# PARALLEL TETRAHEDRAL MESH REFINEMENT

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## ABSTRACT

# PARALLEL TETRAHEDRAL MESH REFINEMENT

The Adaptive Mesh Refinement is one of the main techniques used for the solution of Partial Differential Equations. Since *3*-dimensional structures are more complex, there are few refinement methods especially for parallel environments. On the other hand, many algorithms have been proposed for *2*-dimensional structures. We analyzed the Rivara's *longest-edge* bisection algorithm, studied parallelization techniques for the problem, and presented a parallel methodology for the refinement of non-uniform tetrahedral meshes. The proposed algorithm is practical for real-life applications and it is also *scalable* for large mesh structures. We describe a usable data structure for distributed environments and present a utility using the inter-process communication. The *PTMR* utility is capable of distributing the mesh data among processors and it can accomplish the refinement process within acceptable time limits.

# ÖZET

## PARALEL DÖRTYÜZLÜ ÖRGÜ İYİLEŞTİRME

Uyarlanmış Örgü İyileştirme, Parçalı Türevsel Denklemlerin çözümünde kullanılan ana yöntemlerden biridir. Üç boyutlu sistemler daha karmaşık olduğundan, özellikle paralel ortamlar için az sayıda arıtma/iyileştirme yöntemi mevcuttur. Buna rağmen iki boyutlu yapılar için birçok algoritma önerilmiştir. Rivara'nın en uzun kenar bölme tekniği incelenmiş, problemin paralel algoritma ile çözümü çalışılmış ve düzensiz dörtyüzlü örgüler için paralel yöntem sunulmuştur. Önerilen algoritma gerçek uygulamalar için pratik ve büyük örgü yapıları için ölçeklenebilirdir. Dağıtık sistemler için kullanılabilir bir veri yapısı anlatılmış ve işlemciler arası haberleşmeyi kullanan bir uygulama sunulmuştur. PTMR uygulaması örgü bilgisini işlemcilere dağıtıp kısa zamanda iyileştirme işlemini yapabilmektedir.

# TABLE OF CONTENTS

ACKNOWLEDGEMENTS . . . iii
ABSTRACT . . . iv
ÖZET . . . v
LIST OF FIGURES . . . viii
LIST OF SYMBOLS/ABBREVIATIONS . . . xi
1. INTRODUCTION . . . 1
1.1. Contributions . . . 2
1.2. Outline . . . 3
2. MESH REFINEMENT . . . 4
2.1. Refinement of Unstructured Triangulation . . . 5
2.2. Analysis of the Propagation Path . . . 7
2.3. Skeleton Algorithms . . . 10
2.4. 8-Tetrahedra Longest Edge (*8T-LE*) . . . 12
2.5. *3-D* Sequential Algorithm . . . 15
3. PARALLEL ALGORITHM . . . 16
3.1. The Longest-Edge Propagation Graph . . . 17
3.2. Longest-Edge Refinement . . . 18
3.3. *2-D* versus *3-D* . . . 20
3.4. Algorithm for Distributed Environments . . . 21
4. IMPLEMENTATION DETAILS . . . 27
4.1. Data Structure . . . 29
5. TEST RESULTS . . . 34
6. CONCLUSIONS . . . 42
APPENDIX A: PTMR: Reference . . . 43
A.1. Utilities . . . 44
A.1.1. Global Definitions . . . 45
A.1.2. Common Utilities . . . 46
A.1.3. The Dynamic Pointer Array . . . 46
A.1.4. The Pointer Table . . . 46

A.1.5. Input/Output File Formats . . . . . . . . . . . . . . . . . . . . . 47
A.2. Mesh Distribution . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 49
A.2.1. Processor Ranks . . . . . . . . . . . . . . . . . . . . . . . . . 49
A.2.2. Processor Mapping . . . . . . . . . . . . . . . . . . . . . . . . 50
A.2.3. Mesh Partitioning . . . . . . . . . . . . . . . . . . . . . . . . . 51
A.3. Mesh Operations . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 55
A.3.1. The Point Object . . . . . . . . . . . . . . . . . . . . . . . . . 55
A.3.2. The Edge Object . . . . . . . . . . . . . . . . . . . . . . . . . 57
A.3.3. The Tetrahedron Object . . . . . . . . . . . . . . . . . . . . . 58
A.3.4. The Tetra Bucket . . . . . . . . . . . . . . . . . . . . . . . . . 61
A.4. The Refinement Process . . . . . . . . . . . . . . . . . . . . . . . . . 62
A.4.1. The Structure of a Local Mesh . . . . . . . . . . . . . . . . . . 62
A.5. Communication and the *LEPP* Synchronization . . . . . . . . . . . . . 66
A.5.1. The Communication Array . . . . . . . . . . . . . . . . . . . . 66
A.5.2. The LEPP Facility . . . . . . . . . . . . . . . . . . . . . . . . 68
APPENDIX B: PTMR: Manual . . . . . . . . . . . . . . . . . . . . . . . . 70
B.1. Compilation . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 70
B.2. Libraries . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 73
B.3. Testing . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 74
B.3.1. Samples . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 76
REFERENCES . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 77

# LIST OF FIGURES

Figure 2.1. Longest-Edge Bisection of Triangle $t_0$ . . . . . . . . . . . . . . . . 6

Figure 2.2. Longest-Edge Bisection Algorithm . . . . . . . . . . . . . . . . . . 7

Figure 2.3. Longest-Edge Propagation . . . . . . . . . . . . . . . . . . . . . . 8

Figure 2.4. *4T-LE* Refinement . . . . . . . . . . . . . . . . . . . . . . . . . 8

Figure 2.5. Longest-Edge Propagation Path . . . . . . . . . . . . . . . . . . . 9

Figure 2.6. *2-D* Skeleton . . . . . . . . . . . . . . . . . . . . . . . . . . . . 11

Figure 2.7. Refinement Patterns of 4-Triangle . . . . . . . . . . . . . . . . . . 12

Figure 2.8. 4-Tetrahedra and 8-Tetrahedra . . . . . . . . . . . . . . . . . . . . 13

Figure 2.9. *3-D* Refinement Algorithm . . . . . . . . . . . . . . . . . . . . . 15

Figure 3.1. *LEPP*-Graph . . . . . . . . . . . . . . . . . . . . . . . . . . . . 18

Figure 3.2. *3-D* Skeleton Refinement Algorithm . . . . . . . . . . . . . . . . 19

Figure 3.3. Propagation Path . . . . . . . . . . . . . . . . . . . . . . . . . . . 20

Figure 3.4. Longest-Edge Selection . . . . . . . . . . . . . . . . . . . . . . . . 23

Figure 3.5. Distributed Algorithm . . . . . . . . . . . . . . . . . . . . . . . . 24

Figure 3.6. *LEPP*-Graph Partitioning and Synchronization . . . . . . . . . . . 25

Figure 3.7. The *PTMR* Algorithm . . . . . . . . . . . . . . . . . . . . . . . 26

Figure 4.1. Statistics for the Number of Entities of a Common Mesh . . . . . 30

Figure 4.2. Data Structure of PTMR . . . . . . . . . . . . . . . . . . . . . 31

Figure 4.3. Mesh Refinement Example . . . . . . . . . . . . . . . . . . . . 31

Figure 4.4. Mesh Refinement Example . . . . . . . . . . . . . . . . . . . . 32

Figure 4.5. The Object Relationship of the *PTMR* Utility . . . . . . . . . . . 33

Figure 5.1. 14904 tetrahedra / 4502 vertices . . . . . . . . . . . . . . . . . 35

Figure 5.2. 1803 tetrahedra / 527 vertices . . . . . . . . . . . . . . . . . . 36

Figure 5.3. 6670 tetrahedra / 2021 vertices . . . . . . . . . . . . . . . . . . 37

Figure 5.4. 12586 tetrahedra / 3644 vertices . . . . . . . . . . . . . . . . . 37

Figure 5.5. 99121 tetrahedra / 23351 vertices . . . . . . . . . . . . . . . . 38

Figure 5.6. 70203 tetrahedra / 22568 vertices . . . . . . . . . . . . . . . . 38

Figure 5.7. 79263 tetrahedra / 33098 vertices . . . . . . . . . . . . . . . . 39

Figure 5.8. The Refinement Process According to the *LEPP* Algorithm (elapsed time spent in the gateway node) . . . . . . . . . . . . . . . . . . . 40

Figure 5.9. The Refinement Process According to the *LEPP* Algorithm . . . . 40

Figure 5.10. Overall Time . . . . . . . . . . . . . . . . . . . . . . . . . . . 41

Figure A.1. The Distributed CSR Format . . . . . . . . . . . . . . . . . . . . . . . 51

Figure A.2. The Communication Array . . . . . . . . . . . . . . . . . . . . . . . 67

# LIST OF SYMBOLS/ABBREVIATIONS

| | |
|---|---|
| $\alpha$ | angle |
| $e$ | edge |
| $E_{involved}$ | list of selected edges |
| $\mathsf{T}$ | tetrahedron |
| $f$ | face |
| $LEPP(t)$ | longest-edge propagation path for triangle $t$ |
| $M_1(t)$ | sum of the lengths of longest-edge propagation paths for triangle $t$ |
| $M_2(t)$ | the maximum length of longest-edge propagation paths for triangle $t$ |
| $t$ | triangle |
| $T$ | triangular mesh |
| $\tau$ | tetrahedral mesh |
| $v$ | vertex |
| | |
| *1-D* | 1-dimensional |
| *2-D* | 2-dimensional |
| *3-D* | 3-dimensional |
| *8T-LE* | 8-Tetrahedron Longest-Edge |
| *4T-LE* | 4-Triangle Longest-Edge |
| *AMR* | Adaptive Mesh Refinement |
| *DAG* | Directed-Acyclic Graph |
| *LE* | Longest-Edge |
| *LEPP* | Longest-Edge Propagation Path |
| *MPI* | Message Passing Interface |
| *PDE* | Partial Differential Equation |
| *PTMR* | Parallel Tetrahedral Mesh Refinement |

# 1. INTRODUCTION

Adaptive Mesh Refinement (*AMR*) is a technique used to effectively solve numerical systems of Partial Differential Equations (*PDE*). Instead of processing the uniform mesh in which grid points are evenly spaced, we place more grid points to the areas where local error is large in the solution. The adaptive mesh refinement is the preferred methodology in terms of computational and storage requirements. Refinement algorithms begin with an initial mesh conforming to a particular geometry, and the conformity of the overall structure must be preserved after partitioning an element [1, 2]. Most of the research have focused on mesh components as line segments in *1*-dimension, triangles in *2*-dimension, and tetrahedra in 3-dimension [3, 4].

The triangle refinement process have been studied briefly in recent studies [5, 6, 7]. Since we cannot analyze the elements in a planar view, *3*-dimensional structures yield to complexities and difficulties. Refining using the skeleton structure is the main idea behind the algorithms [1]; *3*-dimension is reduced to *2*-dimension, and then to *1*-dimension. The skeleton structure of meshes in all views should preserve conformity, and partitioning of the original mesh is refined according to the information in previous skeleton structures.

The main intention behind this research is to enhance the 8-Tetrahedra Longest-Edge (*8T-LE*) technique [3] and propose a parallel methodology applicable in real life.

The algorithm can be analyzed in several steps, and the crucial part is to find elements that must be refined to make a conforming mesh. Propagation of the refinement and the relationship between elements of tetrahedra is converted to a graph representation. The proposed data structure enables us to process the refinement operation rapidly with parallel methods and to compute local elements independently. Since the proposed representation is based on the *8T-LE* refinement algorithm, it preserves the required information for refinement and conformity.

Problem size and computational cost grow very rapidly in 3-dimensional refinement algorithms. Since *3-D* structures have complexities both in terms of theoretical limits and the amount of resources required for computation, we propose a novel methodology for inter-process communication environments. The Parallel Tetrahedral Mesh Refinement (*PTMR*) utility handles *longest-edge* bisection locally on each processing node and merges the results to produce the refined mesh data.

Processing large mesh structures is another crucial topic for Finite Element problems. During the process of the Differential Equations, the size of the used memory will increase since the geometry of the mesh should be extracted for computation. We cannot locate all of the required elements on a single machine; therefore, we should distribute both computational power and the amount of stored data among distributed processors. The proposed parallel refinement framework is capable of distributing the mesh structure over processing nodes, and it can scale up to the meshes with excessive size.

The overall mesh structure can be partitioned in order to fit into the local memory of computational nodes, and the Rivara's longest edge bisection technique can be used to process the refinement operation locally; results in each node can be synchronized in a proper and efficient way using an appropriate data structure to handle the refinement process in a parallel manner, in which the resulting mesh data is parallely constructed.

### 1.1. Contributions

This work analyzes the *longest-edge bisection* procedure in details and presents a novel methodology solving the refinement problem in parallel. It also proposes a data structure to store elements efficiently, so refinement and bisection processes can be accomplished by preserving the acceptable time limits. The *PTMR* is a scalable utility implemented for Message-Passing (*MPI*) environments and can handle very complex mesh structures. The major contributions of this study are as follows;

- A refinement framework for Finite Element problems.

- The data structure for the mesh operations in distributed environments.
- A brief study about the parallelization of *3-D* mesh refinement algorithms.

## 1.2. Outline

The organization of this study is as follows. In Chapter 2, we explain the longest-edge bisection algorithms and describe the *skeleton* concept in mesh refinement. In Chapter 3, we present methodologies applicable in parallel mesh refinement and also describe the proposed algorithm in this work. The next chapter is about the implementation details and the utilized data structure. In Chapter 5, we present examples to analyze the performance of the proposed algorithm. Finally, a brief description of the *PTMR* utility is given in Appendices.

# 2. MESH REFINEMENT

Many numerical applications and simulations, solid modeling, and computer graphics require geometric objects to be partitioned into smaller pieces in order to process and solve related problems. Triangulating a set of points is the basic tool in finite element method and computational geometry [8, 9, 10]. Therefore, mesh refinement algorithms have a critical role in adaptive finite elements of numerical computations. Especially, *3*-dimensional structures have difficulties in construction of good quality and adapted to geometry solutions [1, 11, 12, 10, 13, 14, 15, 16]. Problem size and computational cost grow very rapidly in *3*-dimensional refinement algorithms, and refinement techniques are usually generalized to *3-D* structures after resolving with theoretical acceptance for *2*-dimensional problems [3, 4, 15].

Two approaches have been mainly used to overcome the refinement problem in *2-D*. The first approach is the *longest-edge bisection* process which guarantees a good-quality, conforming mesh structure with linear time complexity [1, 9, 15, 17]. The second approach is based on the *Delaunay algorithm*, which can be summarized as adding non-vertex points in the circumcenter of the worst triangles of the current structure [18, 19, 20]. *Delaunay* refinement assures the construction of most equilateral triangulation at the optimum time complexity $O(NlogN)$ for a given mesh structure of $N$ vertices [18]. However, the second method cannot be applied easily to *3*-dimensions, and new approaches are needed for tetrahedral mesh refinement using a *Delaunay* triangulation based construction [18, 21]. Therefore, *Longest-Edge Bisection* method is mostly applied due to its straightforward and common implementation in the refinement process.

In this chapter, we present a brief summary of the known longest-edge bisection methodologies both for *2*-dimensional and *3*-dimensional structures. Skeleton algorithms are described in details since the skeleton concept is the basic idea behind the 4-Triangle Longest Edge (*4T-LE*) and 8-Tetrahedron Longest Edge (*8T-LE*) techniques. We also analyze the longest-edge propagation path (*LEPP*) and explain the

characteristics of the *LEPP* for *2-D* and *3-D* mesh structures. Finally, the sequential algorithm based on the concept of longest-edge bisection is demonstrated.

### 2.1. Refinement of Unstructured Triangulation

Triangular Mesh structures, *2*-dimensional, are basically used for numerical solutions such as surface evaluation of finite element problems that are more regular compared to *3*-dimensional problems. In the *2*-dimensional mesh refinement process, we shall use the *longest-edge refinement* algorithm which always concludes with conforming unstructured triangulation [5, 6, 7, 22]. The algorithm is based on *longest-edge bisection* of triangles in which the unique longest-edge of the mesh element is always bisected initially. A bisected edge influences the neighbor elements and triggers them to be refined. Bisection according to longest-edge supplies the conformity of the overall structure [4, 5, 23].

Intersection of adjacent triangles is either a common vertex or common edge. For any triangular mesh structure $T$, the longest-edge propagation path ($LEPP$) for a triangle $t_0$ is the ordered list of triangles $\{t_0, t_1, t_2, ..\}$, such that $t_i$ is adjacent to triangle $t_{i-1}$ by the longest-edge ($LE$) of $t_{i-1}$ [1, 4, 9, 12, 24]. Evaluating and computing the longest-edge propagation path means regarding the elements that will be bisected before partitioning any of the triangles since the longest-edge bisection method finds out effected neighbor triangles [5, 9, 23, 25].

$LEPP(t_0)$, the longest-edge or longest-side propagation path of triangle $t_0$, is always finite and longest-edges in the list of triangles have always increasing lengths [1, 2, 3, 4, 5, 6, 7, 23] .

The longest-edge of any triangle $t_n$ as the last element in the ordered list of $LEPP(t_0)$, is either on the border of bounded *2*-dimensional geometry with a length greater than the longest-edge of $t_{n-1}$ or it shares a common longest edge with triangle $t_{n-1}$ which is also the adjacent triangle. Two adjacent triangles that share common longest-edges are defined as pair of *terminal triangles* [4, 5, 20]. If the longest-edge of a

triangle is on the boundary of the mesh structure, it is defined as the *terminal boundary triangle* [17, 26]. Both types of triangles are terminal points in the *propagation-path*; that ordered list will not spread from these origins [2, 4, 20, 24, 25, 27].

Two neighboring triangles that share a common longest-edge will be called a *pair*. If a triangle does not belong to any pair of terminal triangles, it will be called a *single* triangle [24].

