ÇUKUROVA UNIVERSITY
INSTITUTE OF NATURAL AND APPLIED SCIENCES

M.Sc. THESIS

Hammad JAMIL

COMPARISON OF SHOOTING METHOD AND COMPLEMENTARY FUNCTIONS METHOD IN LINEAR STABILITY PROBLEMS

DEPARTMENT OF MECHANICAL ENGINEERING

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ABSTRACT

MSc. THESIS

COMPARISON OF SHOOTING METHOD AND COMPLEMENTARY FUNCTIONS METHOD IN LINEAR STABILITY PROBLEMS

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Solution of computationally expensive numerical problems demands hybrid and intelligent algorithms, instead of a single generalized approach. A comparative analysis of well renowned "Shooting method" with another numerical method "Complementary Functions Method" (CFM) is performed on a computationally-cumbersome boundary eigenvalue problem, namely, numerical determination of an eigenvalue (λ) in linear stability problems. Both numerical methods are used for the conversion of a governing boundary value problem into an initial value problem. Shooting method takes a direct approach, starts with the guessing of missing initial condition and lays an iterative scheme based upon the system of equations containing two variables: a missing initial condition and an eigenvalue. Conversely, complementary functions method uses a smart approach, excluding the need of finding the missing initial condition by using general properties of linear differential equations: it produces an iterative scheme of algebraic equations with a single variable, namely, an eigenvalue. Reduction in the number of variables by CFM sets the basis of this comparison between the two numerical methods. Both numerical methods are evaluated in terms of accuracy and efficiency for various cases of linear stability problems: (i) Different cases of Euler's Columns (ii) thermal buckling of a disc restrained along its edges (iii) thermal buckling of a functionally graded (FGM) disc.

Key Words: Shooting Method (SM), Complementary Function Method (CFM), Homogenous Boundary Eigenvalue Problem, Non-Homogenous Boundary Eigenvalue Problem, Linear Boundary Eigenvalue Problem

ÖZ

YÜKSEK LİSANS TEZİ

LİNEER STABİLİTE PROBLEMLERİNDE ATIŞ METODU VE TAMAMLAYICI FONKSİYONLARMETODUNUN KARŞILAŞTIRILMASI

Hammad JAMIL

ÇUKUROVA ÜNİVERSİTESİ
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Hesaplama açısından zaman alan sayısal problemlerin çözümünde, tek bir genel yaklaşım yerine hibrit ve akıllı algoritmalar kullanılır. Bu da mevcut sayısal algoritmalar arasında karşılaştırmalı çalışma ihtiyacını doğurur. İyi bilinen "Atış metodu" nun başka bir sayısal yöntem olan "Tamamlayıcı Fonksiyonlar Metodu" (TFM) ile karşılaştırmalı olarak analizi, zahmetli bir sınır özdeğer problem olanlineerstabilite problemleri özelindeyapılacaktır. Her iki sayısal yöntem, geçerli bir sınır değer probleminin bir başlangıç değer problemine dönüştürülmesi için kullanılır. Atış metodu direkt bir yaklaşım benimser, eksik başlangıç durumunun tahmin edilmesiyle başlar ve iki değişken içeren denklemler sistemine dayanan yinelemeli bir şema oluşturur;tahmini bir başlangıç koşulu ve bir özdeğer ile başlanır. Tersine, TFM, lineer diferansiyel denklemlerin genel özelliklerini kullanarak eksik başlangıç koşulunu bulma ihtiyacı hariç akıllı bir yaklaşım kullanır: Tek bir değişken, yani bir özdeğer ile yinelemeli bir cebirsel denklem şeması üretir. TFM ile değişken sayısındaki azalma, iki sayısal yöntem arasındaki bu karşılaştırmanın temelini oluşturmaktadır. Her iki sayısal yöntem de çeşitli lineer stabilite problemleri için doğruluk ve verimlilik açısından değerlendirilecektir. Test problemleri (i) Farklı Euler Kolonları, (ii) kenarları boyunca tutturulmuş bir diskin termal burkulması (iii) fonksiyonel derecelendirilmiş (FGM) bir diskin termal burkulması.

Anahtar Kelimeler: Atış Metodu (SM), Tamamlayıcı Fonksiyon Metodui (TFM), Homojen Sınır Özdeğer Problemi, Homojen Olmayan Sınır Özdeğer Problemi, Lineer Sınır Özdeğer Problemi

EXPANDED ABSTRACT

The exodus of engineering from the conventional practices of prototyping and experimental-based design verification to validation using numerical simulation and modeling is inevitable. Computational cost is one of the fundamental challenges experienced during the numerical analysis in all engineering problems. The incessant amplification in computational resources alone cannot be translated into the solution of computationally intensive problems. One of the most promising approaches is to generate a hybrid algorithm, comprising various numerical techniques with an ability to decide an optimum solution method. Problem-based selection of a numerical technique and balancing it with computational cost, calls for the detailed comparative analysis between the different numerical methods.

Linear elasticity stability related problems cover a wide range of topics in applied solid mechanics, extending from free or forced vibration problems to buckling cases. Solving a boundary eigenvalue problem is one of the basic requirements while dealing with the elastic stability cases. In the case of the boundary eigenvalue problems (BEVP), quest of the assigned numerical method is not limited to the conversion of a boundary value problem into an initial valuable problem alone. An additional task is introduced with the presence of an eigenvalue, prompting the increase in the computational cost of numerical algorithm. Moreover, in the case of inhomogeneous boundaries, the level of complexity for the employed numerical method is enhanced further.

Wide utilization of the shooting method in solving boundary value problems may be attributed to its straightforward approach. It transforms the linear boundary value problem into a set of linear algebraic equations. Formulation includes a multivariable-function in the terms of missing initial conditions and boundary conditions. It makes use of root finding method iteratively, converging a solution to the correct initial conditions by satisfying both boundary conditions.

Occurrence of an eigenvalue in the boundary eigenvalue problems (BEVP) makes it inadequate for the direct application of root finding method and demands an additional step. Ability of CFM is that it removes the need of guessing initial conditions while solving BVP which makes it an ideal method for solving such problems. Further, it gives the complete solution by separately finding complementary and particular solutions. In the case of eigenvalue BVP, complementary and particular solutions generate a single algebraic equation containing eigenvalue λ as a variable and solving it for roots gives the required solutions.

Cases considered under this study for comparative analysis includes both homogenous and non-homogeneous boundary eigenvalue problems. Homogenous boundary eigenvalue problems consist of the set of boundary conditions equating to zero. On the other hand, non-homogeneous boundary eigenvalue problems comprise a set of boundary conditions equating to non-zero values.

In this study, the role of complementary functions method is extended by applying it first time on the non-homogeneous boundary eigenvalue problems. Further, the presence of similarity among boundary conditions for the different cases of Euler columns generates another interesting challenge while applying the shooting method. Shooting method produces the possible value of missing initial conditions on the basis of a function dependent upon the boundary conditions. Under the above- mentioned facts, the selection of the boundary conditions become an additional task while dealing with the cases having similar boundary conditions. The difficulty of shooting method in the selection of boundary conditions is also shown in this study.

An individually dedicated code is written for both methods on Mathematica software. A simulation is created for each problem accommodating all the required boundary conditions. In order to establish the validity of the code and numerical methods, problems which have analytical or closed form solution are considered. Comparison of numerically calculated results with the analytical

solutions, establishes the criteria of validity for the results calculated by shooting method and CFM. Comparison of both methods is performed on the basis of step size, number of iterations, accuracy and processing time.

A clear distinction is tried to establish in terms of the efficiency among both methods under different cases of linear elastic stability. The last case of this comparative analysis is itself a novel problem in the area of linear elastic stability. A case of thermal buckling for the functionally graded disc, restrained along its edges is modeled and solved numerically by using both numerical methods. Numerical solutions for the different cases of the functionally graded discs and plates can be seen in different research papers, but the problem mentioned above is lacking adequate investigation in the present literature. It further increases the importance of the last case and gives it novelty among other cases.

In order to increase the competitiveness of this comparative analysis, a comparison of CFM with finite element model (FEM) for the thermal buckling of a disc is also considered for future work. Under the above consideration, a governing equation for the thermal buckling of disc is derived and its weak formulation form also included in this thesis. Final results of this comparative analysis have shown the dominance of CFM in efficiency among all the problems of linear stability. Efficiency of complementary functions method (CFM) for solving linear elastic stability problems also gives inspiration for extending this comparative analysis investigation to non-linear problems too.

GENİŞLETİLMİŞ ÖZET

Geleneksel prototip oluşturma ve deneysel tabanlı tasarım doğrulamasında sayısal simülasyon ve modelleme kullanmak kaçınılmazdır. Hesaplama zaman ve maliyeti, tüm mühendislik problemlerinde sayısal analiz sırasında karşılaşılan temel zorluklardan biridir. Yalnızca hesaplama kaynaklarındaki sürekli gelişme hesaplama açısından yoğun sorunların çözümüne yeterli değildir. En umut verici yaklaşımlardan biri, optimum çözüm yöntemine karar verme yeteneğine sahip çeşitli sayısal teknikler içeren bir hibrid algoritma oluşturmaktır. Sayısal bir tekniğin probleme dayalı seçimi ve hesaplama maliyetiyle dengelenmesi, farklı sayısal yöntemler arasında ayrıntılı karşılaştırmalı analiz yapılmasını gerektirir.

Lineer elastik stabilite problemleri, katı uygulamalı mekanikte titreşim problemlerinden burkulmaya kadar uzanan çok çeşitli konuları kapsar. Özdeğer sınır değer problemlerini çözmek, mekanik kararlılık durumlarıyla çalışırken karşılaşılan temel gereksinimlerden biridir . Sınır Özdeğer problemleri (BVP) söz konusu olduğunda, atanan sayısal yöntem arayışı, sınır değer probleminin başlangıç değer problemine dönüştürülmesi ile sınırlı değildir. Sayısal algoritmanın hesaplama maliyetinde artışa neden olan bir Özdeğer varlığı, ek bir görev başlatır. Ayrıca, homojen olmayan sınırlar olması durumunda, kullanılan sayısal yöntem için karmaşıklıkseviyesini arttırır.

Tamamlayıcı Fonksiyonlar Metodu (TFM)'nun avantajı, BVP'yi çözerken başlangıç koşullarını tahmin etme ihtiyacını ortadan kaldırmasıdır. Ayrıca, tamamlayıcı ve özel çözümleri ayrı ayrı bularak genel çözümü sağlar. Özdeğer BVP durumunda; tamamlayıcı ve özel çözümler, özdeğerλ içeren tek bir cebirsel denklem üretir. Bunu kökler için çözmek bize gerekli çözümleri verir.

Karşılaştırmalı analiz için bu çalışma kapsamında ele alınan vakalar, hem

homojen hem de homojen olmayan sınır özdeğer problemlerini içerir. Homojen sınır özdeğer problemleri, sıfıra eşit sınır koşulları kümesinden oluşur. Diğer yandan homojen olmayan sınır özdeğer problemleri, sıfır olmayan değerlere

eşit bir dizi sınır koşulunu içerir. Sonuç olarak, homojen olmayan sınır özdeğer problemlerinin elde edilmesi için TFM uygulanırken yeni bir değişken kısıtlaması üretilir. Ayrıca, Euler kolonunun farklı durumları için sınır koşulları arasında benzerlik olması, atış metodunu uygularken başka bir ilginç zorluk oluşturur.

Atış metodu, sınır koşullarına bağlı bir fonksiyon bazında eksik başlangıç koşullarının olası değerini üretir. Sınır koşullarının seçimini ek bir görev haline getirirken, çok sayıda sınır koşulu olan vakalarla ilgilenir.

Mathematica yazılımında her iki metod için ayrı bir özel kod yazılmıştır. Her bir problem için, her durumda gerekli tüm sınır koşullarını karşılayan bir simülasyon oluşturulmuştur. Kodun ve sayısal yöntemlerin geçerliliğini sağlamak için, analitik çözümü olan problemler ele alınmıştır. Sayısal olarak hesaplanan sonuçların ve analitik çözümlerin karşılaştırılması, atış metodu ve tamamlayıcı Fonksiyonlar metodu (TFM) kullanılarak hesaplanan sonuçların geçerliliğini ortaya koymuştur. Her iki yöntemin karşılaştırılması adım boyutu, tekrarlama sayısı, doğruluk ve işlem süresi bazında yapılmıştır. Lineer elastik stabilitenin farklı problemleri göz önüne alındığında, her iki yöntemin verimliliği arasında çok net bir ayrım ortaya çıkmaktadır.

