ÇUKUROVA UNIVERSITY
INSTITUTE OF NATURAL AND APPLIED SCIENCES

MSc THESIS

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POISSON ALGEBRAS AND THEIR IDEALS

DEPARTMENT OF MATHEMATICS

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ABSTRACT

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A commutative associative algebra P becomes a Poisson algebra if it is a Lie algebra with the Lie bracket $\{,\}$ and if it satisfies the Leibniz rule,

$$\{p_1.p_2, p_3\} = p_1.\{p_2, p_3\} + \{p_1, p_3\}.p_2, \text{ for all } \ p_1, p_2, p_3 \in P.$$

In this thesis, Poisson algebras, Poisson modules, Poisson derivations, Poisson ideals are investigated and some theorems, illustrative propositions and examples about them are given.

Key words: Poisson algebras, Lie algebras, ideals, modules, derivations.

ÖZ

YÜKSEK LİSANS TEZİ

POISSON CEBİRLERİ VE İDEALLERİ

İbrahim BAKARİ

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P değişmeli ve birleşmeli bir cebir olsun. Eğer P, $\{,\}$ braket çarpımı ile bir Lie cebiri ve her $p_1, p_2, p_3 \in P$ için

$$\{p_1.p_2, p_3\} = p_1.\{p_2, p_3\} + \{p_1, p_3\}.p_2$$

Leibniz eşitliği sağlanıyorsa P ye bir Poisson cebiri denir. Bu tezde, Poisson cebirleri, Poisson modülleri, Poisson derivasyonları ve Poisson idealleri incelenmiştir ve bunlarla ilgili bazı teoremler ve örnekler verilmiştir.

Anahtar kelimler: Poisson cebiri, Lie cebiri, ideal, modül, derivasyon.

EXTENDED ABSTRACT

A commutative associative algebra P is a Poisson algebra if it is a Lie algebra with the Lie bracket $\{\}$ Leibniz identity

$$\{p_1.p_2, p_3\} = p_1.\{p_2, p_3\} + \{p_1, p_3\}.p_2$$

holds for all $p_1, p_2, p_3 \in P$.

Poisson algebras are one of the main topics of discussion in the twentieth century. Poisson algebras appear in the investigation of Hamiltonian mechanics and they are in the center of the study of quantum groups. In geometry, manifolds with Poisson structures are called Poisson manifolds. In physics Poisson algebra structure is a fundamental part of covariant canonical quantization. In mathematics they play an important role in Poisson geometry and deformation of commutative associative algebra.

There are some close similarities between free Poisson algebras and free Lie algebras as well with free associative algebras. Even though most of their structures are well known, free Poisson algebras are not expansively known and they are still being investigated.

The origin of Poisson algebra is mainly from the research of Siméone Denis Poisson in the 19th century when he was looking into the structure of celestial mechanics.

In this work, we showed how to construct a Poisson algebra from any arbitrary Lie algebra P. From our investigation we noticed that there are two main type of Poisson algebras : The symplectic algebra S_n,for each n , , S_n is a $K[p_1, ...p_n, q_1, ..., q_n]$ polynomial algebra with its Poisson bracket given as

$$\{p_i, q_j\} = \delta_{ij} \ , \{p_i, p_j\} = \{q_i, q_j\} = 0 \text{ and } \delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j \end{cases},$$

here, $1 \leqslant i, j \leqslant n$ and the algebra of Lie type, where for any Lie algebra P with linear basis given by $e_1, e_2, ..., e_n, ...$ and the symmetric algebra S_P of P given by the bracket defined by

$\{e_i, e_j\} = [e_i, e_j]$ for all i, j where [-,-] is the Lie algebra commutator of P is the Poisson algebra of type P.

With this in mind, Poisson algebra P can be constructed over a given field K as follows;

First, one should observed that as an algebra P is the same with the symmetric algebra $S_{(P)}$ of P and $S_{(P)}$ is identified with the polynomial ring $K[e_1, e_2, ..., e_n]$ where $e_1, e_2, ..., e_n$ is a linear basis of P over K. Next, the Poisson bracket {-,-} of P is defined by $\{e_i, e_j\} = [e_i, e_j]$ for $1 \leqslant i, j \leqslant n$and extend it by linearity using Leibniz's rule on all P. For instance,

- $\{e_i.e_j, e_k\} = e_i.\{e_j, e_k\} + \{e_i, e_k\}.e_j$
- $\{e_i.e_j.e_k, e_w\} = e_ie_j.\{e_k, e_w\} + e_ie_k.\{e_j, e_w\} + e_je_k.\{e_i, e_w\}$

and then we considered the restricted enveloping algebra of P with the linear ordered basis of P and by applying the Poincare- Birkhoff-Witt theorem (Oh et al., 2002) with this, a canonical basis is obtained, more so the restricted enveloping algebra has a natural filtration that provide us with an associative graded algebra that is isomorphic to a truncated polynomial ring $S_{(P)}(P) = gr\{U_P^n(P), n \geqq 0\}$, where $S_{(P)}(P)$ is the restricted symmetric algebra, when $S_{(P)}(P)$ is applied to a defined Poisson bracket, one obtained a Poisson algebra.

After constructing a Poisson algebra from an arbitrary algebra P, we looked into it Poisson subalgebra which is defined for any subspace Q of P. Q is Poisson subalgebra of P if for all $q_1, q_2 \in Q$, $q_1.q_2 \in Q$ and $\{q_1, q_2\} \in Q$ and we provide some examples to illustrate how a subspace Q of P can be a Poisson subalgebra of P.

We also investigated Poisson derivations and we noticed that for a derivation D to be consider a Poisson derivation it has to be a derivation as an associative algebra as well as a derivation as a Lie algebra. That is D is a Poisson derivation if and only if

1. $D(pq) = D(p)q + pD(q)$
2. $D\{p, q\} = \{D(p), q\} + \{p, D(q)\}$

for all $p, q \in P$

We further looked into the concept of Poisson morphism. For a map ϕ to be a Poisson morphism, it has to be a morphism both the morphism of as an associative algebra as well as a morphism as a Lie algebra. That is, For any two Poisson algebra P and Q, the map $\phi : P \longrightarrow Q$ is said to be Poisson morphism if;

1. $\phi(p_1.p_2) = \phi(p_1).\phi(p_2)$
2. $\phi(\{p_1, p_2\}_P) = \{\phi(p_1), \phi(p_2)\}_Q$ for all $p_1, p_2 \in P$

and we gave and example of a map that is a Poisson morphim. In this example we showed the map $\phi_1 : P \longrightarrow P \otimes Q$ that takes $p \rightarrow p \otimes 1$ and $\phi_2 : Q \longrightarrow P \otimes Q$ that takes $q \rightarrow 1 \otimes q$ are Poisson morphisms and $\{p \otimes 1, 1 \otimes q\} = 0$.

In addition we investigated Poisson polynomial ring.This is because by considering the conditions on which some Poisson algebras on a polynomial ring are given by derivation which can be seen as a Poisson type of an anti-symmetric rings of polynomial constructed from an endomorphism β and a β - derivation Oh (2006). This is important cos it gave us an inside on how to use this rings are applied to have a universal property of Poisson algebra when Poisson enveloping algebra is applied on them.

As a case study, modules of Poisson algebras are investigated in the 4th section of our work, In the 5th section we looked into the concept of Poisson enveloping algebras, in the last section we investigated the ideals of a Poisson algebras. In each case study, examples, propositions and theorems are given as well as working out on the proofs of some of the proposition and theorem stated.

This study is done in order to establish and give strong evidence that Poisson algebras are closely related to associative algebras and Lie algebras and some of the basic theorems that hold for them can be used for Poisson algebras as well. For example the Jung Van-Der Kulk theorem (Jung, 1942; Van der Kulk, 1953) which determines the automorphisms of polynomial algebra with two variables. We used this theorem to give some examples of Poisson algebra automorphisms with two variables.

GENİŞLETİLMİŞ ÖZET

P, değişmeli ve birleşmeli bir cebir olsun. Eğer P, $\{,\}$ braket çarpımı ile bir Lie cebiri ve her $p_1, p_2, p_3 \in P$ için

$$\{p_1.p_2, p_3\} = p_1.\{p_2, p_3\} + \{p_1, p_3\}.p_2,\ p_1, p_2, p_3 \in P,$$

Leibniz eşitliği sağlanıyorsa P ye bir Poisson cebiri denir. P bir Poisson cebiri ise $\{,\}$ çarpımı Poisson braket olarak adlandırılır.

Poisson cebirleri 20. yüzyılın önemli araştırma konularından biridir. Poisson cebirleri, Hamilton mekaniği calışmalarında karşımızda çıkar. Ayrıca kuantun grubu ile ilgili çalışmaların merkezinde yer alır. Geometride Poisson yapılarının manifoldları Poisson manifold olarak bilinir. Fizikte, Poisson cebir yapısı kovaryant kanonik ölçümünün, Hamilton mekaniğinin ve topolojik cisimlerin önemli bir parçasıdır. Poisson cebirleri ayrıca, Poisson geometrisinde ve değişmeli birleşmeli cebirlerin deformasyonunda temel bir yere sahiptir.Poisson cebirleri, serbest Lie cebirleri ve ek olarak serbest birleşmeli cebirler arasında bazı benzerlikler vardır. Serbest Lie cebirleri ile serbest birleşmeli cebirlerin bir cok özelliği bilinmesine rağmen serbest Poisson cerbirleri ilgili geniş bir bilgiye sahip olunmayıp bunlarla ilgili çalışmalar yoğun bir şekilde devam etmektedir.

Poisson cebirlerinin çıkış noktası esas olarak yaklaşık iki yüz yil önce Simeone Denis Poisson'un "three body problem on celestial Mechanic" ile ilgili calışmasına dayanmaktadır. Bu tezde herhangi bir Lie cebirinden nasıl bir Poisson cebiri inşa edildiği gösterilmiştir. Poisson cebirleri ilgili araştırmalarda karşımızda iki önemli Poisson cebiri sınıfı çıkar.

Birincisi S_n ile gösterilen simpletik cebirdir. S_n, $2n$ değişkenli

$$K[p_1, ...p_n, q_1, ..., q_n]$$

polinom cebiri olup aşağıda tanımlanan braket işlemi ile bir Poisson cebiri yapısı oluşturur.

$$\{p_i, q_j\} = \delta_{ij} \ , \ \{p_i, p_j\} = \{q_i, q_j\} = 0 \text{ ve } \delta_{ij} = \begin{cases} 1, & \text{eğer } i = j, \\ 0, & \text{eğer } i \neq j. \end{cases}$$

Diğeri ise verilen bir Lie cebirinden inşa edilen Poisson cebiridir. P, lineer bazı $e_1, e_2, ..., e_n, ...$ olan bir Lie cebiri olsun. $S_{(P)}$ simetrik cebiri $K[e_1, e_2, ..., e_n]$ polinom cebiri olup $S_{(P)}$ üzerinde Poisson braketi $\{e_i, e_j\} = [e_i, e_j]$ olarak tanımlanır. $S_{(P)}$ bu komütator işlemi ile bir Lie cebiridir ve Leibniz kuralını sağlar.