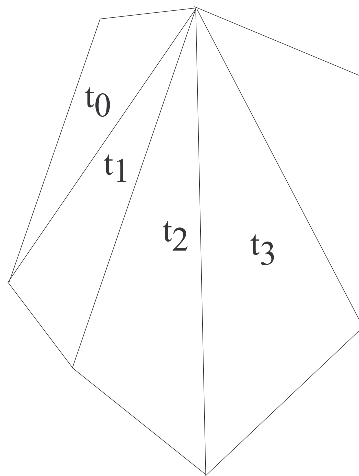



Figure 2.1. Longest-Edge Bisection of Triangle $t_0$

Figure 2.1 shows the $LEPP(t_0)$=$\{t_0, t_1, t_2, t_3\}$, propagation path for triangle $t_0$. Triangulation of refinement problems can be solved via evaluating the longest-edge propagation path. We can compute the *LEPP* and bisect elements in the list to accomplish the refinement process. If the triangle is a terminal boundary triangle, we bisect $t$; otherwise, we bisect the last pair of terminal triangles in the *LEPP* [3, 5, 6, 7, 17, 23]. In the given example, terminal triangle pairs $t_3 - t_2$, $t_2 - t_1$, $t_1 - t_0$ will be bisected in order. This procedure, starting from an initial conforming geometry, will produce a good-quality nested triangulation with linear time complexity [2, 4, 20].

The Longest-edge Propagation Path algorithm, shown in Figure 2.2, can be generalized to *3*-dimensional tetrahedral mesh structures. The *3-D LEPP* for a tetrahedron T, is the set of neighboring tetrahedra that have adjacent longest-edge greater or equal to the preceding tetrahedra in the list [3, 4, 9, 12, 24, 26]. A *terminal tetrahedra set* is the set of tetrahedra that share a common longest-edge [4].

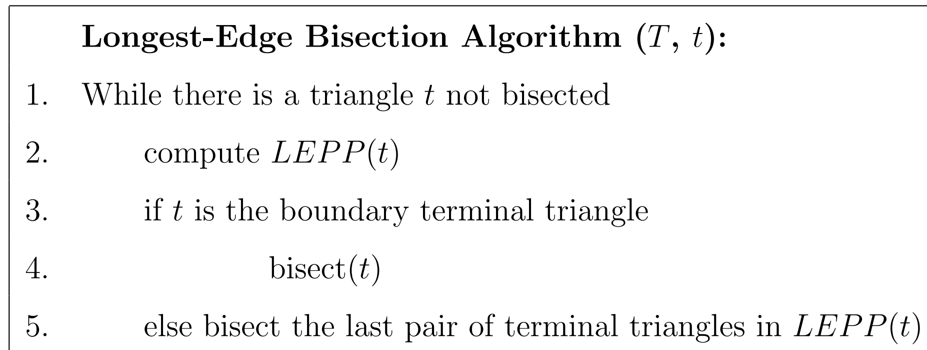

**Longest-Edge Bisection Algorithm ($T$, $t$):**
1. While there is a triangle $t$ not bisected
2. compute $LEPP(t)$
3. if $t$ is the boundary terminal triangle
4. bisect($t$)
5. else bisect the last pair of terminal triangles in $LEPP(t)$

Figure 2.2. Longest-Edge Bisection Algorithm from Rivara [4]

## 2.2. Analysis of the Propagation Path

Numerical experiments have demonstrated asymptotic behavior of the refinement process and some other characteristics of the generated mesh sequences. While analyzing the refinement process, we can describe it basically as inserting former vertices to result in well-shaped and conforming triangles. The produced mesh structure should not lead to unexpected effects in terms of numerical stability and accuracy [2, 7].

Mesh conformity and mesh smoothness can be summarized as [6, 24];

- Intersection of adjacent triangles is either a common vertex or an edge.
- Transition between small and large elements should be gradual.

Longest-edges are bisected progressively so all angles in triangle refinement are greater or equal to half of the smallest angle in the initial mesh geometry [1, 4, 5, 6, 20, 23]. Thus, known longest-edge refinement algorithms guarantee the construction of smooth and conforming structures.

Longest-edge bisection can propagate to the entire mesh in worst cases. Propagation is accomplished by traversing the Longest-edge Neighbor Triangles of triangle $t$. The neighboring triangle of $t$ is the triangle that shares the longest-edge with $t$. Figure 2.3 describes an eccentric situation. However, theoretical results and experiments show that successive processing of unstructured triangular mesh refinement results in mesh structures in which the average propagation path is reduced in each refinement stage

and approaches to the constant of *5* [24].

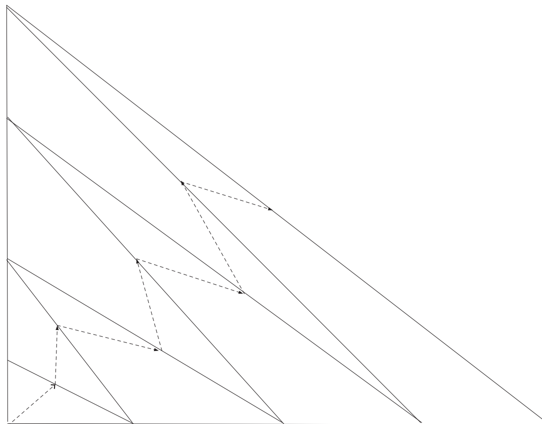


Figure 2.3. Longest-Edge Propagation

Four Triangle Longest Edge Partition(*4T-LE*) divides a triangle $t$ into four sub-triangles such that the original triangle is bisected by its longest edge and then two new resulting triangles are bisected by joining the midpoint of the longest-edge with midpoints of the edges from original triangle $t$. In a single refinement stage, a triangle can be bisected into four sub-triangles [2]. According to the processing of the algorithm and evaluation of *LEPP*, a triangle can be partitioned into lesser sub-triangles. Figure 2.4 shows the possible *4T-LE* if the longest-edge is chosen for bisection.

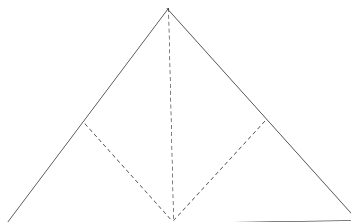


Figure 2.4. *4T-LE* Refinement

Another concept defining the characteristics of the *LEPP* is the balancing degree. If triangular geometry contains $n$ triangles and $m$ of them are in pairs of terminal triangles, then the *LEPP*-balancing degree of triangulation is defined as $\frac{n}{m}$. *LEPP*-balancing $=$ 1 means that all triangles are in terminal pairs and that there is no terminal boundary triangle [24]. $M_1(t)$ is defined as the sum of the lengths of the *LEPP*s' of the neighbors of triangle $t$. $M_2(t)$ is defined as the maximum length of

the *LEPP*s' of the neighbors of triangle $t$. For a triangular mesh structure in which *LEPP*-balancing degree $\simeq 1$, $M_1(t) \simeq 5$ and $M_2(t) \simeq 2$ [24].

For *2*-dimensional mesh refinement, we can restrict the length of the propagation path. This behavior is crucial in terms of analyzing the algorithms and performance of the bisection process. However, such a limitation cannot be stated for *3*-dimensional tetrahedral meshes. It should be noted that this property is the most important difference between *2*-dimensional and *3*-dimensional refinement algorithms; lack of such a limiting definition ($M_1(t) \simeq 5$ in *2-D*) results in the challenge for *3*-dimensional problems.

*LEPP* of a triangle $t$ is always finite and elements in the propagation list have strictly increasing edge lengths [4]. Terminal-edge points to the end of a propagation list in which a refinement path will not propagate anymore. They are utilized in refinement algorithms for *2-D* and *3-D* structures to define stopping points in a propagation graph. The edge which is longest in the near area is supposed to be a terminal-edge, so all terminal edges can be refined safely until the mesh structure conforms specifications. Refining a terminal-edge will not affect conformity of the mesh [24, 26].

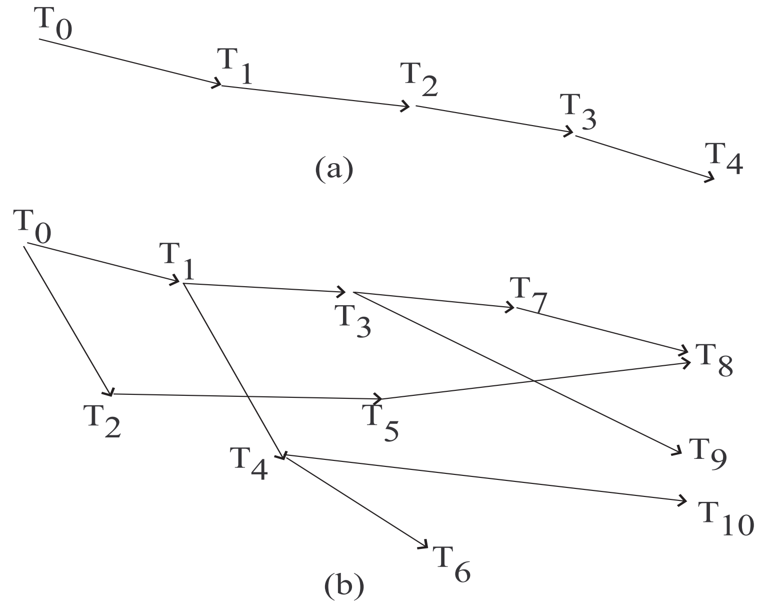



Figure 2.5. Longest-Edge Propagation Path (a) *2-D* (b) *3-D*

In *2-D* meshes, a *LEPP-graph* forms a forest since each edge can only have two neighbor triangles. Each tree in the forest can be used to find the elements that should be refined for a conforming mesh structure [28].

A propagation path for *3-D* mesh forms a directed-acyclic graph (*DAG*) such that each longest edge can be shared by many triangles and refinement operation can propagate in many directions. Therefore, the Longest-edge propagation graphs for *3-D* meshes are *denser*. Figure 2.5 is an example of *LEPP* for *2-D* and *3-D* structures.

The *LEPP* graph can be sequentially processed with $O(n)$ time complexity [1, 2, 6, 7, 29, 30]. Parallel implementation can be handled in $O(logn)$ time if the structure is *2-D* [28]. However, it cannot be stated for *3-D* mesh, since the number of edges in the propagation graph is not linearly related to the number of elements as in a *LEPP* graph of a *2-D* mesh.

On the other hand, numerical experiments demonstrate an asymptotic behavior of mesh sequences for the *4T-LE* refinement. The propagation path extend to a few neighboring adjacent triangles [24]. This information is used to analyze and prepare a refinement algorithm for *2-D* triangle meshes. The number of elements that should be refined is related to the elements in the *LEPP* graph, and this property proves the practical advantages for longest-edge mechanism.

### 2.3. Skeleton Algorithms

The *8T-LE* partition is defined in terms of a polyhedron skeleton concept using a simple edge-midpoint bisection procedure. The *2*-dimensional algorithm, which is also formulated as 4-triangle longest-edge, works over wireframe meshes containing the edges of target triangles and some neighboring triangles to prepare a conforming structure [15, 2, 25, 27].

Information in the lower dimension is used to partition appropriate triangles. The 8-tetrahedra algorithm is the generalization of skeleton algorithms in *3*-dimensions.

Figure 2.6 represents the *2*-dimensional view of a tetrahedron. Volume refinement is based on the partitioned triangular faces of the tetrahedron in *2*-dimension [1, 3, 4]. Four-triangle partitions or partial partitions of neighboring triangles are accomplished by using edge bisection; that is, by refining the wireframe mesh of the *3*-dimensional edges of tetrahedra [12, 25].

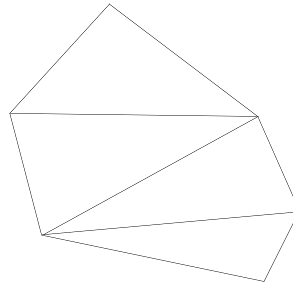


Figure 2.6. *2-D* Skeleton

The 4-Triangle algorithm produces a finite number of distinct triangles that are embedded in the parent. Figure 2.7.a shows the partition patterns of 4-Triangles longest-edge. The refinement process generates triangles that have the smallest angle greater or equal to $\alpha/2$ where $\alpha$ is the smallest angle in the original mesh [6]. Continuing refinement iterations produce a stable molecule around a vertex for a local refinement of a conforming triangulation; angles of the vertex are not divided in the next operations [6, 7, 22]. Figure 2.7.b shows the fractal property of 4-Triangle method.

The skeleton algorithm for 4-Triangle mesh refinement can be analyzed in two steps; bisecting the edges in a *1*-dimensional skeleton and partitioning individual triangles according to the bisected edges [9, 25]. The *3*-dimensional skeleton algorithm is a generalized version of 4-Triangles. If tetrahedral mesh $\tau$ is conforming, then *2-D* skeleton, which is the triangular faces of the elements of $\tau$, is also conforming [3]. Moreover, a *1-D* skeleton of $\tau$ tetrahedral mesh is a conforming wireframe mesh of the elements of $\tau$ [3, 9, 25].

We can basically define the 3D algorithm as;

- Partition edges over *1-D* skeleton.

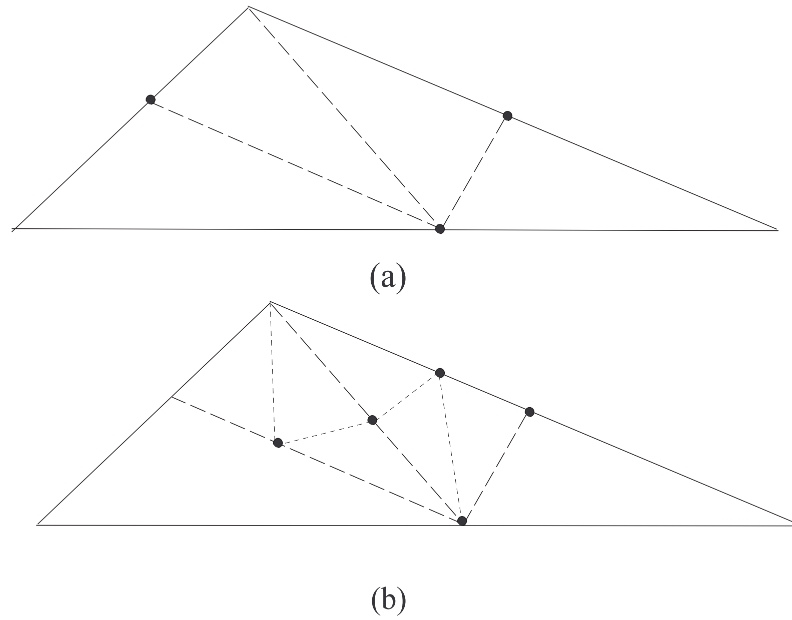



Figure 2.7. (a) Refinement Patterns of 4-Triangle. (b) Stable Molecule Behavior of 4-Triangle.

- Partition faces over *2-D* skeleton that are involved in the refinement. (4-Triangle *LE* )
- Partition involved tetrahedrons according to partition patterns. (8-Tetrahedra *LE*)

The 8-Tetrahedra longest edge partition is a *3-D* algorithm that can be explained by applying 4-Triangle skeleton refinement methodology to the faces of corresponding mesh $\tau$ [3]. Partitioning any tetrahedron T in mesh $\tau$ produces both conforming volume mesh and conforming surface mesh.

## 2.4. 8-Tetrahedra Longest Edge (*8T-LE*)

We need some definitions related to 8-Tetrahedra Longest-edge partition. Two *primary faces* of any tetrahedron T are the faces that share the unique longest edge. The remaining faces are the *secondary faces* of the tetrahedral [3, 9]. The secondary longest edges are the longest edges of secondary faces. The remaining edges are called third-class edges. There may be one secondary longest edge and four third-class edges,

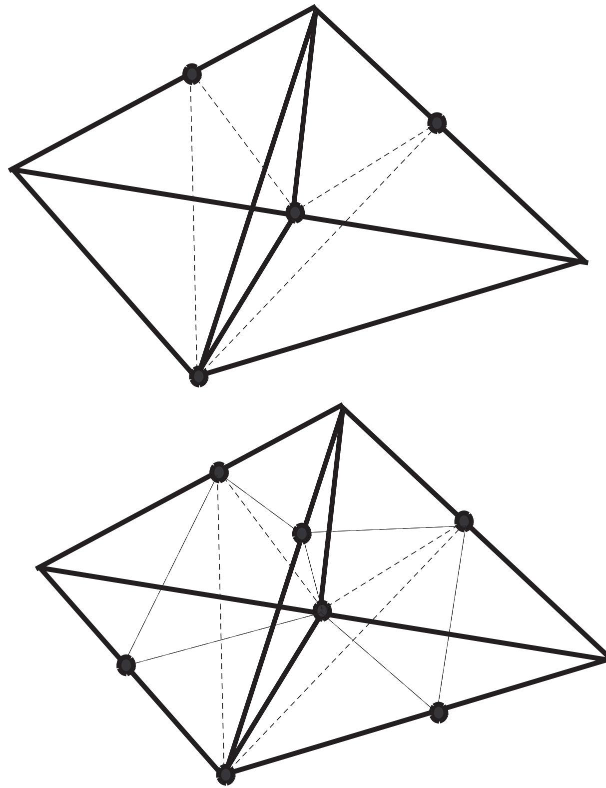


Figure 2.8. 4-Tetrahedra and 8-Tetrahedra

or two secondary longest edges and four third-class edges, a total of 6 edges in the tetrahedron.

We assume there is a unique longest edge that is also the longest edge of two primary faces and that there are unique secondary edges [1, 3, 15]. In order to prevent confusion, a unique selection is required during the progress of the refinement algorithm. The overall structure of the *8T-LE* can be defined as follows;

- Longest edge bisection produces two tetrahedron $T_1$and $T_2$.
- Bisection of $T_1$ and $T_2$ according to the longest secondary edges of $T$ produces $T_{11}$, $T_{12}$ and $T_{21}$, $T_{22}$ (4-Tetrahedra).
- Bisection of each 4 sub-tetrahedra according to the midpoint of the remaining edges of $T$ produces *8* sub-tetrahedra.

