langle i| , \tag{4.1}$$

where $p_i \in [0,1]$ and $\sum_i p_i = 1$. In addition to this, a finite d-dimensional pure coherent state is given by

$$|\psi\rangle = \sum_{j=1}^{d} e^{i\theta_j} \psi_j |j\rangle , \quad \big(0 \leq \theta_j \leq \pi\big), \tag{4.2}$$

where $\{\psi_j\}_{j=1,2,\ldots,d}$ are non-negative real numbers, arranged in non-increasing order ($\psi_j \geq \psi_{j+1} \geq 0$), and satisfying $\sum_{j=1}^{d} \psi_j^2 = 1$. Here, without loss of generality, we can and from now on will assume that $\theta_j = 0$ for all j, as all these complex phases also can be eliminated by diagonal unitaries, which are always assumed to be free operations in any version of the resource theory of coherence.

We will focus on particular quantum operations for which measurement outcomes are retained as stated in [2]. These quantum operations are defined by Kraus operators

$\{K_i\}$ that map incoherent states into incoherent states, i.e., such that $\sum_i K_i^\dagger K_i = I$ and, for all i and $\rho \in \mathcal{I}$,

$$\rho \to \rho_i = \frac{K_i \rho K_i^\dagger}{\mathrm{Tr}[K_i \rho K_i^\dagger]} \in \mathcal{I}. \tag{4.3}$$

Operations of the form as in Eq. (4.3) in which the Kraus operators satisfy the above condition are known as incoherent operations (IO) and can be adopted as the free operations in the context of the resource theory of coherence as defined in [2]. A relevant subset of IO is constituted by strictly incoherent operations (SIO), which are completely positive trace preserving maps whose Kraus operators K_i satisfy both $K_i \mathcal{I} K_i^\dagger \subseteq \mathcal{I}$ and $K_i^\dagger \mathcal{I} K_i \subseteq \mathcal{I}$ [20, 21, 33, 75]. We will demonstrate that, although SIO have a very limited coherence distillation power when mixed input states are concerned [71], they nonetheless suffice for our distillation protocol with pure input states.

We are now ready to analyze the task of one-shot coherence distillation, whose goal is to transform a single copy of the input states given in Eq. (4.2) into a maximally coherent one via (possibly probabilistic) incoherent operations. The state given in Eq. (4.2) is a d-level maximally coherent state for $\{\psi_i\}_{i=1,\ldots,d} = \{\frac{1}{\sqrt{d}}, \ldots, \frac{1}{\sqrt{d}}\}$, i.e.,

$$|\Psi_d\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |i\rangle . \tag{4.4}$$

An optimal local conversion strategy of bipartite entangled pure states was proposed by Vidal [29]. Adapting those results to the case of coherence distillation, one can obtain the maximal probability of transforming the coherent state $|\psi\rangle$ in Eq. (4.2) to the maximally coherent state $|\Psi_d\rangle$ in Eq. (4.4) [30, 76], which is given by

$$p(|\psi\rangle \to |\Psi_d\rangle) = \min_{k\in[1,d]} \left\{ \frac{d \sum_{i=k}^{d} \psi_i^2}{d-k+1} \right\} = d\psi_d^2. \tag{4.5}$$

We construct the explicit Kraus operators to implement the transformations

$$\sum_{i=1}^{d} \psi_i |i\rangle \xrightarrow{\text{SIO}} \left\{ \left(p_q, \frac{1}{\sqrt{q}} \sum_{i=1}^{q} |i\rangle \right) \right\}_{q=1,2,\ldots,d}, \tag{4.6}$$

by means of SIO as defined above. These are given by

$$K_q = \sqrt{p_q} \left(\frac{1}{\sqrt{q}} \sum_{i=1}^{q} \frac{|i\rangle \langle i|}{\psi_i} \right), \tag{4.7}$$

where

$$\begin{aligned} p_d &= d\psi_d^2, \\ p_q &= q\left(\psi_q^2 - \psi_{q+1}^2\right), \quad q = 1,2,\ldots,(d-1). \end{aligned} \tag{4.8}$$

Note that the operation identified by the above Kraus operators is not only incoherent but also strictly incoherent. Observe further that the above Kraus operators satisfy the normalization condition $\sum_{i=1}^{d} K_i^\dagger K_i = I_d$, implying that they define a legitimate quantum channel. By construction, we have that

$$K_q |\psi\rangle = \sqrt{p_q}\,|\Psi_q\rangle, \qquad q = 1,2,\ldots,d, \tag{4.9}$$

i.e., such channel implements the transformations in Eq. (4.6). Observe that the success probability of the transformation $|\psi\rangle \to |\Psi_d\rangle$, denoted by p_d and given by Eq. (4.8), achieves its maximal value as given by Eq. (4.5) (see also [29]). In this sense, the described protocol is optimal. It is not difficult to verify that the probabilities in Eq. (4.8) correctly satisfy the completeness relation, that is $\sum_{i=1}^{d} p_i = 1$. As a result, we initially have the coherent state $\sum_{i=1}^{d} \psi_i |i\rangle$, and after a single-step measurement process with the given Kraus operators in Eq. (4.7) we obtain a q-level ($q = 2,\ldots,d$) maximally coherent state with a certain probability given by Eq. (4.8). This ensures minimal waste of resources in the distillation protocol, as a useful (albeit of smaller dimension) maximally coherent state is obtained even when the desired outcome is not recorded. Such a feature is explored in more quantitative detail in the following section.

4.2.2 Coherence loss

While we know that the degree of coherence cannot increase under IO defined in Eq. (4.3), when quantified by suitable coherence monotones [3], one may wonder how much coherence is lost on average during our protocol. We adopt the l_1 norm of coherence [2], a proper quantifier of coherence fulfilling strong monotonicity under IO, for this study. The l_1 norm of coherence of the state $|\psi\rangle = \sum_{i=1}^{d} \psi_i |i\rangle$ is given by

$$C_{l_1}(\rho_{|\psi\rangle}) = \Big(\sum_{i=1}^{d} \psi_i\Big)^2 - 1, \tag{4.10}$$

and the l_1 norm of coherence of the (maximally coherent) state $|\Psi_q\rangle = \frac{1}{\sqrt{q}} \sum_{i=1}^{q} |i\rangle$ is given by

$$C_{l_1}(\rho_{|\Psi_q\rangle}) = q - 1, \quad (q = 1,2,\ldots,d), \tag{4.11}$$

where $|\Psi_1\rangle = |1\rangle$ is an incoherent state, and therefore, $C_{l_1}(\rho_{|\Psi_1\rangle}) = 0$. Combining Eq. (4.8) with Eq. (4.11) we can obtain the average coherence for the output ensemble, given by

$$\bar{C}_{l_1}(\rho_{\text{out}}) = \sum_{q=1}^{d} p_q C_{l_1}(\rho_{|\Psi_q\rangle}) = 2\sum_{i=1}^{d}(i-1)\psi_i^2. \tag{4.12}$$

Monotonicity (under selective IO on average) yields that $C_{l_1}(\rho_{|\psi\rangle}) \geq \bar{C}_{l_1}(\rho_{\text{out}})$, i.e., $(\sum_{i=1}^{d}\psi_i)^2 - 1 \geq 2\sum_{i=1}^{d}(i-1)\psi_i^2$. Thus, the average loss of coherence for our protocol is found to be

$$C_{l_1}(\rho_{|\psi\rangle}) - \bar{C}_{l_1}(\rho_{\text{out}}) = \Big(\sum_{i=1}^{d}\psi_i\Big)^2 - 2\sum_{i=1}^{d} i\psi_i^2 + 1. \tag{4.13}$$

The quantity in Eq. (4.13) obviously vanishes when the input is already a d-dimensional maximally coherent state, in which case the protocol leaves it invariant with certainty. On the other hand, it can be interesting to investigate classes of states for which there is a large loss of coherence on average during the distillation protocol. One such a class is given by what we may refer to as 'harmonic power states', namely, input states $|\psi\rangle$ with coefficients

$$\psi_i = \frac{1}{i^{\alpha}\sqrt{H_d^{(2\alpha)}}}, \tag{4.14}$$

where $H_d^{(2\alpha)}$ is the d^{th} harmonic number of order 2α, $H_d^{(2\alpha)} = \sum_{j=1}^{d} 1/j^{2\alpha}$, with $\alpha \in [0,\infty)$. These states nearly achieve the minimal $\bar{C}_{l_1}(\rho_{\text{out}})$ for a given $C_{l_1}(\rho_{|\psi\rangle})$, as plotted in Fig. 4.2 for dimension $d = 4$.

4.2.3 No complete waste of resources

While the obtained $(d-1)$ outcomes are maximally resourceful states of their corresponding dimension, an incoherent state–waste–is also obtained with a nonzero probability $p_1 = \psi_1^2 - \psi_2^2$. The above strategy can be improved so as to avoid complete waste of resources with certainty, provided that the initial state satisfies some mild assumptions on the amount of coherence it contains. Namely, if $\psi_1^2 < 1/2$ it is possible to modify the described protocol in such a way as to make $p_1 = 0$, where p_1 is the probability of outputting an incoherent state (the case $q = 1$ in (4.9)). This can be accomplished by first transforming $|\psi\rangle$ into an appropriate intermediate state $|\chi\rangle$, and

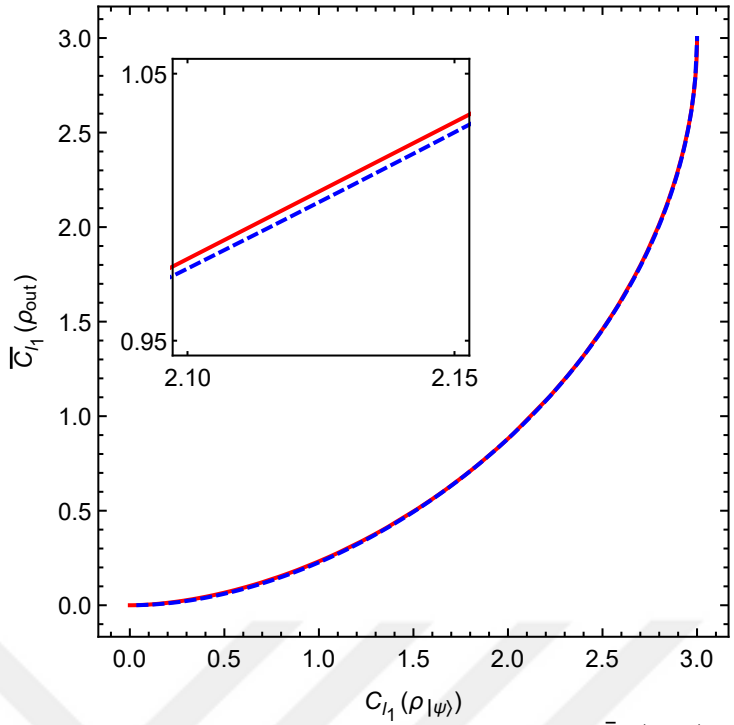

Figure 4.2 : Plot of the output average l_1 norm of coherence $\bar{C}_{l_1}(\rho_{\text{out}})$ versus the input coherence $C_{l_1}(\rho_{|\psi\rangle})$ for states in dimension $d = 4$. The solid (red) line corresponds to harmonic power states defined by Eq. (4.14). The dashed (blue) line corresponds to states maximizing the average coherence loss defined by Eq. (4.13), as obtained by solving numerically the corresponding constrained optimization problem. The small difference between the two lines is better seen in the zoomed-in inset. Even if coherence is decreased on average, our protocol always yields a maximally coherent state (in some dimension $q \leq d$) with nonzero probability, apart from the trivial case $q = 1$ in which an incoherent state is obtained. All the quantities plotted are dimensionless.

by finally applying the original protocol to $|\chi\rangle$. The required state $|\chi\rangle$ takes the form

$$|\chi\rangle = \psi_1 \sum_{i=1}^{k} |i\rangle + \psi'_{k+1} |k+1\rangle + \sum_{i=k+2}^{d} \psi_i |i\rangle , \tag{4.15}$$

where $k > 1$ is any integer such that $k\psi_1^2 + \psi_{k+1}'^2 + \sum_{i=k+2}^{d} \psi_i^2 = 1$ is satisfied for some ψ'_{k+1} subjected to the constraints $\psi_1 \geq \psi'_{k+1} \geq \psi_{k+2} \geq \cdots \geq \psi_d \geq 0$. Using the results in [8, 26, 27], we know that the transformation $|\psi\rangle \to |\chi\rangle$ can be performed deterministically. After attaining the temporary state $|\chi\rangle$, we apply the protocol defined by Eq. (4.7), which outputs the ensemble $|\chi\rangle \to \{(p_q, |\Psi_q\rangle)\}_{q=k,\ldots,d}$. The entire transformation is then given by

$$|\psi\rangle \to |\chi\rangle \to \left\{ \left(p_q, \frac{1}{\sqrt{q}} \sum_{i=1}^{q} |i\rangle \right) \right\}_{q=k,\ldots,d} . \tag{4.16}$$

It should be highlighted that the probability of obtaining the state $|\Psi_d\rangle$, $p_d = d\psi_d^2$, is still maximum. Let us discuss a simple example of the above procedure. Consider the initial state $|\psi\rangle = \sqrt{0.35}\,|1\rangle + \sqrt{0.3}\,|2\rangle + \sqrt{0.25}\,|3\rangle + \sqrt{0.1}\,|4\rangle$ in dimension $d = 4$. We can transform this into the temporary state $|\chi\rangle = \sqrt{0.35}\,|1\rangle + \sqrt{0.35}\,|2\rangle + \sqrt{0.2}\,|3\rangle + \sqrt{0.1}\,|4\rangle$ ($k = 2$) with unit probability. Then, the protocol in Eq. (4.7) yields the states $|\Psi_4\rangle$, $|\Psi_3\rangle$, and $|\Psi_2\rangle$ with probabilities 0.4, 0.3 and 0.3, respectively. Ultimately, by the help of a proper intermediate state $|\chi\rangle$ given in Eq. (4.15), we can obtain an ensemble of maximally coherent q-level ($q = k, \ldots, d$) states, guaranteeing that all the output states have coherence.

As one can easily notice, this strategy can also be adapted to the entanglement distillation by local operations and classical communication. For the initial bipartite pure entangled state $|\phi\rangle = \sum_{i=1}^{d} \phi_i\,|ii\rangle$ (Schmidt coefficients are ordered in non-increasing order as usual), provided that $\phi_1^2 \leq 1/2$ one can find an intermediate state $|\varphi\rangle$ such that

$$|\varphi\rangle = \phi_1\,|11\rangle + \phi_1\,|22\rangle + \phi_3'\,|33\rangle + \sum_{i=4}^{d} \phi_i\,|ii\rangle\,, \tag{4.17}$$

where $\phi_1 \geq \phi_3' \geq \phi_4 \geq \cdots \geq \phi_d \geq 0$. Then, analogously to coherence distillation, using the results in [8, 26] we can obtain the transformation $|\phi\rangle \to |\varphi\rangle$ deterministically in order to avoid producing a separable output with certainty. The complete transformation is then given by

$$|\phi\rangle \to |\varphi\rangle \to \Big\{ \big(p_q, |\Phi_q\rangle\big) \Big\}_{q=2,3,\ldots,d}, \tag{4.18}$$

where $|\Phi_q\rangle = \frac{1}{\sqrt{q}} \sum_{i=1}^{q} |ii\rangle$ and the probability of obtaining the separable state $|\Phi_1\rangle = |11\rangle$ is equal to zero. Here, while the probability of getting $|\Phi_2\rangle$ and $|\Phi_3\rangle$ increases ($p_2 = 2(\phi_1^2 - \phi_3'^2)$) and decreases ($p_3 = 3(\phi_3'^2 - \phi_4^2)$), respectively, the other probabilities p_m of getting $|\Phi_m\rangle$ ($m = 4, 5, \ldots, d$) remain unchanged. It is always possible to find an intermediate state $|\varphi\rangle$ of the form (4.17) for the initial states such that $\phi_3^2 - \phi_4^2 \geq \phi_1^2 - \phi_2^2$. Therefore, if the initial entangled bipartite states $\sum_{i=1}^{d} \phi_i\,|ii\rangle$ satisfy this relation, our results ensure that both the transformations given in Eq. (4.18) can be implemented and hence that no waste of entanglement resources is achieved when only the set of the output states are considered.

Another point that needs to be discussed pertains to the largest amount of distilled entanglement. It is given by

$$\langle E \rangle_{\max} = \sum_{j=1}^{d} (\lambda_j - \lambda_{j+1}) j \ln j, \tag{4.19}$$

for the state $\sum_{j=1}^{d} \sqrt{\lambda_j}\,|jj\rangle$ [58, 60]. Considering the state $|\phi\rangle = \sqrt{0.35}\,|11\rangle + \sqrt{0.3}\,|22\rangle + \sqrt{0.25}\,|33\rangle + \sqrt{0.1}\,|44\rangle$ as the initial bipartite entangled state, one can transform it into the intermediate state $|\varphi\rangle = \sqrt{0.35}\,|11\rangle + \sqrt{0.35}\,|22\rangle + \sqrt{0.2}\,|33\rangle + \sqrt{0.1}\,|44\rangle$ deterministically. Then, the largest amount of distilled entanglement is found to be 1.11821 and 1.09205 for the states $|\phi\rangle$ and $|\varphi\rangle$, respectively. Thus, although the entire transformation $|\phi\rangle \to |\varphi\rangle \to \{p_q, |\Phi_q\rangle\}_{q=2,\ldots,d}$ may provide no complete waste of resources, it may lead to a decreased amount of the largest distilled entanglement. This is resulting from the increase (decrease) of the probability of obtaining lower (higher) dimensional maximally entangled states.

4.3 Conclusion

In this chapter, we presented a simple, practical and efficient strategy for optimal one-shot distillation of quantum coherence from pure input states of arbitrary dimension. The key advantage of our protocol lies in its ability to provide a single map to obtain all q-level ($q = 2, 3, \ldots, d$) maximally coherent pure states starting from a d-level coherent input pure state, as illustrated in Fig. 4.1. In this way, useful degrees of coherence resource are "recycled" even when the maximally resourceful d-dimensional state is not obtained.

The probability of success, defined by the outcome $q = d$, is maximal, confirming optimality of the protocol. On the other hand, our protocol only fails when the trivial outcome $q = 1$ is obtained, in which case no resource is distilled. This makes our protocol preferable to conventional distillation protocols such as the one in [56], which has instead a higher failure probability and produces no useful output in case the desired maximally resourceful output is not obtained. We furthermore showed how to modify the protocol into a two-step strategy which completely nullifies the failure probability, leading to no waste of coherence in the outputs; this is possible for a subclass of input states that we characterize. Our strategy can also be adapted to entanglement distillation.

