T.C.
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A GLOBAL OPTIMIZATION TECHNIQUE USING AUXILIARY FUNCTION METHOD IN DIRECTIONAL SEARCH VIA BÉZIER SURFACE

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A Global Optimization Technique Using Auxiliary Function Method in Directional Search via Bézier Surface submitted by **Idris A. Masoud ABDULHAMID** in partial fulfillment of the requirements for **Doctor Of Philosophy** in **Department of Mathematics,** Graduate School of Natural and Applied Sciences, Süleyman Demirel University by,

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COMMITMENT

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Idris A. Masoud ABDULHAMID

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In the last two decades, deterministic global optimization algorithms have been developed for non convex, nonlinear optimization problems powered by applications spanning science and engineering. The presence of several local minimizers of a non convex objective function implies that global optimization is a significant problem, adding to that moving from a minimizer point to a better one is sometimes a challenge in the search process in addressing continuous global optimization problems. Certain auxiliary functions, namely, filled functions, can be used to address these problems. On the other hand, some algorithms rely on auxiliary function techniques to iteratively decrease extremum points by adding intermediate functional expressions of variables and constraints until each intermediate expression can be externalized with a feasible convex set. In this dissertation, we have developed a new category of auxiliary functions based on the definition of the filled function and the concept of convexity. Namely, (i) we have proposed three new auxiliary functions, which not only preserve the functional features of known traditional functions, but also address difficulties in their computational implementation, for instance, the case of non smoothness. In addition, they are achieving the filled function properties. To demonstrate the applicability of the methods provided, firstly, test functions of 2-50 variables were applied, secondly, we used one of these methods to detect threshold values in images of melanoma in skin cancer. Computational results indicate that the method proposed is strong, reliable, and effective. The best threshold value has been identified for segmentation of the aforementioned lesion. (ii) We introduced a new convex-transforming method, an important class of generalized convex functions which supplies many functional forms often present in non convex problems. The results indicated that the proposed convexification techniques significantly reduced the local minimizers of the given function.

Keywords: Global optimization, Nonlinear optimization, Auxiliary function, Globally convex.

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ÖZET

Doktora

YARDIMCI FONKSIYON METODUNDA BÉZIER EĞRİLERİ YOLU İLE YÖNLÜ ARAMA KULLANARAK YENİ BİR GLOBAL OPTİMİZASYON TEKNİĞİ

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Son yirmi yılda, bilim ve mühendislik uygulamaları sonucunda ortaya çıkan konveks olmayan ve doğrusal olmayan optimizasyon problemleri çözmek için deterministik global optimizasyon algoritmaları geliştirilmiştir. Konveks olmayan bir objektif fonksiyonunun birkaç yerel minimumlaştırıcısın varlığı, küresel optimizasyonun önemli bir problem olduğunu, bir minimumlaştırıcı noktadan daha iyi bir noktaya geçmenin bazen global optimizasyon problemlerini çözümünde arama sürecinde büyük bir zorluk olduğunu gösterir.Yardımcı fonksiyonlar yaklaşımları örneğin; doldurulmuş fonksiyon bu sorunları gidermek için kullanılabilir. Öte yandan, bazı algoritmalar değişkenlerin ve kısıtlamaların ara fonksiyonel ifadelerini ekleyerek her bir ara ifade bir konveks kümeyle harici hale getirilene kadar uç nokta noktalarını yinelemeli olarak azaltmak için yardımcı fonksiyon tekniklerini kullanır. Bu tezde, doldurulmuş fonksiyonun tanımına ve konvekslik kavramına dayanan yeni bir yardımcı fonksiyon kategorisi geliştirilmiştir. Yani, (i) sadece bilinen geleneksel fonksiyonların fonksiyonel özelliklerini korumakla kalmayıp, aynı zamanda (örneğin; düzgün olmama durumu gibi) hesaplama uygulamalarındaki zorlukları da ele alan üç yeni yardımcı fonksiyon önerilmiştir. Ek olarak, doldurulmuş fonksiyon özellikleri elde edilmiştir. Elde edilen yöntemlerin uygulanabilirliğini göstermek için ilk olarak 2-50 değişkenli test fonksiyonlarına uygulamalar yapılmıştır. İkinci olarak, cilt kanserinde melanom görüntülerinde eşik değerlerini tespit etmek için bu yöntemlerden birini kullanılmıştır. Bilgisayar sonuçları önerilen yöntemin güçlü, güvenilir ve etkili olduğunu göstermektedir. En iyi eşik değeri, yukarıda belirtilen lezyonun segmentasyonu için tanımlanmıştır. (ii) Konveks olmayan problemlerin çözümünde sıklıkla kullanılan ve birçok fonksiyonel formu sağlayan yeni bir konveks yöntemi geliştirilmiştir. Sonuçlar, önerilen konveksle tekniklerinin verilen fonksiyonun lokal minimize edicilerini önemli ölçüde azalttığını göstermiştir.

Anahtar Kelimeler: Global optimizasyon, Doğrusal olmayan optimizasyon, Yardımcı fonksiyon, Küresel konveks.

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LIST OF SYMBOLS AND ABBREVIATION

$\mathbb{R}^n$	**:** Space of vectors x to n real components
$x \in \mathbb{R}^n$	**:** Is intended as a column vector
$\mathbb{R}_+$	**:** Set of real non-negatives
(P)	**:** The continuous global optimization problem considered in this dissertation
x^T	**:** Vector obtained as a transposed of x
f	**:** The objective function
n	**:** Dimension of the problem
Λ**,**S	**:** The domain of problem (P)
x_k	**:** May indicate, depending on the case, the $k-th$ component of x, (in which case $x_k \in \mathbb{R}$), or the $k-th$ vector of a sequence
$\{x_k\}$	**:** Sequence formed by the vectors x_k
$\|x_k\|$	**:** Norm of x
$\nabla f(x)$	**:** Gradient of a function $f : \mathbb{R}^n \to \mathbb{R}$ calculated in x
$\nabla^2 f(x)$	**:** Hessian matrix $(n \times n)$ of a function $f : \mathbb{R}^n \to \mathbb{R}$ calculated in x
d_k	**:** Search direction
k	**:** Iteration counter
B_k	**:** The basin of f at x_k^* over Λ
$B(c,r)$	**:** The open ball
x_k^*, x_{k+1}^*	**:** The current and the successor minimizer of f over Λ
x^*	**:** The global minimizer of f over Λ
l_j	**:** Lower bound
u_j	**:** upper bound

1. INTRODUCTION

1.1. Background

Global optimization is the process of finding the best possible solution to a problem of optimization within a given time limit. An optimization problem, in turn, is a problem for which the potential solutions are different. The problem of optimization can be classified as a problem of local optimization or a problem of global optimization. Any possible solution is sufficient according to a local optimization problem, but in global optimization the assignment is to locate the absolute best solution. On the other hand, the challenges of global optimization are the easiest to address, yet they are complex in many practical applications. These applications include but are not limited to the engineering, operational research, and molecular biology sectors (Floudas and Pardalos, 1990, 2014; Horst and Tuy, 2013). In engineering, for example, problems of optimization appear in the design and implementation of engineering. In operations research, the decision is made by improving optimization models to characterize the nature of the problem and therefore a mathematical transaction is applied to solve such models, but the problems appear as models in the molecular framework in molecular biology. Many of these optimization problems may have functions provided as a 'black-box', that is, the function may be a restraint that provides a function estimate in the search range for a given point.

It is important to take into account the mathematical construction of the objective function, the decision variables, and the constraints of the given problem during the resolution of the optimization problem. Continuous or noncontinuous, linear or nonlinear, and convex or non convex may be the objective function. The variables for the decision may be discrete or continuous. The convex or non convex search region can be feasible. These distinctions have a big influence on how to solve the problem of optimization. Furthermore, these distinctions are used to classify the problems of optimization. A problem with linear objective function construction and linear constraints is known as a linear programming problem. The problem that the objective function or whichever of the constraints is nonlinear is known as a nonlinear optimization problem. Overhead two cases, the decision variables are supposed to be continuous. In addition, the problem that the variable should be discrete is called a problem of discrete opti-

mization. The problem is known as a mixed-integer programming problem together with mixed variables, which is both continuous and discrete. In this thesis, we focus on non-linear problems of global optimization with continuous variables. Figure 1.1 shows the important class of Optimization problem.

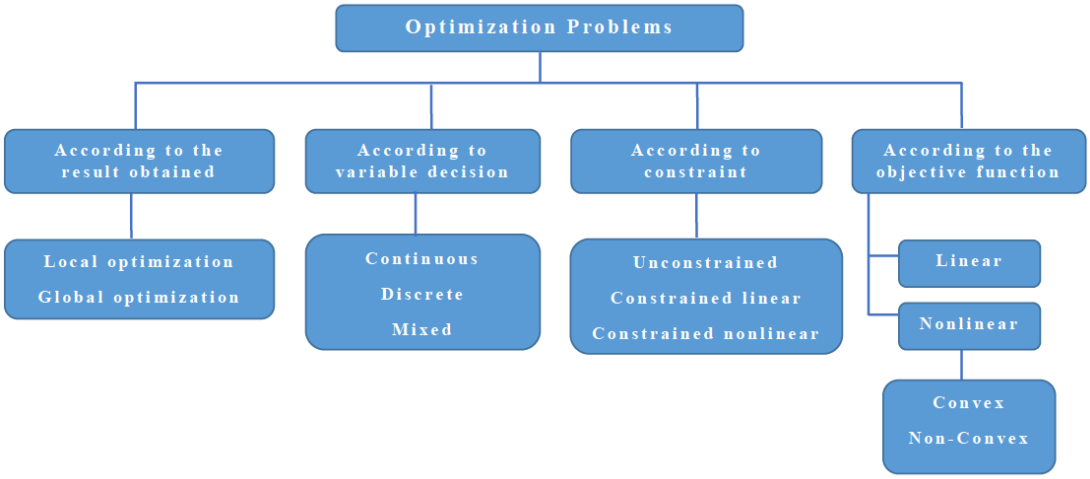


Figure 1.1. Classification of optimization methods

1.2. Local Optimization Approaches

Since there does not exist a suitable algorithmic characterization of global optima, most optimizers are based on local optimization. The most important derivative-free local based methods are pattern search methods and model-based search methods. Pattern search methods perform local exploration of the cost function for a pattern (a set of points conveniently selected). The Hooke–Jeeves (Hooke and Jeeves, 1961) method and the Nelder–Mead simplex method (Nelder and Mead, 1965) are two well-known examples of pattern search local methods.

The method of Hooke and Jeeves consists of a sequence of exploratory moves from a base point in the coordinate directions, which, if successful, are followed by pattern moves. The purpose of an exploratory move is to acquire information about the cost function in the neighborhood of the current base point. A pattern move attempts to speed up the search by using the information already acquired about the function to identify the best search direction.

The Nelder–Mead simplex algorithm belongs to the class of simplex-based direct search methods introduced by Spendley et al. in (Spendley et al., 1962). This class

of methods evolves a pattern set of $n+1$ vectors on an n-dimensional search space that is interpreted as the vertex set of a simplex, i.e., a convex n-dimensional polytope. Different simplex-based methods use different operators to evolve the vertex set. The Nelder–Mead algorithm starts by ordering the vertices of the simplex and uses five operators: reflection, expansion, outer contraction, inner contraction and shrinkage. At each iteration, only one of these operations is performed. As a result, a new simplex is obtained such that either contains a better vertex (a vertex that is better than the worst one and substitutes it) or has a smaller volume. From the point of view of the function value, every operator, except for the shrinkage, is productive and obtains a vertex that improves the worst one. The shrinkage operator reduces the volume of the simplex to guarantee algorithm termination. Unfortunately, the convergence of the Nelder–Mead simplex algorithm to a local minimizer is not guaranteed (Lagarias et al., 1998), and a number of modifications have been introduced to get convergence. Nevertheless, the Nelder–Mead simplex algorithm remains as one of the most popular direct search algorithms.

Model-based search methods are based on the idea of using function evaluations to compute a model of the cost function and to obtain derivative approximations from this model. Many model-based methods are trust-region algorithms (Conn et al., 2000) that interpolate the cost function in some region of appropriate shape and size to obtain a good approximation of the cost function that can later be easily optimized on that region. Typically, the model to be adjusted is a fixed-order polynomial, and the trust region is a sphere in some norm of the search space. The model function is sequentially optimized in the trust region, and the trust region is updated for the next iteration until the approximated model satisfies a first-order optimality condition. Model-based methods also include (Regis and Shoemaker, 2005; Custódio et al., 2010; Conn and Le Digabel, 2013). During the last decade, there has been a considerable amount of work to handle constraints in the field of direct search methods. New algorithms have been developed to deal with bounds and linear inequalities (Lewis and Torczon, 1999, 2000) smooth nonlinear constraints (Lewis and Torczon, 2002; Kolda et al., 2006) or non-smooth constraints (Audet and Dennis Jr, 2004; Abramson et al., 2004; Dennis et al., 2004). In this work, we do not deal with constraints. Our aim here is to develop a simple and efficient algorithm that improves the heuristics-based optimization

methods that are presently used to obtain the global minimum in a multidimensional and multimodal continuous function.

1.3. Global Optimization Approaches

Local optimization methods are the best option for convex or at least unimodal, optimization problems. However, most real-world problems are not convex, and the solution provided by a local method is not guaranteed to be global. Unfortunately, even if derivatives are available, it is not possible in general to prove that a local optimum is global. A rigorous global optimization algorithm for a multimodal cost function requires in-depth exploration of the search space, and this is only possible for functions with a small number of variables, because the computational time for space exploration increases exponentially with the dimension of the search space.

To circumvent the curse of dimensionality, many global optimization algorithms use random search (Solis and Wets, 1981) and heuristics (Pearl, 1984; Michalewicz and Fogel, 2013). Some successful classes of global optimization algorithms are simulated annealing algorithms (Kirkpatrick et al., 1983), evolutionary algorithms (Back, 1996), estimation of distribution algorithms (Larrañaga and Lozano, 2001), particle swarm optimization algorithms (Kennedy and Eberhart, 1995; Poli et al., 2007), differential evolution algorithms (Storn and Price, 1997; Neri and Tirronen, 2010), tabu search algorithms (Glover, 1989; Crainic and Toulouse, 2003) or ant colony optimization algorithms (Dorigo et al., 1996; Dorigo and Stutzle, 2004).

In recent years, there has been a great interest in derivative-free methods for global optimization; see the monographs (Paulavičius and Žilinskas, 2014; Sergeyev et al., 2013; Strongin and Sergeyev, 2013; Žilinskas and Zhigljavsky, 2016; Zhigljavsky, 2012) and the references therein. In particular, there are two classes of algorithms that are of special interest. The first class is Lipschitz optimization methods. Lipschitz methods substitute the cost function by an auxiliary function that underestimates it. The auxiliary function is linear piecewise and is obtained by partitioning the search space and repeatedly using the Lipschitz property (Shubert, 1972; Sergeyev and Kvasov, 2006; Lera and Sergeyev, 2010; Kvasov and Sergeyev, 2013). An interesting Lipschitz method is DIRECT (DIviding RECTangles), which was introduced in (Jones et al., 1993). In this method, the cost function is evaluated at several sample points by using

all possible weights for local versus global search expressed by the Lipschitz constant. The DIRECT algorithm has gained popularity due to its simplicity. However, it has two weaknesses. First, it is slow in reaching the solution with high accuracy, and second, it spends an excessive number of evaluations exploring local minima. Some variants have been developed to improve these weaknesses (Gablonsky and Kelley, 2001; Finkel and Kelley, 2006; Paulavičius et al., 2014; Liu and Cheng, 2014).

1.4. Classification Of Global Optimization Problem

The problem of global optimization can be classified by the properties and constraints of the objective function. The problem is defined as an unconstrained problem without restrictions. Problem with linear (or nonlinear) restrictions and nonlinear objective function is referred to as a linear (and nonlinear) constrained problem. Moreover, the problem with constraints such as limited variables is referred to as a constrained global optimization problem, which often has functions described by one or more of the following constructions:

- The accurate gradient of the objective function (with/without constraints, if existence) is extremely costly to calculate.
- computation of the objective function (with/without constraints, if existence) is time-consuming and/or costly.
- objective functions (with/without constraints, if existing) is noisy.

Many real-life implementations often present these problems. There are no methods that can solve all of these various types of problems. Therefore, various methods have been evolved to tackle different kinds of problems. The major idea behind the evolution of these methods is to improve methods such that:

- working in a broad domain of problems,
- locates the global minimizer with an absolute guarantee,
- is simple to perform, and
- uses quite a little calculation.

In solving general continuous global optimization problems, different approaches have been developed in the last three decades. These approaches can be separated into two streams, stochastic/heuristic and deterministic.

1.4.1. Statistic Approach

Multi-start technique (see,e.g., (Anderssen and Bloomfield, 1975; Mockus, 1989; Jones et al., 1993; Litinetski and Abramzon, 1998; Huyer and Neumaier, 1999)) includes approaches that use stochastic class. Such approaches produce a lot of points in the feasible area from a verified distribution of probabilities and use some local search strategies from these points to find the optimal solution.
Simulation of annealing algorithms (see, e.g., (Pincus, 1970; Kirkpatrick et al., 1983; Szu and Hartley, 1987; Dekkers and Aarts, 1991; Bélisle, 1992)). They are generalizations of a Monte Carlo technique that, together with the physical annealing procedure for finding low-energy solids in a heat bath, generated from an analogy.

Evolutionary / Genetic algorithms (see, e.g., (Holland, 1975; Freeman et al., 1999; Forrest, 1993; Michalewicz, 2013)). These algorithms simulate biological evolution, so the species fitter, the higher the survival rate.Generally, stochastic / heuristic techniques are fast and easy to perform, but there is no guarantee that the solutions are accurate.

1.4.2. Deterministic Approach

In contrast to stochastic methods, deterministic methods are more credible along with the price of costly computational costs. Approaches that employ deterministic methods include, for example:
Branch and bound approach: The widely applied deterministic approach to extremes. The concept of the branch and bound approach is to divide the feasible region into divisions and reject some non-promising partitions using approximate lower boundaries by skipping the objective function in these partitions. In turn, each of the sub-regions is treated and possibly further subdivided in the same manner as their parent region. Different methods for estimating the lower bounds of the target functions indicate different branch types and bound methods. The primal-dual branch and bounding methods, for example, (see, e.g., (Floudas and Visweswaran, 1990, 1993; Maranas and Floudas, 1994; Androulakis et al., 1995; Adjiman et al., 1997)) Use some primal-dual methods to approximate the lower limits, but use the interval branch and the bound methods instead (see, e.g., (Moore, 1966, 1979; Alefeld and Herzberger, 2012; Hansen,

1979)). And so on, it is obvious that the branching and bounding methods only converge between branching and bounding under confirmed unity conditions.

Modulation method of the functions (see, e.g., (Goldstein and Price, 1971; Levy and Montalvo, 1985; Renpu, 1990)). It includes all methods which use an effective improvement of the objective function to achieve the current optimum local solution. Its solution procedure is based on the following three phases:

Phase 1:(Initialization) Choose a suitable initial point x_0 and let the iteration number $k := 0$.

Phase 2:(Local Search) Starting with a certain point and using any common local method to search for the x_k^* of f.

Phase 3:(Global Search) Constructing an auxiliary function to guarantee that its minimizers do not become superior to $f(x_k^*)$ and use it as an initial point for searching about a better one. If this is achieved, we put $k = k+1$ and return to Phase 2. Therefore, if such a point can not be found, we interrupt this process and consider the last established locally optimal solution as a globally optimal solution.

The approach to modification of functions is promising. The numerical performance of most methods using this approach, however, is still far from satisfactory. This consideration induces the study reported in this dissertation. We will give three model classes of methods using the function modification approach in the next passage, known as the tunneling algorithm, the filled function method, and the convexification method. More detailed discussions can be found in (Pardalos et al., 2000) about the other solution approach, recent developments, and global optimization trends.

1.4.3. Tunneling Algorithms

The mathematician Levy introduced the concepts of tunneling algorithms at the 7th Conference on Numerical Analysis in Dundee, Scotland, in 1977. And then such a paper published in SIAM Journal in 1985 (Levy and Montalvo, 1985). The primary aim of tunneling algorithms is to define a possible solution to the problem, the minimizers of which are isolated. The idea of the method is based on determining the best minimizer x_{k+1}^* for a given objective function f from the current x_k^* of f is to draw a tunnel from

x_k^*, so the end of the tunnel, $\acute{x}_{k+1}$, has the same value x_k^*. They construct an auxiliary function for more clarification and minimize it in the global search, this function is later known as (tunneling function). It comes in the form of:

$$T(x) = \frac{f(x) - f(x_k^*)}{\|x - x_k^*\|^{2\eta}} \tag{1.1}$$

where $\eta \leq 1 \in \mathbb{Z}$ is a parameter depends on the problem. There we assume x_k^* is a pole of T and the nil element of T corresponds with the range of $\{x : f(x) = f(x_k^*), \quad x \neq x_k^*\}$. Another pole for such a tunneling function can be set at x_{k+1}^* in the case $\acute{x}_{k+1} \equiv x_{k+1}^*$ is guaranteed as a local minimizer of the function f, repeat the same procedure until all global minimizers of f have been found considering the function values as $f(x_k^*)$. We may find (1.1) the simplest form of the tunneling function provided. In practice, in the event that the minimizers are isolated with the same objective function value, the general form of tunneling would be as follows:

$$T(x) = \frac{f(x) - f(x_k^*)}{\|x - x_m\|^{2\lambda_0} \prod_{j=1}^{J} \|x - x_{k+j}^*\|^{2\eta_j}} \tag{1.2}$$

where $f(x_k^*) = f(x_{k+1}^*) = \cdots = f(x_{k+J}^*)$; $j = 1, 2, \ldots, J$ are parameters dependent on the problem; x_m and λ_0 are prepared in order to smooth out any irrelevant local minimizer of T. Furthermore, in the case $\acute{x}_{k+1}$ failed to be a minimizer of f, it can be utilized as an initial point for minimizing f and found a better minimizer x_{k+1}^* of f. then they have to construct a new tunneling function at x_{k+1}^* and repeat the process.
In 1989, Yao (Yao, 1989) developed a new tunneling algorithm for the neural network framework branch to address the constrained problems of global optimization. Additionally, tunneling algorithm development can be found in (Lucidl and Piccioni, 1989; Cetin et al., 1993; Chowdhury et al., 2000), and tunneling algorithm implementations can be found in (Kanada, 1995; Chowdhury et al., 1998; Shi et al., 2001).