Şimdi K cismi üzerinde tanımlı bir P Lie cebirinden bir Poisson cebiri elde edelim.

İlk olarak P nin bir $e_1, e_2, ..., e_n$ lineer bazı seçilir daha sonra simetrik cebiri $S_{(P)}$ olarak da bilinen $K[e_1, e_2, ..., e_n]$ polinom cebiri ele alınır. $1 \leqslant i, j \leqslant n$ için $\{e_i, e_j\} = [e_i, e_j]$ olarak tanımlanan Poisson braketi, lineer olarak Leibniz kuralına genişletebilir.

örneğin,

- $\{e_i.e_j, e_k\} = e_i.\{e_j, e_k\} + \{e_i, e_k\}.e_j$
- $\{e_i.e_j.e_k, e_w\} = e_ie_j.\{e_k, e_w\} + e_ie_k.\{e_j, e_w\} + e_je_k.\{e_i, e_w\}$

Böylece (P) bu $\{,\}$ çarpım ile bir Poisson cebiridir. Ayrıca bir çok Poisson cebiri örneği verilmiştir. En çok bilinenleri ise polinom cebirlerinden elde edilen Poisson cebiridir.

Genel olarak $K[p_1, ...p_n, q_1, ..., q_n]$, $2n$ değişkenli polinom cebiri

$$\{f, g\} = \sum_{i=1}^{n} \frac{\partial f_i}{\partial p_i}\frac{\partial g_i}{\partial q_i} - \frac{\partial g_i}{\partial p_i}\frac{\partial f_i}{\partial q_i}$$

braket çarpımı ile bir Poisson cebiridir.
Bu genellemeden farklı olarak üç değişkenli $\mathbb{C}[p_1, p_2, p_3]$ polinom cebiri aşağıda tanımlanan braket çarpımı ile bir Poisson cebiridir.
$P = \mathbb{C}[p_1, p_2, p_3]$ ve $f \in P$ olsun, her $g, h \in P$ için

$$\{g, h\}_f = \begin{vmatrix} f_{p_1} & f_{p_2} & f_{p_3} \\ g_{p_1} & g_{p_2} & g_{p_3} \\ h_{p_1} & h_{p_2} & h_{p_3} \end{vmatrix}$$

olarak tanımlansın. O zaman $\mathbb{C}[p_1, p_2, p_3]$ bu çarpım ile bir Poisson cebiridir.

Eğer L, serbest üreteç kümesi X olan bir serbest Lie cebiri ise $S_{(L)}$ serbest üreteç kümesi X olan serbest Poisson cebiridir. Bu tezde ayrıca birçok Poisson cebiri örnekler verilmiştir.
Eğer P bir birleşmeli cebir ise $p, q \in P$ icin $\{p, q\} = p.q - q.p$ olarak tanımlanan braket işlemi ile$P$ bir Poisson cebiridir. Yani her birleşmeli cebir aynı zamanda bir Poisson cebir olarak düsünülebilir.

P bir Poisson cebiri ve Q, P nin bir alt vektör uzayı olsun. Eğer Q, hem birleşmeli cebir hemde Lie cebiri olarak P nin bir alt cebiri ise Q ya P'nin bir Poisson alt cebiri denir.
$p \in P$ olmak üzere p nin P deki merkezi, P nin bir Poisson alt cebiridir. Ayrıca p nin merkezi

$$\text{her } p \in P \text{ için, } C(p) = \{q \in P : \{p, q\} = 0\}$$

olarak tanımlanır ve $C(P)$, P nin bir Poisson alt cebiridir.
P ve Q, K cismi üzerinde tanımlı herhangi iki Poisson cebiri olsun.

$$\phi : (P, \{,\}_P) \longrightarrow (Q, \{,\}_Q)$$

lineer dönüşümü her $p_1, p_2 \in P$ için

$$\phi(p_1.p_2) = \phi(p_1).\phi(p_2) \text{ ve } \phi(\{p_1, p_2\}_P) = \{\phi(p_1), \phi(p_2)\}_Q$$

eşitliklerini sağlıyorsa, ϕ ye bir Poisson homomorfizmi denir. Eğer ϕ aynı zamanda birebir ve örten ise ϕ ye bir izomorfizm denir.

$$\phi_1 : P \longrightarrow P \otimes Q,\ p \to p \otimes 1 \text{ ve } \phi_2 : Q \longrightarrow P \otimes Q,\ q \to 1 \otimes q$$

birer bir Poisson homomorfizmidir ve $\{p \otimes 1, 1 \otimes q\} = 0$.

Jung-van-der-Kulk teoremi $K[p, q]$ polinom cebirinin otomorfizmleri grubunun yapısı belirlenmiştir (Jung, 1942; Van der Kulk, 1953).
Örnek 3.1 ile $K[p, q]$ polinom cebiri üzerinde tanımlı

$$\{f, g\} = \frac{\partial f}{\partial p}\frac{\partial g}{\partial q} - \frac{\partial g}{\partial p}\frac{\partial f}{\partial q}\ ,\ f, g \in K[p, q]$$

braket işlemi ile bir Poisson cebiridir. Jung-Van-der-Kulk teoreminden (Jung, 1942; Van der Kulk, 1953)

$$\tau : p \to p + q^3, q \to q \text{ ve } \psi : p \to p + q \ , \ q \to p + 2q$$

$K[p, q]$ polinom cebirinin birer otomorfizmdir. Bu tezde örnek olarak τ ve ψ nın Poisson cebiri olarak $K[p, q]$ nun bir otomorfizmi olduğu gösterilmiştir.

P bir Poisson cebir olsun. Eger $D : P \longrightarrow P$ dönüşümü D nin hem birleşmeli cebiri olarak hem de Lie cebiri olarak bir derivasyonu ise D ye bir Poisson derivasyon denir.

Yani $D : P \longrightarrow P$ bir lineer dönüşüm olmak üzere her $p_1, p_2 \in P$ için

$$D(p.q) = D(p).q + p.D(q) \text{ ve } D\{p, q\} = \{D(p), q\} + \{p, D(q)\}$$

ise D ye, P nin bir Poisson derivasyonu denir. Poisson derivasyonları 3.5 Bölümde incelenen Poisson polinom halkalarının belirlenmesinde önemli bir rol oynar. P bir Poisson cebir olsun. Bu cebir üzerindeki r değişkenine göre polinomların halkasını $P[r]$ ile gösterelim. Değişmeli ve birleşmeli bir cebir olan $P[r]$ üzerinde Poisson braketinin varlığının Poisson drivasyonları ile ilgili olduğu bilinmektedir.

Şöyleki,

$$\lambda, \beta : P \longrightarrow P$$

bir lineer dönüşümler olsun. Teorem 3.1 de (Oh, 2006), $p, q \in P$ için

$$\{p, q\} = \{p, q\}_P, \{p, r\} = \beta(p)r + \lambda(p)$$

tanımlı çarpımın Poisson braketi olması için gerek ve yeter koşulun β nın bir Poisson derivasyonu, λ nın da aşağıdakı

$$\lambda(\{p,q\}_P) - \{\lambda(p),q\}_P - \{p,\lambda(q)\}_P = \lambda(p)\beta(q) - \beta(p)\lambda$$

eşitliği sağlayan bir derivasyon olduğu gösterilmiştir.
$p,q \in P$ için $D_p(q) = \{p,q\}$ olarak tanımlanan $D_p : P \longrightarrow P$ dönüşümü P nin bir derivasyonudur. Eğer D, P nin bir derivasyonu ise $\{D, D_p\} = D_{D(p)}$ elde edilir. Ayrıca $p,q \in P$ için $\{D_p, D_q\} = D_{\{p,q\}}$ olduğu gösterilmiştir.

4. Bölümde Poisson modülleri incelenmiştir. P bir Poisson cebiri olsun. P yi birleşmeli cebiri olarak düşünelim ve M_P bir P-modül olsun.

$$\circ : P \times M_P \longrightarrow M_P$$

bilineer dönüşümü her $p_1, p_2 \in P$ ve $m \in M_P$ icin

$$p_1 \circ (p_2 m) = \{p_1, p_2\}m + p_2(p_1 \circ m)$$

eşitliğini sağlasın. Eğer P yi bir Lie cebiri olarak düşündüğümüzde, M_P bir P-Lie modül ve

$$(p_1 p_2) \circ m = p_1(p_2 \circ m) + p_2(p_1 \circ m)$$

ise M_P ye bir Poisson P-modül denir. Bu kısımda Poisson modülleri örneklerle verilmiştir ve bazı önermeler ispatlanmıştır. 5. bölumde ise Poisson cebirlerin enveloping cebirini inceledik. Son Bölümde ise Poisson cebirlerinin ideallerinden bahsettik. P bir Poisson cebiri ve $I \subseteq P$ olsun. Eğer I birleşmeli cebir olarak P nin bir ideali ve $i \in I$ ve her $p \in P$ için

$\{i, p\} \in I$ ise I ya P nin bir ideali denir. Yani I, P nin hem birleşmeli cebir olarak hemde Lie cebiri olarak bir ideali ise I, P nin bir Poisson idealidir.

Bu tanımı daha iyi anlamak için $P = C[p_1, p_2, p_3]$ polinom cebirini düşünelim. $f \in P$ olsun. $g, h \in P$ için

$$\{g, h\}_f = \begin{vmatrix} f_{p_1} & f_{p_2} & f_{p_3} \\ g_{p_1} & g_{p_2} & g_{p_3} \\ h_{p_1} & h_{p_2} & h_{p_3} \end{vmatrix}$$

olarak tanımlı $\{,\}_f$ braket çarpımı ile P bir Lie cebiri olup, Leibniz kuralını sağlar. Yani $(P, ., \{,\}\mathbb{C})$ bir Poisson cebiridir.

Daha sonra $fP = \{f.p : p \in P\}$ kümesinin P nin bir Poisson ideali olduğu gösterilmiştir.

Son olarak Poisson asal ideal ve Poisson maksimal ideal kavramlarından söz edilip bir önerme ve ispatı ile tez tamanlanmıştır.

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CONTENTS……………………………………………………………………PAGE

1. INTRODUCTION

A Poisson algebra is an algebra that is considered to have the form of a commutative associative algebras and that of a Lie algebras, which are congruent with each other when we apply the Leibniz identity to their structures. This algebra was introduced by Simeone-Denise Poisson in the 19th century when he investigated the structures of celestial mechanics. Poisson algebras is connected to different branches of mathematics and physics. In mathematics, Poisson algebras, contribute a great deal in Poisson geometry (Vaisman, 2012) and deformation of commutative associative algebra(Gerstenhaber, 1964). In physics Poisson algebra plays a fundamental part in deformation quantization (Kontsevich, 2003), Hamiltonian mechanics and topological field theories (Sneddon et al., 1980). Poisson structures can be used in the investigation of vertex operator algebra.

In this thesis we investigate the notion of Poisson algebras, modules of Poisson algebra, enveloping algebras of a Poisson algebra and Poisson ideals. We will also construct Poisson algebras from an arbitrary algebra. It should be noted that we will mainly work with commutative Poisson algebra.