Bu karşılaştırmalı analizin incelenmesi kapsamında ele alınan son durum, kenarları boyunca tutturulmuş fonksiyonel derecelendirilmiş diskin termal burkulma problemidir. Pek çok havacılık uygulamasında, fonksiyonelderecelendirilmiş malzemelerden oluşan dairesel diskler, diskin termal burkulmasına yol açan yüksek sıcaklığa maruz bırakılır. Fonksiyonel derecelendirilmiş diskler veya plakalar için farklı araştırma makaleleri mevcuttut, ancak yukarıda bahsedilen problem mevcut literatürde yeterli araştırmadan yoksundur. Bu karşılaştırmalı analizi tamamlamakiçin, bir diskin termal burkulması sonlu elemanlarmetodu (FEM) ile de çözülmüştür.

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NOMENCLATURE

L	: The total length of the Euler Column
E	: Elastic Modulus
D	: The diameter of the disc
r	: The Radius of the disc
x, y :	: Independent, dependent variable
u	: guessed initial condition
v	: Poisson Ratio
Λ	: Eigenvalue

Abbreviations

CFM	:	Complementary Functions Method
SM	:	Shooting Method
BVP	:	Boundary Value Problem
IVP	:	Initial Value Problem
BEVP	:	Boundary Eigenvalue Problem
FEM	:	Finite Element Method
FGM	:	Functionally Graded materials

1. INTRODUCTION

The history of using Numerical algorithms for solving mathematical equations dates back to the Egyptian Rhind papyrus (c. 1650 BC), it illustrates a root-finding method for the solution of a very simple low order algebraic equation. In advanced problems of today's world, it's a rare chance to find solutions of the problems in an analytical or closed form. A numerical method forms the basis of all the efficient mathematical models. However, the focus in numerical analysis is mostly limited to the few robust and comprehensive numerical methods such as finite element method (FEM), due to their generalized and versatile approach. The availability of ready-made packaged programs with user friendly interface for carrying out different structural analysis gives further strength to the above narrative. On the other hand, the multiple folds increase in the computing power, the introduction of more complex and sophisticated numerical algorithms went against the narrative of using few generalized methods in all problems. In computational mechanics, solving computationally cumbersome problems with the low computational cost as well as producing results with high accuracy is considered as the important deciding factor. Such deciding factors formulate the need of comparative analysis among different available numerical methods for solving various problems. Except above mentioned factors, stability and easy-to-apply features also carries importance for the superiority of any numerical method on any other method. In the light of above discussion, it won't be wrong to say that, no numerical method can be considered as a perfect choice in all situations. It leads to the creation of hybrid and intelligent algorithms with an ability to choose most suitable numerical method for the given problem.

1.1. Linear Elasticity Stability Problems

In the theory of the linear plane elasticity, stability related problems cover a wide range of solid mechanics problems. It stretches from the cases of free or forced vibration to the thermal buckling problems. In mathematical terms, solving linear elasticity stability problems means dealing with the boundary eigenvalue problems. Form longtime, applied mathematicians have been interested in solving boundary eigenvalue problems by using different numerical methods.

The basic principle behind using any numerical method for solving the boundary value problem is to convert a boundary value problem (BVP) into an initial value problem (IVP). Although in the case of the boundary eigenvalue problems (BEVP), this task is not just limited to convert a boundary value problem into an initial value problem but also an additional task of computing correct Eigenvalues is introduced. These multiple tasks all-together make the BEVP a computationally cumbersome problem. Most of the numerical methods dealing with BEVP proceed by numerically substituting random values in place of eigenvalue from a predicted range. This doubles the task for these numerical methods and also affects significantly their computational cost. This condition makes BEVP a computationally cumbersome problem and demands more efficient methods. A flowchart for the number of variables in BEVP is shown below (Figure 1.1.)

Most of the numerical methods achieved this additional task of determining correct Eigenvalues in boundary eigenvalue problems by initially predicting a range of guessed Eigenvalues, with high probability of containing an eigenvalue. It proves to be a very inefficient approach in terms of the computational cost.

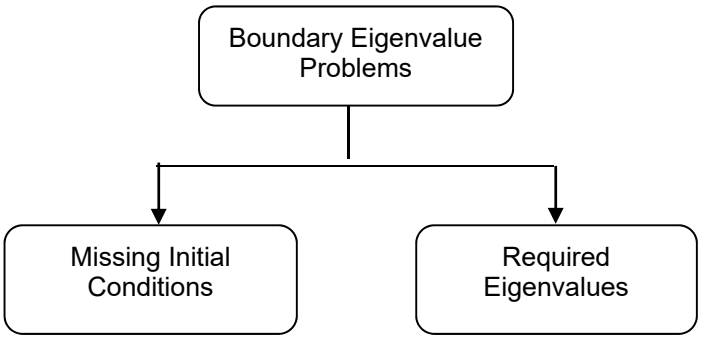

Figure 1.1. Flowchart of the Boundary Eigenvalue Problem variables

One of the important problems comes under the elastic stability cases is Euler's column. All cases differentiate with each other on the basis of the boundary conditions like fixed-fixed ends or fixed-free etc. (Figure 1.2)

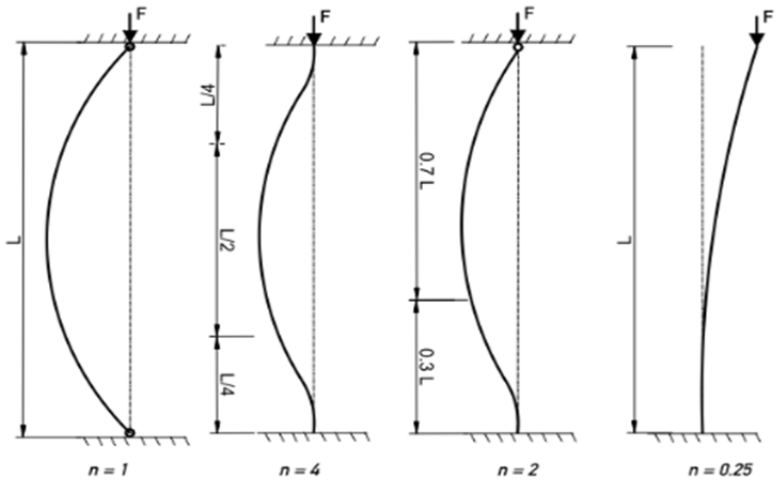

Figure 1.2. Euler's column cases (Timoshenko et. al., (1969))

Another interesting problem of structural mechanics related with linear stability is the buckling of a solid circular disc, fixed along its edges and subjected to a thermal load (Figure 1.3). Restraining the disc along its circumference from radial expansion under high temperature generates a compressive stress, leading to the buckling of the disc. Application of this model can be seen commonly in the engine of an aircraft or an automobile. In aircraft engine, there are many components containing circular discs confined along their circumference and are subjected to extremely high temperature values. Same problem become more complex on considering a buckling of a functionally graded disc. In literature, many problems related with functionally graded disc are solved but the thermal buckling of a functionally graded disc lacks adequate study. Moreover, differentiation of elastic stability cases as homogeneous and non-homogeneous boundary eigenvalue problems also increases the scope of this research.

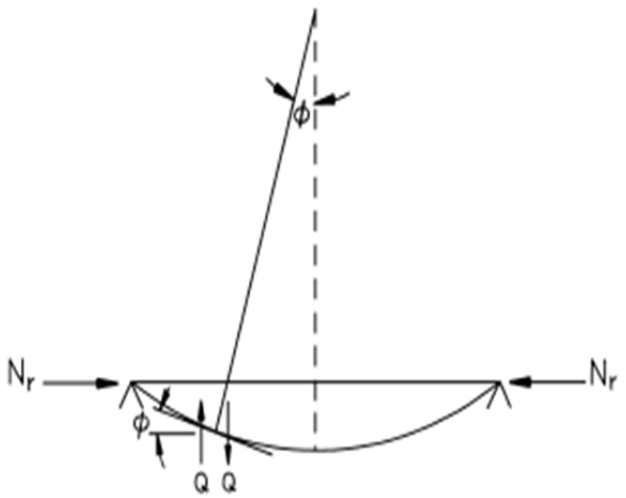

Figure 1.3. Buckling of a disc under axial load (Timoshenko et. al., (1969))

1.2. Shooting Method (SM)

It is a commonly-used numerical method for solving the boundary value problems. In literature of boundary value problems, it is classified as an initial value type numerical technique. The main idea behind the shooting method is based upon the conversion of a boundary value problem into an initial value problem. For this purpose, solution is calculated by shooting different trajectories with the guessed values of missing initial conditions (Figure 1.4). Above process numerically goes on until a particular trajectory generated corresponding to the particular guessed initial value satisfy all the boundary conditions. The methodology of shooting method makes it very simply to apply but also it reduces the stability of SM.

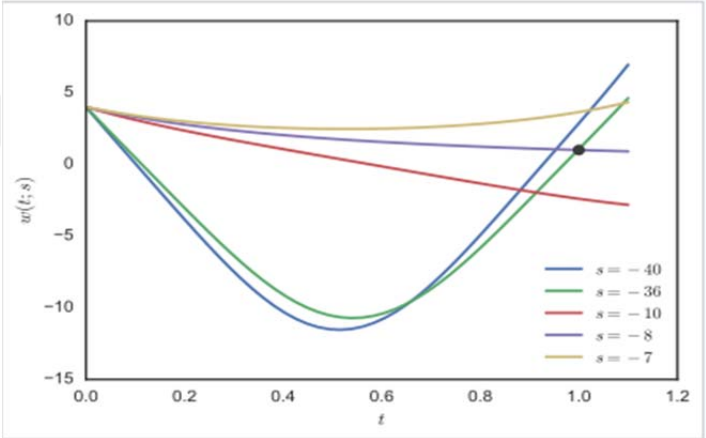

Figure 1.4. Trajectories w(t;s) for s = w'(0) equal to −7, −8, −10, −36, and −40. The point (1,1) is marked with a circle. (Wikipedia, (2020))

For transforming this process into an automated numerical algorithm, an algebraic equation in the terms of the missing initial conditions equated to the 5 require boundary condition. Later that algebraic equation can be solved by any suitable root finding method. Every time the solution is tested for the given boundary conditions and this process continues until the solution satisfies all the boundary conditions. Both linear and non-linear boundary problems can be solved

by using Shooting method. For linear cases, Newton's method can be used as a root finding method while in the case of non-linear boundary value problems; secant method is used as a root finding method.

In the case of boundary eigenvalue problems, a multivariable function based upon the missing initial condition and eigenvalue needs to be considered. It increases the difficult of problem and by making it computationally cumbersome. As a result, there is a significant increase in the computational cost of the shooting method while solving boundary eigenvalue problems. These prospects lead to the need of finding other numerical techniques that can handle this computationally-expensive problem more efficiently. Further, SM also gives the challenge of choosing missing boundary conditions and especially when there are similar boundary conditions among different boundary value problems.

For complex problems, multiple shooting method is generally used due to its ability to shoot in multiple ranges and converge more quickly. Same technique can be use in the case of boundary eigenvalue problems. Multiple intervals for eigenvalues and missing initial conditions are given and algorithm continues to run until it converges. In other words, number of variables to tackle stays the same for the multiple shooting method but it makes searching process faster by working in multiple intervals at same time. At the same time, there is another issue related with shooting method stability in long intervals which may lead to the failure in convergence. It becomes a more severe problem when it is dealing with any computationally expensive problem like BEVP.