1.4.4. Auxiliary Functions Which Achieve Filled Function Properties

Ge proposed the first definition of filled functions in 1983, primarily at the 10th Biennial Conference on Numerical Analysis in Dundee, Scotland, and his paper was published in 1990 in (Renpu, 1990). The basic idea of the filled function (Renpu, 1990) constructs an auxiliary function (called filled function) at the current minimizer point x_k^*

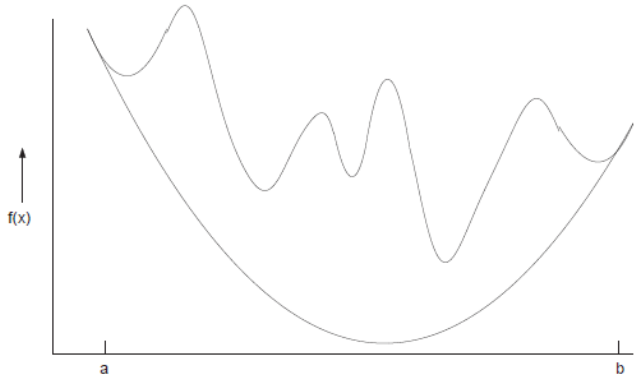


Figure 1.2. A continuous approach using transformation

of the objective function f such that x_k^* is a strict maximizer of the filled function and the current basin B_k is part of a hill of such filled function. By filling in the basin B_k of f at x_k^* and other basins of f higher than B_k, searching for a minimizer of the filled function will make the current minimizer x_k^* of f better than x_{k+1}^* of f better. It has proven that the filled function has no minimizer or saddle point in any basin of f higher than B_k, but it has a minimizer in a basin of f lower than B_k over direction $x - x_k^*$. The previous properties give us a guarantee that an $\acute{x}_{k+1}$ point can be specified in another basin that is less than B_k by minimizing the filled function using any common search method. Again, the $\acute{x}_{k+1}$ point can be used as an initial point to minimize f, and then a better f minimizer can be found.

In 1987, this idea developed by Ge (Ren-Pu, 1987) to solve the constrained problems of global optimization. Instead of immediately solving the (P) problem, Ge suggested it could minimize the exact penalty function of the (P) problem by a filled function method. A great success caused by the application of the filled function methods to solve many practical problems (see, e.g., (Borghi and Fabbri, 1997; Borghi et al., 1998, 1999)). Different algorithms can be found in (Ge and Huang, 1989; Ge, 1990; Jiannan, 1996; Kanzow, 2000).

1.4.5. Convexification Methods

To optimize non-convex continuous functions globally, the objective function is increasingly converted into a smoother function including fewer local minimizers. For a one-dimensional non-convex objective function, this can be clearly shown in Figure 1.2.

For instance, we consider the problem:

$$\min f(x). \tag{1.3}$$

This problem can be transformed into

$$\min U_\upsilon(x) = f(x) + \upsilon g(x), \tag{1.4}$$

where a strictly convex $g(x)$ has been developed, and $\upsilon > 0$ is a parameter. $U_\upsilon(x)$ is convex for sufficiently accurate υ. If $x^*_\upsilon = argmin\{U_\upsilon(x)\}$ is a minimizer of $U_\upsilon(x)$ for a fixed υ. Then x^*_υ approaches $x^* = argmin\{f(x)\}$ as $\upsilon \to 0$. Thus, the function was convexified by adding a convex term $\upsilon g(x)$.

Piela et al. (Piela et al., 1989), provided the principal approaches to construct a transformation into one more smoother include the diffusion equation, the packet annealing method of Shalloway (Shalloway, 1992), and the effective energy simulated annealing method of Coleman, Shalloway and Wu (Coleman et al., 1994). Straub (Straub, 1996) explores certain transformations used for molecular conformation issues. Figure 1.3 provides a three-dimensional representation of a function of two variables and in both cases shows the contours in the x_1, x_2 plane region. The minimizer and maximizer points of the function given can be seen in Figure 1.3 (a), Figure 1.3 (b) illustrates the strict convex form of such function. Finally, Figure 1.4 shows the contours of a strictly convex quadratic function, the ellipses representing the set of the lower level are specified in the outline here. The center at the origin which represents the function's the only global minimizer point.

Moré and Wu (Moré and Wu, 1997) proposed an algorithm for the continuous global optimization approach in 1997, which deals with the problem of distance geometry with almost 100% of success, generally, an arbitrary function could not succeed in the continuation process. The transformation may particularly remove tall, narrow valleys (hills), whereas the global minimizer (maximizer) is found in one of these valleys (hills). More algorithms for the continuous approach using transformation can be found in (Locatelli and Schoen, 2002; Coleman et al., 1994; Kostrowicki et al., 1991)

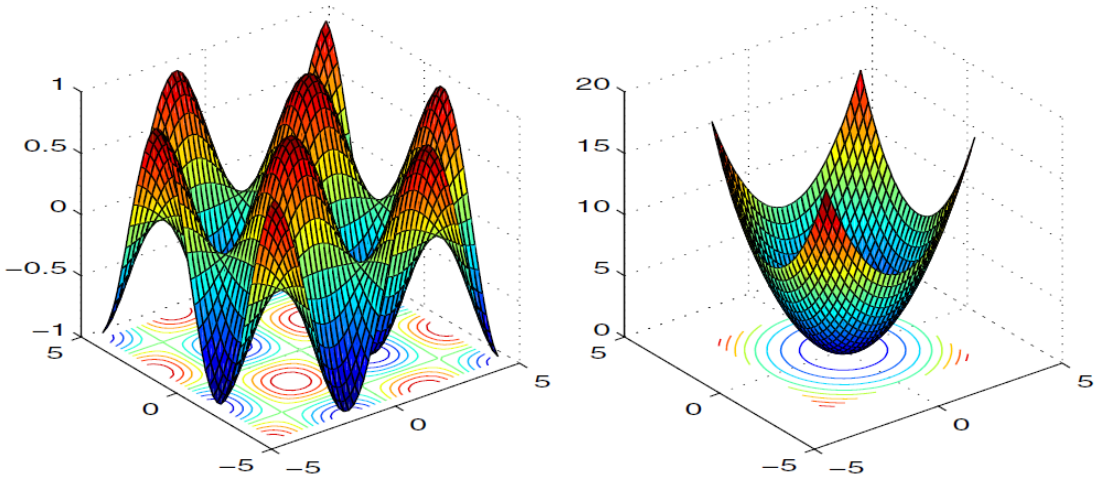


Figure 1.3. Three-dimensional representations and outlines.

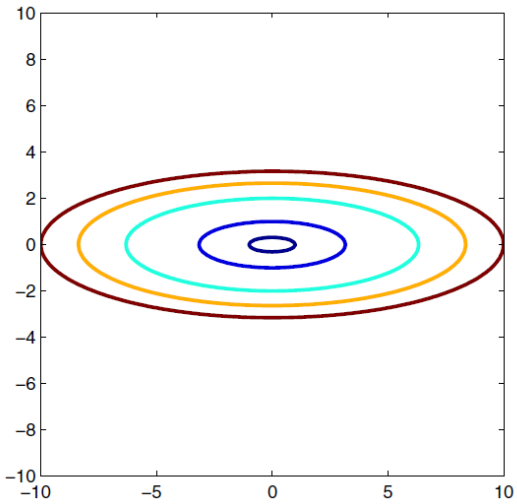


Figure 1.4. Outlines of a strictly convex quadratic function

1.5. Nonsmooth Optimization

Here we introduce a brief idea about the widely used nonsmooth optimization approaches. We take the problem of nonsmooth optimization into account:

$$\min f(x), \quad x \in S \tag{1.5}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is not a continuously derivative objective function, but we assume a given function f is a continuous locally Lipschitz on the set $S \subseteq \mathbb{R}^n$.

There are many ways to tackle nonsmooth problems. Direct application of a gradient centered on such a problem is a simple process, but it could result in failure of convergence, optimum conditions, or gradient approximation. For some of its factors, the

explanations behind those difficulties come from the non-differentiability of the objective function. In comparison, derivative-free dependent approaches may be another way, but traditional models, such as the Powell approach or genetic algorithms may not be reliable and become unreliable as the problem's dimension increases. On the other hand, the convergence of these methods has been demonstrated only for smoothness cases. Furthermore, different smoothing approaches might in some cases achieve satisfactory results but as effective as the direct non-smooth method. Consequently, particular tools to solve nonsmooth problems are required. Some methods for addressing the nonsmooth problems, including the ones were studied in (Bagirov et al., 2014) are assuming that only the objective function value as well as the subgradient are available at each point. In addition, there are some hybrid methods, which combine the characteristics of the methods described above, and discrete gradient methods which may be regarded as semi-derivative free methods for nonsmooth problems. Nonsmooth methods can be used efficiently for smooth problems but not conversely, and we can therefore say that smooth optimization methods address a wider range of issues than smooth optimization.

1.5.1. Significance Of The Study

New algorithms are proposed to solve the global optimization problem based on some auxiliary functional methods and a proposed smoothing approach. Numerical results show that this method can locally resolve the optimal solutions and thus solve the high dimensions benchmark functions by offering a solution closer to the global one. In addition, a variety of filled functional methods and globally convex transformation are given, and some useful variations are suggested to improve efficiency.

1.5.2. Research Objectives

This thesis aims at developing and testing a globally integrated algorithm, including auxiliary function method to provide a calculation optimum value that can solve a general class of continuous optimization problems. In the beginning, we use an efficient auxiliary function algorithm, which can ignore local solutions and therefore offer a more globally acceptable solution. The resulting problem has fewer local minimizers, so a global solution can be accomplished with less computational effort. The threshold

value for medical images was calculated and used in the segmentation of skin lesions by one of the proposed algorithms. Three major tasks in the fields of skin lesion image processing, injury segmentation, lesion dermoscopy, and lesion classification were addressed as a result of this issue. We try to determine the best global solution for the problems that have several local minimizers. Our research goals are summarized as follows:

- to analyze existing auxiliary function approaches and compare their computational performance when addressing the problems of continuous optimization and mixed optimization;
- to develop new auxiliary function algorithms for efficient resolution of problems of continuous optimization;
- to propose that, this problem is decomposed into continuous upper and lower level problem;
- to propose a new method of smoothing that addresses non-smooth problem.
- to determine the nearest global solution to minimize the images pixels for the threshold value finding.

1.5.3. Thesis Organization

The main objective of this thesis is to establish some high-performance methods for solving the general continuous global optimization problem. After this introduction, this dissertation is divided into six sections.

Section 2, which deals with continuous global optimization, first we identified the problem formulation, also described the standard assumptions imposed on the problem. Then, we recall some definitions of differentiability, continuity, functionality, and convexity, the filled function and extend some definitions in continuous global optimization to meet our needs. In Section 3, we have presented concepts for the filled function method, including a demonstration example, a literature review of the completed function method, and the characteristics of the filled function method. Section 4 includes a new class of auxiliary functions with three new filled function methods and a globally convex auxiliary function. Section 5 proposes segmenting melanoma skin lesions as an application for global optimization. Section 6 reported the numerical results for the proposed filled functions. Finally, in Section 7 there was a review of what we have

accomplished with recommendations for further development.

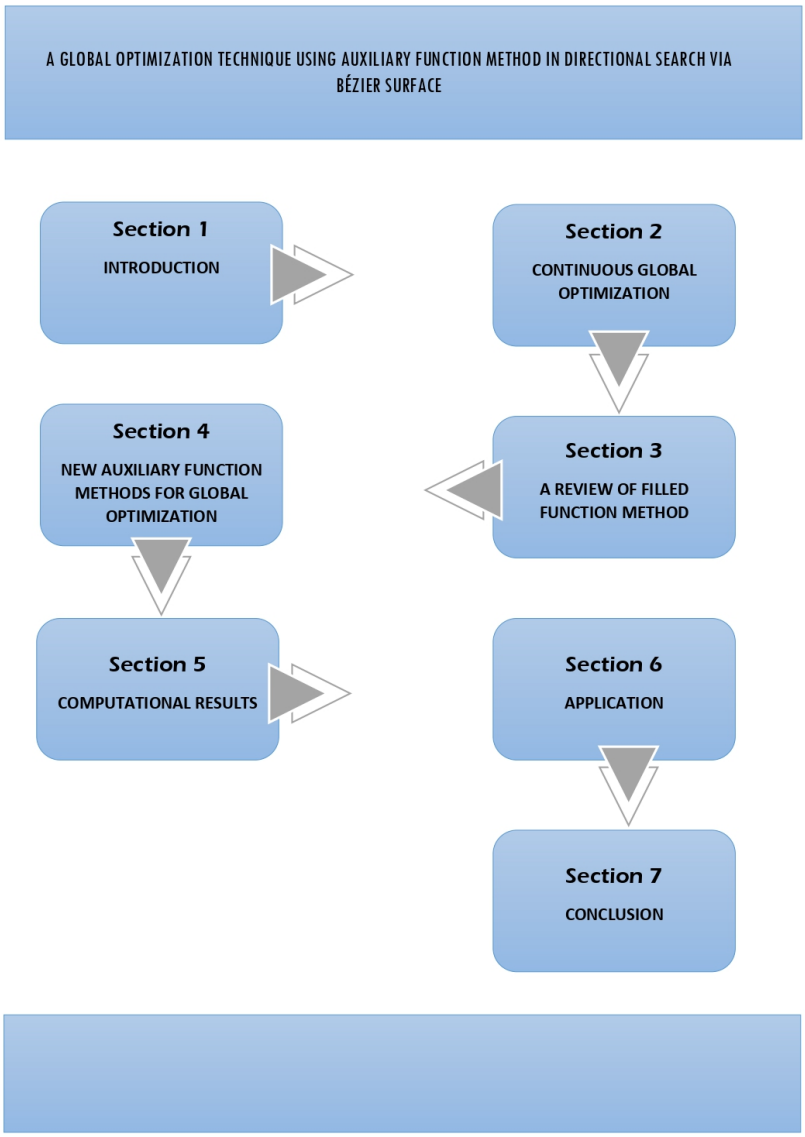


Figure 1.5. Thesis organization

2. CONTINUOUS GLOBAL OPTIMIZATION

2.1. Problem Formulation

We consider the problem of finding the global minimizer of the unconstrained global optimization problem:

$$(P) \quad \min_{x\in\Lambda} f(x) \tag{2.1}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is a nonlinear continuous objective function and x is the decision variable, where $x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n$. The set Λ is the domain of the search space, is defined by assign the upper bound (u_j) and lower bound (l_j) limits of each jth component of x.

The optimization problem (2.1) includes the determination of the best point x in Λ that gives the least value of the objective function f. This point is known as the minimizer of f on Λ. In concern with this optimization problem, one has to distinguish between two kinds of minimizers, local and global minimizers. The point $x_k^* \in \Lambda$ is said to be a local minimizer of f over Λ in a case there exists $\varepsilon > 0$ such that $f(x) \leq f(x_k^*), \forall x \in \Lambda$ and $\|x - x_k^*\| < \varepsilon$. furthermore, the point $x^* \in \Lambda$ is said to be a global minimizer of f over Λ in case that $f(x_k^*) \leq f(x), \forall x \in \Lambda$. therefore, we can say that the global minimizer x^* is the best local minimize in Λ. Without losing generality, we consider only minimization problems for the reason that maximizing f is identical to minimizing $-f$.

Let x_1 and x_2 are any two distinct points in the set Λ, At the rest of this thesis, we have to take in mind the following assumptions:

Assumption 2.1. $\Lambda \subset \mathbb{R}^n$ is a closed bounded set and $int(\Lambda)$ (the interior of Λ). There exists a constant K satisfying

$$1 \leq \max_{x_1, x_2 \in \Lambda} \|x_1 - x_2\| \leq K < \infty \tag{2.2}$$

Assumption 2.2. $0 < L < \infty$ is a constant presented to satisfy

$$\|x_1 - x_2\| \leq L\|f(x_1) - f(x_2)\| \tag{2.3}$$

Assumption 2.3. The function f is differentiable in $\mathbb{R}^n$, and there can be an infinite number of minimizers, Though there is a finite number of different minimizers values. Most auxiliary function methods are prepared to solve box constrained problems. Un-

constrained and more generally constrained problems can be modified into an equivalent box constrained form by adding a penalty term to the objective function f (Ng et al., 2005b), for the following some cases (see, e.g.,(Xu et al., 2019; Sahiner et al., 2016; Yilmaz and Sahiner, 2019))

2.2. Theoretical Background

To illustrate the explanation in the remaining parts of this thesis, it was necessary to recall some definitions in sets as well as in functions and extend some fundamental notations and definitions as required during the thesis. These fundamentals will be for functions and derivatives, convex Analysis, nonsmooth optimization, and continuous global optimization to suit our needs. We recall the formal definition of the filled function as well.

2.2.1. Functions and sets

Definition 2.1. The open ball with center $y \in \mathbb{R}^n$ and radius r is denoted by $B(y,r)$ and defined by:

$$B(y,r) = \{x \in \mathbb{R}^n : \|x-y\| < r\}.$$

Definition 2.2. Given a set $\Lambda \subseteq \mathbb{R}^n$, a point $y \in \Lambda$ is an interior point of Λ if there exists $r > 0$ for which $B(y,r) \subseteq \Lambda$. The set of all interior points of a given set Λ is called the interior of the set and is denoted by $int(\Lambda)$:

$$int(\Lambda) = \{x \in \Lambda : B(x,r) \subseteq \Lambda \; for \; some \; r > 0\}.$$

Definition 2.3. Given a set $\Lambda \subseteq \mathbb{R}^n$, a boundary point of Λ is a point $x \in \mathbb{R}^n$ satisfying the following: any neighborhood of x contains at least one point inΛ and at least one point in its complement Λ^c. The set of all boundary points of a set Λ is denoted by $bd(\Lambda)$ and is called the boundary of Λ.

Definition 2.4.

1. A set $\Lambda \subseteq \mathbb{R}^n$ is called bounded if there exists $M > 0$ for which $\Lambda \subseteq B(0,M)$.
2. A set $\Lambda \subseteq \mathbb{R}^n$ is called compact if it is closed and bounded.

Examples of compact sets are closed balls and line segments. The positive orthant is not compact since it is unbounded, and open balls are not compact since they are not closed.

2.2.2. Differentiability and continuity

Definition 2.5. It is said that the function $f : \mathbb{R}^n \to \mathbb{R}$ is continuous at the point $\hat{x} \in \Lambda$ if for any $\varepsilon > 0$, $\exists \delta > 0$ s.t. for $x \in \Lambda$ and $\|x - \hat{x}\| < \delta$ satisfy $|f(x) - f(\hat{x})| < \varepsilon$.

Definition 2.6. Let $f : \mathbb{R}^n \to \mathbb{R}$. It is said that f has a directional derivative $Df(x,d)$ at a point $x \in \mathbb{R}^n$ along the direction $d \in \mathbb{R}^n$ if the following limit exists:

$$\lim_{t \to 0^+} \frac{f(x+td) - f(x)}{t} = Df(x,d).$$

Definition 2.7. Let $f : \mathbb{R}^n \to \mathbb{R}$. It is said that f has a partial derivative $\partial f(x)/\partial x_j$ at the point $x \in \mathbb{R}^n$ with respect to the variable x_j if the limit exists

$$\lim_{t \to 0^+} \frac{f(x_1, \ldots, x_j + t, \ldots, x_n) - f(x_1, \ldots, x_j, \ldots, x_n)) - f(x)}{t} = \partial f(x)/\partial x_j.$$

If f admits partial derivatives with respect to all components, we will denote by $\nabla f(x)$ the vector (column) n-dimensional of the prime partial derivatives of f at x, i.e.

$$\nabla f(x) = \begin{bmatrix} \frac{\partial f(x)}{\partial x_1} \\ \vdots \\ \frac{\partial f(x)}{\partial x_n} \end{bmatrix}$$

Definition 2.8. Let $f : \mathbb{R}^n \to \mathbb{R}$ and $x \in \mathbb{R}^n$. If there are second partial derivatives $\partial^2 f(x)/\partial x_i \partial x_j$, for $i = 1, \ldots, n$ and $j = 1, \ldots, m$ at x, we define a Hessian matrix of f at x the matrix $n \times n$

$$\nabla^2 f(x) = \begin{bmatrix} \frac{\partial^2}{\partial x_1^2} f(x) & \cdots & \frac{\partial^2}{\partial x_1^2 \partial x_1^n} f(x) \\ \vdots & \ddots & \vdots \\ \frac{\partial^2}{\partial x_n^2 \partial x_1^2} f(x) & \cdots & \frac{\partial^2}{\partial x_n^2} f(x) \end{bmatrix}$$

2.2.3. Global and Local Optima

Although our prime interest in this part is to debate the minimizer and maximizer points of a given function f on the entire space, we will nevertheless present additional common definitions of a global minimizer and maximizer point of a function on a given set Λ.

Definition 2.9. A point $x^* \in \Lambda$ is called a global minimizer point of the function f on Λ if:

$$f(x^*) \leq f(x) \qquad \forall x \in \Lambda,$$

and it is said that $x^* \in \Lambda$ is a strict global minimizer point of f on Λ if:

$$f(x^*) < f(x) \qquad \forall x \in \Lambda, \quad x \neq x^*.$$

Definition 2.10. A point $x_k^* \in \Lambda$ is called a local minimizer point of f on Λ if there exists a neighbourhood $B(x_k^*, \varepsilon)$, with $\varepsilon > 0$ such that

$$f(x_k^*) \leq f(x) \quad \forall x \in \Lambda \cap B(x_k^*, \varepsilon)$$

and it is said that $x_k^* \in \Lambda$ is a strict local minimizer point of f on Λ if there exists a neighbourhood $B(x_k^*, \varepsilon)$, with $\varepsilon > 0$ such that

$$f(x_k^*) < f(x) \qquad \forall x \in \Lambda \cap B(x_k^*, \varepsilon), \quad x \neq x_k^*.$$

Definition 2.11. A local minimizer point $x_k^* \in \Lambda$ is called an isolated local minimizer point of f on Λ if there is a neighborhood $B(x_k^*, \rho)$, with $\rho > 0$ such that x_k^* is the only local minimizer point in $B(x_k^*, \rho) \cap \Lambda$.

Definition 2.12. Let $f : \Lambda \to \mathbb{R}$ is a function defined on an open set $\Lambda \subseteq \mathbb{R}^n$. Suppose that $x_k^* \in int(\Lambda)$ and f is differentiable over some neighborhood of x_k^*. Then x_k^* is called a stationary point of f if

$$\nabla f(x) = 0$$

.

Definition 2.13. Let $f : \Lambda \to \mathbb{R}$ is a function defined on an open set $\Lambda \subseteq \mathbb{R}^n$. and suppose that f is continuously differentiable over Λ. A stationary point x_k^* is said to be a saddle point of f over Λ if it is neither a local minimizer point nor a local maximizer of f over Λ.

Definition 2.14. The set Λ on which the optimization of the objective function f is performed is called the feasible set, and any point $x \in \Lambda$ is called a feasible solution.