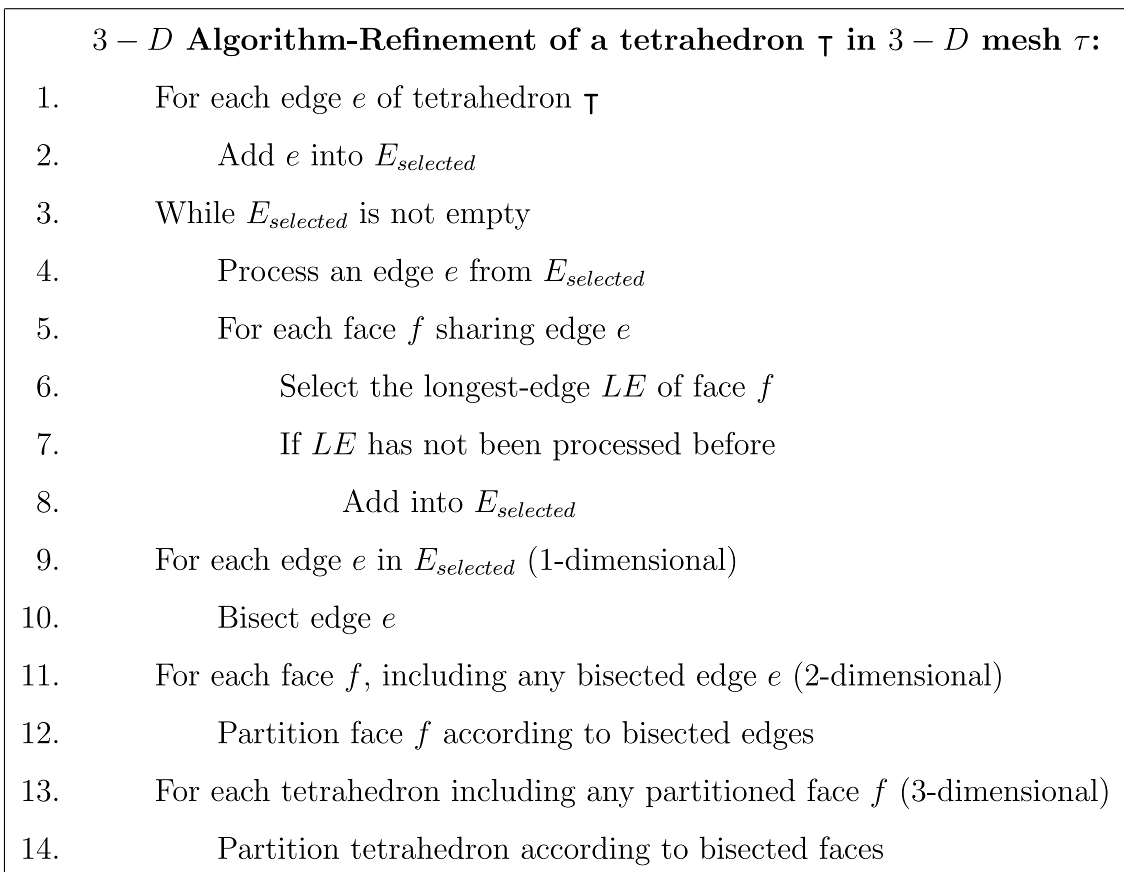
If we bisect the tetrahedron with its longest edge, and continue partitioning with

secondary edges there will be 4 sub-tetrahedra that do have a unique third-class edge of the original tetrahedron $\mathsf{T}$. Such a volume triangulation which is called 4-Tetrahedra will not be conforming only if secondary edges share a vertex and one of those edges is opposite to the longest edge of $\mathsf{T}$. It is claimed that there are four possible translations of 4-Tetrahedra partition [3];

- There is only a single secondary edge, and it is opposite to the longest edge of $\mathsf{T}$.
- Secondary longest edges and longest edge of $\mathsf{T}$ are the three edges of a triangle in the tetrahedron.
- Secondary longest edges are opposite each other, and both share a vertex with the longest edge of $\mathsf{T}$.
- Secondary edges share a vertex and one of them is opposite to the longest edge of $\mathsf{T}$.

Only the first three cases produce conforming triangulation. Therefore, *8T-LE* which supplies overall conformity by bisecting each of four sub-tetrahedra by non-bisected third-class edges of $\mathsf{T}$ is the utilized methodology in *3-D* refinement. It is proven that such a partitioning for all 4 cases produces conforming volume triangulation for any tetrahedron [3]. Figure 2.8 show the 4-Tetrahedra and 8-Tetrahedra longest-edge bisection schemes.

Properties of tetrahedral meshes have been studied by many researchers [3, 4, 9, 12, 11, 13, 14, 15, 16, 17, 24]. The *8T-LE* partition pattern is the latest methodology used in the refinement process [3, 4]. Volume triangulation with 8 new internal tetrahedra occurred and each face has been partitioned into 4 triangles. There is an interior edge from the midpoint of the longest-edge of $\mathsf{T}$ to the midpoint of the edge opposite the longest-edge. Such a triangulation produces conformity both in volume and surface structure. Surface structure is identical to the pattern obtained by 4-Triangle partitioning.



**$3-D$ Algorithm-Refinement of a tetrahedron $\mathsf{T}$ in $3-D$ mesh $\tau$:**

1. For each edge $e$ of tetrahedron $\mathsf{T}$
2. Add $e$ into $E_{selected}$
3. While $E_{selected}$ is not empty
4. Process an edge $e$ from $E_{selected}$
5. For each face $f$ sharing edge $e$
6. Select the longest-edge $LE$ of face $f$
7. If $LE$ has not been processed before
8. Add into $E_{selected}$
9. For each edge $e$ in $E_{selected}$ (1-dimensional)
10. Bisect edge $e$
11. For each face $f$, including any bisected edge $e$ (2-dimensional)
12. Partition face $f$ according to bisected edges
13. For each tetrahedron including any partitioned face $f$ (3-dimensional)
14. Partition tetrahedron according to bisected faces

Figure 2.9. *3-D* Refinement Algorithm

## 2.5. *3-D* Sequential Algorithm

The *3-D* algorithm for refining any tetrahedron $\mathsf{T}$ in a conforming mesh $\tau$ is a generalized version of a *2-D* skeleton refinement algorithm. The volume structure of the refined mesh based on the *8T-LE* also produces the refined *2-D* skeleton surface structure. Moreover, refinement of the faces of $\mathsf{T}$ as a *2-D* skeleton structure in tetrahedra mesh $\tau$ produces a refined *2-D* volume structure.

The sequential algorithm for 3D skeleton refinement is finite with linear complexity $O(n)$ [3]. The longest-edge procedure described in previous sections is used in the refinement algorithm which is shown in Figure 2.9.

# 3. PARALLEL ALGORITHM

Because of the excessive size of mesh structures used in current research projects, developing a parallel refinement algorithm is a crucial topic for Partial Differential Equation (*PDE*) problems. There are many related projects investigating an effective procedure that is applicable to adaptive meshes [29, 31, 32, 33, 30, 34, 35, 36, 37, 38, 39, 40, 41].

The most recent methodology for parallel refinement is based on *terminal edges*, which are defined as edges that do not propagate and do not cause other elements to be refined [26]. The *Terminal-Edge Bisection* procedure has sequential and parallel solutions, and the main idea is to bisect the *terminal-edge* which is the longest among selected edges first and to continue this process until all of required elements are refined [26]. However, there are some drawbacks in such a solution like partitioning more components than required and increasing the number of elements in the resulting *3-D* structure, which already necessitates an extreme number of resources. This solution may not be practical despite the flexibility of the proposed technique.

There are many other solutions for refining *2-D* triangular meshes in a parallel manner, and those methods are rather different from the parallel techniques used for tetrahedral mesh structures [28, 30, 32, 33]. Remote references pointing to elements located in other processors are used for the parallel implementations, and there are also some approximation methods suggested to handle the refinement problem.

Since the refinement process is one of the underlying computational parts of the main *PDE* problem, the solution should be flexible and simple enough to integrate with other components.

In this chapter, we analyze the longest-edge propagation graph for the parallel refinement process and present the proposed algorithm for *3*-dimensional mesh refinement in a parallel manner. First, properties of the propagation graph, directed

acyclic graph, is described. We describe the difficulties for *3*-dimensional algorithms and compare them with *2*-dimensional approaches. Finally, we present the proposed *3-D* algorithm for tetrahedral mesh refinement in distributed environments. The *3-D* algorithm for distributed environments uses the *8T-LE* technique to process and find elements that will be refined.

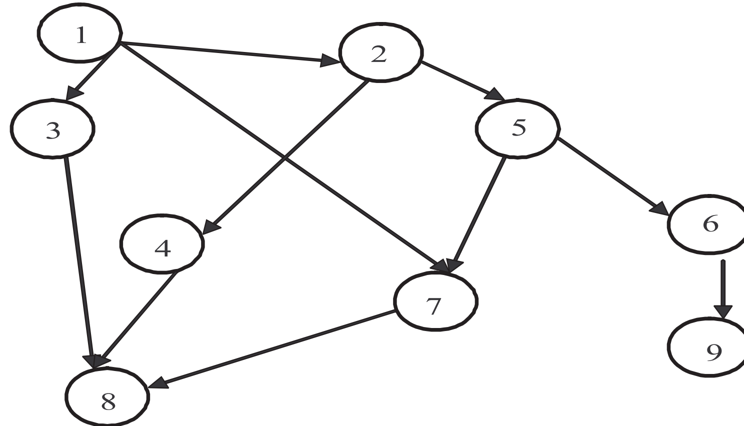
### 3.1. The Longest-Edge Propagation Graph

Initially, we may find the propagation paths of faces and associated tetrahedra. Instead of processing in an recursive manner and deciding whether or not to refine the element in the sequence, the algorithm will select first the components that should be refined, and process them independently. The final mesh structure will also be conformed according to the longest-edge propagation criterion [26, 28]. Therefore, the data structure used in the implementation should provide both relations of faces in terms of neighborhood and shared longest-edges for *3-D* mesh $\tau$. Preparing the propagation graph for the refinement, which is a *Directed-Acyclic Graph* (*DAG*), is the first step of the overall algorithm. It can be parallelized since each component such as edge and face has enough information to form the relationship graph without any dependency or requirement.

All other nodes that can be reached from $v$ in *LEPP*-Graph should be processed in order to accomplish the refinement operation. Since the *LEPP*-Graph is prepared according to the length of the edges, and each node represents one of the longest edges in the initial mesh, when we go through elements in a propagation path, the length of nodes increases. Therefore, one method is to sort nodes according to the length and then compute propagation path. The *Terminal-Edge Bisection* procedure performs the refinement process in a similar manner; however, it only interacts with the last element of the propagation path [26]. After the bisection of the terminal-edge, the previous node in the path will be selected in the next sequence, and the propagation situation will be completed without necessarily computing all reachable edges [26].

Figure 3.1 shows the reachable nodes from a selected element in order to represent

the characteristics of the *LEPP*-Graph for *3-D* structures. On the other hand, storing and computing all of the reachable elements are inappropriate due to required memory and computational requirements.




Longest Edges (with length) in LEPP-Graph

1: 2, 3, 4, 5, 6, 7, 8, 9
2: 4, 5, 6, 7, 8, 9
3: 8
4: 8
5: 6, 7, 8, 9
6: 9
7: 8
8:

Figure 3.1. *LEPP*-Graph

### 3.2. Longest-Edge Refinement

The sequential longest-edge bisection algorithm traverses all edges and select elements that must be refined, and the overall process is handled by first visiting faces that have longer longest-edges. Some studies have proposed algorithms which process bisection after the selection of all components marked to be refined [1, 3, 4, 9, 17, 26]. It has been theoretically proved in recent studies that produced mesh structure is a conforming mesh in the longest-edge refinement. Figure 3.2 presents the bisection algorithm; the LEPP is processed while traversing the overall mesh.

In order to parallelize the algorithm, we should select the longest edge of each face. We assume that each face $f$ has a unique longest edge and also each tetrahedral

**3-D Skeleton Refinement Algorithm** $(\tau, \mathsf{T})$

/* Find involved edges, faces, and tetrahedra */

1. Initialize $S_E, S_F$, and $S_T$;
   respectively sets of involved edges, faces, and tetrahedra.

2. Initialize $P_E$, set of processing edges
3. for each edge $e$ of $\mathsf{T}$
4. add edge $e$ to set $S_E$
5. add edge $e$ to set $P_E$
6. While $P_E \neq \emptyset$, do
7. pick $e$ from $P_E$
8. for each tetrahedron $\mathsf{T}$ sharing edge $e$
9. for each face $f$ of $\mathsf{T}$ having an edge in $S_E$
10. find longest-edge $e$ of $f$
11. if $e$ is not in $S_E$
12. add $e$ to $S_E$
13. add $e$ to $P_E$
14. add $f$ to $S_F$
15. add $t$ to $S_T$

16. Partition involved edges
17. for each edge $e$ in $S_E$
18. create vertex $v$ midpoint of $e$
19. bisect $e$
20. Partition involved faces
21. for each edge F in SF, do
22. partition $f$ according its bisected edges.
23. Partition involved tetrahedra
24. for each tetrahedron $\mathsf{T}$ in $S_T$
25. partition $\mathsf{T}$ according to the partition of its faces.

Figure 3.2. *3-D* Skeleton Refinement Algorithm from Plaza and Rivara [3]

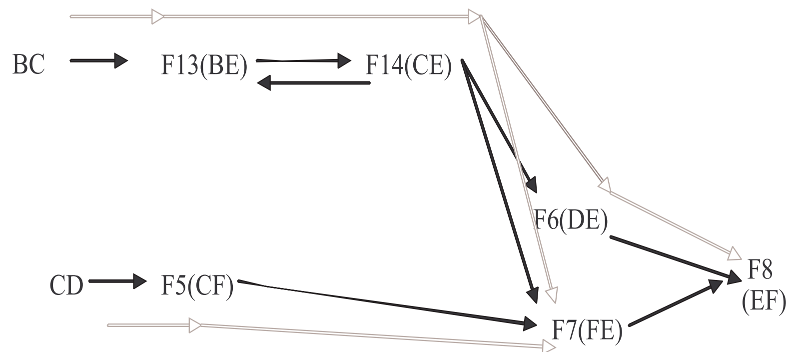



Figure 3.3. Propagation Path

has a unique longest edge according to *8T-LE*; thus, there must be a unique selection procedure. Each face is pointed with its longest-edge and we start from the edges of the tetrahedral that is going to be refined in a conforming mesh. Figure 3.3 and 3.4 presents the idea of longest-edge selection and the concept of traversing the propagation path in *3-D* bisection steps. In this example, selection of edges BC and CD result in propagation to the *longest-edges* of other faces in the mesh structure.

### 3.3. *2-D* versus *3-D*

Theoretical limits for parallel tetrahedral mesh refinement depend on the processing of the *LEPP* graph. It states the elements that are affected after an initial tetrahedron refinement. If we compute and mark those elements to be refined, other parts such as bisection steps are easily handled in a parallel environment since they will not depend on one another.

The propagation graph does not have a specific property that can be used to find reachable elements within a parallel algorithm. On the other hand, there are some approaches for similar methodologies; the *LEPP* graph is a directed-acyclic graph(*DAG*) and the *DAG* can be evaluated in many parallel ways. Most of the related techniques use random algorithms and state effective complexity times [42, 43, 44, 45, 46, 47, 48]. Average complexity may have proper values, but worst case complexity is not accept-

able when compared to sequential algorithms. Those parallel techniques are probably not practically applicable for mesh refinement; the refinement process is another subcomponent of the *PDE* problem and should be simple enough for the implementation.

The sequential algorithm has complexity $O(n)$; when data is distributed and evaluated in a parallel manner, we should deal with the relations between propagation paths. In *3-D*, each element in the *LEPP* graph can have more than one propagation. Thus, the number of edges in the *LEPP* graph, which are keeping the propagation relationship, is *O(n)* [13, 26, 49, 46, 47], if $n$ is the number of elements. Therefore, we can process *LEPP* graphs in *O(logn)* time with $n^2$ processors. For the *2-D* algorithm, elements propagate to only one other element, and the number of edges in the *LEPP* graph is *O(n)* ; thus, the *LEPP* graph can be processed in $O(logn)$ time with $n$ processors.

### 3.4. Algorithm for Distributed Environments

We start from an initial tetrahedron $\tau$, refine according to *LE*-bisection and progress to find other elements that must be partitioned. The second step is selecting all remaining components that should be refined to form a conforming mesh structure. Propagating through the initial components will lead us to all other elements that corrupt the conformity.

The next step is combining components to form a conforming mesh structure. Due to the nature of structural algorithms, explained in the previous chapter, we refine faces according to the refined one-dimensional edges; and refine tetrahedron according to *2-D* faces. Components in former steps represent the edges in the one-dimensional skeleton.

The parallel algorithm can be analyzed in three steps:

- Prepare propagation graph for the refinement, ($DAG$) for *3-D* mesh $\tau$.
- Find components which must be refined.

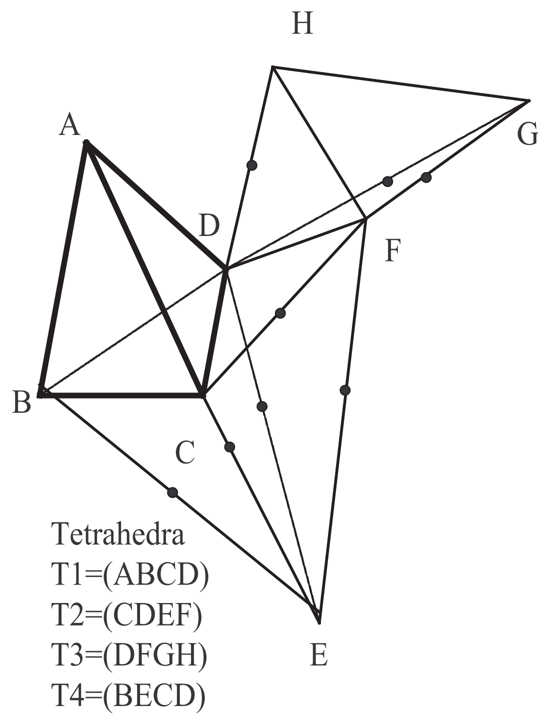
- Partition components according to the longest-edge bisection procedure.