1.3. Complementary Functions Method (CFM)

It is a method which uses basic principles of linear ordinary differential equation while solving boundary value problems. The major benefit of this method is that it removes the need of finding missing initial conditions. This feature makes it a very effective method for solving BEVP. The major obstacle faced by all the other numerical methods in solving boundary eigenvalue problems is the presence

of two unknown variables, solving it leads to increase in their computational cost. On the other hand, CFM becomes extremely effective in this situation and takes a lead on the other method. Already complementary functions method removes the need of finding missing initial conditions and it left us only with the eigenvalues as a variable.

Complementary functions method reduces a BVP into an IVP and gives a solution as a set of complementary and particular solutions. It generates a set of algebraic equations, enforcing boundary conditions on them leads to the final solution. In the case of BEVP, a set of algebraic equations are solved by simply applying a root finding method (Figure 1.5). Ability of CFM to calculate complementary and particular solutions separately makes CFM more robust for solving both homogeneous and non-homogeneous BEVP. A set of algebraic Equation in the terms of eigenvalue representing complementary and particular solutions is obtained. Afterward, basic principles of matrix algebra are use to force this matrix of complementary and particular solutions equal to zero. As a result, any root finding method will be enough to find the solution in the form of required Eigenvalues. Dealing with the only single variable while solving BEVP significantly reduces the computation cost of CFM

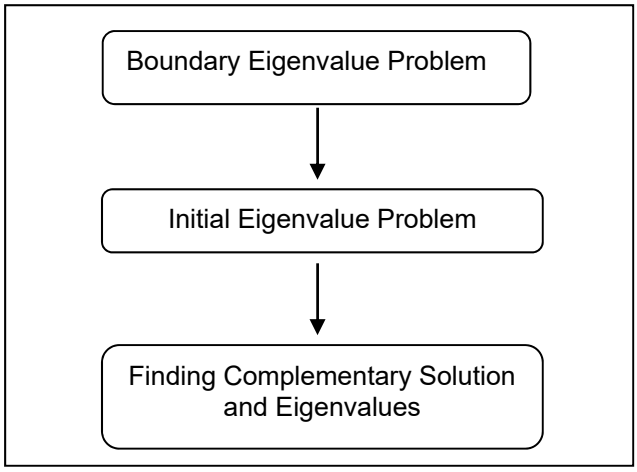

Figure 1.5. Flowchart of the Complementary functions method (CFM) Algorithm

1.4. Scope of Research

The scope of this research is limited to the linear boundary eigenvalue problems. Although the shooting method is readily applicable to nonlinear problems, the extension of this comparative analysis to non-linear cases is also a possibility for future research works.

2. PREVIOUS STUDIES

Numerous studies are available where different numerical methods are employed for solving boundary eigenvalue problems, including shooting method and complementary functions method. Linear stability problems are very frequently come under discussion in the field of solid mechanics. The only missing point observed in previous studies is the comparative analysis among different numerical methods for solving linear elasticity stability problems. This above-mentioned fact further highlights the importance and significance of this thesis topic. In order to give the detail background about previous studies conducted on solving BEVP by using both methods, following studies with brief details are shared below.

Jones (1993) performed a study on the use of SM for calculating eigenvalues in two-point BEVP of fourth order. The purpose of this study is to show that SM which is generally associated with the non-eigenvalue problems can also be used for solving boundary eigenvalue problems. Two steps are used to apply shooting method; in the first step guessed intervals of eigenvalue are computed by simply incrementing the guessed value of λ for 50,000 steps. After that output is refined by using the least square method and intervals with the maximum probability of containing eigenvalue are determined. In the second step, SM is employed on those intervals of λ to calculate the exact location of the required eigenvalues.

Chen (1996) offered an analytical proof for the use of the SM on the BEVP. In his study he tried to show that shooting method can be very effective in solving certain type of BEVP.

Schrodinger equation for the two-dimensional hydrogen atom with logarithmic potential function has been tried to solve by using shooting method with an analytical proof.

Ishikawa (2007) also presented another way of applying shooting method for solving of second order. A use of method named Discretized Matrix Eigenvalue leads to the approximation of the possible interval for the presence of eigenvalues. Further, linear multistep method is run in parallel to the above method for the exact approximation of the eigenvalues. In the last step the correction of eigenvalue takes place by using above two methods together in parallel. Results obtained by the first method are substituted in the next method and so on. In total, three steps are employed in the whole process for solving the problem.

Li (2007) used SM for the free vibration analysis of thermally post buckled polar orthotropic circular disc. Both buckling and free vibration generates separate configuration of two coupled boundary eigenvalue problems. As a result two interdependent mixed boundary eigenvalue problems are solved together with the use of shooting method. In another study, Li et al. (2002) studied the pinned- fixed ended elastic rod under heating for the thermal post buckling. Again SM is employed while eigenvalue is found by applying a myriad of iterations in different given intervals.

Zhu et al. (2011) employed a new technique in the form of a perturbation theory with the shooting method for evaluating eigenvalues. He applied his method on the post-buckling of a nonlinear case of Euler's beam column structure. Perturbation theory is used to find the traces of bifurcation points and it followed by the Newton-Raphson iterations to find the bifurcation solutions. Calculations for the first three bifurcation point are shown and it is compared with other points. Nonlinearities are also considered during the formulation of the boundary eigenvalue problem.

In all the above studies, an additional method is employed with the SM for the approximation of the eigenvalues. Later that step followed by the normal shooting method which includes finding missing initial conditions. These limitations clearly show the decrease in the efficiency of the SM while dealing with

the boundary eigenvalue problems .An additional method is always required to get the approximate values of the eigenvalues.

Ma (2004) further extended the application of the SM to the buckling and free vibration of the functionally graded material (FGM) discs. During all the above cases, SM handled two variables in iteration scheme with a myriad of iterations. It raises a lot of questions, on the efficiency of SM for the stability problems.

Complementary functions method is a vigorously strong numerical method in handling linear elasticity and stability problems. Miele (1968) introduced the complementary functions method to solve the two point linear boundary value problem. A boundary value problem is solved by proposing a solution for the given problem in the form of a combination of the complementary and particular solutions. Later, conversion of given boundary value problem into an initial value problem by allotting any linearly independent initial conditions. All equations integrated form one end to the other boundary condition by using Runge Kutta method of 4^{th} order.

Agarwal (1982) shown in a study that complementary functions method can be used for the non linear boundary value problem. A two point non linear boundary value problem is solved by using CFM. Same methodology which is employed while solving a linear boundary value problem is converted into an iterative for problem. That research produces a significant increase in the scope of CFM as a numerical method.

Murty et. al. (1992) solved a non-linear multi-point boundary value problem by using complementary functions method. He derived a more generalized form for the application of the complementary functions method on non linear and linear boundary value problems. It further increases the scope of shooting method and makes it more robust method.

Tutuncu et. al. (2009) introduced CFM for the stress analysis of pressurized functionally graded cylinders, disks and spheres. Effective use of

complementary functions method is shown in solving a variety of two point boundary value problems. Verification of results is also presented against the benchmark solutions available in literature. Further, Tutuncu et. al. (2007) use CFM in thermoelastic analysis of functionally graded rotating disks.

Yildirim et. al. (2019) employed complementary functions method for solving a rotational elastic instability case for heterogeneous disks with inconsistent thickness under the effect of magnetic field and thermal loading. A complex instability problem is solved easily by the use of CFM. Further, Yildirim et. al. (2018) also used CFM for solving the Inertio-Elastic Instability problem for the disk of Variable-Thickness and composed of Functionally-Graded material. In both problems the reason for using complementary functions method is given as easy to apply and efficient numerical method among different available numerical techniques.

Çalım (2009) applied CFM on non uniform composite beams under free and forced vibration. Ordinary differential equations obtained in scalar form and transformed to the Laplace domain are solved by complementary functions method in order to calculate accurately the dynamic stiffness matrix of the problem. Application of the complementary functions method is extending form buckling to vibrations problems. Even on limiting the scope of CFM to linear elasticity problems, still there are plenty complex solid mechanics problems can be solved easily by using CFM.

Noori et. al. (2018) applied CFM while solving the case of free and forced vibration for functionally graded parabolic arches. A boundary eigenvalue problem in the domain of Laplace is converted into an initial eigenvalue problem. Further by using root finding method for the obtained homogeneous solutions required Eigenvalues can be calculated. He further did comparison among CFM and FEM, presented results showing significant efficiency in computational cost of CFM over FEM.

3. NUMERICAL METHODS AND PROBLEMS

Formulation of both numerical methods involved in comparative analysis is explained under this section. All steps of numerical methods are explained in detail with a solved example. In order to give better idea about the coding of these numerical methods, a flow chart representing the basic coding algorithm is given for both methods. Later, the governing equations for the different linear stability cases are presented. It includes all problems formulation for analytical solution and also numerical solution form for CFM and SM.

3.1. Complementary Functions Method

CFM is a numerical method used to solve linear boundary value problems. From literature review, it is not a new method in the field of pure mathematics and has been under use for many years for solving different type of boundary value problems. In the field of applied solid mechanics, it is relatively a new method and has been undervalued due to the notion of using more generalize methods like finite element method or shooting method. As the world moving towards more hybrid and problem to problem based specific algorithms, availability of high computing power, such specific methods are coming under focus. In this study, complementary functions method is applied on the eigenvalue problems and shown how effective CFM is in handling linear BEVP.

3.1.1. Formulation of CFM

In order to established the basis of the numerical algorithm for complementary functions method, step by step formulation of CFM is given below

Step 1

Solution of n^{th} order linear ordinary differential BVP is composed of c_m complementary solutions with d_m constants and an expression for inhomogeneous solution a. where the boundary conditions B_m are given by g and h

$$f(x) = \sum_{i=1}^{m}\{a_m(x).c_m\} + b(x) \qquad g < x < h \tag{1}$$

Step 2

In this step a set of linearly independent initial conditions are substituted as the initial conditions of a given problem. Generally in order to simplify, 0 and 1 are substituted as the values of initial conditions. Considering a family of linearly independent initial conditions

$$\{a_1(g), \dots., a_m(g), b(g)\}^T = \{1. \dots 0\ \ 0, \dots., 0 \dots .1\ \ 0, 0 \dots 0\ \ 0\}^T$$
$$a_1(g) = \{a_1(g)^0, \dots., a_1(g)^n\}\ ,\ a_m(g) = \{a_m(g)^0, \dots., a_m(g)^n\} \tag{2}$$
$$b(g) = \{b(g)^0, \dots., b(g)^n\}$$

Step 3

Any numerical integration method can be applied to calculate homogeneous and non-homogeneous solutions. In other words, Range Kutta method of 4^{th} order is employed as a numerical integrator to find values of complementary solution ($a_n(h)$) and Particular solutions b(h)at the other end of boundary conditions.

Step 4

In the last step all the generated algebraic equations in the terms of complementary and particular solutions are solved together to find the constants d_m.

Current study extended the role of CFM for solving boundary eigenvalue problems. In the case of BEVP, an additional task of calculation eigenvalue is included while solving it as a boundary value problem. The process of applying CFM method is same like solving any other boundary value problem except the last step. Here below, step by step formulation of complementary functions method for solving BEVP is given.

Step 1

Solution of n^{th} order eigenvalue linear ordinary differential boundary value problem is composed of c_m complementary solutions with d_m constants, and an expression for inhomogeneous solution a. where the boundary conditions B_m are given by g and h.

$$f(x,\lambda) = \sum_{i=1}^{m}\{a_m(x,\lambda).c_m\} + b(x) \qquad g < x < h \tag{3}$$

Step 2

In this step a set of linearly independent initial conditions are substituted as the initial conditions of a given problem. Generally in order to simplify 0 and 1 are substituted as the values of initial conditions. Considering a family of linearly independent initial conditions

$$\{a_1(g),\dots.,a_m(g),b(g)\}^T = \{1.\dots 0\ \ 0,\dots.,0\dots.1\ \ 0,0\dots 0\ \ 0\}^T$$
$$a_1(g) = \{a_1(g)^0,\dots.,a_1(g)^n\}\ ,\ a_m(g) = \{a_m(g)^0,\dots.,a_m(g)^n\} \tag{4}$$
$$b(g) = \{b(g)^0,\dots.,b(g)^n\}$$

Step 3

Any numerical integration method can be applied to calculate homogeneous and non-homogeneous solutions. In other words, Range Kuttamethod of 4th order is employed as a numerical integrator to find values of complementary solution(a_n(h)) and Particular solutions b(h)at the other end of boundary conditions.