A further generalization of our scheme and of the seminal works in [58, 60] to other quantum resource theories [77], beyond coherence and entanglement, would be a worthwhile direction for future investigation.

5. THREE-QUBIT PURE STATE TRANSFORMATIONS

In this chapter, we are interested in two problems under the umbrella of three-qubit pure states transformations. The first one is presented under the title “Canonical operators and the optimal concentration of three-qubit Greenberger-Horne-Zeilinger states", and the second one is presented under the title “Deterministic transformations of three-qubit pure states".

It is well known that quantum states that can be transformed into each other by local unitary transformations are equal from the information theoretic point of view. This defines equivalence classes of states and allows one to write any state with the minimal number of parameters called the canonical form of the state. We define the equivalence classes of local measurements such that local operations which transform states from one equivalence class into another with the same probability are equivalent. This equivalence relation allows one to write the operators with the minimal number of parameters, which we call canonical operators, and hence the use of the canonical operators simplifies the optimal manipulation of quantum states. We use the canonical local operators for the concentration of three-qubit Greenberger-Horne-Zeilinger (GHZ) states and obtain the optimal concentration protocols in terms of the unitary invariants of quantum states, namely, the bipartite concurrences and the three-tangle. The material covered in the first part of this chapter is published in the paper [49][1].

In the second part of this chapter, we propose an explicit protocol for the deterministic transformations of three-qubit pure states by local operations and classical communications. In this protocol we use the fact that a three-qubit pure state can be written in two different sets of parameters in the canonical form [49]. This allows us to construct a protocol such that each party performs a two-outcome measurement where each state after the measurement can be transformed into the other by local unitary transformations. By the application of that protocol we find the set of target states that can be obtained from a general W-type source state

[1] Gokhan Torun and Ali Yildiz, Canonical operators and the optimal concentration of three-qubit Greenberger-Horne-Zeilinger states, *Phys. Rev. A*, 89, 032320, (2014).

with unit probability by measurements on one qubit only, on two qubits, and on all qubits. Using the results, we find the conditions for deterministic transformability of general W-type pure states. We then apply the proposed protocol for the deterministic transformations of the GHZ state. We obtain the set of states that can be obtained from the GHZ state with unit probability, by measurements on one qubit only, on two qubits, and on all qubits. We find that all states, except the state with all bipartite concurrences are nonzero, can be deterministically obtained from the GHZ state by local operations and classical communications. We explicitly find the local operations for these transformations. This work can be thought of as an alternative method for the corresponding problem, i.e., optimal manipulations of the three-qubit pure states.

5.1 Introduction

Quantum entanglement is not only the most fascinating aspect of quantum mechanics, but it is also used as a resource in quantum information processes. The use of quantum entanglement as a resource requires a deep understanding of how the states are transformed under local operations and classical communication (LOCC). Two-qubit maximally entangled Einstein-Podolski-Rosen (EPR) state, $(|00\rangle + |11\rangle)/\sqrt{2}$, is used as a resource in perfect quantum information processes such as teleportation [38] and dense coding [39]. If the resource is a mixed state of two qubits, then the fidelity of the teleportation can be increased by the manipulation of the quantum state by LOCC [64, 78–80]. In the case where the resource is a nonmaximally entangled pure state, $\sqrt{1-a^2}\,|00\rangle + a\,|11\rangle$ $(0 \leq a \leq 1/2)$, it is possible to perform the task with some nonzero probability [41, 81]. An alternative way is concentrating the partial entanglement by local operations and then performing the information task with unit probability [55]. Optimal concentration of an EPR state can be obtained by a measurement of one of the qubits with the positive operator valued measurement (POVM) elements

$$M_1 = \frac{a}{\sqrt{1-a^2}}\,|0\rangle\langle 0| + |1\rangle\langle 1|\,, \quad M_2 = \frac{\sqrt{1-2a^2}}{\sqrt{1-a^2}}\,|0\rangle\langle 0|\,. \tag{5.1}$$

This method is called the Procrustean method [55] as the state is either transformed to an EPR state with a probability of $P = 2a^2$ or else to a separable state. It is also possible to obtain any partially entangled state from an EPR state with unit probability. One can obtain the partially entangled state by the measurement of one of the qubits

of an EPR state with POVM elements

$$M_1 = \sqrt{1-a^2}\,|0\rangle\langle 0| + a\,|1\rangle\langle 1|, \quad M_2 = a\,|0\rangle\langle 0| + \sqrt{1-a^2}\,|1\rangle\langle 1|, \tag{5.2}$$

followed by local unitary transformations.

While the entanglement of two qubits is well understood, many problems still remain unsolved for higher-dimensional systems. One of these problems is the optimal manipulation of states both from a single copy or from many copies. Optimal manipulation of a single copy of a state is of importance from the practical point of view, as experimentalists are not able to perform joint operations on multiple copies of the system in general. Nielsen [26] used the algebraic theory of majorization and obtained necessary and sufficient conditions for the entanglement transformation of two pure states of a bipartite system. An optimal local conversion strategy of these states was proposed by Vidal [29]. Although an extension of Nielsen's theorem to three-qubit pure states has recently been obtained by Tajima [51], there is no straightforward generalization to the multipartite systems. In the three-qubit case, there are two classes of tripartite-entangled states which cannot be converted into each other by stochastic local operations and classical communication (SLOCC), namely, the Greenberger-Horne-Zeilinger (GHZ) and W class states [82,83]. It has been shown that the minimal number of product terms for any given state remains unchanged under SLOCC and the minimal number of product terms is three for W class states. Any GHZ class state, on the other hand, can be written as the sum of two product vectors uniquely in the standard product form

$$|\psi_{\mathrm{GHZ}}\rangle = \sqrt{K}(c_\delta\,|0\rangle\,|0\rangle\,|0\rangle + s_\delta e^{i\varphi}\,|\varphi_A\rangle\,|\varphi_B\rangle\,|\varphi_C\rangle), \tag{5.3}$$

where

$$|\varphi_A\rangle = c_\alpha\,|0\rangle + s_\alpha\,|1\rangle, \quad |\varphi_B\rangle = c_\beta\,|0\rangle + s_\beta\,|1\rangle, \quad |\varphi_C\rangle = c_\gamma|0\rangle + s_\gamma|1\rangle, \tag{5.4}$$

and $K = (1+2c_\delta s_\delta c_\alpha c_\beta c_\gamma c_\varphi)^{-1}$ (c_δ and s_δ stand for $\cos\delta$ and $\sin\delta$, etc.). The ranges for the five parameters are $\delta \in (0,\pi/4], \alpha,\beta,\gamma, \in (0,\pi/2]$, and $\varphi \in [0,2\pi)$. The W state,

$$|W\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle), \tag{5.5}$$

and the GHZ state,

$$|\mathrm{GHZ}\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle), \tag{5.6}$$

are the representatives of the W and GHZ classes, respectively. Any GHZ class state given by Eq. (5.3) can be obtained by the application of the invertible local operators

$$|\psi_{\mathrm{GHZ}}\rangle = \frac{1}{\sqrt{p}}(A\otimes B\otimes C)\,|\mathrm{GHZ}\rangle\,, \tag{5.7}$$

where

$$\begin{aligned} A &= c_\delta\,|0\rangle\langle 0| + s_\delta c_\alpha e^{i\varphi}\,|0\rangle\langle 1| + s_\delta s_\alpha e^{i\varphi}\,|1\rangle\langle 1|\,,\\ B &= |0\rangle\langle 0| + c_\beta\,|0\rangle\langle 1| + s_\beta\,|1\rangle\langle 1|\,,\\ C &= |0\rangle\langle 0| + c_\gamma|0\rangle\langle 1| + s_\gamma|1\rangle\langle 1|\,, \end{aligned} \tag{5.8}$$

and the probability of success, $p = \langle \mathrm{GHZ}|A^\dagger A\otimes B^\dagger B\otimes C^\dagger C\,|\mathrm{GHZ}\rangle$, is not necessarily the maximum probability that can be achieved. The optimal transformation of the GHZ state, given by Eq. (5.6), to any GHZ-type state, given by Eq. (5.3), is still an open problem. Cui *et al.* [84] used the two-term product decomposition of the GHZ class states to obtain the upper bounds for the optimal probability of transformation from a GHZ state to other states of the GHZ class. The complexity of the optimal transformation of multi-qubit systems is mainly due to the large numbers of parameters coming from the general local operators since the general local operators acting on qubits are two-by-two complex matrices in general.

The optimal manipulation of the GHZ and W states is important because they are used as a resource in quantum information processes [85–90]. The optimal distillation protocols for symmetric and asymmetric three-qubit W states were proposed in [44]. Kintas and Turgut obtained an upper bound for the maximum probability of transforming N-qubit W states [91] and the necessary and sufficient conditions to obtain this upper bound were presented by Cui *et al.* [45]. The optimal local transformations of flip and exchange symmetric multiqubit states were obtained in [46]. Sheng *et al.* proposed practical two-step entanglement concentration protocols for some W class states [47]. The first GHZ state distillation protocols [92] consisting of two stages, primary and secondary distillations, were not optimal. Acin *et al.* [43] used one successful branch protocol in the the optimal concentration problem starting with the two-term product decomposition of a GHZ class state and obtained some partial results.

We approach the quantum-state manipulation problem from a different perspective by using the canonical form of states presented in [93, 94] and the equivalence classes of

local operators. The use of equivalence classes reduces the number of parameters and hence simplifies the manipulation problem since local operators which transform the states from one equivalence class into another with the same probability are equivalent. Moreover, the conversion probabilities and measurement operators can be found in terms of the unitary invariants of the quantum state.

The present chapter is organised as follows. We start with the canonical form of a general GHZ class pure state and discuss the equivalence classes of local POVMs (Section 5.2). These equivalence classes are used to obtain the maximum success probabilities and measurement operators for the optimal concentration of the GHZ states (Section 5.3). We then propose an explicit protocol for the deterministic transformations of three-qubit pure states (Section 5.4). We conclude with discussion and summary (Section 5.5).

5.2 Canonical Forms of States and Operations on Three-qubit Pure States

First, we define the notation for the equivalence relations for states and operations. Two states $|\varphi\rangle$ and $|\varphi'\rangle$ are in the same equivalence class ($|\varphi\rangle \sim |\varphi'\rangle$) if they can be transformed into each other by local unitary transformations. Two operators M and M' are in the same equivalence class ($M \equiv M'$) if they both transform states in one equivalence class, $|\psi_i\rangle$, to states in some other equivalence class with the same probability of success, i.e., the operators M and M' are equivalent ($M \equiv M'$) if they satisfy the following relations:

$$\begin{aligned} |\varphi\rangle = \frac{M|\psi_i\rangle}{\sqrt{\langle\psi_i|M^\dagger M|\psi_i\rangle}} \sim |\varphi'\rangle = \frac{M'|\psi_i\rangle}{\sqrt{\langle\psi_i|M'^\dagger M'|\psi_i\rangle}}, \\ \Big(P = \langle\psi_i|M^\dagger M|\psi_i\rangle = \langle\psi_i|M'^\dagger M'|\psi_i\rangle, \forall\ |\psi_i\rangle\Big). \end{aligned} \tag{5.9}$$

Following the approach presented by Acin *et al.* [93, 94], we define the canonical form of the GHZ class states which we are going to use in the concentration. Any three-qubit state

$$|\phi\rangle = \sum_{ijk} t_{ijk}|ijk\rangle \tag{5.10}$$

defines matrices T_0 and T_1 by

$$|\phi\rangle = \sum_{jk} T_{0,jk}|0\rangle|jk\rangle + T_{1,jk}|1\rangle|jk\rangle. \tag{5.11}$$

Under the unitary transformation on the first qubit, the matrices T_0 and T_1 transform as

$$\begin{aligned} T_0' =& u_{00}^A T_0 + u_{01}^A T_1, \\ T_1' =& u_{10}^A T_0 + u_{11}^A T_1, \qquad \Big(u_{tz} = \langle t|U|z\rangle\Big). \end{aligned} \tag{5.12}$$

It is always possible to make $\det T_0' = 0$ and the unitary transformations on the second and third qubits diagonalize T_0' which bring the state to the canonical form given by

$$|\psi\rangle = \lambda_0 |000\rangle + \lambda_1 e^{i\varphi} |100\rangle + \lambda_2 |101\rangle + \lambda_3 |110\rangle + \lambda_4 |111\rangle, \quad \lambda_i \geq 0. \tag{5.13}$$

However, the parameters λ_i and φ do not uniquely determine the state (the equivalence class) because there are two solutions for $\det T_0' = 0$, and this leads to two sets of parameters for the canonical form. The state with the other set of parameters can be found to be

$$|\tilde{\psi}\rangle = \tilde{\lambda}_0 |000\rangle + \tilde{\lambda}_1 e^{i\tilde{\varphi}} |100\rangle + \tilde{\lambda}_2 |101\rangle + \tilde{\lambda}_3 |110\rangle + \tilde{\lambda}_4 |111\rangle, \quad (\tilde{\lambda}_i \geq 0, \sum_{i=0}^{4} \tilde{\lambda}_i^2 = 1), \tag{5.14}$$

where

$$\tilde{\lambda}_0 = \frac{\lambda_0}{\kappa}, \quad \tilde{\lambda}_2 = \lambda_2 \kappa, \quad \tilde{\lambda}_3 = \lambda_3 \kappa, \quad \tilde{\lambda}_4 = \lambda_4 \kappa, \tag{5.15}$$

$$\kappa \equiv \left[\frac{\tau + C_{bc}^2}{4(\lambda_2^2 + \lambda_4^2)(\lambda_3^2 + \lambda_4^2)} \right]^{1/2}, \tag{5.16}$$

$$\begin{aligned} \tilde{\lambda}_1 e^{i\tilde{\varphi}} \;=\; & \frac{\lambda_1}{\kappa} \left[\frac{\lambda_4^2(\lambda_2^2 + \lambda_3^2 + \lambda_4^2) - \lambda_2^2 \lambda_3^2}{(\lambda_2^2 + \lambda_4^2)(\lambda_3^2 + \lambda_4^2)} \cos(\varphi) - i \sin(\varphi) \right] \\ & + \frac{\lambda_2 \lambda_3 \lambda_4 (\lambda_0^2 + \lambda_1^2 - \lambda_2^2 - \lambda_3^2 - \lambda_4^2)}{\kappa (\lambda_2^2 + \lambda_4^2)(\lambda_3^2 + \lambda_4^2)}. \end{aligned} \tag{5.17}$$

The equivalence of the states given by Eqs. (5.13) and (5.14) under local unitary transformations can be explicitly shown as follows: the following unitary operators

$$\begin{aligned} U_A &= \frac{1}{\sqrt{|\lambda_2 \lambda_3 - e^{i\varphi} \lambda_1 \lambda_4|^2 + \lambda_0^2 \lambda_4^2}} \begin{pmatrix} \lambda_2 \lambda_3 - e^{i\varphi} \lambda_1 \lambda_4 & \lambda_0 \lambda_4 \\ -\lambda_0 \lambda_4 & \lambda_2 \lambda_3 - e^{-i\varphi} \lambda_1 \lambda_4 \end{pmatrix}, \\ U_B &= \frac{1}{\sqrt{\lambda_2^2 + \lambda_4^2}} \begin{pmatrix} \lambda_2 & \lambda_4 \\ -\lambda_4 & \lambda_2 \end{pmatrix}, \quad U_C = \frac{1}{\sqrt{\lambda_3^2 + \lambda_4^2}} \begin{pmatrix} \lambda_3 & \lambda_4 \\ -\lambda_4 & \lambda_3 \end{pmatrix} \end{aligned} \tag{5.18}$$

applied to parties A, B and C respectively will transform the state Eq. (5.13) into the state Eq. (5.14). It follows from Eq. (5.15) that $\tilde{\lambda}_0 \tilde{\lambda}_2 = \lambda_0 \lambda_2$ and hence one may

conclude that $\lambda_0\lambda_2$ is invariant under local unitary transformations. In this way, the five unitary invariants can be found to be

$$\begin{gathered}\tilde{\lambda}_0\tilde{\lambda}_2 = \lambda_0\lambda_2 = C_{ac}/2, \quad \tilde{\lambda}_0\tilde{\lambda}_3 = \lambda_0\lambda_3 = C_{ab}/2, \quad \tilde{\lambda}_0\tilde{\lambda}_4 = \lambda_0\lambda_4 = \sqrt{\tau}/2, \\ |\tilde{\lambda}_2\tilde{\lambda}_3 - e^{i\tilde{\varphi}}\tilde{\lambda}_1\tilde{\lambda}_4| = |\lambda_2\lambda_3 - e^{i\varphi}\lambda_1\lambda_4| = C_{bc}/2, \\ \tilde{\lambda}_2\tilde{\lambda}_3 - \tilde{\lambda}_1\tilde{\lambda}_4\cos(\tilde{\varphi}) = \lambda_2\lambda_3 - \lambda_1\lambda_4\cos(\varphi), \\ J_5 = 4\lambda_0^2(|\lambda_1\lambda_4 e^{i\varphi} - \lambda_2\lambda_3|^2 + \lambda_2^2\lambda_3^2 - \lambda_1^2\lambda_4^2) = \lambda_0^2 C_{BC}^2 + \lambda_2^2 C_{AB}^2 - \lambda_1^2\tau \end{gathered} \tag{5.19}$$

where τ is three-tangle [95] and C_{ab} is the concurrence between qubits one (a) and two (b), etc. Concurrence [95] is given by

$$C_{ab} = \max\{0, \alpha_1 - \alpha_2 - \alpha_3 - \alpha_4\}, \tag{5.20}$$

where $\alpha_1 \geq \alpha_2 \geq \alpha_3 \geq \alpha_4$ are the eigenvalues of the matrix $\sqrt{\sqrt{\rho_{ab}}\tilde{\rho}_{ab}\sqrt{\rho_{ab}}}$. Here, $\tilde{\rho}_{ab} = (\sigma_y \otimes \sigma_y)\rho_{ab}^*(\sigma_y \otimes \sigma_y)$ where the density matrix ρ_{ab}^* is the complex conjugate of the density matrix ρ_{ab} (the reduced density matrix of ρ_{abc}, i.e., $\rho_{ab} = \mathrm{Tr}_c[\rho_{abc}]$). We use Eq. (5.15), Eq. (5.16) and Eq. (5.17) to remove the ambiguity in the definition of the canonical form and define the equivalence classes uniquely in terms of the parameters of the canonical form. By using the property

$$\kappa \equiv \frac{\tau + C_{bc}^2}{4(\lambda_2^2 + \lambda_4^2)(\lambda_3^2 + \lambda_4^2)} > 1 (< 1) \leftrightarrow \tilde{\kappa} \equiv \frac{\tau + C_{bc}^2}{4(\tilde{\lambda}_2^2 + \tilde{\lambda}_4^2)(\tilde{\lambda}_3^2 + \tilde{\lambda}_4^2)} < 1 (> 1), \tag{5.21}$$

we choose the state with $\kappa > 1$ as the canonical form. In the case where $\kappa = 1$, we obtain $\tilde{\kappa} = 1$, $\lambda_i = \tilde{\lambda}_i$, and $\tilde{\varphi} = 2\pi - \varphi$, and we choose $0 \leq \varphi \leq \pi$ as the canonical state. If the three-tangle is nonzero, then the three-qubit state is of GHZ class. If three-tangle is zero and the reduced density matrices $\rho_A \equiv \mathrm{Tr}_{BC}|\psi\rangle\langle\psi|$, ρ_B, and ρ_C have rank two, then the state $|\psi\rangle$ is a W-class state.