Definition 2.15. d_k is a descent direction of f at x if $f(x+d_k) < f(x)$.

Definition 2.16. The basin of a given function, f, at a selected stationary point x_1^* to be a related domain include that point x_1^*, and the steepest downhill trajectory converges to x_1^* from any starting point and denoted by B_k^*. Additionally, another point basin, x_2^*, is lower than the x_1^* basin if $f(x_2^*) < f(x_1^*)$, respectively. The hill of $f(x)$ at x_1^* is the basin of $-f(x)$ at its isolated minimizer x_1^*.

Definition 2.17. If a given compact set Λ_k^* is a minimizer of the function f over Λ and B_k^* is the Basin of the function f at Λ_k^* over Λ. S_k is called the simple basin of the function f at Λ_k^* over Λ if $\Lambda_k^* \subset S_k \subset B_k^*$ and satisfying $(x-x_k^*)^T \nabla f(x) > 0$ for any $x \in S_k \setminus \Lambda_k^*, x_k^* \in \Lambda_k^*$.

From the previously introduced characterizations of the descent directions, we derive necessary conditions and sufficient conditions for an assigned point to be a local minimizer point of a real-value function.

Theorem 2.1. Let $x_k^* \in \mathbb{R}^n$ be a local minimizer point of problem 2.1, then there cannot be a decent direction for f at x_k^* .

Proof. If there was a direction d_k of descent in x_k^*, then, by definition 2.15, at every neighborhood of x_k^* it would be possible to find, for $t > 0$ quite small, a point $x_k^* + td_k \in \mathbb{R}^n$ such that $f(x_k^* + td_k) < f(x_k^*)$, which contradicts the hypothesis that x_k^* is a local minimizer point. □

From first-order stated in the previous paragraph immediately follows a necessary

condition of optimality of the first order.

Theorem 2.2. Let $f : \mathbb{R}^n \to \mathbb{R}$ continuously differentiable in the neighborhood of a point $x_k^* \in \mathbb{R}^n$. A necessary condition for x_k^* to be a local minimizer point of f is that x_k^* be a stationary point of f, that is:

$$\nabla f(x_k^*) = 0.$$

Proof. If $\nabla f(x_k^*) \neq 0$, then the direction $d_k = -\nabla f(x_k^*)$ satisfies

$$d_k^T \nabla f(x_k^*) = \|\nabla f(x_k^*)\|^2 < 0.$$

Consequently, there is a descent direction and this contradicts the hypothesis that x_k^* is a local minimizer point. □

If f can be differentiated twice, a necessary condition of the second order is immediately obtained.

Theorem 2.3. Let $f : \mathbb{R}^n \to \mathbb{R}$ be twice continuously differentiable in the neighborhood of a point $x_k^* \in \mathbb{R}^n$. A necessary condition for x_k^* to be a local minimizer point of f is that x_k^* is a stationary point of f and the Hessian matrix is positive semidefinite, that is:
(a) $\nabla f(x_k^*) = 0$;
(b) $y^T \nabla^2 f(x_k^*) y \geq 0$, for each $y \in R^n$.

Proof. (a) follows from Theorem 2.2. If (b) does not apply, y must exist such that

$$y^T \nabla^2 f(x_k^*) y < 0,$$

and therefore, $\nabla f(x_k^*)^T y = 0$, and since $y's$ direction will be a descent direction of f, which contradicts the hypothesis that x_k^* is a local minimizer point. □

2.2.4. Convexity

In this part, we introduce some fundamental concepts on the convexity required in the other sections.

Definition 2.18. A set $C \in \mathbb{R}^n$ is said to be convex if, however, $x_1, x_2 \in C$, the segment $[x_1, x_2]$ joining x_1 and x_2 is all contained in C, ie if:

$$x_1, x_2 \in C,\ \lambda \in \mathbb{R},\ 0 \leq \lambda \leq 1\ imply\ (1-\mu)x_1 + \lambda x_2 \in C$$

.

Definition 2.19. Let $C \in \mathbb{R}^n$ be a convex set and let $f : C \in \mathbb{R}$. It is said that f is convex on C if, however, we fix $x, y \in C$ such that

$$f((1-\lambda)x + \lambda y) \leq (1-\lambda)f(x) + \lambda f(y),$$

for each λ such that $0 \leq \lambda \leq 1$.
It is said that f is strictly convex on C if, however we fix $x, y \in C$ with $x \neq y$ we have

$$f((1-\lambda)x + \lambda y) < (1-\lambda)f(x) + \lambda f(y),$$

for each λ such that $0 < \lambda < 1$.

Definition 2.20. Let $C \in \mathbb{R}^n$ be a convex set. A point $x \in C$ is called a point extreme of C if it cannot be expressed as a convex combination of two points of C distinguished by x or equivalently, if there are no $y, z \in C$ with $z \neq y$ such that

$$x = (1-\lambda)y + \lambda z, with\ 0 < \lambda < 1.$$

Definition 2.21. Let $S \subseteq \mathbb{R}^n$. A convex envelope of S is defined as the set $conv(S)$ of all convex combinations of elements of S.

Proposition 2.1. Let $S \subseteq \mathbb{R}^n$ be a compact set and let f be a continuous function on S. The convex envelope $conv(f(x))$ of $f(x)$ on S satisfy:

i. $conv(f(x))$ is convex on S;

ii. $conv(f(x)) \leq f(x)$ for each $x \in S$;

iii. if $\tilde{f}(x)$ is a convex function on S such that $\tilde{f}(x) \leq f(x)$ for all $x \in S$, then $\tilde{f}(x) \leq convf(x)$ for each $x \in S$.

In the case of convex programming problems, where S is a convex set and $f : S \to \mathbb{R}$ is a convex function, we can establish that all local minimizer points are points of the global minimizer, and we can characterize geometrically the set of optimal solutions .

Proposition 2.2. Let $S \subseteq \mathbb{R}^n$ be a convex set and let f be a convex function on S. Then, every local minimizer point of f on S is also a global minimizer; moreover, the set of points of global minimizer of f on S is a convex set.

Proof. Let x_k^* be a local minimizer point of f on S and be $x \neq x_k^*$ any other point of S. Since x_k^* is assumed to be a local minimizer, there must exist a neighborhood $B(x_k^*; \rho)$ with $\rho > 0$ such that

$$f(x_k^*) \leq f(y), \ for\ each\ y \in B(x_k^*; \rho) \cap S$$

and therefore, taking into account the convexity of S and the convexity of $B(x_k^*; \rho)$, it is possible to find a value λ, with $0 < \lambda \leq 1$, such that the corresponding point of the segment $z = (1-\lambda)x_k^* + \lambda x$ belongs to $B(x_k^*; \rho) \cap S$, which implies $f(x_k^*) \leq f(z)$. For the convexity of f we have moreover

$$f(z) = f((1-\lambda)x_k^* + \lambda x) \leq (1-\lambda)f(x_k^*) + \lambda f(x)$$

.

From the previous inequalities, we get:

$$f(x_k^*) \leq f(z) \leq (1-\lambda)f(x_k^*) + \lambda f(x),$$

from which, since $\lambda > 0$, it follows that $f(x_k^*) \leq f(x)$. Let $X \subseteq S$ be the set of optimal solutions; if $X = \phi$ or $X = x_k^*$ the statement 2.11 is obvious. Suppose then that z, y are two any points of X, i.e., we assume that

$$f(z) = f(y) = \min_{x \in S} f(x)$$

Taking into account the convexity hypotheses, it must be then, for each $\lambda \in [0,1]$

$$f((1-\lambda)z+\lambda y) \leq (1-\lambda)f(z)+\lambda f(y) = f(z),$$

that implies that $[z,y] \subseteq X$, therefore that X is convex. □

2.2.5. Directional Search Based Methods

The unidimensional search methods have essentially the objective of constructively globalizing the optimization algorithms, i.e., of ensuring that a stationary point is reached asymptotically, starting from an arbitrarily chosen initial point.
In the following, we refer to the case in which, for each point assigned $x_k \in \mathbb{R}^n$, the direction $d_k \in \mathbb{R}^n$ is assumed to be known and the one-dimensional search consists in the determination of the step $\alpha_k \in \mathbb{R}$ along d_k and therefore of the point $x_k + \alpha_k d_k$ as shown in Figure 2.1. In this case, we are talking about line search, i.e., searching along the line passing through x_k and parallel to d_k. Suppose that the sequence $\{x_k\}$ generated by the algorithm considered is defined assuming:

$$x_{k+1} = x_k + \alpha_k d_k,\ k = 0,1,... \tag{2.4}$$

The step α_k is determined with reference to the function of a single real variable, $\phi : \mathbb{R} \to \mathbb{R}$ defined (for a fixed k) from:

$$x_{k+1} = x_k + \alpha_k d_k,$$

that characterizes the trend of f along the search direction d_k. We have, of course, in particular, $\phi(0) = f(x_k)$.
Being f differentiable, there exists a derivative of $\phi(\alpha)$ with respect to α, denoted by $\dot{\phi}(\alpha)$. By using the rules of derivation of functions composed of several variables, it is easy to verify that

$$\dot{\phi}(\alpha) = \nabla f(x_k + \alpha d_k)^T d_k, \tag{2.5}$$

and therefore coincide with the directional derivative of f along the direction d_k at the point $x_k + \alpha d_k$. In particular, we have

$$\dot{\phi}(0) = \nabla f(x_k)^T d_k$$

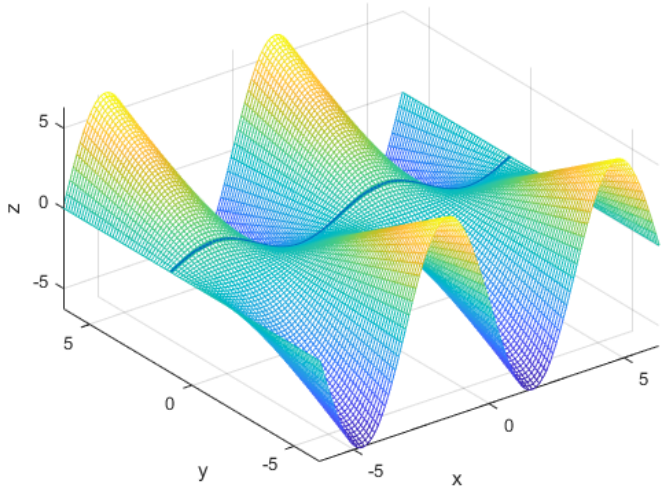


Figure 2.1. Searching in one dimension

and therefore $\dot{\phi}(0)$ is the directional derivative of f at x_k along d_k. If f is twice differentiable, there also exists the second derivative $\ddot{\phi}(\alpha)$ of ϕ with respect to α and we have

$$\ddot{\phi}(\alpha) = d_k^T \nabla^2 f(x_k + \alpha d_k) d_k, \tag{2.6}$$

All the one-dimensional search algorithms allow theoretically to satisfy the condition

$$\lim_{k\to\infty} \frac{\nabla f(x_k)^T d_k}{\|d_k\|} = 0, \tag{2.7}$$

and therefore, under appropriate hypotheses on the search directions, they allow to establish the convergence to stationary points, based on the sufficient conditions considered in the previously.

2.2.6. Bézier Curves

In the sense that there are no angular edges, Bézier curves are a commonly used tool in computer graphics for modeling "smooth" behavioral curves. These are based on a set of control points, often in three (quadratic curves of Bézier) or four (cubic curves of Bézier) ordered and finite. Nevertheless, the curve usually not passes through the control point, but shows some attraction to the "elastic cord", connecting the extreme point with the control point, which results in the flexible deformation. The control points used to generate the Bézier curve include P_0, P_1, P_2. The Q_0 point starts its path P_0P_1 segment direction beginning at P_0 and moves according to the equation

$Q_0 = P_0 + t(P_1 - P_0)$, where $0 \leq t \leq 1$. The parameter t can be considered as a measure of the percentage of the path traveled (actually when multiplied by 100); in the graph $t = 0.25$ corresponds to 25%. Thus, when $t = 0$, we are at $Q_0 = P_0$, when $t = 1$, $Q_0 = P_1$, and for the intermediate values of t, the point Q_0 goes through the different points of the segment P_0P_1. Simultaneously, and in an entirely analogous way, the point Q_1 crosses the segment P_1P_2 according to the equation $P_1 = P_1 + t(P_2 - P_1)$. Finally, for each t, the point B is located on the Q_0Q_1 segment following the same type of linear interpolation $B = Q_0 + t(Q_1 - Q_0)$. The equation of the Bézier curve is therefore given by:

$$\begin{aligned} B(t) &= Q_0 + t(Q_1 - Q_0) = (1-t)Q_0 + tQ_1 \\ &= (1-t)((1-t)P_0 + tP_1) + t((1-t)P_1 + tP_2) \\ &= (1-t)^2P_0 + 2t(1-t)P_1 + t^2P_2. \end{aligned} \tag{2.8}$$

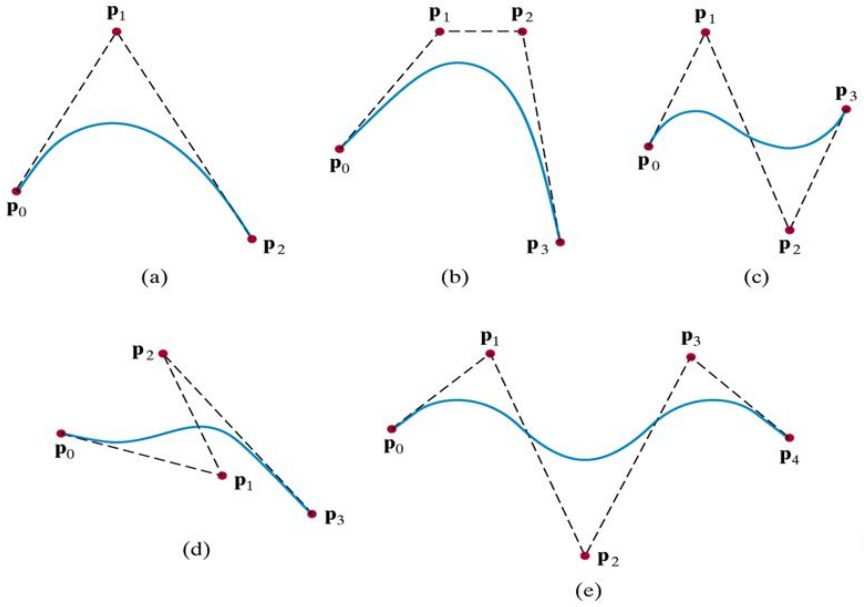


Figure 2.2. Various examples of Bézier curves

The control points used to generate the Bézier curve with four control points include P_0, P_1, P_2, P_3. With the first three points, we proceed as in the previous case to determine a point called R_0 in the graph, and with the last three we do the same to determine a point called R_1 in the graph. Between the two points, the linear interpolation corresponding to the parameter t is performed to determine the point B of the Bézier curve generated

by these control points. Proceeding as in the previous case, the degree three polynomial corresponding to this curve is obtained with four control points.

$$\begin{aligned} B(t) &= (1-t)R_0 + tR_1 \\ &= (1-t)((1-t)_2P_0 + 2t(1-t)P_1 + t^2P_2) + t((1-t)^2P_1 + 2t(1-t)P_2t^2P_3) \\ &= (1-t)^3P_0 + 3t(1-t)^2P_1 + 3t^2(1-t)P_2 + 3(1-t)P_3. \end{aligned} \tag{2.9}$$

The same method may generate higher-grade Bézier curves, which are typically not used due to the computational effort required. Instead, it is preferable to model complex profiles using several low-grade Bézier curves, with groups of control points that are properly close together, so that they link well together. As with Lagrange interpolation. In general, the Bézier curves with n control points ($n = 2,3,4,\ldots,n$) are expressed in the form:

$$B(t) = \sum_{i=0}^{n} P_i b_{i,n}(t), \;\; t \in [0,1], \tag{2.10}$$

where
$$b_{i,n}(u) = \binom{n}{i} t^i(1-t) \; where \; \binom{n}{i} = \frac{n!}{n!(n-i)n!}, \;\; t \in [0,1] \tag{2.11}$$

Sometimes Bézier curves are also considered for which the control points, P_0, P_1, P_2,...,P_n have weights w_0, w_1, w_2,..., w_n (which would correspond to assign different powers of attraction to different control points) and then the formula to consider would be:

$$B(t) = \frac{\sum_{i=0}^{n} b_{i,n}P_i w_i}{\sum_{i=0}^{n} b_{i,n}w_i}. \tag{2.12}$$

3. A REVIEW OF FILLED FUNCTION METHOD

This section begins with the filled function definitions, followed with a generic filled function approach ,then a review of some previous works in the same context. The characteristics of the filled function will be presented at the end of this section.

Definition 3.1. Let x_k^* is a current minimizer of f. Let B_k is the basin of f at x_k^* over Λ. A function $F : \Lambda \to \mathbb{R}$ is said to be a filled function of f at x_k^* if it satisfies the following properties:

C1 x_k^* is a maximizer of F and the whole basin B_k of f at x_k^* over Λ becomes a part of a hill of F;

C2 F has no stationary points in any basin of f higher than B_k;

C3 If f has a basin B_{k+1} at x_{k+1}^* lower than B_k, then there exists a point $x' \in B_{k+1}$ is a minimizer of F.

One of the most recent methods established solutions for the global optimization is the filled function method. This is when the current local minimizer has been identified by any common local search, the task of the filled function is to escape to better one by constructing an auxiliary function with this minimizer. In this thesis, we constructed three new auxiliary functions, such functions achieve the conditions C1-C3 in definition 3.1.

3.1. Generic Filled Function Approach

It is possible to illustrate the basic principles of the filled function method as follows. Firstly, an initial point is chosen and a local search of the objective function f is carried out to determine an appropriate local minimizer. An auxiliary function is installed on such a local minimizer, which is called a filled function, in which the local minimizer of f will be a local maximizer of $FF1$. When minimizing $FF1$, the local minimizer can be identified or the limits of the feasible region can be attained. Typically, the $FF1$ local minimizer represents a new starting point to minimize f to find a better one. Once more, at this enhanced point, a newly filled function $FF2$ is created. This process is continued until no improved local minimizer is being identified for the previously filled

function. The last local minimizer found is therefore considered as a global minimizer approximation. In some cases, after the advanced stages of the search process close the box restricted boundaries of the feasible area, the local minimizer can not be reached, this can be handled by adjusting the parameters of the filled function and then repeating the search process. The parameter refinement proceeds so that the parameters meet their default limits; the best solution found so far is considered as the global minimizer. Note that the number of parameters of the filled function is different from one function to another, therefore, some parameters are single and some are multiparameter, but less parameter-free.

3.2. Literature review of filled function method

3.2.1. Continuous Filled Function Methods

The function 3.1 was the kernel from which the idea of filled function originated. The principle starts by choosing a minimizer point x_1^* and then constructing an auxiliary function at that point. This concept had been proposed in 1983 by Ge (Ren-Pu, 1987) as follows:

$$D(x,x_1^*,) = \frac{1}{\rho + f(x)} \times \exp\left(-\frac{\|x-x_1^*\|^2}{\varsigma^2}\right), \tag{3.1}$$

Where ρ and ς are parameters. Such a function developed by Ge to be rewritten in the same year as:

$$\tilde{D}(x,x_1^*,\rho,\varsigma) = \frac{1}{\rho + f(x)} \times \exp\left(-\frac{\|x-x_1^*\|}{\varsigma^2}\right). \tag{3.2}$$

Qin and Ge (Ge and Qin, 1987) played a central role in the development of this method by introducing a family of filled functions as shown below:

$$\chi_1(x,x_1^*,\rho,\varsigma) = -\varsigma^2 \times \ln(\rho + f(x)) - \|x-x_1^*\|^2, \tag{3.3}$$

$$\chi_2(x,x_1^*,\rho,\varsigma) = -\varsigma^2 \times \ln(\rho + f(x)) - \|x-x_1^*\|, \tag{3.4}$$

$$\Gamma(x,x_1^*,\delta) = [f(x_1^*) - f(x)] \times \exp\left(\delta\|x-x_1^*\|^2\right), \tag{3.5}$$

$$\zeta(x,x_1^*,\delta) = (f(x_1^*) - f(x)) \times \exp\left(\delta\|x-x_1^*\|\right), \tag{3.6}$$

$$\psi(x,x_1^*,\beta) = 2\beta \times (f(x_1^*) - f(x))(x-x_1^*) - \nabla f(x), \tag{3.7}$$

$$\xi(x,x_1^*,\beta) = \beta \times (f(x_1^*) - f(x))\frac{(x-x_1^*)}{\|x-x_1^*\|} - \nabla f(x). \tag{3.8}$$

Instead of the functions (3.7) and (3.8), Wu proposed the following two filled functions because such filled functions do not succeed in the existence of the objective function:

$$F(x,x_1^*,\kappa) = (f(x_1^*) - f(x)) \times (\|x-x_1^*\|^2 + 1)^{\kappa}, \tag{3.9}$$

$$Q(x,x_1^*,\kappa) = -(f(x) - f(x_1^*)) \times (\|x-x_1^*\| + 1)^{\kappa}, \tag{3.10}$$

Sheng and Wang suggested a more filled function in 1992 as follows:

$$L(x,x_1^*,\mu,\nu) = \ln(\frac{1}{\mu + f(x)} + 1) \times \exp(-\frac{\|x-x_1^*\|}{\nu^2}). \tag{3.11}$$

Most of the filled functions provided after the year 2000 were function generalization or simplification of (3.1) and (3.5). In 1994, for instance, Zhuang (Jiannan, 1994) proposed the following function, which is a function generalization of (3.1)

$$\varphi(x,x_1^*,\upsilon,\eta) = \xi(\upsilon + f(x)) \times \exp(-\frac{\|x-x_1^*\|}{\eta^2}), \tag{3.12}$$

where ξ is twice continuously differentiable. Again in 2001, the function (3.10) developed by Xu et al. This resulted in the following filled function:

$$F(x,x_1^*,\gamma,\delta) = \zeta[\gamma + f(x)] \times \exp(\delta\xi(\|x-x_1^*\|^{\kappa})), \tag{3.13}$$

where $\kappa > 0$ and δ Meet conditions and the functions ζ and ξ. In addition to the parameter β.Recently, the function (3.13) developed by Xu and given as follow:

$$\phi(x,x_1^*,\delta) = -\chi(f(x) - f(x_1^*)) - \delta\xi(\|x-x_1^*\|^{\kappa}), \quad \kappa > 0 \in \mathbb{Z}, \tag{3.14}$$

In addition, another amendment of function (3.5) by Kong (Min, 2000) in the year 2000 has become as follows:

$$H(x,x_1^*,\delta) = -\ln[f(x) - f(x_1^*) + 1] \times \exp(\delta\|x-x_1^*\|^2), \tag{3.15}$$

where $\delta > 0$, is sufficiently large. recently, Liu (Liu, 2001a,b, 2002) present the following class of filled functions in one parameter:

$$L_1(x) = \frac{1}{\ln[1 + f(x) - f(x_k^*)]} - q\|x-x_k^*\|^2,\ f(x) > f(x_k^*) - 1, \tag{3.16}$$

$$L_2(x) = \frac{1}{\arctan[f(x) - f(x_k^*)]} - q\|x - x_k^*\|^p,\ p > 0 \in \mathbb{Z}, \tag{3.17}$$

$$L_3(x) = -[f(x) - f(x_k^*)]^{1/m} - q\|x - x_k^*\|^2,\ m > 1 \in \mathbb{Z}, \tag{3.18}$$

$$L_4(x) = \frac{1}{[f(x) - f(x_k^*)]^m} - q\|x - x_k^*\|^p,\ p > 0 \in \mathbb{Z}, m > 1 \in \mathbb{Z}, \tag{3.19}$$

where f is twice continuously differentiable; x_k^* is isolated for all k; $\nabla f(x)$ is positive define for all k; and q is sufficiently large. This class of filled functions generalized as follows

$$L_k(x) = \delta(f(x) - f(x_k^*)) - q\|x - x_k^*\|^p, \tag{3.20}$$

where $p > 0 \in \mathbb{Z}$ is a constant; $q > 0$ is a parameter; and $\delta : \mathbb{R}_{++} \to \mathbb{R}$ is a continuously differentiable and strictly decreasing function.