Step 4

In the last step all the generated algebraic equations in the terms of complementary eigenvalue are solved together to find the required eigenvalues. Since the product of constants and complementary solutions in the terms of λ is equal to zero. It is only possible when the determinant of complementary solutions in the terms of λ is equal to zero.

$$A\,(\lambda).C\ =\ 0 \tag{5}$$

3.1.2. Example Solved by CFM

In this example a linear second order boundary value problem is solved by using complementary functions method (CFM). All the step are given below

Given equation: $Y'' + (1/x^2) * y = x$

Boundary conditions**:** $Y(1) + Y'(1) = 2.936$

$Y(2) = 0$

Step 1

Express the unknown fact y=$y(x)$

as$y(x)$=$y_0(x)$+$b_1y_1(x)$+$b_2y_2(x)$+⋯+$b_ny_n(x)$

Where

$y_0(x)$ = Inhomogeneous solution

$y_1(x)$ = Homogeneous solution

b_1= constants

n = order of governing equation

Step 2

Find $y_0(x)$, $y_1(x)$, … , $y_n(x)$ by solving the following initial value problems with given linearly independent initial conditions (Table 3.1)

Table 3.1. Allotting linearly independent initial conditions for applying CFM

	y_0	y_1	y_2
GE	$Y'' + \left(\frac{1}{x^2}\right)y = x$	$Y'' + \left(\frac{1}{x^2}\right)y = 0$	$Y'' + \left(\frac{1}{x^2}\right)y = 0$
$y(0)$	0	1	0
$y'(0)$	0	0	1

Step 3

In this step, Range Kutta method of 4th order is employed as a numerical integrator to find values of complementary solution and Particular solutions at the other end of boundary conditions**.** Calculated results as y_0 homogeneous solutions, y_1 (homogeneous solution) and y_2 (non-homogeneous solution) are given below in (Table 3.2), (Table 3.3) and (Table 3.4) respectively.

Table 3.2. Example solution for y_0 (homogeneous solution)

1	0	0
1.1	0.00516289	0.104852
1.2	0.021278	0.21894
1.3	0.0492421	0.341769
1.4	0.0899129	0.473034
1.5	0.144124	0.612547
1.6	0.212693	0.760195
1.7	0.296432	0.915915
1.8	0.396144	1.07967
1.9	0.512634	1.25146
2	0.646704	1.43128

Table 3.3. Example solution for y_1 (homogeneous solution)

1	1	0.
1.1	0.995314	-0.0907713
1.2	0.982371	-0.165745
1.3	0.962583	-0.228133
1.4	0.937082	-0.280363
1.5	0.906787	-0.324299
1.6	0.87245	-0.361395
1.7	0.834697	-0.3928
1.8	0.794049	-0.419433
1.9	0.750945	-0.442037
2	0.705755	-0.461219

Table 3.4. Example solution for y_2 (non-homogeneous solution)

1	0	0
1.1	0.0998488	0.995602
1.2	0.198895	0.984388
1.3	0.296574	0.968583
1.4	0.392509	0.949708
1.5	0.486449	0.928824
1.6	0.578232	0.906676
1.7	0.667761	0.883798
1.8	0.754981	0.860572
1.9	0.839873	0.837272
2	0.922439	0.814097

Step 4

In last step all the three equations are solved together simultaneously and boundary conditions are forced on each of them to calculate unknown constants. Any method to solve algebraic equations simultaneously can be applied in order to calculate missing constants. Putting these calculated unknown constants back in the supposed solution will give us the final solution of BVP.

Values of unknown constants b_1 and b_2

b_1=2.936

b_2=−2.936

Final Solution

$y(x)=y0(x)+2.936\ y1(x)-2.936\ y2(x)$

$y'(x)=y0'(x)+2.936\ y1'(x)-2.936\ y2'(x)$

3.1.3. Flowchart for CFM algorithm

An algorithm is created on the basis of the given above details about the formulation of CFM for solving boundary eigenvalue problems. Flowchart of this algorithm is shown below (Figure 3.1)

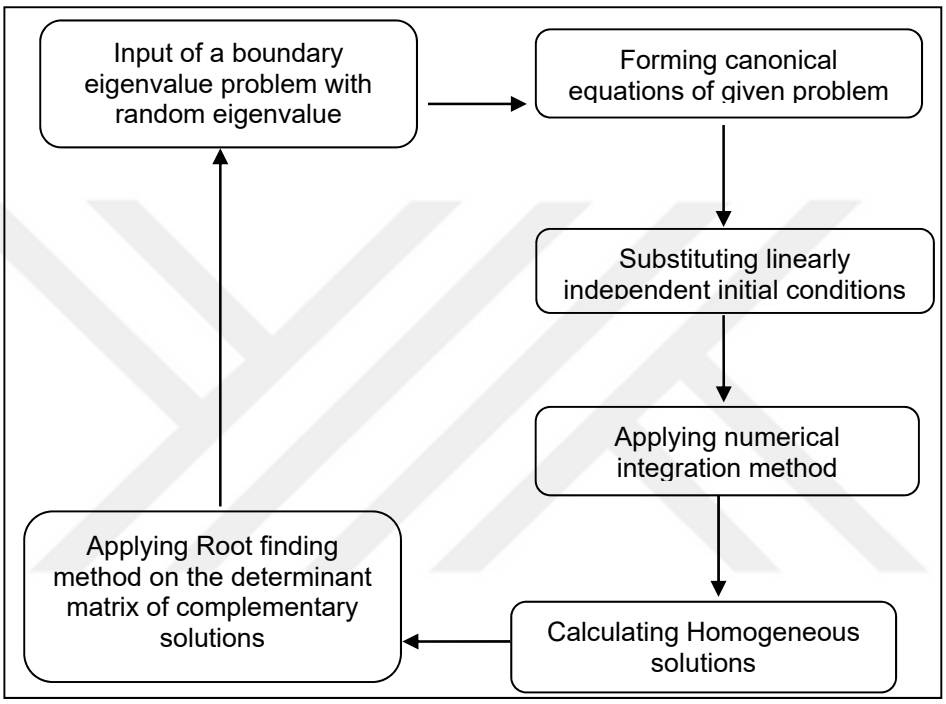

Figure 3.1.Flowchart of CFM algorithm

3.2. Shooting Method

In numerical analysis, the shooting method is an iterative method for solving a boundary value problem by reducing it to the solution of an initial value problem. In simple words, shooting method shoot out trajectories with different initial conditions until the finding of a trajectory that satisfy given boundary values. The shooting method converts BVP into an initial value problem by considering a multi-variable function of one of the given boundary conditions and a missing

initial condition. In the last step, root finding method is used to solve the algebraic equation.

3.2.1. Formulation of SM

In order to established the basis of the numerical algorithm for shooting method, step by step formulation of SM is given below

Step 1

Consider a boundary value problem Q(X, Y) of order n with the boundary conditions **a** and **b**

$$\frac{dY}{dX} = Q(X,Y) \quad , \quad a < X < b \tag{6}$$
$$Y = \{ y_1, y_2, \dots\dots\dots, y_n \}^T$$

Step 2

A function is formed in the terms of unknown initial conditions and one of the boundary conditions. Consider F(X) represents the solution of the above given BVP with initial conditions Y

$$F(b) - b = 0 \tag{7}$$

Step 3

In last step root finding method is applied to solve the above algebraic equation. This process goes on, until initial conditions satisfying above algebraic equation come out.

In the case of BEVP, an additional task of calculation eigenvalue is included while solving it as a boundary value problem. The process of applying shooting method is same like solving any other boundary value problem except that

an additional unknown variable included. Here below step by step formulation of complementary functions method for solving BEVP is given.

Step 1

Consider a boundary value problem Q(X, Y, λ) of order n with the boundary conditions a and b

$$\frac{dY}{dX} = Q(X, \lambda, Y) \quad , \quad a \leq X \leq b$$

$$Y = \{y_1, y_2, \dots\dots\dots, y_n\}^T \tag{8}$$

Step 2

A function is formed in the terms of unknown initial conditions and one of the boundary conditions. Consider F(X) represents the solution of the above given boundary value problem with initial conditions Y

$$F(b, \lambda) - b = 0 \tag{9}$$

Step 3

In last step root finding method is applied to solve the above algebraic equation. This process goes on, until initial conditions satisfying above algebraic equation come out.

3.2.2. Example Solved by SM

In this example a linear second order boundary value problem is solved by using Shooting method (SM). All the step are given below

Given equation $Y'' + (1/x^2) * y = x$

Boundary conditions $y(1) = 2.936$

$y(2) = 0$

Step 1

Reducing the second order differential equation into two first order differential equation

Let $y=z_1$

$y'=z_2$

We reduced 2nd order differential equation into two first order differential equations.

$$z_1' = z_2 \quad (10)$$

$$z_2' = -(1/x^2)\, z_1 \quad (11)$$

Step 2

Solve these equations by using any numerical method. Take one initial condition at any one of the boundary condition and for second initial condition use guessing. The process continues numerically until supposed initial conditions satisfy the give boundary conditions (Figure 3.2)

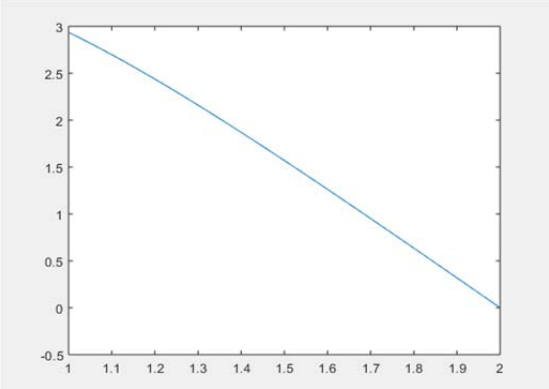

Figure 3.2. Final result of the solved example by shooting method

3.2.3. Flowchart for SM algorithm

An algorithm is created On the basis of the given above details about the formulation of Shooting method for solving BEVP. Flowchart of this algorithm is shown below (Figure 3.3).

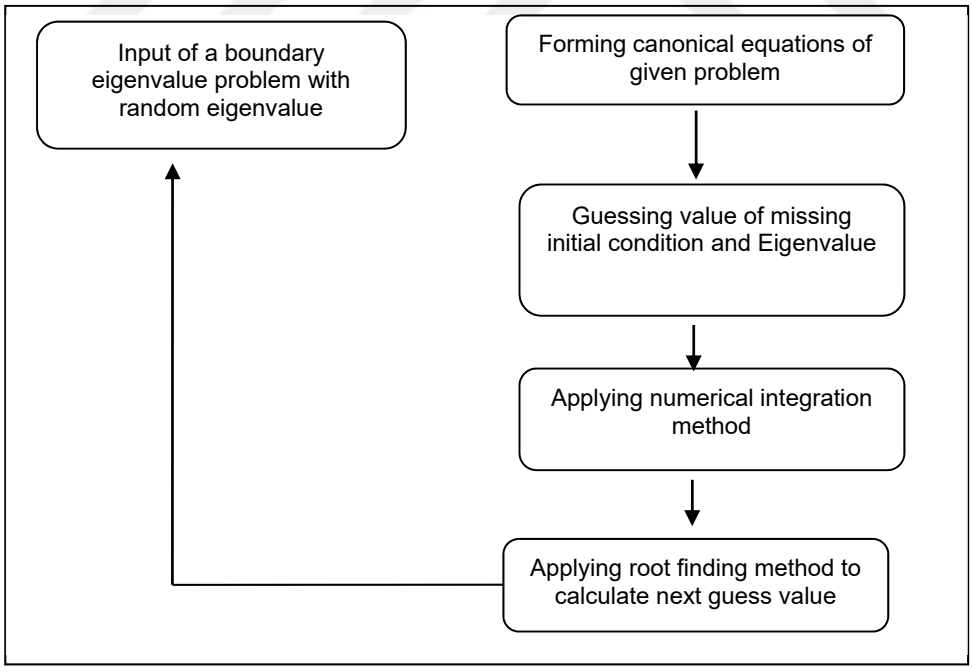

Figure 3.3.Flowchart of SM algorithm

3.3. Homogeneous boundary Eigenvalue Problems

BEVP are termed as homogenous, in case given boundary conditions are equal to zero. This differentiation is important in the comparative analysis of these two numerical methods and gives different results in terms of computational cost. Different cases of homogeneous BEVP are solved by using both numerical methods and step by step formulation of all the problems is given below. Governing differential equations for all the cases are taken from (Timoshenko et.al., (1969))

3.3.1 Buckling of the Fixed- Fixed Euler Column

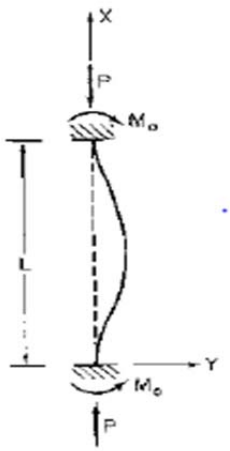

Figure 3.4. Buckling of Fixed – Fixed Euler column (Chajes, (1974))

Consider an axially loaded slender rod as an Euler Column of length “L” with flexural rigidity of “D=EI” under the compressive force “P” (Figure 3.4) Differential boundary equations and analytical expression of eigenvalue for the buckling of Euler Columns under fixed–fixed boundary condition is given below.