Our work has five sections, the first section is basic definitions and theorems, here we recalled some definitions and we tried to discuss and prove some theorems and propositions. This will guide us to understanding the objectives of our work. In the next four sections, definitions, theorems and concepts of Poisson algebras, Poisson modules, Poisson derivations and Poisson ideals are stated and solutions and some proves are given. We assume that all fields throughout this thesis have characteristic zero.

2. BASIC DEFINITIONS AND THEOREMS

Definition 2.1. Let S and T be two vector spaces over F. A function $\varphi : S \longrightarrow T$ is said to be a linear transformation if for any two vectors $s_1, s_2 \in S$ and any scalar $f \in F$ the following axioms hold.

1. $\varphi(s_1 + s_2) = \varphi(s_1) + \varphi(s_2)$
2. $\varphi(fs_1) = f\varphi(s_1)$.

Definition 2.2. For any algebras S and T. A linear map $\varphi : S \longrightarrow T$ is a homomorphism if for all $s_1, s_2 \in S$, $\varphi(s_1 s_2) = \varphi(s_1)\varphi(s_2)$.

Definition 2. A linear transformation $\varphi : S \longrightarrow T$ is an isomorphism if it is a bijection(one-to-one and onto). If there is an isomorphism between S and T we say that S is isomorphic to T and write $S \simeq T$.

The inverse of an isomorphism is an isomorphism and the composition of two isomorphism is also an isomorphism.

Definition 2.4. For a vector space T, a linear transformation $\varphi : T \longrightarrow T$ is an endomorphism of T. The set of all such endomorphism forms a vector space over the field F denoted by $End(T)$.

An endomorphism of T which is also an isomorphism is known as an automorphism.

Definition 2.5. For any two vector spaces S, T, let $\varphi : S \longrightarrow T$ be a linear transformation then

- $Ker(\varphi) = \{s \in S : \varphi(s) = 0\}$
- $Im(\varphi) = \{\ t \in T : t = \varphi(s), s \in S\}$

Definition 2.6. A map $\varphi : T \times T \longrightarrow F$ is a bilinear map if the axioms below hold,

1. $\varphi(t_1 + t_2, t_3) = \varphi(t_1, t_3) + \varphi(t_2, t_3)$ and $\varphi(ft_1, t_2) = f\varphi(t_1, t_2)$
2. $\varphi(t_1, t_2 + t_3) = \varphi(t_1, t_2) + \varphi(t_1, t_3)$ and $\varphi(t_1, ft_2) = f\varphi(t_1, t_2)$

Definition 2.7. An algebra P on a field F is a vector space with a bilinear map

$$P \times P \longrightarrow P,\ (p_1, p_2) \longrightarrow p_1p_2.$$

If we have,

$$p_1, p_2, p_3 \in p,\ p_1(p_2p_3) = (p_1p_2)p_3$$

then we say P is an associative algebra.
If $1_P \in P$ such that $1_Pp = p1_p$, $\forall p \in P$ then P is a unitary algebra.

Definition 2.8. A Vector subspace T of an algebra P is a left ideal if for all $p_1 \in P$ and $t \in T$, $tp_1 \in T$ and a right ideal if for all $t \in T$ and $p_2 \in p$, $p_2t \in T$. If T is at the same time left and right ideal, then we say T is a two-sided ideal.

Definition 2.9. Let T be a two-sided ideal of P, the algebra P/T is known as the quotient algebra of P by T.

Definition 2.10. Let P be an algebra on a F field. An F-linear map $\theta : P \longrightarrow P$ is said to be a derivation of P if it satisfies the the Leibniz's law.

$$\theta(p_1p_2) = \theta(p_1)p_2 + p_1\theta(p_2) \text{ for all } p_1, p_2 \in P.$$

The collection of all such derivations is know as $Der(P)$.

Properties Of Derivation:

If P is an algebra over a vector space F and $\theta : P \longrightarrow P$ is an F-derivation then

- If 1 is the unit of P , then $\theta(1) = \theta(1^2) = 2\theta(1)$ this implies that $\theta(1) = 0$. Generally $\theta(k) = 0$ for all $k \in F$
- If P is commutative then $\theta(p^2) = p\theta(p) + \theta(p)p = 2p\theta(p)$ and $\theta(p^n) = np^{n-1}\theta(p)$ by the Leibniz's Law.

Proposition 2.1. Let D and G be elements of $Der(P)$ then,

- $Der(P)$ is a vector subspace of P
- The composition $D \circ G$ is not a derivation

Proof. We have to show that $Der(P)$ is a vector subspace of P.
Let $D, G \in Der(P)$, $p_1, p_2 \in P$ and $k \in F$

(1)

$$\begin{aligned}(D+G)(p_1p_2) &= D(p_1p_2) + G(p_1p_2)\\ &= (D(p_1)p_2 + p_1D(p_2)) + (G(p_1)p_2 + p_1G(p_2))\\ &= D(p_1)p_2 + p_1D(p_2) + G(p_1)p_2 + p_1G(p_2)\\ &= D(p_1)p_2 + G(p_1)p_2 + p_1D(p_2) + p_1G(p_2)\\ &= (D(p_1) + G(p_1))p_2 + p_1(D(p_2) + G(p_2))\\ &= (D+G)(p_1)p_2 + p_1(D+G)(p_2)\end{aligned}$$

Hence $D + G \in Der(P)$.

(2)

$$\begin{aligned}kD(p_1p_2) &= kD(p_1p_2)\\ &= k(D(p_1)p_2 + p_1D(p_2))\\ &= kD(p_1)p_2 + p_1kD(p_2)\end{aligned}$$

Hence $kD \in Der(P)$

From (1) and (2) we see that $Der(P)$ is a vector subspace of $gl(P)$.

For simplicity, let denote $D \circ G = DG$ where DG is the composition of D and G.

$$\begin{aligned}
DG(p_1p_2) &= D(G(p_1)p_2 + p_1G(p_2)) \\
&= D(G(p_1)p_2) + D(p_1G(p_2)) \\
&= D(G(p_1))p_2 + G(p_1)D(p_2) + D(p_1)G(p_2) + p_1D(G(p_2)) \\
&= (DG)(p_1)p_2 + p_1(DG)(p_2) + G(p_1)D(p_2) + D(p_1)G(p_2) \\
&\neq (DG)(p_1)p_2 + p_1(DG)(p_2)
\end{aligned}$$

Hence DG is not a derivation.

Definition 2.11. An algebra P is a Lie algebra with the product denoted by

$$(p_1, p_2) \to \{p_1, p_2\}$$

(where the bracket $\{-,-\}$is a Lie bracket) if the following are true:

- $\{p_1, p_1\} = 0$
- $\{p_1, \{p_2, p_3\}\} + \{p_2, \{p_1, p_3\}\} + \{p_3, \{p_1, p_2\}\} = 0$

for all $p_1, p_2, p_3 \in P$.

Definition 2.12. For any Lie algebra P and it subspace Q, if for all $q_1, q_2 \in Q$, $\{q_1, q_2\} \in Q$ then Q is a Lie subalgebra of P.

Proposition 2.2. Generally subalgebras, quotient algebras and direct product of Lie algebras are Lie algebras.

Example 2.1. For any associative algebra P the bracket $\{p_1, p_2\} = p_1p_2 - p_2p_1$ defines a Lie algebra structure on P.

Example 2.2. For a vector space S, $End(S)$, which is the endomorphisms of S consists of a structure of an associative algebra and by the multiplication of the composition of maps denoted by $gl(S)$, it is a Lie algebra with the bracket defined as $\{p_1, p_2\} = p_1p_2 - p_2p_1$ for all $p_1, p_2 \in End(S)$.

Definition 2.13. For any two Lie algebra P_1 and P_2, a linear map $\varphi : P_1 \longrightarrow P_2$ is morphism of Lie algebra if for all $p_1, p_2 \in P_1$, $\phi(\{p_1, p_2\}) = \{\phi(p_1), \phi(p_2)\}$

Definition 2.14. Let P be an associative algebra and D be an associative derivation of P, if for all $p_1, p_2 \in P$, $D(\{p_1, p_2\}) = \{p_1, D(p_2)\} + \{D(p_1), p_2\}$ then D is also a Lie algebra derivation.

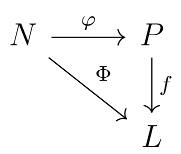
Definition 2.15. Let P be a Lie algebra and let $p \in P$ the map $P \longrightarrow P$ defined by $p \rightarrow \{p_1, p\}$ is known as the adjoint application of p_1 and denoted by adp_1.

Proposition 2.2. let P be a Lie algebra, then $Der(P)$ is a Lie subalgebra $gl(P)$ and also the map $ad : p \longrightarrow Der(P)$ defined by $p \rightarrow adp$ is a Lie algebra homomorphishm.

Proof. Let $D, G \in Der(P)$ and $p_1, p_2 \in P$

$$\{D, G\} = (DG - GD)(p_1 p_2)$$
$$= D(G(p_1 p_2)) - G(D(p_1 p_2))$$
$$= D(G(p_1)p_2 + p_2 G(p_1)) - G(D(p_1)p_2 - p_2 D(p_1))$$
$$= D(G(p_1)p_2) + D(p_2 G(p_1)) - G(D(p_1)p_2) - G(p_2 D(p_1))$$
$$= D(G(p_1))p_2 + G(p_1)D(p_2) - D(p_2)G(p_1) + p_2 D(G(p_1))$$
$$- G(D(p_1))p_2 - D(p_1)G(p_2)) + G(p_2)D(p_1) - p_2 G(D(p_1))$$
$$= D(G(p_1))p_2 - G(D(p_1))p_2 + p_1 D(G(p_2)) - p_1 G(D(p_2))$$
$$= (D(G(p_1)) - G(D(p_1)))p_2 + p_1(D(G(p_2)) - G(D(p_2)))$$
$$= \{D, G\}(p_1)p_2 + p_1\{D, G\}(p_2)$$

Definition 2.1 For any $\varnothing \neq N$, P is a free Lie algebra generated by N if P is a Lie algebra and there exist a map $\varphi : N \longrightarrow P$ satisfying the following properties. For every Lie algebra L and every map $\Phi : N \longrightarrow L$,there exist exactly one morphism $f : P \to L$, such that the following diagram commutes.



3. POISSON ALGEBRAS

In this section of the thesis, we will define the main concept of our work, give some examples and some lemmas and propositions.

Definition 3.1. For any algebra P over a commutative associative ring R, P is a Poisson algebra over R if $(P, .)$ is associative on R and $(P, ., \{-, -\})$ is a Lie R-algebra which are congruent over the Leibniz's rule

$$\{p_1.p_2, p_3\} = p_1.\{p_2, p_3\} + \{p_1, p_3\}.p_3, \text{ for all } p_1, p_2, p_3 \in P.$$

From definition 3.1 we see that Poisson algebra is an algebra that makes commutative associative algebra and Lie algebra to be congruous with each other via the Leibniz's identity.

Associative algebras are Poisson algebras if we consider the zero Poisson bracket $\{p_1, p_2\} = 0$ and Lie algebras are also a Poisson algebras by the zero associative product $p_1.p_2 = 0$. such algebras are called null Poisson algebras.