The *3*-dimensional mesh refinement solution for adaptive structures should be effective in terms of scalability, distributed costs, and partitioned data. Required memory for tetrahedral mesh structures increases if compared to *2*-dimensional structures. Therefore, distributing the computational power with partitioned data structure is crucial if large structures are concerned.

The distributed algorithm accomplishes refinement problems by utilizing the local bisection procedure and synchronizing partitioned tetrahedra. Since propagation-paths are distributed, terminal points for a local mesh may trigger to another *LEPP* globally. In Figure 3.5, we demonstrated the overall algorithm.

The propagation path is distributed among each processor, and they compute local *LEPP*-Graphs independently. After the local refinement process, computing nodes are informed to trigger the refinement if the border element of the local mesh partition is selected to be bisected by another processor. Figure 3.6 presents the logically partitioned *LEPP*-Graph. In the given example, overall structure is partitioned among *3* processors. Node *13* and *14* have a common *longest-edge*; thus, refining Node *13* representing a tetrahedron in the figure results in propagation of the *LEPP* and refinement of Node *14*. The other processor is informed that neighbor node in the border of the local partition should be refined. Therefore, the *LEPP* graph of the local mesh structure is synchronized and the integrity of the overall propagation paths is preserved.

The synchronization process is limited and cannot exceed a few loops due to the conforming structure of input mesh structure. The previous chapter analyzes the *LEPP* features and states that the depth of the propagation graph does not increase dramatically, especially for *2-D* structures. In real-life problems, we usually start to refine some tetrahedra, causing an unacceptable error ratio in a *PDE* problem. Such a situation will not propagate to all other elements of the mesh; thus, handling large mesh structures and distributing them among remote processor to compute at the same time is more important. Figure 3.7 presents the flow-chart representation.




Faces
F1=(ABC)
F2=(ABD)
F3=(ACD)
F4=(BCD)
F5=(CDF)
F6=(CED)
F7=(CFE)
F8=(DFE)
F9= (DFH)
F10=(HGF)
F11=(DFG)
F12=(DHG)
F13=(BCE)
F14=(BDE)

Faces(T1) = F1, F2, F3, F4
Faces(T2) = F5, F6, F7, F8
Faces(T3) = F9, F10, F11, F12
Faces(T4) = F13, F14, F4, F6

Figure 3.4. Longest-Edge Selection

**Distributed Algorithm($P$ processors, Tetrahedral Mesh $\tau$ ):**

1. Distribute the Tetrahedral Mesh Structure $\tau$ among processors,
2. Each processor $P_i$ handles its local mesh structure.

3. Process the Longest-Edge Algorithm locally:
4. foreach edge $e$ of tetrahedron T that needs to be refined
5. add edge $e$ to the list of selected edges $E_{selected}$

6. $LEPP$ algorithm:
7. while all edges in $E_{selected}$ are processed
8. if $LE$ of the face $f$ that edge $e$ belongs is not selected
9. add edge $LE$ to the list of selected edges $E_{selected}$
10. add tetrahedron T that edge $e$ belongs to
    the list of selected tetrahedra $T_{selected}$

11. Synchronize local Propagation Paths:
12. if local terminal point in the $LEPP$ also belongs to another
    local mesh structure $\tau$ owned by processor $P_i$
13. inform processor $P_i$

14. Process the $LEPP$ algorithm after synchronization.

15. Bisect selected edges $E_{selected}$
16. Bisect selected tetrahedra $T_{selected}$
17. Collect local mesh structure from processing nodes, $P_i$s

Figure 3.5. Distributed Algorithm

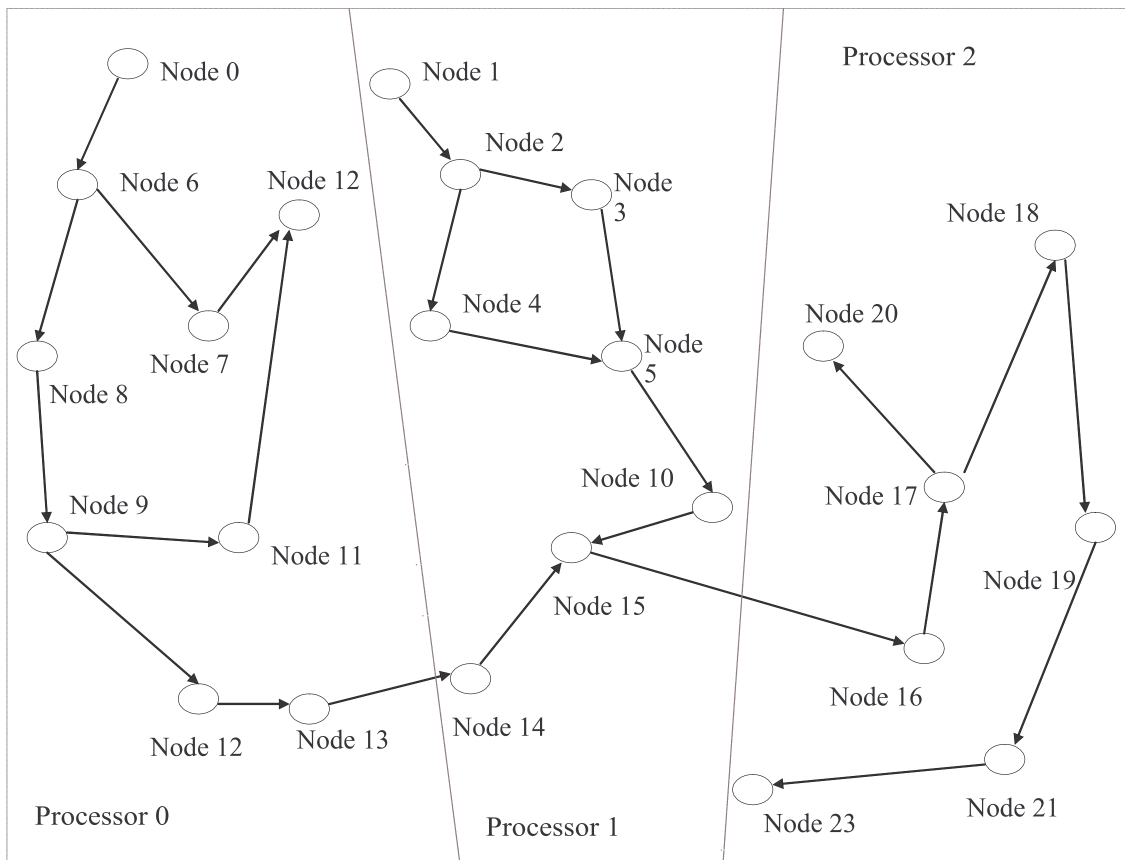



Figure 3.6. *LEPP*-Graph Partitioning and Synchronization

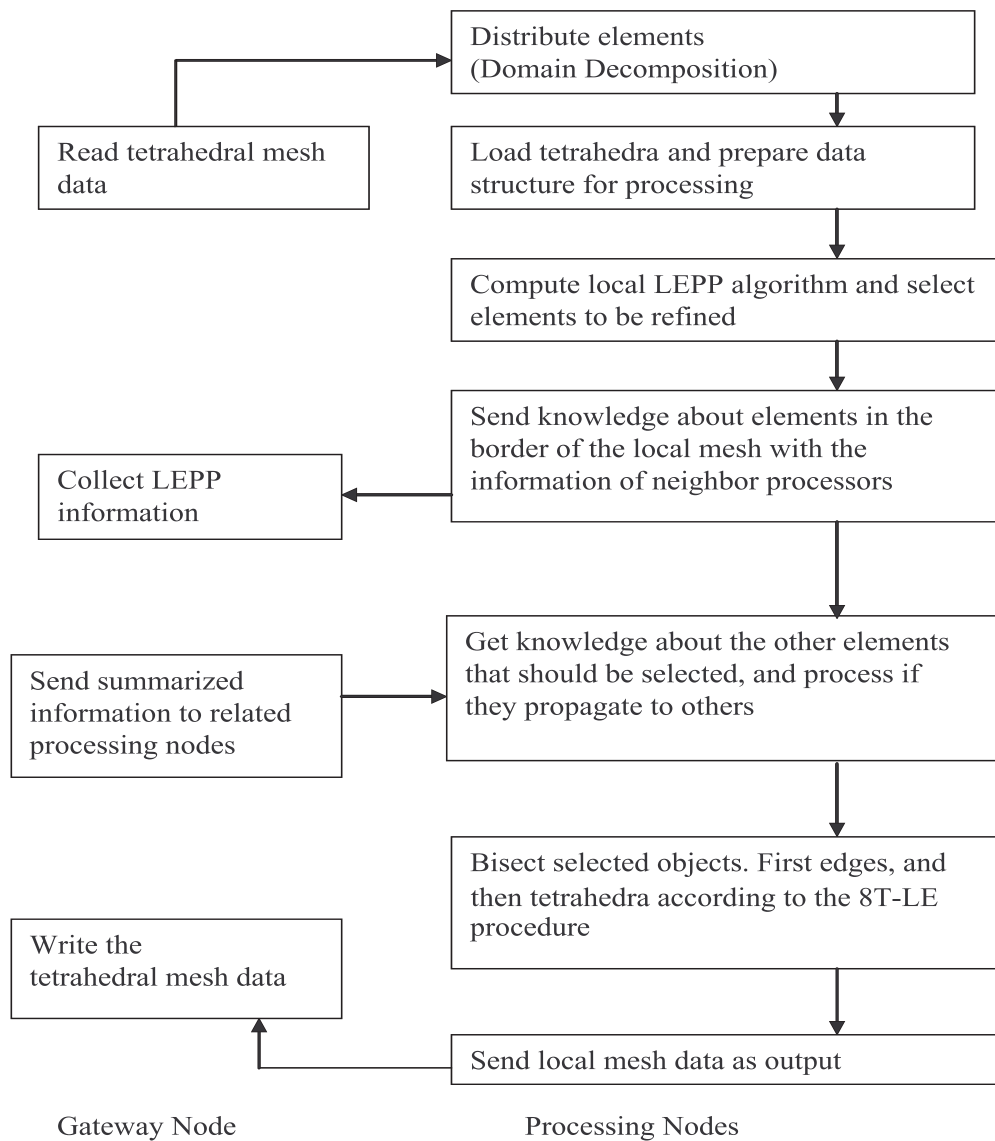



Figure 3.7. The *PTMR* Algorithm

# 4. IMPLEMENTATION DETAILS

Parallel Mesh Refinement algorithms have been studied for *2-D* and *3-D* structures. In *2-D* triangular mesh, we can utilize the *longest-edge* refinement technique to prepare a parallel implementation. Since propagation to neighbors of an element cannot be limited as we can state in *2-D*, tetrahedron mesh refinement has difficulties in terms of parallelization. Most of the proposed parallel models use some approximation techniques to prepare a conforming mesh instead of concluding with the most proper mesh structure. Features of the tetrahedron mesh such as average and maximum interaction of elements in a proper conforming structure have been studied, but those properties are not sufficient to be used for the refinement implementation in parallel.

We have studied theoretical properties of refinement to investigate a proper parallel algorithm with logarithmic complexity for tetrahedron mesh structures. Some models to handle the parallel refinement have been proposed; however, they are not practically applicable due to scalability reasons. Processing of the *propagation-path* with the computation of all reachable elements results in very poor performance compared to those algorithms for the sequential refinement. Therefore, we decided to study practical implementation of the problem in distributed environments.

Parallel Tetrahedral Mesh Refinement (*PTMR*) implementation can be encapsulated and used by other Finite Element programs; thus, it provides a framework for the refinement process. Initially, distribution of the elements among processing nodes is accomplished. The mesh object is loaded locally and prepared for the refinement operation. After finishing the refinement process, master processor collects new vertices and tetrahedra. Another important feature is the profiler; that is, all methods are also capable of collecting elapsed time information in the network communication and computational code segments.

The parallel framework for tetrahedral mesh refinement, *PTMR*, requires *MPI* [50] and *PaRMetis* [51] libraries. Loaded mesh is partitioned in order to have mini-

mum *edge-cut*s in the *LEPP*-Graph. Mesh partitioning is accomplished according to tetrahedra, and each processor keeps only the elements that are assigned. The mesh structure is partitioned fairly concerning the network cost between processing nodes. The overall mesh geometry fits into the memories of each processing nodes; thus, we can handle tetrahedral mesh files with excessive memory requirements. The most important advantage of *PTMR* is the scalability; we can not deploy a very large mesh structure on a single node due to memory limitations, but we can distribute the data and operate in a parallel manner.

Tetrahedrons that should be refined initially are computed, and other elements which will be affected are selected using the *LEPP*. The selection procedure is accomplished by traversing the mesh data in the local processing node. A refined edge in one of the processors can trigger another tetrahedron which is in another processor's local memory. Therefore, related processing nodes are informed if an edge will be refined in the border of a partition.

Procedure of the parallel implementation can be stated as follows:

- Partition and distribute tetrahedral mesh elements.
- Compute initial tetrahedrons that should be refined.
- Prepare the local *LEPP*-Graph sequentially and select the edges that should be refined.
- Inform other processing nodes whether a border element in the local partition is selected.
- Refine according to the *8T-LE* procedure.
- Collect mesh data with recently produced tetrahedrons.

The gateway node is used not only to read the initial mesh data from file but also to prepare communication objects holding the information whether border elements of the local mesh partition require refinement or not. Therefore, the gateway node minimizes the number of network messages, and this situation is an important issue to enhance the performance of an *MPI* program [50, 52]. Each processing node sends

the information about the selected border elements to be refined with the knowledge of neighbor processors that should take action. The gateway node collects the effected elements and informs other processors to start refinement process for the classified element.

### 4.1. Data Structure

Designing a proper data structure is another challenge in the implementation. Some know techniques have been investigated [53, 54, 55, 56, 27, 57] in order to prepare a flexible architecture for the *PTMR*. A mesh structure is formed of vertices, edges, triangles and tetrahedrons. In order to process algorithms, we should be able to evaluate each element and keep relations between them. The *8T-LE* algorithm is a skeleton algorithm, and elements in lower dimensions are required for refinement. It is stated that the number of tetrahedrons in a conforming mesh is much more than the number of other elements; and any tetrahedron is adjacent to many edges and vertices [58]. Figure 4.1 demonstrates the number of elements of a conforming mesh on average. Relations between elements are the related adjacencies; as an example, changing an edge will affect *5* tetrahedra on average [58].

Since the *8T-LE* is not principally parallelizable due to the sequential progress of *Longest-Edge* propagation, we developed a new data structure with a convenient parallel algorithm which is applicable to distributed environments. During the implementation of distributed algorithms in *PTMR* utility, we keep adjacency relationships between edges and tetrahedrons. Each tetrahedron object keeps the list of edges it owns, and edge objects have the list of vertices that form itself. Edge objects also have the list of tetrahedrons which are adjacent, so that, while evaluating the *LEPP*, computation can be handled without searching adjacencies each time.

The data structure of the local mesh object has vertices, edges and tetrahedra. Each edge object has the list of tetrahedra it is owned by. The tetrahedron object keeps the list of edges that form this tetrahedron, and we can calculate the information of faces when required in the propagation path process.

Each vertex object has a single unique identifier which distinguishing them in the global space. Each processor starts from a sequence which will not intersect with other processors. During the operation of the *LEPP* synchronization, a unique identifier which is the smallest number among all other local sequences, is selected as the identifier for the effected neighbor elements in the border of a local structure.

In $8T - LE$ and $4T - LE$, we must select the longest-edge and that should be unique in the tetrahedron or in the triangle. The edge object has a simple methodology to handle the uniqueness for the length comparison. If the length of two edges are equal, identifiers of the first and then the second vertices are compared to select one of them as the longer edge.

Figure 4.2 shows the used data structure skeleton. Figure 4.5 presents a more detailed view of the data model which is also evaluated in Appendix A.

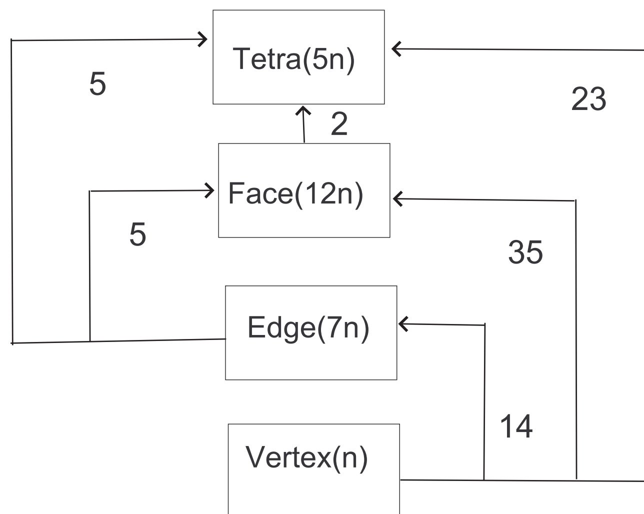



Figure 4.1. Statistics for the Number of Entities of a Mesh from Garimella [58]

Mesh structure is partitioned between each processing node according to cost of interaction between elements. Thus, while synchronizing the *LEPP*s between components, the parallel tetrahedron refinement process minimizes the network and computational costs. Such an implementation also covers the scalability objective. Moreover, the gateway node used in the algorithm enables us to minimize the number of network messages. *PTMR* uses communication objects in the *MPI* environment in order to

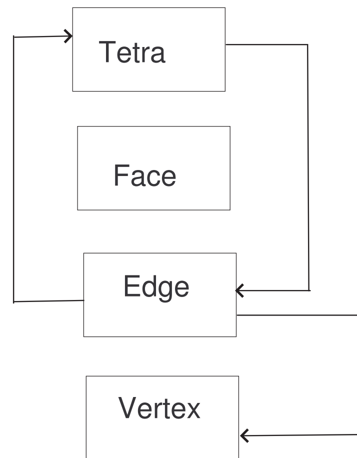



Figure 4.2. Data Structure of PTMR

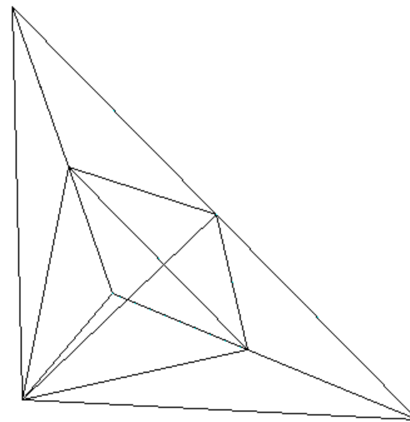


Figure 4.3. Mesh Refinement Example

minimize the network traffic and handle the dynamic data transfer between processors. Each processing node uses the synchronization information to select and start the refinement from received elements if it is required. Synchronizing the borders of propagation graph, which is partitioned among processors as a result of the local mesh processing, is accomplished by summarizing the received bisection information and distributing among the concerned processors. Since the gateway node collects all messages used to synchronize the overall *LEPP*, it prevents duplicate messages in the communication environment.