We consider that the most general local operator

$$A' = e^{i\theta_1}a|0\rangle\langle 0| + e^{i\theta_2}b|0\rangle\langle 1| + e^{i\theta_3}c|1\rangle\langle 0| + e^{i\theta_4}d|1\rangle\langle 1| \quad (a,b,c,d \geq 0) \tag{5.22}$$

acts on the first qubit which transforms the state given by Eq. (5.13) into

$$|\psi'\rangle = \frac{1}{\sqrt{p_A}}(A' \otimes I_B \otimes I_C)|\psi\rangle \tag{5.23}$$

with a probability of $p_A = \langle\psi|\,(A'^{\dagger}A' \otimes I_B \otimes I_C)\,|\psi\rangle$ and then the resulting state can be brought into the canonical form

$$\begin{aligned}|\psi'\rangle =& \frac{1}{\sqrt{p_A}}\Bigg[\lambda_0 \frac{\left|e^{i(\theta_1+\theta_4)}ad - e^{i(\theta_2+\theta_3)}bc\right|}{\sqrt{b^2+d^2}}\,|000\rangle \\ &+ \left(\lambda_0 \frac{e^{i(\theta_1-\theta_2)}ab + e^{i(\theta_3-\theta_4)}cd}{\sqrt{b^2+d^2}} + e^{i\varphi}\lambda_1\sqrt{b^2+d^2}\right)|100\rangle \\ &+ \sqrt{b^2+d^2}\Big(\lambda_2\,|101\rangle + \lambda_3\,|110\rangle + \lambda_4\,|111\rangle\Big)\Bigg],\end{aligned} \tag{5.24}$$

by local unitary transformations. Using the fact that the action of the POVM operator

$$\begin{aligned}A =& \frac{\left|e^{i(\theta_1+\theta_4)}ad - e^{i(\theta_2+\theta_3)}bc\right|}{\sqrt{b^2+d^2}}\,|0\rangle\langle 0| \\ &+ \left(\frac{e^{i(\theta_1-\theta_2)}ab + e^{i(\theta_3-\theta_4)}cd}{\sqrt{b^2+d^2}}\right)|1\rangle\langle 0| + \sqrt{b^2+d^2}\,|1\rangle\langle 1|\end{aligned} \tag{5.25}$$

on the first qubit transforms the state (5.13) into the state (5.24) with the same probability p_A, we conclude that the operators given by Eqs. (5.22) and (5.25) are in the same equivalence class ($A' \equiv A$). Hence, the canonical local operator on the first qubit can be taken as

$$A = a_1\,|0\rangle\langle 0| + e^{i\alpha_1}c_1\,|1\rangle\langle 0| + d_1\,|1\rangle\langle 1| \quad (a_1, c_1, d_1 \geq 0,\ \ 0 \leq \alpha_1 < 2\pi) \tag{5.26}$$

The fact that the probability of any transformation should be less than or equal to unity ($P = \langle\psi|A^{\dagger}A\,|\psi\rangle \leq 1$) implies that the eigenvalues of $A^{\dagger}A$ should be less than or equal to one. This leads to a constraint on the parameters given by

$$a_1^2 + c_1^2 + d_1^2 + \sqrt{\left((a_1-d_1)^2 + c_1^2\right)\left((a_1+d_1)^2 + c_1^2\right)} \leq 2. \tag{5.27}$$

The parameters a_1, c_1, and d_1 satisfying the inequality (5.27) and α_1 uniquely determine the equivalence class. Now we consider that the most general transformation

$$B' = e^{i\beta_1}q\,|0\rangle\langle 0| + e^{i\beta_2}r\,|0\rangle\langle 1| + e^{i\beta_3}s\,|1\rangle\langle 0| + e^{i\beta_4}t\,|1\rangle\langle 1| \quad (q, r, s, t \geq 0) \tag{5.28}$$

is performed on the second qubit and the state given by Eq. (5.13) is transformed into the state

$$|\psi''\rangle = \frac{1}{\sqrt{p_B}}(I_A \otimes B' \otimes I_C)\,|\psi\rangle \tag{5.29}$$

with probability $p_B = \langle\psi|\,(I_A \otimes B'^{\dagger} B' \otimes I_C)\,|\psi\rangle$. Then the resulting state can be brought into the canonical form

$$\begin{aligned}|\psi''\rangle =& \frac{1}{\sqrt{p_B}}\Bigg[\lambda_0\sqrt{q^2+s^2}\,|000\rangle \\ &+\left(\lambda_1 e^{i\varphi}\sqrt{q^2+s^2}+\lambda_3\frac{e^{i(\beta_2-\beta_1)}qr+e^{i(\beta_4-\beta_3)}st}{\sqrt{q^2+s^2}}\right)|100\rangle \\ &+\left|\lambda_2\sqrt{q^2+s^2}+\lambda_4\frac{e^{i(\beta_2-\beta_1)}qr+e^{i(\beta_4-\beta_3)}st}{\sqrt{q^2+s^2}}\right||101\rangle \\ &+\left|\frac{e^{i(\beta_1+\beta_4)}qt-e^{i(\beta_2+\beta_3)}rs}{\sqrt{q^2+s^2}}\right|\Big(\lambda_3\,|110\rangle+\lambda_4\,|111\rangle\Big)\Bigg],\end{aligned} \tag{5.30}$$

by local unitary transformations. Using the result that the operator

$$\begin{aligned}B =& \sqrt{q^2+s^2}\,|0\rangle\langle 0| + \left(\frac{e^{i(\beta_2-\beta_1)}qr+e^{i(\beta_4-\beta_3)}st}{\sqrt{q^2+s^2}}\right)|0\rangle\langle 1| \\ &+\left|\frac{e^{i(\beta_1+\beta_4)}qt-e^{i(\beta_2+\beta_3)}rs}{\sqrt{q^2+s^2}}\right||1\rangle\langle 1|\end{aligned} \tag{5.31}$$

on the second qubit transforms the state given by Eq. (5.13) into the state given by Eq. (5.30) with the same probability p_B we conclude that B and B' are in the same equivalence class ($B' \equiv B$). Hence, the canonical operator on the second qubit is given by

$$B = a_2\,|0\rangle\langle 0| + e^{i\alpha_2} b_2\,|0\rangle\langle 1| + d_2\,|1\rangle\langle 1| \quad (a_2, b_2, d_2 \geq 0,\ \ 0 \leq \alpha_2 < 2\pi). \tag{5.32}$$

It can similarly be shown that the canonical operator on the third qubit is of the form

$$C = a_3\,|0\rangle\langle 0| + e^{i\alpha_3} b_3\,|0\rangle\langle 1| + d_3\,|1\rangle\langle 1| \quad (a_3, b_3, d_3 \geq 0,\ \ 0 \leq \alpha_3 < 2\pi). \tag{5.33}$$

The condition that eigenvalues of $B^{\dagger}B$ and $C^{\dagger}C$ should be less than or equal to one leads to the constraint

$$a_i^2 + b_i^2 + d_i^2 + \sqrt{\left((a_i-d_i)^2+b_i^2\right)\left((a_i+d_i)^2+b_i^2\right)} \leq 2, \quad i = 2,3. \tag{5.34}$$

The parameters a_i, b_i, and d_i satisfying the inequality (5.34) and α_i uniquely determine the equivalence classes.

5.3 Optimal Manipulation and Concentration of GHZ States

In this section, we discuss the optimal manipulation of three-qubit pure states using *one successful branch protocol* (OSBP). In this protocol, we maximize local success probabilities and the state is either transformed into the desired one or else the particle is disentangled from other particles. Maximization of the probability imposes some conditions on the operators. Two operators A and $\tilde{A} = \alpha A$ (α: constant) make the same transformations on states

$$|\psi'\rangle = \frac{1}{\sqrt{p}} A |\psi\rangle = \frac{1}{\sqrt{\tilde{p}}} \tilde{A} |\psi\rangle , \tag{5.35}$$

with probabilities p and $\tilde{p} = |\alpha| p$, respectively. By choosing α, it is possible to increase or decrease the transformation probability for any transformation. However, the value of α and hence the probability is restricted by the condition that the greater eigenvalue of $A^\dagger A$ can not exceed one. Hence, the transformation probability is maximum when the greater eigenvalue is one, i.e., $\det(I_A - A^\dagger A) = 0$. In this case, for a two-outcome POVM with elements A and $\bar{A}$, satisfying $\bar{A}^\dagger \bar{A} + A^\dagger A = I_A$, the rank of $\bar{A}$ is one, and hence the state $(\bar{A} \otimes I_B \otimes I_C) |\psi\rangle$ is a product state in the first particle. Similar considerations can be done for the measurement of the second and third particles with the POVM elements B, $\bar{B}$ and C, $\bar{C}$ satisfying $\bar{B}^\dagger \bar{B} + B^\dagger B = I_B$ and $\bar{C}^\dagger \bar{C} + C^\dagger C = I_C$. If the transformation $(A \otimes B \otimes C) |\psi\rangle$ gives the desired state then the probability maximization of this transformation requires

$$\det(I_A - A^\dagger A) = 0,\ \det(I_B - B^\dagger B) = 0,\ \det(I_C - C^\dagger C) = 0, \tag{5.36}$$

which imply that the state is either transformed into the desired state or otherwise disentangled (i.e., we are using OSBP).

The problem is the optimal transformation of the generic state given by Eq. (5.13) into the state

$$|\psi'\rangle = \lambda_0' |000\rangle + \lambda_1' e^{i\varphi'} |100\rangle + \lambda_2' |101\rangle + \lambda_3' |110\rangle + \lambda_4' |111\rangle , \quad \left(\lambda_i' \geq 0\right), \tag{5.37}$$

by the action of all three parties

$$
\begin{aligned}
|\psi'\rangle =& \frac{(A\otimes B\otimes C)\,|\psi\rangle}{\sqrt{P}} \\
=& \frac{1}{\sqrt{P}}\Bigg[\lambda_0 a_1 a_2 a_3\,|000\rangle + \Big((\lambda_0 c_1 e^{i\alpha_1} + \lambda_1 e^{i\varphi} d_1) a_2 a_3 + \lambda_2 d_1 a_2 b_3 e^{i\alpha_3} \\
& + \lambda_3 d_1 b_2 e^{i\alpha_2} a_3 + \lambda_4 d_1 b_2 b_3 e^{i(\alpha_2+\alpha_3)}\Big)\,|100\rangle \\
& + \Big|(\lambda_2 d_1 a_2 d_3 + \lambda_4 d_1 b_2 e^{i\alpha_2} d_3)\Big|\,|101\rangle \\
& + \Big|(\lambda_3 d_1 d_2 a_3 + \lambda_4 d_1 d_2 b_3 e^{i\alpha_3})\Big|\,|110\rangle + \lambda_4 d_1 d_2 d_3\,|111\rangle\Bigg],
\end{aligned}
\tag{5.38}
$$

where $P = \langle\psi|A^\dagger A\otimes B^\dagger B\otimes C^\dagger C\,|\psi\rangle$. We now impose that the transformed state given by Eq. (5.38) is the GHZ state and obtain the conditions

$$
\begin{aligned}
& c_1 e^{i\alpha_1} = \frac{\lambda_2\lambda_3 - e^{i\varphi}\lambda_1\lambda_4}{\lambda_0\lambda_4} d_1, \quad b_2 = \frac{\lambda_2}{\lambda_4} a_2, \quad b_3 = \frac{\lambda_3}{\lambda_4} a_3, \\
& \alpha_2 = \alpha_3 = \pi, \quad \lambda_0 a_1 a_2 a_3 = \lambda_4 d_1 d_2 d_3,
\end{aligned}
\tag{5.39}
$$

and the probability of success turns out to be

$$
P = 2\lambda_0^2 a_1^2 a_2^2 a_3^2. \tag{5.40}
$$

The maximization of the local probabilities given by Eq. (5.36) leads to the following constraints

$$
\begin{aligned}
(1-a_1^2)(1-d_1^2) =& c_1^2 = \frac{C_{bc}^2}{\tau} d_1^2, \\
(1-a_2^2)(1-d_2^2) =& b_2^2 = \frac{C_{ac}^2}{\tau} a_2^2, \\
(1-a_3^2)(1-d_3^2) =& b_3^2 = \frac{C_{ab}^2}{\tau} a_3^2.
\end{aligned}
\tag{5.41}
$$

The problem of optimal concentration of the GHZ state using OSBP is reduced to the problem of the maximization of the probability given by Eq. (5.40) subject to the constraints given by Eqs. (5.27), (5.34), (5.39) and (5.41). The solution for the most general case, where all bipartite entanglements C_{ab}, C_{ac}, and C_{bc} are nonzero, requires numerical calculations. However we find the analytical solutions in terms of the concurrences and the three tangle in the cases where at least one of the concurrences is zero. In these cases, complex phases in the states can be eliminated using local unitary transformations. For example, in the case $C_{ab} = 0$ ($\lambda_3 = 0$) the general state turns out to be

$$
|\psi\rangle = \lambda_0\,|000\rangle + e^{i\varphi}\lambda_1\,|100\rangle + \lambda_2\,|101\rangle + \lambda_4\,|111\rangle\,. \tag{5.42}
$$

However, the complex phase can be eliminated by the local unitary transformations

$$U = (|0\rangle\langle 0| + e^{-i\varphi}|1\rangle\langle 1|) \otimes I_B \otimes (|0\rangle\langle 0| + e^{i\varphi}|1\rangle\langle 1|). \tag{5.43}$$

(*i*) *States with no bipartite entanglement* ($C_{ab} = C_{ac} = C_{bc} = 0$): In this case the canonical form of the state is given by

$$|\psi\rangle = \lambda_0|000\rangle + \lambda_4|111\rangle \quad (\lambda_0 \geq \lambda_4). \tag{5.44}$$

The local operation $\frac{\lambda_4}{\lambda_0}|0\rangle\langle 0| + |1\rangle\langle 1|$ on one of the qubits is sufficient to obtain the GHZ states with the maximum success probability given by

$$P_{max} = 1 - \sqrt{1-\tau}. \tag{5.45}$$

Since the three tangle, τ, is an entanglement monotone, so is P_{max}.

(*ii*) *States with only one nonzero bipartite entanglement* (e.g., $C_{ab} = C_{ac} = 0$, $C_{bc} \neq 0$): In this case the canonical form of the state is given by

$$|\psi\rangle = \lambda_0|000\rangle + \lambda_1|100\rangle + \lambda_4|111\rangle. \tag{5.46}$$

We find that the concentration of the GHZ state depends solely on the party A: The parties B and/or C cannot obtain the GHZ state, but only party A has this privilege. The solutions for the local operators are found to be

$$A = \frac{\sqrt{P_{max}}}{\sqrt{2}\lambda_0}|0\rangle\langle 0| - \frac{\lambda_1\sqrt{P_{max}}}{\sqrt{2}\lambda_0\lambda_4}|1\rangle\langle 0| + \frac{\sqrt{P_{max}}}{\sqrt{2}\lambda_4}|1\rangle\langle 1|,\ B = I,\ C = I, \tag{5.47}$$

where the maximum probability is found to be

$$P_{max} = 1 - \sqrt{1-\tau}. \tag{5.48}$$

We note that if the bipartite entanglement exists only between two qubits, the concentration probability depends only on the three-tangle, not on the bipartite concurrence as given by Eq. (5.48). From symmetry it is straightforward to find the optimal distillation protocols for the other states in this class: The states

$$\begin{aligned} |\psi\rangle =& \lambda_0|000\rangle + \lambda_2|101\rangle + \lambda_4|111\rangle, \\ |\psi\rangle =& \lambda_0|000\rangle + \lambda_3|100\rangle + \lambda_4|111\rangle, \end{aligned} \tag{5.49}$$

can be optimally transformed to the GHZ state by the operators

$$\begin{aligned} A =& I,\ B = \frac{\sqrt{P_{max}}}{\sqrt{2}\lambda_0}|0\rangle\langle 0| - \frac{\lambda_2\sqrt{P_{max}}}{\sqrt{2}\lambda_0\lambda_4}|0\rangle\langle 1| + \frac{\sqrt{P_{max}}}{\sqrt{2}\lambda_4}|1\rangle\langle 1|,\ C = I, \\ A =& I,\ B = I,\ C = \frac{\sqrt{P_{max}}}{\sqrt{2}\lambda_0}|0\rangle\langle 0| - \frac{\lambda_3\sqrt{P_{max}}}{\sqrt{2}\lambda_0\lambda_4}|0\rangle\langle 1| + \frac{\sqrt{P_{max}}}{\sqrt{2}\lambda_4}|1\rangle\langle 1|, \end{aligned} \tag{5.50}$$

respectively, with the maximum probability given by Eq. (5.48).