3.2.2. Discrete Filled Function Methods

In 1998, Zhu (Zhu, 1998) was the first researcher who adjusted the continuous filled function directly to solve discrete optimization problems. Therefore, he defined a filled function containing two parameters as follows:

$$Z(x, x_k^*, \delta, \upsilon) = \frac{1}{\delta + f(x)} \exp\left(\frac{\|x - x_k^*\|^2}{-\upsilon^2}\right), \tag{3.21}$$

where the parameters δ and υ are adjusted to satisfies $0 < \delta + f(x_k^*) < \xi$ and $\upsilon^2 \ln(\frac{\delta + \hat{f}}{\delta + f(x_k^*)}) < 1$, where $\hat{f}$ is the upper bound of f over X and $\xi \leq \min\{|f(x_1) - f(x_2)| : f(x_1) \neq f(x_2, x_i \in X, i = 1, 2\}$. Zhu proposes that the algorithm should terminate in the case all searches for a minimizer of $Z(x, x_k^*, \delta, \upsilon)$ beginning in some $N(x_k^*)$ stop at vertices without finding an improved point of f.

In 2005, Ng et al.(Ng et al., 2005b) proposed a new discrete filled function with improved theoretical properties, the discrete filled function is as follows:

$$G_1(x, x_k^*, \mu, \rho) = f(x_k^*) - \min\{f(x_k^*), f(x)\} - \rho\|x - x_k^*\|^2 + \mu\{max[0, f(x) - f(x_k^*)]\}^2, \tag{3.22}$$

where ρ and μ are parameters which satisfy certain properties.

The function 3.22 was improved by Ng et al., and presented as follows:

$$G_2(x, x_k^*, \mu, \rho) = W_\mu(f(x) - f(x_k^*)) - \rho\|x - x_k^*\|, \tag{3.23}$$

$$W_\mu(z) = z.\mu[(1-\upsilon)(\frac{1-\upsilon\mu}{\mu-\upsilon\mu})^{-z/\omega} + \upsilon],$$

where $\omega > 0$ is a sufficiently small number and $0 < \upsilon \leq 1$ is a constant. Additional samples of discrete filled functions can be found in Yongjian and Yumei (2007); Shang and Zhang (2008); Yang and Zhang (2007); Gu and Wu (2006); Yang et al. (2008).

3.3. Characteristic Of The Filled Function

Similarly to the Tunneling function, the filled function has been introduced to exit the attraction zone of the local minimizer and to succeed through local minimization in determining a new stationary points corresponding to the values of the lower objective function. The first filled function date to the late 1980s. Recently, there has been a growing interest in these functions, which led to the definition of many new filled functions. The common feature of all the proposed filled functions is the fact that they are composed of two terms $\eta(x)$ and $\phi(x)$, which beyond the particular expressions, always play similar roles:

- role of the term $\eta(x)$ is to prevent the point x_k^* from being a stationary point of the filled function;
- role of the term $\phi(x)$ is to eliminate the stationary points of the objective function that correspond to the higher values of the function $f(x_k^*)$.

In the various filled functions, these terms are combined by multiplying them (multiplicative filled functions) or adding them (additive filled functions). In the following, we will limit ourselves to considering only the additive filled functions that have been recently introduced and that have shown to be more efficient from the computational point of view. Most of the filled functions can be grouped into two classes:

- filled functions that transform the minimizer point x_k^* into a local maximizer;
- filled functions that have compact level sets.

3.3.1. Filled functions that transform the minimizer point into a maximizer

The filled functions belong to this class have as a characteristic the term $\eta(x)$ transforms the stationary point x_k^* into a local maximizer of the filled function, a widely used choice is the following:

$$\eta(x) = exp(\frac{-\|x-x^*\|^2}{\gamma^2}), \tag{3.24}$$

where $\gamma > 0$, is a fixed constant. The term $\phi(x) > 0$, which may depend on one or more parameters, tries to eliminate the stationary points of the function $f(x)$ that have a value of an objective function higher than or equal to $f(x_k^*)$, and ensure that the filled function has a minimizer value lower than $f(x_k^*)$. A very simple and relatively intuitive choice is the following:

$$\phi(x,\tau,\delta) = \tau \min\{0, f(x) - f(x_k^*) + \delta\}^3,$$

where $\delta > 0$ and $\tau > 0$ are parameters. This term is null in all the points in which $f(x) \geq f(x_k^*) - \delta$, while in other points it assumes increasing values in the form as the value of the parameter τ increases (Lucidi and Piccialli, 2002). Using the previous choices, the following additive filled function is reached:

$$Q(x,\tau,\delta) = exp(\frac{-\|x-x^*\|^2}{\gamma^2}) + \tau \min\{0, f(x) - f(x_k^*) + \delta\}^3. \tag{3.25}$$

3.3.2. Filled functions that have compact level sets

The filled functions belong to this class differ from the previous ones in the choice of the term $\eta(x)$. In fact, in these filled functions, the term $\eta(x)$ mainly has the role of ensuring that the filled functions have compact level sets. To be able to obtain this property, we must renounce the fact that the point x_k^* is transformed into a maximizer, but we are content to transform it into a non-stationary point of the filled function (Lucidi and Piccialli, 2002). A very simple choice is the following:

$$\eta(x) = \|x - \tilde{x}\|^2, \tag{3.26}$$

where $\tilde{x}$ is any point in $\mathbb{R}^n$. The role of the term $\phi(x)$ for these functions is the same as the filled functions considered previously. In particular, the additive filled function

belonging to this class is as the following:

$$V(x,\tau,\delta) = \|x-\tilde{x}\|^2 + \tau\ min\{0, f(x) - f(x_k^*) + \delta\}^3 \tag{3.27}$$

4. NEW AUXILIARY FUNCTION METHODS FOR GLOBAL OPTIMIZATION

In this section, a new class of auxiliary functions proposed, most of them achieve the properties C1-C3 of the filled function in definition 3.1. Based on that, we tried to extend the approach proposed by Ge and Qin (1990), since, in our opinion, the particular filled functions there introduced show interesting theoretical properties.

4.1. A New Filled Function Method with Two Parameters

Structure of this filled function based in two phases:

Phase one: *Transforming the problem into one-dimensional*

Directional search method is based on the directions d_k, $k = 1,...,m$. If we have an objective function $f(x)$ with n-dimensions, we can use the line $l_\alpha = x_0 + \alpha d_k, \alpha \in R$, to construct a one-dimensional problem $L(\alpha)$. Moreover, we might want to choose α_k^* as the answer of

$$min_\alpha L(\alpha) = f(x_0 + \alpha d_k), \tag{4.1}$$

that means α_k^* in the direction d_k can be a result of a one-dimensional minimization problem. We obtain a local minimizer α_k^1 of $L(\alpha)$, then we construct the filled function on $L(\alpha)$, next, we take an initial point as starting to find the second minimizer α_k^2 of $L(\alpha)$. By repeating the above process, we will obtain the global minimizer α_k^* in the direction d_k as a solution of one-dimensional problem $L(\alpha)$, and by using $\hat{x}_k = x_0 + \alpha_k^* d_k$ we can minimize $f(x)$ when we use $\hat{x}_k$ as a starting point. Consequently, by comparing all minimizer points $\hat{x}_k$, $k = 1,...,m$ with each other, we will obtain the global minimizer of the problem f.

Phase two: *Constructing the function*

We suppose that the point α_k^1 is a local minimizer of the function $L(\alpha)$ that can be determined by any efficient method. Reducing the objective function from a multi-dimensional as a one-dimensional function making the minimization process easier and more efficient. For this purpose, we offer a new filled function as follows:

$$I_\alpha(\alpha, \alpha_k^1) = G(L(\alpha) - L(\alpha_k^1))U(|\alpha - \alpha_k^1|^2), \tag{4.2}$$

and

$$G(u) = \begin{cases} cos(\beta u), & u < 0; \\ 1, & \text{othewise}, \end{cases}$$

where $u = L(\alpha) - L(\alpha_k^1)$, $\beta > 1$, and $U(\kappa)$ is an escape function. The private form of $U(\kappa)$ presented in the literature into several forms, for instance, κ^ρ, $exp(\kappa^\rho)$, and $arctan(\kappa^\rho)$, where ρ is a positive integer. In the proposed paper, the function U will be selected as $-\rho|\alpha - \alpha_k^1|^2$, $\rho > 0$, that is, the final form of the filled function will be as following:

$$I_\alpha(\alpha, \alpha_k^1) = -\rho|\alpha - \alpha_k^1|^2 G(L(\alpha) - L(\alpha_k^1)), \tag{4.3}$$

where the parameters β and ρ require to be adjusted appropriately. The proposed filled function is continuously differentiable with two parameters. the new idea and advantages of the proposed algorithm are: First, this algorithm converts the objective function from multi-dimensional as a one-dimensional function, this allows us to obtain the global minimizer easier. Second, the trigonometric function $cos(\beta u)$ allows to add many stationary points in the lower basin, this idea has many advantages, for example, it helps to reduce the time and function evaluation which are essential in cases like this as we see can clearly in the experimental results. Now, let α_k^1 be the current local minimizer of $L(\alpha)$, then we can define:

$$L_{S_1} = \{\alpha | L(\alpha) \geq L(\alpha_k^1), \alpha \in R, \alpha \neq \alpha_k^1\}, \text{ and } L_{S_2} = \{\alpha | L(\alpha) < L(\alpha_k^1), \alpha \in R\}.$$

The next theorems show that the function $I_\alpha(\alpha, \alpha_k^1)$ achieve the conditions C1-C3 in Definition 3.1.

Theorem 4.1. Let α_k^1 be a local minimizer of $I_\alpha(\alpha, \alpha_k^1)$, then α_k^1 is a strictly local maximizer of $I_\alpha(\alpha, \alpha_k^1)$.

Proof. Since α_k^1 is a local minimizer of $L(\alpha)$, there exists a neighborhood $N(\alpha_k^1, \varepsilon^*)$ of α_k^1, $\varepsilon^* > 0$ such that $L(\alpha) \geq L(\alpha_k^1)$ for all $\alpha \in N(\alpha_k^1, \varepsilon^*)$. Then, for all $\alpha \in N(\alpha_k^1, \varepsilon^*)$, $\alpha \neq \alpha_k^1$, we have:
$I_\alpha(\alpha, \alpha_k^1) = -\rho|\alpha - \alpha_k^1|^2 < 0 = I_\alpha(\alpha_k^1, \alpha_k^1)$.
Thus, α_k^1 is a strict local maximizer of $I_\alpha(\alpha, \alpha_k^1)$. □

Theorem 4.2. Assume that α_k^1 is a local minimizer of $L(\alpha)$ and α is any point in L_{S_1} then $I_\alpha(\alpha, \alpha_k^1)$ has no stationary point on L_{S_1} .

Proof. Since $L(\alpha) \geq L(\alpha_k^1)$ and $\alpha \neq \alpha_k^1$, we have:
$I_\alpha(\alpha, \alpha_k^1) = -\rho|\alpha - \alpha_k^1|^2, \nabla I_\alpha(\alpha, \alpha_k^1) = -2\rho(\alpha - \alpha_k^1)$.
This means that $\nabla I_\alpha(\alpha, \alpha_k^1) \neq 0$, i.e., α is not a stationary point of $I_\alpha(\alpha, \alpha_k^1)$. □

Theorem 4.3. Suppose α_k^1 is a local minimizer of $L(\alpha)$ but not a global minimizer, and $L_{S_2} = \{\alpha | L(\alpha) < L(\alpha_k^1), \alpha \in R\}$ is not empty, then there exists a point $\alpha' \in L_{S_2}$ is a local minimizer of $I_\alpha(\alpha, \alpha_k^1)$.

Proof. Let $L_{S_3} = \{\alpha | L(\alpha) \leq L(\alpha_k^1), \alpha \in R\}$ and $\partial L_{S_2} = \{\alpha | L(\alpha) = L(\alpha_k^1), \alpha \in R\}$, then $L_{S_3} = L_{S_2} \bigcup \partial L_{S_2}$, which means ∂L_{S_2} is the boundary of the sets L_{S_2} and L_{S_3}. Since $L(\alpha)$ is continuous, then ∂L_{S_2} and L_{S_3} are bounded and closed sets.
Now for any $\alpha \in \partial L_{S_2}$, we have

$$I_\alpha(\alpha, \alpha_k^1) = -\rho|\alpha - \alpha_k^1|^2,$$

also, for any $\alpha \in L_{S_2}$ we have

$$I_\alpha(\alpha, \alpha_k^1) = -\rho|\alpha - \alpha_k^1|^2 \cos(\beta(L(\alpha) - L(\alpha_k^1))).$$

Since $I_\alpha(\alpha, \alpha_k^1)$ is continuously differentiable and has the term $\cos(\beta(L(\alpha) - L(\alpha_k^1)))$, $\beta > 1$, then there is at least one point exists, $\alpha' \in L_{S_2}$ is a minimizer of the function $I_\alpha(\alpha, \alpha_k^1)$. □

4.2. New Auxiliary Functions Smoothed Via Bézier Curves

We consider the problem of finding the global minimizer point of a given nonsmooth unconstrained objective function. The algorithms of such methods are based on the smoothing of an auxiliary function that is constructed using a known local minimizer and smoothed by using Bézier curves. The functions achieve all filled function properties. In this aspect, we have prepared two auxiliary functions which we will present as follows:

Algorithm 1

According to the investigation and hypotheses in the earlier section, a new algorithm for obtaining the global minimizer of the function $f(x)$ will be proposed, and the experimental results will be provided as follows.

Step 1 (Initialization) Determine the parameters $\beta > 1$ and $\rho > 0$, choose a starting point $x_0 \in S$, generate the direction d_k,$k = 1,2,...,m$, and set $\varepsilon = 10^{-2}$;

Step 2 Create $L(\alpha) = f(x_0 + \alpha d_k)$ as a one-dimensional function;

Step 3
1. Obtain the local minimizer α_k^i of $L(\alpha)$ starting from α_0 and then choose $\rho = -1$.
2. Construct the filled function $I_\alpha(\alpha, \alpha_k^i)$ at α_k^i;
3. Start from $\alpha_0 = \alpha_k^i + \rho\varepsilon$ to find a minimizer υ_1 of $I_\alpha(\alpha, \alpha_k^i)$;
4. If υ_1 in S, go to (5), otherwise go to (7);
5. Minimize $L(\alpha)$ start from υ_1 to obtain α_{k+1}^i, and then, go to (6);
6. If the point α_{k+1}^i in S, let $\alpha_k^i = \alpha_{k+1}^i$, and go to (2).
7. If $\rho = 1$, terminate the iteration and give $\alpha_k^* = \alpha_k^i$ otherwise; let $\rho = 1$, go to(3).

Step 4 Calculate $\hat{x}_k$ using $\hat{x}_k = x_0 + \alpha_k^* d_k$, and consequently, find x_k^* of $f(x)$ by using $\hat{x}_k$ as the initial point.

Step 5 If $k < m$, let $k = k+1$ and produce d_{k+1} as a new search direction and go to (Step 2) otherwise; go to (Step 6).

Step 6 Picks out the global minimizer of $f(x)$ using :

$$x^* = \min\{f(x_1^*), f(x_2^*), ..., f(x_m^*)\}.$$

4.2.1. The First Auxiliary Function And Its Properties

In this section, we propose a new auxiliary function for the problem 2.1 at a local minimizer x_k^* of f over Λ as follows:

$$\begin{aligned} \tilde{\gamma}(x, x_k^*, \rho, \mu, a) &= f(x_k^*) - [f(x_k^*) - f(x)]\tilde{\psi}_{\Lambda_1}(\tau, \rho, \mu) \\ &\quad + a\chi(\|x - x_k^*\|^2). \end{aligned} \tag{4.4}$$

The characteristic function $\tilde{\psi}_{\Lambda_1}(\tau, \rho, \mu)$ is the smoothing form of $\psi_{\Lambda_1}(x)$ where

$$\psi_{\Lambda_1}(x) = \begin{cases} 1 & x \in \Lambda_1 \\ 0 & otherwise \end{cases} \tag{4.5}$$

and

$$\tilde{\psi}_{\Lambda_1}(\tau, \rho, \mu) = \begin{cases} 0, & \tau > \mu, \\ \Gamma_2, & \mu \geq \tau > 0, \\ \Gamma_1, & 0 \geq \tau > \frac{-\mu}{\rho}, \\ 1, & \tau \leq \frac{-\mu}{\rho}. \end{cases} \tag{4.6}$$

the functions Γ_1 and Γ_2 constructed by using Bézier curves and define as follows:

$$\begin{aligned}\Gamma_1 &= (2\mu-2\rho+2((\mu-1)(\rho^2\tau-2\rho\tau+\mu \\ &\quad -1))^{1/2}-2\rho\mu-2\rho\tau+\rho^2\mu+\rho^2\tau \\ &\quad -2\rho((\mu-1)(\rho^2\tau-2\rho\tau+\mu-1))^{1/2} \\ &\quad +2)/(\rho-2)^2, \end{aligned} \tag{4.7}$$

and

$$\begin{aligned}\Gamma_2 &= (\rho\mu-\mu+(\mu(\rho^2\tau-2\rho\tau+ \\ &\quad \mu))^{1/2})^2/(\mu(\rho-2)^2), \end{aligned} \tag{4.8}$$

ρ and μ are two parameters adjusted enough, and $\tau = f(x)-f(x_k^*)$. The term χ is called the scape function and can be written as $\frac{1}{1+\|x-x_k^*\|^2}$ and satisfies the following properties:

a) $\chi(\tau)>0$,

b) $\chi^{'}(\tau)<0$,

c) $\lim_{\tau\to\infty}\chi(\tau)=0$.

Theorem 4.4. Suppose x_k^* is the local minimizer of f and the parameters ρ and μ are defined as above, then we have

$$0\le\tilde{\gamma}(x,x_k^*,\rho,\mu)-\gamma(x,x_k^*)\le\max\{\mu^2,\frac{\mu-\mu^2}{\rho}\} \tag{4.9}$$

for all $x\in\Lambda$ where $\gamma(x,x_k^*)=f(x_k^*)-[f(x_k^*)-f(x)]\psi_{\Lambda_1}(x)$.

Proof. From the definitions of $\tilde{\gamma}(x,x_k^*,\rho,\mu)$ and $\gamma(x,x_k^*)$, we have

$$\begin{aligned}\tilde{\gamma}(x,x_k^*,\rho,\mu)-\gamma(x,x_k^*) &= (f(x)-f(x_k^*))(\tilde{\psi}_{\Lambda_1}(\tau,\rho,\mu) \\ &\quad -\psi_{\Lambda_1}(x)). \end{aligned} \tag{4.10}$$

According to the states of τ and parameters ρ and μ, we consider the following four cases:

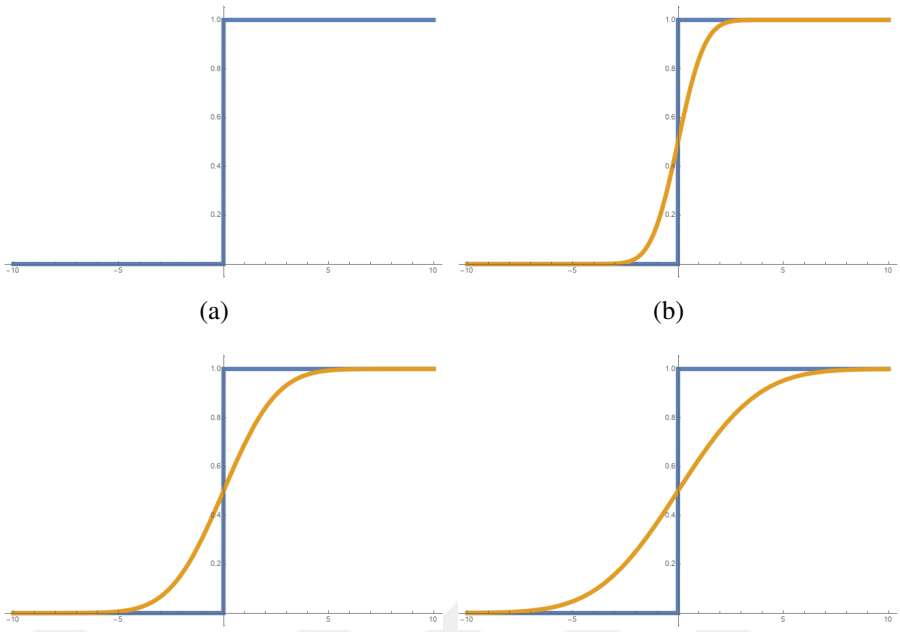


Figure 4.1. (a)the characteristic function $\psi_{\Lambda_1}(x)$. (b)-(d)The characteristic function $\tilde{\psi}_{\Lambda_1}(x,\rho,\mu)$ with deferent values of the parameters τ ,ρ and μ

Case 1: If $\tau > \mu$, this gives

$$\tilde{\gamma}(x,x_k^*,\rho,\mu) - \gamma(x,x_k^*) = 0,$$

for $x \in \Lambda$.

Case 2: If $\mu \geq \tau > 0$, we have

$$\tilde{\gamma}(x,x_k^*,\rho,\mu) - \gamma(x,x_k^*) \leq \mu^2,$$

for $x \in \Lambda$.