Example 3.1 Let consider the bracket $\{p_1, p_2\} = p_1p_2 - p_2p_1$ on an associative algebra P, P becomes a Poisson algebra.

Solution: Let $p_1, p_2, p_3 \in P$, we have that

$$\begin{aligned}\{p_1p_2, p_3\} &= (p_1p_2)p_3 - p_3(p_1p_2)\\ &= p_1(p_2p_3) - p_1(p_3p_2) + (p_1p_3)p_2 - (p_3p_1)p_2\\ &= p_1.\{p_2, p_3\} + \{p_1, p_3\}.p_2\end{aligned}$$

Example 3.2. The commutative ring $K[p,q]$ of polynomial has a Poisson structure given by

$$\{f, g\} = \tfrac{\partial f}{\partial p}\tfrac{\partial g}{\partial q} - \tfrac{\partial g}{\partial p}\tfrac{\partial f}{\partial q}$$

for all $f, g \in K[p,q]$.

Solution: This can be shown by showing that the bracket defined at Example 3.2 holds for all the axioms of Poisson algebra.

Let $f, g, h \in K[p,q]$ and $k, r \in K$,

$$\begin{aligned}\{f, kg + rh\} &= \frac{\partial f}{\partial p}\frac{\partial(kq + rh)}{\partial q} - \frac{\partial(kg + rh)}{\partial p}\frac{\partial f}{\partial q}\\ &= \frac{\partial f}{\partial p}\frac{\partial(kg)}{\partial q} + \frac{\partial f}{\partial p}\frac{\partial(rh)}{\partial q} - \frac{\partial(kg)}{\partial p}\frac{\partial f}{\partial q} - \frac{\partial(rh)}{\partial p}\frac{\partial f}{\partial q}\\ &= k\left(\frac{\partial f}{\partial p}\frac{\partial g}{\partial q} - \frac{\partial g}{\partial p}\frac{\partial f}{\partial q}\right) + r\left(\frac{\partial f}{\partial p}\frac{\partial h}{\partial q} - \frac{\partial h}{\partial p}\frac{\partial f}{\partial q}\right)\\ &= k\{f, g\} + r\{f, h\}\end{aligned}$$

That is $\{f, kg + rh\} = k\{f, g\} + r\{f, h\}$ $\forall k, r \in K$ and $f, g \in K[p,q]$. Hence the bracket is bilinear.

$\forall f, g, h \in K[p, q]$, is $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$?

$$\begin{aligned}
&\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} \\
&= \{f, \frac{\partial g}{\partial p}\frac{\partial h}{\partial q} - \frac{\partial h}{\partial p}\frac{\partial g}{\partial q}\} + \{g, \frac{\partial h}{\partial p}\frac{\partial f}{\partial q} - \frac{\partial f}{\partial p}\frac{\partial h}{\partial q}\} + \{h, \frac{\partial f}{\partial p}\frac{\partial g}{\partial q} - \frac{\partial g}{\partial p}\frac{\partial f}{\partial q}\} \\
&= (\frac{\partial f}{\partial p}\frac{\partial A}{\partial q} - \frac{\partial A}{\partial p}\frac{\partial f}{\partial q}) + (\frac{\partial g}{\partial p}\frac{\partial B}{\partial q} - \frac{\partial B}{\partial p}\frac{\partial g}{\partial q}) + (\frac{\partial h}{\partial p}\frac{\partial C}{\partial q} - \frac{\partial C}{\partial p}\frac{\partial h}{\partial q}) \\
&= (\frac{\partial f}{\partial p}\frac{\partial A}{\partial q} + \frac{\partial g}{\partial p}\frac{\partial B}{\partial q} + \frac{\partial h}{\partial p}\frac{\partial C}{\partial q}) - (\frac{\partial A}{\partial p}\frac{\partial f}{\partial q} + \frac{\partial B}{\partial p}\frac{\partial g}{\partial q} + \frac{\partial C}{\partial p}\frac{\partial h}{\partial q}) \\
&= 0
\end{aligned}$$

This proves the Jacobi $\forall f, g, h \in K[p, q]$. This is done as such because our algebra is assumed to be commutative and hence the partial differentials,

$$\tfrac{\partial f}{\partial p}\tfrac{\partial g}{\partial q} = \tfrac{\partial g}{\partial q}\tfrac{\partial f}{\partial p}$$

For simplicity we take

$$\begin{aligned}
\tfrac{\partial g}{\partial p}\tfrac{\partial h}{\partial q} - \tfrac{\partial h}{\partial p}\tfrac{\partial g}{\partial q} &= A\ , \\
\tfrac{\partial h}{\partial p}\tfrac{\partial f}{\partial q} - \tfrac{\partial f}{\partial p}\tfrac{\partial h}{\partial q} &= B\ , \\
\tfrac{\partial f}{\partial p}\tfrac{\partial g}{\partial q} - \tfrac{\partial g}{\partial p}\tfrac{\partial f}{\partial q} &= C
\end{aligned}$$

3) Leibniz's identity: Let $f, g, h \in K[p, q]$,

$$\begin{aligned}\{fg,h\} &= \frac{\partial(fg)}{\partial p}\frac{\partial h}{\partial q} - \frac{\partial h}{\partial p}\frac{\partial(fg)}{\partial q}\\ &= (g\frac{\partial f}{\partial p} + f\frac{\partial g}{\partial p})\frac{\partial h}{\partial q} - \frac{\partial h}{\partial p}(g\frac{\partial f}{\partial q} + f\frac{\partial g}{\partial q})\\ &= g(\frac{\partial f}{\partial p}\frac{\partial h}{\partial q} - \frac{\partial h}{\partial p}\frac{\partial f}{\partial q}) + (\frac{\partial g}{\partial p}\frac{\partial h}{\partial q} - \frac{\partial h}{\partial p}\frac{\partial g}{\partial q})f\\ &= g\{f,h\} + \{g,h\}f\end{aligned}$$

From 1) , 2) and 3) we see that the ring $K[p,q]$ structure is that of Poisson algebra defined by $\{f,g\} = \frac{\partial f}{\partial p}\frac{\partial g}{\partial q} - \frac{\partial g}{\partial p}\frac{\partial f}{\partial q}$. Hence $K[p,q]$ is forms a Poisson algebra.

Generally the polynomial ring $K[p_1, ...p_n, q_1, ..., q_n]$ forms a Poisson algebra if we define its Poisson bracket as follows,

$$\{f,g\} = \sum_{i=1}^{n} \frac{\partial f_i}{\partial p_i}\frac{\partial g_i}{\partial q_i} - \frac{\partial g_i}{\partial p_i}\frac{\partial f_i}{\partial q_i}$$

Example 3.3. Let $P = \mathbb{C}[p,q,r]$ and $f \in P$, and by defining the partial differentials of f with respect to p, q, r as follows,

$$\{q,r\} = f_p \ , \ \{r,p\} = f_q \ , \{p,q\} = f_r$$

then there is an exact Poisson bracket $\{-,-\}$ on P giving by

$$\{g,h\} = \begin{vmatrix} f_p & f_q & f_r \\ g_p & g_q & g_r \\ h_p & h_q & h_r \end{vmatrix} \ , \{f,-\} = 0$$

for all $g, h \in P$.

We are going to show that the bracket $\{g, h\}_f$ is a Poisson bracket by brute calculation and thus show that the complex polynomial ring $P = \mathbb{C}[p, q, r]$ is a Poisson algebra.

Solution:

1) Let $s, t, c \in \mathbb{C}[p, q, r]$, and $\eta, \mu \in \mathbb{C}$

$$\{s, \eta t + \mu c\} = \begin{vmatrix} f_p & f_q & f_r \\ s_p & s_q & s_r \\ (\eta t + \mu c)_p & (\eta t + \mu c)_q & (\eta t + \mu c)_r \end{vmatrix}$$

$$= \begin{vmatrix} f_p & f_q & f_r \\ s_p & s_q & s_r \\ \eta t_p + \mu c_p & \eta t_q + \mu c_q & \eta t_r + \mu c_r \end{vmatrix}$$

$$= \begin{vmatrix} f_p & f_q & f_r \\ s_p & s_q & s_r \\ \eta t_p & \eta t_q & \eta t_r \end{vmatrix} + \begin{vmatrix} f_p & f_q & f_r \\ s_p & s_q & s_r \\ \mu c_p & \mu c_q & \mu c_r \end{vmatrix}$$

$$= \eta \begin{vmatrix} f_p & f_q & f_r \\ s_p & s_q & s_r \\ t_p & t_q & t_r \end{vmatrix} + \mu \begin{vmatrix} f_p & f_q & f_r \\ s_p & s_q & s_r \\ c_p & c_q & c_r \end{vmatrix}$$

This shows that $\{s, \eta t + \mu c\} = \eta\{s, t\} + \mu\{s, c\}$ which proofs the Bilinearity of the bracket

2) By the straightforward but a little bit long calculations, Jacobi identity holds on the complex ring $P = \mathbb{C}[p, q, r]$.

3) Leibniz's identity holds

$$\{st, c\}_f = \begin{vmatrix} f_p & f_q & f_r \\ (st)_p & (st)_q & (st)_r \\ c_p & c_q & c_r \end{vmatrix}$$

$$= \begin{vmatrix} f_p & f_q & f_r \\ s(t)_p & s(t)_q & s(t)_r \\ c_p & c_q & c_r \end{vmatrix} + \begin{vmatrix} f_p & f_q & f_r \\ t(s)_p & t(s)_q & t(s)_r \\ c_p & c_q & c_r \end{vmatrix}$$

$$= s\begin{vmatrix} f_p & f_q & f_r \\ t_p & t_q & t_r \\ c_p & c_q & c_r \end{vmatrix} + t\begin{vmatrix} f_p & f_q & f_r \\ a_p & a_q & a_r \\ c_p & c_q & c_r \end{vmatrix}$$

= t{s,c} + {t,c}s

Hence the bracket is a Poisson algebra bracket.

3.1. The Construction of a Poisson Algebra

In this section, Poisson algebra from is built from an arbitrary associative algebra P.
There are two main categories of Poisson algebras:

1. *The symplectic algebra.* For each n, S_n as an algebra is a polynomial algebra $k[p_1, ...p_n, q_1, ..., q_n]$ with its Poisson bracket defined as
$$\{p_i, q_j\} = \delta_{ij} \ , \{p_i, p_j\} = \{q_i, q_j\} = 0 \text{ and } \delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$
where $1 \leqslant i, j \leqslant n$

2. *Algebra of Lie class.* For a P Lie algebra that have the linear basis given by $s_1, s_2, ..., s_n, ...$ and $S_{(P)}$ the symmetric algebra of P given by the Poisson bracket defined by $\{s_i, s_j\} = [s_i, s_j]$ for all i, j where [-,-] is the commutator of Lie algebra P. This algebra becomes Lie class algebra of P.