(*iii*) *States with only one vanishing bipartite entanglement* (e.g., $C_{ab} = 0,\ C_{ac} \neq 0,\ C_{bc} \neq 0$): In this case, the canonical form of the state is given by

$$|\psi\rangle = \lambda_0 |000\rangle + \lambda_1 |100\rangle + \lambda_2 |101\rangle + \lambda_4 |111\rangle . \tag{5.51}$$

We find that the condition $C_{ab} = 0$ together with the maximization of the local probabilities given by Eq. (5.41) leads us to the result that $a_3 = d_3 = 1,\ b_3 = 0$, and any operation other than the unitary transformation on the third qubit reduces the probability of obtaining the GHZ state. By using the transformation given by Eq. (5.38), it can be shown that neither party A nor party B can concentrate the GHZ state alone, but the combined action of both parties A and B is necessary to obtain the GHZ state. We find the solution for the maximum success probability given by

$$P_{max} = 1 + \frac{C_{ac}C_{bc}}{\sqrt{\tau}} - \sqrt{\left(1 + \frac{C_{ac}C_{bc}}{\sqrt{\tau}}\right)^2 - \tau}. \tag{5.52}$$

To prove that no concentration protocol can give a greater probability, one needs to show that the inequality

$$P(|\psi\rangle) \geq \sum_i p_i P(|\psi_i\rangle) \tag{5.53}$$

is satisfied for any sequence of local quantum operations that transform $|\psi\rangle$ into $|\psi_i\rangle$ with a probability p_i. The right-hand side of the inequality (5.53) is the average probability of obtaining the GHZ state using several branches, whereas the left-hand side is the probability for OSBP. By taking into account that any POVM can be decomposed into a sequence of two-outcome POVMs [43], it is sufficient to show

$$P(|\psi\rangle) \geq p_1 P(|\psi_1\rangle) + p_2 P(|\psi_2\rangle), \tag{5.54}$$

where $|\psi_1\rangle$ and $|\psi_2\rangle$ are obtained by the most general POVMs on one of the qubits. We start with a two outcome POVM with operators

$$\begin{aligned} A_1 =& a_1 |0\rangle\langle 0| + c_1 e^{i\alpha_1} |1\rangle\langle 0| + d_1 |1\rangle\langle 1| , \\ A_2 =& f_1 |0\rangle\langle 0| + g_1 e^{i\beta_1} |1\rangle\langle 0| + h_1 |1\rangle\langle 1| , \end{aligned} \tag{5.55}$$

acting on the first qubit and satisfying $A_1^\dagger A_1 + A_2^\dagger A_2 = I$. The states

$$\begin{aligned}|\psi_1\rangle =& \frac{1}{\sqrt{p_1}}\Big(\lambda_0 a_1 |000\rangle + (\lambda_0 c_1 e^{i\alpha_1} + \lambda_1 d_1)|100\rangle \\ &+ \lambda_2 d_1 |101\rangle + \lambda_4 d_1 |111\rangle\Big), \\ |\psi_2\rangle =& \frac{1}{\sqrt{p_2}}\Big(\lambda_0 f_1 |000\rangle + (\lambda_0 g_1 e^{i\beta 1} + \lambda_1 h_1)|100\rangle \\ &+ \lambda_2 h_1 |101\rangle + \lambda_4 h_1 |111\rangle\Big),\end{aligned} \tag{5.56}$$

are obtained with probabilities $p_i = \langle\psi| A_i^\dagger A_i \otimes I_B \otimes I_C |\psi\rangle$. The complex phases in Eq. (5.56) can be eliminated by local unitary transformations and then OSBP is used on the states $|\psi_1\rangle$ and $|\psi_2\rangle$ to give the maximum probabilities $P(|\psi_1\rangle)$ and $P(|\psi_2\rangle)$ for the concentration of the GHZ state. To check if the inequality (5.54) is satisfied, we maximize $p_1 P(|\psi_1\rangle) + p_2 P(|\psi_2\rangle)$ and find that the maximum is obtained for $P(|\psi_1\rangle) = 0$ or $P(|\psi_2\rangle) = 0$, which proves that no concentration protocol can produce a higher probability of success than the OSBP we present.

5.4 Deterministic Transformations of Three-qubit Pure States

While deterministic transformations of some classes of multi-qubit states have been widely studied [50–52, 91], it is a general manner to keeping doors always open for alternative protocols. In the following, from this point of view, the problem of the deterministic transformations of three-qubit pure states will be discussed.

5.4.1 Deterministic transformation of W-type states

We will present the quantum operations to succeeding the transformations between three-qubit W-type states. The general form of three-qubit W-type states is given by

$$|\psi_W\rangle = \lambda_0 |000\rangle + \lambda_1 |100\rangle + \lambda_2 |101\rangle + \lambda_3 |110\rangle, \quad \Big(\lambda_i \geq 0, \sum_{i=0}^{3} \lambda_i^2 = 1\Big), \tag{5.57}$$

where the pure $|W\rangle$ state is of the form $|W\rangle = \frac{1}{\sqrt{3}}(|100\rangle + |010\rangle + |001\rangle)$. The state with the other set of parameters (second solution of $\det(T_0' = 0)$) is given by

$$|\psi_W'\rangle = \lambda_0' |000\rangle + \lambda_1' |100\rangle + \lambda_2' |101\rangle + \lambda_3' |110\rangle, \quad \Big(\lambda_i' \geq 0, \sum_{i=0}^{3} \lambda_i'^2 = 1\Big), \tag{5.58}$$

where $\lambda_i' = \lambda_i$ $(i = 0, 2, 3)$ and $\lambda_1' = \pm\lambda_1$ $(\varphi' = 0, \pi)$. The states $|\psi_W\rangle$ and $|\psi_W'\rangle$ are equivalent ($|\psi_W\rangle \sim |\psi_W'\rangle$) under local unitary transformations, i.e., local unitary

(LU) equivalent. Thus, unitary invariants are found to be $C_{ab} = 2\lambda_0\lambda_3 = 2\lambda_0'\lambda_3'$, $C_{ac} = 2\lambda_0\lambda_2 = 2\lambda_0'\lambda_2'$, $C_{bc} = 2\lambda_2\lambda_3 = 2\lambda_2'\lambda_3'$ and $J_5 = C_{ab}C_{ac}C_{bc}$.

We now start with a general three-qubit W-type state with all concurrences C_{ab}, C_{ac} and C_{bc} are nonzero. This state is given by [91]

$$|\chi\rangle = x_0|000\rangle + x_1|100\rangle + x_2|010\rangle + x_3|001\rangle. \tag{5.59}$$

The state (5.59) can be transformed into the canonical form

$$|\chi\rangle = x_1|000\rangle + x_0|100\rangle + x_2|110\rangle + x_3|101\rangle, \tag{5.60}$$

with the unitary transformation σ_x on the first qubit, that is $|0\rangle_a \leftrightarrow |1\rangle_a$. Then, Alice (party a) performs a general two-outcome measurement on her respective part of the state (5.60) with canonical measurement operators

$$M_k^{(a)} = \sqrt{p_k}\frac{\lambda_0}{x_1}|0\rangle\langle 0| + \sqrt{p_{2^{2-k}}}\sqrt{1-\frac{\lambda_0^2}{x_1^2}}|1\rangle\langle 0| + \sqrt{p_k}|1\rangle\langle 1|, \tag{5.61}$$
$$\Big(\sum_{k=1}^{2} {M_k^{(a)}}^\dagger M_k^{(a)} = I\Big).$$

We then have the states

$$\begin{aligned}|\psi_k\rangle =& \frac{(M_k^{(a)}\otimes I^{(b)}\otimes I^{(c)})|\chi\rangle}{\sqrt{p_k}} \\ =& \lambda_0|000\rangle + (-1)^{k-1}\lambda_1|100\rangle + x_3|101\rangle + x_2|110\rangle,\end{aligned} \tag{5.62}$$

for $k = 1,2$. Here, the probabilities are found to be $p_1 = (\lambda_1 + x_0)/2\lambda_1$ and $p_2 = (\lambda_1 - x_0)/2\lambda_1$. The unitary transformation $\sigma_z = |0\rangle\langle 0| - |1\rangle\langle 1|$ applied by the parties a, b and c will transform the state $|\psi_2\rangle$ into the state $|\psi_1\rangle$, i.e., $(\sigma_z\otimes\sigma_z\otimes\sigma_z)|\psi_2\rangle = |\psi_1\rangle$. What we finally have is the following:

$$\begin{aligned}&|\chi\rangle = x_0|000\rangle + x_1|100\rangle + x_2|010\rangle + x_3|001\rangle \\ &\downarrow \left\{\Big(M_k^{(a)}, p_k = \frac{1}{2} + \frac{(-1)^{k-1}x_0}{2\lambda_1}\Big)\right\}_{k=1,2}, \quad \Big(x_1 > \lambda_0, \quad x_0 < \lambda_1\Big) \\ &|\psi\rangle = \lambda_1|000\rangle + \lambda_0|100\rangle + x_2|010\rangle + x_3|001\rangle.\end{aligned} \tag{5.63}$$

We note that the state $|\psi\rangle$ given in Eq. (5.63) is LU equivalent with the state $|\psi_1\rangle$. The state $|\psi\rangle$ given in Eq. (5.63) is the most general state that can be deterministically obtained by the measurements on the first qubit (the party a) of the source state $|\chi\rangle$ given in Eq. (5.60). Furthermore, if Bob (party b) performs a two-outcome

measurement on the second qubit of the state $|\psi\rangle$ given in Eq. (5.63) with canonical measurement operators given by

$$M_k^{(b)} = \sqrt{p_k}\,|0\rangle\langle 0| + \sqrt{p_{2^{2-k}}}\sqrt{1-\frac{\mu_2^2}{x_2^2}}\,|0\rangle\langle 1| + \sqrt{p_k}\frac{\mu_2}{x_2}\,|1\rangle\langle 1|, \qquad \Big(\sum_{k=1}^{2} {M_k^{(b)}}^\dagger M_k^{(b)} = I\Big), \tag{5.64}$$

then the resulting state will be one of the states

$$\begin{aligned} |\phi_k\rangle =& \frac{(I^{(a)}\otimes M_k^{(b)}\otimes I^{(c)})\,|\psi\rangle}{\sqrt{p_k}} \\ =& (-1)^{k-1}\mu_0\,|000\rangle + \lambda_0\,|100\rangle + \mu_2\,|010\rangle + x_3\,|001\rangle, \end{aligned} \tag{5.65}$$

where the probabilities are found to be $p_1 = (\mu_0+\lambda_1)/2\mu_0$ and $p_2 = (\mu_0-\lambda_1)/2\mu_0$, respectively. The states $|\phi_1\rangle$ and $|\phi_2\rangle$ are LU equivalent, and the unitary transformation $-\sigma_z$ applied by party a, and unitary transformation σ_z applied by party b will transform the state $|\phi_2\rangle$ into the state $|\phi_1\rangle$, i.e., $(-\sigma_z\otimes\sigma_z\otimes I)\,|\phi_2\rangle = |\phi_1\rangle$. What we finally have is the following:

$$\begin{aligned} &|\psi\rangle = \lambda_1\,|000\rangle + \lambda_0\,|100\rangle + x_2\,|010\rangle + x_3\,|001\rangle \\ &\downarrow \left\{\Big(M_k^{(b)},\, p_k = \frac{1}{2} + \frac{(-1)^{k-1}\lambda_1}{2\mu_0}\Big)\right\}_{k=1,2}, \quad \Big(x_2 > \mu_2, \quad \lambda_1 < \mu_0\Big) \\ &|\phi\rangle = \mu_0\,|000\rangle + \lambda_0\,|100\rangle + \mu_2\,|010\rangle + x_3\,|001\rangle. \end{aligned} \tag{5.66}$$

Alice and Bob (the parties a and b) together can transform the state $|\chi\rangle$ given in Eq. (5.60) to a state $|\phi\rangle$ given in Eq. (5.66) with unit probability (by combining the Eq. (5.63) and Eq. (5.66)). In a similar way, we can find how Charlie (party c) transforms a W-type state into another W-type state, i.e., we start with the general measurement operators and impose conditions for deterministic transformations and solve for the equations. One can find that a two-outcome measurement on the third qubit (of the state $|\psi\rangle$ given in Eq. (5.66)) with measurement operators

$$M_k^{(c)} = \sqrt{p_k}\,|0\rangle\langle 0| + \sqrt{p_{2^{2-k}}}\sqrt{1-\frac{\alpha_3^2}{x_3^2}}\,|0\rangle\langle 1| + \sqrt{p_k}\frac{\alpha_3}{x_3}\,|1\rangle\langle 1|, \qquad \Big(\sum_{k=1}^{2} {M_k^{(c)}}^\dagger M_k^{(c)} = I\Big), \tag{5.67}$$

gives one of the states

$$\begin{aligned} |\varphi_k\rangle =& \frac{(I^{(a)}\otimes I^{(b)}\otimes M_k^{(c)})\,|\phi\rangle}{\sqrt{p_k}} \\ =& (-1)^{k-1}\alpha_0\,|000\rangle + \lambda_0\,|100\rangle + \mu_2\,|010\rangle + \alpha_3\,|001\rangle, \end{aligned} \tag{5.68}$$

with probabilities $p_1 = (\alpha_0 + \mu_0)/2\alpha_0$ and $p_2 = (\alpha_0 - \mu_0)/2\alpha_0$, respectively. The states $|\varphi_1\rangle$ and $|\varphi_2\rangle$ are LU equivalent, and the unitary transformation $-\sigma_z$ on the first qubit, and unitary transformation σ_z on the second qubit transform the state $|\varphi_2\rangle$ into the state $|\varphi_1\rangle$, i.e., $(-\sigma_z \otimes \sigma_z \otimes I)|\varphi_2\rangle = |\varphi_1\rangle$. What we finally have is the following:

$$\begin{aligned}
&|\phi\rangle = \mu_0|000\rangle + \lambda_0|100\rangle + \mu_2|010\rangle + x_3|001\rangle \\
&\downarrow \left\{ \left(M_k^{(c)}, p_k = \frac{1}{2} + \frac{(-1)^{k-1}\mu_0}{2\alpha_0}\right) \right\}_{k=1,2}, \quad \left(x_3 > \alpha_3, \quad \mu_0 < \alpha_0\right) \\
&|\varphi\rangle = \alpha_0|000\rangle + \lambda_0|100\rangle + \mu_2|010\rangle + \alpha_3|001\rangle .
\end{aligned} \tag{5.69}$$

To sum up, the solutions for the deterministic transformations of three-qubit pure W-type state by the measurements on all three qubits are given by

$$\begin{aligned}
&|\chi\rangle = x_0|000\rangle + x_1|100\rangle + x_2|010\rangle + x_3|001\rangle \\
&\downarrow \left\{ \left(M_k^{(a)}, p_k = \frac{1}{2} + \frac{(-1)^{k-1}x_0}{2\lambda_1}\right) \right\}_{k=1,2}, \quad \left(x_1 > \lambda_0, \quad x_0 < \lambda_1\right) \\
&|\psi\rangle = \lambda_1|000\rangle + \lambda_0|100\rangle + x_2|010\rangle + x_3|001\rangle \\
&\downarrow \left\{ \left(M_k^{(b)}, p_k = \frac{1}{2} + \frac{(-1)^{k-1}\lambda_1}{2\mu_0}\right) \right\}_{k=1,2}, \quad \left(x_2 > \mu_2, \quad \lambda_1 < \mu_0\right) \\
&|\phi\rangle = \mu_0|000\rangle + \lambda_0|100\rangle + \mu_2|010\rangle + x_3|001\rangle \\
&\downarrow \left\{ \left(M_k^{(c)}, p_k = \frac{1}{2} + \frac{(-1)^{k-1}\mu_0}{2\alpha_0}\right) \right\}_{k=1,2}, \quad \left(x_3 > \alpha_3, \quad \mu_0 < \alpha_0\right) \\
&|\varphi\rangle = \alpha_0|000\rangle + \lambda_0|100\rangle + \mu_2|010\rangle + \alpha_3|001\rangle .
\end{aligned} \tag{5.70}$$

In this subsection, using the canonical operators, we showed that any given initial W-type state $|\chi\rangle$, Alice (party a) can deterministically transforms $|\chi\rangle$ into states $|\psi\rangle$. The party a can only decreases the value of the coefficient of $|100\rangle$ and increases the value of the coefficient of $|000\rangle$ while it can not deterministically change the coefficients of the basis vectors $|010\rangle$ and $|001\rangle$ by LOCC. The party b (Bob), on the other hand, decreases the value of the coefficient of $|010\rangle$ and increases the value of the coefficient of $|000\rangle$, but can not change the coefficients of the vectors $|100\rangle$ and $|001\rangle$ in a deterministic transformation. The party c (Charlie) decreases the value of the coefficient of $|001\rangle$ and increases the value of the coefficient of $|000\rangle$, but can not change the coefficients of the vectors $|100\rangle$ and $|010\rangle$. This implies that the states that can be deterministically obtained by the cooperation of the three parties are the states with greater coefficients of the basis vectors $|100\rangle$, $|010\rangle$ and $|001\rangle$. These results are consistent with the fact that local operations on some particles can not increase (may decrease) the average entanglement between the particle and its environment.

5.4.2 Deterministic transformation of GHZ-type states

After discussing the deterministic transformations of three-qubit W-type states in the previous subsection, we now go ahead with the deterministic transformations of three-qubit GHZ-type states. Any state in the GHZ class can be written as

$$|\psi_{GHZ}\rangle = \lambda_0 |000\rangle + \lambda_1 e^{i\varphi} |100\rangle + \lambda_2 |101\rangle + \lambda_3 |110\rangle + \lambda_4 |111\rangle, \tag{5.71}$$

where $\lambda_0\lambda_4 \neq 0$, $\lambda_i \geq 0$, $\sum_{i=0}^{4} \lambda_i^2 = 1$ and $\varphi \in [0,\pi]$. The state with the other set of parameters (second solution of $\det(T_0' = 0)$) is given by

$$|\psi'_{GHZ}\rangle = \lambda'_0 |000\rangle + \lambda'_1 e^{i\varphi'} |100\rangle + \lambda'_2 |101\rangle + \lambda'_3 |110\rangle + \lambda'_4 |111\rangle, \tag{5.72}$$

where $\lambda'_0\lambda'_4 \neq 0$, $\lambda'_i \geq 0$, $\sum_{i=0}^{4} \lambda_i'^2 = 1$. We also have

$$\begin{aligned}
\lambda'_0 = \frac{\lambda_0}{\kappa}, \quad \lambda'_2 = \lambda_2\kappa, \quad \lambda'_3 = \lambda_3\kappa, \quad \lambda'_4 = \lambda_4\kappa, \quad \kappa \equiv \left[\frac{\tau + C_{BC}^2}{4(\lambda_2^2+\lambda_4^2)(\lambda_3^2+\lambda_4^2)}\right]^{1/2}, \\
\lambda'_1 e^{i\varphi'} = \frac{\lambda_1}{\kappa}\left[\frac{\lambda_4^2(\lambda_2^2+\lambda_3^2+\lambda_4^2) - \lambda_2^2\lambda_3^2}{(\lambda_2^2+\lambda_4^2)(\lambda_3^2+\lambda_4^2)}\cos(\varphi) - i\sin(\varphi)\right] \\
+ \frac{\lambda_2\lambda_3\lambda_4(\lambda_0^2+\lambda_1^2-\lambda_2^2-\lambda_3^2-\lambda_4^2)}{\kappa(\lambda_2^2+\lambda_4^2)(\lambda_3^2+\lambda_4^2)}.
\end{aligned} \tag{5.73}$$

We originally have the three-qubit pure GHZ state given by Eq. (5.6). The unitary invariants appertaining to this state ($(|000\rangle + |111\rangle)/\sqrt{2}$) are $\tau = 1$, $C_{ab} = C_{ac} = C_{bc} = 0$ and $J_5 = 0$. We are going to find the quantum operations for the deterministic transformations of the GHZ state given by Eq. (5.6) by performing measurements on one party only, two parties only and all three parties.