Parallel implementation for distributed environment has been prepared and tested

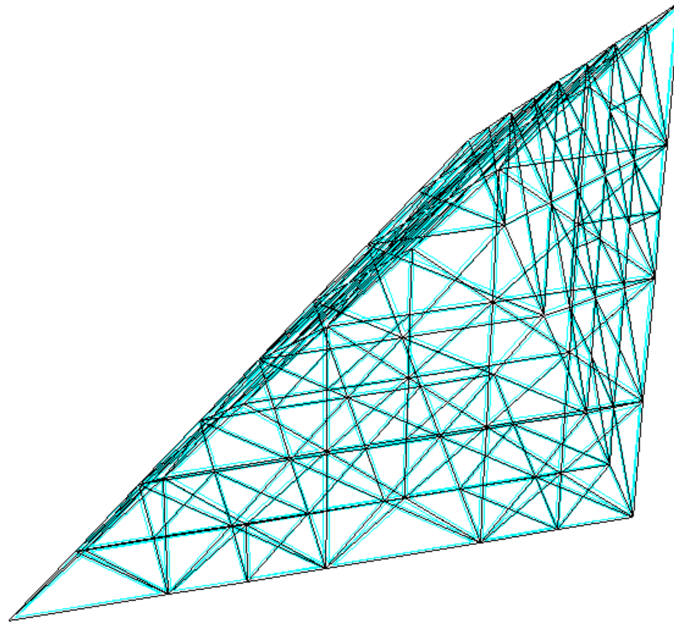


Figure 4.4. Mesh Refinement Example

for homogenous platforms. Utilized data-structure can also handle heterogeneous environments and is capable of adapting to data distribution. Test results about the proposed methodology of refinement process are presented in the next chapter. Since data is properly distributed, we can separate and reduce the overall computation and memory cost of the refinement process. Figure 4.3 and 4.4 show produced tetrahedral mesh results.

**Vertex ID**
**x,y,z coordinates**
**parent , parent 2**
*Reference to parents if created by bisecting an edge*

Point Object

**Point 1, Point 2**
**Sub-Edge 1, Sub-Edge 2**
*Reference to sub-edges after bisection*
**Mid-point**
*Reference to the mid-point produced after selecting this* edge
**Tetrahedra List**
*List of tetrahedral that own this edge*

Edge Object

**Edge 1, Edge 2, Edge 3, Edge 4, Edge 5, Edge 6**
**parent Tetra-Bucket**
*Reference to Tetra-bucket that tetrahedron resides*

Tetrahedron Object

**Tetrahedra List**
*List of tetrahedra that are in this bucket object*
**Neighbour processors**
*Neighbour processor that also own tetrahedral of this object*

Tetra-Bucket

**Tetra-Bucket List**

***List of Edges to be refined***
***List of Tetrahedra to be refined***

***Communication Object***

Mesh Object (located on each processor)

Figure 4.5. The Object Relationship of the *PTMR* Utility

# 5. TEST RESULTS

Mesh input files from the *GAMMA* project [59] were used as examples for testing the *PTMR* utility. The aim of the *GAMMA* project is to analyze and develop mesh generation algorithms which are suitable for Finite Element Computations. Some research topics of the project are Mesh Generation Algorithms, Error Estimation, Parallel Computing, and Data-structures [59].

We also used mesh generation tools to understand the accuracy of the methodology. Some of the mesh inputs were generated by *TetGen* [60], which is a program for generating tetrahedral meshes for arbitrary *3-D* domains. Mesh generation of the *TetGen* program is based on *Delaunay* methods, and the tool was written in ANSI C++ [60, 61].

The cluster system with 128 nodes from High Performance Computing Center of *ULAKBİM* [62] was used to run test scenarios. Sun Grid Engine (*SGE*) [63, 64] is the Job Management System of the cluster. It is a multi-user environment and two execution queues were defined for each node; thus, the job scheduler might assign a node for the execution in which any other process was also running. Moreover, two processes of a single test job can be executed in one processor; without any network communication cost between them. Since processes of the *PTMR* were competing for computing cycles with another processes in the system, we did sometimes encounter unexpected results. Configuration of the system in which test scenarios were executed was not proper to evaluate the performance of the *PTMR*. Therefore, results for elapsed time values in test routines did not reflect the actual performance. However, they did provide an estimation and show that mesh refinement can be processed in a few minutes by using the *PTMR* utility.

The following figures present some of the generated mesh files which were produced by refining sample input structures. Mesh structures were viewed by *Medit* [65]

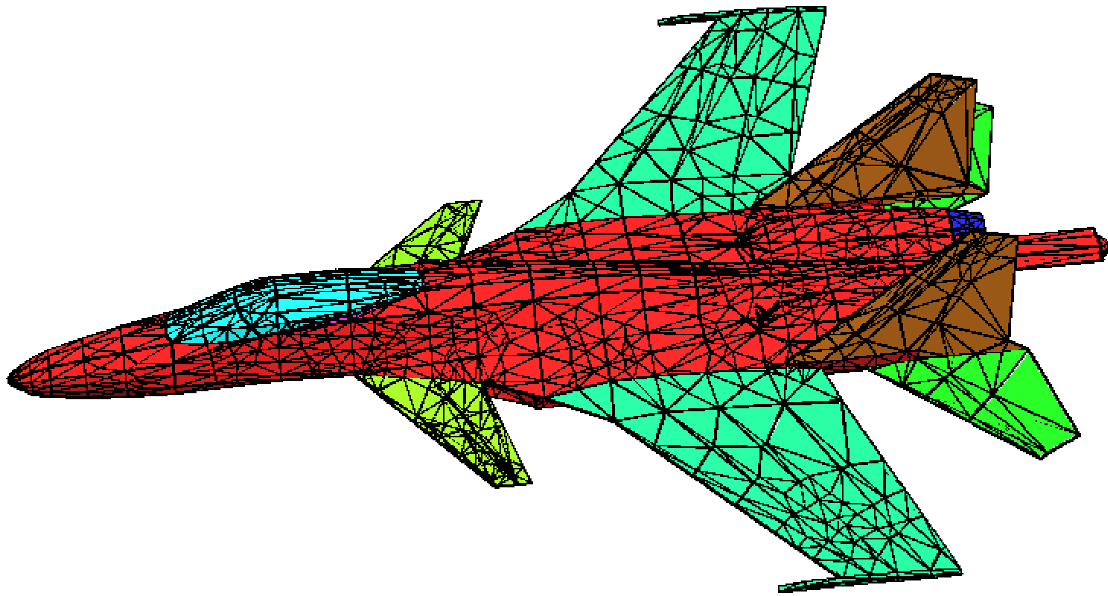


Figure 5.1. 14904 tetrahedra / 4502 vertices

and *Tetview* [66] which are graphic programs for viewing tetrahedral meshes.

Refining the tetrahedra mesh in Figure 5.2, which has 1803 tetrahedra and 527 vertices, will produce a new structure with 13506 tetrahedra and 3387 vertices. After refining the resulting tetrahedral mesh we produced the *3-D* formation shown in Figure 5.5 that has 99121 tetrahedra and 23351 vertices. Refinement of mesh in Figure 5.1, 14904 tetrahedra, and 4502 vertices produced 70203 tetrahedra and 22568 vertices, as shown in Figure 5.6. Refinement of the resulting mesh second time resulted in 235941 tetrahedra and 22568 vertices.

Refinement of the mesh with 12586 tetrahedra and 3644 vertices in Figure 5.4 produced the *3-D* geometry of 79263 tetrahedra and 33098 vertices as shown in Figure 5.7. Other test results generated from the same source produced mesh structures with 450573 tetrahedra and138842 vertices, and 770882 tetrahedra and 215017 vertices.

Information about elapsed time for the steps of the program can be viewed via defining the debugging parameter of the *PTMR* utility. Figures from 5.8 to 5.10 show the graphic of performance counts. Figure 5.8 shows the time spent in the gateway

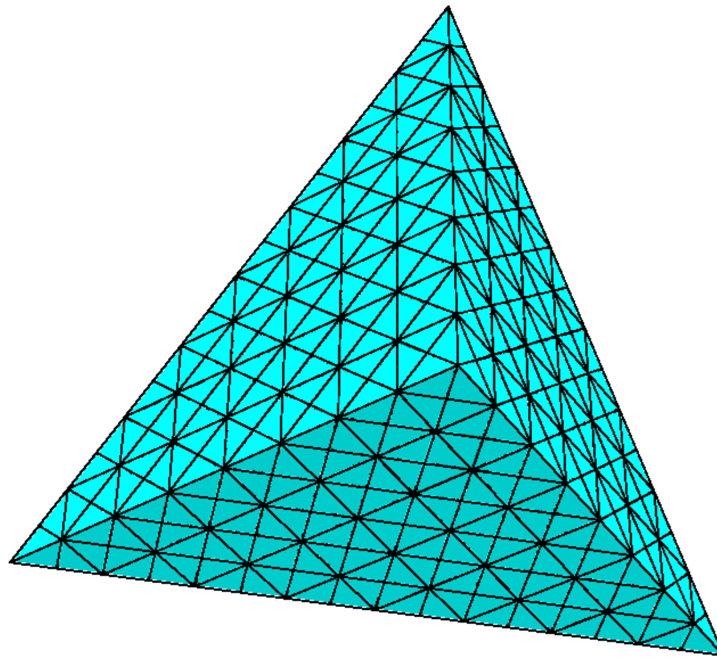


Figure 5.2. 1803 tetrahedra / 527 vertices

node while synchronizing the *LEPP* graph. According to the Figure 5.8, elapsed time does not increase as the number of processor increases; thus, the gateway node is not the bottleneck of the overall algorithm. Mesh structures with 1803, 12586, and 74178 tetras are used for the test case, and the resulting meshes have 74178, 13281, and 450573 tetras in the given order. Figure 5.9 presents the maximum amount of time spent in a single processor among processing nodes during the refinement process. As the number of processors increases, maximum time spent decreases, since data is distributed among processing nodes.

In Figure 5.10, overall time of the PTMR utility is shown. As can be seen from the figure, the increase in the number of elements can be handled by using more processing nodes.