Governing differential equation

$$y'' + K^2 y = M_0 / EI \; ; \quad K^2 = P/EI \tag{12}$$

Canonical equations and analytical solution of equation (12)

$$\frac{dY}{dX} = Q(X, \lambda, Y) \quad , \qquad 0 < X < L \tag{13}$$
$$Y = \{ y_1, y_2 \}^T$$

Where X and Y represents axial coordinate and deflection of rod respectively. Where $\lambda = P/D$ and bending moment is given by M_0.

$$Q = \{y_2, -\lambda^2 y_1 + \frac{M_0}{D}\}^T \quad y1(0) = y2(0) = y1(1) = y2(1) = 0 \tag{14}$$

$$\lambda = \frac{2n\pi}{L} \tag{15}$$

The solution method by shooting method includes the following steps

By taking missing initial conditions as a function of given boundary conditions; the boundary eigenvalue problem for fixed –fixed Euler column is converted into an initial value problems.

From equation (10) – (12) the initial value problem for fixed-fixed ended is given as

$$\begin{cases} \frac{dY}{dX} = Q(X, \lambda, Y) \\ Y(0) = \{ 0, \beta \}^T \end{cases} \quad , \quad 0 < X < L \tag{16}$$

Here the value for the missing initial condition is given by β. As a result, for a definite value of β and λ, denote the solution of equation (10) as $y(X, \lambda, \beta)$ such that the value of β and λ satisfy the following algebraic equation

$$(L;\lambda, \beta) = 0 \tag{17}$$

Both above equation (13) and equation (14) are needed to be solved simultaneously in order to find such condition that guessed value of missing initial condition and eigenvalue satisfy the all boundary conditions.

An iteration system is introduced by using any numerical integration method for the integration of the Eq. (13) and secant method to find the root β of Eq.(11). Analytical continuation is employed on λ by increasing it values on each step from an infinitesimal value. Similarly for other Euler columns an iterative scheme is formulated.

The solution method by CFM includes the following steps

CFM converts given boundary value problem into an initial value problem by producing an algebraic equation containing only a single variable. Directly applying any root finding algorithm on given equation will determine eigenvalues. On following the numerical scheme mentioned under the heading of formulation of complementary functions method, a single variable algebraic equation is generated for fixed-fixed column case.

$$A\,(\lambda).C\ =\ 0 \qquad (18)$$

$$\text{A}\,(\lambda) = \{1\ \ 0\ \ 1,\ 0\ \ 1\ \ 0,\ a_1(1)a_2(1)\ \ 1\ \}^T, \quad C = \{\ c_1, c_2, c_3\}^T$$

For a non trivial solution C ≠ 0, hence

$$Det|A(\lambda)| = 0$$

Both above equation (13) and equation (14) are needed to be solved simultaneously in order to find the correct eigenvalue. Since only one variable present in both equations it makes it far easier to solve than solving it by SM

3.3.2 Buckling of the Hinged - Hinged Euler Column

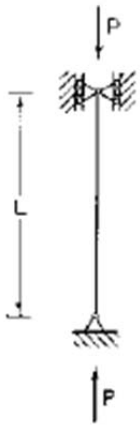

Figure 3.5. Buckling of the Hinged - Hinged Euler column (Chajes, (1974))

Consider an axially loaded slender rod as an Euler Column of length “L” with flexural rigidity of “D=EI” under the compressive force “P” (Figure 3.5) Differential boundary equations and analytical expression of eigenvalue for the buckling of Euler Columns under Hinged - Hinged boundary condition is given below.

Governing differential equation

$$y'' + K^2 y = 0 \quad ; \quad K^2 = P/EI \tag{19}$$

Canonical equations and analytical solution of equation (19)

$$\frac{dY}{dX} = Q(X, \lambda, Y) \quad , \quad 0 < X < L \tag{20}$$

$$Y = \{ y_1, y_2 \}^T$$

Where X and Y represents axial coordinate and deflection of rod respectively. Where $\lambda = P/D$ and bending moment is given by M_0.

$$Q = \{y_2, -\lambda^2 y_1\}^T \qquad y_1(0) = y_1(1) = 0 \tag{21}$$

$$\lambda = \frac{n\pi}{L} \tag{22}$$

The solution method by shooting method includes the following steps

From equation (20) – (21) the initial value problem for fixed-fixed ended is given as

$$\begin{cases} \frac{dY}{dX} = Q(X, \lambda, Y) \\ \mathrm{Y}(0) = \{ 0, \beta \}^T \end{cases} , \quad 0 < X < L \tag{23}$$

Here the value for the missing initial condition is given byβ. As a result, for a definite value of β and λ, denote the solution of equation (10) as $y(X, \lambda, \beta)$ such that the value of β and λ satisfy the following algebraic equation

$$(L;\lambda, \beta) = 0 \tag{24}$$

Both above equation (19) and equation (20) are needed to be solved simultaneously in order to find such condition that guessed value of missing initial condition and eigenvalue satisfy the all boundary conditions.

The solution method by CFM includes the following steps

On following the numerical scheme mentioned under the heading of formulation of CFM, a single variable algebraic equation is generated for hinged-hinged column case.

$$A(\lambda).C = 0 \tag{25}$$

$$A(\lambda) = \{1 \;\; 0, \; a_1(1)a_2(1)\}^T, \quad C = \{c_1, c_2\}^T$$

For a non trivial solution $C \neq 0$, hence

$$Det|A(\lambda)| = 0$$

Both above equation (20) and equation (21) are needed to be solved simultaneously in order to find the correct Eigenvalue.

3.3.3 Buckling of the Fixed - Hinged Euler Column

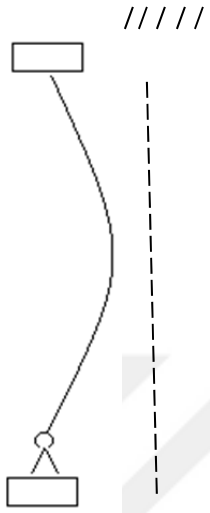

Figure 3.6. Buckling of the Fixed – Hinged Euler column

Consider an axially loaded slender rod as an Euler column of length "L" with flexural rigidity of "D=EI" under the compressive force "P" (Figure 3.6) Differential boundary equations and analytical expression of eigenvalue for the buckling of Euler Columns under Fixed - Hinged boundary condition is given below.

Governing differential equation

$$y'' + K^2 y = M_0 x / EIL \; ; \quad K^2 = P/EI \tag{26}$$

Canonical equations and analytical solution of equation (26)

$$\frac{dY}{dX} = Q(X, \lambda, Y) \quad , \qquad 0 < X < L \tag{27}$$

$$Y = \{ y_1, y_2 \}^T$$

Where X and Y represents axial coordinate and deflection of rod respectively. Where $\lambda = P/D$ and bending moment is given by M_0.

$$Q = \{y_2, -\lambda^2 y_1\}^T \qquad y_1(0) = y_1(1) = 0 \tag{28}$$

$$\lambda = \frac{4.49n}{L} \tag{29}$$

The solution method by shooting method includes the following steps

By taking missing initial conditions as a function of given boundary conditions; the boundary eigenvalue problem for hinged –fixed Euler column is converted into an initial value problems.

From equation (27) – (28) the initial value problem for hinged-fixed ended is given as

$$\begin{cases} \frac{dY}{dX} = Q(X, \lambda, Y) \\ Y(0) = \{0, \beta\}^T \end{cases}, \quad 0 < X < L$$

$$\begin{cases} \frac{dY}{dX} = Q(X, \lambda, Y) \\ Y(0) = \{0, \beta\}^T \end{cases}, \quad 0 < X < L \tag{30}$$

Here the value for the missing initial condition is given byβ. As a result, for a definite value of β and λ, denote the solution of equation (10) as $y(X, \lambda, \beta)$ such that the value of β and λ satisfy the following algebraic equation

$$(L;\lambda, \beta) = 0 \tag{31}$$

Both above equation (25) and equation (26) are needed to be solved simultaneously in order to find such condition that guessed value of missing initial condition and eigenvalue satisfy the all boundary conditions.

The solution method by CFM includes the following steps

On following the numerical scheme mentioned under the heading of formulation of Complementary functions method, a single variable algebraic equation is generated for fixed- hinged column case.

$$A\ (\lambda).C\ =\ 0 \tag{32}$$

$$\mathrm{A}(\lambda) = \{1\ 0\ 0, \mathrm{a}_1^1(1)\ \mathrm{a}_2^1(1)\ \ 1\ ,\ \ \mathrm{a}_1(1)\mathrm{a}_1(1)\ \ 1\ \}^T,$$

$$C = \{\ \mathrm{c}_1, \mathrm{c}_2, \mathrm{c}_3\}^T$$

For a non trivial solution C ≠ 0, hence

$$Det|A(\lambda)| = 0$$

Both above equation (31) and equation (32) are needed to be solved simultaneously in order to find the correct eigenvalue.

3.3.4 Thermal Buckling of a circular disc with Constant Temperature Distribution

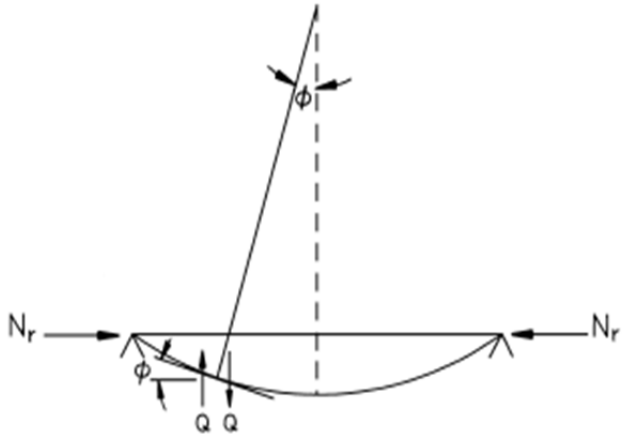

Figure 3.7. Thermal buckling of a circular disc with constant temperature distribution (Timoshenko et. al., (1969))

Governing differential equation

$$r^2 \frac{d^2\phi}{dr^2} + r \frac{d\phi}{dr} - \phi = -\frac{Qr^2}{D} \quad (33)$$

Where,

$$D = \frac{Et^3}{12(1-v^2)}$$

E = Modulus of elasticity

Q = Shear

Analytical solution

Consider a thin circular disk of radius r, restrained or fixed along the outer edge is subjected to a radial temperature variation, $\theta(r) = T(r) - T_0$ with T_0 being the

reference temperature (Figure 3.7). The differential equation for the bending of circular disc under thermal radial load N_t

$$\begin{cases} \frac{d\phi}{dr} = F(r, \beta, \theta(r)) \quad , \quad 0 \le r \le a \\ \phi = \{\phi_1, \phi_2\}^T \\ F(\phi, r, \theta(r)) = \{\phi_2, (-r\phi_2 + \phi_1 - \frac{Q\,r^2}{D})/\,r^2\}^T \\ \int \phi_1(a)\,dr = \phi_1(a) = 0; \phi_1(0) = 0 \end{cases} \tag{34}$$

