OPTIMALITY CONDITIONS AND SOLUTION METHODS FOR SOME CLASSES OF NONCONVEX AND NONDIFFERENTIABLE OPTIMIZATION PROBLEMS

PhD Dissertation

Mohamed Muhumed HASSAN

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Mohamed Muhumed HASSAN

PhD DISSERTATION

Department of Mathematics
Programme in Applied Mathematics
Supervisor: Prof. Dr. Refail KASIMBEYLİ

Eskişehir
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FINAL APPROVAL FOR THESIS

This thesis titled OPTIMALITY CONDITIONS AND SOLUTION METHODS FOR SOME CLASSES OF NONCONVEX AND NONDIFFERENTIABLE OPTIMIZATION PROBLEMS has been prepared and submitted by Mohamed Muhumed HASSAN in partial fulfillment of the requirements in "Eskişehir Technical University Directive on Graduate Education and Examination" for the Degree of PhD in Mathematics Department and has been examined and approved on 03/07/2025.

Committee Members	**Title, Name and Surname**	**Signature**
Member (Supervisor)	: Prof. Dr. Refail KASIMBEYLİ	......
Member	: Doç. Dr. İlknur ATASEVER GÜVENÇ	......
Member	: Doç. Dr. Gülçin DİNÇ YALÇIN	......
Member	: Doç. Dr. Ali Hakan TOR	......
Member	: Dr. Öğr. Üyesi Temel ERMİŞ	......

....
Prof. Dr. Harun BÖCÜK
Director of the Institute of Graduate Programs

03/07/2025

SUPERVISOR APPROVAL

PhD student Mohamed Muhumed HASSAN, under my supervision, has completed this thesis titled OPTIMALITY CONDITIONS AND SOLUTION METHODS FOR SOME CLASSES OF NONCONVEX AND NONDIFFERENTIABLE OPTIMIZATION PROBLEMS. According to my inspections, the work is scientifically and ethically appropriate for the student to the thesis defense exam.

Supervisor
Prof. Dr. Refail KASIMBEYLİ

ABSTRACT

OPTIMALITY CONDITIONS AND SOLUTION METHODS FOR SOME CLASSES OF NONCONVEX AND NONDIFFERENTIABLE OPTIMIZATION PROBLEMS

Mohamed Muhumed HASSAN
Department of Mathematics
Programme in Applied Mathematics
Eskişehir Technical University, Institute of Graduate Programs, July 2025
Supervisor : Prof. Dr. Refail KASIMBEYLİ

This dissertation investigates optimality conditions and solution methods for specific classes of nonconvex and nondifferentiable optimization problems. It introduces novel modifications to Zoutendijk's feasible direction method and the Frank–Wolfe algorithm by incorporating the concept of radial epiderivatives in place of classical derivatives.

The study begins with a foundational overview of optimization principles, followed by an in-depth exploration of radial epiderivatives, including new theoretical developments and illustrative examples. Subsequent chapters detail the proposed algorithmic modifications and provide rigorous convergence analyses. This work aims to enhance the theoretical framework and practical applications of optimization techniques in complex, nonconvex, and nondifferentiable contexts.

Keywords: Nonsmooth and nonconvex optimization, Frank—Wolfe algorithm, Radial epiderivative, Zoutendijk method.

ÖZET

BAZI SINIF DIŞBÜKEY OLMAYAN VE TÜREVLENEMEYEN OPTİMİZASYON PROBLEMLERİ İÇİN EN İYİLİK KOŞULLARI VE ÇÖZÜM YÖNTEMLERİ

Mohamed Muhumed HASSAN
Matematik Anabilim Dalı
Uygulamalı Matematik Bilim Dalı
Eskişehir Teknik Üniversitesi, Lisansüstü Eğitim Enstitüsü, Temmuz 2025
Danışman : Prof. Dr. Refail KASIMBEYLİ

Bu tez, belirli sınıflardaki konveks olmayan ve türevlenemeyen optimizasyon problemleri için optimalillik koşullarını ve çözüm yöntemlerini araştırmaktadır. Klasik türevlerin yerine radyal epitürev kavramını entegre ederek Zoutendijk'in uygulanabilir yön yöntemi ve Frank–Wolfe algoritmasına yenilikçi değişiklikler getirmektedir.

Bu çalışma, optimizasyon prensiplerinin temel bir incelemesiyle başlayıp, radyal epiderivatifler üzerine derinlemesine bir keşif sunarak yeni teorik gelişmeler ve açıklayıcı örnekler içermektedir. Sonraki bölümler, önerilen algoritmik değişiklikleri ayrıntılı olarak ele almakta ve titiz yakınsama analizleri sağlamaktadır. Bu çalışma, karmaşık, konveks olmayan ve türevlenemeyen bağlamlarda optimizasyon tekniklerinin teorik çerçevesini ve pratik uygulamasını geliştirmeyi amaçlamaktadır.

Anahtar Sözcükler : Düzgün ve konveks olmayan optimizasyon, Frank– Wolfe algoritması, Radyal epitürev, Zoutendijk yöntemi.

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Finally, amidst the relentless demands of a PhD—the late nights, the unanswered questions, the pursuit of precision—I am reminded daily that life's simplest moments often hold the deepest meaning. To my little ones, Amira and Akram, whose laughter kept me grounded and whose bedtime stories were a welcome break from equations: this journey would have been a shadow of itself without your light.

To all who supported me, knowingly or unknowingly: this work stands as a testament to your generosity.

Mohamed Muhumed HASSAN

03/07/2025

STATEMENT OF COMPLIANCE WITH ETHICAL PRINCIPLES AND RULES

I hereby truthfully declare that this thesis is an original work prepared by me; that I have behaved in accordance with the scientific ethical principles and rules throughout the stages of preparation, data collection, analysis and presentation of my work; that I have cited the sources of all the data and information that could be obtained within the scope of this study, and included these sources in the references section; and that this study has been scanned for plagiarism with "scientific plagiarism detection program" used by Eskişehir Technical University, and that "it does not have any plagiarism" whatsoever. I also declare that, if a case contrary to my declaration is detected in my work at any time, I hereby express my consent to all the ethical and legal consequences that are involved.

Mohamed Muhumed HASSAN

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GLOSSARY OF SYMBOLS AND ABBREVIATIONS

$\mathbb{R}$	:	The set of all real numbers
$\mathbb{R}_+$	:	The set of non-negative real numbers
c, λ	:	Non-negative constants
$u,\ v$	:	Vectors in $\mathbb{R}^n$
$f^r(\overline{x}; x)$	:	Radial epiderivative of f at $\overline{x}$ in the direction of x
min	:	Minimize
max	:	Maximize
b_2	:	Vector of nonbinding constraints' right-hand sides.
A_2	:	Nonbinding constraints coefficient matrix.
FWA	:	Frank–Wolfe Algorithm
Ex.	:	Example

1. INTRODUCTION

1.1. Basics of Optimization

1.1.1. Historical background

Mathematical optimization is the science and art of choosing the best possible element from a range of possible candidates based on certain criteria. Egyptian and Greek mathematicians were the first to solve optimization problems in their geometrical studies. For example, Euclid, the father of geometry, investigated the concepts of the minimal distance between a line and a given point. Before calculus was invented (or discovered) in 17th century by G. W. Leibniz and I. Newton, mathematicians investigated only trivial mathematical optimization scenarios such as the well-known secretary problem and optimal dimensions of a geometric figure. In the 19th century, the earliest algorithms for solving optimization problems were presented. The following list highlights the most prominent figures who have profoundly influenced the field of optimization through their groundbreaking work.

- Around 300 BC, *Euclid* investigated optimization problems such as "among all rectangles with fixed perimeter, which rectangle has the greatest area?"
- In the 17th century, *J. Kepler* computed the optimal dimensions for maximum volume of a wine barrel.
- A few decades after Kepler, *Isaac Newton* examined the body with minimal resistance.
- In the 18th century, *J. L. Lagrange* formulated the problem of minimal surfaces (also known as Plateau's problem).
- In 1724, *G. Monge* studied the transportation problem.
- In the 19th century, *J. Fourier* introduced linear programming problems.
- In 1905, *J. Jensen* introduced the concept of convexity of functions.
- *H. Hancock* wrote the first textbook on optimization titled *Theory of Maxima and Minima* in 1917.
- In 1939, *L.V. Kantorovich* proposed an LP-problem and suggested a method on how to solve it.

- *J. Neumann* developed the field of operations research and the concept of duality.
- *G. Dantzig* formulated the simplex method for solving linear programming In 1947.
- In 1950s, *J. Nash* introduced the concept of Nash equilibrium in game theory.
- In 1953, *R. Bellman* developed dynamic programming.
- in 1939, *W. Karush* derived the KKT conditions, which were later rediscovered by *H. Kuhn* and *A. Tucker* in 1951.
- In 1984, *N. Karmarkar* proposed the interior-point methods for LP problems.
- *Y. Nesterov* proposed accelerated gradient methods in 1983.
- In the 1970s, *R. Rockafellar* laid the groundwork for convex and variational analysis.
- In the 2000s, *R. kasimbeyli*, developed concepts ranging from radial epiderivatives to weak subgradients, cone separation theorems, and more.

It should be noted that the names on the list above are but a few among the most prominent mathematicians in the field. For more details, see [12, 18, 21, 27, 29, 35, 38, 39, 40, 42].

1.1.2. Basic terminology

- Objective function: A function $f : \mathbb{R}^n \to \mathbb{R}$ to be optimized. To minimize (or maximize) is to find a point x^* in $\mathbb{R}^n$ that satisfies $f(x^*) \leq f(x)$ (or $f(x^*) \geq f(x)$) for all x.
- Decision variables: These are the variables $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$ that we are trying to solve for.
- Constraints: The set S of rules that the decision variables must satisfy.

$$S = \{x \in \mathbb{R}^n \mid g_i(x) \leq 0 \ (i = 1, \cdots, m),\ h_j(x) = 0 \ (j = 1, \cdots, p)\}.$$

- Feasible Region: The set S of all points that satisfy each and every constraint.
- Optimal solution: A point $x^* \in S$ such that $f(x^*) \leq f(x) \ \ \forall x \in S$ (for minimization) and $f(x^*) \geq f(x) \ \forall x \in S$ (for maximization).
- Local minimum: A point x^* such that $f(x^*) \leq f(x) \ \forall x \in S \cap \mathcal{N}(x^*, \epsilon)$, where $\mathcal{N}(x^*, \epsilon)$ is the $\epsilon-$neighborhood of x^*.

- Global minimum: A point x^* that satisfies $f(x^*) \leq f(x)$ for all $x \in S$.
- Convex set: A set S where $\forall x_1, x_2 \in S$ and $\lambda \in [0,1]$ we have $\lambda x_1 + (1-\lambda)x_2 \in S$.
- Convex function: A function $f(x)$ where

$$f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda)f(x_2) \;\; \forall x_1, x_2 \in S \text{ and } \;\lambda \in [0,1].$$

- Concave function: A function $f(x)$ where

$$f(\lambda x_1 + (1-\lambda)x_2) \geq \lambda f(x_1) + (1-\lambda)f(x_2) \;\; \lambda \in [0,1] \quad \text{and} \quad \forall x_1, x_2 \in S.$$

In mathematical optimization, we usually maximize or minimize an objective function based on specific criteria. In this thesis, we investigate and examine ways to optimize the following problem:

$$\begin{array}{rl} \text{Min/Max} & f(x) \\ \text{Subject to} & x \in S \end{array} \tag{1.1}$$

where $f(x)$ is a nondifferentiable and nonconvex function with a linear and norm term, whereas S is a set of compact polyhedra.

Despite the wealth of literature on optimization methods, the problem of minimizing or maximizing nondifferentiable and nonconvex objective functions—particularly those involving linear and norm terms—remains a significant challenge. These types of problems frequently arise in practical applications such as machine learning, signal processing, and resource allocation. This thesis contributes to this field by proposing novel theoretical tools (via radial epiderivatives) and algorithmic approaches that can effectively handle such complex problems.

1.2. Literature Review

The mathematical optimization of a linear function over a compact polyhedron originated in linear programming. One of the most basic and earliest concave linear programming problems is structured as follows:

$$\begin{array}{rl} \text{Min/Max} & \langle v, x \rangle \\ \text{Subject to} & x \in S \end{array} \tag{1.2}$$

George Dantzig's simplex method [26] was the first algorithm to systematically solve a linear programming problem by traversing the vertices of the feasible set. Mathematicians believed — some even tried to prove — that the time complexity of the simplex algorithm was polynomial. However, Klee *&* Minty [4] demonstrated through illustrative examples that the simplex method could — in the worst-case scenario — visit every single vertex before reaching the optimal point. An example of these pathological problems is described in equation (1.2).

$$
\begin{array}{ll}
\min & -2^{n-1}x_1 - 2^{n-2}x_2 - 2^{n-3}x_3 - \cdots - 2x_{n-1} - x_n \\
\text{s.t.} & x_1 \leq 5 \\
 & 4x_1 + x_2 \leq 25 \\
 & 8x_1 + 4x_2 + x_3 \leq 125 \\
 & \vdots \\
 & 2^n x_1 + 2^{n-1}x_2 + 2^{n-3}x_3 + \cdots + 4x_{n-1} + x_n \leq 5^n \\
 & x_i \geq 0 \quad \forall\ i = 1, 2, \cdots, n.
\end{array}
\tag{1.3}
$$

Linear programming problems, such as problem (1.3), showed that the time complexity of the simplex algorithm is, in fact, exponential.

In order to understand what “exponential time complexity” means, let us consider problem (1.3) with 100 variables. This means that the algorithm would visit $2^{100} - 1 \approx 1.3 \times 10^{30}$ vertices before arriving at the optimal point. If a computer were to operate continuously, executing one quintillion (a billion billion or 10^{18}) iterations every second, it would require approximately

$$\frac{1.3 \times 10^{30}}{10^{18} \cdot 3 \times 10^{7}} = 43,333 \text{ years.}$$

In the search for faster algorithms with polynomial time complexity, *Khachiyan* published his ellipsoid method, and *Karmarkar* proposed his interior point method [10]. Even though the interior point methods and ellipsoid method had polynomial-time complexity in practice, Dantzig’s simplex method usually outperforms them for most problems. Subsequent contributions by mathematicians A. Nemirovski and Y. Nesterov [16] enhanced the computational efficiency of Karmarkar’s interior-point algorithm.

1.2.1. Concave minimization

Concave optimization is a vast field with various applications in economics, computer science, and mathematics. The study of concave optimization can be traced back to early works in nonlinear programming. Consider the problem

$$\begin{array}{cc} \min & f(x) \\ \text{Subject to} & x \in S \end{array} \tag{1.4}$$

where $x = (x_1, x_2, \cdots, x_n)$ and $S = \{x \in \mathbb{R}^n \mid Ax \leq b,\ x \geq 0\}$. The function $f(x)$ is under the assumption that it is concave, well-defined for all $x \in S$, and the constrained optimal value of $f(x)$ is finite.

The optimal solution of (1.4) occurs at an extreme point of S. However, problem (1.4) is particularly challenging due to the existence of multiple local minima.