Case 3: If $0 \geq \tau > \frac{-\mu}{\rho}$, we have

$$\tilde{\gamma}(x,x_k^*,\rho,\mu) - \gamma(x,x_k^*) \leq \frac{\mu - \mu^2}{\rho},$$

for $x \in \Lambda$.

Case 4: If $\tau \leq \frac{-\mu}{\rho}$, we have

$$\tilde{\gamma}(x,x_k^*,\rho,\mu) - \gamma(x,x_k^*) = 0,$$

for $x \in \Lambda$. □

The next theorems show that the function $\tilde{\gamma}(x, x_k^*, \rho, \mu)$ achieves conditions C1-C3 in Definition 3.1.

Theorem 4.5. Suppose x_k^* is a local minimizer of f and $\tilde{\gamma}(x, x_k^*, \rho, \mu, a)$ is defined by (4.4), then the point x_k^* is a local maximiser of $\tilde{\gamma}(x, x_k^*, \rho, \mu, a)$.

Proof. As x_k^* is a local minimizer of the function f, there exists $\kappa > 0$. $\chi = N(x_k^*, \kappa)$ is a neighbourhood of x_k^* such that $f(x) \geq f(x_k^*)$ for any $x \in \chi$. When $x \neq x_k^*$, then

$$\frac{\tilde{\gamma}(x, x_k^*, \rho, \mu, a)}{\tilde{\gamma}(x_k^*, x_k^*, \rho, \mu, a)} = \frac{f(x_k^*) + \frac{a}{1+\|x-x_k^*\|^2}}{f(x_k^*) + a} < 1. \tag{4.11}$$

$$\tilde{\gamma}(x, x_k^*, \rho, \mu, a) < \tilde{\gamma}(x_k^*, x_k^*, \rho, \mu, a). \tag{4.12}$$

Therefore, we have

Thus, x_k^* is a local maximiser of $\tilde{\gamma}(x, x_k^*, \rho, \mu, a)$. □

Theorem 4.6. Suppose x_k^* is a local minimizer of f, then $\tilde{\gamma}(x, x_k^*, \rho, \mu, a)$ has no stationary point for $x \in \Lambda_2$, where $\Lambda_2 = \{x \in \Lambda | f(x) \geq f(x_k^*), x \neq x_k^*\}$.

Proof. In the case $\tau = f(x) - f(x_k^*) > \mu$, we have

$$\tilde{\gamma}(x, x_k^*, \rho, \mu, a) = f(x_k^*) + a\chi(\|x - x_k^*\|^2). \tag{4.13}$$

For any x satisfying $f(x) \geq f(x_k^*)$, we have

$$\nabla\tilde{\gamma}(x, x_k^*, \rho, \mu, a) = a\nabla\chi(\|x - x_k^*\|^2). \tag{4.14}$$

However, $\|a\nabla\chi(\|x - x_k^*\|^2)\| > 0$ for any $x \in \Lambda_2$, i.e., $\tilde{\gamma}(x, x_k^*, \rho, \mu, a)$ does not have stationary point at $x \in \Lambda_2$. □

Theorem 4.7. Suppose x_k^* is a local minimizer point of f but not global and f has a lower minimizer than x_k^*, then $\tilde{\gamma}(x, x^*, \rho, \mu, a)$ has a stationary point in $\Lambda_1 = \{x \in \Lambda | f(x) < f(x_k^*)\}$ if $a = |a| \leq \frac{L}{T}$, where $\|\nabla(a\chi(\|x - x_k^*\|^2))\| \leq |a|T$ and $\|\nabla f\| \leq L$.

Proof. Selecting the parameters ρ and μ to be sufficiently small, our smoothed function $\tilde{\gamma}(x, x^*, \rho, \mu, a)$, can be obtained as

$$\tilde{\gamma}(x, x^*, \rho, \mu, a) = f(x) + a\chi(\|x - x_k^*\|^2)$$

in the most part of the lower basin. As the norm of the gradient of the function,

$$\chi(\|x-x_k^*\|^2) = \frac{1}{1+\|x-x_k^*\|^2},$$

is bounded, there exists a number $T > 0$ such that

$$\|\nabla(a\chi(\|x-x_k^*\|^2))\| \leq |a|T.$$

Hence, the following inequality,

$$\|\nabla(a\chi(\|x-x_k^*\|^2))\| \leq |a|T \leq \|\nabla f\| \leq L,$$

enables f to increase faster than the escape function χ decreases, providing a stationary point in Λ_1, which implies

$$|a| \leq \frac{L}{T}.$$

□

The idea of the algorithm can be described in the following three important steps:

a) The first step is to reduce the objective function to a one-dimensional function in each search direction, d_k. The one-dimensional functions are found using $F_{d_k}(\delta) = f(x_0 + \delta d_k)$ as a function of δ.

b) The second step is to construct an auxiliary function $\tilde{\gamma}(\delta, \delta_k^*, \rho, \mu, a)$, at δ_k^i. Point δ_k^i is any arbitrary local minimum of $F_{d_k}(\delta)$, and it is used as an initial point to search for the global minimizer in direction d_k.

c) The third step is to repeat the above phases for all search directions to obtain the local minimizers of $f(x)$. The lowest among these minimizers is the required global minimizer of $f(x)$.

4.2.2. The Second Auxiliary Function And Its Properties

In this section, a further auxiliary function was designed to solve the problem of continuous global optimization 2.1. The idea of this auxiliary function starts when the following equation is used to remove as many local minimizers as possible:

Algorithm 2

A) Initialization Step:

1. Select parameters ρ and μ; select and adjust $a > 0$ according to Theorem 4.
2. Select an initial point $x_0 \in \Lambda$.
3. Generate direction d_k, $k = 1, \ldots, \eta$.
4. Set $\varepsilon = 10^{-3}$.

B) Main Step:

1. Construct a one-dimensional function for the first direction, d_k:

$$F_{d_k}(\delta) = f(x_0 + \delta d_k).$$

2. Starting from any arbitrary initial point δ_0, find δ_k^i which is a local minimizer of F_{d_k}, and select $\vartheta = -1$.
3. Construct the auxiliary function $\tilde{\gamma}(\delta, \delta_k^*, \rho, \mu, a)$, at δ_k^i.
4. Using $\delta_0 = \delta_k + \vartheta\varepsilon$, find the minimizer, δ_γ, of $\tilde{\gamma}(\delta, \delta_k^*, \rho, \mu, a)$ at δ_k^i .
5. If $\delta_\gamma \in \Lambda$, go to (6); otherwise, go to (8).
6. Minimise F_{d_k} again starting from δ_γ to find δ_k^{i+1} lower than δ_k^i, and go to (7).
7. If $\delta_k^{i+1} \in \Lambda$, put $\delta_k^i = \delta_k^{i+1}$ and go to (1).
8. If $\vartheta = 1$, stop and put $\delta_k^i = \delta_k^*$; otherwise, take $\vartheta = 1$ and go to (4).
9. Use $\grave{x}_k = x_0 + \delta_k^* d_k$ to find $\grave{x}_k$.
10. Find x^* of $f(x)$ using $\grave{x}_k$ as the initial point.
11. If $k < \eta$, set $k = k + 1$ and create a new search direction, d_{k+1}, and then, go to (1); otherwise, go to (7).
12. Select the best global minimizer of the function $f(x)$.

$$\psi(x, x_k^*) = \min\{f(x), f(x_k^*)\}.$$

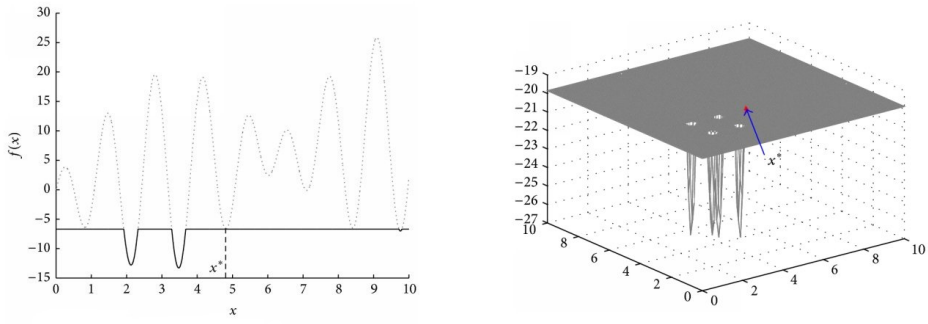
An example of the 2D and multi-dimensional function $\psi(x, x_k^*)$ is shown in Figure 4.2. By adding the $\mathcal{D}_{\mathcal{U}_k}(\mu)$ characteristic equation and using the μ parameter, and defining $\mathcal{D}_{\mathcal{U}_k}(\mu) : \mathbb{R}^n \to \mathbb{R}$, we are able to rewrite the $\psi(x, x_k^*)$ function as following:

$$\psi(x, x_k^*) = f(x_k^*) + (f(x) - f(x_k^*))\mathcal{D}_{\mathcal{U}_k}(\mu), \tag{4.15}$$

where

$$\mathcal{D}_{\mathcal{U}_k}(\mu) = \begin{cases} 1, & \mu \geq 0 \\ |\mu^{3/2}|, & \mu < 0. \end{cases} \tag{4.16}$$

By applying the Bézier curves seen in section **??**, we can obtain the smoothed form of the $\mathcal{D}_{\mathcal{U}_k}(\mu)$ term to make the $\psi(x, x_k^*)$ equation smooth. We can obtain by this process:



(a) (b)

Figure 4.2. a) Example of eliminating the minimizers of the 2D function at the current local minimizer x_k^*. b) Example of eliminating minimizers of the 3D function at the current local minimizer x_k^*.

$$\tilde{\mathcal{D}}_{\mathcal{U}_k}(\mu,\lambda) = \begin{cases} 1, & \mu > 0, \\ \eta_1(\mu,\lambda), & 0 \geq \mu > -\lambda, \\ \eta_2(\mu,\lambda), & -\lambda \geq \mu > -0.5-\lambda, \\ 1, & \mu \leq -0.5-\lambda, \end{cases} \tag{4.17}$$

where $\mu = f(x) - f(x_k^*)$ and $\lambda > 0$, and the polynomials $\eta_1(\mu,\lambda)$ and $\eta_2(\mu,\lambda)$ defined as follows:

$$\eta_1(\mu,\lambda) = \left(\frac{\frac{2\left(3\lambda-\frac{6}{5}\right)^2}{9\left(\lambda-\frac{3}{10}\right)^2}-2}{y_1} + 2y_1 + \frac{2\lambda-\frac{4}{5}}{\lambda-\frac{3}{10}} + 1 \right) \left(\frac{\lambda-\frac{2}{5}}{\lambda-\frac{3}{10}} + \frac{y_3}{y_1} + y_1 - 1 \right)^2,$$

and

$$\eta_2(\mu,\lambda) = \left(y_4 - \frac{3}{4y_4} + \frac{1}{2} \right)^2 \left(\frac{3}{2y_4} - 2y_4 + 2 \right),$$

and

$$y_1 = \left(\frac{(3\lambda-\frac{6}{5})^3}{27(\lambda-\frac{3}{10})^3} - \frac{\mu}{2\lambda-\frac{3}{5}} + \sqrt{(\frac{\mu}{2\lambda-\frac{3}{5}} - \frac{(3\lambda-\frac{6}{5})^3}{27(\lambda-\frac{3}{10})^3} + y_2)^2 - y_3^3} - y_2 \right)^{\frac{1}{3}},$$

$$y_2 = \frac{(3\lambda-\frac{6}{5})(3\lambda-\frac{9}{10})}{6(\lambda-\frac{3}{10})^2},$$

$$y_3 = \frac{(3\lambda-\frac{6}{5})^2}{9(\lambda-\frac{3}{10})^2} - 1,$$

$$y_4 = \left(\sqrt{(\frac{5\lambda}{2} + \frac{5\mu}{2} + \frac{5}{8})^2 + \frac{27}{64}} - \frac{5\mu}{2} - \frac{5\lambda}{2} - \frac{5}{8} \right)^{\frac{1}{3}}.$$

Figure 4.3 shows the behaviour of the functions $\eta_1(\mu,\lambda)$ and $\eta_2(\mu,\lambda)$.

The function $\psi(x,x_k^*)$ after smoothed $\mathcal{D}_{\mathcal{U}_k}(\mu)$ becomes:

$$\tilde{\psi}(x,x_k^*) = f(x_k^*) + (f(x) - f(x_k^*))\tilde{\mathcal{D}}_{\mathcal{U}_k}(\mu,\lambda), \tag{4.18}$$

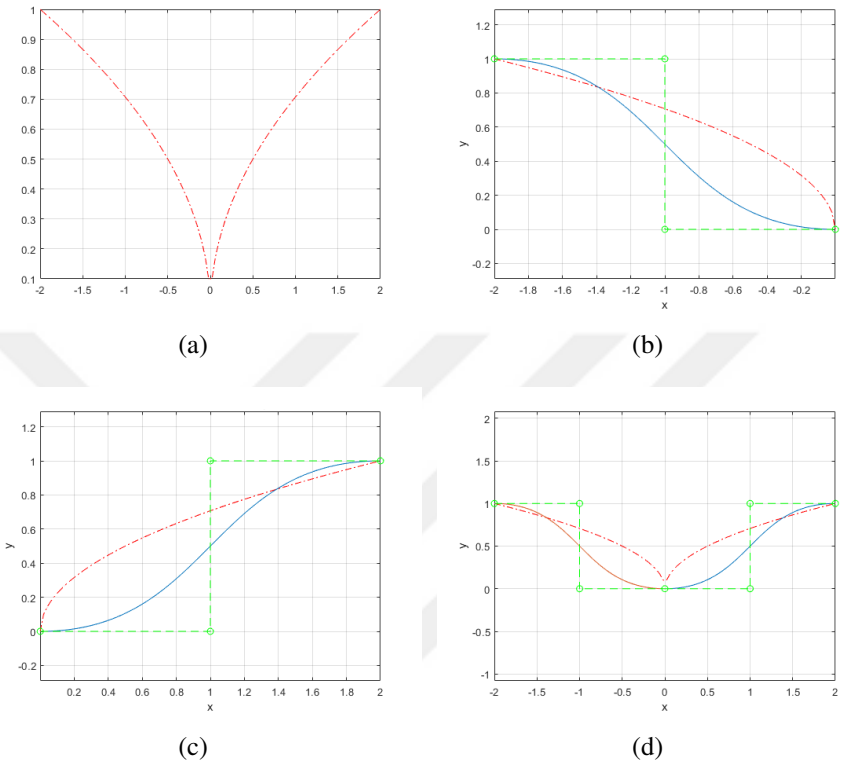


Figure 4.3. Behaviour of the functions $\eta_1(\mu,\lambda)$ and $\eta_2(\mu,\lambda)$, the original curves with (red), and the smoothed ones are in (blue).

After eliminating the minimizers not better than the one chosen so far will result in a loss of data, the explanation is the flatland caused by the elimination process, to treat this we have to add a term, this term plays a role in helping the auxiliary function constructed as well to scape from the existing minimizer to a further better one. This term is used in the literature with multiform (see, e.g., (Renpu (1990); Ren-Pu (1987); Jiannan (1996); Ng et al. (2005a); Ge and Qin (1987); Jiannan (1994); Min (2000); Liu (2001a))). Therefore after adding this term, the final form of the auxiliary function will be as follows:

$$\tilde{\psi}(x,x_k^*,\lambda,\nu) = f(x_k^*) + (f(x) - f(x_k^*))\tilde{\mathcal{D}}_{\mathcal{U}_k}(\mu,\lambda) + \exp\left(\frac{-\|x - x_k^*\|^2}{\nu}\right). \tag{4.19}$$

The next theorems show that the auxiliary function 4.19 achieves conditions C1-C3 in

Definition 3.1.

Theorem 4.8. Suppose x_k^* is a local minimizer of f and $\tilde{\psi}$ is defined by (4.19), then the point x_k^* is a local maximiser of $\tilde{\psi}$.

Proof. As x_k^* is a local minimizer of function f, there exists $\upsilon > 0$. $\chi = \eta(x_k^*, \upsilon)$ be a neighbourhood of x_k^* so that $f(x) \geq f(x_k^*)$ for any $x \in \chi$. When $x \neq x_k^*$, then

$$\frac{\tilde{\psi}(x, x_k^*, \lambda, \nu)}{\tilde{\psi}(x_k^*, x_k^*, \lambda, \nu)} = \frac{f(x_k^*) + \frac{a}{1+\|x-x_k^*\|^2}}{f(x_k^*) + a} < 1. \tag{4.20}$$

Therefore, we have

$$\tilde{\psi}(x, x_k^*, \lambda, \nu)) < \tilde{\psi}(x_k^*, x_k^*, \lambda, \nu). \tag{4.21}$$

It shows that the x_k^* minimizer is a $\tilde{\psi}$ local maximizer. □

Theorem 4.9. Suppose x_k^* is a local minimizer of f, then the function 4.19 has no stationary point for $x \in \Lambda_2$.

Proof. We can say that for $f(x) - f(x_k^*) > \lambda$, we have $\tilde{\psi}(x, x_k^*, \lambda, \nu)) = f(x_k^*) + \exp\left(\frac{-\|x-x_k^*\|^2}{\nu}\right)$. Thus for $f(x) \geq f(x_k^*)$ and $x \neq x_k^*$ the following is implies:

$$\nabla\tilde{\psi}(x_k^*, x_k^*, \lambda, \nu) = -2\frac{(x - x_k^*)}{\nu} \exp\left(\frac{-\|x - x_k^*\|^2}{\nu}\right).$$

Accordingly, $\nabla\tilde{\psi} \neq 0$. As a result, $\nabla\tilde{\psi}$ can't have a stationary point. □

Theorem 4.10. Suppose x_k^* seem to be a local $f(x)$ minimizer, and not a global minimizer. Then, there's a point $\hat{x} \in \Lambda_2$, so that $\hat{x}$ is a local minimizer of $\tilde{\psi}(x, x_k^*, \lambda, \nu)$.

Proof. Firstly, we show that $\tilde{\psi}(x, x_k^*, \lambda, \nu)$ includes a local minimizer point. To do so, suppose that f has x_{k+1}^* lower than x_k^*. Therefore, the subset. $\Pi = \{x : f(x_k^*) \geq f(x), x \in \Lambda\}$ is not empty and closed due to the continuity and limitation of f. Then $\tilde{\psi}$ has a local minimizer on Π.

Assume x_{k+1}^* can be a local minimizer of f lower than x_k^*, and $\hat{x}$ is a local minimizer of

$\tilde{\psi}$, therefore, $\nabla\tilde{\psi} = 0$. Accordingly,

$$\exp\left(\frac{-\|x-x_k^*\|^2}{\nu}\right) + f(\hat{x}) = 0.$$

Thus,

$$\nabla f(\hat{x})(\hat{x}-x_k^*) = (\hat{x}-x_k^*)\left(-\exp\left(\frac{-\|x-x_k^*\|^2}{\nu}\right)\right) > 0.$$

Since $(x_{k+1}^* - x_k^*) \cong (\hat{x}-x_k^*)$ and $\|x_{k+1}^* - x_k^*\| \leq \|\hat{x}-x_k^*\|$, we can recognize they're in a certain direction. That's evidence $(\hat{x}-x_k^*)(-\exp(\frac{-\|x-x_k^*\|^2}{\nu})) > 0$. □

Algorithm 3

Step 0 Let $k = 0$, $\nu = 0.5$, $\lambda = 0.5$, $\varepsilon = 0.001$ and select an initial point $x_0 \in \Lambda$.

Step 1. Let M be the number of directions d_k for $k = 1, 2, 3, ..., M$.

Step 2. Construct the one-dimensional function $Y(\theta) = f(x_0 + \theta d_k)$.

Step 3.
1. From any starting point θ_0, find a local minimizer θ_k^j for $Y(\theta)$ and select $\delta = -1$.
2. Construct the auxiliary function $\tilde{\psi}_\theta(\theta, \theta_k^j, \beta, \tau)$ at θ_k^j.
3. Beginning from $\theta_0 = \theta_k^j + \delta\varepsilon$, find the minimizer θ_ψ of $\tilde{\psi}_\theta(\theta, \theta_k^j, \beta, \tau)$.
4. If $\theta_\psi \in \Lambda$, go to (5); otherwise go to (7).
5. Beginning from θ_H, minimize $Y(\theta)$ to find θ_k^{j+1} lower than θ_k^j and go to (6).
6. If $\theta_k^{j+1} \in \Lambda$, put $\theta_k^j = \theta_k^{j+1}$ and go to (2).
7. If $\delta = 1$, stop and put $\theta_k^* = \theta_k^j$; otherwise, take $\lambda = 1$ and go to (3).

Step 4. Using $\hat{x}_k = x_0 + \theta_k^* d_k$, calculate $\hat{x}_k$.

Step 5. Using $\hat{x}_k$ as the initial point, find x^* of the function $f(x)$.

Step 6. If $k < M$, let $k = k+1$; create a new search direction d_{k+1}, and go back to Step 2; otherwise, go to Step 7.

Step 7. Select the global minimizer of $f(x)$ which is determined by:

$$f(x^*) = \min\{f(x_k^*)\}_{k=1}^M.$$

4.3. A New Globally Convex Auxiliary Function

We consider the following unconstrained minimization problem:

$$\begin{cases} \min f(x), \\ s.t.\ x \in S, \end{cases} \tag{4.22}$$

where $f(x) : \mathbb{R}^n \to \mathbb{R}$ is non-convex with multi local minimizers and S is a compact set in R^n,i.e., $S = \{x : l_j \leq x_j \leq u_j, j = 1, \ldots, n\}$ and $l_j, u_j \in \mathbb{R}$. Suppose that x_1^* is the

current local minimizer of f which is obtained following any common local optimization method. We construct the problem

$$\begin{cases} H(x), \\ s.t.\ x \in S, \end{cases} \tag{4.23}$$

where $H(x)$ is a globally convexificated auxiliary function of problem 4.22 constructed as follows:

$$H(x) = U(x, x_k^*) + \min\{f(x), f_k^*\} - f_k^*, \tag{4.24}$$

where $U(x)$ is a continuously differentiable convex function defined by applying a Bézier curve at the points l_j, x_1^*, u_j as follows:

$$U(t) = (1-t)^2 l_j + 2t(1-t)x_1^* + t^2 u_j,\ t \in [0,1] \tag{4.25}$$

and

$$U(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} (1-t)^2 x_1 + 2t(1-t)x_b + t^2 x_2 \\ (1-t)^2 y_1 + 2t(1-t)y_b + t^2 y_2 \end{pmatrix}, \quad t \in [0,1]. \tag{4.26}$$

By using a computer algebra system to eliminate the parameter t from the parametric equations $x(t)$ and $y(t)$, we can obtain the quadratic function $U(x)$.