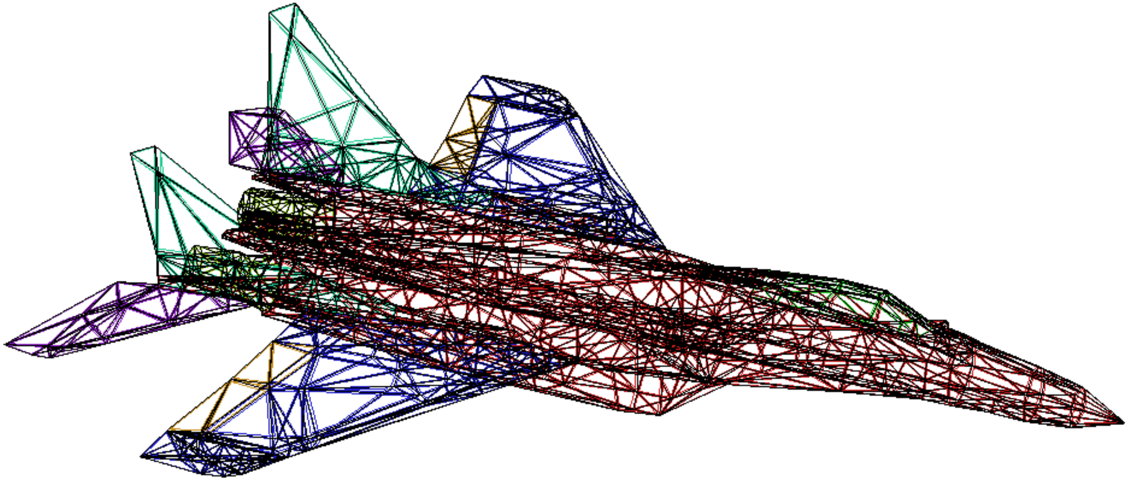


Figure 5.3. 6670 tetrahedra / 2021 vertices

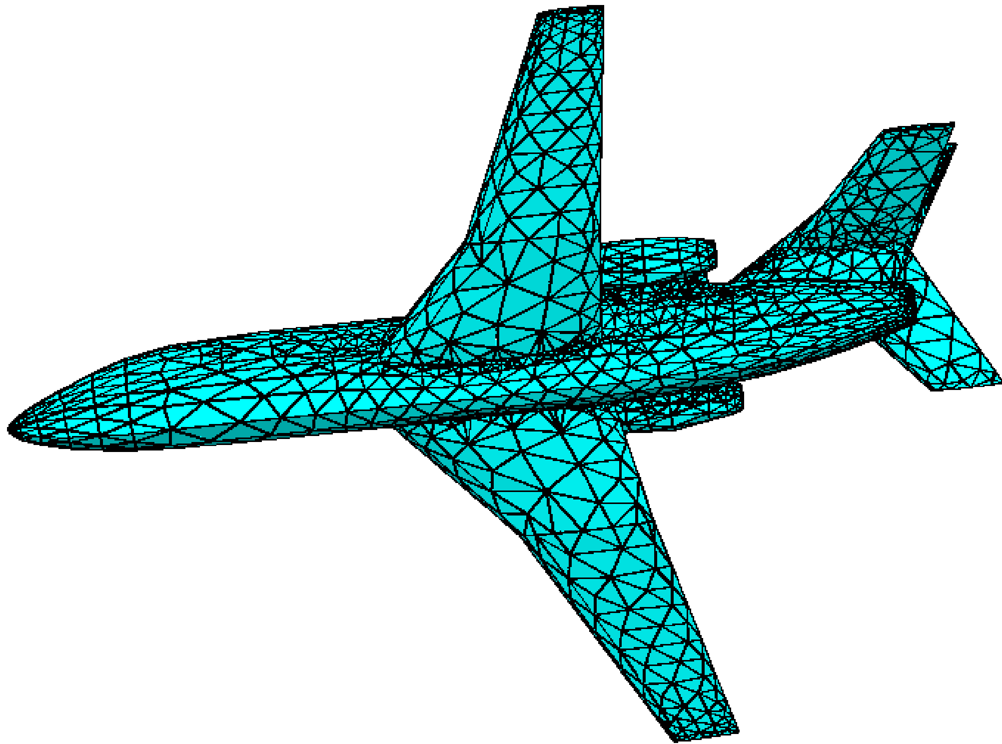


Figure 5.4. 12586 tetrahedra / 3644 vertices

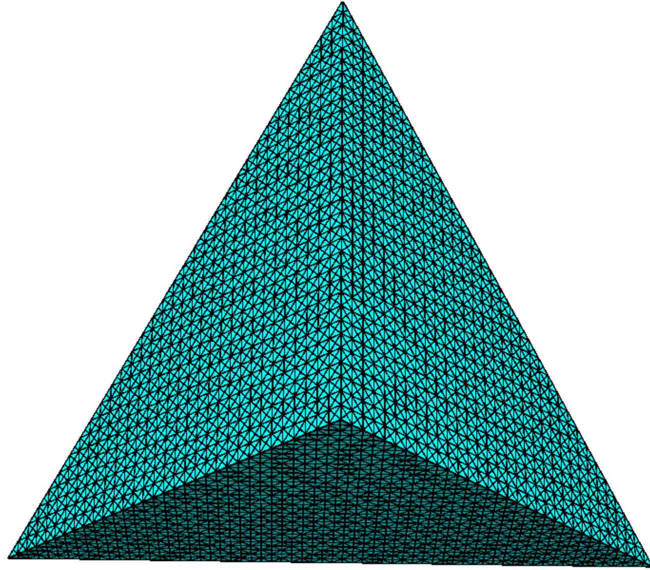


Figure 5.5. 99121 tetrahedra / 23351 vertices

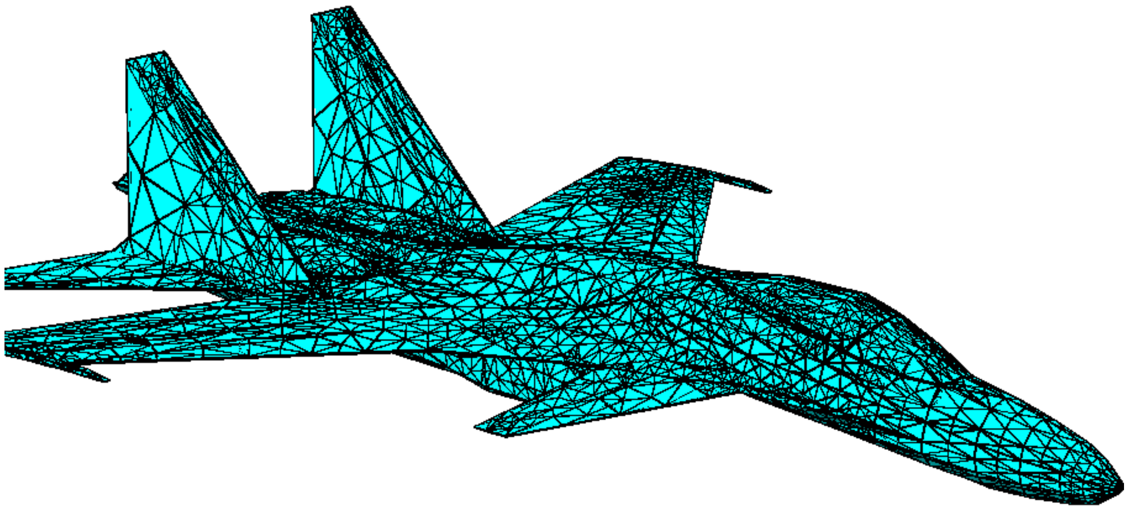


Figure 5.6. 70203 tetrahedra / 22568 vertices

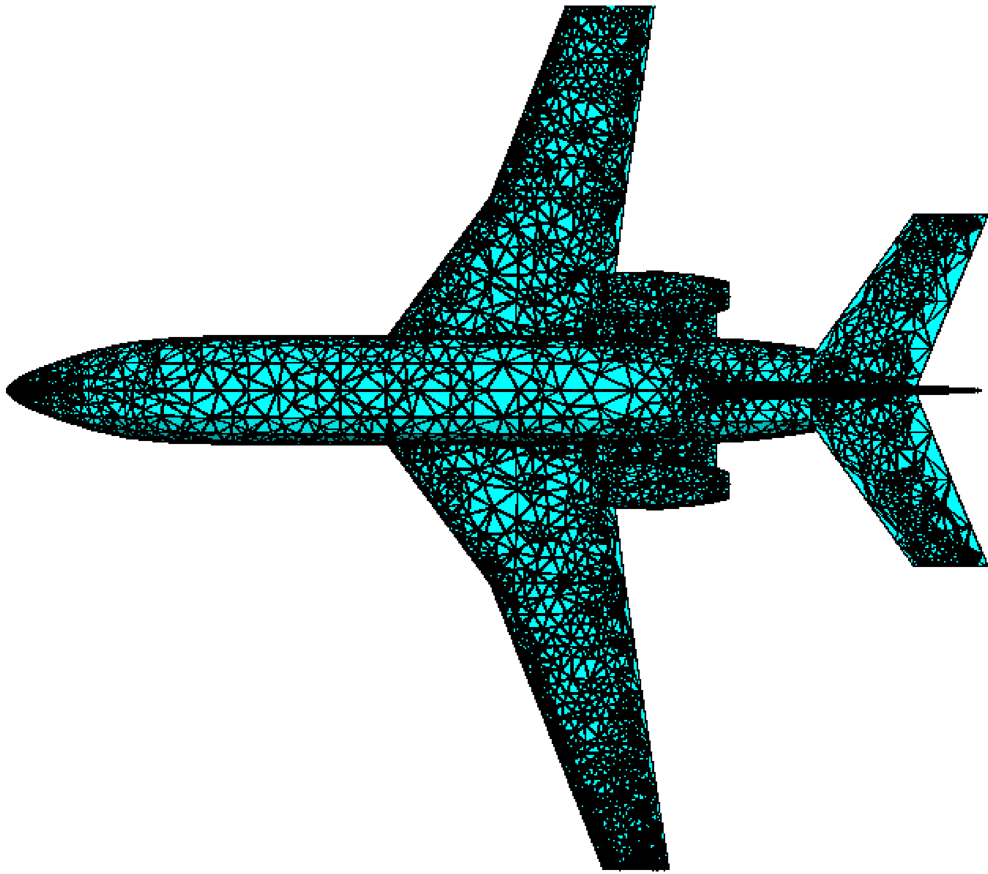


Figure 5.7. 79263 tetrahedra / 33098 vertices

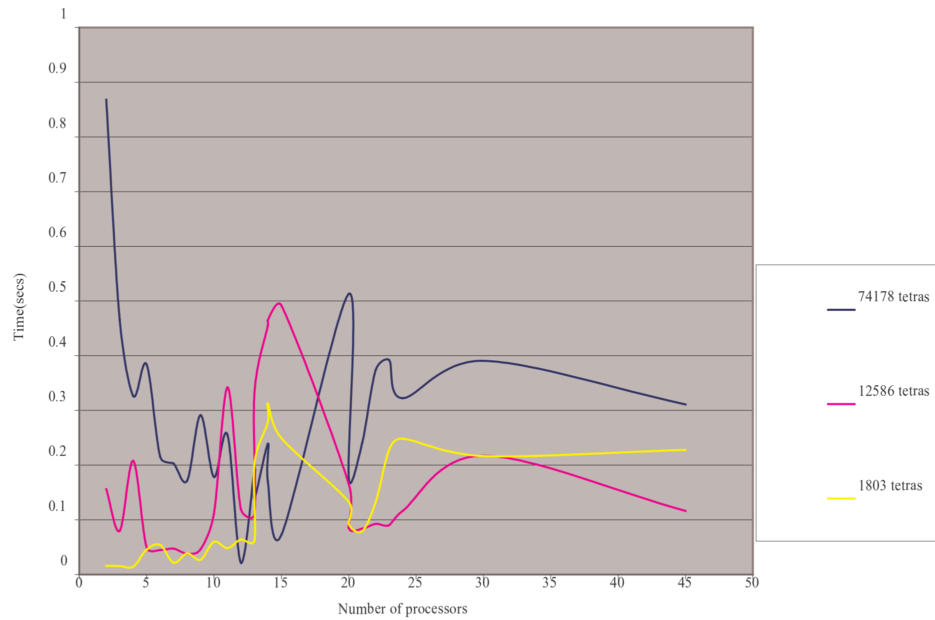



Figure 5.8. The Refinement Process According to the *LEPP* Algorithm (elapsed time spent in the gateway node)

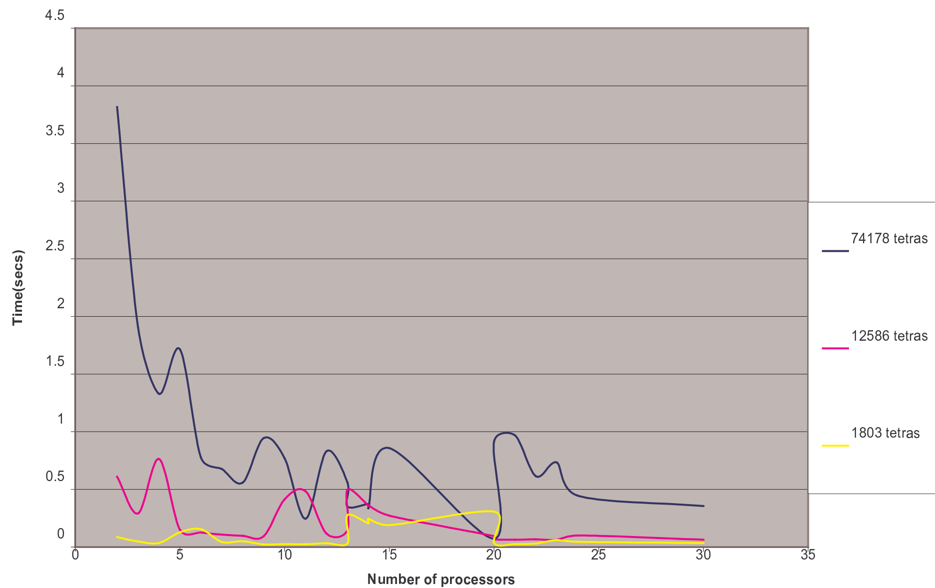



Figure 5.9. The Refinement Process According to the *LEPP* Algorithm

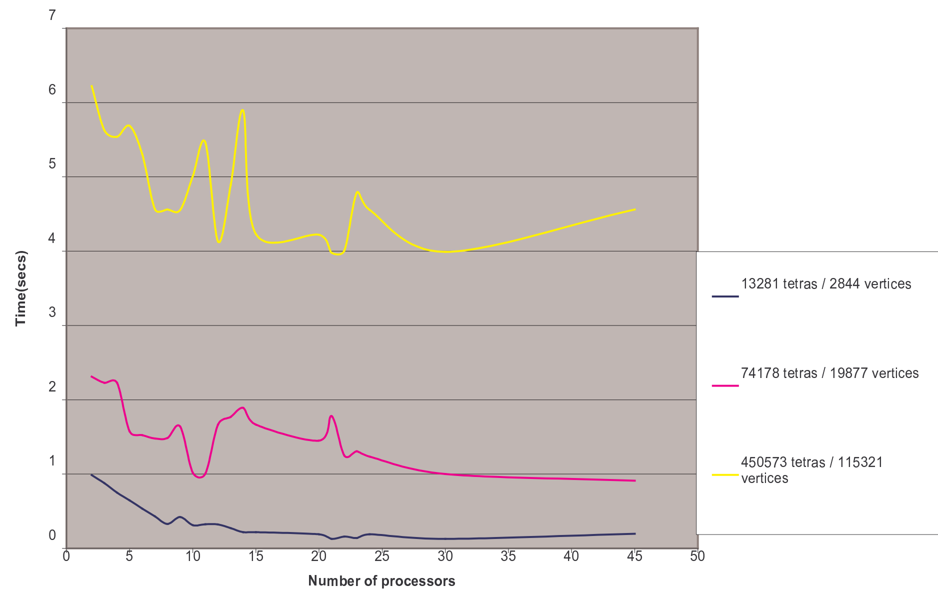



Figure 5.10. Overall Time

# 6. CONCLUSIONS

A parallel mesh refinement algorithm for distributed environments is proposed, in which each processing node works over its local elements sequentially and synchronizes changes to update the overall mesh structure.

We analyzed the *longest-edge* bisection algorithm and presented details about the parallel refinement process. *3-D* structures have difficulties especially in processing the *propagation path* of selected tetrahedra. We presented a practical and scalable methodology which is capable of solving refinement problem within acceptable time limits. Representation of the mesh is also crucial; the data structure must be compact not to consume so much memory, but it should be flexible and simple for computation. We explained the proposed objects to accomplish the construction of mesh topology.

We also present a parallel utility, *PTMR*, that can distribute the mesh data and process in an inter-process communication environment; thus, clusters of ordinary nodes can be used to process very large mesh structures.

# APPENDIX A: PTMR: Reference

Parallel Tetrahedral Mesh Refinement(*PTMR*) implementation, build for Distributed Environments, consists of modules that can be encapsulated and used by other Finite-Element programs; thus, it provides a framework for the refinement process. *PTMR* has the following source files and each of them contains independent modules.

- local.h, utils.h, utils.c, table.c table.h, list.h, fmesh.h
- tmesh.h, tmesh.c, map.h, map.c, rank.h, rank.c
- lPoint.h, lPoint.c, lEdge.h, lEdge.c, lTetra.h, lTetra.c, T_bucket.h, T_bucket.c
- lMesh.h, lMesh.c
- comm.h, comm.c, lp.h, lp.c
- ptmr.c

There are also some facilities used to handle some implementation issues and they enabled us to prepare useful programming for other code segments of the mesh partitioning. The overall framework is classified as mesh partitioning, local mesh processing and refinement according to the longest-edge algorithm. We describe each component in details and specify their important methods and functions.

Initially, distribution of the elements among processing nodes is accomplished. The mesh object is loaded locally and prepared for the refinement operation. After finishing the refinement process, master processor collects new vertices and tetrahedra.

```
MPI_Init(&argc, &argv);
................................................
 rank=rank_form();
 process_args(argc, argv,rank_globalRank(rank));
 map_new(map, rank, "create map_ object");
     tmesh_new(tmesh, "create tmesh_ object ");
     tmesh_dist(rank,map,tmesh);  // initial distribution
```

```
     lmesh=lMesh_new_load(rank,map,tmesh);
     tmesh_destroy(tmesh,"destroy tmesh_ object");
     if(rank_processingNode(rank)){
       // Processing Node
       lMesh_process(lmesh, rank);
       lMesh_send_points(lmesh,rank);
       lMesh_send_tetras(lmesh,rank);
     }else{
        // Gateway Node
        lMesh_gateway(lmesh,rank);
        lMesh_recv_points(lmesh,rank);
        lMesh_recv_tetras(lmesh,rank);
        fmesh=lMesh_2fmesh(lmesh);
        write_fmesh(fmesh,get_args_file_name());
       }
       lMesh_free(lmesh);
       fmesh_destroy(fmesh, "destroy fmesh_ object");
 map_destroy(map, "destroy map_ object");
 rank_destroy(rank,"destroy rank_ object");
MPI_Finalize();
```

The following sections explain the implementation issues, clarify some important functions and present structure of modules used in this work. All methods presented are also capable of collecting elapsed time information in the network communication and computational code segments. Output of steps and results of concerned functions can be viewed whether appropriate *debug* and *timing* definitions are set.

### A.1. Utilities

Those procedures consist of general definitions and input/output file formats of the tetrahedral mesh structure.

### A.1.1. Global Definitions

**local.h** includes global definitions for all modules. It defines header files that are required by used libraries; sets and organizes definitions of *timing* and *debug* levels.

```
/* LOCAL.H */
#ifndef __PTMR__LOCAL_H__
#define __PTMR__LOCAL_H__
/* headers */
#include <stdio.h>
#include <string.h>
#include <stdlib.h>
#include <errno.h>
#include <math.h>
#include <glib.h>
#include <mpi.h>
#include <parmetis.h>
.......................................
#ifdef _TIMING_LEVEL_3
        #ifndef _TIMING_LEVEL_2
             #define _TIMING_LEVEL_2
        #endif
        #ifndef _TIMING_LEVEL_1
             #define _TIMING_LEVEL_1
        #endif
#endif
#ifdef _TIMING_LEVEL_2
        #ifndef _TIMING_LEVEL_1
             #define _TIMING_LEVEL_1
        #endif
```

```
#endif
```

### A.1.2. Common Utilities

**utils.h** consists of global definitions and utility functions. It includes some encapsulated *macro* definitions for memory allocation and other common procedures.

### A.1.3. The Dynamic Pointer Array

**list.h** defines the *List_* object used to hold pointers of the selected edges and tetrahedra elements in the refinement process.

- *List_new( List_ pointer, definition text)*
- *List_getsize( List_ pointer)*
- *List_setsize( List_ pointer, size(int))*
- *List_get( List_ pointer, index(int))*
- *List_add( List_ pointer, data pointer)*
- *List_del( List_ pointer, data pointer)*
- *List_print_param( List_ pointer, print function, parameter)*
- *List_free( List_ pointer, definition text)*

### A.1.4. The Pointer Table

**Table.[ch]** defines the data-structure which is using Binary-Balanced Tree. The *Table_* object is used to keep the relationship between points(vertices) and edges.

- *Table_get( Table_ pointer, index 1(int), index 2(int), data pointer(RETURN)):*
  Returns the corresponding element if exits, else returns NULL.

- *Table_set( Table_ pointer, index 1(int), index 2(int) , data pointer):*
  Sets indices for the corresponding element.

### A.1.5. Input/Output File Formats

**fmesh.**[**ch**] defines the initial mesh structure that is read from an input file. Output of the overall process, which is the new tetrahedral mesh, is also an *fmesh_* object. The *fmesh_* object is used to read and write data from or into a file.

```
/* initial mesh structure that is read from the file */
typedef struct fmesh_ {
  int n_point;    // the number of points in the mesh read.
  int n_tetra;    // the number of tetras in the mesh read.
  float * points; // coordinates of the points.
  int * tetras;   // point id's of the tetras.
}fmesh_;
```

Methods of the *fmesh_* print debug information such as the number of elements read and the total time elapsed in the file operations. It searches for *filename*.ele that vertex ids' of each tetrahedron is written and *filename*.node that coordinates of vertices is written.

*write_fmesh* and *read_fmesh* are the main functions of this object; the local mesh structure(the *lMesh_* object) has conversion methods to map the data of the geometry in order to optimize the processing of this input structure.

The file format suitable for an *fmesh_* object is:

**.ele** *files:*

```
First line:
     <# of tetrahedra> <nodes per tetrahedron (4)>
Remaining lines list # of tetrahedra:
     <tetrahedron #> <node> <node> ... <node>
```

**.ele** *file example:*

```
4 4 0
    1 7 2 3 5
    2 7 2 6 1
    3 4 2 6 5
    4 7 6 2 5
```

**.node** *files*:

```
First line:
     <# of points> <dimension (3)> <# of attributes>
Remaining lines list # of points:
     <point #> <x> <y> <z> [attributes] [boundary marker]
```

.**node** *files* example:

```
7 3 0 1
   1 100 200 100 1
   2 100 100 100 1
   3 200 100 100 1
   4 100 100 200 1
   5 150 100 150 1
   6 100 150 150 1
   7 150 150 100 1
```

• *write_fmesh(fmesh pointer, filename):*

Read the fmesh_ object from a file.

•*read_fmesh( fmesh pointer, filename):*

Write the fmesh_ object into a file.

## A.2. Mesh Distribution

The mesh structure is partitioned fairly concerning the network cost between processing nodes. The following modules and data structures are responsible for the distribution of the mesh data; the network cost should be minimized and computational cost should be balanced. The overall mesh geometry is partitioned and it fits into the memories of each processing nodes; thus, we can handle tetrahedral mesh files with huge memory requirements.