Hoang Tuy [2], proposed a method for solving (1.4) by identifying local minima using hyperplane cuts (Tuy's cut) to exclude non-promising regions. This work was further extended through the development of branch-and-bound techniques and cutting plane methods, which remain central to solving concave minimization problems today.

In [6], A. Taha introduced a computational methodology for solving problem (1.4), structured around a branch-and-bound framework integrated with a tailored cutting plane strategy to refine convergence.

Benson and Sayin [17] introduced a finite algorithm that combines a neighbor generation process with branch-and-bound search. Mangasarian and Meyer [28], and Mangasarian [30, 31] examined existence of solutions for NP-hard absolute value equaitons (AVEs).

Current methods for addressing problem (1.4) typically use either branch-and-bound, outer approximation, inner approximation, extreme point ranking, cutting planes, or a combination of these approaches see [2, 5, 8, 11, 13, 14, 15, 17, 37].

Concave and nonconvex optimization problems frequently arise in real-world scenarios. For instance, deep learning involves navigating nonconvex loss functions; portfolio optimization must often account for piecewise and nondifferentiable risk metrics; and network resource allocation often involves minimizing concave utility functions.

These applications motivate the need for more robust theoretical tools and algorithms capable of tackling such problems effectively.

This dissertation is structured as follows:

Section 1: This section lays the groundwork by introducing optimization fundamentals. It begins with a historical overview and clear definitions of key concepts, while also reviewing existing literature with a focus on concave minimization. Together, these subsections provide a solid base of knowledge that informs the more advanced discussions in later sections.

Section 2: This section explores the concept of radial epiderivatives in depth. It starts by revisiting previously known results from a variety of sources, then systematically introduces new theorems, corollaries, and examples. The discussion further extends to cover the radial epiderivative for pointwise minimum and maximum functions, piecewise continuous functions, sums and differences, convex combinations, and compositions, offering a comprehensive analysis of the subject.

Section 3: Dedicated to proposed methods, this section introduces a modified version of Zoutendijk's method, complete with numerous illustrative examples. It also details a slightly altered Frank–Wolfe algorithm, adapted to be effective for nonconvex and nondifferentiable functions, thereby broadening the practical applications of these optimization techniques.

Section 4: The final section is devoted to the convergence analysis of the newly developed proposed methods. It rigorously examines the performance and theoretical underpinnings of these algorithms, ensuring that the advancements presented in the earlier sections are both sound in theory and effective in practice.

2. RADIAL EPIDERIVATIVE THEORY

2.1. Introduction

The notion of the radial epiderivative has emerged as a significant tool in nonconvex mathematical optimization. Early developments arose in distinct frameworks: Flores-Bazán [23, 25] introduced the concept within a topological setting, whereas Jahn and Rauh [19] proposed a related approach through contingent epiderivatives. Although Flores-Bazán (2001) offered a rigorous theoretical generalization of derivative-based optimality conditions for nonconvex set-valued optimization, the resulting formulations remained largely abstract, with limited immediate applicability to computational or algorithmic implementations.

In contrast, R. Kasimbeyli [32] pioneered an independent framework for radial epiderivatives, introducing definitions and calculation techniques tailored to nonconvex optimization. His work diverged fundamentally from earlier settings, providing a rigorous mathematical foundation that resolved ambiguities in prior formulations and enabled direct applications in numerical analysis. Kasimbeyli's approach systematically developed radial epiderivatives as standalone tools, equipping them with properties essential for algorithmic implementation. Building on this foundation, Kasimbeyli and G. D. Yalcin [44, 45] established explicit connections between radial epiderivatives and nonsmooth optimization concepts—such as weak subdifferentials, Clarke's directional derivatives, Rockafellar's subderivatives, and classical directional derivatives—thereby bridging theoretical advances with practical computational methodologies.

We begin by presenting foundational definitions for radial epiderivative theory, including the closed radial cone, the graph and domain of a set-valued function, and the epigraph in partially ordered spaces.

Definition 2.1. *[33, Definition 1.2] Let U be a non-empty subset of a real normed space $(Z, \|\cdot\|_Z)$. The closed radial cone $R(U, \overline{z})$ at a point $\overline{z} \in cl(U)$ consists of all vectors $z \in Z$ satisfying the following: there exists a sequence of positive scalars $\{\lambda_n\}$*

and a sequence $\{z_n\} \subset Z$ *such that*

$$\lim_{n\to\infty} (z_n - \overline{z}) = z,$$

where convergence is understood in the norm topology of Z.

The closed radial cone $R(U, \overline{z})$ can equivalently be characterized as

$$R(U, \overline{z}) = cl(\text{cone}(U - \overline{z})),$$

where $\text{cone}(U - \overline{z})$ denotes the conic hull of the translated set $U - \overline{z}$, defined as the minimal closed conical set containing $U - \overline{z}$. Here, $\overline{\cdot}$ represents the topological closure operator in the normed space Z.

Definition 2.2. *[33, Definition 1.3] Consider two real normed spaces* $(X, \|\cdot\|_X)$ *and* $(Y, \|\cdot\|_Y)$, *a non-empty subset* $S \subseteq X$, *and a set-valued mapping* $f \colon S \to Y$. *The following notions are defined:*

(1) The graph of f, *denoted* $\text{graph}(f)$, *is defined as the collection of all pairs*

$$(x, y) \in X \times Y \quad \textit{such that} \quad x \in S \textit{ and } y \in f(x).$$

(2) The domain of f, *written* $\text{dom}(f)$, *is the set of all points* $x \in X$ *where* $f(x)$ *is non-empty:*

$$\text{dom}(f) = \Big\{x \in X \;\Big|\; f(x) \neq \emptyset\Big\}.$$

If $\text{dom}(f) \neq \emptyset$, *the mapping* f *is classified as proper.*

(3) Assume Y *is equipped with a partial ordering induced by a convex cone* $C \subseteq Y$. *The epigraph of* f, *denoted* $\text{epi}(f)$, *is characterized by*

$$\text{epi}(f) = \Big\{(x, y) \in X \times Y \;\Big|\; x \in S,\ y \in f(x) + C\Big\}.$$

With these preliminaries in place, we now define the radial epiderivative and explore the foundational results that establish its existence and key properties.

Definition 2.3. *[32, Definition 1.6] Let* $(X, \|\cdot\|_X)$ *and* $(Y, \|\cdot\|_Y)$ *be real normed spaces, where* Y *is partially ordered by a convex cone* $C \subseteq Y$. *Let* $S \subseteq X$ *be a nonempty* *subset and $f\colon S \to Y$ a set-valued mapping. For a point $(\overline{x}, \overline{y}) \in \operatorname{graph}(f)$, the radial epiderivative of f at $(\overline{x}, \overline{y})$, denoted $f^r(\overline{x}; \overline{y})\colon X \to Y$, is the unique single-valued mapping satisfying*

$$\operatorname{epi}\left(f^r(\overline{x}; \overline{y})\right) = R\left(\operatorname{epi}(f), (\overline{x}, \overline{y})\right),$$

where $R(\cdot)$ denotes the closed radial cone to the epigraph of f at $(\overline{x}, \overline{y})$.

We now present the existence condition for the radial epiderivative proved by Kasimbeyli in [33].

Theorem 2.1. *[33, Theorem 3.2] Let $(X, \|\cdot\|_X)$ be a real normed space, $D \subseteq X$ a non-empty subset, and $\overline{x} \in X$. Let $f\colon D \to \mathbb{R} \cup \{+\infty\}$ be a proper function. Suppose there exist functions $g_1, g_2\colon X \to \mathbb{R}$ satisfying the inclusions:*

$$\operatorname{epi}(g_1) \subseteq R\left(\operatorname{epi}(f); (\overline{x}, f(\overline{x}))\right) \subseteq \operatorname{epi}(g_2),$$

where $R(\cdot)$ denotes the closed radial cone. Then, the radial epiderivative $f^r(\overline{x}, f(\overline{x}))\colon X \to \mathbb{R}$ is explicitly given by

$$f^r(\overline{x}, f(\overline{x}))(x) = \min\left\{y \in \mathbb{R} \,:\, (x, y) \in R\left(\operatorname{epi}(f), (\overline{x}, f(\overline{x}))\right)\right\}, \quad \forall x \in X.$$

Proposition 2.2. *[44, Theorem 1] Let $(X, \|\cdot\|_X)$ be a real normed space, and let $\overline{x} \in X$. Suppose $f\colon X \to \mathbb{R}$ admits a radial epiderivative at $\overline{x}$. Then, the radial epiderivative $f^r(\overline{x}; h)$ can alternatively be expressed as*

$$f^r(\overline{x}; h) = \inf_{t>0} \liminf_{u \to h} \frac{f(\overline{x} + tu) - f(\overline{x})}{t}, \quad \forall h \in X,$$

where the limit inferior is taken over sequences $u \to h$ in X, and the infimum is computed over all $t > 0$.

Theorem 2.3. *Consider a real normed space $(X, \|\cdot\|_X)$ and a proper function $f\colon X \to \mathbb{R} \cup \{+\infty\}$ that is finite at $\overline{x} \in X$. If f satisfies a local lower Lipschitz condition at $\overline{x}$, then f admits a radial epiderivative at $\overline{x}$. Furthermore, when $X = \mathbb{R}^n$, this lower Lipschitz property is both necessary and sufficient for the existence of the radial epiderivative at $\overline{x}$.*

Theorem 2.4. *[44, Theorem 8] Let $(X, \|\cdot\|_X)$ be a normed space, S be a nonempty subset of X, and let $f : S \to \mathbb{R} \cup \{+\infty\}$ be a proper function. If f is radially epidifferentiable at $\overline{x} \in X$ then f is lower semicontinuous at $\overline{x}$.*

Note that the converse of the above theorem is not necessarily true. For example, $f(x) = -\sqrt[3]{|x|^2}$ is (lower semi) continuous at $x = 0$ but not radially epidifferentiable $x = 0$.

Theorem 2.5. *[45, Theorem 5] Let $f : \mathbb{R}^n \to \mathbb{R}$ be a proper function that is finite at a point y. The function f is radially epidifferentiable at y if and only if f is lower Lipschitz at y. This means there exists a constant $L > 0$ such that for all $x \in \mathbb{R}^n$, the following inequality holds:*

$$f(x) - f(y) \geq -L\|x - y\|.$$

Theorem 2.6. *[45, Theorem 11] Let $(X, \|\cdot\|_X)$ be a real normed space, and let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper function. Suppose f is radially epidifferentiable at a point $\bar{x} \in X$. Then, a vector $h \in X$ is a descent direction for f at $\bar{x}$ if and only if the radial epiderivative $f^r(\bar{x}; h)$ satisfies $f^r(\bar{x}; h) < 0$.*

Corollary 2.7. *[45, Corollary 3] Let $(X, \|\cdot\|_X)$ be a real normed space, and let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper function. Suppose f is radially epidifferentiable at a point $\overline{x} \in X$. Then, f achieves a global minimum at $\overline{x}$ if and only if the radial epiderivative $f^r(\overline{x}; h)$ attains its minimum at $h = 0_X$ (the zero vector in X).*

2.2. Main Results

In this subsection, we introduce new theoretical and computational results in the study of radial epiderivatives for classes of functions that exhibit both nonconvexity and nondifferentiability. These results provide critical insights into the behavior of such functions under radial epidifferentiability frameworks.

Theorem 2.8 (Norm-linear function). *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function such that $f(x) = \langle v, x\rangle - c\,\|x\|$, then f is radially epidifferentiable and its radial epiderivative at*

$\overline{x}$ in the direction of x is given

$$f^r(\overline{x};x) = \langle v, x\rangle - c\|x\| = f(x)$$

Proof. For all $\overline{x}, x \in \mathbb{R}^n$, we have

$$\begin{aligned} f^r(\overline{x};x) &= \inf_{t>0} \liminf_{u\to x} \frac{f(\overline{x}+tu) - f(\overline{x})}{t} \\ &= \inf_{t>0} \liminf_{u\to x} \frac{\langle v, \overline{x}+tu\rangle - c\|\overline{x}+tu\| - [\langle v, \overline{x}\rangle - c\|\overline{x}\|]}{t} \end{aligned}$$

Since $t > 0$ and by the triangle inequality $\|\overline{x}+tu\| \geq t\|u\| - \|\overline{x}\|$

$$\begin{aligned} &\leq \inf_{t>0} \liminf_{u\to x} \frac{\langle v, \overline{x}\rangle + t\langle v, u\rangle + c\|\overline{x}\| - ct\|u\| - \langle v, \overline{x}\rangle + c\|\overline{x}\|}{t} \\ &= \inf_{t>0} \left(\langle v, x\rangle - c\|x\| + \frac{2c\|\overline{x}\|}{t} \right) \\ &= \langle v, x\rangle - c\|x\| = f(x) \end{aligned}$$

In a similar fashion, we can verify that $f^r(\overline{x};x) \geq f(x)$ since for $\overline{x}, x \in \mathbb{R}^n$ we have

$$f^r(\overline{x};x) = \inf_{t>0} \liminf_{u\to x} \frac{f(\overline{x}+tu) - f(\overline{x})}{t}$$

$$= \inf_{t>0} \liminf_{u\to x} \frac{\langle v, \overline{x}+tu\rangle - c\|\overline{x}+tu\| - [\langle v, \overline{x}\rangle - c\|\overline{x}\|]}{t}$$

for $t > 0$ and by the triangle inequality $\|\overline{x}+tu\| \leq \|\overline{x}\| + t\|u\|$, we have

$$\begin{aligned} &\geq \inf_{t>0} \liminf_{u\to x} \frac{\langle v, \overline{x}\rangle + t\langle v, u\rangle - c\|\overline{x}\| - ct\|u\| - \langle v, \overline{x}\rangle + c\|\overline{x}\|}{t} \\ &= \langle v, x\rangle - c\|x\| = f(x) \end{aligned}$$

Since $f^r(\overline{x};x) \leq f(x)$ and $f^r(\overline{x};x) \geq f(x)$, it follows that $f^r(\overline{x};x) = f(x)$. ☐

Theorem 2.9 (Norm-affine function). *If $f(x) = \langle v, x \pm a\rangle - c\|x \pm a\| + \alpha$, then f is radially epidifferentiable and its radial epiderivative at $\overline{x}$ in the direction of x is given*

$$f^r(\overline{x};x) = \langle v, x\rangle - c\|x\|$$

Proof. For all $\overline{x}, x \in \mathbb{R}^n$, we have

$$\begin{aligned} f^r(\overline{x}; x) &= \inf_{t>0} \liminf_{u\to x} \frac{f(\overline{x}+tu) - f(\overline{x})}{t} \\ &= \inf_{t>0} \liminf_{u\to x} \frac{\langle v, \overline{x} \pm a + tu\rangle - c\|\overline{x} \pm a + tu\| \pm \alpha - [\langle v, \overline{x} \pm a\rangle - c\|\overline{x} \pm a\| \pm \alpha]}{t} \end{aligned}$$

Since $t > 0$ and by the triangle inequality $\|\overline{x} \pm a + tu\| \geq t\|u\| - \|\overline{x} \pm a\|$

$$\begin{aligned} &\leq \inf_{t>0} \liminf_{u\to x} \frac{\langle v, \overline{x} \pm a\rangle + t\langle v, u\rangle + c\|\overline{x} \pm a\| \pm \alpha - ct\|u\| - \langle v, \overline{x} \pm a\rangle + c\|\overline{x} \pm a\| \mp \alpha}{t} \\ &= \inf_{t>0} \left(\langle v, x\rangle - c\|x\| + \frac{2c\|\overline{x} \pm a\|}{t} \right) \\ &= \langle v, x\rangle - c\|x\| \end{aligned}$$