The function $H(x)$ is a continuous function and is called a globally convexified auxiliary function of the function $f(x)$ at the point x_1^*. This function has the following properties:

1. problem 4.22 has no stationary point in the domain $S_1 = \{x \in S : f(x) \geq f(x_1^*)\}$.
2. problem 4.22 has a minimizer point in the domain $S_2 = \{x \in S : f(x) < f(x_1^*)\}$.

Furthermore, such a problem has a stationary point $z \in S$ that satisfies the conditions

$$\begin{cases} \frac{\partial H(z)}{\partial x_j} \geq 0,\ if\ z_j = l_j \\ \frac{\partial H(z)}{\partial x_j} \leq 0,\ if\ z_j = u_j \\ \frac{\partial H(z)}{\partial x_j} = 0,\ if\ l_j < z_j < u_j \end{cases} \tag{4.27}$$

This problem has common properties that can be seen from the following theorems.

Theorem 4.11. Assume that f_g^* is the global minimizer value of Problem 4.22; then Problem 4.23 has a unique global minimizer x_g^*.

Proof. Since f_g^* is the value of the global minimizer of f, then $f(x) > f(x_g^*)$ for all $x \in S$, and $\min\{f(x), f_g^*\} = f_g^*$; therefore, $H(x) = U(x, x_g^*)$. Thus, the function U has a unique minimizer is x_g^*. This leads to Problem 4.23 having only one minimizer, namely, x_g^*. □

Theorem 4.12. Let x_1^* be a local minimizer of $f(x)$; then x_1^* is a local minimizer of $H(x)$.

Proof. Since x_1^* is a local minimizer of $f(x)$, we have $\min\{f(x), f(x_1^*)\} = f(x_1^*)$. Then $H(x) = U(x, x_1^*)$.

The derivatives of $H(x)$ are

$$H'(x_1^*) = U'(x, x_1^*) = 0, \tag{4.28}$$

$$H''(x) = U''(x, x_1^*) > 0. \tag{4.29}$$

and

It can be seen that x_1^* is a local minimizer of $H(x)$. □

Theorem 4.13. Let x_1^* be a local minimizer of $f(x)$; then $H(x)$ has no stationary point in $S_1 = \{x \in \mathbb{R} : f(x) > f(x_1^*)\}$ except x_1^*.

Proof. For any $x \in S_1$, we have

$$H(x) = U(x, x_1^*). \tag{4.30}$$

Since $H'(x) = U'(x, x_1^*)$ and $U'(x, x_1^*) \neq 0$ except for x_1^*, then $H'(x) \neq 0$ in $S_1 \setminus \{x_1^*\}$. Therefore, the proof is complete. □

Algorithm 4

Step 1. Determine the lower bound *lb* and the upper bound *ub* of $f(x)$ in $S \subset \mathbb{R}$, select the starting point x_0 and let $k = 1$.

Step 2. Use the starting point x_0 and any local solver to obtain the local minimizer x_k^* of $f(x)$ and set $f(x_k^*) = f_k^*$.

Step 3. Taking *lb*, x_k^* and *ub* as control points, we construct the continuously differentiable function $U(x, x_k^*)$ using Bezier curves.

Step 4. Construct the auxiliary function

$$H(x) = U(x, x_k^*) + \min\{f(x), f_k^*\} - f_k^*$$

Step 5. If $H(x) = U(x, x_k^*)$, go to Step7, otherwise go to Step6.

Step 6. Minimize the function $H(x)$ on S to obtain the better local minimizer $\tilde{x}$ of $H(x)$ and set $x_0 = \tilde{x}$ and $k = k + 1$; go to Step3.

Step 7. Stop the algorithm and but $x^* = x_k^*$, where x^* is the unique approximated global minimizer point.

4.4. A Numerical Example

In this section, we apply our algorithm to solve the following one dimensional example.

Example 4.1.: We define the following constrained optimization problem:

$$\min_{x\in\mathbb{R}n} f(x) = -\sum_{j=1}^{6} \frac{a_j}{1+\|x-b_j\|_2^2},$$

where the values of b_j and $a_j \in \mathbb{R}$ for $j = 1,2,\ldots,6$ are presented in Table4.1.

Table 4.1. Values of a_j and b_j Example1 4.1.

j	1	2	3	4	5	6
a_j	1.8	1.2	4.8	3.5	2.8	2.1
b_j	-7	2	-3.5	6	-1	4

The problem can be defined as the global minimization of the continuously differentiable multi-modal function. It has local minimizers at the point b_j with magnitude a_j for $j = 1,2,\ldots,6$. The graph of the function f is shown in Figure. 4.4 (a).

We first apply a local search to find the first local minimizer as in Algorithm 4. We select $x_0 = -8$ as a starting point, and we find the first local minimizer as $x_1^* = -6.9337$ with the corresponding value $f_1^* = -2.2978$. Then, we construct our auxiliary function at this point. The construction of the auxiliary function consists of two phases. In the first phase, we find the Bézier curve (function) with respect to the boundary and the first local minimizer. Second, we consider the function $\min\{f(x), f_1^*\}$ to eliminate the upper local minimizers. The graphs of these functions are shown in Figure 4.4 (b-c). After summing these two functions, the construction of the auxiliary function is completed. The graph of the auxiliary function is shown in Figure 4.4 (d).

According to Algorithm 4, we apply the local solver to find the local minimizer of the auxiliary function that will be the starting point for the next local minimizer of the objective function. The process is continued until the same local minimizers are found. The global minimizer is found as $x^* = -3.4756$ and the corresponding value $f^* = -5.4382$. The final stage is shown in Figure 4.4 (e-f).

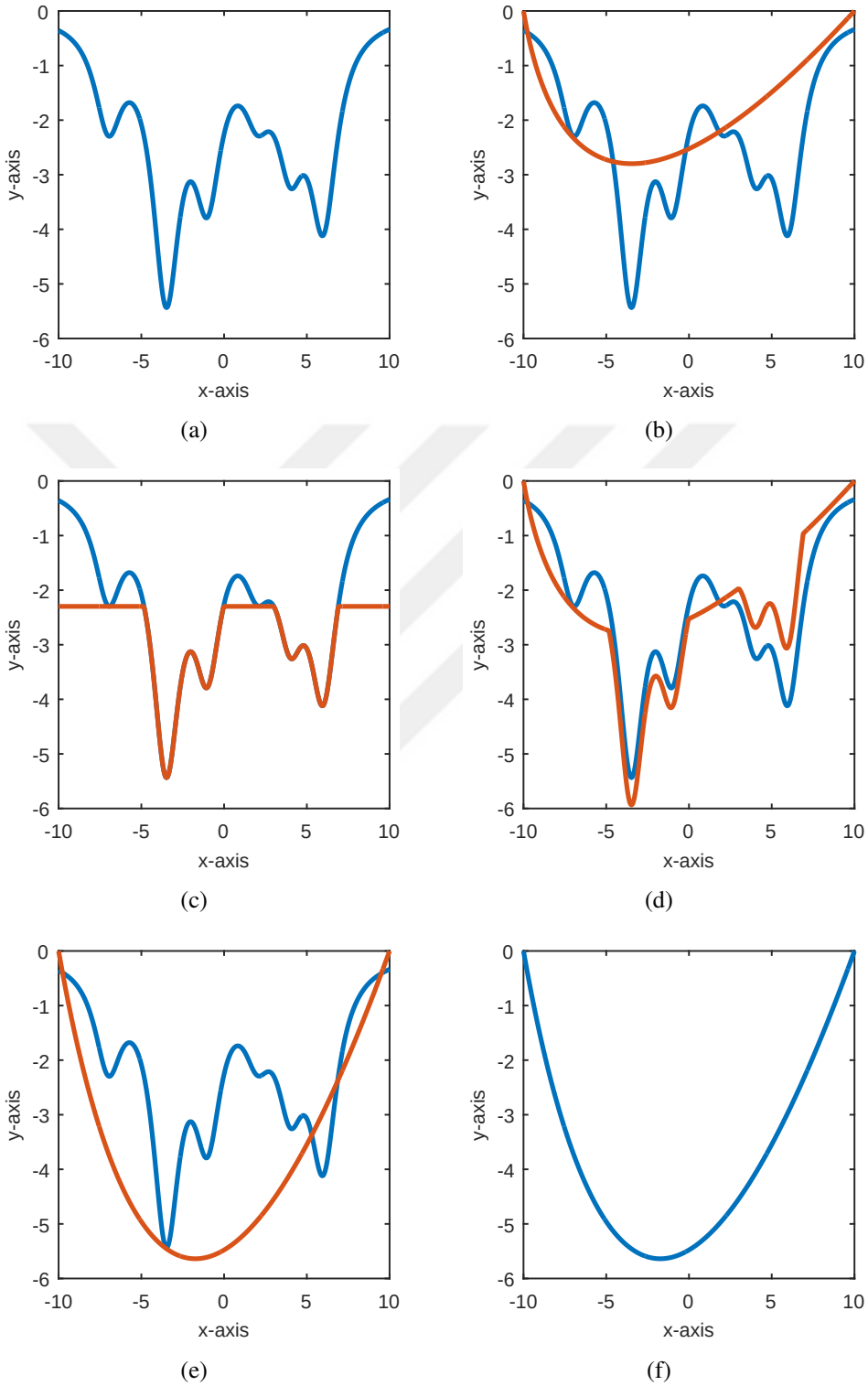


Figure 4.4. Convexification steps

5. COMPUTATIONAL RESULTS

In this section, we selected the filled function methods based on their theoretical features and Algorithms 1, 2, and 3, respectively, from those mentioned in the previous section. Such functions were evaluated on several benchmark problems, The selected benchmark functions with thier knowledge could be seen in Table 5.1. The computational tests were performed by Matlab 6.5.1, within Windows XP, and Intel(R) CPU speed 1.60 GMHZ and 4 GB RAM. The initial local minimizer was found by using "fminsearch" that for specifying the direction search, and for the step size, Armijo line search is applied. The objective is to compare the effectiveness of our methods without necessarily resolving high dimensional problems. For such test problems, we will report the computational results of the proposed method with the mean time of sum running T, the total number of functions evaluations f_v, the mean of the best value in the 10 runs f_m, the best value in 10 runs f_b, the rate of successfully obtaining the true optimal solution among 10 runs R.

5.1. Demonstrative Example

For an evident picture of the filled function method, we apply algorithm 2 to 3-humpback camel function. 3-humpback camel function given as the following:

$$min f(x) = 2x_1^2 - 1.05x_1^4 + \frac{x_1^6}{6} + x_1x_2 + x_2^2, \tag{5.1}$$

where $x_i \in [-5,5],\ i = 1,2$. We already know the global minimizer of this function $(0,0)$. The first step we set a starting point, $x_0 = (1.5,1.5)$ at $f(x_0) = 1.0828$, then by using "$fminsearch$" the initial point $x_1^* = (1.748,0.874)$ is found at $f(x_1^*) = 0.2986$. after that, we constructed the filled function $(\tilde{\gamma}(x,x_1^*,\rho,\mu,a))$, at x_1^*. By using x_1^* as an initial point of such filled function, we can find $\hat{x} = (0.302,0.535)$ at $f(\hat{x}) = 0.2984$ as a local minimizer of such a filled function. Since $f(\hat{x}) < f(x_1^*)$, we have to minimized f again starting from $\hat{x}$, we found $x_2^* = (0,0)$, at $f(x_2^*) = 0$, and we constructed the filled function $\tilde{\gamma}(x,x_2^*,\rho,\mu,a)$ again at x_2^*, then we minimized it, we get $\hat{x} = (2,1.5)$ is a vertex and $f(\hat{x}) > f(x_2^*)$. We used an additional searches in the lower basins starting from $(0,1)$, $(-1,0)$, and $(0,-1)$, respectively.

As none of the certain yields an improved point, the parameters of the filled function 4.4

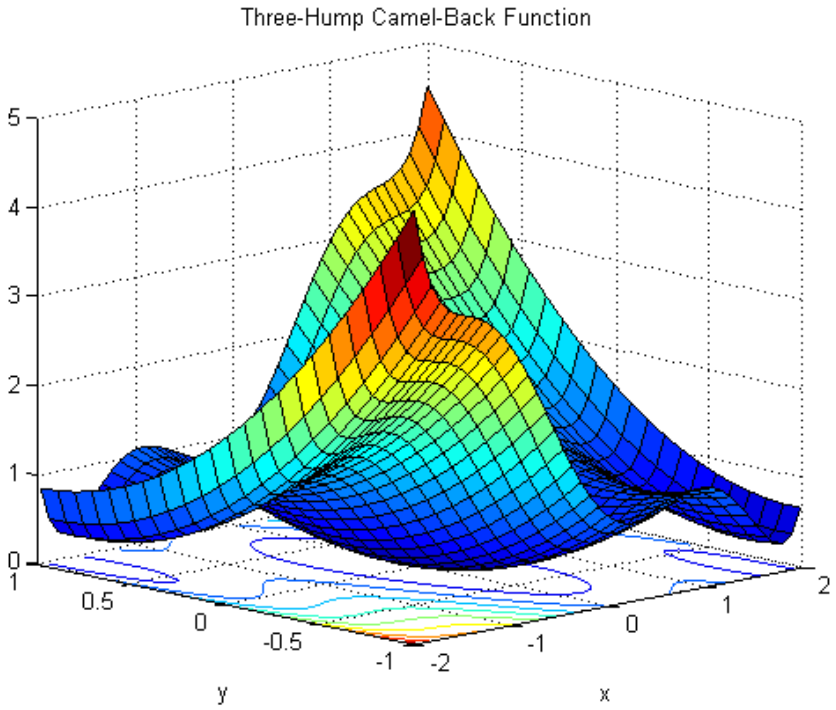


Figure 5.1. 3-Hump Back Camel Function.

are adjusted. The improved function is thereafter minimized once more starting from these points in turn. when no local minimizer found in a lower basin and the termination criteria is satisfied, $x_k^* = (0,0)$ is considered the global minimizer of f.

The performance of such Algorithms has been evaluated with common benchmarks with a few to thousand local minimizers and 2-50 variables. The computational results of the three algorithms are summarized in Tables 5.2, 5.3, and 5.4. These tables indicate that some of these functions are considered moderate, whereas the remainder are hard to resolve. Our methods have been successful in finding a global minimizer for all benchmark functions with different efficiencies. Since the methods represent a deterministic optimization algorithm, the convergence speed and efficiency of the global minimizer finding can be impacted with the initial points. On the other hand, the existence of proper stopping criteria is one of the benefits of optimization algorithms. If the algorithm is running successfully, it finds the objective function value very close to the known global optimum, f^*. Here, an algorithm is considered an achievement if the objective function value identified by all benchmark functions is$\leq [f^* + 1e-5]$.

Speed and accuracy of convergence were measured by using both the average number of objective function evaluations (f_v) and the success rate (R). f_v is a reliable indicator

Table 5.1. Test problems

N	Problem	n	f^*
1	2-dimensional test function($c = 0.2$)	2	0.0
2	2-dimensional test function($c = 0.5$)	2	0.0
3	2-dimensional test function ($c = 0.05$)	2	0.0
4	3-hump back Camel function	2	0.0
5	6-hump back camel function	2	-1.0316
6	Treccani Function	2	0.0
7	Goldstein-Price Function	2	3.0
8	Shubert I function	2	-186.7309
9	Shekel function ($m = 5$)	4	-10.1532
10	Shekel function ($m = 7$)	4	-10.1532
11	Shekel function($m = 10$)	4	-10.5364
12	Sin-square function	10	0.0
13	Beale function	2	0.0
14	Bohachevsky-2 test function	2	0.0
15	Booth function	2	0.0
16	Baranin Function	2	0.397887
17	Easom function	2	-1.0
18	Matyas function	2	0.0
19	Hartman function	3	-3.86278
20	Colville function	4	0.0
22	Ackley function	2, 10, 30, 50	0.0
23	Dixon Price function	2, 10, 25, 50	0.0
24	Griewank function	2, 5, 10, 20	0.0

of computational efficiency because the function evaluation includes a comprehensive computations in application problems. It is also independent of the computer and software interface used and thus useful for researchers to compare. The algorithm reliability is expressed in T, which is the number of times that the algorithm is optimally placed globally, according to the defined accuracy of 10 runs. For the above purposes, they were used to compare the approaches as shown in Table 5.5. Moreover, the Figures 5.2 and 5.3 provide a comparison between (times) and (times against problems dimensions), respectively.

The results in Table 5.3 show that, out of the 32 benchmark functions tested in this study achieved 100% R for 22 benchmark functions compared to 100% R for 17 benchmark functions in Table 5.2 and 10 in Table 5.4. It is evident that algorithm 2 is more robust to benchmark functions than algorithm 1 and algorithm 3.

Table 5.2. Results of Algorithm 1

Problem	n	T_1	f_m	f_b	f_v	R
1	2	0.4461	$3.1596e-15$	$3.8227e-16$	182	80
2	2	0.4784	$1.2357e-14$	$2.7141e-15$	154	90
3	2	0.3577	$3.4436e-15$	$7.3088e-17$	126	100
4	2	0.3906	$4.9389e-16$	$2.3545e-16$	98	100
5	2	0.4194	-1.0316	-1.0316	182	100
6	2	0.3815	$4.4911e-16$	$1.4944e-18$	154	100
7	2	0.5016	3.0000	3.0000	154	100
8	2	0.3521	-186.7309	-186.7309	168	80
9	4	0.7441	-10.1532	-10.1532	182	100
10	4	0.7333	-10.4029	-10.4029	201	80
11	4	0.7636	-10.5321	-10.5321	212	100
12	10	0.9523	$2.9532e-13$	$6.4330e-15$	144	80
13	2	0.7225	$4.1440e-14$	$2.3652e-14$	72	100
14	2	0.9630	$2.9022e-13$	$9.4369e-16$	240	100
15	2	0.5760	$2.7005e-15$	$9.0137e-18$	96	100
16	2	0.7103	0.3979	0.3979	168	100
17	2	0.6206	-1.0000	-1.0000	24	100
18	2	0.3786	$1.2341e-15$	$2.5430e-16$	42	100
19	3	0.3474	-3.8628	-3.8628	84	90
20	4	0.5543	$9.3413e-11$	$1.6849e-12$	168	100
21	2	0.8938	$6.6613e-15$	$8.8818e-16$	220	80
22	10	0.9946	$3.5527e-15$	$3.5527e-15$	100	80
23	30	1.0838	$4.4409e-15$	$8.8818e-16$	140	90
24	50	1.2657	$2.0724e-15$	$8.8818e-16$	140	80
25	2	0.6108	$3.8192e-15$	$1.3543e-15$	72	100
26	10	1.1863	$6.4333e-08$	$1.1211e-13$	140	70
27	25	1.3091	$5.9254e-10$	$2.8998e-10$	140	90
28	50	1.3788	$7.5085e-11$	$3.2650e-11$	180	90
29	2	0.6489	$9.7145e-17$	0.0000	200	80
30	5	0.7181	$1.4036e-13$	$6.1062e-15$	120	90
31	10	1.1363	$6.3283e-16$	0.0000	164	100
32	20	1.1559	$2.2204e-17$	0.0000	108	100

Table 5.3. Results of Algorithm 2

Problem	n	T_2	f_m	f_b	f_v	R
1	2	0.6729	$7.4499e-16$	$3.6520e-16$	162	100
2	2	0.7472	$1.1625e-14$	$1.9024e-15$	200	90
3	2	0.6261	$3.7276e-15$	$2.9513e-16$	220	100
4	2	0.6756	$1.0460e-15$	$2.0716e-16$	120	100
5	2	0.7609	-1.0316	-1.0316	120	100
6	2	0.6424	$6.0941e-17$	$-1.7525e-15$	200	100
7	2	0.7770	3.0000	3.0000	120	100
8	2	0.9906	-186.7309	-186.7309	234	90
9	4	0.6120	-10.1532	-10.1532	260	80
10	4	0.6073	-10.4029	-10.4029	120	100
11	4	0.5993	-10.5321	-10.5321	180	100
12	10	0.9622	$1.0533e-15$	$1.9403e-16$	120	90
13	2	0.9816	$3.5716e-14$	$2.8002e-16$	66	100
14	2	0.8764	$1.2564e-12$	$2.4980e-15$	264	100
15	2	0.5751	$2.7353e-15$	$1.6968e-16$	80	100
16	2	0.6769	0.3979	0.3979	132	100
17	2	0.6250	-1.0000	-1.0000	20	100
18	2	0.6101	$2.0278e-16$	$1.6629e-18$	60	100
19	3	0.7028	-3.8628	-3.8628	154	80
20	4	0.8111	$1.9569e-11$	$8.1572e-14$	140	100
21	2	0.8183	$1.2331e-16$	$1.1284e-16$	276	90
22	10	0.8187	$6.3002e-17$	$8.8818e-16$	128	100
23	30	0.9060	$2.6645e-15$	$8.8818e-16$	104	100
24	50	1.1061	$1.4098e-16$	$1.3817e-16$	62	90
25	2	0.4243	$5.3849e-15$	$1.1796e-15$	42	100
26	10	1.1211	$7.5055e-08$	$1.1211e-13$	140	80
27	25	1.0141	$5.9254e-10$	$2.8998e-10$	140	100
28	50	1.1843	$7.5085e-11$	$3.2650e-11$	180	90
29	2	0.7650	$2.6698e-13$	$2.2204e-16$	152	80
30	5	0.9327	$4.0140e-13$	$4.4409e-16$	108	100
31	10	1.0485	$4.1722e-12$	$3.2196e-15$	180	100
32	20	1.2995	$1.7446e-12$	$4.4409e-16$	96	100