With this in mind, given a Lie algebra on a K-field, a Poisson algebra P can be constructed as follows:

First, we should observed that P as a space of vectors, is isomorphic to the symmetric algebra $S_{(P)}$ of P and $S_{(P)}$ is the same as the ring $K[s_1, s_2, ..., s_n]$ where $s_1, s_2, ..., s_n$ are linear basis of P on K.
Next, Poisson bracket $\{-, -\}$ of P is defined by $\{s_i, s_j\} = [s_i, s_j]$ and expand it by linearity using Leibniz's axioms on all P. For instance,

- $\{s_i.s_j, s_k\} = s_i.\{s_j, s_k\} + \{s_i, s_k\}.s_j$
- $\{s_i.s_j.s_k, s_w\} = s_is_j.\{s_k, s_w\} + s_is_k.\{s_j, s_w\} + s_js_k.\{s_i, s_w\}$

With this expansion, we obtain a Poisson algebra.

Example 3.1.1. In this Example we are going to construct a Poisson algebra from an almost commutative algebra U, that is filtered associative algebra $U^0 \subseteq U^1 \subseteq U^2 \subseteq ...$ and $U^iU^j \subseteq U^{ij}$ such that the $gr(U) = \oplus_{i=1}^{\infty} = U_i/U_{i-1}$ is commutative and let u^* be the class of $u \in U^i$ and it Poisson bracket is given by,

$\{u^*, v^*\} := u^*v^* - v^*u^* \in gr(U)$, with this bracket, U is a Poisson Algebra.

Solution: First we will have to show that the bracket is bilinear and it is a Lie algebra bracket then show that the Leibniz's axiom holds.

For all u^*, v^*, w^* in the classes of u, v, w and $k_1, k_2 \in K$.

$$
\begin{aligned}
\{u^*, k_1v^* + k_2w^*\} &= u^*(k_1v^* + k_2w^*) - (k_1v^* + k_2w^*)u^* \\
&= u^*(k_1v^*) + u^*(k_2w^*) - (k_1v^*)u^* + (k_2w^*)u^* \\
&= u^*(k_1v^*) - (k_1v^*)u^* + (u^*(k_2w^*) - (k_2w^*)u^*) \\
&= k_1(u^*v^* - v^*u^*) + k_2(u^*w^* - k_2w^*u^*) \\
&= k_1\{u^*, v^*\} + k_2\{u^*, w^*\}
\end{aligned}
$$

Here we show that $\{u^*, \{s^*, w^*\}\} + \{v^*, \{w^*, u^*\}\} + \{w^*, \{u^*, v^*\}\} = 0$

$$\begin{aligned}
&\{u^*, \{v^*, w^*\}\} + \{v^*, \{w^*, u^*\}\} + \{w^*, \{u^*, v^*\}\} \\
&\quad = \{u^*, v^*w^* - w^*v^*\} + \{v^*, u^*w^* - w^*u^*\} + \{w^*, v^*u^* - v^*u^*\} \\
&\quad = -\{v^*w^*, u^*\} + \{w^*v^*, u^*\} - \{u^*w^*, v^*\} + \{w^*u^*, v^*\} \\
&\qquad - \{u^*v^*, w^*\} + \{v^*u^*, w^*\} \\
&\quad = -v^*.\{w^*, u^*\} - \{v^*, u^*\}.w^* + w^*.\{v^*, u^*\} + \{w^*, u^*\}.v^* \\
&\qquad - u^*.\{w^*, v^*\} - \{u^*, v^*\}.w^* + w^*.\{u^*, v^*\} + \{w^*, v^*\}.u^* \\
&\qquad - u^*.\{v^*, w^*\} - \{u^*, w^*\}.v^* + v^*.\{u^*, w^*\} + \{v^*, w^*\}.u^*
\end{aligned}$$

=0

We are going to show that the identity $\{u^*.v^*, w^*\} = u^*.\{v^*, w^*\} + \{u^*, w^*\}.v^*$ holds

$$\begin{aligned}
\{u^*.v^*, w^*\} &= (u^*v^*)w^* - w^*(u^*v^*) \\
&= u^*(v^*w^*) - u^*(w^*v^*) + (u^*w^*)w^* - (w^*(u^*)v^*) \\
&= (u^*(v^*w^*) - u^*(w^*v^*)) + ((u^*w^*)v^* - (w^*(u^*)v^*)) \\
&= u^*.\{v^*, w^*\} + \{u^*, w^*\}.v^*
\end{aligned}$$

from the Solution: above, we see that U becomes a Poisson algebra.

3.2. Poisson Subalgebra

Definition 3.2.1. For any Poisson algebra P, a subspace Q of P is a Poisson subalgebra of P if $\forall q_1, q_2 \in Q$, $q_1.q_2 \in Q$ and $\{q_1, q_2\} \in Q$.

Definition 3.2.2. For a Poisson algebra P, if for every $p \in P$ the set of element $C_P(p) = \{p_1 \in P : \{p, p_1\} = 0\}$ is called the Poisson centralizer of p in P.

Example 3.2.1. $C_P(p)$ is a Poisson subalgebra of P.

Solution: For $p_1, p_2 \in C_P(p)$, $\{p_1, p_2\} \in C_P(p)$ by the Jacobi identity and

$$\begin{aligned} \{p_1 p_2, p\} &= p_1\{p_2, p\} + \{p_1, p\}p_2 \\ &= 0 \end{aligned}$$

Hence $C_P(p)$ is a Poisson subalgebra of P.

Example 3.2.2. Let P be a Poisson algebra,

$$Cas(P) = \{p_1 \in P : \forall p \in P, \{p_1, p\} = 0\}$$

is a Poisson subalgebra of P. This is obvious from Leibniz's identity and Jacobi identity. This subalgebra is known as Casimir subalgebra and its elements are called Casimir elements. (Caressa, 2000)

3.3 Poisson Morphisms

Definition 3.3.1. For any two Poisson algebra P and Q, the map $\phi : (P, \{,\}_P) \longrightarrow (Q, \{,\}_Q)$ is said to be Poisson morphism if;

1. $\phi(p_1.p_2) = \phi(p_1).\phi(p_2)$
2. $\phi(\{p_1, p_2\}_P) = \{\phi(p_1), \phi(p_2)\}_Q$ for all $p_1, p_2 \in P$

Proposition 3.3.1. Let P and Q be Poisson algebras. Then we can construct a Poisson algebra structure on their tensor product $P \otimes Q$ (Ciccoli and l Witkowski, 2006) by defining its Poisson bracket as follows:

$$\{p_1 \otimes q_1, p_2 \otimes q_2\} = \{p_1 p_2\}_A \otimes q_1 q_2 + p_1 p_2 \otimes \{q_1, q_2\}_B$$

for all $p_1, p_2 \in P$ and $q_1, q_2 \in Q$.

Example 3.3.1. From the Proposition 3.3.1 we see that the map $\phi_1 : P \longrightarrow P \otimes Q$ that takes $p \to p \otimes 1$ and $\phi_2 : Q \longrightarrow P \otimes Q$ that takes $q \to 1 \otimes q$ are Poisson morphisms and $\{p \otimes 1, 1 \otimes q\} = 0$.

Solution: Now let show that ϕ_1 is a Poisson morphism.

$$\begin{aligned}\phi_1(p_1 p_2) &= (p_1 p_2) \otimes 1 \\ &= (p_1 \otimes 1)(p_2 \otimes 1) \\ &= \phi_1(p_1)\phi_1(p_2)\end{aligned}$$

then, ϕ_1 is as an associative algebra morphism and since we have

$$\begin{aligned}\phi_1(\{p_1,p_2\}_P) &= \{p_1,p_2\}_P \otimes 1\\ &= \{p_1\otimes 1, p_2\otimes 1\}_{P\otimes Q}\\ &= \{\phi_1(p_1),\phi_1(p_2)\}_{P\otimes Q}\end{aligned}$$

ϕ_1 is also a Lie algebra morphism.
This shows that ϕ_1 is both an associative algebra morphism and a Lie algegra morphism. Hence ϕ_1 is a Poisson algebra morphism. By the same logic ϕ_2 is also a Poisson algebra morphism. **Proposition 3.3.2.** Lets consider the map τ defined by

$$\tau : p \to p + q^3, q \to q$$

we know from Jung -van der Kulk theorem (Jung, 1942; Van der Kulk, 1953) that τ is an automorphism of the polynomial ring $K[p, q]$. If we consider $K[p, q]$ as the Poisson algebra defined in Example 3.2, we claim that τ becomes the automorphism of $K[p, q]$ as a Poisson algebra.

Solution: For this, we are going to show that $\tau(\{p, q\}) = \{\tau(p), \tau(q)\}$.

From the Definition of Poisson bracket in $K[p, q]$, $\{p, q\} = 1$.
Applying τ on this bracket we have $\tau(\{p, q\}) = \tau(1) = 1$.
Moreover we have $\{\tau(p), \tau(q)\} = \{p + q^3, q\} = \{p, q\} = 1$
This shows that $\tau(\{p, q\}) = \{\tau(p), \tau(q)\}$

Proposition 3.3.3. Consider the automorphism group $K[p,q]$ of polynomial algebra determined by the Jung-Van der Kulk theorem (Jung, 1942; Van der Kulk, 1953). By this theorem we know that the map given by

$$\psi : p \to p+q,\ q \to p+2q$$

is an automorphism of the polynomial algebra $K[p,q]$ and from Example 3.2 we also know that $K[p,q]$ is a Poisson algebra,then ψ is a Poisson algebra automorphism of $K[p,q]$.

Solution: We just have to show that $\psi(\{p,q\}) = \{\psi(p), \psi(q)\}$
From the Definition of Poisson bracket in $K[p,q]$, $\{p,q\} = 1$.
If we apply ψ on this bracket we have $\psi(\{p,q\}) = \psi(1) = 1$.
On the other hand we have that $\{\psi(p), \psi(q)\} = \{p+q, p+2q\} = \{p,q\} =$
1
This shows that $\psi(\{p,q\}) = \{\psi(p), \psi(q)\}$.

Definition 3.3.2. For P and Q be a Poisson algebras and the transformation $\phi : P \longrightarrow Q$ be a Poisson morphism

- ϕ is a Poisson isomorphism if it is bijective
- If $P = Q$ then, ϕ is a Poisson algebra endomorphism
- If $P = Q$ and ϕ is bijective then ϕ is a Poisson algebra automorphism

3.4 Poisson Derivations

Definition 3.4.1. Let P be a Poisson algebra and the map $D : P \longrightarrow P$ is a Poisson derivation of P if D holds on following;

1. $D(pq) = D(p)q + pD(q)$
2. $D\{p, q\} = \{D(p), q\} + \{p, D(q)\}$

$\forall p, q \in P$

From Definition 3.4.1 we see that for a derivation D to be a Poisson derivation of a Poisson algebra P, D has to be both a derivation of P as an associative algebra and that of P as a Lie algebra.

Proposition 3.4.1. Let P be a Poisson algebra, for every $p \in P$ if we defined a map $D_p : P \longrightarrow P$ such that $D_p(q) = \{p, q\}$ then D_p is a Poisson algebra derivation.