In a similar manner, it can be proved that $f^r(\overline{x}, x) \geq f(x)$. Since for $\overline{x}, x \in \mathbb{R}^n$ we have:

$$\begin{aligned} f^r(\overline{x}; x) &= \inf_{t>0} \liminf_{u\to x} \frac{f(\overline{x}+tu) - f(\overline{x})}{t} \\ &= \inf_{t>0} \liminf_{u\to x} \frac{\langle v, \overline{x} \pm a + tu\rangle - c\|\overline{x} \pm a + tu\| \pm \alpha - [\langle v, \overline{x} \pm a\rangle - c\|\overline{x} \pm a\| \pm \alpha]}{t} \end{aligned}$$

for $t > 0$ and by the triangle inequality $\|\overline{x} \pm a + tu\| \leq \|\overline{x} \pm a\| + t\|u\|$ we have

$$\begin{aligned} &\geq \inf_{t>0} \liminf_{u\to x} \frac{\langle v, \overline{x} \pm a\rangle + t\langle v, u\rangle - c\|\overline{x} \pm a\| - ct\|u\| \pm \alpha - \langle v, \overline{x} \pm a\rangle + c\|\overline{x} \pm a\| \mp \alpha}{t} \\ &= \langle v, x\rangle - c\|x\| \end{aligned}$$

Since $f^r(\overline{x}; x) \leq \langle v, x\rangle - c||x||$ and $f^r(\overline{x}; x) \geq \langle v, x\rangle - c||x||$ then $f^r(\overline{x}; x) = \langle v, x\rangle - c||x||$. ☐

Example 2.1. *If* $f(x_1, x_2) = (x_1 + 3) - 3(x_2 - 1) - 2\,(\,|x_1 - 1| + |x_2 - 2|\,)$, *then* f^r *at* $\overline{x}$ *is*

$$f^r(\overline{x}; (x_1, x_2)) = x_1 - 3x_2 - 2\,(\,|x_1| + |x_2|\,)$$

Corollary 2.10. *Let* $f(x) = \langle v, x - a\rangle + c\|x - b\| + \beta$. *Then*

$$f^r(b; x) = \langle v, x\rangle + c\|x\|.$$

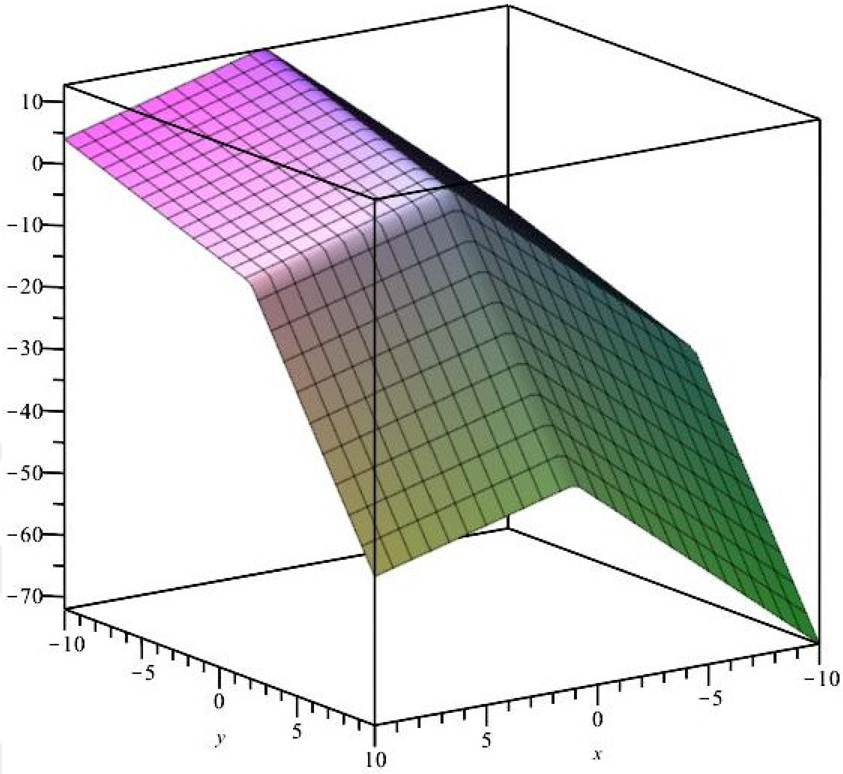

Figure 2.1. *Graph of f and its radial epiderivative from Example 2.1*

2.2.1. Radial epiderivative of min of functions

Theorem 2.5 establishes that the point-wise minimum and maximum defined as

$$f^1(x) = \min\{f_1(x), f_2(x)\} \quad \text{and} \quad f^2(x) = \max\{f_1(x), f_2(x)\}$$

are both radially epidifferentiable at any point x since both f^1 and f^2 satisfy the local lower Lipschitz condition. Here

$$f_i(x) = \langle v_i, x - a_i\rangle - c_i\|x - b_i\| + \alpha_i \qquad i = 1, 2.$$

Theorem 2.11. *Let $g : \mathbb{R} \to \mathbb{R}$ be defined as follows:*

$$g(x) = \min_{i\in\{1,2\}}\{\langle v_i, x - a_i\rangle - c_i|x - b_i| + \alpha_i\}$$

where $f_i(x) = \langle v_i, x - a_i\rangle - c_i|x - b_i| + \alpha_i$. Then $g(x)$ is radially epidifferentiable at each point $\overline{x} \in \mathbb{R}$ and each direction $x \in \mathbb{R}$, then $g^r(\overline{x}; x) = ax - b|x|$ where a and b

can be obtained by solving the following system: $a + b = \max\{v_1 + c_1,\ v_2 + c_2\}$*, and* $a - b = \min\{v_1 - c_1,\ v_2 - c_2\}$

Proof. For any $x, \overline{x} \in \mathbb{R}$, we consider the radial epiderivative of the pointwise minimum of two functions:

$$g^r(\overline{x}; x) = \inf_{t>0} \frac{\min\{f_1(\overline{x} + tx), f_2(\overline{x} + tx)\} - \min\{f_1(\overline{x}), f_2(\overline{x})\}}{t}.$$

We analyze this expression by considering two main cases based on whether $\overline{x} \geq b_i$ or $\overline{x} < b_i$.

Case 1. If $\overline{x} \geq b_i$, then depending on which function is active (i.e., achieves the minimum), straightforward algebra shows:

$$g^r(\overline{x}; x) = \begin{cases} (v_1 - c_1)x, & \text{if } f_1 \text{ is active,} \\ (v_2 - c_2)x, & \text{if } f_2 \text{ is active.} \end{cases}$$

Case 2. If $\overline{x} < b_i$, similar reasoning yields:

$$g^r(\overline{x}; x) = \begin{cases} (v_1 + c_1)x, & \text{if } f_1 \text{ is active,} \\ (v_2 + c_2)x, & \text{if } f_2 \text{ is active.} \end{cases}$$

Combining both cases, we conclude:

$$g^r(\overline{x}; x) = \begin{cases} \min\{v_1 - c_1,\, v_2 - c_2\} \cdot x, & \text{if } x \geq b_i, \\ \max\{v_1 + c_1,\, v_2 + c_2\} \cdot x, & \text{if } x < b_i. \end{cases} \tag{2.1}$$

If we now define constants a and b by

$$a - b = \min\{v_1 - c_1,\, v_2 - c_2\}, \quad a + b = \max\{v_1 + c_1,\, v_2 + c_2\},$$

then the radial epiderivative simplifies to the compact form:

$$g^r(\overline{x}; x) = ax - b|x|.$$

□

Remark 1. *In Equation (2.1), when* $x < b_i$*, the term* $\max\{v_1 + c_1, v_2 + c_2\}$ *is used instead of* $\min\{v_1 + c_1, v_2 + c_2\}$*. This is because* $v_1 + c_1$ *and* $v_2 + c_2$ *represent lines with negative slopes. In this case, the larger the slope, the lower the graph of the function lies. Therefore, the maximum value corresponds to the lower boundary of the function in this region.*

Example 2.2. *Let*

$$f_1(x) = 3(x-1) - 4|x-2|,$$
$$f_2(x) = 6(x-4) - 5|x-5|.$$

From Theorem 2.11, we know that the radial epiderivative of $f(x) = \min\{f_1, f_2\}$ *is* $f^r(\overline{x}; x) = ax - b|x| where Solving the system we get a=5 and b=6, thus f^r(\overline{x}; x) = 5x - 6|x|$

The following corollary is a generalization of Theorem 2.11 to any number of functions in $\mathbb{R}$.

Corollary 2.12. *Let* $g : \mathbb{R} \to \mathbb{R}$ *be defined as follows:*

$$g(x) = \min_{\{i=1,2,\cdots,m\}} \{\langle v_i, x - a_i\rangle - c_i|x - b_i| + \alpha_i\}.$$

Then $g(x)$ *is radially epidifferentiable at each* $\overline{x}$ *and each direction* x *both in* $\mathbb{R}$ *and*

$$g^r(\overline{x}; x) = ax - b|x|$$

where a *and* b *can be obtained by solving the following system:*

$$a + b = \max\{v_1 + c_1,\ v_2 + c_2,\ v_3 + c_3, \cdots,\ v_m + c_m\}$$
$$a - b = \min\{v_1 - c_1,\ v_2 - c_2,\ v_3 - c_3, \cdots,\ v_m - c_m\}$$

Proof. The proof of this corollary is straightforward and follows a similar approach to that of Theorem 2.11. For brevity, it is omitted. □

Example 2.3. *If*

$$f_1(x) = -(x-1) - 3|x-2|$$
$$f_2(x) = 2(x-2) - 5|x-1|$$
$$f_3(x) = 5(x-1) - |x+1|$$
$$f_4(x) = 3(x+2) - 4|x-3|,$$

then we have

$$a + b = \max\{2,\ 7,\ 6,\ 7\} = 7$$
$$a - b = \min\{-4,\ -3,\ 4,\ -1\} = -4$$

solving the system we get $a = \dfrac{3}{2}$ *and* $b = \dfrac{11}{2}$. *Thus the radial epiderivative of*

$g(x) = \min\{f_1, f_2, f_3, f_4\}$

$$\therefore g^r(\overline{x}; x) = \frac{3}{2}x - \frac{11}{2}|x|.$$

The radial epiderivative, in general, does not satisfy additivity over arbitrary functions. However, for the class of piecewise norm-linear functions—specifically those of the form $g_i(x_i) = v_i x_i - c|x_i|$—the radial epiderivative exhibits a rare additive property. The following lemma formalizes this observation, demonstrating that the radial epiderivative of a separable sum of such functions decomposes into a sum of component-wise radial epiderivatives. This property is critical for the proofs of the subsequent theorems and corollaries.

Lemma 2.13. *Let $g_i : \mathbb{R}^m \to \mathbb{R}$ be functions defined by*

$$g_i(x_i) = v_i x_i - c|x_i| \quad \text{for } i = 1, 2, \ldots, m,$$

where $v_i \in \mathbb{R}$ and $c \geq 0$ are constants. Define the aggregate function $f : \mathbb{R}^m \to \mathbb{R}$ as

$$f(x) = \sum_{i=1}^{m} g_i(x).$$

Then for functions of this specific form, the radial epiderivative of the sum equals the sum of the radial epiderivatives:

$$\left(\sum_{i=1}^{m} g_i\right)^r = \sum_{i=1}^{m} (g_i)^r .$$

Where $(\cdot)^r$ represents radial epiderivative.

Theorem 2.14. *Let $g : \mathbb{R}^2 \to \mathbb{R}$ be defined by*

$$g(x) \ = \ \min\Big\{f_1(x), \ f_2(x)\Big\},$$

where each

$$f_i(x) = v_{i1}(x_1 - a_{i1}) + v_{i2}(x_2 - a_{i2}) - c_i\left(|x_1 - b_{i1}| + |x_2 - b_{i2}|\right) + \alpha_i.$$

Then g is radially epidifferentiable at every $\overline{x} \in \mathbb{R}^2$. Moreover, its radial epiderivative is *given by*

$$g^r(\overline{x};x) \;=\; a_1\,x_1 \;+\; a_2\,x_2 \;-\; b_1\,|x_1| \;-\; b_2\,|x_2|,$$

where for each coordinate $j = 1, 2$,

$$\begin{cases} a_j \;+\; b_j \;=\; \max\{\, v_{1j} + c_1,\; v_{2j} + c_2\}, \\ a_j \;-\; b_j \;=\; \min\{\, v_{1j} - c_1,\; v_{2j} - c_2\}. \end{cases}$$

Proof. By Lemma 2.13 and by the fact that the function $\|x\|_1$ separates into $|x_1|+|x_2|$, the radial epiderivative f_i is equal to sum of the radial epiderivates of $v_{i1}(x_1-a_{i1})-c_i|x_1-b_{i1}|$ and $v_{i2}(x_2-a_{i2})-c_i|x_2-b_{i2}|$. This means that the radial epiderivative (of $\langle v,x\rangle \pm \|x\|$) in two dimensions is simply the sum of the contributions from each coordinate:

Case analysis for x_1.

By Theorem 2.8, the radial epiderivative of $\langle v_{i1}, x_1\rangle - c_i|x_1|$ is

$$a_1\,x_1 \;-\; b_1\,|x_1| \tag{2.2}$$

where

$$\begin{cases} a_1 \;+\; b_1 \;=\; \max\{\, v_{11} + c_1,\; v_{21} + c_2\}, \\ a_1 \;-\; b_1 \;=\; \min\{\, v_{11} - c_1,\; v_{21} - c_2\}. \end{cases}$$

Case analysis for x_2.

By the same reasoning, define a_2 and b_2 via

$$\begin{cases} a_2 \;+\; b_2 \;=\; \max\{\, v_{12} + c_1,\; v_{22} + c_2\}, \\ a_2 \;-\; b_2 \;=\; \min\{\, v_{12} - c_1,\; v_{22} - c_2\}. \end{cases}$$

Then the radial epiderivative of $\langle v_{i2}, x_2\rangle - c_i|x_2|$ is

$$a_2\,x_2 \;-\; b_2\,|\,x_2\,|. \tag{2.3}$$

Combining (2.2) and (2.3), we get

$$g^r(\overline{x};x) \;=\; \Big[a_1\,x_1 - b_1\,|\,x_1|\Big] \;+\; \Big[a_2\,x_2 - b_2\,|\,x_2|\Big] \;=\; a_1\,x_1 \;+\; a_2\,x_2 \;-\; b_1\,|x_1| \;-\; b_2\,|x_2|.$$

Thus we obtain the claimed formula for $g^r(\overline{x};x)$. ☐

Example 2.4. *Consider*

$$\begin{aligned} f_1(x_1,x_2) &= 3\,(x_1-2)\;+\;2\,(x_2+1)\;-\;5(|x_1-1|\;+\;|x_2-3|),\\ f_2(x_1,x_2) &= 2\,(x_1-1)\;+\;2\,(x_2-2)\;-\;4|x_1-3|\;-3\,|x_2-1|. \end{aligned}$$

Define $g=\min\{f_1,f_2\}$. We compute for each coordinate:

$$For x_1:\quad \begin{cases} a_1+b_1=\max\{\,3+5,\;2+4\}=8,\\ a_1-b_1=\min\{\,3-5,\;2-4\}=-2.\end{cases} \implies a_1=3,\quad b_1=5.$$

$$For x_2:\quad \begin{cases} a_2+b_2=\max\{\,2+5,\;2+4\}=7,\\ a_2-b_2=\min\{\,2-5,\;-2-3\}=-3.\end{cases} \implies a_2=2,\quad b_2=5.$$

Hence,

$$g^r(x_1;x_2)\;=\;3\,x_1\;+\;2\,x_2-5\,(|x_1|+|x_2|).$$

The next corollary generalizes Theorem 2.14 to any number of functions.