ϕ, Angle exists between the axis of revolution and any normal to the plate. In the case of only in-plane radial forces, shearing force per unit length Q will be given by

$$\begin{cases} Q = N_t.\phi \\ D = \frac{Et^3}{12(1\text{-}v^2)} \end{cases} \tag{35}$$

Where D represents the flexural rigidity of a disc with Elastic modulus E, Poisson ratio$v$, and thickness t. Thermal stresses and displacement components are functions of the radius, and assuming the plane stress condition, the equilibrium equation is

$$d\sigma rr/dr + (\sigma rr - \sigma\varphi\varphi)/r = 0 \tag{36}$$

Strain-displacement relations are

$$\{\varepsilon_x, \varepsilon_y\}^T = \{u', u/r\}^T \tag{37}$$

Stress –strain relationship in polar coordinates for the plane stress condition

$$\begin{cases} \sigma rr = \frac{E}{1-\nu^2}[\varepsilon rr + \nu\varepsilon\varphi\varphi - (1+\nu)\alpha\theta] \\ \sigma\varphi\varphi = \frac{E}{1-\nu^2}[\varepsilon\varphi\varphi + \nu\varepsilon rr - (1+\nu)\alpha\theta] \end{cases} \quad (38)$$

Upon the introduction of equation (37) Into equation (38) and then into equation (36), the equilibrium equation in terms of displacements becomes

$$\frac{d}{dr}\left[\frac{1}{u}\frac{d(ur)}{dr}\right] = (1+\nu)\alpha\frac{d\theta}{dr} \quad (39)$$

Integrating this equation, the radial displacement for a disk of radius **r** becomes

$$\mathrm{u} = (1+\nu)\frac{\alpha}{\mathrm{r}}\int_0^{\mathrm{r}}\theta r dr + C_1 r + \frac{C_2}{\mathrm{r}} \quad (40)$$

Since the displacement must be finite at r = 0, it follows that C_2 must be zero and C_1 can be found from boundary condition that u(r) = 0, equation (3), (4) & (5) gives radial stress and thermal load

$$\begin{aligned} \sigma rr &= \frac{E\alpha\theta}{-1+\upsilon} \\ \mathrm{Nr} &= \frac{E\alpha\theta h}{-1+\upsilon} \end{aligned} \quad (41)$$

The solution method of shooting method includes the following steps

Shooting method devised the initial value form of a given boundary value problem by substituting a multivariable function for the missing initial conditions. Later, iteration scheme is applied on a function of missing initial condition such that it satisfies given boundary conditions. The respective system of iterative algebraic equations, for all the three cases under the shooting method is given below

$$\begin{cases} \frac{d\boldsymbol{\phi}}{dr} = F(r,\beta,\boldsymbol{\theta}) \ , \ 0 \le r \le a \\ \qquad \boldsymbol{\phi}(0) = \{0,\beta\}^{\mathrm{T}} \end{cases} \tag{42}$$

The guessed value of the missing initial condition is represented by β. Particular value of β andθ, denote equation (10) solution as $\phi(r,\beta,\theta)$ such that the value of β and θ satisfy the following algebraic equation

$$(a;\theta,\beta) = 0 \tag{43}$$

The solution method of CFM includes the following steps

CFM make use of the linear ordinary differential equation properties to compose the initial value form of a given boundary value problem. An iterative single variable algebraic equation is solved directly by suitable root finding method. From equation (1) and (2) the solution of BEVP is given by

$$\begin{cases} f(x,\lambda) = \sum_{i=1}^{m}\{\mathrm{a_m(x,\lambda).c_m}\} + \mathrm{b(x)} \qquad \mathrm{g} < x < h \\ \mathrm{A}\,(\lambda).\mathrm{C} = 0 \ ; \quad \mathrm{C} \neq 0 \quad \Rightarrow \quad \mathrm{Det|A}(\lambda)| = 0 \end{cases} \tag{44}$$

From equation (34), (35), (41) and (44)

$$\begin{cases} \mathrm{A}\,(\lambda).\mathrm{C} = 0 \\ \mathrm{A}\,(\lambda) = \{1 \ \ 0, \ \mathrm{a_1(1)a_2(1)}\}^{T}, \quad C = \{\mathrm{c_1,c_2}\}^{T} \end{cases} \tag{45}$$

3.3.5 Thermal Buckling of a circular disc with Radial Temperature Distribution

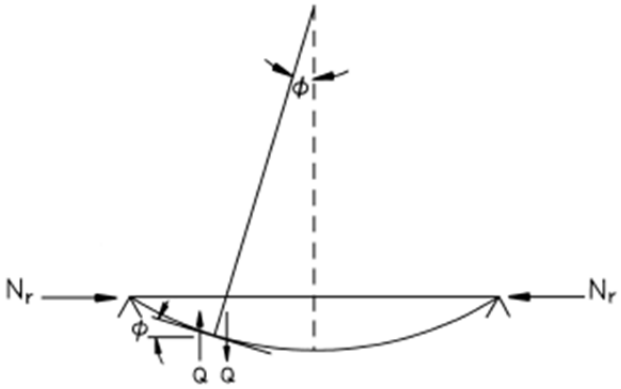

Figure 3.8. Thermal buckling of a circular disc with radial temperature distribution (Timoshenko et. al., (1969))

Governing differential equation

$$r^2 \frac{d^2\phi}{dr^2} + r \frac{d\phi}{dr} - \phi = -\frac{Qr^2}{D} \quad (46)$$

Where,

$$D = \frac{Et^3}{12(1-v^2)}$$

E = Modulus of elasticity

Q = Shear

Analytical Solution

Consider a thin circular disk of radius r, restrained or fixed along the outer edge is subjected to a radial temperature variation, $\theta(r) = T(r) - T_0$with T_0 being the reference temperature. (Figure 3.7).The differential equation for the bending of circular disc under thermal radial load N_t from (Timoshenko, 2009)

$$\begin{aligned} \frac{d\phi}{dr} &= F(r,\beta,\theta(r)) \quad , \quad 0 \le r \le a \\ \phi &= \{\phi_1, \phi_2\}^T \\ F(\phi, r, \theta(r)) &= \{\phi_2, (-r\phi_2 + \phi_1 - \frac{Q\,r^2}{D})/\,r^2\}^T \\ \int \phi_1(a)\,dr &= \phi_1(a) = 0; \phi_1(0) = 0 \end{aligned} \tag{47}$$

ϕ, Angle exists between the axis of revolution and any normal to the plate. In the case of only in-plane radial forces, shearing force per unit length Q will be given by

$$\begin{aligned} Q &= N_t.\phi \\ D &= \frac{Et^3}{12(1-\nu^2)} \end{aligned} \tag{48}$$

Where D represents the flexural rigidity of a disc with Elastic modulus E, Poisson ratio ν, and thickness t. Thermal stresses and displacement components are functions of the radius, and assuming the plane stress condition, the equilibrium equation is

$$d\sigma rr/dr + (\sigma rr - \sigma\varphi\varphi)/r = 0 \tag{49}$$

Strain-displacement relations are

$$\{\varepsilon_x, \varepsilon_y\}^T = \{u', u/r\}^T \tag{50}$$

Stress –strain relationship in polar coordinates for the plane stress condition

$$\begin{aligned} \sigma rr &= \frac{E}{1-\nu^2}[\varepsilon rr + \nu\varepsilon\varphi\varphi - (1+\nu)\alpha\theta] \\ \sigma\varphi\varphi &= \frac{E}{1-\nu^2}[\varepsilon\varphi\varphi + \nu\varepsilon rr - (1+\nu)\alpha\theta] \end{aligned} \tag{51}$$

Upon the introduction of equation (29) into equation (30) and then into equation (1), the equilibrium equation in terms of displacements becomes

$$\frac{d}{dr}\left[\frac{1}{u}\frac{d(ur)}{dr}\right] = (1+\nu)\alpha\frac{d\theta}{dr} \tag{52}$$

Integrating this equation, the radial displacement for a disk of radius **r** becomes

$$\mathrm{u} = (1+\nu)\frac{\alpha}{\mathrm{r}}\int_0^{\mathrm{r}}\theta r dr + C_1 r + \frac{C_2}{\mathrm{r}} \tag{53}$$

Since the displacement must be finite at r = 0, it follows that C_2 must be zero and C_1 can be found from boundary condition that u(r) = 0, equation (3), (4) & (5) gives radial stress and thermal load

$$\theta\,(r) = b.r \tag{54}$$

$$\begin{cases} \sigma rr(r) = \frac{2bEr\alpha}{3(-1+\upsilon)} \\ \mathrm{Nr(r)} = \frac{2bEr\alpha h}{3(-1+\upsilon)} \end{cases} \tag{55}$$

The solution method of shooting method includes the following steps

From equation (45) and (46), the initial value form under shooting method for above problem is give by

$$\begin{cases} \frac{d\phi}{dr} = F(r,\beta,\theta(r)) \;,\; 0 \le r \le a \\ \phi(0) = \{0,\beta\}^{\mathrm{T}} \end{cases} \tag{56}$$

The guessed value of the missing initial condition is represented by β. Particular value of β andθ, denote equation (10) solution as $\phi(r,\beta,\theta)$ such that the value of β and θ satisfy the following algebraic equation

$$(a;\theta,\beta) = 0 \tag{57}$$

Complementary Functions Method

From equation (1) and (2) the solution of BEVP is given by

$$\begin{cases} f(x,\lambda) = \sum_{i=1}^{m}\{\mathrm{a_m(x,\lambda).c_m}\} + \mathrm{b(x)} \quad g < x < h \\ \mathrm{A}(\lambda).\mathrm{C} = 0\ ; \quad \mathrm{C} \neq 0 \quad \Rightarrow \quad \mathrm{Det}|\mathrm{A}(\lambda)| = 0 \end{cases} \tag{58}$$

From equation (48), (49), (57) and (58)

$$\begin{cases} \mathrm{A}(\lambda).\mathrm{C} = 0 \\ \mathrm{A}(\lambda) = \{1 \ \ 0,\ \mathrm{a_1(1)a_2(1)}\}^T, \quad C = \{\mathrm{c_1,c_2}\}^T \end{cases} \tag{59}$$

3.3.6 Thermal Buckling of a FGM circular disc with Constant Temperature Distribution

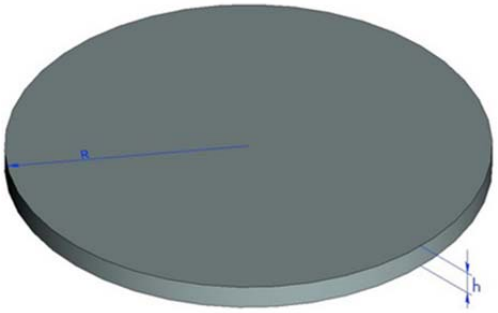

Figure 3.9. Thermal Buckling of a functionally graded disc

Governing differential equation

$$r^2 \frac{d^2\phi}{dr^2} + r \frac{d\phi}{dr} - \phi = -\frac{Qr^2}{D} \quad (60)$$

Where,

$$D = \frac{Et^3}{12(1-v^2)}$$

E = Modulus of elasticity

Q = Shear

Consider a thin functionally graded circular disk in of radius r, restrained or fixed along the outer edge. The material is assumed to be functionally graded in the radial direction. Variations in the material properties such as Young's modulus and Poisson's ratio may be arbitrary functions of the radial coordinate is subjected to a radial temperature variation, $\theta(r) = T(r) - T_0$ with T_0 being the reference temperature (Figure 3.8). The differential equation for the bending of circular disc under thermal radial load N_t from (Timoshenko et. al., (1969))

$$\begin{gathered} \frac{d\phi}{dr} = F(r, \beta, \theta(r)) \quad , \quad 0 \le r \le a \\ \phi = \{\phi_1, \phi_2\}^T \\ F(\phi, r, \theta(r)) = \{\phi_2, (-r\phi_2 + \phi_1 - \frac{Q r^2}{D})/ r^2\}^T \\ \int \phi_1(a)\, dr = \phi_1(a) = 0; \phi_1(0) = 0 \end{gathered} \quad (61)$$