### A.2.1. Processor Ranks

**rank.**[**ch**] defines the rank of a processing node in the overall Processor Cell.

```
/* rank object */
typedef struct rank_ {
         int global_rank;   // the rank in the MPI_COMM_WORLD.
         int process_rank;  // the rank in the processing node.
         int global_size;   // the number of processors in the
...........................// MPI_COMM_WORLD.
         int process_size;  // the number of processing nodes.
         int gateway;       // the global rank of the GATEWAY node.
         MPI_Comm pcomm;    // the communication object.
} rank_ ;
```

The *Gateway* node is specialized to read input from the input file and prepare the initial synchronization of the data mapping and distribution. Therefore, we require a different communication object originated from the *MPI_COMM_WORLD* [50]. The *rank_* object returns the ranking of the node and classifies processors as processing

nodes and gateway node.

### A.2.2. Processor Mapping

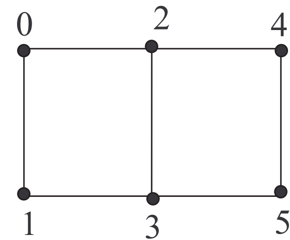
**map.**[**ch**] handles the distribution and the mapping of elements to processors.

```
/* MAPPING */
typedef struct map_{
 int n_element;      // number of tetras in the overall mesh.
 int * pdist;        // distribution array, among processors.
 int * edist;        // used for redistribution and keeps new id's.
 int * parts;        // map elements to processors.
                     // only gateway node keeps all elements
                     // for computing the initial mapping.
} map_ ;
```

The *sync_map* function forms the initial distribution of processors and keeps the partitioning information. It operates according to the order of execution switch and works coordinately with the *ParMetis* [51] methods.

```
/* sync_map todo switch
 _SYNC_MAP_PDIST_INITIAL           // form the initial distribution.
 _SYNC_MAP_PDIST_SEND              // send the pdist object.
 _SYNC_MAP_PARTS_CREATE            // allocate memory for map->parts.
 _SYNC_MAP_PARTS_GATHER            // gather map->parts objects.
 _SYNC_MAP_EDIST_CREATE            // allocate memory for map->edist.
 _SYNC_MAP_PARTS_PROCESS           // RE-calculate pdist and edist.
 _SYNC_MAP_EDIST_SEND              // send the edist object.
 _SYNC_MAP_PARTS_DESTROY           // destroy the parts object.
*/
```

*map→parts*, *tmesh→adj*, and *tmesh→adjx* are originated from the known *CSR* format [51]. An example for the Distributed *CSR* format is shown in Figure A.1.




Processor 0:

| | |
|---|---|
| adj | 0 2 4 7 |
| adjx | 1 2 0 3 0 3 4 |
| parts | 0 3 6 |

Processor 1:

| | |
|---|---|
| adj | 0 3 5 7 |
| adjx | 1 2 5 2 5 3 5 |
| parts | 0 3 6 |

Figure A.1. The Distributed CSR Format

•*sync_map(rank pointer, map pointer, todo switch - integer):*
process according to the "SYNC_MAP todo switch".

### A.2.3. Mesh Partitioning

**tmesh.**[**ch**] defines the mesh object that holds the local elements and distributes the elements among processors.

```
/* TMESH */
typedef struct tmesh_ {
        int g_point;        // the number of points in the overall mesh.
```

```
        int g_tetra;        // the number of tetras in the overall mesh.
        int n_tetra;        // the number of local tetras.
        int * tetras;       // tetras in the local processor.
        float * points;     // all points.
        int * adj;          // adj and adjx keep adjacency
        int * adjx;         // information between elements.

} tmesh_ ;
```

The *send_tmesh* function has three operation flags. Initially, a tetrahedra structure is partitioned without any restriction and send to the processing nodes to compute the ideal partitioning information.

The partitioning function *ipar*, uses *ParMETIS_V3_PartMeshKway* from *ParMetis* library [51] to compute the new mapping. The data of the tetrahedral mesh is dispatched according to the final mapping. Moreover, the *ParMETIS_V3_Mesh2Dual* procedure is used to prepare the adjacency relationship between neighbor elements in different processing nodes. The *ipar_adj* function prepares the *DUAL* graph [51], and the required information is preserved locally in each node that we build for other components.

```
* SEND_TMESH - The todo switch
     _SEND_TMESH_INIT            //initialize the tmesh_ object.
     _SEND_TMESH_TETRAS          //send initial tetras.
                                 // to prepare the partitioning.
     _SEND_TMESH_SEND            //distribute the MESH data.
*/
```

The *ParMETIS_V3_PartMeshKway* routine is used to compute a $k$-way partitioning of a mesh on $p$ processors. The mesh can contain elements of different types. Parameters of the *ParMETIS_V3_PartMeshKway*;

- elmdist array describing how the elements of the mesh are distributed among the processors.
- eptr, eind arrays specifing the elements that are stored locally at each processor.
- elmwgt array storing the weights of the elements.
- wgtflag used to indicate if the graph is weighted.
- numflag used to indicate the numbering scheme.
- ncon used to specify the number of weights that each vertex has.
- ncommonnodes degree of connectivity among the vertices in the dual graph.
- tpwgts used to specify the fraction of vertex weight that should be distributed to each sub-domain for each balance constraint.
- ubvec array of size ncon that is used to specify the imbalance tolerance for each vertex weight.
- options array of integers that is used to pass parameters to the routine.
- edgecut the number of edges that are cut by the partitioning.
- part partition vector of the locally-stored vertices.
- comm *MPI* communicator.

The *ParMETIS_V3_Mesh2Dual* routine is used to construct a distributed graph given a distributed mesh. Parameters of the *ParMETIS_V3_Mesh2Dual*;

- elmdist array describing how the elements of the mesh are distributed among the processors.
- eptr, eind arrays specifying the elements that are stored locally at each processor.
- numflag used to indicate the numbering scheme.
- ncommonnodes degree of connectivity among the vertices in the dual graph.
- xadj, adjncy adjacency information disccussed in previous sections.
- comm *MPI* communicator.

The *tmesh_dist* function is the base procedure for the initial distribution of the mesh structure. It prepares the mapping of objects to processing nodes and distributes

elements according to this information. The *ParMetis* also supports heterogeneous distribution of the elements. Therefore, we can assign different weight degrees to different nodes and partition data according to the capacity of each node.

```
/* TMESH_DIST */
sync_map(rank,map,_SYNC_MAP_PDIST_INITIAL+_SYNC_MAP_PDIST_SEND+\
_SYNC_MAP_PARTS_CREATE);
send_tmesh(rank,map,fmesh,tmesh,_SEND_TMESH_INIT+_SEND_TMESH_TETRAS);
//fmesh_ is NULL for processing nodes.

if( rank_processingNode(rank))
                         ipar(rank,map,tmesh); //parmetis
sync_map(rank,map,_SYNC_MAP_PARTS_GATHER+_SYNC_MAP_PARTS_PROCESS);
sync_map(rank,map,_SYNC_MAP_PDIST_SEND);
send_tmesh(rank,map,fmesh,tmesh,_SEND_TMESH_SEND);

sync_map(rank,map,_SYNC_MAP_PARTS_DESTROY);

if(rank_processingNode(rank))
                         ipar_adj(rank,map,tmesh);
```

The *tmesh_* object supports the redistribution of elements among processors. Initially, elements of the mesh data are distributed fairly according to the load metrics. After a refinement step, processing nodes that produced many new tetrahedra, will handle more elements than others. Therefore, the distribution and the mapping can be recomputed observing the network cost metric; and then, mesh components can be resynchronized according to new the distribution to preserve the balanced partitioning. The *T_bucket_* object has suitable methods to export mesh components and import elements located in other processing nodes during such a redistribution operation.

•*send_tmesh(rank_ pointer, map_ pointer, fmesh_ pointer, tmesh_ pointer, todo switch):*

Sends and initializes the tmesh according to the todo switch.

•*ipar(rank_ pointer, map_ pointer, tmesh_ pointer):*

Runs in processing nodes only, prepares the map→parts.

•*ipar_adj(rank_ pointer, map_ pointer, tmesh_ pointer):*

Prepares the DUAL graph from the mesh, fills tmesh_→adj and tmesh_→adjx.

•*tmesh_dist(rank_ pointer, map_ pointer, tmesh_ pointer):*

Reads the data and distributes among processors.

## A.3. Mesh Operations

The data structure of the local mesh object has vertices, edges and tetrahedra. Each edge object has the list of tetrahedra it is owned by. The tetrahedron object keeps only the list of edges that form this tetrahedron, and it does not cover the face information. We can calculate the information of faces quickly when required in the propagation path process.

### A.3.1. The Point Object

**lPoint.**[**ch**] defines the vertex object of a *3-D* tetrahedral mesh structure. The *lPoint_* object keeps the coordinates and the identification number of points stored locally in each processor.

```
typedef struct lPoint_ lPoint_;
struct lPoint_ {
         int id;             // id of the Point.
        float x,y,z;         // x y z coordinates.
        float value;         // local value for the Point.
```

```
        int referrer_count;   // edges referring this object.
        lPoint_ * parent1;
        lPoint_ * parent2;
} ;
```

*lPoint_* objects are referred by the edge elements; they have the coordinates and the value of the vertex at that point for the given differential equation. Since, more than one edge can refer to a single vertex, the *lPoint_* object is removed when there is no referrer pointing to the vertex. It also keeps the parents if created in a refinement processing. Each *lPoint_* object has a single unique identifier which distinguishes them in the global space. Each processor starts from a sequence which will not intersect with other processors. During the operation of the *LEPP* synchronization, a unique identifier which is the smallest number among all other local sequences, is selected as the identifier for the effected neighbor elements in the border of a local structure.

Local vertex information is initially loaded from a *tmesh_* object, and processing nodes uses this information while defining new edges and midpoints. Only the gateway node keeps a list of pointers referring to points of the overall mesh. There are also methods to export the data into the communication object and import from a communication objects. Only the element in the border of a local mesh partition is synchronized if it is processed in the *LEPP* algorithm. Each processor works over its own elements without concerning the overall mesh structure. When refinement process is finished, the *gateway* node collects all new points from processing nodes and operates on them to prepare the final list of vertices of the total mesh structure.

- *lPoint_load_initial(rank, number of points,points a[0]=x,a[1]=y,a[2]=z, ..):*

  Loads the initial lPoint_ object from the array of coordinates.

- *lPoint_point2comm(comm_ pointer, lPoint_ LIST):*

  Exports all data to communication object to transmit to the gateway node.

- *lPoint_comm2point(comm_ pointer, table_ pointer, lPoint_ LIST):*

Gets points from a comm_ object and process them.

### A.3.2. The Edge Object

**lEdge.[ch]** defines the object that holds edge information in the local mesh structure.

```
typedef struct lEdge_ lEdge_;
struct lEdge_ {
       lPoint_ *point1;    // Point A.
       lPoint_ *point2;    // Point B.
       lPoint_ *midpoint; // midpoint between A and B.
       lEdge_  *subedge1; // sub-edges.
       lEdge_  *subedge2;
       List_   *tetras;   // tetras that are neighbors of this edge.
       int status;        // STATUS of the EDGE object.
      float length;       // length of the EDGE.
};
```

The *lEdge_* object keeps the tetrahedra which are adjacent and own this edge. Each edge object has a list of pointers to the neighbor tetrahedra. According to the *LEPP* algorithm, edges are selected and partitioned concerning the skeleton concept.

During the refinement process and other mesh operations, we need to obtain the correct edge between two points. Thus, the table of pointers is used to find and reserve the address of a *lEdge_* object. In *8T-LE* and *4T-LE,* we must select the longest-edge and that should be unique in the tetrahedron or in the triangle. The edge object has a simple methodology to handle the uniqueness for the length comparison If the length of two edges are equal, identifiers of the first and then the second vertices are compared to select one of them as the longer edge.

- *lEdge_select ( lEdge_ pointer, LIST newpoints):*

  Changes the status if selected, and creates the midpoint;
  assigns an unique id for the midpoint.

- *lEdge_divide( lEdge_ pointer):*

  Divides the Edge, two new sub-edges are created.

- *lEdge_get( table_ pointer, lPoint_ pointer point 1, lPoint_ pointer point 2 ):*

  Finds and returns the edge between point 1 and point 2, creates the edge if not found.

- *lEdge_list_divide_all( lEdge_ LIST, starting index):*

  Divides all edges in the list starting from a given index.

### A.3.3. The Tetrahedron Object

**lTetra.**[**ch**] defines the object that holds tetrahedra elements of the mesh structure.

```
typedef struct lTetra_ {
         T_bucket_ * tbucket;  // pointer to the T_bucket_ object.
         lEdge_    * edges[6]; // edges of the tetrahedron.
         int status;           // status of the tetrahedron.
                    // (_TETRA_NEW, _TETRA_SELECTED, _TETRA_DIVIDED)
}lTetra_ ;
```

Tetrahedra are stored in a tetra bucket (The *T_bucket_* object), and each tetrahedron has a pointer referring to its owner bucket. In order to simulate the Finite Element problem situation, we calculate the volume of each element and select the tetrahedron which has a larger volume than the specified. Selected elements will be marked to be refined and this information will be passes to the longest-edge refinement process as the initial input.

During the executing the *LEPP*-algorithm, we should propagate to the other elements that must be refined in order to conform the proper mesh structure. We should cover the *2-D* structure and check longest-edges of neighbor faces. Initially, selected edges are the edges of the selected tetrahedra; after the propagation process, new tetrahedra will be selected to be refined, and their addresses will be stored to be partitioned.

```
inline void lTetra_lepp_face( lTetra_ * lTetra_ptr, lEdge_ * edge,
List_ * edge2refine,List_* newpoints){
.......................................
 lTetra_2points_except(lTetra_ptr, &pointA, &pointB,
(edge)->point1, (edge)->point2);
 lTetra_edge_FIND(lTetra_ptr, pointA, (edge)->point1, edgeA);
    lTetra_edge_FIND(lTetra_ptr, pointA, (edge)->point2, edgeB);

    if(!lEdge_if_selected(edgeA) && !lEdge_if_selected(edgeB)){
        le= lEdge_compare_res(edge, edgeA);
        le= lEdge_compare_res(le, edgeB);

        if(le != (edge)){
              lEdge_list_ADD(edge2refine, le,newpoints);
.......................................
```

The information of faces is computed according to the longest-edge algorithm for *2-D* structures. The new edge marked to be refined has also the information of effected tetrahedra that own this element.

While partitioning a tetrahedron according to the *8T-LE*, we divide each element into two subcomponents; and then, we continue partitioning process in those new sub-elements.

```
inline void lTetra_divide_2( lTetra_ * lTetra_ptr, lTetra_ ** tetra1,
 lTetra_ **tetra2 , Table_ * table){
....................................
(*tetra1)= lTetra_new_points((lTetra_ptr)->tbucket, pointA,
pointB,longestEdge->point1,longestEdge->midpoint,table);

(*tetra2)= lTetra_new_points((lTetra_ptr)->tbucket, pointA,
pointB,longestEdge->point2,longestEdge->midpoint,table);
....................................

inline void lTetra_divide( lTetra_ * lTetra_ptr, Table_ * table){

....................................
lTetra_divide_2(tetra[di], & tetra1, & tetra2, table);
if(tetra1){
lTetra_free(tetra[di]);
tetra[di]=tetra1;
tetra[last]=tetra2;
last++;
.....................................
```

- *lTetra_lepp_face( lTetra_ pointer, lEdge_ pointer, List_ edge2refine, newpoints list):*

  Checks if other edges of neighbour face need to be refined.

- *lTetra_divide_2( lTetra_ pointer, lTetra_ pointer 1 -subtetra -address, lTetra_ pointer 2 - subtetra - address , Table_ pointer):*

  Divides tetrahedron into two new sub-elements.

- *lTetra_divide( lTetra_ pointer, Table_ pointer, List_ newtetralist):*

  Bisects a tetrahedron (8T-LE).

- *lTetra_list_divide_all(List_ tetra2refine list, starting index, Table_ pointer, List_ newtetralist):*

  Divides all tetras in the list.

### A.3.4. The Tetra Bucket

**T_bucket.[ch]** (The Tetra Bucket) keeps list of tetrahedra that are formed by refining an initial tetrahedron. New tetrahedra is formed after partitioning, and they are stored in the *T_bucket_* object. It also has a flag that is used to specify whether this bucket has element or not – this flag is checked while searching for elements that need to be processed for the refinement.

```
typedef struct T_bucket_{
 List_ * tetras;  //   the list of tetrahedrons within this bucket.
 List_ * neigh;   //   neighbor T_Buckets of this element.
                  //  (T_bucket has a single tetrahedron initially).
  int  status;    //  (_T_BUCKET_NEW_, _T_BUCKET_DONTCHECK_)
}T_bucket_;
```

Each tetrahedron bucket has a specific data structure that is the list of other elements having adjacent components. This information is crucial while synchronizing the overall structure, and it is used to inform the other processors about the state of the propagation path produced locally. While selecting edges and tetrahedra to be partitioned, we also prepare a communication object that will be used in the distributed *LEPP*-process.

The *T_bucket* has import and export facilities to transmit and gather objects from other computing nodes.

•*T_bucket_NEW(T_bucket_ pointer, index, tmesh_ pointer, map_ pointer, rank_ pointer, lPoint_ List, Table_ pointer):*
Creates a new T_bucket object, adds the tetrahedron to the list of elements, and prepares the neighbourhood information.

•*T_bucket_list_load(tmesh_ pointer, map_ pointer, rank_ pointer, lPoint_ list, Table_ pointer):*
Loads the initial tetrahedra for the refinement process.

•*T_bucket_tetra2comm( comm_ pointer, T_bucket_ List ):*
Exports the T_bucket object into the communication object for transmitting them.

•*T_bucket_comm2tetra( comm_ pointer, Table_ pointer, lPoint_ List, tetra array, number of tetras in the array):*
Imports the tetrahedron from a communication object in order to receive elements produced by computing nodes.

## A.4. The Refinement Process

Each of the mesh partition is stored by a local data structure that is optimized and prepared for the refinement according to longest-edge bisection in distributed environments. Computing nodes execute the *LEPP*-algorithm locally and synchronize the resulting propagation path to accomplish the overall mesh conformity.