Corollary 2.15. *Let $g:\mathbb{R}^2\to\mathbb{R}$ be defined as:*

$$g(x)=\min_{\{i=1,2,\cdots,m\}}\{f_1(x),f_2(x),\cdots,f_m(x)\},$$

where each $f_i(x)$ has the form:

$$f_i(x)\;=\;\langle v_i,\;x-a_i\rangle\;-\;c_i\,\|\,x-b_i\|_1\;+\;\alpha_i,$$

with $v_i=(v_{i1},v_{i2})$, $a_i=(a_{i1},a_{i2})$, $b_i=(b_{i1},b_{i2})$, $x=(x_1,x_2)\in\mathbb{R}^2$, $c_i,\alpha_i\in\mathbb{R}$, and $\|\,x-b_i\|_1\;=\;|\,x_1-b_{i1}\,|\;+\;|\,x_2-b_{i2}\,|$. Then, $g(x)$ is radially epidifferentiable at any $\overline{x}\in\mathbb{R}^2$, and its radial epiderivative is:

$$g^r(\overline{x};x)=a_1x_1+a_2x_2-b_1|x_1|-b_2|x_2|,$$

where the coefficients a_j,b_j $(j=1,2)$ satisfy:

$$\begin{cases} a_j+b_j=\max\{v_{1j}+c_1,v_{2j}+c_2,\cdots,v_{mj}+c_m\},\\ a_j-b_j=\min\{v_{1j}-c_1,v_{2j}-c_2,\cdots,v_{mj}+c_m\}.\end{cases}$$

Proof. The proof of this corollary follows immediately from Theorem 2.14. □

Example 2.5. *Consider*

$$\begin{aligned}
f_1(x_1,x_2) &= 2\,x_1 \;+\; 3\,x_2 \;-\; 4\Big(|x_1| \;+\; |x_2|\Big),\\
f_2(x_1,x_2) &= 5\,x_1 \;-\; 2\,x_2 \;-\; 3\Big(|x_1| \;+\; |x_2|\Big),\\
f_3(x_1,x_2) &= 3\,x_1 \;+\; 4\,x_2 \;-\; 2\Big(|x_1| \;+\; |x_2|\Big),\\
f_4(x_1,x_2) &= -\,x_1 \;-\; 5\,x_2 \;-\; 6\Big(|x_1| \;+\; |x_2|\Big),\\
f_5(x_1,x_2) &= 4\,x_1 \;+\; 2\,x_2 \;-\; 4\Big(|x_1| \;+\; |x_2|\Big).
\end{aligned}$$

Define $g = \min\{f_1, f_2, f_3, f_4, f_5\}$. *We compute for each coordinate:*

For x_1 :

$$\begin{cases} a_1 + b_1 = \max\{\,6,\ 8,\ 5,\ 5,\ 8\} = 8, \\ a_1 - b_1 = \min\{\,-2,\ 2,\ 1,\ -7,\ 0\} = -7. \end{cases} \qquad \Longrightarrow \; a_1 = \frac{1}{2}, \quad b_1 = \frac{15}{2}.$$

For x_2 :

$$\begin{cases} a_2 + b_2 = \max\{\,7,\ 1,\ 6,\ 1,\ 6\} = 6, \\ a_2 - b_2 = \min\{\,-1,\ -5,\ 2,\ -11,\ -2\} = -11. \end{cases} \qquad \Longrightarrow \; a_2 = -\frac{5}{2}, \quad b_2 = \frac{17}{2}.$$

Hence,

$$g^{r}(x_1;x_2) \;=\; \frac{1}{2}\,x_1 - \frac{15}{2}\,|x_1| \;-\; \frac{5}{2}\,x_2 - \frac{17}{2}\,|x_2|.$$

Theorem 2.16 extends Theorem 2.14 to $\mathbb{R}^n$.

Theorem 2.16. *Let* $g : \mathbb{R}^n \to \mathbb{R}$ *be defined as:*

$$g(x) = \min\{f_1(x), \cdots, f_m(x)\},$$

where each $f_i(x)$ *has the form:*

$$f_i(x) = \langle v_i, x - a_i\rangle - c_i\|x - b_i\|_1 + \alpha_i,$$

with v_i, $a_i, b_i \in \mathbb{R}^n$, $c_i \geq 0$, *and* $\|x - b_i\|_1 = \sum_{j=1}^{n} |x_j - b_{ij}|$.

Then, $g(x)$ *is radially epidifferentiable at any* $\overline{x} \in \mathbb{R}^n$, *and its radial epiderivative is:*

$$g^{r}(\overline{x}; x) = \sum_{j=1}^{n} \left(a_j x_j - b_j |x_j|\right),$$

where the coefficients a_j, b_j are determined by solving, for each dimension j:

$$\begin{cases} a_j + b_j = \max\{v_{1j} + c_1, v_{2j} + c_2, , \cdots , v_{mj} + c_m\}, \\ a_j - b_j = \min\{v_{1j} - c_1, v_{2j} - c_2, , \cdots , v_{mj} + c_m\}. \end{cases}$$

Proof. The theorem can be demonstrated in a manner similar to theorem 2.14, by examining the individual dimensions. $\square$

Example 2.6. *Let f_1 $f_2 : \mathbb{R}^3 \to \mathbb{R}$ such that*

$$\begin{aligned} f_1(x) = 2x_1 + 3x_2 - x_3 - 4\|x\|_1 \\ f_2(x) = 5x_1 - 2x_2 + 4x_3 - 3\|x\|_1. \end{aligned}$$

Define $g(x) = \min\{f_1, f_2\}$, then

$$\text{For } x_1 : \begin{cases} a_1 + b_1 = \max\{6, 8\} = 8, \\ a_1 - b_1 = \min\{-2, 2\} = -2, \end{cases} \Longrightarrow a_1 = 3,\ b_1 = 5.$$

$$\text{For } x_2 : \begin{cases} a_2 + b_2 = \max\{7, 1\} = 7, \\ a_2 - b_2 = \min\{-1, -5\} = -5, \end{cases} \Longrightarrow a_2 = 1,\ b_2 = 6.$$

$$\text{For } x_3 : \begin{cases} a_3 + b_3 = \max\{3, 7\} = 7, \\ a_3 - b_3 = \min\{-5, 1\} = -5, \end{cases} \Longrightarrow a_3 = 1,\ b_3 = 6.$$

Thus the radial epiderivative of $g(x)$ is:

$$g^r(\overline{x}; x) = 3x_1 - 5|x_1| + x_2 - 6|x_2| + x_3 - 6|x_3|.$$

2.2.2. Radial epiderivative of max of functions

The results established in Theorem 2.11 and Corollary 2.15 are equally applicable to *maximum function* of several functions, as shown in Theorem 2.17 and Corollary 2.18.

Theorem 2.17. *Let $g : \mathbb{R} \to \mathbb{R}$ be defined as follows:*

$$g(x) = \max_{\{i=1,2\}} \{\langle v_i, x - a_i\rangle - c_i|x - b_i| + \alpha_i\}$$

where $f_i(x) = \langle v_i, x - a_i\rangle - c_i|x - b_i| + \alpha_i$. Then $g(x)$ is radially epidifferentiable at each point x and $\overline{x} \in \mathbb{R}$ and

$$g^r(\overline{x}; x) = ax - b|x|$$

where a and b can be obtained by solving the following system:

$$\begin{aligned} a+b &= \min\{v_1+c_1,\ v_2+c_2\} \\ a-b &= \max\{v_1-c_1,\ v_2-c_2\} \end{aligned}$$

Proof. For any $x, \overline{x} \in \mathbb{R}$, we consider the radial epiderivative of the pointwise maximum of two functions:

$$g^r(\overline{x}; x) = \inf_{t>0} \frac{\max\{f_1(\overline{x}+tx), f_2(\overline{x}+tx)\} - \max\{f_1(\overline{x}), f_2(\overline{x})\}}{t}.$$

We evaluate this by considering whether $\overline{x} \geq b_i$ or $\overline{x} < b_i$, and which of the two functions attains the maximum.

Case 1. If $\overline{x} \geq b_i$, then:

$$g^r(\overline{x}; x) = \begin{cases} (v_1-c_1)x, & \text{if } f_1 \text{ dominates,} \\ (v_2-c_2)x, & \text{if } f_2 \text{ dominates.} \end{cases}$$

Case 2. If $\overline{x} < b_i$, we similarly obtain:

$$g^r(\overline{x}; x) = \begin{cases} (v_1+c_1)x, & \text{if } f_1 \text{ dominates,} \\ (v_2+c_2)x, & \text{if } f_2 \text{ dominates.} \end{cases}$$

By combining the outcomes of all subcases, the epiderivative becomes:

$$g^r(\overline{x}; x) = \begin{cases} \max\{v_1-c_1,\ v_2-c_2\} \cdot x, & \text{if } x \geq b_i, \\ \min\{v_1+c_1,\ v_2+c_2\} \cdot x, & \text{if } x < b_i. \end{cases}$$

Now, define constants a and b by

$$a+b = \min\{v_1+c_1,\ v_2+c_2\}, \quad a-b = \max\{v_1-c_1,\ v_2-c_2\}.$$

Then the radial epiderivative simplifies to the compact expression:

$$g^r(\overline{x}; x) = ax - b|x|.$$

□

Example 2.7. *Let*

$$\begin{aligned} f_1(x) &= 2(x-2) - 3|x-1|, \\ f_2(x) &= 5(x-3) - 2|x-3|, \end{aligned}$$

From Theorem 2.14, we know that the radial epiderivative of $f(x) = \max_x\{f_1, f_2\}$ *is* $f^r(\overline{x}; x) = ax - b|x|$ *where*

$$\begin{aligned} a + b &= \min\{2+3, 5+2\} = 5 \\ a - b &= \max\{2-3, 5-2\} = 3 \end{aligned}$$

Solving the system we get $a = 4$ *and* $b = 1$*, thus* $f^r(\overline{x}; x) = 4x - |x|$.

Corollary 2.18 generalizes Theorem 2.14 for any number of functions.

Corollary 2.18. *Let* $g : \mathbb{R} \to \mathbb{R}$ *be defined as follows:*

$$g(x) = \max_{\{i=1,2,\cdots,m\}}\{\langle v_i, x - a_i\rangle - c_i|x - b_i| + \alpha_i\}.$$

Then $g(x)$ *is radially epidifferentiable at each point* x *and* $\overline{x} \in \mathbb{R}$ *and*

$$g^r(\overline{x}; x) = ax - b|x|$$

where a *and* b *can be obtained by solving the following system:*

$$\begin{aligned} a + b &= \min\{v_1 + c_1,\ v_2 + c_2,\ v_3 + c_3, \cdots,\ v_m + c_m\} \\ a - b &= \max\{v_1 - c_1,\ v_2 - c_2,\ v_3 - c_3, \cdots,\ v_m - c_m\} \end{aligned}$$

Example 2.8. *If*

$$\begin{aligned} f_1(x) &= 2(x-1) - 3|x-2| + 3 \\ f_2(x) &= 3(x-2) - 5|x-4| \\ f_3(x) &= -(x-3) - 2|x-1| - 2 \\ f_4(x) &= 5(x-5) - 7|x-7| \end{aligned}$$

then we have

$$\begin{aligned} a + b &= \min\{5,\ 8,\ 1,\ 12\} = 1 \\ a - b &= \max\{-1,\ -2,\ -3,\ -2\} = -1 \end{aligned}$$

solving the system, we get $a = 0$ *and* $b = 1$. *The radial epiderivative of* $f(x) =$

$\max\{f_1, f_2, f_3, f_4\}$ *is*

$$f^r(\overline{x}; x) = -|x|$$

Theorem 2.19. *Let* $g : \mathbb{R}^2 \to \mathbb{R}$ *be defined by*

$$g(x) \; = \; \max\Big\{f_1(x), \; f_2(x)\Big\},$$

where each $f_i(x)$ *has the form:*

$$f_i(x) \; = \; \langle v_i, \; x - a_i \rangle \; - \; c_i \, \|x - b_i\|_1 \; + \; \alpha_i,$$

with $v_i = (v_{i1}, v_{i2})$*,* $a_i, b_i \in \mathbb{R}^2$*,* $c_i \geq 0$*, and*

$$\|x - b_i\|_1 \; = \; |x_1 - b_{i1}| \; + \; |x_2 - b_{i2}|.$$

Then g *is radially epidifferentiable at every* $\overline{x} \in \mathbb{R}^2$*, and its radial epiderivative is:*
$g^r(\overline{x}; x) \; = \; a_1 x_1 + a_2 x_2 - b_1|x_1| - b_2|x_2|,$
where for each coordinate $j = 1, 2$*:*

$$\begin{cases} a_j + b_j \; = \; \min\Big\{v_{1j} + c_1, \; v_{2j} + c_2\Big\}, \\ a_j - b_j \; = \; \max\Big\{v_{1j} - c_1, \; v_{2j} - c_2\Big\}. \end{cases}$$

Proof. The proof follows the same logic as Theorem 2.14. ☐

Corollary 2.20 (Generalization to m Functions for Maximum). *Let* $g : \mathbb{R}^2 \to \mathbb{R}$ *be defined as:*

$$g(x) = \max_{\{i=1,2\}} \Big\{f_i(x)\Big\},$$

where each $f_i(x)$ *has the form:*

$$f_i(x) = v_{i1}(x_1 - a_{i1}) + v_{i2}(x_2 - a_{i2}) - c_i\left(|x_1 - b_{i1}| + |x_2 - b_{i2}|\right) + \alpha_i.$$

Then, $g(x)$ *is radially epidifferentiable at any* $\overline{x} \in \mathbb{R}^2$*, and its radial epiderivative is:*

$$g^r(\overline{x}; x) = a_1 x_1 + a_2 x_2 - b_1|x_1| - b_2|x_2|,$$

where the coefficients a_j, b_j $(j = 1, 2)$ *satisfy:*

$$\begin{cases} a_j + b_j = \min\Big\{v_{1j} + c_1, v_{2j} + c_2, \dots, v_{mj} + c_m\Big\}, \\ a_j - b_j = \max\Big\{v_{1j} - c_1, v_{2j} - c_2, \dots, v_{mj} - c_m\Big\}. \end{cases}$$