The first case, measurements on one party only: First, we start with the general two-outcome measurement on the first qubit of the state $(|000\rangle + |111\rangle)/\sqrt{2}$ with the measurement operators given by

$$\begin{aligned}
M_1^{(a)} =& \lambda_0 |0\rangle\langle 0| + \lambda_1 |1\rangle\langle 0| + \lambda_4 |1\rangle\langle 1|, \\
M_2^{(a)} =& \lambda'_0 |0\rangle\langle 0| - \lambda'_1 |1\rangle\langle 0| + \lambda'_4 |1\rangle\langle 1|, \quad \Big(\sum_{k=1}^{2} M_k^{(a)\dagger} M_k^{(a)} = I\Big).
\end{aligned} \tag{5.74}$$

The state after the measurement will be one of the states

$$\begin{aligned}
|\psi_1\rangle &= \lambda_0 |000\rangle + \lambda_1 |100\rangle + \lambda_4 |111\rangle, \quad p_1 = 1/2, \\
|\psi_2\rangle &= \lambda'_0 |000\rangle - \lambda'_1 |100\rangle + \lambda'_4 |111\rangle, \quad p_2 = 1/2,
\end{aligned} \tag{5.75}$$

where $p_k = \langle GHZ| M_k^{(a)\dagger} M_k^{(a)} |GHZ\rangle$ for $k = 1,2$. The states given in Eq. (5.75) are LU equivalent, and the local unitary transformations $U^{(a)} = \frac{1}{\sqrt{\lambda_0^2+\lambda_1^2}}\big(-\lambda_1 I - i\lambda_0\sigma_y\big)$,

$U^{(b)} = -\sigma_x$ and $U^{(c)} = i\sigma_y$, on qubits a, b and c, respectively, will transform the state $|\psi_2\rangle$ into the state $|\psi_1\rangle$, i.e., $(U^{(a)} \otimes U^{(b)} \otimes U^{(c)})|\psi_2\rangle = |\psi_1\rangle$. Here, $\sigma_x = |0\rangle\langle 1| + |1\rangle\langle 0|$, $\sigma_y = -i|0\rangle\langle 1| + i|1\rangle\langle 0|$ and $\sigma_z = |0\rangle\langle 0| - |1\rangle\langle 1|$ are the Pauli matrices. The second and the third qubits will be entangled ($C_{bc} \neq 0$), while the first qubit will stay unentangled with the other two qubits (C_{ab}=C_{ac}=0) by the measurement on the first qubit. This is consistent with the fact that any local operation on any particle cannot increase average entanglement between the particle and other systems.

Second, we start with the general two-outcome measurement on the second qubit of the GHZ state with the measurement operators given by

$$\begin{aligned} M_1^{(b)} =& \lambda_0|0\rangle\langle 0| + \lambda_2|0\rangle\langle 1| + \lambda_4|1\rangle\langle 1|, \\ M_2^{(b)} =& \lambda_0'|0\rangle\langle 0| - \lambda_2'|0\rangle\langle 1| + \lambda_4'|1\rangle\langle 1|, \quad \Big(\sum_{k=1}^{2} {M_k^{(b)}}^\dagger M_k^{(b)} = I\Big). \end{aligned} \tag{5.76}$$

The state after the measurement will be one of the states

$$\begin{aligned} |\psi_1\rangle &= \lambda_0|000\rangle + \lambda_2|101\rangle + \lambda_4|111\rangle, \quad p_1 = 1/2, \\ |\psi_2\rangle &= \lambda_0'|000\rangle - \lambda_2'|101\rangle + \lambda_4'|111\rangle, \quad p_2 = 1/2, \end{aligned} \tag{5.77}$$

where $p_k = \langle GHZ|{M_k^{(b)}}^\dagger M_k^{(b)}|GHZ\rangle$ for $k = 1,2$. The states given in Eq. (5.77) are LU equivalent, and the local unitary transformations $U^{(a)} = -\sigma_x$, $U^{(b)} = \frac{1}{\sqrt{\lambda_2^2+\lambda_4^2}}\big(\lambda_2 I - i\lambda_4\sigma_y\big)$ and $U^{(c)} = i\sigma_y$, on the qubits a , b and c, respectively, will transform the state $|\psi_2\rangle$ into the state $|\psi_1\rangle$. The first and the third qubits will be entangled ($C_{ac} \neq 0$), while the second qubit will stay unentangled with the other two qubits (C_{ab}=C_{bc}=0) by the measurement on the second qubit. This is consistent with the fact that any local operation on any particle cannot increase average entanglement between the particle and other systems.

Third, similarly, we start with the general two-outcome measurement on the third qubit of the GHZ state with canonical measurement operators given by

$$\begin{aligned} M_1^{(c)} =& \lambda_0|0\rangle\langle 0| + \lambda_3|0\rangle\langle 1| + \lambda_4|1\rangle\langle 1|, \\ M_2^{(c)} =& \lambda_0'|0\rangle\langle 0| - \lambda_3'|0\rangle\langle 1| + \lambda_4'|1\rangle\langle 1|, \quad \Big(\sum_{k=1}^{2} {M_k^{(c)}}^\dagger M_k^{(c)} = I\Big). \end{aligned} \tag{5.78}$$

The state after the measurement will be one of the states

$$\begin{aligned} |\psi_1\rangle &= \lambda_0|000\rangle + \lambda_3|110\rangle + \lambda_4|111\rangle, \quad p_1 = 1/2, \\ |\psi_2\rangle &= \lambda_0'|000\rangle - \lambda_3'|110\rangle + \lambda_4'|111\rangle, \quad p_2 = 1/2, \end{aligned} \tag{5.79}$$

where $p_k = \langle GHZ| {M_k^{(c)}}^\dagger M_k^{(c)} |GHZ\rangle$ for $k = 1,2$. The states given in Eq. (5.79) are LU equivalent, and the local unitary transformations $U^{(a)} = -\sigma_x$, $U^{(b)} = i\sigma_y$ and $U^{(c)} = \frac{1}{\sqrt{\lambda_3^2+\lambda_4^2}}\left(\lambda_3 I - i\lambda_4\sigma_y\right)$, on the qubits a, b and c, respectively, will transform the state $|\psi_2\rangle$ into the state $|\psi_1\rangle$. The first and the second qubits will be entangled ($C_{ab} \neq 0$), while the third qubit will stay unentangled with the other two qubits (C_{ac}=C_{bc}=0) by the measurement on the third qubit. This is consistent with the fact that any local operation on any particle cannot increase average entanglement between the particle and other systems. Similarly, three-tangle cannot be increased by any local operation. In the case discussed above the initial three tangle is τ_i=1 and the final three tangle is τ_f=$4\lambda_0^2\lambda_4^2 < 1$ as expected.

The second case, measurements on any two of three parties: We now discuss the deterministic transformations that can be obtained by the measurements of two parties. First, we will consider the first two qubits (parties a and b). As discussed above the state that can be obtained by measurements on the first qubit is given by $|\psi\rangle = \lambda_0|000\rangle + \lambda_1|100\rangle + \lambda_4|111\rangle$. In that case, if Alice applies the measurement operators

$$\begin{aligned} M_1^{(a)} &= \lambda_0|0\rangle\langle 0| + \lambda_1|1\rangle\langle 0| + \frac{1}{\sqrt{2}}|1\rangle\langle 1|, \\ M_2^{(a)} &= \lambda_0'|0\rangle\langle 0| - \lambda_1'|1\rangle\langle 0| + \frac{1}{\sqrt{2}}|1\rangle\langle 1|, \quad \Big(\sum_{k=1}^{2} {M_k^{(a)}}^\dagger M_k^{(a)} = I\Big), \end{aligned} \tag{5.80}$$

to the state $(|000\rangle + |111\rangle)/\sqrt{2}$ it gives the state $|\psi\rangle = \lambda_0|000\rangle + \lambda_1|100\rangle + \frac{1}{\sqrt{2}}|111\rangle$ deterministically. Then, if Bob applies the measurement operators

$$\begin{aligned} M_1^{(b)} &= \frac{\mu_0}{\sqrt{2}\lambda_0}|0\rangle\langle 0| + \mu_2|0\rangle\langle 1| + \mu_4|1\rangle\langle 1|, \\ M_2^{(b)} &= \frac{\mu_0'}{\sqrt{2}\lambda_0}|0\rangle\langle 0| - \mu_2'|0\rangle\langle 1| + \mu_4'|1\rangle\langle 1|, \quad \Big(\sum_{k=1}^{2} {M_k^{(b)}}^\dagger M_k^{(b)} = I\Big), \end{aligned} \tag{5.81}$$

to the state $\lambda_0|000\rangle + \lambda_1|100\rangle + \frac{1}{\sqrt{2}}|111\rangle$, one of the states

$$\begin{aligned} |\phi_1\rangle &= \mu_0|000\rangle + \mu_1|100\rangle + \mu_2|101\rangle + \mu_4|111\rangle, \\ |\phi_2\rangle &= \mu_0'|000\rangle + \mu_1'|100\rangle - \mu_2'|101\rangle + \mu_4'|111\rangle, \end{aligned} \tag{5.82}$$

is obtained with probabilities $p_1 = p_2 = 1/2$, respectively. The following unitary operators

$$U^{(a)} = \frac{-\mu_1\sigma_z + \mu_0\sigma_x}{\sqrt{\mu_0^2+\mu_1^2}}, \quad U^{(b)} = \frac{\mu_2 I - i\mu_4\sigma_y}{\sqrt{\mu_2^2+\mu_4^2}}, \quad U^{(c)} = -i\sigma_y, \tag{5.83}$$

applied to parties a, b and c, respectively, will transform the state $|\phi_2\rangle$ into the state $|\phi_1\rangle$ where $\mu_0' = \frac{\mu_0}{\kappa}$, $\mu_1' = \frac{\mu_1}{\kappa}$, $\mu_2' = \kappa\mu_2$, $\mu_4' = \kappa\mu_4$, and $\kappa^2 = \frac{\mu_0^2+\mu_1^2}{\mu_2^2+\mu_4^2}$. Consequently, deterministic transformations of the GHZ state by measurements by parties A and B can be summarized as follows:

$$\begin{aligned}
&|GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle) \\
&\downarrow \left\{ M_k^{(a)}, p_k = \frac{1}{2} \right\}_{k=1,2} \\
&|\psi\rangle = \lambda_0 |000\rangle + \lambda_1 |100\rangle + \frac{1}{\sqrt{2}} |111\rangle \\
&\downarrow \left\{ M_k^{(b)}, p_k = \frac{1}{2} \right\}_{k=1,2}, \quad \left(\lambda_0\mu_1 = \lambda_1\mu_0\right) \\
&|\phi\rangle = \mu_0 |000\rangle + \mu_1 |100\rangle + \mu_2 |101\rangle + \mu_4 |111\rangle .
\end{aligned} \tag{5.84}$$

We note that for the final state $|\phi\rangle$ given in Eq. (5.84), we have $C_{ab} = 0$, $C_{ac} \neq 0$ and $C_{bc} \neq 0$, i.e., there is no bipartite entanglement between the first and the second qubits.

Second, we will consider the first and third qubits (parties a and c). We know that if Alice applies the measurement operators given in (5.80) to GHZ state it gives the state $|\psi\rangle = \lambda_0 |000\rangle + \lambda_1 |100\rangle + \frac{1}{\sqrt{2}} |111\rangle$ deterministically. Then, if Charlie applies the measurement operators

$$\begin{aligned}
M_1^{(c)} &= \frac{\mu_0}{\sqrt{2}\lambda_0} |0\rangle\langle 0| + \mu_3 |0\rangle\langle 1| + \mu_4 |1\rangle\langle 1| , \\
M_2^{(c)} &= \frac{\mu_0'}{\sqrt{2}\lambda_0} |0\rangle\langle 0| - \mu_3' |0\rangle\langle 1| + \mu_4' |1\rangle\langle 1| , \quad \Big(\sum_{k=1}^{2} {M_k^{(c)}}^\dagger M_k^{(c)} = I \Big),
\end{aligned} \tag{5.85}$$

to the state $\lambda_0 |000\rangle + \lambda_1 |100\rangle + \frac{1}{\sqrt{2}} |111\rangle$, one of the states

$$\begin{aligned}
|\phi_1\rangle &= \mu_0 |000\rangle + \mu_1 |100\rangle + \mu_3 |110\rangle + \mu_4 |111\rangle , \\
|\phi_2\rangle &= \mu_0' |000\rangle + \mu_1' |100\rangle - \mu_3' |110\rangle + \mu_4' |111\rangle ,
\end{aligned} \tag{5.86}$$

is obtained with probabilities $p_1 = p_2 = 1/2$, respectively. The following unitary operators

$$U^{(a)} = \frac{-\mu_1\sigma_z + \mu_0\sigma_x}{\sqrt{\mu_0^2+\mu_1^2}}, \quad U^{(b)} = -i\sigma_y, \quad U^{(c)} = \frac{\mu_3 I - i\mu_4\sigma_y}{\sqrt{\mu_3^2+\mu_4^2}}, \tag{5.87}$$

applied to parties a, b and c, respectively, will transform the state $|\phi_2\rangle$ into the state $|\phi_1\rangle$ where $\mu_0' = \frac{\mu_0}{\kappa}$, $\mu_1' = \frac{\mu_1}{\kappa}$, $\mu_3' = \kappa\mu_3$, $\mu_4' = \kappa\mu_4$, and $\kappa^2 = \frac{\mu_0^2+\mu_1^2}{\mu_3^2+\mu_4^2}$. Consequently, deterministic transformations of the GHZ state with measurements by parties a and c

can be summarized as follows:

$$
\begin{aligned}
&|GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle) \\
&\downarrow \left\{ M_k^{(a)}, p_k = \frac{1}{2} \right\}_{k=1,2} \\
&|\psi\rangle = \lambda_0 |000\rangle + \lambda_1 |100\rangle + \frac{1}{\sqrt{2}} |111\rangle \\
&\downarrow \left\{ M_k^{(c)}, p_k = \frac{1}{2} \right\}_{k=1,2}, \quad \left(\lambda_0 \mu_1 = \lambda_1 \mu_0\right) \\
&|\phi\rangle = \mu_0 |000\rangle + \mu_1 |100\rangle + \mu_3 |110\rangle + \mu_4 |111\rangle .
\end{aligned}
\tag{5.88}
$$

We note that for the final state $|\phi\rangle$ given in Eq. (5.88), we have $C_{ac} = 0$, $C_{ab} \neq 0$ and $C_{bc} \neq 0$, i.e., the local measurements on the first and the third qubits create bipartite entanglements between the first and the second qubits, between the second and the third qubits, but do not create an entanglement between the first and the third qubits.

Third, we will consider the second and third qubits (parties b and c). As discussed above the state that can be obtained by measurements on the second qubit is given by $|\psi\rangle = \lambda_0 |000\rangle + \lambda_2 |101\rangle + \lambda_4 |111\rangle$. In that case, if Bob applies the measurement operators

$$
\begin{aligned}
M_1^{(b)} &= \frac{1}{\sqrt{2}} |0\rangle\langle 0| + \lambda_2 |1\rangle\langle 0| + \lambda_4 |1\rangle\langle 1| , \\
M_2^{(b)} &= \frac{1}{\sqrt{2}} |0\rangle\langle 0| - \lambda_2' |1\rangle\langle 0| + \lambda_4' |1\rangle\langle 1| , \quad \Big(\sum_{k=1}^{2} {M_k^{(b)}}^\dagger M_k^{(b)} = I \Big),
\end{aligned}
\tag{5.89}
$$

to GHZ state $(|000\rangle + |111\rangle)/\sqrt{2}$ it gives the state $|\psi\rangle = \frac{1}{\sqrt{2}} |000\rangle + \lambda_2 |101\rangle + \lambda_4 |111\rangle$ deterministically. Then, if Charlie applies the measurement operators

$$
\begin{aligned}
M_1^{(c)} &= \mu_0 |0\rangle\langle 0| + \frac{\mu_1}{\sqrt{2}\lambda_2} |0\rangle\langle 1| + \frac{\mu_4}{\sqrt{2}\lambda_4} |1\rangle\langle 1| , \\
M_2^{(c)} &= \mu_0' |0\rangle\langle 0| - \frac{\mu_1'}{\sqrt{2}\lambda_2} |0\rangle\langle 1| + \frac{\mu_4'}{\sqrt{2}\lambda_4} |1\rangle\langle 1| , \quad \Big(\sum_{k=1}^{2} {M_k^{(c)}}^\dagger M_k^{(c)} = I \Big),
\end{aligned}
\tag{5.90}
$$

to the state $\frac{1}{\sqrt{2}} |000\rangle + \lambda_2 |101\rangle + \lambda_4 |111\rangle$, then one of the states

$$
\begin{aligned}
|\phi_1\rangle &= \mu_0 |000\rangle + \mu_1 |100\rangle + \mu_2 |101\rangle + \mu_3 |110\rangle + \mu_4 |111\rangle , \\
|\phi_2\rangle &= \mu_0' |000\rangle - \mu_1' |100\rangle + \mu_2' |101\rangle - \mu_3' |110\rangle + \mu_4' |111\rangle ,
\end{aligned}
\tag{5.91}
$$

is obtained with probabilities $p_1 = p_2 = 1/2$, respectively. The following unitary operators

$$
U^{(a)} = -i\sigma_y, \quad U^{(b)} = \frac{\mu_2 \sigma_z + \mu_4 \sigma_x}{\sqrt{\mu_2^2 + \mu_4^2}}, \quad U^{(c)} = \frac{\mu_3 I - i\mu_4 \sigma_y}{\sqrt{\mu_3^2 + \mu_4^2}},
\tag{5.92}
$$

applied to parties a, b and c, respectively, will transform the state $|\phi_2\rangle$ into the state $|\phi_1\rangle$ where $\mu_0' = \frac{\mu_0}{\kappa}$, $\mu_2' = \kappa\mu_2$, $\mu_3' = \kappa\mu_3$ and $\mu_4' = \kappa\mu_4$. Consequently, deterministic transformations of the GHZ state by measurements by parties b and c can be summarized as follows:

$$\begin{aligned}
&|GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle) \\
&\downarrow \left\{ M_k^{(b)}, p_k = \frac{1}{2} \right\}_{k=1,2} \\
&|\psi\rangle = \frac{1}{\sqrt{2}}|000\rangle + \lambda_2|101\rangle + \lambda_4|111\rangle \\
&\downarrow \left\{ M_k^{(c)}, p_k = \frac{1}{2} \right\}_{k=1,2}, \quad \left(\lambda_4\mu_2 = \lambda_2\mu_4, \quad \lambda_4\mu_1 = \lambda_2\mu_3, \quad \mu_1\mu_4 = \mu_2\mu_3\right) \\
&|\phi\rangle = \mu_0|000\rangle + \mu_1|100\rangle + \mu_2|101\rangle + \mu_3|110\rangle + \mu_4|111\rangle
\end{aligned} \tag{5.93}$$

We note that for the final state $|\phi\rangle$ given in Eq. (5.93), we have $C_{bc} = 0$, $C_{ab} \neq 0$ and $C_{ac} \neq 0$, i.e., the party b is not entangled with the party c while the parties a and b, and a and c are entangled.