Proof. Let $q, r \in P$, from Leibniz's law we have that,

$$\begin{aligned} D_p(qr) &= \{p, qr\} \\ &= q\{p, r\} + r\{p, q\} \\ &= qD_p(r) + rD_p(q) \end{aligned}$$

and from Jacobi identity

$$\begin{aligned} D_p(\{q, r\}) &= \{\{p, q\}, r\} + \{q, \{p, r\}\} \\ &= \{D_p(q), r\} + \{q, D_p(r)\} \end{aligned}$$

Leibniz's identity shows that D_p is a derivative of an associative algebra P and by using the Jacobi identity on D_p we see that D_p is the Derivative of Lie algebra P. Hence D_P is a Poisson algebra derivation.

Result: For every $p \in P$, $D_p : q \to \{p, q\}$ is canonical, that is, it is both associative product and the Poisson bracket derivation.

Example 3.4.1. The map $adp : P \longrightarrow P$, $p \to \{p, q\}$ is a derivation of P as Poisson algebra.

Solution: From Leibniz's law we have,

$$\begin{aligned} adp(qr) &= \{p, qr\} \\ &= q\{p, r\} + r\{p, q\} \\ &= q.adp(r) + r.adp(q) \end{aligned}$$

which is the derivation of Q as an associative algebra. From Jacobi identity, we have that,

$$\begin{aligned} adp\{q, r\} &= \{p, \{q, r\}\} \\ &= \{\{p, q\}, r\} + \{q, \{p, r\}\} \\ &= \{adp(q), r\} + \{q, adp(r)\} \end{aligned}$$

which a derivation of P as a Lie algebra. Therefore adp is a Poisson algebra derivation.

Definition 3.4.2. The associative derivation of a Poisson algebra is known as "Hamiltonian". It is defined by

$$ham(p) : P \longrightarrow P$$

such that;

$$ham(p)q = \{p, q\}$$

$\forall p \in P$. The set of all such derivation is denoted by $Ham(P)$.

Proposition 3.4.2. Let D be a derivation of P and $p \in P$ then $\{D, D_p\} = D_{D(p)}$.

Proof. For an element q of P

$$\begin{aligned}
\{D, D_p\}(q) &= (DD_p - D_pD)q \\
&= (DD_p)q - (D_pD)q \\
&= D(D_p(q)) - D_p(D(q)) \\
&= D(\{p, q\}) - \{p, D(q)\} \\
&= \{D(p), q\} + \{p, D(q)\} - \{p, D(q)\} \\
&= \{D(p), q\} \\
&= D_{D(p)}(q)
\end{aligned}$$

from the prove above we see that $Ham(P)$ is a Lie ideal of Poisson derivation of P.

Proposition 3.4.3. For all $p, q \in P$, $\{D_p, D_q\} = D_{\{p,q\}}$.

Proof. For an element $r \in P$

$$\begin{aligned}
\{D_p, D_q\}(r) &= (D_p D_q - D_q D_p) \\
&= (D_p D_q)(r) - (D_q D_p)(r) \\
&= (D_p(D_q(r)) - (D_q(D_p(r)) \\
&= D_p(\{q, r\}) - D_q(\{p, r\}) \\
&= \{p, \{q, r\}\} - \{q, \{p, r\}\} \\
&= \{\{p, q\}, r\} \\
&= D_{\{p,q\}}(r)
\end{aligned}$$

Then we get $\{D_p, D_q\} = D_{\{p,q\}}$. We note that $Ham(P)$ is a Lie subalgebra of $Der(P)$.

3.5 Poisson Polynomial Rings

When we consider the conditions on which some Poisson algebras on a polynomial ring are given by derivation which can be seen as a Poisson type of an anti-symmetric rings of polynomial constructed from an endomorphism β and a β - derivation Oh (2006). With this in mind, we know that for any Poisson algebra P, a k-linear map $\beta : P \longrightarrow P$ is a Poisson derivation if β is both a derivation as an associative algebra and a derivation as a Lie algebra. In this part we used the articled written by Oh (Oh, 2006).

Theorem 3.1.

Let β and λ be k-linear map on a Poisson algebra P with Poisson bracket $\{.,.\}_P$, Then the ring $P[r]$ of polynomial is a Poisson algebra with a bracket given as

$$\{p, q\} = \{p, q\}_P, \{p, r\} = \beta(p)r + \lambda(p) \tag{1}$$

for all $p, q \in P$ if and only if β is a Poisson derivation and λ is a derivation so that

$$\lambda(\{p, q\}_P) - \{\lambda(p), q\}_P - \{p, \lambda(q)\}_P = \lambda(p)\beta(q) - \beta(p)\lambda \tag{2}$$

for all $p, q \in P$ see (Oh, 2006). Here we can denote the Poisson algebra $P[r] = P[r; \beta; \lambda]$

Proof. Let $P[r]$ be a Poisson algebra with the Poisson bracket given by (1), we have

$$\begin{aligned}
\{pq, r\} &= \beta(pq)r + \lambda(pq) \\
p\{q, r\} + \{p, r\}q &= p(\beta(q)r + \lambda(q)) + (\beta(p)r + \lambda(p))q \\
&= (p\beta(q) + \beta(p)q)r + (p\lambda(q) + \lambda(p)q) \\
&= (p\beta(q))r + (\beta(p)q)r + p\lambda(q) + \lambda(p)q
\end{aligned}$$

Since $\{pq, r\} = p\{q, r\} + \{p, r\}q$ then by the polynomial identity we have $\beta(pq) = p\beta(q) + \beta(p)q$ and $\lambda(pq) = p\lambda(q) + \lambda(p)q$ for all $p, q \in P$. This shows that both β and λ are associative derivations on P. In addition, since the Poisson bracket satisfy the Jacobi, we have

$$
\begin{aligned}
0 &= \{\{p,q\},r\} + \{\{q,r\},p\} + \{\{r,p\},q\} \\
&= \{\{p,q\}_P,r\} + \{\beta(q)r + \lambda(q),p\} + \{q,\beta(p)r + \lambda(p)\} \\
&= \beta(\{p,q\}_P)r + \lambda(\{p,q\}_P) + \{\beta(q)r,p\} \\
&\quad + \{\lambda(q),p\} + \{q,\beta(p)r\} + \{q\lambda(p)\} \\
&= \beta(\{p,q\}_P)r + \lambda(\{p,q\}_P) + \{\beta(q),p\}r \\
&\quad + \{t,a\}\beta(b) + \{\lambda(b),a\} + \{b,\beta(a)\}t + \{b,t\}\beta(a) + \{b,\lambda(a)\} \\
&= \beta(\{p,q\}_P)r + \lambda(\{p,q\}_P) + \{\beta(q),p\}r - (\beta(p)r \\
&\quad + \lambda(p))\beta(q) + \{\lambda(q),p\} + \{q,\beta(p)\}t + (\beta(q)t \\
&\quad + \lambda(q))\beta(p) + \{q,\lambda(p)\} \\
&= \beta(\{p,q\}_P)r + \lambda(\{p,q\}_P) + \{\beta(p),q\}r - \beta(q)\beta(p)r - \beta(q)\lambda(p) \\
&\quad + \{\lambda(q),p\} + \{q,\beta(p)\}r + \beta(p)\beta(q)r \\
&\quad + \beta(p)\lambda(q) + \{q,\lambda(p)\} \\
&= (\beta(\{p,q\}_P) - \{\beta(p),q\}_P - \{p,\beta(q)\}_P)r \\
&\quad + \lambda(\{p,q\}_P) - \{\lambda(p),q\}_P - \{p,\lambda(q)\} - \lambda(p)\beta(q) + \beta(q)\lambda(p)
\end{aligned}
$$

for all $p,q \in P$ and this shows that β is a Poisson derivation and λ is a derivations such that (β,λ) satisfies
$(\lambda\{p,q\}_P) - \{\lambda(p),q\}_P - \{p,\lambda(q)\} = \lambda(p)\beta(q) - \beta(b)\lambda(a)$
That is, equation (2) holds.
On the other hand, by assuming that β is a Poisson derivation and λ is a derivation such that (β,λ) satisfies equation (2) and by defining a

k-linear map $\{.,.\} : P[r] \times P[r] \longrightarrow P[r]$ such that for all monomials $pr^i, qr^j \in P[r]$ we have

$$\{pr^i, qr^j\} = (\{p,q\}_P + jq\beta(p) - ip\beta(q))r^{i+j} + (jq\lambda(p) - ip\lambda(q))r^{i+j-1} \quad (3)$$

By taking the case where $i = 0, j = 1$ and $b = 1$ we get

$\{p, r\} = \beta(p)r + \lambda(p)$, that is equation (3) becomes equation (1) . From equation (3), we have that for all $f, g \in P[r]$ $\{f, g\} = -\{g, f\}$ and that for a fixed element $f \in P[r]$, the k-linear map

$$\{f, .\} : P[r] \longrightarrow P[r] \ , \ g \to \{f, g\}$$
$$\{., f\} : P[r] \longrightarrow P[r] \ , \ g \to \{g, f\}$$

are derivation on $P[r]$ because β and λ are derivations. Now, by checking that for $pr^i, qr^j, tr^k \in P[r]$ the Jacobi identity

$$\{\{at^i, bt^j\} \ , ct^k\} + \{\{bt^j, ct^k\} \ , at^i\} + \{\{ct^k, at^i\} \ , bt^j\} = 0 \qquad (4)$$

We are going to use induction on i, j, k to proof that the identity in equation (4) holds for $pr^i, qr^j, tr^k \in P[r]$ Case 1 : Let $i = j = k = 0$, in equation (4), the result is trivial.

Case 2 : Let $p = 1, i = 1, j = 0, k = 0$ then equation (4) becomes $0 = \{\{r^1, qr^0\} \ , tr^0\} + \{\{qr^0, tr^0\} \ , r^1\} + \{\{tr^0, r^1\} \ , qr^0\} = \{\{r, q\} \ , t\} + \{\{q, t\} \ , r\} + \{\{t, r\} \ , q\}$. This result is obvious from equation (2).

Now lets supposed that for $j = 0$ and $k = 0$, equation (4) holds

Case 3 : for $i + 1, j = K = 0$ we have $\{\{pr^{i+1}, q\}, t\} + \{\{q, t\}, pr^{i+1}\} +$

$\{\{t, pr^{i+1}\}, q\}$
$= (\{\{pr^i, q\}, t\} + \{\{q, t\}, pr^i\} + \{\{t, pr^i\}, q\})r$
$+pr^i(\{\{r, q\}, t\} + \{\{q, t\}, r\} + \{\{t, r\}, q\})$
$= 0$ We get this by the Leibniz's law and the induction hypothesis. Thus for the case $j = k = 0$ equation (4) holds. From here we see that by respectively using induction on k and j, the proof is complete.

Result of Theorem 3.1

Let β and λ a derivation on a Poisson algebra P generated by a set S as an algebra.