Example 2.9. *Consider*

$$\begin{aligned} f_1(x_1, x_2) &= 2x_1 + 3x_2 - 4(|x_1| + |x_2|), \\ f_2(x_1, x_2) &= 5x_1 - 2x_2 - 3(|x_1| + |x_2|). \end{aligned}$$

Define $g = \max\{f_1, f_2\}$. *Compute coefficients:*
For x_1*:*

$$\begin{cases} a_1 + b_1 = \min\{2+4, 5+3\} = \min\{6, 8\} = 6, \\ a_1 - b_1 = \max\{2-4, 5-3\} = \max\{-2, 2\} = 2, \end{cases} \quad \Longrightarrow \; a_1 = 4, \; b_1 = 2.$$

For x_2*:*

$$\begin{cases} a_2 + b_2 = \min\{3+4, -2+3\} = \min\{7, 1\} = 1, \\ a_2 - b_2 = \max\{3-4, -2-3\} = \max\{-1, -5\} = -1, \end{cases} \quad \Longrightarrow \; a_2 = 0, \; b_2 = 1.$$

Thus, the radial epiderivative is:

$$g^r(\overline{x}; x) = 4x_1 - 2|x_1| - |x_2|.$$

Theorem 2.21. *Let* $g : \mathbb{R}^n \to \mathbb{R}$ *be defined as:*

$$g(x) = \max_{\{i=1,2,\cdots,m\}} \Big\{f_i(x)\Big\},$$

where each $f_i(x)$ *has the form:*

$$f_i(x) = \langle v_i, x - a_i \rangle - c_i \|x - b_i\|_1 + \alpha_i,$$

with $v_i,\ a_i,\ b_i \in \mathbb{R}^n$, $c_i \geq 0$, *and* $\|x - b_i\|_1 = \sum_{j=1}^{n} |x_j - b_{ij}|$. *Then,* $g(x)$ *is radially epidifferentiable at any* $\overline{x} \in \mathbb{R}^n$, *and its radial epiderivative is:*

$$g^r(\overline{x}; x) = \sum_{j=1}^{n} \left(a_j x_j - b_j |x_j|\right),$$

where the coefficients a_j, b_j satisfy:

$$\begin{cases} a_j + b_j = \min\Big\{v_{1j} + c_1, v_{2j} + c_2, \ldots, v_{mj} + c_m\Big\}, \\ a_j - b_j = \max\Big\{v_{1j} - c_1, v_{2j} - c_2, \ldots, v_{mj} - c_m\Big\}. \end{cases}$$

Example 2.10. *Let*

$$\begin{aligned} &f_1(x) = -(x_1 - 2) + 3(x_2 - 1) + 2(x_3 + 1) - 5\|x\|_1 \\ &f_2(x) = 3(x_1 + 1) - (x_2 - 3) + 5(x_3 - 2) - 3|x_1| - 2|x_2| - 6|x_3| \\ &f_3(x) = 2x_1 - 2(x_2 - 2) + 3(x_3 - 3) - 4|x_1| - 3|x_2| - 4|x_3|. \end{aligned}$$

Define $g = \max\{f_1, f_2, f_3\}$. Compute coefficients:

For x_1:

$$\begin{cases} a_1 + b_1 = \min\{4, 6, 6\} = 6, \\ a_1 - b_1 = \max\{-6, 0, -2\} = 0, \end{cases} \quad \Longrightarrow \; a_1 = 2,\; b_1 = 2.$$

For x_2:

$$\begin{cases} a_2 + b_2 = \min\{8, 1, 1\} = 1, \\ a_2 - b_2 = \max\{-2, -3-, 5\} = -2, \end{cases} \quad \Longrightarrow \; a_2 = -{}^{1}\!/_{2},\; b_2 = {}^{3}\!/_{2}.$$

For x_3:

$$\begin{cases} a_3 + b_3 = \min\{7, 11, 7\} = 7, \\ a_3 - b_3 = \max\{-3, -1, -1\} = -1, \end{cases} \quad \Longrightarrow \; a_3 = 3,\; b_3 = 4.$$

Thus, the radial epiderivative is:

$$g^r(\overline{x}; x) = 2x_1 - 2|x_1| - \frac{1}{2}x - \frac{3}{2}|x_2| + 3\,x_3 - 4|x_3|.$$

2.2.3. Radial epiderivative of piecewise functions

Theorem 2.22. *If $f_1(x) = v_1 x - c_1|x|$, $f_2(x) = v_2 - c_2|x|$, then the functions*

$$f(x) = \begin{cases} f_1(x), & x \leq 0 \\ f_2(x), & x > 0 \end{cases}$$

is radially epidifferentiable and its radial epidifferivative is $f(x)$ itself and can be written as: $f^r(\overline{x}; x) = ax - b|x|$, where $a + b = v_1 + c_1$, and $a - b = v_2 - c_2$.

Proof. For every $\overline{x}$ and x we have:

$$f^r(\overline{x}; x) = \inf_{t>0} \lim_{u \to x} \frac{f(\overline{x} + tu) - f(\overline{x})}{t}$$

Case 1. $\overline{x} \leq 0$, then $\overline{x} + tx \leq 0$ for $t > 0$, we have

$$\begin{aligned} f^r(\overline{x}; x) &= \inf_{t>0} \frac{v_1(\overline{x} + tx) + c_1(\overline{x} + tx) - v_1\overline{x} - c_1\overline{x}}{t} \\ &= (v_1 + c_1)x \end{aligned}$$

Case 2. $\overline{x} > 0$, then $\overline{x} + tx > 0$ for $t > 0$, we have

$$\begin{aligned} f^r(\overline{x}; x) &= \inf_{t>0} \frac{v_2(\overline{x} + tx) - c_2(\overline{x} + tx) - v_2\overline{x} + c_2\overline{x}}{t} \\ &= (v_2 - c_2)x \end{aligned}$$

□

Example 2.11. *If*

$$f(x) = \begin{cases} 3(x-1) - 3|x-1| + 2, & x \leq 0 \\ x - 1 - 4|x+1| + 3, & x > 0 \end{cases}$$

then the radial epiderivative of $f(x)$ can be obtained by solving

$$a + b = 6, \qquad a - b = -3$$

here $a = \dfrac{3}{2}$ and $b = \dfrac{9}{2}$, thus $f^r(\overline{x}; x) = \dfrac{3}{2}x - \dfrac{9}{2}|x|$ Similarly, if

$$f(x) = \begin{cases} -2|x+3| + 2, & x \leq 0 \\ -3|x+1| - 3, & x > 0 \end{cases}$$

then $\qquad f^r(\overline{x}; x) = -\dfrac{1}{2}x - \dfrac{5}{2}|x|.$

Theorem 2.23. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be*

$$f(x) = \begin{cases} -c_1\|x + a\| + \alpha, & x \leq 0 \\ -c_2\|x + b\| + \beta, & \textit{Otherwise}, \end{cases}$$

then the radial epiderivative of $f(x)$ is

$$f^r(\overline{x}; x) = \begin{cases} -c_1\|x\|, & x \leq 0 \\ -c_2\|x\|, & \textit{Otherwise}. \end{cases}$$

Proof. We will show the proof for the case when $\overline{x} \leq 0$

$$f^r(\overline{x}; x) = \inf_{t>0} \frac{-c_1\|\overline{x} + tx + a\| + c_1\|\overline{x} + a\|}{t}$$

since $t > 0$ and by the triangle inequality $\|\overline{x} + tx + a\| \leq \|\overline{x} + a\| + \|tx\|$

$$\begin{aligned} &\geq \frac{-c_1\|\overline{x} + a\| - c_1\|tx\| + c_1\|\overline{x} + a\|}{t} \\ &= -c_1\|x\|. \end{aligned}$$

In a similar fashion we can show that $f^r \leq -c_1\|x\|$.

$$f^r(\overline{x}; x) = \inf_{t>0} \frac{-c\|\overline{x} + tx + a\| + c_1\|\overline{x} + a\|}{t}$$

since $t > 0$ and by the triangle inequality $\|\overline{x} + tx + a\| \geq \|tx\| - \|\overline{x} + a\|$

$$\begin{aligned} &\geq \frac{c_1\|\overline{x} + a\| - c_1\|tx\| + c_1\|\overline{x} + a\|}{t} \\ &= \left(\inf_{t>0} -c_1\|x\| + \frac{2c_1\|\overline{x} + a\|}{t}\right) \\ &= -c_1\,\|x\|. \end{aligned}$$

The proofs of the remaining cases can be obtained in a similar fashion. ☐

2.2.4. Radial epiderivative of sum and difference

Theorem 2.24. *let $f_1 = v_1x \pm c_1|x|$ and $f_2 = v_2x \pm c_2|x|$, and*

$$\begin{aligned} f^1 &= f_1 + f_2 = (v_1 + v_2)x \pm (c_1 \pm c_2)|x| \\ f^2 &= f_1 - f_2 = (v_1 - v_2)x \pm (c_1 \mp c_2)|x|. \end{aligned}$$

Then the radial epiderivative of f^1 is identical to f^1 itself, and the same is true for f^2.

Proof. This theorem is a direct consequence of Theorem 2.8. ☐

Remark 2. *For functions of the form $f(x) = vx \pm c|x|$:*

- *The sum of two convex (or two concave) functions will always remain convex (or concave).*
- *The difference between two such functions (both convex or both concave) may reverse their convexity or concavity, potentially becoming concave (or convex).*

2.2.5. Radial epiderivative of a convex combination

Theorem 2.25. *Let*

$$f_1(x) = v_1x - c_1|x| \ \ and \ \ f_2(x) = v_2x - c_2|x|,$$

where $v_1, c_1, v_2, c_2 \in \mathbb{R}$*. For any convex combination*

$$f(x) = \lambda f_1(x) + (1-\lambda)f_2(x) \ with \ \lambda \in [0,1],$$

the radial epiderivative is $f^r(\overline{x}; x) = ax - b|x|$ *where*

$$\begin{cases} a \ = \ \lambda\, v_1 \ + \ (1-\lambda)\, v_2 \\ b \ = \ \lambda\, c_1 \ + \ (1-\lambda)\, c_2 \end{cases}$$

Proof.

$$f(x) = \lambda(v_1x - c_1|x|) + (1-\lambda)(v_2x - c_2|x|) = [\lambda v_1 + (1-\lambda)v_2]\, x - [\lambda c_1 + (1-\lambda)c_2]\, |x|.$$

Thus, $f(x) = a\ x - b\ |x|$, where $a \ = \ \lambda v_1 \ + \ (1-\lambda)\, v_2$ and $b \ = \ \lambda c_1 \ + \ (1-\lambda)\ c_2$. By Theorem 2.8 we know that the radial epiderivative of $f(x) \ = a\ x - \ b\ |x|$ is $f(x)$ itself. ☐

Example 2.12. *If* $f_1(x) = 3x - 2|x|$ *and* $f_2(x) = 4x - 2|x|$*, then radial epiderivative of their convex combination is given by* $f^r(\overline{x}; x) = (4-\lambda)x - 2|x|$

Corollary 2.26. *Let* $f_1(x) = v_1x + c_1|x|$ *and* $f_2(x) = v_2x + c_2|x|$*, where* $v_1, c_1, v_2, c_2 \in \mathbb{R}$*. For any convex combination*

$$f(x) = \lambda f_1(x) + (1-\lambda)f_2(x) \ with \ \lambda \in [0,1],$$

the radial epiderivative is $f^r(\overline{x}; x) = ax + b|x|$ *where*

$$\begin{cases} a \ = \ \lambda\, v_1 \ + \ (1-\lambda)\, v_2 \\ b \ = \ \lambda\, c_1 \ + \ (1-\lambda)\, c_2 \end{cases}$$

Theorem 2.27 establishes a result concerning the radial epiderivative of a convex combination of functions in the Euclidean space $\mathbb{R}^n$.

Theorem 2.27. *Let*

$$f_i(x) = v_{i1}\, x_1 \,+\, v_{i2}\, x_2 \,+\, \cdots + \, v_{in}\, x_n \,-\, c_i \|(x_1, x_2, \cdots, x_n)\| \quad \text{for } \; i = 1, 2.$$

For any convex combination $f(x) = \lambda f_1(x) + (1-\lambda) f_2(x)$ *with* $\lambda \in [0,1]$*, the radial epiderivative* $f^r(\overline{x}; x)$ *is given by:*

$$f^r(\overline{x}; x) \; = \; a_1\, x_1 + a_2\, x_2 + \cdots + \; a_n\, x_n - c \,\|\, (x_1, x_2, \cdots, x_n)\|$$

where

$$\begin{cases} a_1 \;=\; \lambda\, v_{11} \;+\; (1-\lambda)\, v_{21} \\ a_2 \;=\; \lambda\, v_{21} \;+\; (1-\lambda)\, v_{22} \\ \qquad\quad \vdots \\ a_n \;=\; \lambda\, v_{1n} \;+\; (1-\lambda)\, v_{2n} \\ c \;\;=\; \lambda\, c_1 \;\;+\; (1-\lambda)\, c_2 \end{cases}$$

Proof.

$$\begin{aligned} f(x) &= \lambda\,[v_{11}\, x_1 + v_{12} x_2 + \cdots + v_{1n} x_n - c_1 \|(x_1, x_2, \cdots, x_n)\|\,] \\ &\quad + (1-\lambda)\,[v_{21}\, x_1 + v_{22} x_2 + \cdots + v_{2n} x_n - c_2 \|(x_1, x_2, \cdots, x_n)\|\,] \\ &\quad = [\,\lambda\, v_{11} + (1-\lambda)\, v_{21}\,] x_1 \;+\; [\,\lambda\, v_{12} + (1-\lambda)\, v_{22}\,] x_2 \;+\; \cdots \\ &\quad + [\,\lambda\, v_{1n} + (1-\lambda)\, v_{2n}] x_n \;-\; [\lambda\, c_1 + (1-\lambda)\, c_2]\, \|(x_1, x_2, \cdots, x_n)\|. \end{aligned}$$

If we set $a_1 = \lambda\, v_{11} + (1-\lambda)\, v_{21}$, $\;\; a_2 = \lambda\, v_{12} + (1-\lambda)\, v_{22}$, $\cdots$, $\;\; a_n = \lambda\, v_{1n} + (1-\lambda)\, v_{2n}$, and $c = \lambda\, c_1 + (1-\lambda)\, c_2$, then

$$f(x) \; = \; a_1\, x_1 + a_2\, x_2 + \cdots + \; a_n\, x_n - c \,\|\, (x_1, x_2, \cdots, x_n)\|.$$

By previous theorems, we know that $f(x)$ is identical its own radial epiderivative. □

Example 2.13. *If* $f_1(x) = 4x_1 - x_2 - 3x_3 - 2\|\,(x_1, x_2, x_3)\,\|$*,* $f_2(x) = -x_1 + 2x_2 + x_3 - 3\,\|\,(x_1, x_2, x_3)\,\|$*, then the radial epiderivative of the convex combination* $f(x) = \lambda f_1 + (1-\lambda) f_2$ *(with* $\lambda = 1/3$*) is*

$$f^r(\overline{x}; x) = \frac{2}{3} x_1 + x_2 - \frac{1}{3} x_3 - \frac{8}{3}\,\|\,(x_1, x_2, x_3)\,\|.$$

2.2.6. Radial epiderivative of composition of functions

Theorem 2.28. *Let* $f_1(x) = v_1\,x \;-\; c_1\,|x|$ *and* $f_2(x) = v_2\,x \;-\; c_2\,|x|$*, then the radial epiderative of* $f_1 \circ f_2$ *is given by*

$$(f_1 \circ f_2)^r(\overline{x}; x) = v_1\,(\,v_2\,x \;-\; c_2\,|x|\,) - c_1\,(a\,x \;+\; b\,|x|\,)$$

where

$$a = \frac{|v_2 - c_2| - |v_2 + c_2|}{2} \quad \textit{and} \quad b = \frac{|v_2 - c_2| + |v_2 + c_2|}{2}$$

Proof. First, let us simplify the compotion of f_1 and f_2

$$(f_1 \circ f_2)^r = v_1\,(\,v_2\,x \;-\; c_2\,|x|\,) - c_1\,|\,v_2\,x \;-\; c_2\,|x|\,|$$

Let $|\,v_2\,x \;-\; c_2\,|x|\,| = a\,x \;+\; b\,|x|$.