The third case, measurements on three parties: We will now discuss the deterministic transformations of the GHZ state by the measurements performed by all three parties. We know that the state $|\phi\rangle = \mu_0|000\rangle + \mu_1|100\rangle + \mu_2|101\rangle + \mu_4|111\rangle$ can be deterministically obtained by measurements on the first and second qubit (see Eq. (5.84)). Then, if we perform a two-outcome measurement by the measurement operators $M_1^{(c)} = a_3|0\rangle\langle 0| + b_3|0\rangle\langle 1| + d_3|1\rangle\langle 1|$ and $M_2^{(c)} = a_3'|0\rangle\langle 0| + b_3'|0\rangle\langle 1| + d_3'|1\rangle\langle 1|$ on the third qubit of the state given in Eq. (5.84), we obtain one of the states

$$\begin{aligned}
|\varphi_1\rangle =& \frac{1}{\sqrt{p_1}}\Big(a_3\mu_0|000\rangle + (a_3\mu_1 + b_3\mu_2)|100\rangle + b_3\mu_4|110\rangle \\
&+ d_3\mu_2|101\rangle + d_3\mu_4|111\rangle\Big) \\
=& \alpha_0|000\rangle + \alpha_1|100\rangle + \alpha_2|101\rangle + \alpha_3|110\rangle + \alpha_4|111\rangle,
\end{aligned} \tag{5.94}$$

$$\begin{aligned}
|\varphi_2\rangle =& \frac{1}{\sqrt{p_2}}\Big(a_3'\mu_0|000\rangle + (a_3'\mu_1 + b_3'\mu_2)|100\rangle + b_3'\mu_4|110\rangle \\
&+ d_3'\mu_2|101\rangle + d_3'\mu_4|111\rangle\Big) \\
=& \alpha_0'|000\rangle + \alpha_1'|100\rangle + \alpha_2'|101\rangle - \alpha_3'|110\rangle + \alpha_4'|111\rangle,
\end{aligned} \tag{5.95}$$

with probabilities p_1 and p_2, respectively. For the equivalence of the states $|\varphi_1\rangle$ and $|\varphi_2\rangle$ under local unitary transformations we impose the following conditions

$$\alpha_0' = \frac{\alpha_0}{\kappa}, \quad \alpha_2' = \alpha_2\kappa, \quad \alpha_3' = \alpha_3\kappa, \quad \alpha_4' = \alpha_4\kappa,$$
$$\alpha_1' = \frac{\alpha_1}{\kappa}\left[\frac{\alpha_4^2(\alpha_2^2+\alpha_3^2+\alpha_4^2)-\alpha_2^2\alpha_3^2}{(\alpha_2^2+\alpha_4^2)(\alpha_3^2+\alpha_4^2)}\right]+\frac{\alpha_2\alpha_3\alpha_4(\alpha_0^2+\alpha_1^2-\alpha_2^2-\alpha_3^2-\alpha_4^2)}{\kappa(\alpha_2^2+\alpha_4^2)(\alpha_3^2+\alpha_4^2)},$$
$$\kappa \equiv \left[\frac{\tau+C_{BC}^2}{4(\alpha_2^2+\alpha_4^2)(\alpha_3^2+\alpha_4^2)}\right]^{1/2}. \tag{5.96}$$

Also, for the case of deterministic transformation, it is required that $\sum_{k=1}^{2} {M_k^{(c)}}^\dagger M_k^{(c)} = I$. After some calculations, it is not hard to find that

$$C_{ab}C_{ac}C_{bc} = 0, \tag{5.97}$$

where $C_{ab} = 2\alpha_0\alpha_3$, $C_{ac} = 2\alpha_0\alpha_2$ and $C_{bc} = 2|\alpha_2\alpha_3 - \alpha_1\alpha_4|$ ($\alpha_0 \neq 0$). Hence, deterministic transformation from the pure GHZ to a GHZ-type state is possible if the target state satisfies the condition $C_{ab}C_{ac}C_{bc} = 0$. In other words, the GHZ state can not be deterministically transformed to a state with all bipartite entanglements are nonzero. All other states can be obtained with unit probability using the protocol we present.

5.5 Conclusion and Discussion

One of the main difficulties in the optimal manipulation problem of multipartite entangled states is that there are too many parameters coming from the general forms of the states and local operations. The use of equivalence classes significantly reduces the number of parameters as the states that can be transformed into each other by local unitary transformations are equal from the information theoretic point of view. In addition to the equivalence classes of states, we define the equivalence classes of local measurements such that local operations which transform states from one equivalence class into another with the same probability are equivalent. This approach does not only simplify the concentration problem, but also the results arise in terms of the local unitary invariants of the quantum states, namely, bipartite concurrences and the three-tangle.

We showed that any given initial W-type state $|\chi\rangle$, Alice (party a) can deterministically transforms $|\chi\rangle$ into states $|\psi\rangle$. The party a can only decreases the value of the

coefficient of $|100\rangle$ and increases the value of the coefficient of $|000\rangle$ while it can not deterministically change the coefficients of the basis vectors $|010\rangle$ and $|001\rangle$ by LOCC. The party b (Bob), on the other hand, decreases the value of the coefficient of $|010\rangle$ and increases the value of the coefficient of $|000\rangle$, but can not change the coefficients of the vectors $|100\rangle$ and $|001\rangle$ in a deterministic transformation. The party c (Charlie) decreases the value of the coefficient of $|001\rangle$ and increases the value of the coefficient of $|000\rangle$, but can not change the coefficients of the vectors $|100\rangle$ and $|010\rangle$. This implies that the states that can be deterministically obtained by the cooperation of the three parties are the states with greater coefficients of the basis vectors $|100\rangle$, $|010\rangle$ and $|001\rangle$.

We found that when there is no bipartite entanglement between particles ($C_{ab} = C_{ac} = C_{bc} = 0$), the optimal concentration of the GHZ state can be obtained by a single measurement on any one of the particles. In the case where there is bipartite entanglement between only two particles (e.g., $C_{ab} = C_{ac} = 0,\ C_{bc} \neq 0$), the optimal concentration of the GHZ state can only be done by a measurement on the particle which has no bipartite entanglement with the other two as given by Eq. (5.47). We obtained the interesting result that the maximum success probabilities for both cases, states with no bipartite entanglement and states with only one nonzero bipartite entanglement, depend only on the three-tangle as given by Eqs. (5.45) and (5.48). We also found that the optimal concentration of states with only one vanishing bipartite entanglement (e.g., $C_{ab} = 0,\ C_{ac} \neq 0,\ C_{bc} \neq 0$) can only be obtained by local measurements performed by the parties A and B with the maximum success probability given by Eq. (5.52). We may conclude that the use of equivalence classes of local measurement operators simplifies the optimal manipulation problem and may be used for the manipulation of other multi-partite states.

6. QUANTUM DISCORD

In this chapter, we introduce three procedures (all are fairly similar) to obtain the separable Werner states which have nonzero discord. The general idea is simple, and the following. In the first stage, we have a classically correlated state in our hand. Then, we perform some local operations, and our primary aim is to obtain these local operations, on subsystems of the initial state. After completing the whole process, a nonzero discord Werner state is acquired. This work can be thought of as an alternative method for the corresponding problem, i.e., obtaining the separable Werner states which have nonzero discord.

6.1 Introduction

Quantum discord, one of the most popular candidates for general non-classical correlations was first introduced by Zurek [4] and independently by Vedral [5], attempts to quantify non-classical correlations. It has been experimentally shown that separable states with nonzero quantum discord may have high probability of success in various quantum processes [96, 97]. It is accurate to conclude that the investigation of the behaviour of quantum correlations, i.e., quantum discord, under local operations is of paramount importance to hold key for the development of quantum computation and quantum information science. The reader is referred to [98, 99] for many other applications of quantum discord.

We are going to obtain separable Werner state, $\rho_{W_{1/3}}$, which has nonzero quantum discord and may have high probability of success in remote state preparation, by local operations where we initially have a classically correlated state, i.e., a zero discord state.

The present chapter is organised as follows. We expound on the basic notions which are necessary to follow the corresponding work (Section 6.2). We recall the definition of quantum discord, and local operations which are focussed on (Section 6.3). We then discuss local creation of quantum discord (Section 6.4). We first give the definition of

correlation rank (Subsection 6.4.1) which has crucial importance for our problem and set of (non)zero discord state (Subsection 6.4.2). We finally introduce three fairly similar methods to obtain the separable Werner states which have nonzero discord (Subsection 6.4.3).

6.2 Basic Notions

6.2.1 Shannon entropy & von Neumann entropy

In classical information theory the amount of information contained in a random variable X is quantified as the Shannon entropy. The Shannon entropy associated with probability distribution, p_x, is defined by the following formula

$$H(X) \equiv H(p_1, ..., p_n) \equiv -\sum_x p_x \log p_x, \tag{6.1}$$

where logarithms indicated by log are taken to base two. Then, it is conventional to say that entropies are measured in bits with this convention for the logarithm. The Shannon entropy measures the uncertainty associated with a classical probability distribution. Quantum states are described in a similar fashion, with density operators replacing probability distributions. Von Neumann defined the entropy of a quantum state ρ by the formula

$$S(\rho) \equiv -tr(\rho \log \rho). \tag{6.2}$$

In this formula logarithms are taken to base two, as usual. If λ_x are the eigenvalues of ρ then von Neumann's definition can be re-expressed as $S(\rho) = -\sum_x \lambda_x \log \lambda_x$ similar to the Shannon entropy.

6.2.2 Mutual information

Consider two random variables A and B. The mutual information content of A and B, measures how much information A and B have in common. It is defined as

$$J(A:B) = H(A) - H(A|B) \quad \big(J(B:A) = H(B) - H(B|A)\big), \tag{6.3}$$

where $H(A)$ $(H(B))$ is the information content of the subsystem A (B), and $H(A|B)$ $(H(B|A))$ is the uncertainty in measurement of A (B) when B (A) is known. In other words, $H(A|B) = \sum_b p_b H(A|b)$ $(H(B|A) = \sum_a p_a H(B|a))$ is the conditional entropy of

A (B) given B (A) where p_b (p_a) is the probability that the random variable B (A) takes the value b (a), and $H(A|b)$ ($H(B|a)$) is the entropy of the variable A (B) conditioned on the variable B (A) taking the value b (a), $H(A|b) = -\sum_a p_{a|b} \log(p_{a|b})$ ($H(B|a) = -\sum_b p_{b|a} \log(p_{b|a})$), and $p_{a|b}$ ($p_{b|a}$) is the probability of a (b) given b (a). Hence, the mutual information measures the average decrease of entropy on A when B (on B when A) is found out. Using the Bayes rule, $p_{A|b} = p_{A,b}/p_b$, a classically equivalent form of mutual information can be derived. Starting from the definition of $H(A,B)$, the information content of the composite system, one can obtain

$$\begin{aligned} H(A,B) &= -\sum_{a,b} p_{a,b} \log(p_{a,b}) \\ &= -\sum_{a,b} p_{a,b} \log(p_{a|b} p_b) \\ &= -\sum_{a,b} p_{a,b} \log(p_{a|b}) - \sum_{a,b} p_{a,b} \log(p_b) \\ &= -\sum_{a,b} p_b \big(p_{a|b} \log(p_{a|b})\big) - \sum_b p_b \log(p_b) \\ &= H(A|B) + H(B). \end{aligned} \tag{6.4}$$

The equation $H(A|B) = H(A,B) - H(B)$ leads to another classically equivalent expression for the mutual information:

$$I(A:B) = H(A) + H(B) - H(A,B). \tag{6.5}$$

6.2.3 Quantum and classical correlations

A state shared by two parties, Alice and Bob, is called separable if and only if the density operator ρ can be written as a convex combination of pure product states

$$\rho = \sum_i p_i \left|\psi_i\right\rangle \left\langle\psi_i\right|_A \otimes \left|\phi_i\right\rangle \left\langle\phi_i\right|_B, \tag{6.6}$$

where $\left|\psi_i\right\rangle_A$ and $\left|\phi_i\right\rangle_B$ are state vectors on the spaces $\mathcal{H}_A$ and $\mathcal{H}_B$ of subsystems A and B, respectively, and the probabilities p_i are nonnegative and sum up to one, $\sum_i p_i = 1$. Otherwise the state is an entangled state. A mixed state is called classically correlated if it can be written as

$$\rho_{cc} = \sum_{i,j} p_{ij} \left|i\right\rangle \left\langle i\right|^a \otimes \left|j\right\rangle \left\langle j\right|^b, \tag{6.7}$$

where $\{\left|i\right\rangle^a\}$ are orthogonal states on Alice's Hilbert space $\mathcal{H}_A$ and $\{\left|j\right\rangle^b\}$ are orthogonal states on Bob's Hilbert space $\mathcal{H}_B$. The probabilities p_{ij} are nonnegative and

sum up to one, $\sum_{i,j} p_{ij} = 1$. On the other hand, a separable state $\rho_{\text{sep}} = \sum_i p_i |a_i\rangle \langle a_i| \otimes |b_i\rangle \langle b_i|$ is not necessarily classically correlated only, since the states $\{|a_i\rangle\}$ and $\{|b_i\rangle\}$ do not have to be orthogonal (see subsection 6.4.2).

6.3 Quantum Discord and Local Operations

The two classically identical expressions for the mutual information, Eq. (6.3) and Eq. (6.5), differ in the quantum case. In the quantum analog, the definitions of mutual information, given by Eq. (6.3) and Eq. (6.5), take form $J(A:B) = S(\rho_B) - S(B|A)$ and $I(A:B) = S(\rho_A) + S(\rho_B) - S(\rho_{AB})$, respectively. In contradistinction to the classical case, in the quantum analog there are many different measurements that can be performed on a system, and measurements generally disturb the quantum state. The state of B, after the outcome corresponding to M_a has been obtained, is $\rho_{B|a} = \text{tr}_A[(M_a \otimes I)\rho_{AB}(M_a^\dagger \otimes I)]/p_a$ with probability $p_a = \text{tr}((M_a \otimes I)\rho_{AB}(M_a^\dagger \otimes I))$. This allows us to define a classical-quantum version of the conditional entropy $S(B|A) = \sum_a p_a S(\rho_{B|a})$. The quantum discord [4], a measure of the quantumness of correlations, of a state ρ_{AB} under a measurement $\{M_a\}$ is defined as a difference between the quantum forms of mutual information,

$$\begin{aligned} \mathcal{D} &= I(A:B) - J(A:B) \\ &= S(\rho_A) - S(\rho_{AB}) + \min_{\{M_a\}} \sum_a p_a S(\rho_{B|a}), \end{aligned} \tag{6.8}$$

where the minimization is over all local projectors. Quantum discord was only calculated for some special states due to the complexity of the minimization problem, and it still needs to be developed. Besides this, there is a different measure of quantum discord, namely geometric measure. Dakic *et al.* introduced the following geometric measure of quantum discord [100]:

$$D(\rho) = \min_{\chi \in \Omega_0} \|\rho - \chi\|^2, \tag{6.9}$$

where Ω_0 denotes the set of zero-discord states and $\|X - Y\|^2 = \text{Tr}(X - Y)^2$ is the square norm in the Hilbert-Schmidt space. For two qubit systems geometric measure of quantum discord can be evaluated analytically. Any two qubit state can be written in Bloch representation as

$$\rho = \frac{1}{4}\Big(I \otimes I + \sum_{i=1}^{3} x_i \sigma_i \otimes I + \sum_{i=1}^{3} y_i I \otimes \sigma_i + \sum_{i,j=1}^{3} T_{ij} \sigma_i \otimes \sigma_j\Big), \tag{6.10}$$

where $x_i = \mathrm{Tr}[\rho(\sigma_i \otimes I)]$, $y_i = \mathrm{Tr}[\rho(I \otimes \sigma_i)]$ are components of the local Bloch vectors, $T_{ij} = \mathrm{Tr}\rho(\sigma_i \otimes \sigma_j)$ are components of the correlation tensor, and σ_i, $i \in \{1,2,3\}$ are the three Pauli matrices. Then, the geometric measure of discord is given by [100,101]

$$D_A(\rho) = \frac{1}{4}\left(\|\vec{x}\|^2 + \|T\|^2 - k_{max}\right), \tag{6.11}$$

where k_{max} is the largest eigenvalue of matrix $K = \vec{x}\vec{x}^T + TT^T$. We will use both Eq.(6.8) and Eq.(6.11) for the measure of quantum discord.