1. If $\beta(s)$=$\lambda(s)$ for all $s \in S$, then $\beta = \lambda$
2. If $s\lambda(s) = \lambda(s)s$ for all $s \in S$ then $s\lambda = \lambda s$
3. If β satisfies $\beta(\{s_1, s_2\}) = \{\beta(s_1), s_2\} + \{s_1\beta(s_2)\}$ for all $s_1, s_2 \in S$ then β is a Poisson derivation
4. If β and λ satisfy equation (2) for all elements in S then β and λ satisfy equation (2) for all elements in P

Proof:

1) Let $\beta(s)$=$\lambda(s)$. Since β and λ are derivations we have

$$\beta(s_1s_2) = s_2\beta(s_1) + \beta(s_2)s_1 \text{ and } \lambda(s_1s_2) = s_2\lambda(s_1) + \lambda(s_2)s_1.$$

This implies that

$s_2\beta(s_1) + \beta(s_2)s_1 = s_2\lambda(s_1) + \lambda(s_2)s_1$

$s_2\beta(s_1) - s_2\lambda(s_1) + \beta(s_2)s_1 - \lambda(s_2)s_1 = 0$

$s_2(\beta(s_1) - \lambda(s_1)) + (\beta(s_2) - \lambda(s_2))s_1 = 0$

Since $s_1 \neq 0$ and $s_2 \neq 0$ we see that

$\beta(s_1) - \lambda(s_{s1}) = 0$ and $\beta(s_2) - \lambda(s_2) = 0$

Hence for all $s \in S$, $\beta(s)$=$\lambda(s)$

2) Let $s\lambda(s) = \lambda(s)s$

Since s can be any element of S we see that $s\lambda = \lambda s$ as required.

4. MODULES FOR POISSON ALGEBRAS

In this part, we entirely study the article by Farkas (Farkas, 2000) to understand the Poisson module structure.

Definition 4.1. For any Poisson algebra P and P-module M_P, M_P has a Poisson action given that there is k-bilinear map

$$\circ : P \times M_P \longrightarrow M_P$$

such that

$$p_1 \circ (p_2 m) = \{p_1, p_2\}m + p_2(p_1 \circ m) \tag{5}$$

for all $p_1, p_2 \in P$ and $m \in M_P$. Then, M_P is a Poisson a P-module if there is an action which can make M_P into a Lie algebra module over P and if it can satisfy the Leibniz's rule

$$(p_1 p_2) \circ m = p_1(p_2 \circ m) + p_2(p_1 \circ m) \tag{6}$$

Example 4.1. Let $M_P = D(P)$ where $D(P)$ is the ring over P with differential operators. We know that all d-derivations of P lies in $D(P)$; let $P.Ham(P)$ be the P-module generated by $Ham(P)$, we can say that

$P.Hamp \subseteq D(P)$. Therefore we can define

$$(p \circ m) = ham(p)m$$

This has a Poisson argument because

$$ham(p_1)p_2 = p_2 ham(p_1) + \{p_1, p_2\} \tag{7}$$

It is a Lie algebra argument because

$$\{ham(p_1), ham(p_2)\} = ham\{p_1, p_2\} \tag{8}$$

and finally

$$ham(p_1 p_2) = p_1 ham(p_2) + p_2 ham(p_1) \tag{9}$$

so the Leibniz's rule hold and thus the $D(P)$ module inherits the structure of a Poisson P-module.

Example 4.2. Assume that P is an integral domain with the zero bracket $\{P, P\} = 0$ and that we have a d-linear derivation $D \in Der_d(P)$ of P which is not zero. There are many such derivations when P is a

polynomial algebra in at least one indeterminate.
Let P to have an action given as follows

$$p_1(p_2 \circ m) = D(p_1)p_2m$$

Then

$$\begin{aligned} p_1(p_2 \circ m) &= D(p_1)(p_2 \circ m) \\ &= p_2(D(p_1) \circ m) + \{p_1, p_2\}m \\ &= p_2(p_1 \circ m) + \{p_1, p_2\}m \end{aligned}$$

for all $p_1, p_2 \in P$ and $m \in M_P$.
Here we see that $\{p_1, p_2\} = 0$, this is a Lie algebra action because P is a commutative associative algebra. The left Leibniz's rule holds because D is an associative algebra derivation. This action made P into Poisson module over itself, and there exist some $p \in P$ with $[p, -]$ different on the zero map, and yet we have that $ham(p) = 0$.

Example 4.3. All of the Poisson structures of a P-module can be be determined by a Poisson derivation of P on the associative module P itself.
If $p_1, p_2 \in P$ then, any Poisson module bracket satisfies

$$(p_1 \circ p_2) = \{p_1, p_2\} + p_2(p_1 \circ 1)$$

From here we can understand that every Poisson module action is given by a function,

$$\pi : P \longrightarrow P \ , \ \pi(p) = p \circ 1.$$

By basic computation we see that π is a derivation of both an associative algebra and that of a Lie algebra on itself. More so, every Poisson derivation φ of P gives a Poisson P-module with an associative P- module by

$$(p_1 \circ p_2) = \{p_1, p_2\} + p_2\varphi(p_1)$$

and denote by P_φ.

Proposition 4.1. Let P be a Poisson algebra and M_P be a P-module with a Poisson action then $M_S = Hom_p(M_P, P)$ has a Poisson action given by,

$$(p \circ s)(m) = \{p, s(m)\} - s(p \circ m)$$

for all $p \in P$, $s \in M_S$ and $m \in M_P$

Proof. Firstly, whenever $p \in P$ and $s \in M_S$ we verify that $(p \circ s) \in M_S$. It is obvious that $(p \circ s)$ is additive.

Now for all $p_1 \in P$,

$$\begin{aligned}
(p \circ s)(p_1 m) &= \{p, s(p_1 m)\} - s(p \circ (p_1 m)) \\
&= \{p, p_1 s(m)\} - s(\{p, p_1\}m + p_1(p \circ m) \\
&= \{p, p_1\}s(m) + p_1\{p, s(m)\} - (\{p, p_1\}s(m) - p_1 s(p \circ m) \\
&= p_1\{p, s(m)\} - p_1 s(p \circ m) \\
&= p_1(\{p, s(m)\} - s(p \circ m)) \\
&= p_1((p \circ s)(m))
\end{aligned}$$

Next we verify the property of a Poisson action. If $p_2 \in P$ then

$$\begin{aligned}
(p \circ (p_2 s))(m) &= \{p, (p_2 s)(m)\} - (p_2 s)(p \circ m) \\
&= \{p, p_2\}s(m) + p_2\{p, s(m)\} - p_2 s(p \circ m) \\
&= \{p, p_2\}s(m) + (p_2\{p, s(m)\} - p_2 s(p \circ m)) \\
&= \{p, p_2\}s(m) + p_2(p \circ s)m
\end{aligned}$$

Hence $(p \circ (p_2 s)) = \{p, p_2\}s + p_2(p \circ s)$

5. ENVELOPING ALGEBRA OF POISSON ALGEBRAS

In this section we benefit from the article written by Sei-Qwon Oh in (Oh, 1999).

Definition 5.1. For any P=$(P, ., \{,\})$ Poisson algebra on a field F, the set (R, θ, π) here, R is an algebra, θ is an algebra homomorphism from $(P, .)$ into R and π is a Lie homomorphism from $P, \{,\}$ into R_L so that for all $p_1, p_2 \in P$

$$\begin{gathered} \theta(\{p_1, p_2\}) = \pi(p_1)\theta(p_2) - \theta(p_2)\pi(p_1) \text{ and} \\ \theta(p_1, p_2) = \theta(p_1)\pi(p_2) + \theta(p_2)\pi(p_1) \end{gathered}$$

In this case R is known as Poisson enveloping algebra of P if the following is true: If B is an algebra, ξ is an algebra homomorphism from $(P, .)$ into B and η is a Lie homomorphism from $(P, \{,\})$ into B_L such that for all $p_1, p_2 \in P$

$$\begin{gathered} \xi(\{p_1, p_2\}) = \eta(p_1)\xi(p_2) - \xi(p_2)\eta(p_1) \text{ and} \\ \eta(p_1 p_2) = \xi(p_1)\eta(p_2) + \xi(p_2)\eta(p_1) \end{gathered}$$

Then there exist a unique algebra homomorphism w from R into B such that $w\theta = \xi$ and $w\pi = \eta$ (Oh, 1999).

$$\begin{array}{ccc} P & \xrightarrow{\theta,\pi} & R \\ & \searrow^{\xi,\eta} & \downarrow w \\ & & B \end{array}$$

Example 5.1. If we consider the Weyl algebra P_2 generated by the elements p_1, p_2, q_1, q_2 under the relations

$$p_1p_2 = p_2p_1$$
$$q_1q_2 = q_2q_1$$
$$q_ip_j - p_jq_i = \delta_{ij} \text{ for } i, j = 1, 2$$

see (Oh, 1999)
and we know that the commutative $k[p, q]$ ring of polynomial has a Poisson structure giving by

$$\{f, g\} = \frac{\partial f}{\partial p}\frac{\partial g}{\partial q} - \frac{\partial g}{\partial p}\frac{\partial f}{\partial q}$$

for all $f, g \in k[p, q]$ see Example 3.2.

We can see that there is an algebra homomorphism

$$\theta : k[p, q] \longrightarrow P_2 \text{ such that } p \mapsto q_2,\ q \mapsto p_1$$

The map above is well defined because $p_1 \in P_2$ commutes with $q_2 \in P_2$. Now by defining a k-linear map $\pi : k[p, q] \longrightarrow P_2$ given by

$$\theta(p^iq^j) = i\theta(p^{i-1}q^j)q_1 + j\theta(p^iq_{j-1})p_2$$

for all positive integers i, j. From here we can see that

$$\pi((p^iq^j)(p^rq^s)) = \theta(p^iq^j)\pi(p^rq^s) + \theta(p^rq^s)\pi(p^iq^j)$$
$$\theta(\{p^iq^j, p^rq^s\}) = \pi(p^iq^j)\theta(p^rq^s) - \theta(p^rq^s)\pi(p^iq^j)$$
$$\pi(\{p^iq^j, p^rq^s\}) = \pi(p^iq^j)\pi(p^rq^s) - \pi(p^rq^s)\pi(p^iq^j)$$

for all positive integers i, j, r, s, this implies that π is a Lie algebra homomorphism from $(k[p, q], \{,\})$ into $(P_2)_L$ such that

$$\pi(\{u, v\}) = \pi(u)\theta(v) - \theta(v)\pi(v) \text{ and } \pi(uv) = \theta(v)\pi(u) + \pi(v)\theta(u)$$

for all $u, v \in k[p, q]$.

Now, we are going to show that $(P_2), \theta, \pi$ is a Poisson enveloping algebra of $(k[u, v], ..\{,\})$.