In material properties of functionally graded disc, consider elastic modulus E as an algebraic function of radius r. From the literature review, following function is considered foe elastic modulus E

$$E(r) = b.r^2 \quad (62)$$

The solution by shooting method includes the following steps

From equation (61) and (62), the initial value form under shooting method for above problem is give by

$$\begin{aligned} \frac{d\phi}{dr} &= F(r,\beta,\theta) \ , \ 0 \le r \le a \\ \phi(0) &= \{0, \beta\}^{\mathrm{T}} \end{aligned} \tag{63}$$

The guessed value of the missing initial condition is represented by β. Particular value of β andθ, denote equation (63) solution as $\phi(r,\beta,\theta)$ such that the value of β and θ satisfy the following algebraic equation

$$(a;\theta,\beta) = 0 \tag{64}$$

The solution by CFM includes the following steps

From equation (1) and (2) the solution of BEVP is given by

$$\begin{aligned} f(x,\lambda) &= \sum_{i=1}^{m}\{\mathrm{a_m}(\mathrm{x},\lambda).\mathrm{c_m}\} + \mathrm{b(x)} \qquad \mathrm{g} < x < h \\ \mathrm{A}\,(\lambda).\mathrm{C} &= 0 \ ; \quad \mathrm{C} \neq 0 \quad \Rightarrow \quad \mathrm{Det}|\mathrm{A}(\lambda)| = 0 \end{aligned} \tag{65}$$

From equation (61), (62) and (65)

$$\begin{cases} \mathrm{A}\,(\lambda).\mathrm{C} = 0 \\ \mathrm{A}\,(\lambda) = \{1 \ \ 0, \ \mathrm{a_1(1)a_2(1)}\}^{T}, \quad C = \{\mathrm{c_1}, \mathrm{c_2}\}^{T} \end{cases} \tag{66}$$

3.4. Non-homogeneous boundary eigenvalue problems

BEVP are termed as non-homogenous, in case given boundary conditions are not equal to zero. This differentiation is important in comparative analysis of these two numerical methods and produced different results. Different cases of

non-homogeneous boundary Eigenvalue Problems are solved by using both numerical methods and formation of all these problems step by step is given below.

3.4.1 Buckling of the Fixed- Free Euler Column

Figure 3.10. Buckling of the Fixed - Hinged Euler column (Chajes, (1974))

Consider an axially loaded slender rod as an Euler Column of length "L" with flexural rigidity of "D=EI" under the compressive force "P" (Figure 3.9). Differential boundary equations and analytical expression of eigenvalue for the buckling of Euler Columns under Fixed - Free boundary condition is given below.

Governing differential equation

$$y'' + K^2 y' = 0 \quad ; \quad K^2 = P/EI \tag{67}$$

Canonical equations and analytical solution of equation (67)

$$\frac{dY}{dX} = Q(X, \lambda, Y) \quad , \qquad 0 < X < L \tag{68}$$
$$Y = \{ y_1, y_2 \}T$$

Where X and Y represents axial coordinate and deflection of rod respectively. Where $\lambda = P/D$ and bending moment is given by M_0.

$$Q = \{y_2, -\lambda^2 (y_1 + \delta)\}^{\mathrm{T}} \quad y_1(0) = y_1(1) = 0 \, ; y_1(1) = \delta \tag{69}$$

$$\lambda = \frac{n\pi}{2L} \tag{70}$$

The solution by shooting Method includes the following steps

From equation (68) – (69) the initial value problem for fixed-free ended is given as

$$\begin{cases} \frac{dY}{dX} = Q(X, \lambda, Y) \\ \mathrm{Y}(0) = \{ 0, \beta \}^{\mathrm{T}} \end{cases} , \quad 0 < X < 1 \tag{71}$$

$$(1; \lambda, \beta) - \delta = 0 \tag{72}$$

Here the value for the missing initial condition is given by β. As a result, for a definite value of β and λ, denote the solution of equation (10) as $\mathrm{y}(X, \lambda, \beta)$ such that the value of β and λ satisfy the following algebraic equation

$$(\mathrm{L}; \lambda, \beta) = 0 \tag{73}$$

The solution by CFM includes the following steps
From equation (1) and (2) the solution of BEVP is given by

$$\begin{cases} f(x,\lambda) = \sum_{i=1}^{m}\{\mathrm{a_m(x,\lambda).c_m}\} + \mathrm{b(x)} & \mathrm{g} < x < h \\ \mathrm{A}\,(\lambda).\mathrm{C} = 0\ ; \quad \mathrm{C} \neq 0 \quad \Rightarrow \quad \mathrm{Det|A(\lambda)|} = 0 \end{cases} \tag{74}$$

On following the above mentioned numerical scheme, a set of single variable algebraic equations is generated for the fixed-free Column case.

$$A\,(\lambda).C = S \tag{75}$$

$\mathrm{A}\,(\lambda) = \{1 \;\; 0 \;\; 1, \; 0 \;\; 1 \;\; 0, \; \mathrm{a_1(1)a_2(1)} \;\; 1\}^T$,

$C = \{\mathrm{c_1, c_2, c_3}\}^T$, $S = \{0,0,\delta\}^T$

3.4.2 Fixed- Free Euler Column buckling with load through a fixed point

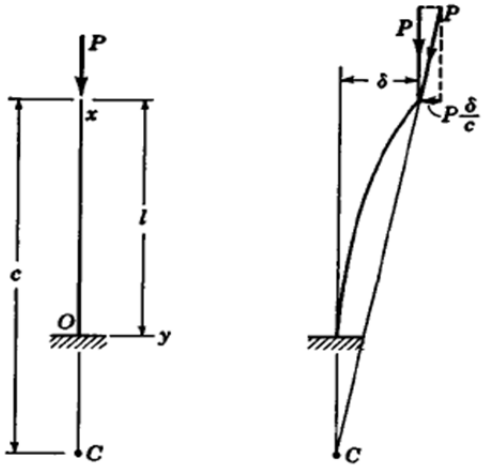

Figure 3.11. Buckling of the Fixed - Free Euler column with a load through a fixed point (Timoshenko et. al.,1969)

Consider a fixed-free Euler Column of length "l", flexural rigidity of "D=EI" and Eigenvalue $\lambda = P/D$ under the compressive force "P". Force "P" passes through a particular point C, causes deflection "δ" (Figure 3.9). the

differential boundary equation and analytical expression of the eigenvalue for the fixed-free column with load through a fixed point.

Governing differential equation

$$y''' + K^2 y' = K^2 \delta/c \quad ; \qquad K^2 = P/EI \tag{76}$$

$$\begin{cases} \frac{dY}{dX} = Q(X,\lambda,Y) \ , \quad 0 < X < l \\ \mathrm{Y} = \{\mathrm{y}_1, \mathrm{y}_2, \mathrm{y}_3\}^{\mathrm{T}} \end{cases} \tag{77}$$

$$\begin{cases} \mathrm{Q}(X,\lambda,Y) = \{\mathrm{y}_2, \mathrm{y}_3, -\lambda^2 (\mathrm{y}_2 + \delta/\mathrm{l})\}^{\mathrm{T}} \\ \lambda = (\tan^{-1} kL(1 - \frac{c}{l})/l \\ \mathrm{y}_1(0) = \mathrm{y}_2(0) = 0; \quad \mathrm{y}_3(\mathrm{l}) = 0; \quad \mathrm{y}_1(\mathrm{l}) = \frac{-\mathrm{P}\delta}{\mathrm{c}} \end{cases} \tag{79}$$

The solution by shooting method includes the following steps

From equation (78) & (79), the initial value problem for fixed-free Euler column through a fixed point is given as

$$\begin{cases} \frac{dY}{dX} = Q(X,\lambda,Y) \ , \quad 0 < X < l \\ \mathrm{Y} = \{\mathrm{y}_1, \mathrm{y}_2, \mathrm{y}_3\}^{\mathrm{T}} \end{cases} \tag{80}$$

$$(l; \lambda, \beta) - \delta = 0 \tag{81}$$

The solution by CFM includes the following steps

From equation (1) and (2) the solution of BEVP is given by

$$\text{A}(\lambda) \quad \text{C} = \text{S} \qquad (82)$$

$$\begin{matrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & 1 \end{matrix} \quad \begin{matrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{matrix} \quad = \quad \begin{matrix} 0 \\ 0 \\ 0 \\ \delta \end{matrix}$$

3.5 Formation of FEM solution for the thermal buckling of disc

In future prospective**,** another important comparative analysis can be run between Complementary functions method (CFM) and Finite element method (FEM) for the boundary eigenvalue problems. It can provide further clarification about the cost effectiveness of CFM in linear stability problems. Finite element form for the thermal buckling of disc is given below.

Basic differential form of the governing equation for the stability of disc under thermal buckling is given from (Timoshenko, 1985)

$$\frac{d^3y}{dr^3} * \text{r} + \frac{d^2y}{dr^2} - [\frac{1}{r} - \text{r}\,(\frac{N_t}{D})] * \frac{dy}{dr}$$

Where N_t = Critical Temperature

r = Radius of the disc

D = Flexural rigidity

y = vertical deflection in disc

The weak form of the above governing stability equation is given by

$$\int_{r1}^{r2} \{\frac{d^3y}{dr^3} [\left(\frac{d^2y}{dr^2} r\right) + \frac{d^2y}{dr^2} \left(\frac{1}{r} - \text{r}\,\frac{N_t}{D}\right) * \frac{dy}{dr}\ ] \}\ \text{dr}$$

4. RESULTS AND DISCUSSION

Calculations of numerical results is performed on an operating system with following specifications "Intel(r) Core(TM) i7-6500u CPU @ 2.50 GHz & 2.60 GHz with 8GB Ram". Same numerical integration method and similar values for the initial guesses of Eigenvalues are selected in order to develop the neutrality in the comparative analysis. "Runge Kutta Method of 4^{th} Order" (RK4) as a numerical integrator and "Secant Method" as a root finding method are employed for both numerical methods. In all three cases, **0** is substituted as initial guessed value for the eigenvalue. Generally shooting method is used with an additional method while dealing with boundary eigenvalue problem. The role of additional method is to find interval where the probability of finding eigenvalue is higher. Under this comparative analysis, both numerical methods are applied without any additional technique. The objective behind above mentioned statement is to make sure that standards for the comparison are equal among both methods.

Availability of limited computational processing power makes us choose different levels of accuracy and precision for both methods. Higher level of accuracy $\mathbf{10^{-6}}$and precision of **5000** steps is considered for Complementary functions method due to its high efficiency. On the other hand, low level of accuracy $\mathbf{10^{-3}}$and precision of 100 steps is considered for shooting method because of its low efficiency even under low standards, shooting method takes hours for finding the required results. In comparison, high level standards are selected for complementary functions method owing to its high efficiency.

All the basic standards which are undertaken while performing this comparative analysis are shown below (Table 4.1).

Table 4.1. Comparative Analysis Standards for both methods

Comparative Analysis Standards	Shooting Method	Complementary Functions Method
Numerical Integration Method	Runge Kutta Of 4^{th} Order	Runge Kutta Of 4^{th} Order
Root Finding Method	Secant Method	Secant Method
Initial Guessed Value	0	0
Accuracy	10^{-3}	10^{-6}
Number of steps	100	5000

4.1. Results of Homogeneous boundary Eigenvalue Problems

Boundary Eigenvalue Problems are termed as homogenous, in case given boundary conditions are equal to zero. This differentiation is important in comparative analysis of these two numerical methods and produced different results. Different cases of homogeneous BEVP are solved by using both numerical methods and results are given below.

Comparative analysis for all the problems is performed under above given standards. It can be clearly observed that complementary functions method is way more efficient than shooting method. Even under high level of comparative standards, the processing time taken by complementary functions method is extremely less than shooting method.

4.1.1 Results for the buckling of the Fixed- Fixed Euler Column

Computational cost in terms of the processing time is calculated by using both numerical methods. Under same initial guesses and tolerance values, iterations are carried out for finding first order (n=1) eigenvalue.