After giving sufficient information about quantum discord, we give details on the quantum operations that we will apply it in our problem. The action of a quantum operation $\{M_i\}$ on a density operator ρ is given by

$$\Lambda(\rho) = \sum_i M_i \rho M_i^\dagger, \tag{6.12}$$

where $\sum_i M_i^\dagger M_i = I$. The corresponding operators M_i are called Kraus operators. For composite systems, local quantum operations can be defined in the same way as it was done for local measurements. The importance of quantum operations lies in the fact that they describe the most general change of a quantum state possible. The classically correlated composite state consisting of the subsystems given by

$$\rho_{cc} = \sum_{i=0}^{n} p_i \left|i\right\rangle \left\langle i\right|_A \otimes \left|i\right\rangle \left\langle i\right|_B. \tag{6.13}$$

can be transformed to other states by local operations. If a local operation is performed on Alice's subsystem, then the subsystem of Bob remains unchanged. In this case, the Kraus operators have the form $M_i = M_i^A \otimes I^B$. That is, $\Lambda^a(\rho_{cc}) = \sum_i (M_i^A \otimes I^B)\rho_{cc}((M_i^A)^\dagger \otimes I^B)$, where $\sum_i (M_i^A)^\dagger M_i^A = I$. For $M_i^A = \left|\psi_i\right\rangle \left\langle i\right|_A$, we obtain

$$\Lambda^a(\rho_{cc}) = \sum_i p_i \left|\psi_i\right\rangle \left\langle \psi_i\right|_A \otimes \left|i\right\rangle \left\langle i\right|_B = \rho_{qc}. \tag{6.14}$$

Similarly, Kraus operators corresponding to local operations on Bob's subsystem have the form $M_i = I^A \otimes M_i^B$. That is, $\Lambda^b(\rho_{cc}) = \sum_i (I^A \otimes M_i^B)\rho_{cc}(I^A \otimes (M_i^B)^\dagger)$, where $\sum_i (M_i^B)^\dagger M_i^B = I$. For $M_i^B = \left|\phi_i\right\rangle \left\langle i\right|_B$, we obtain

$$\Lambda^b(\rho_{cc}) = \sum_i p_i \left|i\right\rangle \left\langle i\right|_A \otimes \left|\phi_i\right\rangle \left\langle \phi_i\right|_B = \rho_{cq}. \tag{6.15}$$

6.4 Local Creation of Quantum Discord

It is well known that entanglement cannot be created by local operations, i.e., separable states remain separable under local operations. However, it is possible to obtain non-classically correlated states using classically correlated states and local operations. Before discussing the local creation of quantum discord, it will be useful to give information about correlation rank, set of zero and nonzero discord states and remote state preparation.

6.4.1 Correlation rank

We briefly recall the definition and the main properties of the correlation rank due to its crucial role on the problem that we are interested in. A given state ρ can be represented in terms of arbitrary bases of Hermitian operators $\{A_i\}$ and $\{B_i\}$ of the two Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively:

$$\rho = \sum_{i=1}^{d_A^2} \sum_{j=1}^{d_B^2} r_{ij} A_i \otimes B_j. \tag{6.16}$$

The real-valued $d_A^2 \times d_B^2$ correlation matrix $R = (r_{ij})$ can be decomposed into its singular value decomposition with the aid of orthogonal matrices U and V as $R = U\text{diag}(c_1, c_2, ..., c_L, 0, 0, .., 0)V^T$, where c_i are the nonzero singular values and L denotes the rank of the correlation matrix of ρ. Introducing the matrices $S_i = \sum_j u_{ji} A_j$ and $F_i = \sum_j v_{ji} B_j$, where $U = u_{ij}$ and $V = v_{ij}$, the state ρ can be written in a diagonal representation as $\rho = \sum_{i=1}^{L} c_i S_i \otimes F_i$ [100]. The state ρ is then characterized by two different properties. On the one hand, L determines how many product operators are needed to represent ρ and thereby allows to draw conclusions about the total correlations [102]. On the other hand, the commutation relations of the local operators S_i (F_i) determine the local quantumness of the state in the subsystem $\mathcal{H}_A$ ($\mathcal{H}_B$), indicated by the quantum discord [100].

6.4.2 Set of (non)zero discord states

The state ρ is of zero discord state on the first subsystem if and only if there exists a von Neumann measurement $\{\Pi_k = |\psi_k\rangle\langle\psi_k|\}$ on the first subsystem such that [100, 103]

$$\sum_k (\Pi_k \otimes I_B)\rho(\Pi_k \otimes I_B) = \rho. \tag{6.17}$$

In other words, the zero-discord state is of the form $\rho = \sum_k p_k |\psi_k\rangle\langle\psi_k| \otimes \rho_k$, where $\{|\psi_k\rangle\}$ is some orthonormal basis set, ρ_k are the quantum states in B, and p_k are nonnegative numbers such that $\sum_k p_k = 1$. As explained in the subsection 6.4.1, introducing the matrices $S_i = \sum_j u_{ji} A_j$ and $F_i = \sum_j v_{ji} B_j$, where $U = u_{ij}$ and $V = v_{ij}$, the state ρ can be written in a diagonal representation as $\rho = \sum_{i=1}^{L} c_i S_i \otimes F_i$. Then, the necessary and sufficient condition (6.17) becomes $\sum_{n=1}^{L} c_n (\sum_k \Pi_k S_n \Pi_k) \otimes F_n = \sum_{n=1}^{L} c_n S_n \otimes F_n$ and it is equivalent to the set of conditions:

$$\sum_k \Pi_k S_n \Pi_k = S_n, \quad k = 1, ..., L, \tag{6.18}$$

or equivalently $[S_n, \Pi_k] = 0$ for all k, n. This means that the set of operators $\{S_n\}$ has a common eigenbasis defined by the set of projectors $\{\Pi_k\}$. Therefore, the set $\{\Pi_k\}$ exists if and only if [100]

$$[S_n, S_m] = 0, \quad n, m = 1, ..., L. \tag{6.19}$$

In order to identify a zero discord state we have to check at most $L(L-1)/2$ commutators, where $L = \mathrm{R} \leq \min\{d_A^2, d_B^2\}$. Now, recall that the state of zero discord is of the form $\rho = \sum_{k=1}^{d_A} p_k \Pi_k \otimes \rho_k$; therefore, it is a sum of at most d_A product operators. This bounds the rank of the correlation tensor to $L \leq d_A$. Thus, the rank of the correlation tensor is a simple discord witness: If $L > d_A$, the state has a nonzero discord [100].

6.4.3 The transformations $\rho_{\mathbf{sep}} \to \rho_{\mathbf{nd}}$

The 2×2 dimensional Werner state, lies in a joint Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$ spanned by two qubits a (Alice) and b (Bob), is given by the following density operator

$$\rho_{W_z} = z|\psi^-\rangle\langle\psi^-| + (\frac{1-z}{4})I_4, \quad z \in [0, 1], \tag{6.20}$$

where I_4 is the identity operator on $\mathcal{H}_A \otimes \mathcal{H}_B$ and $|\psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$ is one of the Bell states. For $z \in (\frac{1}{3}, 1]$ the state given in Eq. (6.20) is entangled, and for $z \in [0, \frac{1}{3}]$ it is a separable but a nonzero discord state. The Fig.6.1 clearly shows some relevant correlations for the state ρ_{W_z} given in Eq. (6.20).

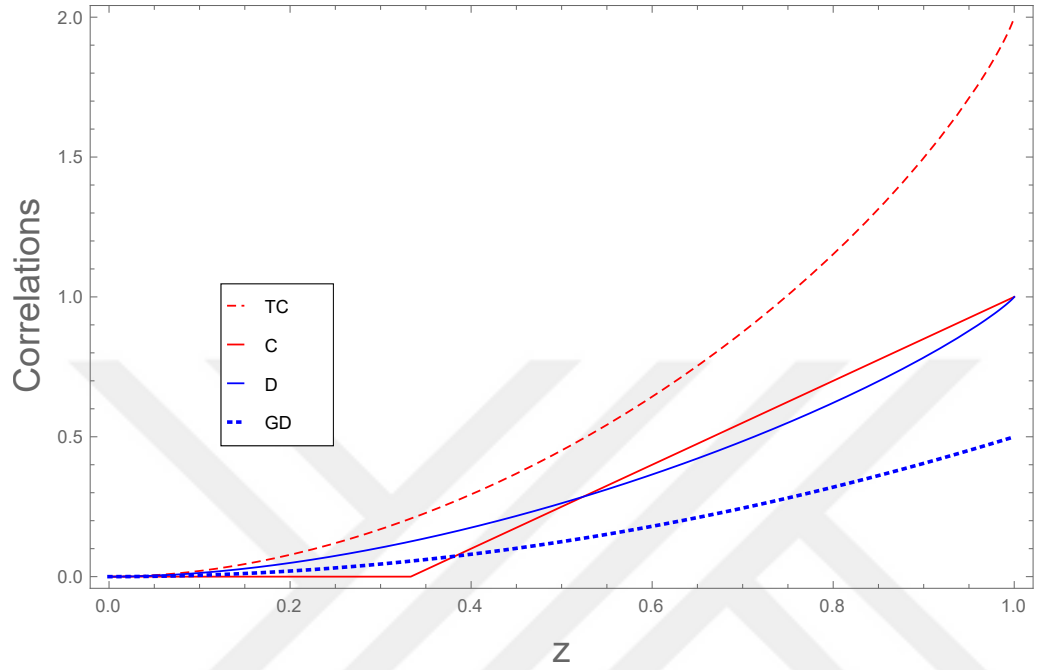

Figure 6.1 : The graph shows the values of Total Correlation (TC), Concurrence (C), Discord (D) and Geometric Discord (GD) for Werner state, $\rho_{W_z} = z|\psi^-\rangle\langle\psi^-| + (\frac{1-z}{4})I_4$, as a function of z.

We consider the problem of the transformations $\rho_{\text{sep}} \to \rho_{\text{nd}}$ where the state ρ_{sep} is our initial separable (or classically correlated) state and the state ρ_{nd} is our target state which is a nonzero discord state, e.g. Werner state for $z = 1/3$.

At this point we firstly need to find the rank of the correlation matrix of the state $\rho_{W_{1/3}}$, L_W, to establish the rank of classically correlated states that we manipulate to obtain the separable Werner state by local operations. From (6.16) and (6.20) one has

$$\begin{aligned}
\rho_{W_{1/3}} &= \frac{1}{3}|\psi^-\rangle\langle\psi^-| + \frac{1}{6}I_4 \\
&= \sum_{i=1}^{4}\sum_{j=1}^{4} r_{ij} A_i \otimes B_j \\
&= \frac{1}{4}A_1 \otimes B_1 + \frac{1}{12}A_2 \otimes B_2 + \frac{1}{12}A_3 \otimes B_3 + \frac{1}{12}A_4 \otimes B_4, \qquad (6.21)
\end{aligned}$$

where $A_1 = I_2, A_2 = -\sigma_x, A_3 = -\sigma_y, A_4 = -\sigma_z, B_1 = I_2, B_2 = \sigma_x, B_3 = \sigma_y$ and $B_4 = \sigma_z$. These choices imply that the nonzero entries of the correlation matrix are found to be

$r_{11} = 1/4$, $r_{22} = 1/12$, $r_{33} = 1/12$, $r_{44} = 1/12$, that is

$$R = \begin{pmatrix} r_{11} & \cdots & r_{14} \\ \vdots & \ddots & \vdots \\ r_{41} & \cdots & r_{44} \end{pmatrix} = \begin{pmatrix} 1/4 & 0 & 0 & 0 \\ 0 & 1/12 & 0 & 0 \\ 0 & 0 & 1/12 & 0 \\ 0 & 0 & 0 & 1/12 \end{pmatrix}. \tag{6.22}$$

The matrix $R^T R$ (or RR^T) have the eigenvalues of $\lambda_1 = 1/16$, $\lambda_2 = \lambda_3 = \lambda_4 = 1/144$. The singular values on the diagonal of Σ, where $R = U\Sigma V^T$, are the square roots of the nonzero eigenvalues of RR^T (or $R^T R$). Then, the singular values are found to be $(c_1, c_2, c_3, c_4) = (1/4, 1/12, 1/12, 1/12)$. This implies that rank of the correlation matrix is four, $L_W = 4$. Thus, the correlation rank L of the classically correlated state that we originally have must be greater or equal to four.

To obtain the separable Werner states from classically correlated state we need to express it in terms of product states: $\rho = \sum_i |\psi_i\rangle\langle\psi_i| \otimes |\phi_i\rangle\langle\phi_i|$. This facilitates the process of investigating the necessary quantum operations. For this purpose, we start with the Werner states given in [104, 105]

$$\rho_{W_z} = \sum_{i=0}^{3} |\omega_i\rangle\langle\omega_i|, \tag{6.23}$$

where

$$\begin{aligned} |\omega_0\rangle &= \frac{1}{2}\left(e^{i\theta_1}|x_1\rangle + e^{i\theta_2}|x_2\rangle + e^{i\theta_3}|x_3\rangle + e^{i\theta_4}|x_4\rangle\right), \\ |\omega_1\rangle &= \frac{1}{2}\left(e^{i\theta_1}|x_1\rangle + e^{i\theta_2}|x_2\rangle - e^{i\theta_3}|x_3\rangle - e^{i\theta_4}|x_4\rangle\right), \\ |\omega_2\rangle &= \frac{1}{2}\left(e^{i\theta_1}|x_1\rangle - e^{i\theta_2}|x_2\rangle + e^{i\theta_3}|x_3\rangle - e^{i\theta_4}|x_4\rangle\right), \\ |\omega_3\rangle &= \frac{1}{2}\left(e^{i\theta_1}|x_1\rangle - e^{i\theta_2}|x_2\rangle - e^{i\theta_3}|x_3\rangle + e^{i\theta_4}|x_4\rangle\right), \end{aligned} \tag{6.24}$$

with $|x_1\rangle = \frac{\sqrt{1+3z}}{2i}|\psi^-\rangle$, $|x_2\rangle = \frac{\sqrt{1-z}}{2}|\psi^+\rangle$, $|x_3\rangle = \frac{\sqrt{1-z}}{2}|\phi^-\rangle$, $|x_4\rangle = \frac{\sqrt{1-z}}{2i}|\phi^+\rangle$. Here, $|\psi^\pm\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle)$ and $|\phi^\pm\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle)$. Also, we have $\langle\omega_i|\omega_k\rangle = z/4$ for $i,k = 0,1,2,3$ $(i \neq k)$, and $\langle\omega_i|\omega_i\rangle = 1/4$. The parameters θ_i satisfy the following equation

$$e^{-2i\theta_1}(1+3z) + (e^{-2i\theta_2} + e^{-2i\theta_3} + e^{-2i\theta_4})(1-z) = 0, \tag{6.25}$$

where two of phase can be chosen as $\theta_1 = 0$, $\theta_2 = \pi/2$ for simplicity. By choosing such that we have $\cos\theta_3 = \sqrt{\frac{1-3z}{2(1-z)}}$, $\sin\theta_3 = \sqrt{\frac{1+z}{2(1-z)}}$, $\cos\theta_4 = -\sqrt{\frac{1-3z}{2(1-z)}}$ and $\sin\theta_4 = \sqrt{\frac{1+z}{2(1-z)}}$. We are now ready to find the separable states $|\omega_i\rangle$ for $z = 1/3$ via the details

given above. If we explicitly write the state $|\omega_0\rangle$, we get

$$\begin{aligned}|\omega_0\rangle &= \frac{1}{2}\left(|x_1\rangle + i\,|x_2\rangle + i\,|x_3\rangle + i\,|x_4\rangle\right)\\ &= \frac{1}{2}\Big(-\frac{i}{\sqrt{2}}\,|\psi^-\rangle + \frac{i}{\sqrt{6}}\,|\psi^+\rangle + \frac{i}{\sqrt{6}}\,|\phi^-\rangle + \frac{1}{\sqrt{6}}\,|\phi^+\rangle\Big)\\ &= \frac{1}{2}\big(\psi_0\,|0\rangle + \psi_1\,|1\rangle\big)\otimes\big(\phi_0\,|0\rangle + \phi_1\,|1\rangle\big)\\ &= \frac{1}{2}\,|\Psi_0\rangle\otimes|\Phi_0\rangle\,,\end{aligned} \tag{6.26}$$

where

$$|\Psi_0\rangle = -ic\,|0\rangle + \frac{1}{\sqrt{6}c}e^{-i\pi/4}\,|1\rangle\,,\quad |\Phi_0\rangle = -\frac{1}{\sqrt{6}c}e^{-i\pi/4}\,|0\rangle + c\,|1\rangle\,. \tag{6.27}$$

Similarly, other states can also be written in terms of product states: $|\omega_i\rangle = \frac{1}{2}\,|\Psi_i\rangle\otimes|\Phi_i\rangle$. Thus, we obtain

$$\begin{aligned}|\Psi_m\rangle &= -ic\,|0\rangle + \frac{1}{\sqrt{6}c}e^{-i\pi/4}e^{im\pi}\,|1\rangle\,,\\ |\Phi_m\rangle &= -\frac{1}{\sqrt{6}c}e^{-i\pi/4}e^{im\pi}\,|0\rangle + c\,|1\rangle\,,\\ |\Psi_n\rangle &= -\frac{i}{\sqrt{6}c}\,|0\rangle - ce^{i\pi/4}e^{in\pi}\,|1\rangle\,,\\ |\Phi_n\rangle &= -ce^{i\pi/4}e^{in\pi}\,|0\rangle + \frac{1}{\sqrt{6}c}\,|1\rangle\,,\end{aligned} \tag{6.28}$$

for $m = 0,1$ and $n = 2,3$. Here, $c = \frac{1}{\sqrt{3+\sqrt{3}}}$, $\langle\Psi_i|\Phi_i\rangle = 0$ $(i = 0,1,2,3)$, $\langle\Psi_i|\Psi_k\rangle \neq 0$ and $\langle\Phi_i|\Phi_k\rangle \neq 0$. All in all, we rewrite the separable Werner state for $z = 1/3$ such that

$$\rho_{W_{1/3}} = \frac{1}{4}\sum_{i=0}^{3}|\Psi_i\rangle\langle\Psi_i|\otimes|\Phi_i\rangle\langle\Phi_i|\,. \tag{6.29}$$

We turn now to the problem we are interested in. We introduce three fairly similar schemes to obtain the state $\rho_{W_{1/3}}$ given in Eq. (6.29). First, suppose Alice (a) and Bob (b) share a $4\otimes 4$ classically correlated state given by

$$\gamma \;=\; \frac{1}{4}\sum_{i=0}^{3}|i\rangle_a\langle i|\otimes|i\rangle_b\langle i|\,. \tag{6.30}$$

Two auxiliary particles, a' and b', can be incorporated into systems a and b, respectively (see Fig. 6.2). Then, the state illustrated in Fig. 6.2 can be written such that

$$\begin{aligned}\sigma &= \gamma\otimes|0\rangle_{a'}\langle 0|\otimes|0\rangle_{b'}\langle 0|\\ &= \sum_{i=0}^{3}\frac{1}{4}\big(|i\rangle\langle i|\otimes|0\rangle\langle 0|\big)_{aa'}\otimes\big(|i\rangle\langle i|\otimes|0\rangle\langle 0|\big)_{bb'}.\end{aligned} \tag{6.31}$$

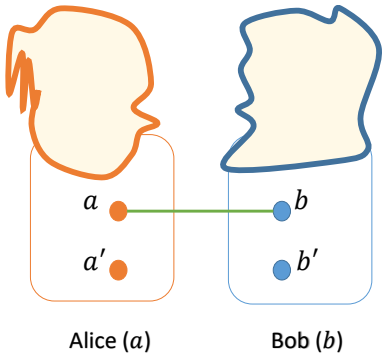

Figure 6.2 : While two 4-level particles, a and b, are classically correlated ($a-b$), the ancillary particles, a' and b', are uncorrelated.