Let U be an algebra , ξ to be an algebra homomorphism from $(k[u, v], ., \{,\})$ into U and η to be a Lie homomorphism from $(k[u, v], ..\{,\})$ into U_L such that

$$\xi(\{u, v\}) = \eta(u)\xi(v) - \xi(u)\eta(v) \text{ and } \eta(uv) = \xi(u)\eta(v) + \xi(v)\eta(u)$$

for all $u, v \in U$, then there is a unique algebra homomorphism $w : P_2 \longrightarrow U$ such that

$$w(p_1) = \xi(v),\ w(p_2) = \eta(v),\ w(q_1) = \eta(u),\ w(q_2) = \xi(u).$$

the equalities above hold because the following are true.

$$\begin{aligned}
w(p_1)w(p_2) - w(p_2)w(p_1) &= \xi(v)\eta(v) - \eta(v)\xi(v) = -\xi(\{v, v\}) = 0 \\
w(q_1)w(q_2) - w(q_2)w(q_1) &= \xi(u)\eta(u) - \eta(u)\xi(u) = \xi(\{u, u\}) = 0 \\
w(q_1)w(p_1) - w(p_1)w(q_1) &= \xi(u)\eta(v) - \eta(v)\xi(v) = \xi(\{v, u\}) = 1 \\
w(q_2)w(p_2) - w(p_2)w(q_2) &= \xi(v)\eta(u) - \eta(u)\xi(v) = -\xi(\{u, v\}) = 1 \\
w(q_1)w(p_2) - w(p_2)w(q_1) &= \eta(u)\eta(v) - \eta(v)\eta(u) = \eta(\{u, v\}) = 0 \\
w(q_2)w(p_1) - w(p_1)w(q_2) &= \xi(u)\xi(v) - \xi(v)\xi(u) = \xi(uv - vu) = 0
\end{aligned}$$

Thus (P_2, θ, π) is a Poisson enveloping algebra of $(k[u, v], ., \{,\})$

Proposition 5.1 Let P be a Poisson algebra and $E(P)= (E(P), \theta, \pi)$ be Poisson enveloping algebra of P and if M is a k-vector space, then M is a Poisson P-module if and only if M is a left $E(P)$-module. Oh (Oh, 1999)

Note. A P-vector space M_P is a Poisson P-module if and only if there exist an algebra homomophism θ and a Lie homomophism π from P into the linear map $End(M_P)$ such that,

$$\theta(\{p, q\}) = \pi(p)\theta(q) - \theta(q)\pi(p) \text{ and } \theta(p, q) = \theta(p)\pi(q) + \theta(q)\pi(p)$$

for all $p, q \in P$.

Proof. Let M_P be a Poisson P-module, from Note 1 we remark that there exist an algebra homomophism $\xi : P \longrightarrow End_k(M_P)$ and a Lie homomorphism $\eta : A \longrightarrow End_k(M_P)$ such that

$$\xi(\{p, q\}) = \eta(p)\xi(q) - \xi(p)\eta(q) \text{ and } \eta(pq) = \xi(p)\eta(q) + \xi(p)\eta(q)$$

for all $p, q \in P$ and from this there exist an algebra homomorphism $h : E(P) \longrightarrow End_k(M_P)$ such that

$$h \circ \theta = \xi \text{ , } h \circ \pi = \eta$$

We see that M_P has a left $E(P)$-module through h.

6. IDEALS OF POISSON ALGEBRAS

In this section, we will present ideals under Poisson algebra and give some lemmas in relation to Poisson ideals. The material used in this section is mostly from the work of Sasom (Sasom, 2012).

Definition 6.1. Let the algebra P be a Poisson algebra and $I \subseteq P$, an ideal I of P given by$\{i, p\} \in I$ for all $i \in I$ and all $p \in P$ is a Poisson ideal.

Definition 6.2. Let S be a Poisson algebra ideal of a Poisson algebra P. If S is both a prime ideal and a Poisson ideal, then we say that S is a Poisson prime ideal. From this Definition, we can also say that S is a Poisson ideal, and for all Poisson ideals $I, J \subseteq P$

$$\text{if } IJ \subseteq S \Longrightarrow I \subseteq S \text{ or } J \subseteq S$$

Definition 6.3. An ideal I of a Poisson algebra P is a Poisson maximal ideal if I is both a maximal ideal of P and a Poisson ideal.
we say that I is a maximal Poisson ideal of P if

$$J \text{ is a Poisson ideal and } I \nsubseteq J \text{ then } J = P$$

It should be noted that Poisson maximal ideal is not equivalent to maximal Poisson ideal. For Example, let $P = \mathbb{C}\{p, q\}$ which is a Poisson algebra with a Poisson bracket given by $\{p, q\} = 1$ Then 0 is a maximal Poisson ideal but not a Poisson maximal ideal.

Example 6.1 Let consider the Poisson algebra $P = C[p_1, p_2, p_3]$ given Example 3.3, that is for an element $f \in P$ the Poisson bracket is defined by

$$\{g, h\}_f = \begin{vmatrix} f_{p_1} & f_{p_2} & f_{p_3} \\ g_{p_1} & g_{p_2} & g_{p_3} \\ h_{p_1} & h_{p_2} & h_{p_3} \end{vmatrix}$$

for all $g, h \in P$.

From this we claim that $fP = \{f.p : p \in P\}$ is a Poisson ideal of P.

Solution: It is enough to show that $\{fg, h\} \in fP$ for all $g, h \in P$.

$$\{fg, h\} = \begin{vmatrix} f_{p_1} & f_{p_2} & f_{p_3} \\ (fg)_{p_1} & (fg)_{p_2} & (fg)_{p_3} \\ h_{p_1} & h_{p_2} & h_{p_3} \end{vmatrix}$$

$$= f_{p_1}((fg)_{p_2}h_{p_3} - h_{p_2}(fg)_{p_3}) - f_{p_2}((fg)_{p_1}h_{p_3} - h_{p_1}(fg)_{p_3}) +$$
$$f_{p_3}((fg)_{p_1}h_{p_2} - h_{p_1}(fg)_{p_2})$$
$$= f_{p_1}(f_{p_2}gh_{p_3} + fg_{p_2}h_{p_3} - h_{p_2}f_{p_3}g - h_{p_2}fg_{p_3}) -$$
$$fp_2(f_{p_1}gh_{p_3} + fg_{p_1}h_{p_3} + h_{p_1}f_{p_3}g - h_{p_1}fg_{p_3})$$
$$+fp_3(f_{p_1}gh_{p_2} + fg_{p_1}h_{p_2} - h_{p_1}f_{p_2}g - h_{p_1}fg_{p_2})$$
$$= f(f_{p_1}g_{p_2}h_{p_3} - f_{p_1}h_{p_2}g_{p_3} - f_{p_2}g_{p_1}h_{p_3} +$$
$$f_{p_2}h_{p_1}g_{p_3} + f_{p_3}g_{p_1}h_{p_2} - f_{p_3}h_{p_1}g_{p_2})$$

This shows that fP is a Poisson ideal of P because

$$f(f_{p_1}g_{p_2}h_{p_3}-f_{p_1}h_{p_2}g_{p_3}-f_{p_2}g_{p_1}h_{p_3}+f_{p_2}h_{p_1}g_{p_3}+f_{p_3}g_{p_1}h_{p_2}-f_{p_3}h_{p_1}g_{p_2}) \in fP$$

Definition 6.4. Let N_R be a left module over a ring R. Given any subset $Y \subseteq N_R$, the annihilator of Y is the set

$$ann_R(Y) = \{r \in R : ry = 0, \forall y \in Y\}$$

which is a left ideal of P

Proposition 6.1. Let P be a Poisson algebra and M_P be a Poisson P-module.
(a) The annihilator $ann_P(M_P)$ is a Poisson ideal of P;
(b) if M_P is a simple Poisson module, then $ann_P(M_P)$ is a Poisson prime ideal of P;
(c) if M_P is a finite dimensional simple Poisson module then $ann_P(M_P)$ is a Poisson maximal ideal of P

Proof:
(a) Let $pM_P = 0$ for some $p \in P$. It follows from Definition 4.1 that

$$0 = p \circ (qm) = \{p, q\}m \text{ for all } p \in P.$$

This shows that $\{p, q\} \in ann_P(M_P)$ thus $ann_P(M_P)$ is a Poisson ideal of P.

(b) Let M_P a simple Poisson module and let I, J be Poisson ideals of P . Suppose that $IJ \subseteq ann_P(M_P)$ this implies that $IJM_P = 0$. Now we are going to show that JM_P is a Poisson submodule of M_P. Let $j \in J$

and $m \in M_P$. For $p \in P$, $p \circ (jm) = j(p \circ m) + \{p, j\}m \in JM_P$ which implies that JM_P is a Poisson submodule of M_P. Considering that M_P is a simple Poisson module, then

$$JM_P = 0 \text{ or } JM_P = M_P$$

If $JM_P = M_P$, this implies that $IM_P = IJM_P = 0$, this shows that

$$I \subseteq ann_P(M_P) \text{ or } J \subseteq ann_P(M_P)$$

Hence $ann_P(M_P)$ is a Poisson prime ideal of P

(c) Let M_P be a finite dimensional simple Poisson module and I, J be Poisson ideals of P. By (b) , $ann_P(M_P)$ is a Poisson prime ideal of P. With this, M_P is a faithful $P/ann_P(M_P)$-module. Let φ be the transformation from $P/ann_P(M_P)$ to the endomorphism ring of M_P $End_C(M_P)$ given by

$$\varphi(a + ann_P(M_P))= pm$$

for all $p \in P$ and $m \in M_P$. We consider φ to be an injective $\mathbb{C}$-homomorphism

(i) Let $p, q \in P, p \neq q$ and $m \in M_P$ then $\varphi(p + ann_P(M_P)) = pm$ and $\varphi(q + ann_P(M_P)) = qm$

Now, let $\varphi(p + ann_P(M_P))$=$\varphi(+ann_P(M_P))$, this implies that

$$pm = qm$$
$$\Rightarrow pm - qm = 0$$
$$\Rightarrow (p - q)m = 0$$
$$\text{since } m \neq 0$$
$$\Rightarrow p = q$$

Hence φ is well defined.

(ii) Let $p, q \in P$ then $\varphi((p + ann_P(M_P))(q + ann_P(M_P))) = pqm$

$\Rightarrow pqm = \varphi(p + ann_P(M_P))qm$

$\Rightarrow pqm = \varphi(p + ann_P(M_P))\varphi(q + ann_P(M_P))m$ thus

$\varphi((p + ann_P(M_P))(q + ann_P(M_P))) = \varphi(p + ann_P(M_P))\varphi(q + ann_P(M_P))$ hence φ is a C-homomorphism.

(iii) Let $p+ann_P(M_P) \in P/ann_P(M_P)$ be such that $\varphi(P+ann_P(M_P)) = 0$. This implies that $pm = 0$ for all $p \in ann_P(M_P)$ and all $m \in M_P$, with this, we have $ker\varphi = 0$. Hence φ is one-to-one

$dim_{\mathbb{C}}(M_P) < \infty$

$dim_{\mathbb{C}}(P/ann_P(M_P)) \leqslant dim_{\mathbb{C}}End_{\mathbb{C}}(M_P) = (dim_{\mathbb{C}}(M_P))^2$.

Since $P/ann_P(M_P)$ is a finite dimensional algebra over $\mathbb{C}$, then $P/ann_P(M_P)$ is an Artinian ring (A ring that satisfies the descending chain condition on ideals; that is there is no infinite descending sequence of ideals) . From (b) we saw that $P/ann_P(M_P)$ is a prime ideal as well thus $ann_P(M_P)$ is a prime ring. Thus $A/ann_A(M)$ is simple hence $A/ann_A(M)$ is a maximal.

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CURRICULUM VITAE

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