Table 5.4. Results of Algorithm 3

Problem	n	T_3	f_m	f_b	f_v	R
1	2	0.5928	$1.3607e-15$	$5.5535e-16$	220	100
2	2	0.8171	$2.9594e-15$	$5.1707e-16$	180	80
3	2	0.7107	$4.4549e-16$	$5.3076e-17$	185	100
4	2	0.6915	$1.9513e-14$	$5.5425e-17$	120	100
5	2	0.6876	-1.0316	-1.0316	140	100
6	2	0.5995	$2.0503e-15$	$-1.3679e-15$	160	90
7	2	0.8032	3.0000	3.0000	140	80
8	2	0.8646	-186.7309	-186.7309	180	80
9	4	0.5886	-10.1532	-10.1532	260	90
10	4	0.5863	-10.4029	-10.4029	120	80
11	4	0.5839	-10.5321	-10.5321	180	90
12	10	1.1094	$4.8040e-14$	$1.9403e-16$	120	100
13	2	0.8825	$4.1388e-14$	$3.4670e-14$	60	100
14	2	0.8376	$2.6804e-13$	$9.4369e-16$	180	90
15	2	0.5155	$6.4909e-15$	$1.0840e-15$	72	100
16	2	0.6062	0.3979	0.3979	140	100
17	2	0.5612	-1.0000	-1.0000	240	90
18	2	0.6547	$2.0278e-16$	$1.6629e-18$	60	90
19	3	0.5767	-3.8628	-3.8628	162	80
20	4	0.8356	$1.9569e-11$	$8.1572e-14$	140	100
21	2	0.9230	$9.8244e-08$	$8.8818e-16$	180	80
22	10	0.8068	$2.2511e-06$	$1.4157e-06$	80	80
23	30	1.3476	$4.4617e-07$	$8.8818e-16$	208	80
24	50	1.0623	$1.0964e-06$	$1.0964e-06$	224	80
25	2	0.4747	$9.0161e-14$	$4.0756e-16$	48	100
26	10	0.9261	$7.4461e-09$	$7.1006e-13$	140	80
27	25	1.0025	$6.3039e-10$	$6.3039e-10$	112	90
28	50	1.1072	$1.8079e-09$	$1.8079e-09$	112	90
29	2	0.6873	$9.7145e-17$	0.0000	200	80
30	5	0.5173	$3.9210e-13$	$2.6645e-15$	84	90
31	10	0.8095	$2.5068e-12$	$1.3323e-15$	126	80
32	20	1.0365	$6.3918e-13$	$3.2196e-15$	156	80

Table 5.5. Comparison between Algorithm 1, 2 and 3

		Algorithm 1		Algorithm 2		Algorithm 3	
Problem	n	T_1	f_v	T_2	f_v	T_3	f_v
1	2	0.4461	182	0.6729	162	0.5928	220
2	2	0.4784	154	07472	200	0.8171	180
3	2	0.3577	126	06261	220	0.7107	185
4	2	0.3906	98	0.6756	120	0.6915	120
5	2	0.4194	182	0.7609	120	0.6876	140
6	2	0.3815	154	0.6424	200	0.5995	160
7	2	0.5016	154	0.7770	120	0.8032	140
8	2	0.3521	168	0.9906	234	0.8646	180
9	4	0.7441	182	0.6120	260	0.5886	260
10	4	0.7333	201	0.6073	120	0.5863	120
11	4	0.7636	212	0.5993	180	0.5839	180
12	10	0.9523	144	0.9622	120	1.1094	120
13	2	0.7225	72	0.9816	66	0.8825	60
14	2	0.9630	240	0.8764	264	0.8376	180
15	2	0.5760	96	0.5751	80	0.5155	72
16	2	0.7103	168	0.6769	132	0.6062	140
17	2	0.6206	24	0.6250	20	0.5612	240
18	2	0.3786	42	0.6101	60	0.6547	60
19	3	0.3474	84	0.7028	154	0.5767	162
20	4	0.5543	168	0.8111	140	0.8356	140
21	2	0.8938	220	0.8183	276	0.9230	180
22	10	0.9946	100	0.8187	128	0.8068	80
23	30	1.0838	140	0.9060	104	1.3476	208
24	50	1.2657	140	1.1061	62	1.0623	224
25	2	0.6108	72	0.4243	42	0.4747	48
26	10	1.1863	140	1.1211	140	0.9261	140
27	25	1.3091	140	1.0141	140	1.0025	112
28	50	1.3788	180	1.1843	180	1.1072	112
29	2	0.6489	200	0.7650	152	0.6873	200
30	5	0.7181	120	0.9327	108	0.5173	84
31	10	1.1363	164	1.0485	180	0.8095	126
32	20	1.1559	108	1.2995	96	1.0365	156

We also implemented Algorithms 2 and 3 for the following global optimization problems with the non-smooth objective functions which are non-smooth versions of Ackleys, Griewank, and the first Levy functions.

Problem 5.1. Non-smooth version of Ackleys function

$$f(x) = -20\exp(-0.2\sqrt{n^{-1}\sum_{i=1}^{n}|x_i|}) - \exp\left(n^{-1}\sum_{i=1}^{n}\cos(2\pi x_i)\right) + 20 + e,$$

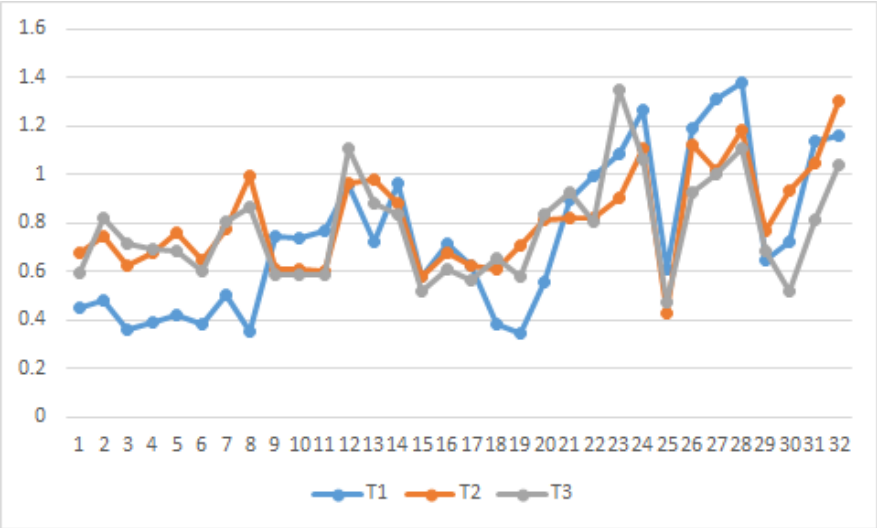


Figure 5.2. Comparative study of the CPU time of Algorithm 1 (blue), Algorithm 2 (orange) and Algorithm 3 (gray) for solving 32 problems.

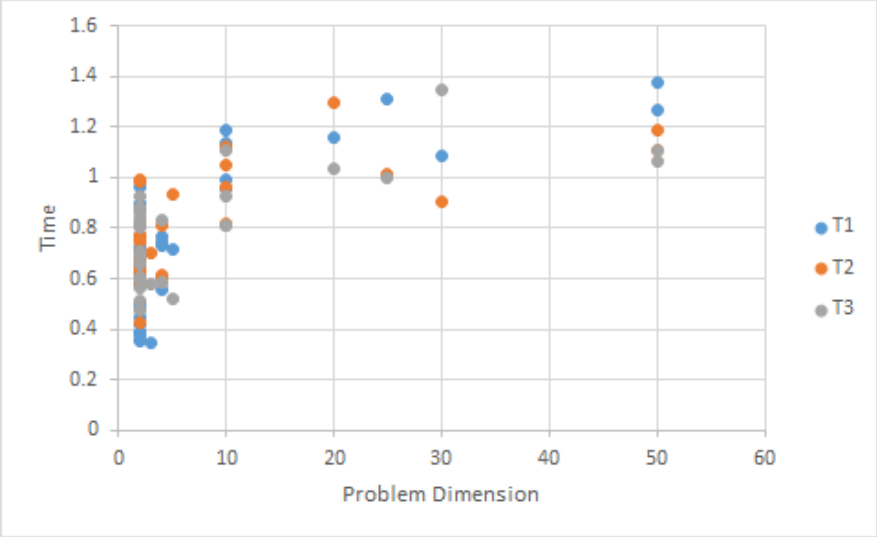


Figure 5.3. Comparative study of the CPU time of Algorithm 1 (blue), Algorithm 2 (orange) and Algorithm 3 (gray) with the dimensions of the 32 problems.

$$-32.768 \leq x_i \leq 32.768, \quad i = 1,2,\ldots,n$$

The global minimum: $x^* = (0,\ldots,0), \quad f(x^*) = 0$

Problem 5.2. Non-smooth version of Griewank function

$$f(x) = \frac{1}{d}\sum_{i=1}^{n} |x_i| - \prod_{i=1}^{n} \cos\left(\frac{x_i}{\sqrt{i}}\right) + 1,$$

$$-500 \leq x_i \leq 500, \quad d = 400, \quad i = 1,\ldots,n$$

The global minimum: $x^* = (0,\ldots,0), \quad f(x^*) = 0$

Problem 5.3. Non-smooth version of first Levy function

$$f(x) = \frac{\pi}{n}\left(10\left|\sin(\pi y_1)\right| + \sum_{i=1}^{n-1}|y_i - 1|\left(1 + 10\left|\sin(\pi y_{i+1})\right|\right) + |y_n - 1|\right),$$

$$y_i = 1 + \frac{x_i - 1}{4}, \quad -10 \le x_i \le 10, \quad i = 1,\ldots,n$$

The global minimum: $x^* = (0,\ldots,0), \quad f^* = 0$

Results of numerical experiments for problems with non-smooth objective functions are presented in Table 5.6. In this table, the best objective function values (f_b), the average objective function values (f_m), the average CPU time T (in seconds), and Success rate R. These results were compared with the Algorithm in (Bagirov et al., 2009). Results from Table 5.6 show that the proposed algorithm is still effective for solving non-smooth global optimization problems. However, the both algorithm 2 and 3 require more computational effort to solve them. We used the same initial points as in the case of the smooth version of these problems. One can see that the algorithms require more CPU time for higher dimensional functions. This is because the algorithms need to compute a number of complex computational process with Bezier polynomials to get a good value of the auxiliary functions. Despite the fact that these functions are nonsmooth versions of the known test functions, the structure of their local minima are quite different. The overall success rate for non-smooth problems is 87%.

Table 5.6. Results of numerical experiments for non-smooth problems.

Prob.	n	Algorithm 2				Algorithm 3				Algorithm (Bagirov et al., 2009)			
		f_b	f_m	T	R	f_b	f_m	T	R	f_b	f_m	T	R
5.1	2	0.0000	0.0000	0.050	100	0.0000	0.0000	0.130	100	0.0000	0.0000	0.21	100
5.1	10	0.0000	0.0000	0.370	100	0.0000	0.0005	0.310	100	0.0000	0.0000	0.26	100
5.1	30	0.0000	0.0018	1.330	93	0.0000	0.0197	2.200	89	0.0000	0.1115	5.60	90
5.1	50	0.0000	0.0254	2.600	88	0.0000	0.0925	3.165	87	0.0000	0.0727	14.19	88
5.2	2	0.0000	0.0000	0.410	100	0.0000	0.0011	0.590	99	0.0000	0.0674	0.42	38
5.2	5	0.0000	0.0010	0.571	91	0.0000	0.0098	0.91	85	0.0000	0.0085	4.43	78
5.2	10	0.0000	0.0035	0.502	92	0.0000	0.0592	0.854	79	0.0000	0.0824	4.57	72
5.2	30	0.0000	0.0552	1.024	87	0.0000	0.0245	3.004	85	0.0000	0.1011	6.48	42
5.2	50	0.0000	0.0541	1.620	78	0.0000	0.0245	3.745	71	0.0000	0.0922	18.44	44
5.3	2	0.0000	0.0000	0.301	100	0.0000	0.0031	0.628	100	0.0000	0.1385	0.37	98
5.3	5	0.0000	0.0086	0.577	100	0.0000	0.0264	0.802	91	0.0000	0.1007	0.72	82
5.3	10	0.0000	0.0000	0.130	95	0.0000	0.0233	0.421	100	0.0000	0.1884	0.31	60
5.3	30	0.0000	0.0095	2.075	87	0.0000	0.0490	2.542	76	0.0000	0.0439	13.12	64
5.3	50	0.0000	0.0081	2.652	89	0.0000	0.0541	4.792	79	0.0000	0.0402	37.46	58

6. APPLICATION

6.1. Melanoma Segmentation Based On Global Optimization Methods

Over recent years, skin cancer has become a great scientific problem. Melanoma has been the most violent of the types of skin cancers. It is able to expand to other organs too fast; therefore, even with its low incidence, it is the reason for the highest death incidence (Celebi et al., 2007; Cheng et al., 2018).The high incidence of this lesion is due to excessive sun exposure, Especially in people having fair skin, besides leading premature aging may cause pre-malignant and malignant lesions. Melanoma (see Figure 6.1) is the least common and most disturbing of other types of skin cancer, even if the lesion seem to be small (Stoecker et al., 1992) due to its high ability to spread through other parts of the body.

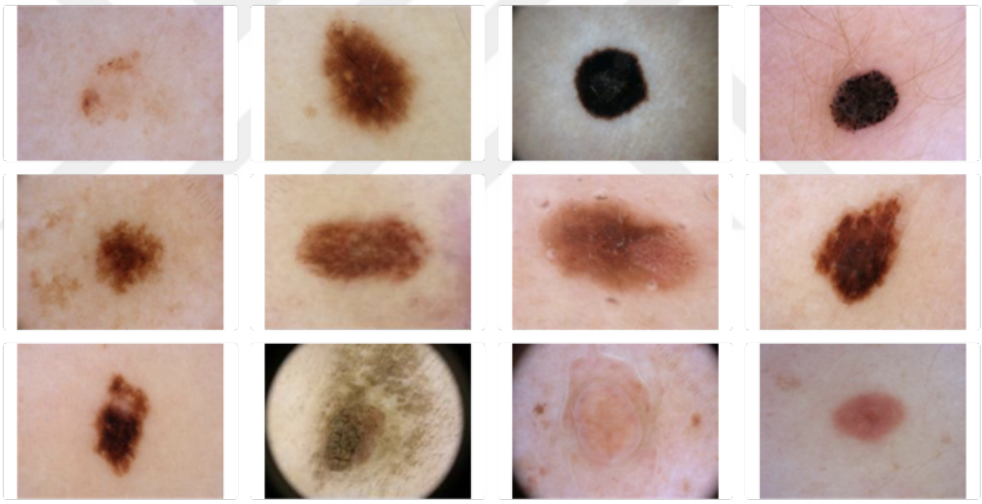


Figure 6.1. Different images of the lesion (Dey et al., 2018; Casari et al., 2018).

In the first step, a prior diagnosis is needed because there is more room for the remedy, However, human visual triggers, such as visual exhaustion, are hindering this and sometimes contribute to demarcation failure. Automatic segmentation of the area in interest of the image is necessary to establish a contour accuracy which reduces the error occurrence; Often important for the detection and diagnosis of other skin lesion processes. How an image is processed is essential because it allows the diagnosis of a given case in time for a possible cure (Stoecker et al., 1992; Erkol et al., 2005; Stolz, 1994). Digital imaging techniques may help define an approach to this type of skin cancer. This method provides those who know the technology with diagnostic

confidence and agility. Requirements for digital image processing increasingly need methods to highlight the information contained in the images for human interpretation and analysis (Kasmi and Mokrani, 2016). Early diagnosis, that is, early detection, is associated with a stronger prognosis of the disease. The lesion at a more advanced stage, that is, after its penetration into the lower layers of the skin, leads to a great likelihood of metastasis with almost no cure. This results in the highest mortality rate for skin cancer relative to other non-melanoma tumors (Erkol et al., 2005; Stolz, 1994; Kasmi and Mokrani, 2016).

Throughout this section, algorithm 2 implemented for solving the problem of global optimization for segmentation of melanoma skin cancer. This method of optimization used to locate threshold values in images of melanoma skin cancer. The proposed algorithm for melanoma segmentation is implemented on the PH2 dataset, the results show that the proposed algorithm exhibits high precision, sensitivity, and specificity compared to other approaches.

6.2. Materials And Methods

6.2.1. Methodology for Detection of Melanoma Skin Cancer

Due to numerous facts, The detection of melanomas in dermatoscopic images is an important and difficult task. The transition between the lesion and the skin that is avoided is smooth. The pictures taken from digital cameras include shadows that show as the skin lesions associated with masking. Dermoscopic imagery involves objects such as bulb shapes, multiple colored patches, specular patterns, presence of eyes, erratic light, and noise. The method of global-optimization-based melanoma detection is explained step by step.

Step 1: The images are imported from the study database.

Step 2: The ground truth images are imported from the study database.

Step 3: The gray image is processed using two transformations, grayscale image conversion to RGB, and the other is median filtering conversion. This transition reduces color and contrast, thus retains luminance. Medium filtration eliminates emissions such as noise from salt and pepper.

Step 4: The histogram of the median image is determined after median filtering. This

collect data on the strength of the median picture distribution together with the distribution position.

Step 5: The suggested global optimization algorithm is applied to the median image histogram using Otsu.

Step 6: In addition, the global optimization approach provides the melanoma detection threshold value. Compared to the median filter image, this threshold value is. If an image point is higher than the threshold, the point will be converted to white, otherwise, the point will be converted to black.

Step 7: Once the image has been turned into black and white, an image overlay function is used to mask the image's corner. After the masking, the image will be produced and compared to the image of the ground truth. For efficient comparison, sensitivity, specificity, predicted positive value, predicted negative value, accuracy, and computation time will be analyzed.

The study methodology examines the possibility of isolation the lesion from the surrounding skin while its color and geometrical dimensions are preserved. The steps of this procedure are described as illustrated in Figure 6.2.

This database formation consists of multi-source images (sites other than melanoma pictures). Data on the means of capture was not available during the formation of base. Therefore, they are forming a fundamental for specific types of resolution, lighting, etc., Therefore, that some images have been rejected with complicated detection. The suggested methodology was tested in the public datasets images PH2 (Dey et al., 2018), ISBI2016 challenge (Casari et al., 2018), and ISBI 2017 challenge (Berseth, 2017).

6.2.2. Results and Discussion

For the performance evaluation, the International Skin Imaging Collaboration advises several metrics. These metrics show the performance of each algorithm and technique, which properly categorizes the pixels. The study measures precision (Acc %), accuracy (Sep %), positive predictive value (PPV), negative predictive value (NPV), and responsiveness (Sen %). The following equations give the quantitative analysis of the results of all the techniques that were applied by the various parameters:

$$AC = \frac{N_t p + N_t n}{N_t p + N_f p + N_t n + N_f n} \tag{6.1}$$

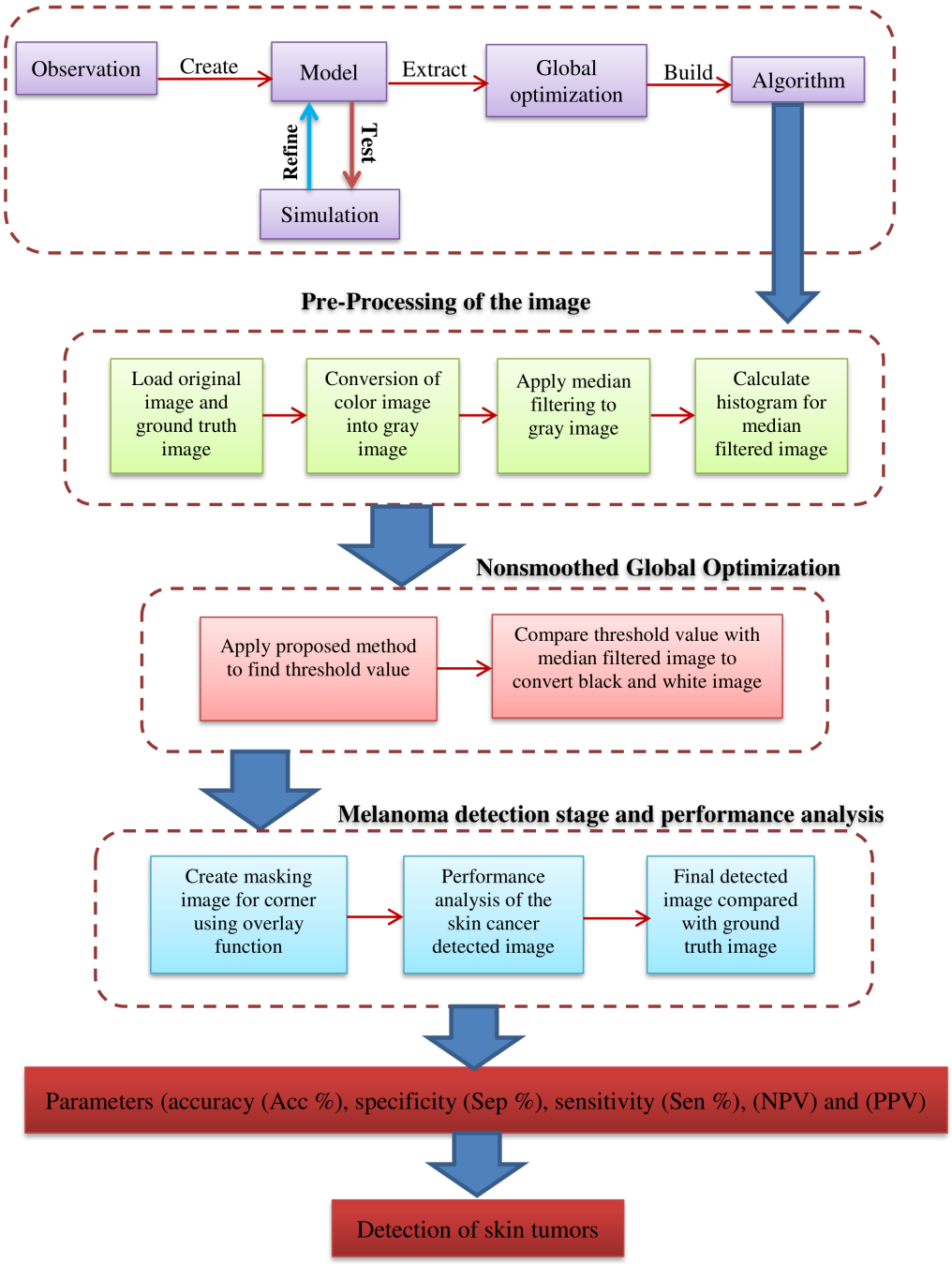


Figure 6.2. Steps of global optimization method for skin cancer segmentation

$$JA = \frac{N_t p}{N_t p + N_f n + N_f p} \tag{6.2}$$

$$DI = \frac{2 \times N_t p}{2 \times N_t p + N_f n + N_f p} \tag{6.3}$$

$$SE = \frac{N_t p}{N_t p + N_f p} \tag{6.4}$$

$$SP = \frac{N_t n}{N_t n + N_f p} \tag{6.5}$$

Figure 6.3 and 6.4 indicates the images of the processed melanoma used to demonstrate the efficacy of the suggested algorithm.