The local unitary transformations on both sides transform the initial state to the state given by

$$\begin{aligned}\tilde{\sigma} &= \left(U_{aa'}\otimes U_{bb'}\right)\sigma\left(U_{aa'}^{\dagger}\otimes U_{bb'}^{\dagger}\right)\\ &= \sum_{i=0}^{3}\left(|\psi_{i+1}\rangle\langle\psi_{i+1}|\otimes|i\rangle\langle i|\right)_{aa'}\otimes\sum_{i=0}^{3}\left(|\phi_{i+1}\rangle\langle\phi_{i+1}|\otimes|i\rangle\langle i|\right)_{bb'},\end{aligned}\tag{6.32}$$

where $U_{aa'}$ is given by

$$\begin{aligned}U_{aa'} &= \Big(\sum_{i=0}^{3}|\psi_{i+1}\rangle\langle i|\otimes|i\rangle\langle 0|+\sum_{i=0}^{3}|\psi_{i+1}^{\perp}\rangle\langle i|\otimes|i\rangle\langle 1| &(6.33)\\ &\quad+\sum_{i=0}^{3}|\tilde{\psi}_{i+1}\rangle\langle i|\otimes|i\rangle\langle 2|+\sum_{i=0}^{3}|\tilde{\psi}_{i+1}^{\perp}\rangle\langle i|\otimes|i\rangle\langle 3|\Big)_{aa'}. &(6.34)\end{aligned}$$

Here, $|\psi_i\rangle=\alpha_i|0\rangle+\beta_i|1\rangle$, $|\psi_i^{\perp}\rangle=-\beta_i^*|0\rangle+\alpha_i^*|1\rangle$ ($\langle\psi_i\,|\,\psi_i^{\perp}\rangle=0$), $|\tilde{\psi}_i\rangle=\alpha_i|2\rangle+\beta_i|3\rangle$, $|\tilde{\psi}_i^{\perp}\rangle=-\beta_i^*|2\rangle+\alpha_i^*|3\rangle$ ($\langle\tilde{\psi}_i\,|\,\tilde{\psi}_i^{\perp}\rangle=0$). For the unitary transformation $U_{bb'}$ replace ψ, α and β with ϕ, λ and δ, respectively. We also note that the unitarity conditions are satisfied, that is $U_{aa'}^{\dagger}U_{aa'}=I$ and $U_{bb'}^{\dagger}U_{bb'}=I$. By tracing out ancillary particles, $\mathrm{Tr}_{a'b'}(\tilde{\sigma})$, we obtain

$$\mathrm{Tr}_{a'b'}[\tilde{\sigma}]=\frac{1}{4}\sum_{i=0}^{3}|\psi_i\rangle_a\langle\psi_i|\otimes|\phi_i\rangle_b\langle\phi_i|.\tag{6.35}$$

By choosing $|\psi_i\rangle_a=|\Psi_i\rangle$ and $|\phi_i\rangle_b=|\Phi_i\rangle$, we get the target state $\rho_{W_{1/3}}$. We can also express the above transformation in the Kraus operator formalism. The transformation with Kraus operators, $M_0^a=|\Psi_0\rangle\langle 0|$, $M_1^a=|\Psi_1\rangle\langle 1|$, $M_2^a=|\Psi_2\rangle\langle 2|$ and $M_3^a=|\Psi_3\rangle\langle 3|$ on Alice's subsystem, and $M_0^b=|\Phi_0\rangle\langle 0|$, $M_1^b=|\Phi_1\rangle\langle 1|$, $M_2^b=|\Phi_2\rangle\langle 2|$ and $M_3^b=|\Phi_3\rangle\langle 3|$ on Bob's subsystem of the given state in Eq.(6.30) gives Werner state that we aim to obtain, $\rho_{W_{1/3}}$, that is

$$\Lambda(\gamma)=\frac{1}{4}\sum_{i=0}^{3}(M_i^a\otimes M_i^b)\gamma(M_i^{a\dagger}\otimes M_i^{b\dagger})=\frac{1}{4}\sum_{i=0}^{3}|\Psi_i\rangle\langle\Psi_i|\otimes|\Phi_i\rangle\langle\Phi_i|.\tag{6.36}$$

Second, consider two systems, Alice (a) and Bob (b), are prepared locally, and both consist of two qubits where qubits a_1-b_1 and a_2-b_2 are classically correlated (green lines), and qubits a_1 and b_1 are uncorrelated with qubits a_2 and b_2, respectively (see Fig. 6.3). The density matrix of the total system possess by two parties Alice and Bob

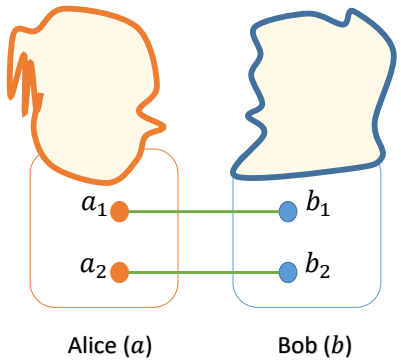

Figure 6.3 : Two systems, Alice (a) with qubits a_1, a_2 and Bob (b) with qubits b_1, b_2, are prepared locally where qubits a_1-b_1 and a_2-b_2 are classically correlated.

as illustrated in Fig. 6.3 is written such that

$$\begin{aligned}\rho &= \frac{1}{2}\Big(\sum_{i=0}^{1}|i\rangle\langle i|\otimes|i\rangle\langle i|\Big)_{a_1b_1}\otimes\frac{1}{2}\Big(\sum_{i=0}^{1}|i\rangle\langle i|\otimes|i\rangle\langle i|\Big)_{a_2b_2}\\ &= \frac{1}{4}\Big(\sum_{i,j=0}^{1}|ij\rangle\langle ij|\otimes|ij\rangle\langle ij|\Big)_{a_1a_2b_1b_2}.\end{aligned} \tag{6.37}$$

Then, the local operations on Alice's subsystem $(a_1\otimes a_2)$

$$\begin{aligned}&M_0^a=|0\Psi_0\rangle\langle 00|, \quad M_1^a=|0\Psi_1\rangle\langle 01|,\\ &M_2^a=|1\Psi_2\rangle\langle 10|, \quad M_3^a=|1\Psi_3\rangle\langle 11|, \quad \Big(\sum_{i=0}^{3}(M_i^a)^\dagger M_i^a=I_4\Big),\end{aligned} \tag{6.38}$$

and the local operations on Bob's subsystem $(b_1\otimes b_2)$

$$\begin{aligned}&M_0^b=|0\Phi_0\rangle\langle 00|, \quad M_1^b=|0\Phi_1\rangle\langle 01|,\\ &M_2^b=|1\Phi_2\rangle\langle 10|, \quad M_3^b=|1\Phi_3\rangle\langle 11|, \quad \Big(\sum_{i=0}^{3}(M_i^b)^\dagger M_i^b=I_4\Big),\end{aligned} \tag{6.39}$$

are applied on their respective systems to give

$$\begin{aligned}\tilde{\rho} &= \frac{1}{4}\Big(|0\Psi_0\rangle\langle 0\Psi_0|\otimes|0\Phi_0\rangle\langle 0\Phi_0|+|0\Psi_1\rangle\langle 0\Psi_1|\otimes|0\Phi_1\rangle\langle 0\Phi_1|\\ &\quad+|1\Psi_2\rangle\langle 1\Psi_2|\otimes|1\Phi_2\rangle\langle 1\Phi_2|+|1\Psi_3\rangle\langle 1\Psi_3|\otimes|1\Phi_3\rangle\langle 1\Phi_3|\Big)_{a_1a_2b_1b_2},\end{aligned} \tag{6.40}$$

where $|\Psi_i\rangle$ and $|\Phi_i\rangle$ are given in Eq.(6.28). Then, taking partial trace of the qubits a_1 and b_1 of the state $\tilde{\rho}$ in Eq. (6.40), i.e., $\mathrm{Tr}_{a_1b_1}[\tilde{\rho}]$, gives the target Werner state:

$$\mathrm{Tr}_{a_1b_1}[\tilde{\rho}]=\frac{1}{4}\sum_{i=0}^{3}|\Psi_i\rangle\langle\Psi_i|\otimes|\Phi_i\rangle\langle\Phi_i|=\rho_{W_{1/3}}. \tag{6.41}$$

Third, consider two systems, Alice (a) and Bob (b), are prepared locally distant from each other. Furthermore, both contain three qubits where qubits a_1-b_1, a_2-b_2 and a_3-b_3 are classically correlated (green lines, see Fig. 6.4). The state of the total

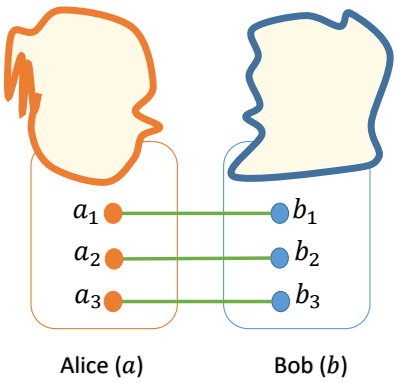

Figure 6.4 : Two systems, Alice (a) with qubits a_1, a_2, a_3 and Bob (b) with qubits b_1, b_2, b_3, are prepared locally. Here, qubits a_k-b_k $(k=1,2,3)$ are classically correlated.

system possess by two parties Alice and Bob, illustrated in Fig. 6.4, is of the form

$$\rho = \rho_{a_1b_1} \otimes \rho_{a_2b_2} \otimes \rho_{a_3b_3}, \tag{6.42}$$

where

$$\rho_{a_kb_k} = \frac{1}{2}\Big(\sum_{i=0}^{1} |i\rangle\langle i| \otimes |i\rangle\langle i|\Big)_{a_kb_k}, \tag{6.43}$$

for $k=1,2,3$. Then, Alice performs the unitary transformation

$$\begin{aligned} U_A \equiv & U_{a_1a_2a_3} \\ = & \sum_{i,j=0}^{1} |ij\rangle\langle ij| \otimes \Big(|\Psi_{2i+j}\rangle\langle 0| + |\Phi_{2i+j}\rangle\langle 1|\Big), \end{aligned} \tag{6.44}$$

on her subsystems. The unitarity condition on U_A, $U_A^*U_A = U_AU_A^* = I$, gives the constraints $\langle\Psi_k|\Phi_k\rangle = 0$ $(k=0,1,2,3)$ where these are attested by Eq. (6.28). Similarly, the unitary transformation U_B, of Bob's subsystem, is taken such that

$$\begin{aligned} U_B \equiv & U_{b_1b_2b_3} \\ = & \sum_{i,j=0}^{1} |ij\rangle\langle ij| \otimes \Big(|\Phi_{2i+j}\rangle\langle 0| + |\Psi_{2i+j}\rangle\langle 1|\Big), \end{aligned} \tag{6.45}$$

where the unitarity condition on U_B gives the same constraints with U_A. After performing the unitary transformations on both Alice's and Bob's respective

subsystems, we get

$$\begin{aligned}\tilde{\rho} &= (U_{a_1a_2a_3}\otimes U_{b_1b_2b_3})\big(\rho_{a_1b_1}\otimes\rho_{a_2b_2}\otimes\rho_{a_3b_3}\big)(U^{\dagger}_{a_1a_2a_3}\otimes U^{\dagger}_{b_1b_2b_3})\\ &= \frac{1}{8}\Big(\sum_{i,j=0}^{1}|ij\rangle|\Psi_{2i+j}\rangle\langle ij|\langle\Psi_{2i+j}|\otimes|ij\rangle|\Phi_{2i+j}\rangle\langle ij|\langle\Phi_{2i+j}|\\ &\quad+|ij\rangle|\Phi_{2i+j}\rangle\langle ij|\langle\Phi_{2i+j}|\otimes|ij\rangle|\Psi_{2i+j}\rangle\langle ij|\langle\Psi_{2i+j}|\Big)_{a_1a_2a_3b_1b_2b_3}. \quad (6.46)\end{aligned}$$

Then, tracing out the qubits a_1, a_2, b_1 and b_2 of the state $\tilde{\rho}$ given in Eq. (6.46), we obtain

$$\begin{aligned}\mathrm{Tr}_{a_1a_2b_1b_2}[\tilde{\rho}] &= \frac{1}{2}\Big(\frac{1}{4}\sum_{i=0}^{3}|\Psi_i\rangle\langle\Psi_i|\otimes|\Phi_i\rangle\langle\Phi_i|+\frac{1}{4}\sum_{i=0}^{3}|\Phi_i\rangle\langle\Phi_i|\otimes|\Psi_i\rangle\langle\Psi_i|\Big)\\ &= \frac{1}{2}\Big(\rho_{W_{1/3}}+\rho_{W_{1/3}}\Big)\\ &= \rho_{W_{1/3}}. \quad (6.47)\end{aligned}$$

6.5 Discussion and Conclusion

Quantum discord was first identified as a resource for computation by Datta et al. [103], where they took a separable state as the resource and showed that the computation was better than using a classical state. In other words, it has been experimentally shown that separable states with nonzero quantum discord may have high probability of success in some quantum processes. In this astonishing case, we obtained separable Werner state, $\rho_{W_{1/3}}$, which have nonzero quantum discord and have high probability of success in remote state preparation (when we compare it with separable but zero discord states) by optimal local operations.

7. CONCLUSION AND FUTURE DIRECTIONS

We proposed alternative and experimentally friendly protocols for the deterministic and (optimal) probabilistic transformations of the non-classically correlated states: Entanglement, Quantum Coherence and Quantum Discord.

In summary, we presented two explicit protocols for the deterministic transformations of coherent states under incoherent operations. We first introduced a permutation-based protocol which provides a method for the single-step transformation of d-dimensional coherent states. We obtained generalized solutions of this protocol for some special cases of d-level systems. Then, we presented an alternative protocol where we used d'-level ($d' < d$) subspace solutions of the permutation-based protocol to achieve the complete transformation as a sequence of coherent state transformations. Moreover, we adapted the same results to the deterministic transformations of $d \otimes d$ bipartite entangled pure states. We gave various illustrative examples both to demonstrate the adaptability of our results to the deterministic transformations of entanglement and to enhance the intelligibility of our methods. Then, we presented a simple, practical and efficient strategy for optimal one-shot distillation of quantum coherence from pure input states of arbitrary dimension. Inspired by a specific entanglement distillation protocol, our main result yields a strictly incoherent operation that produces one of a family of maximally coherent states of variable dimension from any pure quantum state. We expanded this protocol to the case where it is possible, for some initial states, to avert any waste of resources as far as the output states are concerned, by exploiting an additional transformation into a suitable intermediate state. Another important problem is the manipulations of the three-qubit pure states. We discussed both (optimal) probabilistic and deterministic transformations of three-qubit pure states. We defined the equivalence classes of local measurements such that local operations which transform states from one equivalence class into another with the same probability are equivalent. We used the canonical local operators for the concentration of three-qubit

GHZ states and obtain the optimal concentration protocols in terms of the unitary invariants of quantum states, namely, the bipartite concurrences and the three-tangle. Finally, we introduced three procedures to obtain the separable Werner states which have nonzero discord.

One future direction for pure state transformations can be the following. The majorization condition is necessary and sufficient for a pure-to-pure transformation to be implementable via SIO and IO. However, we simply do not know whether the same condition is necessary also under DIO. Additionally, we can show that a coherent state can be transformed into another one (for which majorization condition is satisfied) via DIO. Let us consider the transformation

$$\sqrt{\frac{5}{8}}\,|0\rangle + \sqrt{\frac{3}{8}}\,|1\rangle \xrightarrow{\mathcal{E}} \sqrt{\frac{3}{4}}\,|0\rangle + \sqrt{\frac{1}{4}}\,|1\rangle\,. \tag{7.1}$$

As it seen majorization condition is satisfied. We then define Kraus operators such that

$$\begin{aligned} \mathcal{E} : K_1 &= \frac{\sqrt{3}}{2}\,|0\rangle\langle 0| + \frac{1}{\sqrt{20}}\,|0\rangle\langle 1| + \sqrt{\frac{3}{5}}\,|1\rangle\langle 1|\,, \\ K_2 &= \frac{1}{2}\,|0\rangle\langle 0| - \sqrt{\frac{3}{20}}\,|0\rangle\langle 1| + \frac{1}{\sqrt{5}}\,|1\rangle\langle 1|\,. \end{aligned} \tag{7.2}$$

One can easily check that the operation given in Eq. (7.2) is DIO and not IO. Then, if we perform the operators K_1 and K_2 on the state $|\psi\rangle = \sqrt{\frac{5}{8}}\,|0\rangle + \sqrt{\frac{3}{8}}\,|1\rangle$ we get

$$K_1\,|\psi\rangle = \sqrt{q_1}\Big(\sqrt{\frac{3}{4}}\,|0\rangle + \sqrt{\frac{1}{4}}\,|1\rangle\Big) = \sqrt{q_1}\,|\varphi_1\rangle\,, \tag{7.3}$$

$$K_2\,|\psi\rangle = \sqrt{q_2}\Big(\sqrt{\frac{1}{4}}\,|0\rangle + \sqrt{\frac{3}{4}}\,|1\rangle\Big) = \sqrt{q_2}\,|\varphi_2\rangle\,, \tag{7.4}$$

where the probabilities are found to be $q_1 = 9/10$ and $q_2 = 1/10$. Here, the state $|\varphi_1\rangle$ is already equal to the target state $|\phi\rangle = \sqrt{\frac{3}{4}}\,|0\rangle + \sqrt{\frac{1}{4}}\,|1\rangle$, and the state $|\varphi_2\rangle$ is equal to the target state with the permutation $|0\rangle \leftrightarrow |1\rangle$. Therefore, we obtain $\mathcal{E}(|\psi\rangle\langle\psi|) = |\phi\rangle\langle\phi|$ where the operation $\mathcal{E}$ is a dephasing-covariant incoherent operation (DIO). However, obtaining a general and explicit form of DIO for such transformations is still an open problem that we are interested in.

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