### A.4.1. The Structure of a Local Mesh

**lMesh.[ch]** defines the object storing the local Mesh structure

```
typedef struct lMesh_ lMesh_ ;
struct lMesh_ {
  // USED by processing NODE
  List_ * points_new;   // The list of midpoints.
```

```
  Table_ * E2P;          // The Table that holds the
                         // relationship between Points and Edges.
  List_ * t_buckets;     // The list of Tetrahedron Buckets.
  List_ * edges2refine;  // Edges that will be refined.
  int   e2ref_index;     // current edge index,
                         // while processing LEPP.
  List_ * tetras2refine;// tetras that will be refined.
  comm_ *   __comm_edge;// communication object for the LEPP.

  // USED by the gateway NODE
  List_ * points;        // List of points in the Local Mesh.
.........................................
```

The *lMesh_* object is the main data-structure that holds the Parallel Tetrahedral Mesh Refinement object. It stores local tetrahedra, and local vertices, and also selected elements to be refined. The local *LEPP* procedure is accomplished by propagating elements sequentially.

```
inline void lMesh_lepp_tetra(lMesh_ * lmesh,rank_ * rank){
....................................
for(e2ref=(lmesh)->e2ref_index; e2ref < \
lEdge_list_getsize(lmesh->edges2refine);e2ref++){
    edge=lEdge_list_get((lmesh)->edges2refine, e2ref);
    for(t2ref=0;t2ref< lEdge_tetra_getsize(edge);t2ref++){
      tetra=lEdge_tetra_get(edge,t2ref);
      lTetra_list_ADD((lmesh)->tetras2refine,tetra);
      lMesh_edge_comm_ADD(lmesh, tetra, edge,rank_processingRank(rank));
 lTetra_lepp_face(tetra,edge,(lmesh)->edges2refine,(lmesh)->points_new);
.....................................
```

First, elements that need to be refined is selected, and the local propagation path is prepared according to bisection procedures. While propagating and computing the longest-edge propagation, we also select the other elements that will be affected, so they should be refined. After gathering the synchronization information from the gateway node, we finalize the selection operation for the refinement. Finally, we bisect edges and tetrahedra; and then, we delete former elements and create new objects.

The gateway node is used not only to read and distribute the mesh data; it is also used to prepare communication objects holding the information whether border elements of the local mesh partition require refinement or not. Therefore, the gateway node minimizes the number of network messages, and this situation is an important issue to enhance the performance of an MPI program [50]. Each processing node sends the information about the border element with the knowledge of neighboring processors that should take action. It specifies the breaking points for the overall propagation graph. The gateway node collects the effected elements and informs other processors to start refinement process for the classified element. Essential parts of the computation, preparing the neighbor processors of the local element in the partition border and selecting corner elements of the propagation path, is accomplished by processing nodes.

```
void lMesh_process(lMesh_ * lmesh, rank_ * rank){
 lMesh_refine_init(lmesh);
 retprocessval=
lTetra_check_volume(lmesh->t_buckets,lmesh->edges2refine,
lmesh->points_new);
    lMesh_lepp_process(lmesh,rank);
.....................................

void lMesh_lepp_process(lMesh_ * lmesh, rank_ * rank){
.....................................
 do{
 lMesh_lepp_tetra(lmesh,rank);
```

```
...................................
}while( lMesh_edge_comm_sync_process(lmesh,rank) );

lEdge_list_divide_all(lmesh->edges2refine,0);
lTetra_list_divide_all(lmesh->tetras2refine,0,lmesh->E2P,NULL);
....................................
```

•*lMesh_new_load(rank_ pointer, map_ pointer, tmesh_ pointer):*
Creates the local Mesh object, loads points for all nodes,
loads tetrahedra elements transmitted for processing.

•*lMesh_edge_comm_ADD(lMesh_ pointer,lTetra_ pointer, lEdge_ pointer, rank of this processor):*
Prepares a communication object for the propagation path synchronization.

•*lMesh_lepp_tetra( lMesh_ pointer, rank_ pointer):*
Executes the local longest-edge bisection algorithm.

•*lMesh_edge_comm_sync_process(lMesh_ pointer, rank_ pointer):*
Communicates with the gateway node to finish synchronization,
and finalize the propagation path.

•*lMesh_lepp_process(lMesh_ pointer, rank_ pointer):*
Bisects selected elements of the mesh which are tetrahedra and edges.

•*lMesh_send_points(lMesh_ pointer,rank_ pointer):*
Sends new points to the gateway node after the refinement operation.

• *lMesh_recv_points(lMesh_ pointer, rank_ pointer):*
collects new points from computing nodes

• *lMesh_send_tetras(lMesh_ pointer, rank_ pointer)*

Send new tetrahedra after bisection operation to the gateway node.

• *lMesh_recv_tetras(lMesh_ pointer, rank_ pointer):*

Collects new tetrahedra from computing nodes.

•*lMesh_2fmesh(lMesh_ pointer):*

Converts the local Mesh object to an fmesh_ object,

so it can be written to the output file.

## A.5. Communication and the *LEPP* Synchronization

*PTMR* uses communication objects in the *MPI* environment in order to minimize the network traffic and handle the dynamic data transfer between processors. The flexible structure of this object supplies ease of implementation and performance gain. Instead of transmitting many pieces of small network packets, sending fewer messages containing data larger in size increases the efficiency [50]. Thus, the number of messages is reduced in Inter-process communication environments.

### A.5.1. The Communication Array

**comm.[ch]** defines the communication array with the inital size of elements that will stored in this object. It organizes the inter-process communication and memory allocation. The required space for the data array is pre-allocated, and the used space is increased according to the frequency of the element addition in order to avoid many memory allocation calls. There are also some pre-defined signals to inform the receiving nodes to declare the end of waiting actions.

```
typedef struct comm_ {
  int size;   // size of elements in the array.
  int msize;  // the number of elements in the array.
  int * data; // data array.
```

```
  int inc;    // increment count.
}comm_;
```

The communication object used to transfer the information about the refined edges of a remote processor. The refinement information has four elements; id of points, id of the mid-point, and the ranks of the affected node that are the neighbors of this element. During the transmission of vertices, we use three elements, values of the coordinates; fours element for transfering a tetrahedron, ids of the points forming the tetrahedron. Figure A.2 presents the communication array. In the given example, the edge between vertices with *id* 203 and 19 will be bisected, and the *id* of the selected mid-point is 10034; the processing node with rank 18 has also this element and should also refine it

Propagation Path Information

| ID of point1 | ID of point2 | ID of the midpoint | rank of the neighbor processor |
|---|---|---|---|

exp:

203 19 10034 18

Vertex Information

| value of the x-coordinate | value of the y-coordinate | value of the z-coordinate |
|---|---|---|

Tetrahedron Information

| ID of point1 | ID of point2 | ID of point3 | ID of point4 |
|---|---|---|---|

Figure A.2. The Communication Array

•*comm_add(comm_ pointer, data ):*

Adds elements to the comm_ object that is defined with the size of an element.

•*comm_send(comm_ pointer, processor id):*

Sends the object to the processor with all elements inside.

•*comm_recv(comm_ pointer, processor id):*

Receives elements from the processor given.

•*comm_set_signal_process(comm_ pointer):*

Sets or defines a signal inside the comm_ object.

### A.5.2. The LEPP Facility

**lp.[ch]** defines the object that is used to synchronize and inform the propagation of the *LEPP* process to relevant computing nodes.

```
typedef struct lp_{
 int id1;    // point id 1.
 int id2;    // point id 2.
 int midpoint;  // the midpoint between points.
 char * neigh;  // the list of processors that own this edge.
..........................
} lp_ ;
```

The gateway node collects the information about the results of the local propagation from processing nodes. This data contains the elements that are in the border of the local mesh partition. It also includes the identifiers of nodes that own those marked elements, and this data is packed within the *lp_* object; and then, a collection of elements sent to the computing nodes using the communication object *comm_*. Each processing node uses this information to select and start the refinement from received elements if it is required. Synchronizing the borders of propagation graph, which is partitioned among processors as a result of the local mesh processing, is accomplished by summarizing the received bisection information and distributing among the concerned

processors.

```
void lp_process(lMesh_ * lmesh, rank_ * rank){
.........................................
      count=lp_comm_list_recv(lmesh->__COMM_RECV);
      if(count <= 0){
      lp_comm_list_set_signal(lmesh->__COMM_SEND);
      lp_comm_list_send(lmesh->__COMM_SEND);
.........................................
    lp_import(lmesh,rank,iturn);  //collect
    lp_export(lmesh,rank);        // summarize
    lp_comm_list_send(lmesh->__COMM_SEND);
..........................................
```

• *lp_import(lMesh_ pointer, rank_ pointer):*

Initializes the lp_ object and collects information from communication nodes.

• *lp_export(lMesh_ pointer, rank_ pointer):*

Summarizes and matches edges; the relevant processors should be acknowledged.

• *lp_process(lMesh_ pointer, rank_ pointer):*

Receives all of the communication data about the LEPP syncronization,

and finishes the process by signalling nodes.

# APPENDIX B: PTMR: Manual

The *PTMR* utility has two main components. The first component contains modules that implement the mesh refinement according to the parallel *LEPP* algorithm. The refinement is accomplished locally according to longest-edge bisection, and propagation paths are synchronized in each affected processing nodes. The second component includes modules that are responsible for partitioning the initial mesh structure in such a way that communication cost will be minimized. The network communication is used while processing the refinement algorithm and synchronizing processing nodes. Moreover, the *PTMR* utility includes some auxiliary objects and specialized data structure.

The directory hierarchy of the package contains source files and job submission scripts for cluster environments. The source code and all other resources can be download from the web address (*http://cct.lsu.edu/˜balman/PTMR*).

**src/** source files

**sbin/** executable scripts used to submit test jobs, install required libraries, assign parameters and set surrounding environment variables

**Mesh/** example files of tetrahedral mesh structure

**test/** output of the test results

**lib/** required libraries used by *PTMR*

**bin/** binary files including sequential and multi-threaded versions

## B.1. Compilation

The program is tested with *GNU C* [67] compiler; the Makefile inside the package is capable of producing different executable versions. There are three *debug* and *timing* levels and the amount of ouput results about the execution steps changes according to these parameters. Debugging parameters are defined during the compilation;

```
make "DEBUG_LEVEL=-D_DEBUG_LEVEL_3=1
"TIMING_LEVEL=-D_TIMING_LEVEL_3=1"    ptmr
```

*PTMR* has been analyzed with the memory debugger *electric-fence* [68] and the *GNU* profiler *gproff* [69] to optimize the implementation. We can build an executable which is able to output the profile information, so the code can be examined.

```
compile: $(SRCS)
   $(CC) -Wall $(CFLAGS_DEFAULT) -c $?
debugcc: $(SRCS)
   $(CC) -Wall $(CFLAGS_DEBUG) -c $?
ptmr:  compile
   $(CC) $(CFLAGS_DEFAULT) -o $@ $(OBJS) $(LDFLAGS_DEFAULT)
ptmrdebug: debugcc
   $(CC) $(CFLAGS_DEBUG) -o $@ $(OBJS) $(LDFLAGS_DEBUG)
```

- gcc
  The GNU Compile Collection (gcc) includes compilers for C, C++, Fortran and JAVA.
  (http://gcc.gnu.org/).
- electric-fence
  Electric Fence is used for the debugging of memory allocations for C programs (http://linux.maruhn.com/sec/electricfence.html).
- gproff
  The GNU profiler (gproff) is one of the common profilers collecting information during the execution the program.
  (http://www.cs.utah.edu/dept/old/texinfo/as/gprof_toc.html).

Each component of the program is modularized with seperate *.C* and *.H* files. Objects of the *PTMR* have independent fuctions and methods, so higher-level methods can facilitate them without writing similar code segments. The utility needs to finish

the execution within acceptable time limits; therefore, *C macros* and *inline*d function calls are used for frequently referenced modules in order to conform performance limitations.

```
/* LTETRA.H */
/* 3-D TETRAHEDRAL MESH -- TETRAHEDRON OBJECT */
#ifndef __PTMR__LTETRA_H__
   #define __PTMR__LTETRA_H__
/* headers */
#include "local.h"
#include "utils.h"
#include "list.h"
#include "table.h"
#include "comm.h"
#include "lPoint.h"
#include "lEdge.h"
#include "T_bucket.h"
........................................
........................................

/* LTETRA.C */
/* 3-D TETRAHEDRAL MESH -- TETRAHEDRON OBJECT */
/* headers */
#include "local.h"
#include "utils.h"
#include "list.h"
#include "table.h"
#include "comm.h"
#include "lPoint.h"
#include "lEdge.h"
#include "lTetra.h"
```

```
#include "T_bucket.h"
.....................................
```

## B.2. Libraries

Some common data structures, such as binary balanced tree and pointer arrays are used from the *Glib 2.6.0* [70] library. In order to accomplish the initial distribution of the mesh structure, domain decomposition technique of *ParMetis 3.1* [51] is used. The *PTMR* framework is implemented with *C* language according to *MPICH2* [52] standards for inter-process communication, and dependent libraries for the execution of program are:

- Glib 2.6.0

  GLib is a lower-level library that provides many useful definitions and functions, and those are most commonly used but not limited to creation of GDK [71] and GTK [72] applications.

  (*http://www.gtk.org/download)*
- ParMetis 3.1

  Parmetis is a family of programs for partitioning unstructured graphs and hypergraphs and computing fill-reducing orderings of sparse matrices. It provides Static and Dynamic Graph partitioning and Static Mesh Partitioning.

  (*http://www-users.cs.umn.edu/˜karypis/metis/)*
- mpich2-1.0.2

  MPICH2 is an implementation of the Message-Passing Interface (*MPI*) providing important features for different platforms, including clusters, SMPs, and massively parallel processors.

  (*http://www-unix.mcs.anl.gov/mpi/mpich2)*

## B.3. Testing

Since debugging programs for distributed environments has many difficulties, we also implemented code segments for this purpose. Moreover, total time elapsed wihin the referred procedure is also collected. It has *three* different *debug* and *timing* levels defining the amount of information to be printed. We define the action of each function in header files.

```
/* LMESH_EDGE_COMM_SYNC_PROCESS */
/*_LMESH_EDGE_COMM_SYNC_PROCESS_DEBUG_1 : print num of edges processed
 _LMESH_EDGE_COMM_SYNC_PROCESS_DEBUG_3  : print received comm_ object
 _LMESH_EDGE_COMM_SYNC_PROCESS_TIMING   : time elapsed during the
send/receive operation and the overall time */
#ifdef _DEBUG_LEVEL_2
 #define _LMESH_EDGE_COMM_SYNC_PROCESS_DEBUG_1
#endif
#ifdef _DEBUG_LEVEL_3
 #define _LMESH_EDGE_COMM_SYNC_PROCESS_DEBUG_2
#endif
#ifdef _TIMING_LEVEL_3
 #define _LMESH_EDGE_COMM_SYNC_PROCESS_TIMING
#endif
/* check */
#ifdef _LMESH_EDGE_COMM_SYNC_PROCESS_DEBUG_2
 #ifndef _LMESH_EDGE_COMM_SYNC_PROCESS_DEBUG_1
 #define _LMESH_EDGE_COMM_SYNC_PROCESS_DEBUG_1
 #endif
#endif
........................................
```

Each function or method executes the debug or timing segment according to the given parameters set during the compilition of the program.

```
........................................
#ifdef _LMESH_EDGE_COMM_SYNC_PROCESS_DEBUG_2
 lp_edge_comm_print(lmesh,"lMesh_edge_comm_sync: comm_ received");
#endif
........................................
/* DEBUG */
#ifdef _LMESH_EDGE_COMM_SYNC_PROCESS_DEBUG_1
 printf("lMesh_edge_comm_sync_process: %d edges(comm_) processed \
( process rank = %d) %d new edge selected\n",lp_edge_comm_getsize\
(lmesh),rank_processingRank(rank),
lEdge_list_getsize(lmesh->edges2refine)\
-(lmesh->e2ref_index));
#endif
/* TIMING */
#ifdef _LMESH_EDGE_COMM_SYNC_PROCESS_TIMING
 printf("+ lMesh_edge_comm_sync_process: [%f sec ] %d edges(comm_)\
 processed  %d new edge selected\n",
 timing_get("lMesh_edge_comm_sync_process"),\
lp_edge_comm_getsize(lmesh),lEdge_list_getsize(lmesh->edges2refine)\
-(lmesh->e2ref_index));
#endif
........................................
```

PTMR is organized to scale and manage huge amount of data in inter-process communication environments. Source package includes job submission scripts seting the parameters and preparing the environment for cluster machines in which *MPI* jobs can be executed.

**prep_lib.sh** installs required libraries if them are not found in the cluster system.

**prep_bin.sh** compiles source files and forms executable files.

**testsub.sh** creates a test submission script according to given parameters and submit job to the job queue.

**processRES.sh** processes the output of test results and organizes them for comparison of different metrics.

*PTMR* takes four parameters and uses default values if they are not defined. First one is the filename of the input file and format of it is specified in Appendix A. Rest of the parameters are the value of the maximum valume that is used to simulate the initial tetrahedron selection and maximun number of loops which is set if we want to run the refinement steps more than once to generate larger mesh outputs.

### B.3.1. Samples

We used the *Tetgen* [60, 61] to generate input files containing conforming mesh structures.

- tetgen

  Tetgen is a mesh generation utility for 3D domains.

  (http://tetgen.berlios.de/)
- tetview

  Tetview is a graphic program used to view tetrahedral meshes.

  (http://tetgen.berlios.de/tetview.html).
- medit

  Medit is an another vizualization tool.

  (http://www.ann.jussieu.fr/˜frey/logiciels/medit.html)

*Tetview* [66] and *Medit* [65] are the used programs to view the consequences of mesh outputs after the refinement process.

# REFERENCES


1. Plaza, A., M. Padron, and G. Carey, "A 3D Refinement/Derefinement Algorithm for Solving Evolution Problems", *15th IMACS World Congress in Scientific Computing, Modelling and Applied Mathematics*, Vol. 32, pp. 401–418, 1997.

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