For simplicity, Consider L = D = M = 1. Accuracy of numerical results is given with respect to analytical results. Processing time for CFM is calculated for

10^{-6} accuracy and for Shooting Method it's restricted to 10^{-3} due to high number of iterations.

Values of all above mentioned physical constants are considered in SI units. Verification of numerically calculated Eigenvalues is done by comparison with analytical solutions.

Table 4.2. Numerical results for the Fixed-Fixed Euler Column

	Shooting Method	CFM
Step Size	100	5000
No of Iterations	$7x10^4$	8
Accuracy	10^{-3}	10^{-6}
Processing Time	35 min	0.2 Sec

A gigantic difference is clearly visible among the computational cost of both methods, tested under similar configurations (Table 4.2). The main reason giving the CFM, a stark edge over the shooting method is related with the number of variables. Shooting method is dealing with an algebraic equation containing two variables while CFM reduced it to a single variable algebraic equation. As a result for shooting method, the direct application of root finding method is out of option and left with the vast range of trail values for an eigenvalue. In normal practice, shooting method is not subjected directly to such a generalized interval for λ, instead of it, successive attempts with lower accuracy sort out the traces of eigenvalues and it followed by high accuracy localized intervals around those eigenvalue traces. In order to balance the comparison field among both numerical methods, regardless of the conventional practices, no additional method or technique is used in both methods.

Verification of numerically calculated Eigenvalues is done by comparison with analytical solutions. It's one of the reasons for selecting such kinds of BEVP which have existing analytical solution.

4.1.2 Results for the buckling of the Hinged- Hinged Euler Column

For simplicity, Consider L = D = 1. Accuracy of numerical results is given with respect to analytical results. Processing time for CFM is calculated for 10^{-6} accuracy and for Shooting Method it's restricted to 10^{-3} due to high number of iterations.

Values of all above mentioned physical constants are considered in SI units. Verification of numerically calculated Eigenvalues is done by comparison with analytical solutions (Table 4.3)

Table 4.3. Numerical results for the Hinged-Hinged Euler Column

	Shooting Method	CFM
Step Size	100	5000
No of Iterations	$3x10^4$	6
Accuracy	10^{-3}	10^{-6}
Processing Time	28 min	0.1 Sec

Solving first two cases Hinged-Hinged and Fixed-Fixed by shooting method gives another challenge in selection of known initial conditions. In cases of giving common initial conditions for both cases from equation (10)

$$Y(0) = \{0, \beta\}^T$$

Solution shape for F-F doesn't satisfy boundary conditions $y_2(0) = y_2(1) = 0$. Unless, a set of uncommon initial conditions is consider for F-F from equation (11)

$$Y(0) = \{\beta, 0\}^T$$

It signifies another drawback of using shooting method for stability problems. It is not only difficult to find missing initial conditions but also the task of choosing correct initial conditions from the given boundary conditions.

4.1.3 Results for the buckling of the Fixed - Hinged Euler Column

Computational cost in terms of processing time is calculated by using both numerical methods. Same initial guesses and tolerance values, iterations are carried out for finding first order (n=1) eigenvalue.

For simplicity, Consider L = D = M = 1. Accuracy of numerical results is given with respect to analytical results. Processing time for CFM is calculated for 10^{-6} accuracy and for Shooting Method it's restricted to 10^{-3} due to high number of iterations.

Values of all above mentioned physical constants are considered in SI units. Verification of numerically calculated Eigenvalues is done by comparison with analytical solutions.

Shooting method gives another challenge in selection of known initial conditions. In cases of giving common initial conditions for both cases

Table 4.4. Numerical results for the Fixed-Hinged Euler Column

	Shooting Method	CFM
Step Size	100	5000
No of Iterations	$5x10^4$	6
Accuracy	10^{-3}	10^{-6}
Processing Time	29 min	0.2 Sec

(Table 4.4), depicts that shooting method needed a massive number of iterations along with huge processing time to find the missing initial conditions and eigenvalue with low accuracy of 10^{-3}. Conversely, CFM requires fewer numbers of iterations and insignificant amount of processing time for calculating eigenvalue with double accuracy of 10^{-6}.

In all the above cases of Euler columns a clear difference can be seen among the performance of both methods. Complementary functions method proved to be far more efficient in handling boundary eigenvalue problems as compare to the shooting method. It gives more accurate results with very low computational cost as compared to the shooting method.

4.1.4 Results for the Thermal buckling of a circular disc with Constant Temperature distribution

In order to establish a neutral medium for comparison 0 is substituted as initial value for eigenvalue. Iterations are carried out for finding first order (n=1) eigenvalue with the accuracy of 10^{-6} under step size of 5000.

Consider a = 1, E = $2*10^9$, $\nu = 0.3$ and t = 0.01. Values of all above mentioned physical constants are considered in SI units. Verification of numerically calculated Eigenvalues is done by comparison with analytical solutions (Table 4.5)

Table 4.5. Numerical results of a circular disc with Constant temperature Distribution

	Shooting Method	CFM
Step Size	100	5000
No of Iterations	$3x10^4$	6
Accuracy	10^{-3}	10^{-6}
Processing Time	90 min	32 Sec

In the case the value of the eigenvalue lies away from the input of 0 and as a result no of iterations and processing time increases for shooting method. On the hand it has no effect on the processing time of complementary functions method.

4.1.5 Results for the Thermal buckling of a circular disc with radial Temperature distribution

In order to establish a neutral medium for comparison 0 is substituted as initial value for eigenvalue. Iterations are carried out for finding first order (n=1) eigenvalue with the accuracy of 10^{-6} under step size of 5000.

Consider a = 1, E = $2*10^9$, $\nu = 0.3$ and t = 0.01. Values of all above mentioned physical constants are considered in SI units. Verification of numerically calculated Eigenvalues is done by comparison with analytical at different values of radius; r = 0.01, r = 0.05 and r = 0.1 in (Table 4.6), (Table 4.7) and (Table 4.8) respectively

Table 4.6. Numerical results of a circular disc with radial temperature distribution (for radius of disc = r = 0.01)

	Shooting Method	CFM
Step Size	100	5000
No of Iterations	$3x10^{4}$	6
Accuracy	10^{-3}	10^{-6}
Processing Time	30 min	32 Sec

In all the case when the value of the eigenvalue lies away from the input of 0, no of iterations and processing time increases for shooting method. On the hand it has no effect on the processing time of CFM.

Table 4.7. Numerical results of a circular disc with radial temperature distribution (for radius of disc = r = 0.05)

	Shooting Method	CFM
Step Size	100	5000
No of Iterations	$3x10^{4}$	6
Accuracy	10^{-3}	10^{-6}
Processing Time	60	32 Sec

Table 4.8. Numerical results of a circular disc with radial temperature distribution (for radius of disc = r = 0.1)

	Shooting Method	CFM
Step Size	100	5000
No of Iterations	$3x10^4$	6
Accuracy	10^{-3}	10^{-6}
Processing Time	92 min	32 Sec

4.1.6 Results for the functionally graded circular disc

A functionally graded disc is considered in which elastic modulus is varying along the radius of the disc.

$$E(r) = a.r^2$$

In order to establish a neutral medium for the comparison 0 is substituted as initial value for eigenvalue. Iterations are carried out for finding first order (n=1) eigenvalue with the accuracy of 10^{-6} under step size of 5000.

Consider a = 1, E = $2*10^9$, $\nu = 0.3$ and t = 0.01. Values of all above mentioned physical constants are considered in SI units. Verification of numerically calculated Eigenvalues is done by comparison with analytical solutions (Table 4.9)

Table 4.9. Numerical results of Functionally graded disc

	Shooting Method	CFM
Step Size	100	5000
No of Iterations	$3x10^4$	6
Accuracy	10^{-3}	10^{-6}
Processing Time	110 min	32 Sec

4.2. Results of Non-homogeneous boundary eigenvalue problems

Boundary Eigenvalue Problems are termed as non-homogenous, in case given boundary conditions are not equal to zero. This differentiation is important in comparative analysis of these two numerical methods and produced different results. Different cases of non-homogeneous boundary Eigenvalue Problems are solved by using both numerical methods and results are given below.

Comparative analysis for all the problems is performed under above given standards. It can be clearly observed that complementary functions method is way more efficient than shooting method. Even under high level of comparative standards the processing time taken by complementary functions method is extremely less than shooting method.

4.2.1. Results for the Fixed-Free Euler Column

In fixed-free case since given boundary conditions are not homogeneous, from equation (38)

$$y_1(0) = y_1(1) = 0\,;\; y_1(1) = \delta$$

As a result, here CFM gives a non-homogenous system of algebraic equations, from equation (38)

$$A(\lambda).C = \delta$$

For simplicity, Consider L = δ = M = 1. In order to solve the above non-homogenous system of algebraic equations, directly root finding method cannot be employed. Hence, equation (38) is solved iteratively for the successive values of λ until both sides become equal and this process in literature is called as analytical expansion (Table 4.10). It leads to increase in the number of iterations and processing time for CFM but still its performance is better than shooting method.

Table 4.10. Numerical results of fixed-free Euler Column

	Shooting Method	CFM
Step Size	100	5000
No of Iterations	$2x10^4$	$2x10^3$
Accuracy	10^{-3}	10^{-6}
Processing Time	29 min	11 min

A massive difference is clearly visible among the computational cost of both methods, tested under similar configurations. The main reason giving the CFM, a stark edge over the shooting method is related with the number of variables

4.2.2. Results for the Fixed-Free Euler Column with load through a fixed point

In fixed-free case since given boundary conditions are not homogeneous, from equation (38)

$$y_1(0) = y_2(0) = 0\,;\; y_3(1) = 0\,;\; y_1(1) = \delta$$

As a result, here CFM gives a non-homogenous system of algebraic equations, from equation (38)

$$A(\lambda).C = \delta$$

For simplicity, Consider L = δ = M = 1. In order to solve the above non-homogenous system of algebraic equations, directly root finding method cannot be employed. Hence, equation (38) is solved iteratively for the successive values of λ until both sides become equal and this process in literature is called as analytical expansion (Table 4.11). It leads to increase in the number of iterations and processing time for CFM but still its performance is better than shooting method.

Table 4.11. Numerical results of fixed-free Euler Column with load through a fixed point

	Shooting Method	CFM
Step Size	100	5000
No of Iterations	$2x10^4$	$2x10^3$
Accuracy	10^{-3}	10^{-6}
Processing Time	75 min	30 min

5. CONCLUSION

A comparative investigation is carried out based upon the results accuracy and cost effectiveness, between the two numerical methods namely, Shooting Method and Complementary Function Method respectively, for finding the eigenvalues in different cases of linear elastic stability. Although SM is an effective direct approach for solving Ordinary boundary value problems, but in case of eigenvalue problems due to the introduction of another variable "Eigenvalue" (λ), causes the significant increase in its number of iterations and processing time. On the other hand, CFM solves the same problem with double accuracy as of shooting method under fewer numbers of iterations and insignificant processing time.

Most of the problems considered under this study from linear stability cases can be solve analytically too. The reason behind considering such problems is to prove the validity of the CFM. It makes use of the linear ordinary differential equation properties to compose the initial value form of a given boundary value problem. An iterative single variable algebraic equation is solved directly by suitable root finding method. The main reason giving the CFM, a stark edge over the shooting method is related with the number of variables. Shooting method is dealing with an algebraic equation containing two variables while CFM reduced it to a single variable algebraic equation. As a result for shooting method, the direct application of root finding method is out of option and left with the vast range of trail values for an eigenvalue. In current study, it is shown that by considering methods on the basis of problem to problem and designing hybrid algorithm, computationally expensive problems can be solve very efficiently.

In future prospects, the role of complementary functions method can be extend to solve non linear boundary eigenvalue problems. On the basis of its properties, CFM can make a computationally cumbersome problem very easy to

solve. A comparison of CFM method with finite element method for solving boundary value problems can be another way to test the efficiency of CFM..

In this thesis, the notion of using generalized methods like shooting SM or FEM is came under question. It leads us towards considering an algorithm which is dependent on more than one numerical method.

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CURRICULUM VITAE

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