- $x \geq 0 \implies (a + b\,)\,x \;=\; |\,v_2 - c_2\,|\,x$
- $x < 0 \implies (a - b\,)\,x \;=\; -\,|\,v_2 + c_2\,|\,x$

solving for a and b we get

$$a = \frac{|v_2 - c_2| - |v_2 + c_2|}{2} \quad \text{and} \quad b = \frac{|v_2 - c_2| + |v_2 + c_2|}{2}$$

which means

$$(f_1 \circ f_2)^r(\overline{x}; x) = v_1\,(\,v_2\,x \;-\; c_2\,|x|\,) - c_1\,(a\,x \;+\; b\,|x|\,) \tag{2.4}$$

Obviously the function in (2.4) is equal to its own radial epiderivative. □

Example 2.14. *Consider* $f_1(x) \;=\; 4x \;-\; 3\,|x|$ *and* $f_2(x) \;=\; 2\,x \;-\; 4\,|x|$*, then by theorem 2.28* $(f_1 \circ f_2)^r(\overline{x}; x) = 14\,x \;-\; 28\,|x|$ *and* $(f_2 \circ f_1)^r(\overline{x}; x) = 20\,x \;-\; 22\,|x|$.

3. PROPOSED METHODS

3.1. Introduction

Traditional optimization techniques often struggle when dealing with nondifferentiable and nonconvex functions, making classical derivative-based approaches ineffective. In this section, we explore alternative methods designed to address these limitations, focusing on the objective function

$$f(x) = \langle v, x \;\pm\; a\rangle - c\,\|x \pm b\,\| \pm \alpha \tag{3.1}$$

which lacks both convexity and differentiability.

A key concept in our discusion is the radial epiderivative, a generalized derivative that extends the classical derivative to deal with nondifferentiable functions. By using this tool, we overcome limitations and barriers of traditional calculus, opening doors to more adaptable algorithms.

In this discussion, we will present the Zoutendijk feasible direction method and Frank–Wolfe algorithm examined through the lens of radial epiderivatives.

Theorem 3.1 establishes that the maximum value of a convex function over a polyhedral compact set is attained at one of its extreme points.

Theorem 3.1 (Maximizing a convex function). *[29, Theorem 3.4.7] Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function, and let S be a nonempty compact polyhedral set in $\mathbb{R}^n$. Consider the problem to maximize $f(x)$ subject to $x \in S$. An optimal solution $\bar{x}$ to the problem then exists, where $\bar{x}$ is an extreme point of S.*

Since $\min f(x) = -\max\{-f(x)\}$, Corollary 3.2 follows from Theorem 3.1.

Corollary 3.2. *Let f be a concave function and let S be a nonempty compact polyhedral set in $\mathbb{R}^n$, then the optimal solution $\overline{x}$ to the problem*

$$\begin{array}{rl} \textit{Minimize} & f(x) \\ \textit{Subject to} & x \in S \end{array}$$

is an extreme point of S.

3.2. Modified Method of Zoutendijk

This section introduces an optimization algorithm that mimics Zoutendijk's feasible direction method. The proposed method operates as follows: beginning at an initial feasible point x^0, it iteratively computes both a feasible search direction and an optimal step size to improve the objective function. Key features include:

- Descent Direction Calculation: The algorithm employs the *radial epiderivative* and Theorem 2.6 to identify descent direction that simultaneously satisfy feasibility constraints and guarantee improvement in the objective function.
- Adaptive Step Size: Determines the maximal step size $\lambda > 0$ that maintains feasibility while optimizing $f(x)$ in direction obtained earlier.
- Termination Criterion: The algorithm stops iterations when the computed step size becomes negligible ($\lambda \approx 0$), even if a theoretical descent direction exists.

The Zoutendijk method is designed to address constrained optimization problems of the form:

$$\begin{array}{rll} \text{Minimize:} & f(x) & \\ \text{Subject to:} & Ax \leq b & \text{(Linear inequalities)} \\ & Qx = q & \text{(Linear equalities)} \end{array}$$

where:

$A \in \mathbb{R}^{m \times n}$ is the inequality constraint matrix,

$Q \in \mathbb{R}^{l \times n}$ is the equality constraint matrix,

$b \in \mathbb{R}^m$ is the inequality bound vector,

$q \in \mathbb{R}^l$ is the equality target vector.

Lemma 3.3. *[29, Lemma 10.1.2] Assume A_1 and b_1 represent the binding (active) constraints. Then a direction d is improving if it keeps the binding constraints satisfied ($A_1\, d \leq 0$), adheres to $Q\, d = 0$, and yields a descent in the objective $f^r(\overline{x}; x) < 0$.*

3.2.1. The algorithm

3.2.2. Numerical examples

Example 3.1. *Minimize the following function*

Algorithm 1 Zoutendijk Modified Algorithm with Radial Epiderivative

```
Input: Matrix $A$, vector $b$, matrix $Q$, vector $q$, objective function $f$
Initialization: Find a feasible point $x^1$ such that $Ax^1 \leq b$ and $Qx^1 = q$
Set $k \leftarrow 1$ while stopping criterion not met do
    Identify binding constraints at $x^k$: Decompose $A$ into $(A_1, A_2)$ and $b$ into $(b_1, b_2)$ such that
        $A_1 x^k = b_1$, $A_2 x^k \leq b_2$
Solve the direction-finding problem:
Find $d^k$ that solves
    $$\begin{aligned} \text{minimize} \quad & f^r(x^k, d) \\ \text{subject to} \quad & A_1 d \leq 0 \\ & Qd = 0 \\ & -1 \leq d_j \leq 1 \quad \text{for } j = 1, \ldots, n \end{aligned}$$
Compute $\hat{b} \leftarrow b_2 - A_2 x^k$
Compute $\hat{d} \leftarrow A_2 d^k$ if $\hat{d}_i > 0$ for some $i$ then
    $\lambda_{\max} \leftarrow \min \left\{ \frac{\hat{b}_i}{\hat{d}_i} : \hat{d}_i > 0 \right\}$ else
    $\lambda_{\max} \leftarrow \infty$
Solve the line search problem:
    $$\begin{aligned} \text{minimize} \quad & f(x^k + \lambda d^k) \\ \text{subject to} \quad & 0 \leq \lambda \leq \lambda_{\max} \end{aligned}$$
Let $\lambda^k$ be the optimal solution
Update $x^{k+1} \leftarrow x^k + \lambda^k d^k$
Update active set: identify new binding constraints at $x^{k+1}$ to redefine $A_1$, $A_2$
$k \leftarrow k + 1$
Output: Approximate solution $x^k$
```

$f_1 = (x_1 + 3) - 3(x_2 - 1) - 2(|x_1 - 1| + |x_2 - 2|)$
Subject to

$$\begin{aligned} -x_1 + x_2 &\leq 2 \\ -x_1 + 2x_2 &\leq 6 \\ 3x_1 + x_2 &\leq 21 \\ x_1 + x_2 &\leq 9 \\ -x_1 &\leq 0 \\ -x_2 &\leq 0 \end{aligned}$$

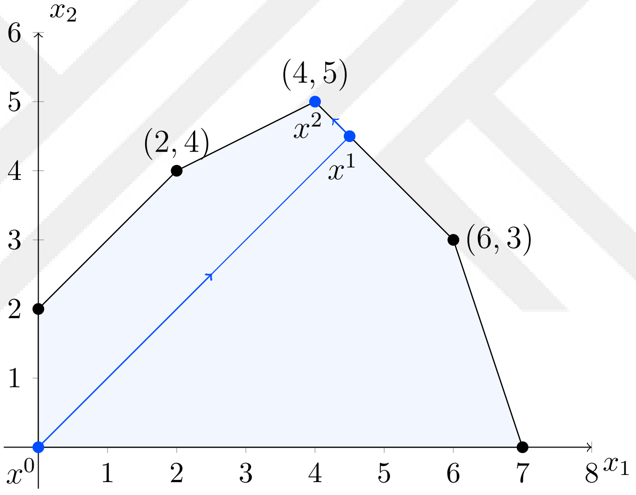

Figure 3.1. *Feasible region for the constraints in Example 3.1*

Now let us solve this problem using the radial epiderivative descent direction and adopting a similar approach as Zoutendijk's procedure with $x^0 = (0,0)^T$. Note that

$$f_1^r(\overline{x}; d) = d_1 - 3d_2 - 2(|d_1| + |d_2|).$$

Iteration 1.

Search Direction. At $x^0 = (0,0)$ the set of binding constraints is $I = \{5,6\}$. To find the descent direction we need to solve the following subproblem:

$$\text{Minimize} \quad d_1 \quad - \quad 3d_2 \quad - \quad 2(\,|d_1| + |d_2|\,)$$

subject to

$$\begin{array}{ccccc} 0 & \leq & d_1 & \leq & 1 \\ 0 & \leq & d_2 & \leq & 1 \end{array}$$

We can readily verify that $d^1 = (1,1)$ is the optimal solution for this direction-finding subproblem.

Line Search. We need to obtain a feasible point starting from $(0,0)$ along the direction $d^1 = (1,1)$ with a minimum value of $f_1(x)$. Every point in this direction can be written $x^1 = x^0 + \lambda^1 d^1 = (\lambda, \lambda)^T$ and

$$f_1(x^1) = -2(\lambda + |\lambda - 1| + |\lambda - 2| - 3).$$

The maximum value of λ^1 for which $x^0 + \lambda^1 d^1$ is feasible can be obtained by

$$\lambda_{\max} = \begin{cases} \min\{\hat{b}_i/\hat{d}_i : \hat{d}_i > 0\} & \text{if } \hat{d} > 0 \\ \infty & \text{if } \hat{d} \leq 0 \end{cases}$$

$$\hat{b} = b_2 - A_2 x^0 = \begin{pmatrix} 2 \\ 6 \\ 21 \\ 9 \end{pmatrix} - \begin{pmatrix} -1 & 1 \\ -1 & 2 \\ 3 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 6 \\ 21 \\ 9 \end{pmatrix}$$

$$\hat{d} = A_2 d^1 = \begin{pmatrix} -1 & 1 \\ -1 & 2 \\ 3 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 4 \\ 2 \end{pmatrix}$$

$$\lambda_{\max} = \left\{ \frac{6}{1}, \frac{21}{4}, \frac{9}{2} \right\} = \frac{9}{2}$$

The value of λ^1 can be obtained by solving:

$$\begin{array}{ll} \text{Minimize} & -2\lambda + 6 - 2|\lambda - 1| - 2|\lambda - 2| \\ \text{subject to} & 0 \leq \lambda \leq \dfrac{9}{2} \end{array}$$

Since the objective function is concave, the unconstrained minimum occurs at ∞

and $\lambda^1 = \frac{9}{2}$, therefore

$$x^1 = x^0 + \lambda^1 d^1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \frac{9}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 9/2 \\ 9/2 \end{pmatrix}.$$

Iteration 2.

Search Direction. At $x^1 = (9/2, 9/2)$ the binding constraint is $I = \{4\}$. To determine which direction we need to move, we have to solve the following direction-finding problem:

$$\begin{array}{lccccc} \text{Minimize} & d_1 & - & 3d_2 & - & 2(\,|d_1| + |d_2|\,) \\ \text{subject to} & & & & & \end{array}$$

$$\begin{array}{ccccccc} & & d_1 & + & d_2 & \leq & 0 \\ - & & 1 & \leq & d_1 & \leq & 1 \\ - & & 1 & \leq & d_2 & \leq & 1 \end{array}$$

By the graphical solution method, it is obvious that $d^2 = (-1, 1)$ is the optimal solution for this direction-finding subproblem.

Line Search. We need to obtain a feasible point starting from $(9/2, 9/2)$ along the direction $d^2 = (-1, 1)$ with a minimum value of $f_1(x_1, x_2)$. Every point in this direction can be written $x^2 = x^1 + \lambda^2 d^2 = (9/2 - \lambda, 9/2 + \lambda)^T$ and $f_1(x^2) = -4\lambda - |2x + 5| - |2x - 7| - 3$. The maximum value of λ^2 for which $x^1 + \lambda^2 d^2$ is feasible can be obtained by

$$\lambda_{\max} = \begin{cases} \min\{\hat{b}_i / \hat{d}_i : \hat{d}_i > 0\} & \text{if } \hat{d} > 0 \\ \infty & \text{if } \hat{d} \leq 0 \end{cases}$$

$$\hat{b} = b_2 - A_2 x^1 = \begin{pmatrix} 2 \\ 6 \\ 21 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} -1 & 1 \\ -1 & 2 \\ 3 & 1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 9/2 \\ 9/2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3/2 \\ 3 \\ 9/2 \\ 9/2 \end{pmatrix}$$

$$\hat{d} = A_2 d^2 = \begin{pmatrix} -1 & 1 \\ -1 & 2 \\ 3 & 1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ -2 \\ 1 \\ -1 \end{pmatrix}$$

$$\lambda_{\max} = \left\{ \frac{2}{2}, \frac{3/2}{3}, \frac{9/2}{1} \right\} = \frac{1}{2}$$

The value of λ^2 can be obtained by solving

$$\begin{array}{ll} \text{Minimize} & -4\lambda - |2x+5| - |2x-7| - 3 = f_1(x^2) \\ \text{subject to} & 0 \leq \lambda \leq \dfrac{1}{2} \end{array}$$

Since the objective function is concave, the unconstrained minimum occurs at ∞ and $\lambda^2 = \dfrac{1}{2}$, therefore

$$x^2 = x^1 + \lambda^2 d^2 = \begin{pmatrix} 9/2 \\ 9/2 \end{pmatrix} + 1/2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$$

The objective function value is $f_1(4,5) = -17$.

Iteration 3

Search Direction. At $x^2 = (4,5)$ the binding constraint is $I = \{2,4\}$. To determine which direction we need to move, we have to solve the following direction-finding subproblem:

$$\begin{array}{lccccc} \text{Minimize} & d_1 & - & 3d_2 & - & 2(\,|d_1| + |d_2|\,) \\ \text{subject to} & & & & & \end{array}$$

$$\begin{array}{cccccc} - & d_1 & + & 2d_2 & \leq & 0 \\ & d_1 & + & d_2 & \leq & 0 \\ - & 1 & \leq & d_1 & \leq & 1 \\ - & 1 & \leq & d_2 & \leq & 1 \end{array}$$

By the graphical solution method, it is obvious that $d^3 = (-1, -1/2)$ is the optimal solution for this direction-finding subproblem.