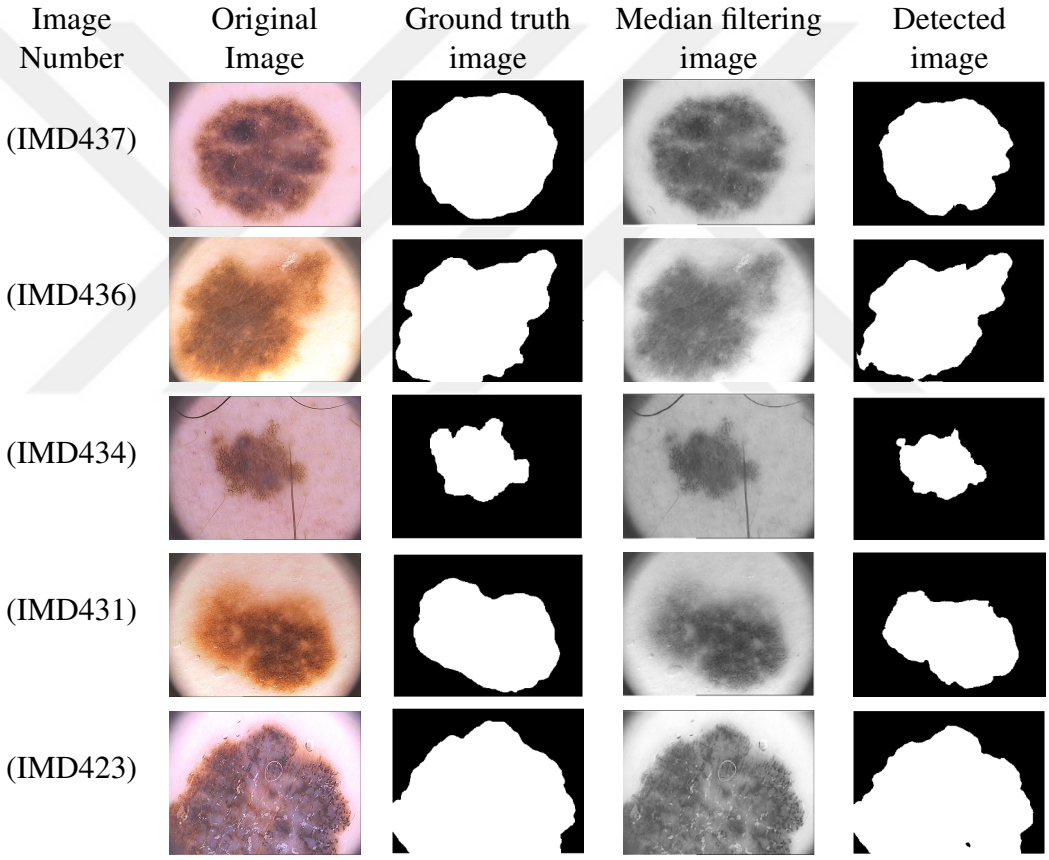


Figure 6.3. Set of images for melanoma detection using global optimization algorithm

Comparing the proposed method with a newly developed method, i.e., the ASLM method* (details available at https:/doi.org/10.1016/j.com-pmedimag.2016.05 .002) proves its efficacy. Six melanoma images are selected at random, and detection is performed using the proposed method and ASLM method. The images appear in Figure 6.5. The final images collected through the proposed method are more precise

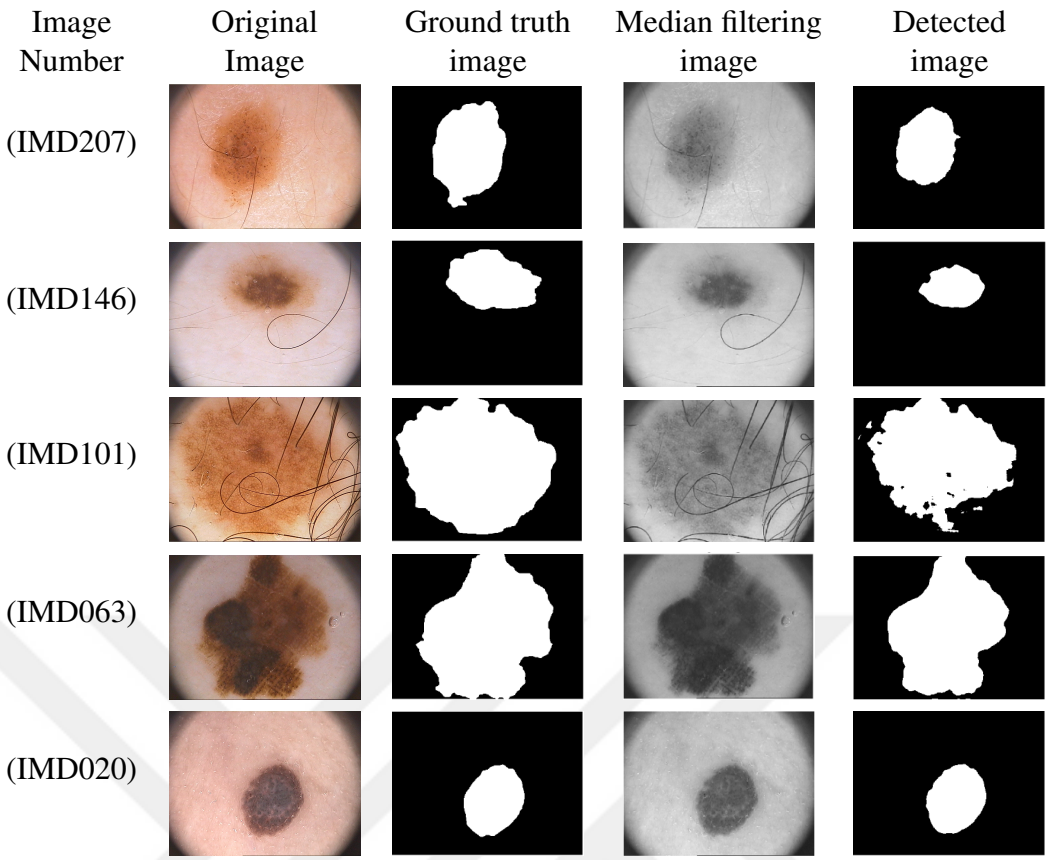


Figure 6.4. Another set of images for melanoma detection using the global optimization algorithm

than the ones produced through the ASLM process. Table 6.1 provides a comparison of the global optimization algorithm's performance parameters with JSEG, SRM, KPP, Kmeans, Otsu, Level Set, and ASLM for 200 image detections. On the basis of these results, the approach suggested yields superior parameters, with a specificity of 0.9928 and a precision of 0.9011. Except for the ALSM method, the sensitivity of the proposed method is higher than that of the other methods.

Results obtained using the proposed global optimization method on the PH2 database images of malignant lesions (melanomas) show IMD088 (blue-white veil, streaks and regression areas), IMD284, IMD405, and IMD419 (blue-white veil) for the second, third, and fourth rows, IMD424 (blue-white veil and streaks) for the fifth row, and IMD425 (blue-white veil and regression areas) for the sixth row.

The computational speeds are set out in Table 6.2 for all processing steps.

All output parameters are determined using a stratified cross validation approach that divides the PH2 database into three subgroups, each with around 40 unhealthy moles and 160 healthy moles for skin cancer lesions. Table 6.3 demonstrates some performance

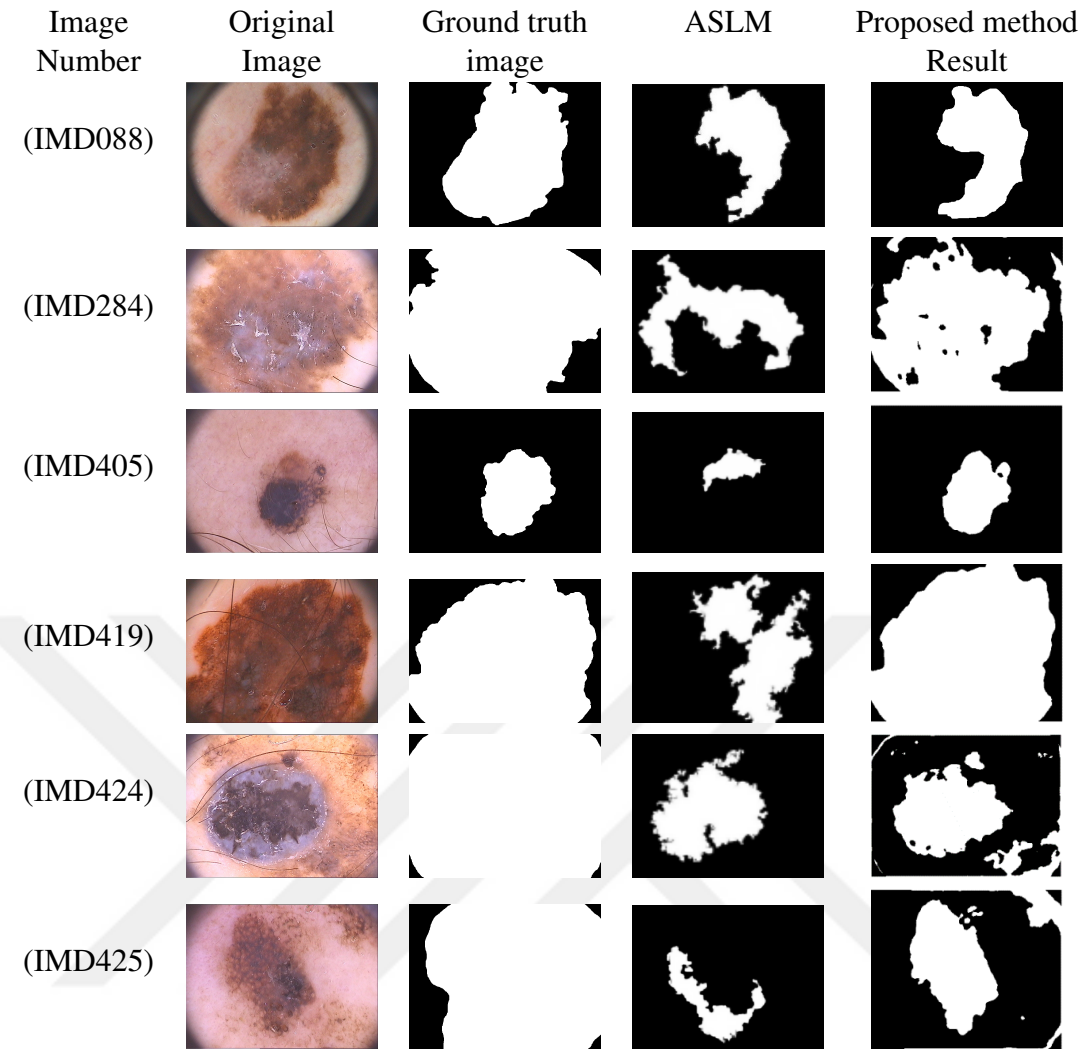


Figure 6.5. Comparison of the proposed method and ASLM method.

parameters of the six processed images, randomly chosen from the PH2 database.

From the ISBI 2016 and ISBI 2017 databases, the most complicated images (i.e., images with dermoscopes, bubbles, hair, and multiple colors) have been used in their original dimensions. These databases contain 8-bit dermoscopic RGB images from $540 * 722$ to $4499 * 6748$ pixels, with different image dimensions. In addition, both databases included the original images coupled with the boundaries of lesion segmentation, which have been recorded by professional dermatologists.

Figure 6.6 lists the segmentation results, where the measurement of the comparative methods was taken from the original papers. Deeper learning methods achieve better results than the method in (Xie et al., 2016), and with the proposed method the highest Jac and Dic values are obtained. Therefore, the proposed method outperforms comparable methods and segments of skin lesions effectively.

Table 6.1. Performance parameter results for 200 image detections from the PH2 database.

Method	Sensitivity	Specificity	Accuracy
Otsu (MATLAB 2014a)	0.5221	0.7064	0.6518
SRM (Boltz, 2010)	0.6518	08757	0.6766
KPP (MATLAB 2014a)	0.4147	0.9581	0.7815
Level Set (Crandall, 2000)	0.7188	0.8003	0.7842
K-means (Open CV2.4)	0.7291	0.8430	0.8249
JSEG (Zhao, 2001)	0.7108	0.9714	0.8947
ASLM	0.8024	0.9722	0.8966
Global optimization algorithm	0.8892	0.9933	0.932

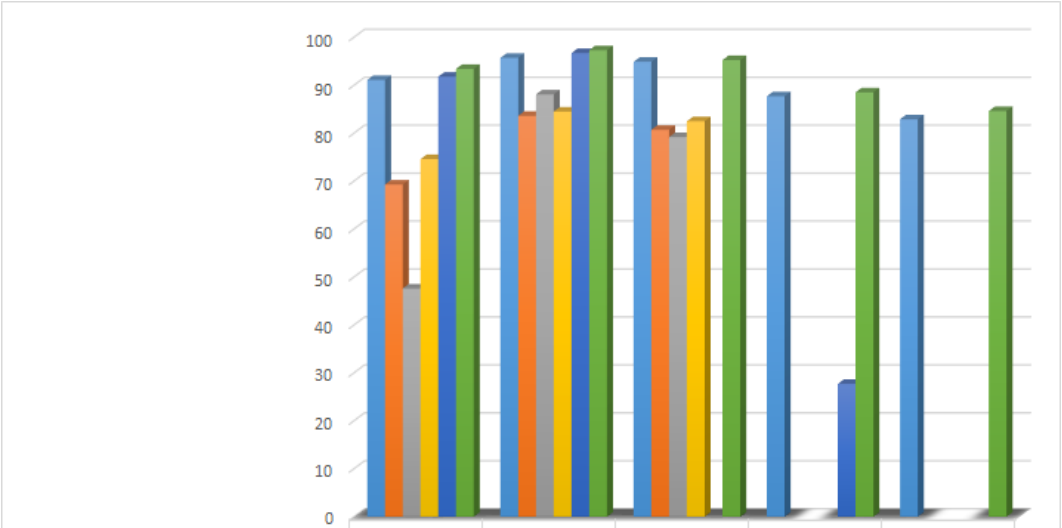
Table 6.2. Total time consumed by the proposed method for six images selected randomly from 200 images in the PH2 database.

Proposed Method (global optimization)	Total time consuming (sec)
Image processing	19.25 ± 2.02
Classification	$(18.27 \pm 12.13) \times 10^{-3}$

Figure 6.6 also compares, in terms of Sen, Spe, Acc, Dic, and Jac, the proposed approach and other common methods employed on the ISBI 2017 database.

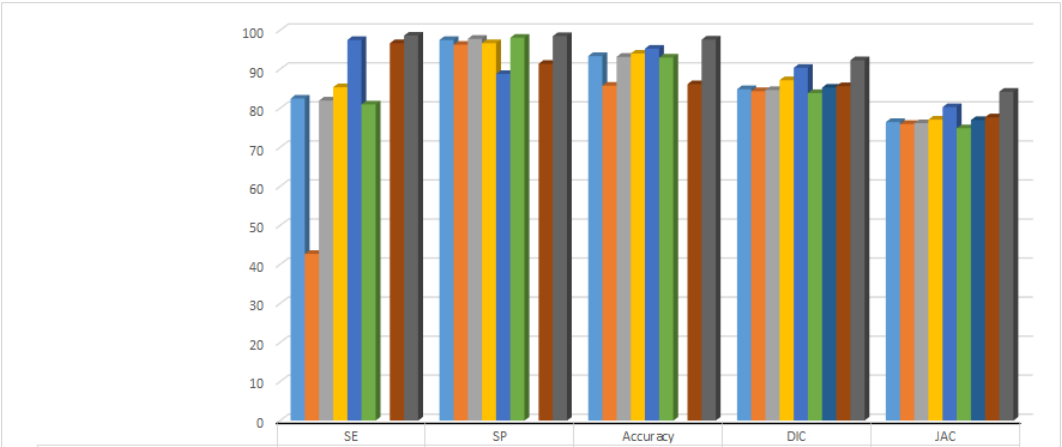
According to uneven skin patches, such as freckles, dermoscopic image repositories could have several small objects. Such small objects can be screened out by using the median filter. In some cases, all impurities are not removed and the filter has an impact on the algorithm work and results.

Figure 6.8 shows images that fail in the segment. In the figure, rows (1-3) PH2, ISBI 2016 and ISBI 2017 datasets, respectively, columns (a) original image, (b) manual segmentation image (ground truth), (c) proposed method segmentation image, and (d) result of the proposed method (green) and ground truth (red). The color contrast of the skin lesion and underlying skin in rows 2 and 3 is identical. In this case, the optimal threshold value is difficult to determine because of color differences between the skin lesions and the background with no noticeable pigmentation in the lesions. If the contrast between the skin lesion and the skin around is not sufficient, the gray threshold algorithm of global optimization takes in a large part of the skin around the image and stretches to the edge of the image. It also causes a segmentation failure as the mask reaches the image's edge. Considering that the primary skin lesion is an adjacent area, segmented images should first involve only one adjacent context with no isolated



	SE	SP	ACC	DIC	JAC
Yu et al. (Yu et al., 2016)	91.1	95.7	94.9	87.7	82.9
Codella et al. (Codella et al., 2017)	69.3	83.6	80.7		
Menegola et al. (Menegola et al., 2017)	47.6	88.1	79.2		
Vasconcelos et al. (Vasconcelos, 2017)	74.6	84.5	82.5		
Oliveira et al. (Oliveira et al., 2019)	91.8	96.7		27.7	
Global optimization algorithm	93.42	97.31	95.24	88.51	84.61

Figure 6.6. Comparison of earlier studies with the current method on ISBI 2016 database (Yu et al., 2016; Codella et al., 2017; Menegola et al., 2017; Vasconcelos and Vasconcelos, 2017; Oliveira et al., 2019).

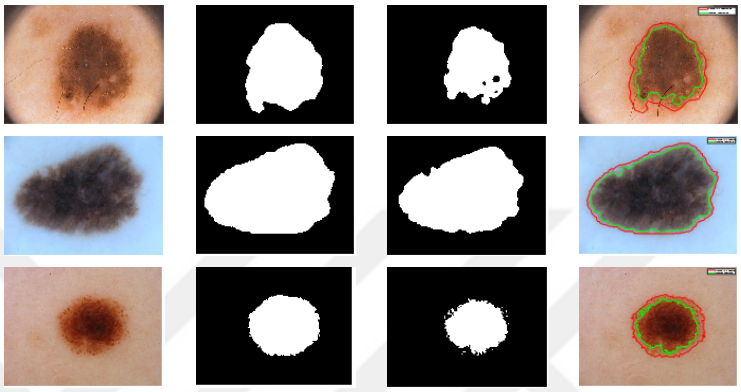


	SE	SP	Accuracy	DIC	JAC
Yuan et al. (Hance et al., 1996)	82.5	97.5	93.4	84.9	76.5
Bi et al. (Bi et al., 2017)	42.7	96.3	85.8	84.4	76
Li & Shen (Li and Shen, 2018)	82	97.8	93.2	84.7	76.2
Al-masni et al. (Al-Masni et al., 2018)	85.4	96.69	94.03	87.2	77.11
Guo et al. (Guo et al., 2018)	97.5	88.8	95.3	90.38	80.32
Jahanifar et al. (Jahanifar et al., 2018)	81	98.1	93	83.9	74.9
Tschandl et al. (Tschandl et al., 2019)				85.3	77
Bi et al. (Bi et al., 2019)	96.7	91.4	86.2	85.7	77.7
Global optimization algorithm	98.64	98.52	97.61	92.31	84.26

Figure 6.7. Comparison of earlier studies with the current method on ISBI 2017 database(Hance et al., 1996; Bi et al., 2017; Li and Shen, 2018; Al-Masni et al., 2018; Guo et al., 2018; Jahanifar et al., 2018; Tschandl et al., 2019; Bi et al., 2019) .

Table 6.3. Comparison of performance parameters for the five processed images selected randomly from the PH2 database.

Image No	PPV	NPV	Computation Time(sec)
1	0.901184	0.999942	3.312
2	1	0.988564	3.407
3	0.984021	1	3.489
4	0.970134	0.997304	3.579
5	0.924157	0.998954	3.357
6	0.9327	0.9898	3.217



(a) (b) (c) (d)

Figure 6.8. A variety of segmentation failure cases.

elements or troughs. The first area (including the image) can be of any size and possibly bordering the image. This is why the image failed in row 1.

Table 6.4. Comparison of the proposed algorithm with state-of-the-art methods.

	Number of cases				
References	Benign	Melanoma	Sen %	Spe %	Acc %
D'Amico & Stanganelli (d'Amico et al., 2004)	927	50	96.41	87.16	91.78
Tanaka et al.(Tanaka et al., 2004)	181	70	90.00	98.30	94.00
Maglogiannis & Kosmopoulosb(Maglogiannis and Kosmopoulos, 2006)	14	20	90.00	93	94.00
Marques et al. (Marques et al., 2012)	146	17	94.10	77.40	85.50
Olszewska & Semantic (Olszewska, 2013)	24	24	**100**	66.66	83.33
Zagrouba & Barhoumi (Zagrouba and Barhoumi, 2005)	160	40	71.30	93.50	82.00
Aljanabi et al.(Aljanabi et al., 2018)	160	40	95.50	98.40	96.02
Proposed method	160	40	95.93	98.99	96.28

The comparison of state-of-the-art methods and the global optimization algorithm is provided in Table 6.4. If sensitivity is considered as the most significant component, which implies that every patient with skin cancer is detected, the global optimization algorithm has the highest performance. Thus, the global optimization algorithm is suitable for finding and segmenting melanoma skin cancer.

7. CONCLUSION

Linear problems often arise in man-made environments, whereas nonlinear problems with several optima are omnipresent. Global optimization is an important tool for exploiting nature. The auxiliary function idea is a promising way of tackling general problems of global optimization. Nevertheless, only dozens of variables could be dealt with conventional auxiliary function methods.

In this dissertation, we provide a variety of contributions to advancing the state-of-the-art on global optimization, including both theoretical and empirical aspects. Consequently, four new auxiliary function methods for dealing with unconstrained continuous global optimization problems have been developed.

In Section 4.1, a new filled function is proposed with in one dimensional, which is continuously differentiable and proved to satisfy all conditions of the definition of the filled function. In addition, this filled function is sensitive to parameters, and the overflow for this function can not occur.

Since some of the existing auxiliary functions have the disadvantages of discontinuity, non-differentiability, or sensitivity to parameters that limit their efficacy, in Section 4.2, new filled function definitions are developed to overcome such limitations under the preceding description. Such filled functions have three parameters and two parameters, respectively, to enhance the numerical computation performance. Two filled function algorithms are presented on the basis of these analyses. They are global optimization methods which used Bézier curves to modify the objective function as a filled function and gradually find a better local minimizer by optimizing the filled function based on the previously found minimizer.

In Section 4.3, we have offered a new globally convex approach for unconstrained continuous global minimization issues. This method was developed to transform the specified objective function into an auxiliary function with a lower local minimizer number by using the suggested control points treated with Bézier curves. With one variable case, we have solved some illustrated problems. These numerical results also indicate the high performance of the proposed solution algorithm.

In Section 6, Algorithm 2 has been suggested and implemented from certain public databases to the skin lesion images, and a threshold value is determined based on

the results of the image histogram, and then the segmented image is extracted using the threshold value taken from such algorithm. To determine the best method for segmenting these images, this approach was compared with other previous methods.

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