Line Search. We need to obtain a feasible point starting from $(4,5)$ along the direction $d^1 = (-1, -1/2)$ with a minimum value of $f_1(x_1, x_2)$. Every point in this direction can be written $x^3 = x^2 + \lambda^3 d^3 = (4 - \lambda, 5 - \frac{\lambda}{2})^T$ and $f_1(x^3) = \frac{1}{2}\lambda - 2\,|\lambda - 3| - |\lambda - 6| - 5$. The maximum value of λ^3 for which $x^2 + \lambda^3 d^3$ is feasible can be obtained by

$$\hat{b} = b_2 - A_2 x^3 = \begin{pmatrix} 2 \\ 21 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} -1 & 1 \\ 3 & 1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 4 \\ 5 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ 4 \\ 5 \end{pmatrix}$$

$$\hat{d} = A_2 d^3 = \begin{pmatrix} -1 & 1 \\ 3 & 1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -1 \\ -1/2 \end{pmatrix} = \begin{pmatrix} 1/2 \\ -7/2 \\ 1 \\ 1/2 \end{pmatrix}, \quad \lambda_{\max} = \left\{ \frac{1}{1/2}, \frac{4}{1}, \frac{5}{1/2} \right\} = 2$$

Table 3.1. *Modified Zoutendijk method steps (Ex. 3.2)*

			Search Direction			Line Search		
k	x_k	$f_2(x_k)$	f_2^r	I	d_k	$\lambda_{\max}$	λ_k	x_{k+1}
1	$(0,0)$	-13.7	-7.354	$\{2-4\}$	$(0.5,1)$	2.6	2.6	$(1.3,2.6)$
2	$(1.3,2.6)$	-16.7	0	$\{1,2\}$	$(0,0)$	0	0	$(1.3,2.6)$

The value of λ^3 can be obtained by solving

$$\begin{array}{ll} \text{Minimize} & \frac{1}{2}\lambda - 2\,|\lambda-3| - |\lambda-6| - 5 = f_1(x^3) \\ \text{subject to} & \quad 0 \le \lambda \le 2 \end{array}$$

The optimal value is $\lambda^3 = 0$, therefore $x^3 = (4,5)^T$.

Example 3.2. *Minimize the function*

$f_2(x) = -2(x_1+2) - 3(x_2-1) - 3\|\,(x_1,x_2)-(3,3)\,\|$

subject to the constraints

$$\begin{aligned} x_1 + x_2 &\le 4 \\ 2x_1 - x_2 &= 0 \\ x_1 &\le 0 \\ -x_2 &\le 0 \end{aligned}$$

The computational outcomes of the first two iterations required for convergence in the Zoutendijk method are summarized in the table below.

In some optimization problems, variables may remain feasible even as they grow indefinitely, leading to unbounded objective values. Example 3.3 demonstrates this phenomenon through an unbounded feasible region.

Example 3.3. *Minimize*

$f_3(x) = -3(x_1-1) + 2(x_2-2) - 3\|\,(x_1,x_2)-(2,3)\,\|$

Subject to

$$\begin{aligned} -x_1 + x_2 &\leq 0 \\ -x_1 + 3x_2 &\leq 3 \\ -x_1 \leq 0, -x_2 &\leq 0. \end{aligned}$$

Table 3.2 presents the key computational results from the first iteration of the Zoutendijk method, which were sufficient to achieve convergence.

Table 3.2. *Key steps of the modified Zoutendijk method (Ex. 3.3)*

			Search Direction			Line Search		
Iter. k	x_k	$f_3(x_k)$	$f_3^r(x_k; d^k)$	I	d_k	$\lambda_{\max}$	λ_k	x_{k+1}
1	$(0,0)$	-11.8	0	$\{1,3,4\}$	$(1,0)$	∞	∞	$(\infty,0)$

Example 3.4 demonstrates a case where a bounded objective value arises from an unbounded feasible region.

Example 3.4. *Minimize* $f_4(x) = -x_1 + 3x_2 - 2(|x_1 - 1| + |x_2 - 2|)$

Subject to $x_1 - x_2 \leq 4,\quad x_1 \leq 8,\quad -x_1 \leq 0,\quad -x_2 \leq 0.$

A summary of the main computational steps for the first three iterations (required by Zoutendijk's method to find the optimal point of the problem) is provided in Table 3.3.

Example 3.5. *Let*

$$\begin{aligned} f_1(x_1,x_2) &= \quad 4(x_1-1) - 2(x_2-2) - 2(\,|x_1-1| + |x_2+2|\,) \\ f_2(x_1,x_2) &= -3(x_1-2) + (x_2-1) - (\,|x_1+1| + |x_2-1|\,). \end{aligned}$$

Solve the following optimization problem:

Minimize $f(x) = \min\{f_1(x),\ f_2(x)\}$

subject to

Table 3.3. *Steps of the revised Zoutendijk algorithm (Ex. 3.4)*

			Search Direction			Line Search		
Iter. k	x_k	$f_5(x_k)$	$f_5^r(x_k; d^k)$	I	d_k	$\lambda_{\max}$	λ_k	x_{k+1}
1	$(0,0)$	-6	-3	$\{3,4\}$	$(1,0)$	4	4	$(4,0)$
2	$(4,0)$	-14	-2	$\{1,4\}$	$(1,1)$	4	4	$(8,4)$
3	$(8,4)$	-14	-1	$\{1,2\}$	$(-1,0)$	8	0	$(8,4)$

$$\begin{array}{rrrrrl}
- & 2x_1 & + & x_2 & \leq & 2 \\
- & x_1 & + & x_2 & \leq & 3 \\
 & 2x_1 & + & x_2 & \leq & 9 \\
 & 3x_1 & + & x_2 & \leq & 12 \\
- & x_1 & & & \leq & 0 \\
 & & - & x_2 & \leq & 0
\end{array}$$

By Theorem 2.14, we know the radial epiderivative of $f(x)$ is $f^r(\overline{x}; x) = x_1 - x_2 - 5|x_1| - 3|x_2|$.

The main computational results of the first four iterations, needed for obtaining the optimal solution, are summarized in Table 3.4.

Algorithm 1 is well-suited for optimization problems in $\mathbb{R}^2$, where convergence is generally reliable. However, for higher-dimensional spaces ($\mathbb{R}^n$ where $n \geq 3$), the method may encounter convergence issues. Specifically, in many cases, the algorithm either stagnates with a step size $\lambda = 0$ or computes a null search direction $d = \mathbf{0}$, indicating a failure to identify a viable descent direction.

Example 3.6. *Minimize*

$$2(x_1 - 2) + (x_2 - 1) - 2(x_3 - 2) - 5(|x_1 - 1| + |x_2 - 3| + |x_3 - 2|)$$

Table 3.4. *Iterations of the modified Zoutendijk algorithm (Ex. 3.5)*

			Search Direction			Line Search		
k	x_k	$f(x_k)$	f^r	I	d_k	$\lambda_{\max}$	λ_k	x_{k+1}
1	$(0,0)$	-6	-2	$\{5,6\}$	$(1,1)$	3	3	$(3,3)$
2	$(3,3)$	-8	-10	$\{3,4\}$	$(-1,1)$	$3/2$	$3/2$	$(1.5,4.5)$
3	$(1.5,4.5)$	-17	-2	$\{2\}$	$(1,1)$	$1/2$	$1/2$	$(2,5)$
4	$(2,5)$	-18	-6	$\{2,3\}$	$(-1,-1)$	1	0	$(2,5)$

subject to the constraints

$$\begin{aligned} x_1 + x_2 - x_3 &\leq 2 \\ 3x_1 - 2x_2 &\leq 1 \\ -x_1 + 3x_2 + x_3 &\leq 3 \\ -x_1, -x_2, -x_3 &\leq 0 \end{aligned}$$

Iteration 1. Let $x^0 = (0,0,0)$ at x^0 only 4th, 5th and 6th constraints are active. To obtain a descent direction from x^0 along $d^1 = (d_1, d_2, d_3)$ solve the following subproblem:

$$\begin{aligned} &\text{Minimize} \qquad 2d_1 + d_2 - 2d_3 - 5(|d_1| + |d_2| + |d_3|) \\ &\text{subject to} \\ &\qquad\qquad 0 \quad \leq \quad d_1, d_2, d_3 \quad \leq \quad 1 \end{aligned}$$

The minimum value of the objective function of the above subproblem is attained at $d^1 = (1,1,1)$. Now, let's find λ (the step size) from (0,0,0) along d^1 with the smallest possible value for $f(x^1)$ where $x^1 = x^0 + \lambda d^1 = (\lambda, \lambda, \lambda)^T$

$$f(x^1) = \lambda - 5(|\lambda - 1| + |\lambda - 2| + |\lambda - 3|) - 1$$

To determine step size we need to find λ_{max} by first finding

$$\widehat{b} = b_2 - A_2 x^0 = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}, \qquad \widehat{d} = A_2 d^1 = \begin{pmatrix} 1 & 1 & -1 \\ 3 & -2 & 0 \\ -1 & 3 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$$

$$\therefore \lambda_{max} = \min\{2/1, 1/1, 3/3\} = 1.$$

Thus the step size (λ) is obtained by solving

$$\begin{array}{ll} \text{Minimize} & f^r(x^1) = \lambda - 5(|\lambda - 1| + |\lambda - 2| + |\lambda - 3|) - 1 \\ \text{subject to} & \qquad\qquad 0 \leq \lambda \leq 1 \end{array}$$

The objective function obtains its minimum at $\lambda = 0$ and $x^1 = (0,0,0)^T$.

The optimization problem in Example 3.6 has its optimal solution at $(0,0,3)$. However, the modified Zoutendijk algorithm fails to converge to this point, instead exhibiting *jamming behavior* near $(0,0,0)$. This phenomenon occurs because the step sizes along the computed search directions diminish to zero (or vanish entirely), preventing further progress toward the true optimum. Such stagnation at non-optimal points represents a key limitation of the method in certain problem settings.

We now shift our focus to the Frank-–Wolfe algorithm, a projection-free method that contrasts with Zoutendijk in its treatment of the feasible set.

3.3. Revised Version of Frank–Wolfe Algorithm

This adaptation of the Frank–Wolfe algorithm replaces classical gradients with *radial epiderivatives*, a generalized derivative concept from nonsmooth analysis. The modified algorithm proceeds as follows:

1. At the current iterate x_k, compute the radial epiderivative $f^r(x^k; d^k)$;
2. Solve the direction-finding subproblem:

$$\min_{d \in S} f^r(x_k; d^k)$$

   to obtain a feasible descent direction d^k;
3. Update via $x^{k+1} = x^k + \alpha_k(d^k - x^k)$, where $\alpha_k \in [0,1]$ is the step size.

By employing radial epiderivatives in the direction-finding subproblem, the algorithm extends to functions that may lack differentiability or convexity, while maintaining the original method's simplicity of solving linear subproblems.

3.3.1. Algorithm

Algorithm 2 Revised Frank–Wolfe Algorithm

```
Input: $f(x)$ (nonconvex, nondifferentiable), compact polyhedral set $S \in \mathbb{R}^n$.
Initialization: Choose $x^0 \in S$                                ▷ Initial feasible point
Set tolerance $\epsilon > 0$, iteration counter $k = 0$
Main Loop: while True do
    Step 1: Determine descent direction $d^k$
    Solve the direction-finding subproblem:
        $d^k = \arg\min_{d \in S} f^r(x^k; d^k)$
    Step 2: Compute step size $\alpha^k$
    Solve the line search:
        $\alpha^k = \arg\min_{0 \le \alpha \le 1} f\left(x^k + \alpha(d^k - x^k)\right)$
    Update iterate:
        $x^{k+1} = x^k + \alpha^k(d^k - x^k)$
    if $\|x^{k+1} - x^k\| < \epsilon$ then
        Stop                                                      ▷ Convergence achieved else
        $k \leftarrow k + 1$                                      ▷ Continue to next iteration
Output: $x^{k+1}$ is the optimal point
```

The Frank–Wolfe algorithm offers two step size strategies:

- Solving a subproblem to compute λ_k, or
- Applying the fixed step size formula $\lambda_k = \dfrac{k}{k+2}$.

Algorithm 2 employs the line search approach (solving a subproblem method), while Algorithm 3 uses the fixed step size approach.

Let $S \subset \mathbb{R}^n$ be a compact polyhedron set and $f : S \to \mathbb{R}$ a nonconvex and nondifferentiable function. The revised version of Frank–Wolfe algorithm solves: $\min_{x \in S} f(x)$.

Algorithm 3 Revised Frank–Wolfe Algorithm

```
Input: $f(x)$ (nonconvex, nondifferentiable), compact polyhedral set $S \in \mathbb{R}^n$.
Initialization: Choose $x^0 \in S$                                ▷ Initial feasible point
Set tolerance $\epsilon > 0$, iteration counter $k = 0$
Main Loop: while True do
    Step 1: Determine descent direction $d^k$
Solve the direction-finding subproblem:
        $d^k = \arg\min_{d \in S} f^r(x^k; d^k)$
    Step 2: Compute step size $\alpha^k$
        $\alpha^k = \frac{k}{k+2}$                                ▷ Predetermined step size
Update iterate:
        $x^{k+1} = x^k + \alpha^k(d^k - x^k)$
    if $\|x^{k+1} - x^k\| < \epsilon$ then
    Stop                                                         ▷ Convergence achieved
    else
    $k \leftarrow k + 1$                                         ▷ Continue to next iteration
Output: $x^{k+1}$ is the optimal point
```

3.3.2. Illustrative examples

Example 3.7. *Minimize*

$$f(x_1, x_2) = (x_1 + 3) - 3(x_2 - 1) - 2(|x_1 - 1| + |x_2 - 2|)$$

subject to

$$\begin{gathered} -x_1 + x_2 \leq 2, -x_1 + 2x_2 \leq 6 \\ 3x_1 + x_2 \leq 21, \quad x_1 + x_2 \leq 9 \\ -x_1, x_2 \leq 0 \end{gathered}$$

We know from Theorem 2.8 that

$$f^r(\overline{x}; d) = d_1 - 3d_2 - 2|d_1| - 2|d_2|$$

Following the framework outlined in Algorithm 2, we proceed to solve the optimization problem.

Iteration 1.

Let $x^0 = (0,0) \in S$.

1. Find $d^1 = (d_1, d_2)$ by solving

$$\begin{array}{cc} \text{Minimize} & d_1 - 3d_2 - 2|d_1| - 2|d_2| \\ \text{Subject to} & d_1, d_2 \in S \end{array}$$

Checking the value of objective function of the subproblem we can conclude that $d^1 = (4,5)$ Therefore,

$$x^1 = x^0 + \alpha(d^1 - x^1) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \alpha \left[\begin{pmatrix} 4 \\ 5 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right] = \begin{pmatrix} 4\alpha \\ 5\alpha \end{pmatrix}$$

2. The step size can be determined by solving

$$\begin{array}{cc} \text{Minimize} & f(x^1) = -11\alpha + 6 - 2(|4\alpha - 1| + |5\alpha - 2|) \\ \text{Subject to} & 0 \leq \alpha \leq 1 \end{array}$$

Solving this we obtain $\alpha = 1$ and

$$x^1 = x^0 + \alpha(d^1 - x^0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + 1 \cdot \left[\begin{pmatrix} 4 \\ 5 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right] = \begin{pmatrix} 4 \\ 5 \end{pmatrix}.$$

Iteration 2.

We have $x^1 = (4,5)$ which is feasible, now we search for a descent direction.

1. Find $d^2 = (d_1, d_2)$ by solving

$$\begin{array}{cc} \text{Minimize} & d_1 - 3d_2 - 2|d_1| - 2|d_2| \\ \text{Subject to} & d_1, d_2 \in S \end{array}$$

The minimizer of this subproblem is $d^2 = (4,5)$. Therefore,

$$x^2 = x^1 + \alpha(d^2 - x^1) = \begin{pmatrix} 4 \\ 5 \end{pmatrix} + 1 \cdot \left[\begin{pmatrix} 4 \\ 5 \end{pmatrix} - \begin{pmatrix} 4 \\ 5 \end{pmatrix} \right] = \begin{pmatrix} 4 \\ 5 \end{pmatrix} = x^1$$

There is no need to continue with the iteration because no matter what value of α we obtain, it will not affect x^2. This means that $x^* = x^1 = (4,5)^T$.

When revisiting Example 3.7 using Algorithm 3, we employ a predetermined step size rule, setting $\lambda_k = \dfrac{k}{k+2}$, eliminating the need to solve an additional subproblem for step size determination. The initial iterations of this approach, as illustrated in Table 3.5, demonstrate that the sequence of solutions progressively converges to the optimal point.

Table 3.5. *Convergence of the revised FWA with fixed step sizes*

Iteration	d^k	x_k	α_k	$f(x_k)$	x_{k+1}
1	$(4,5)$	$(0,0)$	$2/3$	0	$(8/3,10/3)$
2	$(4,5)$	$(8/3,10/3)$	$1/2$	-7.33	$(10/3,25/6)$
3	$(4,5)$	$(10/3,25/6)$	$2/5$	-12.166	$(18/5,9/2)$
4	$(4,5)$	$(18/5,9/2)$	$1/3$	-14.1	$(56/15,14/3)$
5	$(4,5)$	$(56/15,14/3)$	$2/7$	-15.066	$(80/21,100/21)$
6	$(4,5)$	$(80/21,100/21)$	$1/4$	-15.964	$(27/7,135/28)$
7	$(4,5)$	$(27/7,135/28)$	$2/9$	-16.194	$(35/9,175/36)$
8	$(4,5)$	$(35/9,175/36)$	$1/5$	-16.355	$(176/45,44/9)$
$\vdots$	$(4,5)$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	$(4,5)$	$\cdots$	0	-17	$x^* = (4,5)$

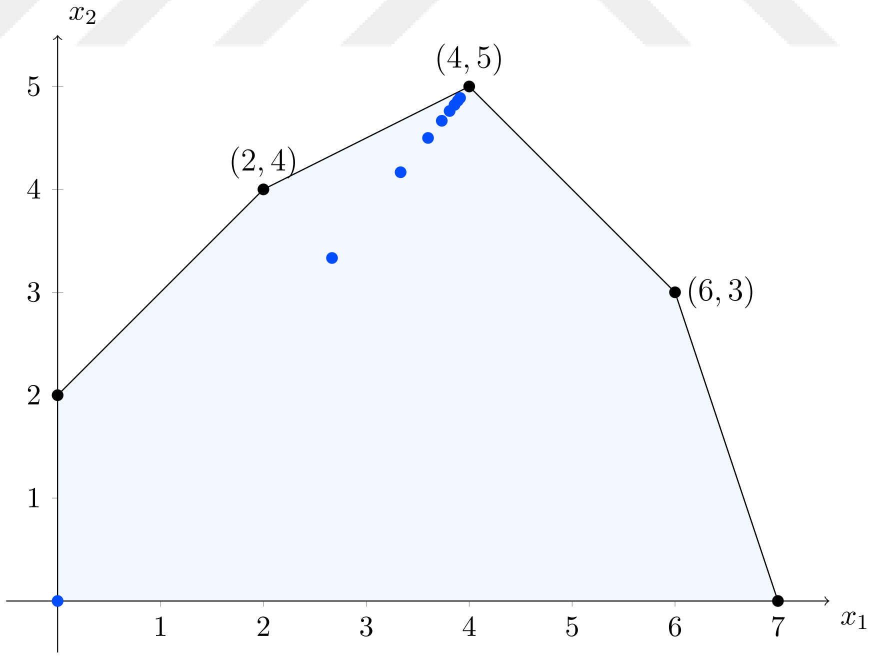

Figure 3.2. *Sequence $x_1, x_2, \cdots \to x^* = (4,5)$ generated by FWA*

Example 3.8. *Minimize*

$$\begin{aligned} f(x_1, x_2, x_3, x_4) = {} & 5(x_1 - 1) + 3(x_2 - 2) + 4(x_3 - 2) - (x_4 - 1) \\ & - 2\left(|x_1 - 2| + |x_2 + 2| + |x_3 - 1| + |x_4|\right) \end{aligned}$$

Subject to

$$\begin{aligned} 5x_1 + 2x_2 - 3x_3 + 2x_4 &\leq 12 \\ -x_1 + 2x_2 - 5x_3 - 8x_4 &\leq 8 \\ 2x_1 - x_3 + x_4 &\leq 20 \\ 3x_1 + x_2 + x_3 + 3x_4 &\leq 16 \\ x_1,\ x_2,\ x_3,\ x_4 &\geq 0. \end{aligned}$$

Table 3.6. *Frank–Wolfe Iteration Summary of Ex. 3.8*

k	x_k	$f(x_k)$	f^r	d_k	α_k	x_{k+1}
0	$(0, 0, 0, 0)$	-25	-31.1	$(4.2, 0, 3.1, 0)$	1	$(4.2, 0, 3.1, 0)$
1	$(4.2, 0, 3.1, 0)$	-38.1	-31.1	$(4.2, 0, 3.1, 0)$	1	$(4.2, 0, 3.1, 0)$

3.3.3. Comparison of the two FWA variants

The Frank–Wolfe algorithm's convergence behavior depends on the step size strategy used.

- **Exact Line Search (Algorithm 2):** This variant may converge in a single iteration under favorable conditions (e.g., polyhedral feasible sets) by solving a line search subproblem at each step. However, the extra optimization required per iteration can be computationally expensive.
- **Predetermined Step Size (Algorithm 3):** Using a rule like $\lambda_k = \frac{k}{k+2}$, this approach avoids subproblem solving and ensures asymptotic convergence at rate $O(1/k)$ for convex problems. It is more scalable for large problems despite slower convergence.

In practice, Algorithm 3 is often preferred for its simplicity and lower per-iteration cost, while Algorithm 2 offers faster convergence at higher computational expense. The choice depends on problem structure and performance trade-offs.

4. CONVERGENCE ANALYSIS

4.1. Modified Method of Zoutendijk

The Zoutendijk feasible direction method is an iterative approach used in constrained optimization. Starting from an initial feasible point x^0 in a compact convex polyhedron $S \subset \mathbb{R}^n$, the method generates a sequence $\{x^k\}$ by moving in a feasible descent direction d^k. At each iteration, a step size λ_k is determined such that the new iterate

$$x^{k+1} = x^k + \lambda_k d^k$$

remains in S. The step size is chosen as the maximum value for which the motion does not violate feasibility, typically stopping when a new constraint becomes active. In practice, the algorithm terminates when $\lambda_k = 0$ (or falls below a tolerance ϵ), indicating that further progress is not possible in the computed descent direction. However, a major drawback of this method is the *jamming* phenomenon: due to selective handling of constraints (using only binding constraints to determine d^k and only nonbinding constraints to bound the step size), the algorithm may zigzag around a nonoptimal point with diminishing step sizes, potentially converging to a point that is not the optimal point. Modifications such as the Topkis–Veinott algorithm have been proposed to address this issue by considering all constraints in the direction-finding subproblem.

4.1.1. Convergence analysis

We now present a formal result on the finite termination of a modified Zoutendijk method that incorporates a bookkeeping mechanism for face visits.

Theorem 4.1 (Finite Termination of the Modified Zoutendijk Method). *Assume that:*

1. *$S \subset \mathbb{R}^n$ is a compact convex polyhedron with finitely many faces.*
2. *At each iteration k, a feasible descent direction d^k is computed such that $f^r(x^k; d^k) < 0$.*

3. The step size λ_k is chosen as the maximum value satisfying

$$x^{k+1} = x^k + \lambda_k d^k \in S,$$

and a bookkeeping procedure ensures that each face of S is visited at most twice. Then, the modified Zoutendijk feasible direction algorithm terminates in a finite number of iterations.

Proof. Face Visitation Argument:

1. *Polyhedral Face Structure*: Since S is a compact convex polyhedron, it has finitely many faces by Minkowski's theorem.
2. *Iterate Transition*: At each iteration k, the maximal step size λ_k is determined by the first inactive constraint becoming active, forcing the transition:

$$x^{k+1} \in \mathcal{F}_j \subset S \setminus \mathcal{F}_i$$

   where $\mathcal{F}_i : , \mathcal{F}_j$ are distinct faces of S.
3. *Anticycling Mechanism*: Using Bland's rule from linear programming, no face is visited more than twice, satisfying:

$$\sum_{m=1}^{M} (\mathcal{F}_m \text{ visited}) \leq 2M \quad (M = \text{total faces})$$

Termination Guarantee: Assume infinite iterations. Then:

$$\exists \text{ infinite sequence } \{\mathcal{F}_k\} \subset S \quad \text{with} \quad \mathcal{F}_k \neq \mathcal{F}_{k+1}$$

But this contradicts the finite face structure ($M < \infty$) and Bland's anticycling condition. Therefore, $\exists K < \infty$ where $\lambda_K < \epsilon$, forcing termination. □

Remark 3. *Finite termination does not necessarily imply that the limit point is optimal. The algorithm may converge to a non-KKT point in practice due to the* jamming *phenomenon. To mitigate this, practical implementations often include:*

- *A tolerance threshold ϵ so that if $\lambda_k < \epsilon$, the algorithm stops.*
- *Modifications in the direction-finding subproblem (e.g., incorporating all constraints) to avoid jamming.*

This method does not guarantee convergence to optimal solutions, nor can it be relied upon to avoid premature termination at non-optimal points, mirroring the convergence failure of classical Zoutendijk methods.

4.2. Revised Version of Frank–Wolfe Algorithm

The Frank–Wolfe algorithm, also known as the conditional gradient method, is another widely used technique in constrained convex optimization. Given a convex objective function f and a compact convex feasible set $S \subset \mathbb{R}^n$, the algorithm proceeds by linearizing the objective function at the current iterate x^k and solving the linear subproblem:

$$d^k = \arg\min_{d \in S} f^r(x^k, d^k).$$

The next iteration is then formed by taking a convex combination of the current point x^k and direction d^k:

$$x^{k+1} = x^k + \gamma_k (d^k - x^k),$$

where the step size γ_k may be chosen via line search or by a predefined rule such as $\gamma_k = \dfrac{2}{k+2}$. The hallmark of the Frank–Wolfe method is that it uses the entire constraint set in its linear minimization oracle, which ensures that both the descent direction and the step size account for all constraints.

4.2.1. Convergence analysis

The convergence rate of the Frank–Wolfe algorithm has been well established under standard smoothness assumptions.

Theorem 4.2 (Modified Convergence of Radial Frank–Wolfe). *Let $f : S \to \mathbb{R}$ be a proper, lower semicontinuous, and radially epidifferentiable function over a compact convex set $S \subset \mathbb{R}^n$. Suppose there exists a constant $C > 0$ such that for all*

$x \in S$, $d \in S$, and $\gamma \in [0,1]$, the inequality holds:

$$f(x + \gamma(d - x)) \leq f(x) + \gamma f^r(x; d - x) + C\gamma^2 \tag{4.1}$$

Let the iterates be generated by:

$$d_k \in \arg\min_{d \in S} f^r(x_k; d - x_k), \quad x_{k+1} = x_k + \gamma_k(d_k - x_k), \quad \gamma_k = \frac{2}{k+2}.$$

Then:

$$\min_{0 \leq i \leq k} f^r(x_i; d_i - x_i) \leq \frac{2(f(x_0) - f_{\inf})}{k+2}, \quad \textit{where } f_{\inf} := \inf_{x \in S} f(x).$$

Proof. Sketch of Convergence Analysis. Following a standard technique, we define the step size $\gamma_k = \frac{2}{k+2}$ and track the optimality gap $\Delta_k = f(x_k) - f_{\inf}$. Using a recursive inequality involving the radial epiderivative and summing over iterations, we obtain a bound on the minimum value encountered.

$$\min_{0 \leq k \leq N} f^r(x_k; d_k - x_k) \leq \frac{2(f(x_0) - f_{\inf})}{N+2}.$$

The full proof can be derived by imitating the approach in [36]. □

Remark 4. *For nearly linear functions, where the curvature constant is nearly zero, the Frank–Wolfe algorithm may converge in a single iteration. This is because the linear approximation of f becomes almost exact, and the solution of the linear subproblem is optimal for the original problem.*

4.3. Comparison Between the Two Methods

Constraint Handling: The key difference between the modified Zoutendijk method and the revised version of Frank–Wolfe algorithm lies in how they handle constraints:

- Zoutendijk Method: Uses only the binding constraints at x^k to determine the descent direction and uses the nonbinding constraints to set an upper bound for the step size.
- Frank–Wolfe Algorithm: Solves a linear subproblem over the entire feasible set S,

thus incorporating both binding and nonbinding constraints in its determination of the search direction and step size.

Convergence Behavior:

- The Zoutendijk method, even in modified form, is sensitive to the number of constraints, with iteration counts potentially growing polynomially due to the polyhedral structure.
- The Frank-Wolfe algorithm has a sublinear convergence rate of $O(1/k)$ under smoothness assumptions, independent of the number of constraints. For nearly linear objectives, it can converge quickly, sometimes in a single iteration.

Practical Implications: In practical implementations, the choice between these methods depends on problem structure and performance requirements:

- For problems with a large number of constraints or where the structure of S is complex, the full constraint treatment of the Frank–Wolfe algorithm may yield more robust and efficient performance.
- The modified Zoutendijk method, while guaranteeing finite termination under strict bookkeeping, may require additional modifications to avoid jamming.

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CURRICULUM VITAE

ORCID ID : 0000-0002-4658-4377
Full Name : Mohamed Muhumed HASSAN
Foriegn Languages : English, Turkish, Arabic

Education and Professional Background:

- 2021—2025, Eskişehir Technical University, Institute of Graduate Programs, Department of Mathematics, PhD in Applied Mathematics, Eskişehir, Türkiye.
- 2017—2020, Anadolu University, Institute of Graduate Programs, Department of Mathematics, MSc in Applied Mathematics, Eskişehir, Türkiye.
- 2011—2015, Mogadishu University, Faculty of Education and Humanities, Department of Mathematics, BEd in Mathematics *and* Physics, Mogadishu, Somalia.